Thermal vs Gibbs-Preserving operations: A Dynamical Maps Perspective

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1 Introduction

What are the minimal resources required to implement general Gibbs-preserving maps using physically reasonable free operations in a one-shot thermodynamic framework?

2 Dynamical maps

Any qubit state can be written in the affine basis as:

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \tag{1}$$

where $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.
For a qubit Hamiltonian $H = \frac{\omega}{2}\sigma_z$, the Gibbs state at inverse temperature $\beta = 1/(k_B T)$ is:

$$\rho_G = \frac{e^{-\beta H}}{Z} = \frac{1}{2}(I + r_G \sigma_z) \tag{2}$$

where the thermal polarization is:

$$r_G = \tanh\left(\frac{\beta\omega}{2}\right) \tag{3}$$

2.1 Thermal Operations

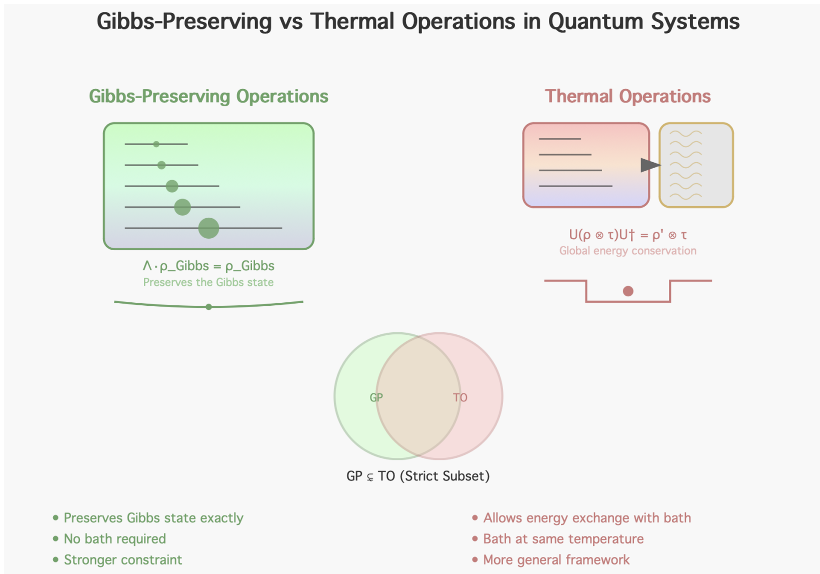
Thermal operations Φ_{th} are defined by:

$$\Phi_{\text{th}}(\rho) = \text{Tr}_B[U(\rho \otimes \tau_B)U^\dagger] \tag{4}$$

$$\text{where } \tau_B = \frac{e^{-\beta H_B}}{Z_B}, \quad [U, H_S + H_B] = 0 \tag{5}$$

Key properties:

- Always time-covariant (phase-covariant): $\Phi_{\text{th}}(e^{-iHt}\rho e^{iHt}) = e^{-iHt}\Phi_{\text{th}}(\rho)e^{iHt}$
- Preserve Gibbs state: $\Phi_{\text{th}}(\tau_S) = \tau_S$ where $\tau_S = e^{-\beta H_S}/Z_S$
- Cannot create coherence between energy eigenstates



2.2 Gibbs-Preserving Maps

A completely positive trace-preserving map Φ is Gibbs-preserving if:

$$\Phi(\tau_S) = \tau_S \quad \text{where } \tau_S = \frac{e^{-\beta H_S}}{Z_S} \tag{6}$$

Crucially, Gibbs-preserving maps can be:

- Time-covariant: $\Phi(e^{-iHt}\rho e^{iHt}) = e^{-iHt}\Phi(\rho)e^{iHt}$
- Non-time-covariant: Can create coherence between energy eigenstates

2.3 Set Relationships

$$\{\text{Thermal Operations}\} \subset \{\text{Time-covariant Gibbs-preserving}\} \subset \{\text{All Gibbs-preserving}\} \tag{7}$$

2.4 Affine-form

Any qubit channel can be written in affine form as:

$$\vec{r'} = T\vec{r} + \vec{t} \tag{8}$$

where T is a $3 \otimes 3$ matrix and $\vec{t}$ is a translation vector.

For the map to preserve the Gibbs state, we require:

$$\Lambda(\rho_G) = \rho_G \tag{9}$$

$$\Lambda_t[\,\cdot\,] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda(t)\cos[\phi] & \lambda(t)\sin[\phi] & 0 \\ 0 & -\lambda(t)\sin[\phi] & \lambda(t)\cos[\phi] & 0 \\ \tau_z(t) & 0 & 0 & \lambda_z(t) \end{pmatrix}$$

$$\frac{1}{2}(1 + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z) \xrightarrow{\Lambda_t[\cdot]} \frac{1}{2}(1 + \lambda(t)a_x\sigma_x + \lambda(t)a_y + (\tau_z(t) + \lambda_z(t)a_z)\sigma_z)$$

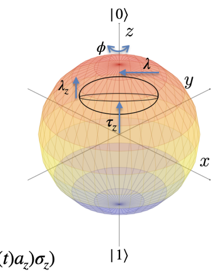


Figure 1: Phase covariant dynamical map and its effect on a qubit bloch sphere. PC maps are defined by isotropic stretching in the x and y direction and anisotropic stretching in z alongwith a shift in the z direciton.

