MIPT QuNetBootCamp

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1 Introduction

1.1 Entanglement Growth in Random Unitary Circuits

In isolated quantum many-body systems, entanglement typically grows from initial product states toward highly entangled states. This process is fundamental to understanding thermalization and quantum information dynamics.
For a system evolving under unitary dynamics with Hamiltonian H

$$\psi(t)\rangle = e^{-iHt/\hbar}\psi(0)\rangle \tag{1}$$

However, studying generic dynamics is challenging. Random unitary circuits provide a tractable model that captures universal features of chaotic quantum dynamics.

1.1.1 Random Unitary Circuits

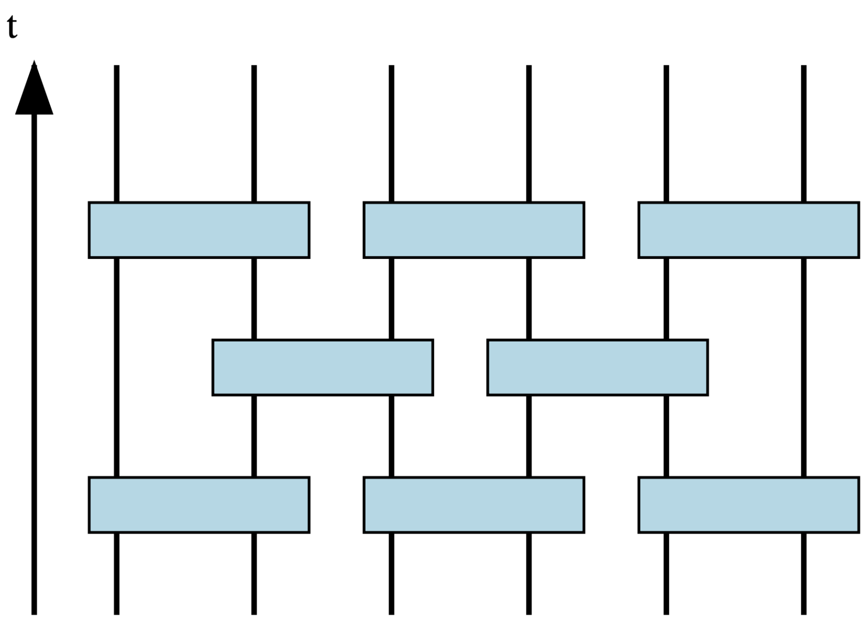
A random unitary circuit consists of:

- Discrete time steps
- Local unitary gates applied to neighboring qubits
- Gates chosen randomly from the Haar measure

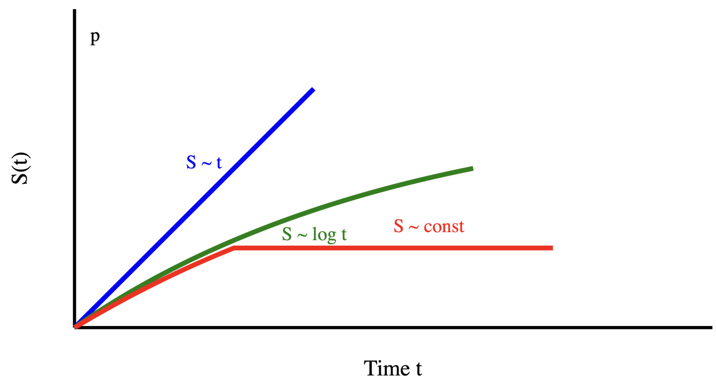
You can also construct constrained dynamics using gates that preserve some quantity like PSWAP Gates that preserve excitation number.
While random unitaries provide maximal scrambling, constrained dynamics like PSWAP (partial-swap) gates offer insight into how conservation laws affect entanglement growth. The PSWAP gate acts on two qubits as:

$$PSWAP_{ij} = c|01\rangle\langle 10| + c^*|10\rangle\langle 01| \tag{2}$$

This preserves particle number while allowing controlled exchange.



