MIPT QuNetBootCamp

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1 Why should I care bout correlations? Seems like a stats thing...

Correlations between subsystems within a closed quantum system can be signatures of the class of dynamics the system is undergoing. Both the value distribution and the temporal evolution depend on the global constraints and the local operations. Correlations are also of mathematical and physical interest because they are not necessarily accessible by finite sized observers or observations. For example, consider two two-qubit universes. We index each of the qubit A and B. Let one be in a pure Bell state $\psi_1 = \frac{1}{2}(|01\rangle + |10\rangle)$ that has maximal entanglement and the other in an uncorrelated state where each qubit is maximally mixed. The density matrices would be,

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 0,0,0,0 \\ 0,1,1,0 \\ 0,1,1,0 \\ 0,0,0,0 \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} \frac{1}{2},0 \\ 0,\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2},0 \\ 0,\frac{1}{2} \end{pmatrix} \tag{1}$$

If observers only had access to local subsystems then both in the universe 1 and universe 2 the local subsystems would look like the density matrix is

$$\rho_{A,B} = \begin{pmatrix} \frac{1}{2},0 \\ 0,\frac{1}{2} \end{pmatrix} \tag{2}$$

This would mean that if the observers made their predictions based on this "local" information, their theories might be misleading. For example, universe 1 has coherence and entanglement that it can use to perform rotations, get extractable work that are impossible for universe 2.
In the systems we have seen this can appear in this way. Consider two qubit dynamical map between two uncorrelated thermal state qubits with z-Bloch vector $a_{1,2}$ would look like

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & 0 \\ a_2 \sin^2\theta & 0 & 0 & \cos^2\theta \end{pmatrix} \tag{3}$$

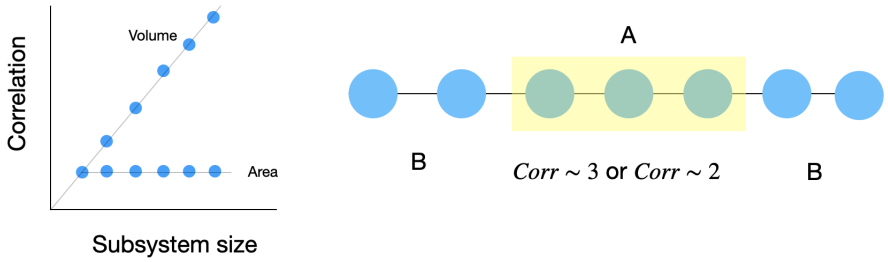


Figure 1: If correlation was volume law then it would grow or be proportional to the number of subsystems inside ie 3. If it is area law it would depend on how many links connect at the boundary of the surface which in this case is 2.

However, if the qubits had correlation in the equal energy subspace, the map gets transformed such that the shift, i.e. the lower, leftmost component becomes $\tau_z = a_j \sin^2\theta \pm C \sin 2\theta$. So if in your universe you only had access to one qubit measurements and you built the universe out of that, you may end up getting dynamics that cannot be explained by your physics.
This should convince you that correlations are important to understand.

1.1 Classes of correlation growth

How correlation spreads and develops can be a diagnostic for the type of evolution and late time state of your system. If you have evolution that is unconstrained, the expectation is that with sufficiently many random unitary operations, most subsystems start looking maximally mixed.
Furthermore, categorizing correlations into classed allows us to describe how quantum correlation is distributed in a system. If you know how where the correlations reside, you can reduce the size of the subspace to evolve and track and therefore reduce simulation complexity.

Studying entanglement or correlation dynamics can also help understand the question of thermalization that asks if quantum system that is completely isolated from its environment will be able to "act as its own bath". Consider a very large system, which we will divide into a macroscopic subsystem A and its (macroscopic) complement B. Suppose that the entire system is in an energy eigenstate ψ with energy E. One can associate a temperature T with this eigenstate by saying that one would get the same energy (in expectation value) from a classical Boltzmann distribution if the temperature had a specific value T. Such a classical Boltzmann distribution has a corre- sponding density matrix $\rho_{eq}(T)$ that is boring and classical: its diagonal entries are proportional to $e^(Ei/kBT)$, where Ei is the energy of the eigenstate i, and the off-diagonal entries of $\rho_{eq}(T)$ are all zero. The ETH asserts that, if the energy eigenstate is "thermalized",

then the reduced density matrix ρ_A for the subsystem A satisfies

$$\rho_A = \mathrm{Tr}_B(\rho)$$

(4)

In other words, tracing out everything outside of the region A leaves you with something that looks like the usual classical thermodynamics. But this statement of ETH can only be satisfied if the subsystem A has extensive entanglement with its environment, since an equilibrium distribution at finite temperature has extensive entropy.