This translates to the constraint:

$$T\vec{r}_G + \vec{t} = \vec{r}_G \tag{10}$$

where $\vec{r}_G = (0, 0, r_G)$.

Expanding the Gibbs-preserving constraint:

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r_G \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r_G \end{pmatrix} \tag{11}$$

This gives us:

$$T_{13}r_G + t_x = 0 \quad \Rightarrow \quad t_x = -T_{13}r_G \tag{12}$$
$$T_{23}r_G + t_y = 0 \quad \Rightarrow \quad t_y = -T_{23}r_G \tag{13}$$
$$T_{33}r_G + t_z = r_G \quad \Rightarrow \quad t_z = r_G(1 - T_{33}) \tag{14}$$

Therefore, the general form becomes for the gibbs preserving map is:

$$\begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} + \begin{pmatrix} -T_{13}r_G \\ -T_{23}r_G \\ r_G(1 - T_{33}) \end{pmatrix} \tag{15}$$

The translation vector is completely determined by the off-diagonal elements T_{13}, T_{23} and diagonal element T_{33} The thermal state acts as a "attractor" with strength determined by r_G

2.5 Constraints on map

2.5.1 Complete Positivity Constraints

For a qubit channel to be completely positive, the matrix T must satisfy:

$$1 + \lambda_1 + \lambda_2 + \lambda_3 \geq 0 \tag{16}$$
$$1 + \lambda_1 - \lambda_2 - \lambda_3 \geq 0 \tag{17}$$
$$1 - \lambda_1 + \lambda_2 - \lambda_3 \geq 0 \tag{18}$$
$$1 - \lambda_1 - \lambda_2 + \lambda_3 \geq 0 \tag{19}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of T.

2.5.2 Trace Preservaiton

The translation vector must satisfy:

$$|\vec{t}| \leq 1 - \|T\|_1 \tag{20}$$

where $\|T\|_1$ is the trace norm of T.

2.5.3 Bloch Ball

For all unit Bloch vectors $\vec{r}$ with $|\vec{r}| \leq 1$:

$$|T\vec{r} + \vec{t}| \leq 1 \tag{21}$$

2.6 Constraints

These reduce to:

$$|\lambda_i| \leq 1 \quad \text{for all eigenvalues} \tag{22}$$
$$|T_{13}|, |T_{23}| \leq \sqrt{1 - r_G^2} \tag{23}$$
$$|t_x|^2 + |t_y|^2 + |t_z|^2 \leq (1 - \max_i |T_{ii}|)^2 \tag{24}$$

2.7 General Thermal Channel

$$T = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} 0 \\ 0 \\ \tau_z \end{pmatrix} \tag{25}$$

3 The Resource Gap

The above discussion tells us that thermal operations are phase covariant where as Gibbs are not. Here Φ to be non-time-covariant, it must satisfy:

$$\Phi(e^{-iHt}\rho e^{iHt}) \neq e^{-iHt}\Phi(\rho)e^{iHt} \tag{26}$$

for some ρ and t. Then, how can we produce non-time-covariant Gibbs-preserving maps using only thermal operations plus minimal additional resources.

4 Breaking Phase covariance

The condition for phase covariance arises also in systems with time translational symmetry. For the Bloch vector this results in the following transformation: For PC evolution we want:

$$a_z\sigma_z \otimes \mathbf{1} \longrightarrow \lambda_z a_z \otimes \mathbf{1} + \tau_z \tag{27}$$

$$a_y\sigma_y \otimes \mathbf{1} \longrightarrow \lambda a_y\sigma_y \otimes \mathbf{1} \tag{28}$$

$$a_x\sigma_x \otimes \mathbf{1} \longrightarrow \lambda a_x\sigma_x \otimes \mathbf{1} \tag{29}$$

which implies no existence of $\sigma_x \otimes \mathbf{1}$ or $\sigma_y \otimes \mathbf{1}$ other than from the initial state values of $a_{y/x}$ and no shift in x, y. These imply that initial $a_y(\sigma_y \otimes \mathbf{1})$ cannot $\longrightarrow \sigma_z \otimes \mathbf{1}$ term.
Time translation also results in two conditions,