Random Unitary Circuit (Brickwork)



2 Entanglement Growth as a Function of Time

2.0.1 Entanglement Entropy

For a bipartite system divided into regions A and B, the entanglement entropy is:

$$S_n(A) = \frac{1}{1n} log_{2[Tr(\rho_A^n)]}(3)$$

where $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$is the reduced density matrix. The von Neumann entropy (n→1) is:

$$S_1(A) = Tr(\rho_A log_2 \rho_A) \tag{4}$$

In random unitary circuits without measurements:

- Short times (tL): Linear growth $S(t) \sim v_E t$
- Long times $(\gg L)$: Volume law saturation $S \sim L$

The entanglement velocity v_E depends on the specific dynamics eg whether the gates are constrained or not.

3 1+1 Dimensions...because its easy

Consider a 1D chain of L qubits evolving under:

- Unitary dynamics: Random two-qubit gates applied in a brickwork pattern
- Measurements: Each qubit measured with probability p after each unitary layer

Then we have a case where there is a competition between the unitary evolution that generates entanglement and projective measurements that destroy entanglement. This leads to a phase transition at critical measurement rate p_c

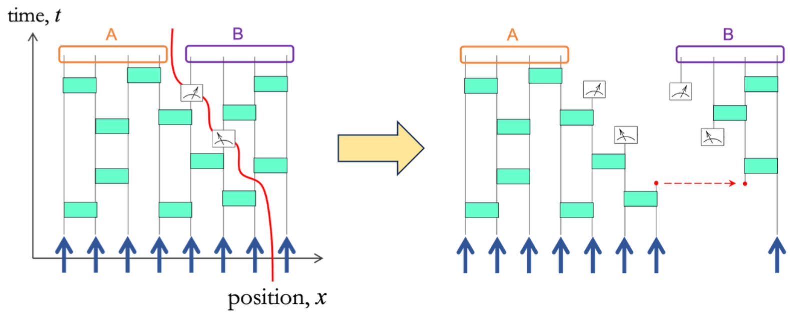


Figure 1: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

3.1 How Measurements Reduce Entanglement

To understand this we have to understand what projective measurements do.
A projective measurement in the computational basis: Projects the qubit into $|0\rangle$ or $|1\rangle$. The outcome probabilities for a single qubit would be given by: $p_0 = |\langle 0|\psi\rangle|^2$ and $p_1 = |\langle 1|\psi\rangle|^2$. After measurement the post-measurement state

$$|\psi'\rangle = P_i|\psi\rangle/\sqrt{p_i} \tag{5}$$

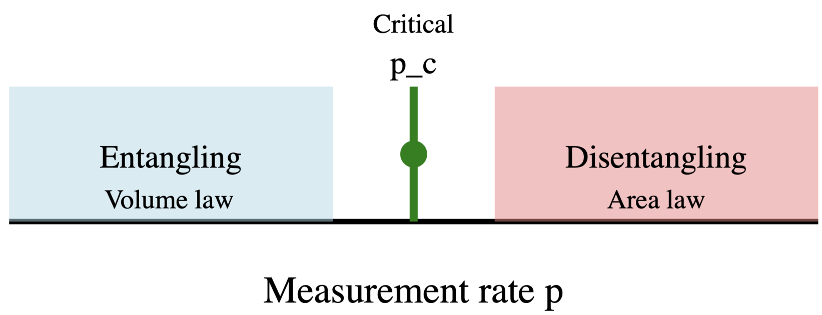
So what does this do to the entanglement? Lets take an example.
Consider a Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Before measurement the entanglement between the qubits is maximal and the state of each individual qubit is maximally mixed.
After measuring qubit 1: if outcome is 0:$|\psi'\rangle = |00\rangle$ and the entanglement is now 0.
Hence, the measurement destroys entanglement by collapsing superpositions. This is actually a crucial ingredient of MIPT. After every measurement you pick on the the pure state options and work with that.

4 Measurement Protocol: Pure State Evolution vs. Mixed States

Track individual quantum trajectories, not ensemble averages. For a specific measurement record $\{m_i\}; |\psi_{\{m_i\}}\rangle = (normalized state after measurements)$. The density matrix averaged over outcomes:

$$\rho_{avg} = \sum_{\{m_i\}} p(\{m_i\})|\psi_{\{m_i\}}\rangle\langle\psi_{\{m_i\}} \tag{6}$$

MIPT and the phase transition appears in individual trajectories, not in ρ_{avg}!



5 Phase transition

At the critical point p_c: Entanglement grows logarithmically: $S(t) \sim A \log t$. We expect that the there are scale-invariant correlations, which means that the entanglement entropy of 2 qubit subsytems, 3 qubit subsytems, 4, 5...and so on look the same. This would also correspond to a decay of mutual information with distance. The expected correlation length is $\xi \sim |p - p_c|^{-\nu}$

6 Fun math trick: Mapping to KPZ Universality

6.1 The Minimal Cut Picture

The zeroth Rényi entropy $S_0 = \log_2(N)$ (where N = number of nonzero eigenvalues) can be computed via minimal cut (MC). MC says that compute the minimum number of bonds that must be cut to separate regions A and B. This is to say that measurements act as "broken bonds" in the percolation problem.

7 Classical Percolation Mapping

We can now list the scaling at different Phases:

- Entangling phase ($p < p_c$)$\rightarrow$ Minimal cut must traverse O(t) unbroken bonds $\rightarrow S_0(t) \sim v_0 t$
- Critical point ($p = p_c$)$\rightarrow$Scale-invariant percolation clusters$\rightarrow S_0(t) \sim A \ln t$
- Disentangling phase ($p > p_c$)$\rightarrow$Minimal cut finds paths through broken bonds$\rightarrow S_0(t) \sim O(1)$

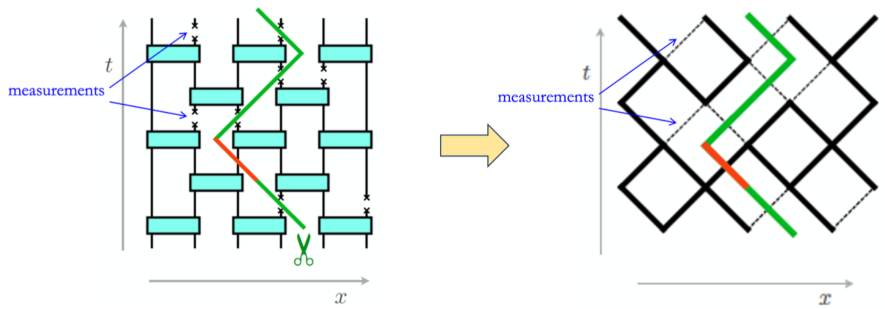


Figure 2: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

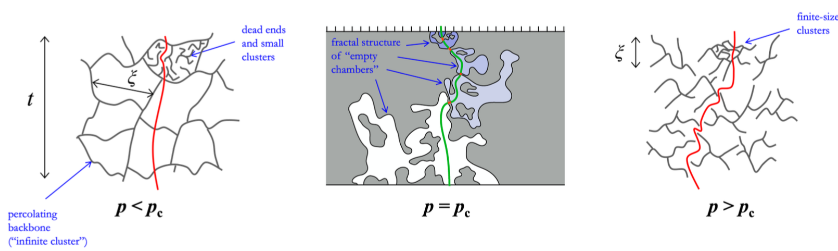


Figure 11: Pictures of the minimal cut in a random network for different measurement regimes.

Figure 3: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

8 References

- Lecture Notes: Introduction to random unitary circuits and the measurement-induced entanglement phase transition