$$(i)[H_E, \rho_E] = 0 \tag{30}$$

$$(ii)[H_S + H_E, H_{SE}] = 0 \tag{31}$$

$$\rho = \sum p_i O_i \qquad\qquad H = \sum \alpha_i O_i \tag{32}$$

$$\rho_s = \sum s_i\gamma_i \qquad\qquad \rho_E = \sum \xi_i E_i \tag{33}$$

s.t $\{\gamma_i \otimes E_j\}$ form $\{O_i\}$ then we can write the system Hamiltonian as, $H_s = \sum_j h_j\gamma \otimes \nVdash$, $H_E = \sum_i k_i \nVdash \otimes E_i$ From the first phase covariance condition we get,

$$\Rightarrow \quad [k_i \nVdash \otimes E_i, \xi_j E_j] = 0 \tag{34}$$

$$\Rightarrow \quad k_i\xi_j[E_i, E_j] = 0 \tag{35}$$

This can result in two cases

1. $E_iE_j = 0$ if $i = j$ (ie same/commute)

2. else the combination of k_i and ξ_j can appear such that they cancel each other out. Consider an exampe of a 1 qubit environment. Let $H_E = k_x\sigma_x + k_y\sigma_y$ and let $\rho_E = \xi_x\sigma_x + \xi_y\sigma_y$ then if $k_x\xi_y = k_y\xi_x$ then the condition will hold.

For the second condition, we have to be a little more thorough.

$$\begin{aligned}
[H_{SE}, H_s + H_E] &= [\alpha_i O_i, h_j\gamma_j \otimes \mathbf{1} + k_i \mathbf{1} \otimes E_i] \\
&= [\alpha_i(O \otimes E_p), h_j\gamma_j \otimes \mathbf{1} + k_i \mathbf{1} \otimes E_i] \\
&= \alpha_i h_j[\gamma_\alpha \otimes E_p, \gamma_j \otimes \mathbf{1}] + \alpha_i k_L[O \otimes E_p, \mathbf{1} \otimes E_i] \\
&= \alpha_i h_j(\gamma_\alpha\gamma_j \otimes E_p\mathbf{1} - \gamma_j\gamma_\alpha \otimes \mathbf{1}E_p) + \alpha_i k_L(\gamma\mathbf{1} \otimes E_pE_i - \mathbf{1}\gamma \otimes E_iE_p) \\
&= \alpha_i h_j([\gamma_\alpha, \gamma_j] \otimes E_p) + \alpha_i k_L(\gamma \otimes [E_p, E_i])
\end{aligned}$$

This gives us ways to break or retain phase covariance depending on the commutation relation and presence of operators in the many body system.

1. $[\gamma_\alpha, \gamma_j] \otimes E_p = 0 \Rightarrow \gamma \otimes [E_p, E_i] = 0$

   $\Rightarrow$ terms # interaction affecting system commute with system Ham terms

   or $\alpha_i h_j$ along with operators cancel out

   or

2. LH term cancels RH.

Note that presence of system environment correlation was not important for determining phase covariance of the local qubit map. How ever correlations in the environment could affect the dynamics.

Lets start by writing the state of the system

$$\begin{aligned}
\rho_{tot} &= \frac{1}{2}(\mathbf{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z) \otimes \frac{z_i}{z_i} E_i && (36)\\
&= \frac{z_i}{2}(\mathbf{1} \otimes E_i) + \frac{a_x z_i}{2}(\sigma_x \otimes E_i) + \frac{a_y z_i}{2}(\sigma_y \otimes E_i) && (37)\\
&\quad + \frac{a_z z_i}{2}(\sigma_z \otimes E_i) && (38)
\end{aligned}$$

$\Rightarrow$ any shift can come from $\frac{\xi_i}{2}(\mathbf{1} \otimes E_i)$

$$\begin{aligned}
\Rightarrow \quad U\frac{\xi_i}{2}(\mathbf{1} \otimes E_i)U^\dagger &= \text{corr terms} + I_X(\sigma_X \otimes \mathbf{1}) && (39)\\
&\quad + I_Y(\sigma_Y \otimes \mathbf{1}) + I_Z(\sigma_Z \otimes \mathbf{1}) && (40)\\
&\quad + (\mathbf{1} \otimes \ldots) && (41)
\end{aligned}$$

If commute $e^{iH_s + H_E + H_{SE}t} = e^{iH_s}e^{iH_E} + e^{iH_{SE}t}$

Breaking PC if environment has terms that don't commute with env. Ham $\Rightarrow$ Shift $\Rightarrow$ Shift can only come from env. Ham terms. P_E

If 1^{st} interaction commutes so $[H_s + H_E, H_{SE}] \neq 0$

If commute $(e^{iH_s} \otimes e^{H_E})e^{H_{SE}} \longrightarrow$ if stationary matrix happen

$e^{H_{SE}}\mathbf{1} \otimes e^{H_E}E_i e^{H_E}\frac{H_{SE}}{C}$ happens