We know of many systems that satisfy ETH. So the question of how a system thermalizes – how it goes from something with little entanglement to something with a sufficiently extensive entanglement to satisfy ETH – is a question about entanglement dynamics. The random unitary circuit provides us with a tool to study some kind of "generic entanglement dynamics", where entanglement forms more or less as quickly as allowed by spatial locality. Of course, in some particular situation this generic dynamics may or may not be impeded by the formation of some conserved quantity that bottlenecks the growth of entanglement.
Clasically, we expect that the entanglement entropy (another term for correlation between a subsystem and its complement. More on this soon.) is an extensive property that scales with the volume or the amount of matter in the system under consideration. If there are more things inside, then there are more ways they can be connected to the outside so there should be a proportionality there. Turn out, in quantum mechanics that is not always the case and you can in fact have systems where the correlation measure grows sub-volume rate.

2 Entanglement Entropy

Entropy in classical physics and information theory characterizes the uncertainty in the state of a physical system. I prefer the operational definition that says, entropy quantifies the amount of new information learnt after identifying a particular microstate that is compatible with the macrostate of the system. This can be given by the von Neumann entropy

$$S(\rho_A) = -\mathrm{Tr}[\rho_A ln(\rho_A)] \tag{5}$$

From this pov, the entanglement between a region and its complement is a measure of how much the system is correlated to its complement. If the entanglement entropy as a function of subsystem size is **area law**, then the correlation grows proportionally to the boundary (surface area) of the region. Physically you can think of this as the entanglement primarily being generated by interactions near

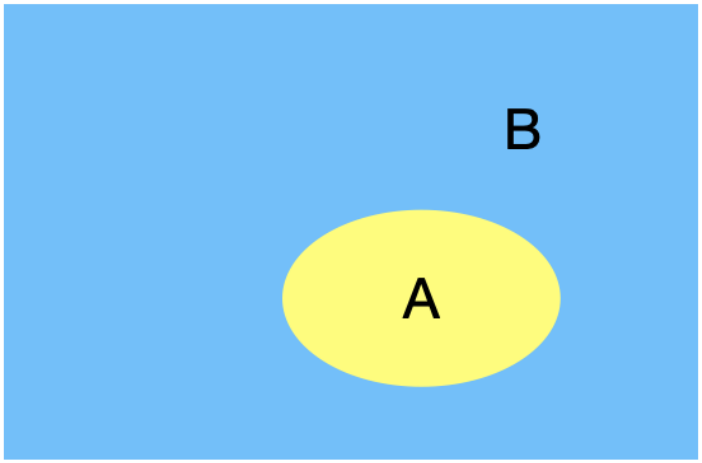


Figure 2: How much is A correlated with B

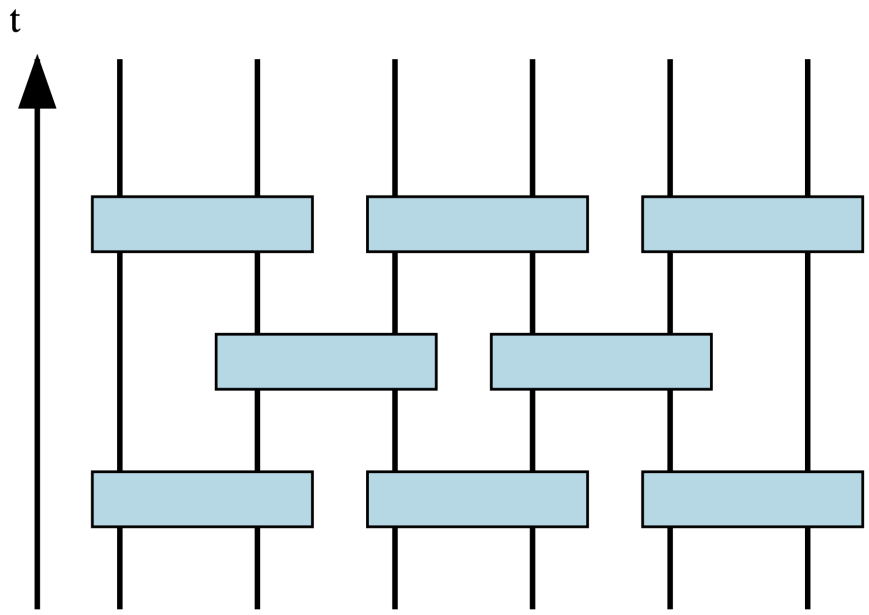
the boundary. If the entanglement entropy as a function of subsystem size is **volume law** then the entanglement between a region and its complement grows proportionally to the volume of the region. This suggests that the entanglement is distributed throughout the entire region.

2.1 Entanglement Growth in Random Unitary Circuits

In isolated quantum many-body systems, entanglement typically grows from initial product states toward highly entangled states. This process is fundamental to understanding thermalization and quantum information dynamics. The faster, wider and strongly the entanglement grows the more the system gets scrambled.
For a system evolving under unitary dynamics with Hamiltonian H

$$\psi(t)\rangle = e^{-iHt/\hbar}\psi(0)\rangle \tag{6}$$

However, studying generic dynamics is challenging. Random unitary circuits provide a tractable model that captures universal features of chaotic quantum dynamics.



Random Unitary Circuit (Brickwork)

2.1.1 Random Unitary Circuits

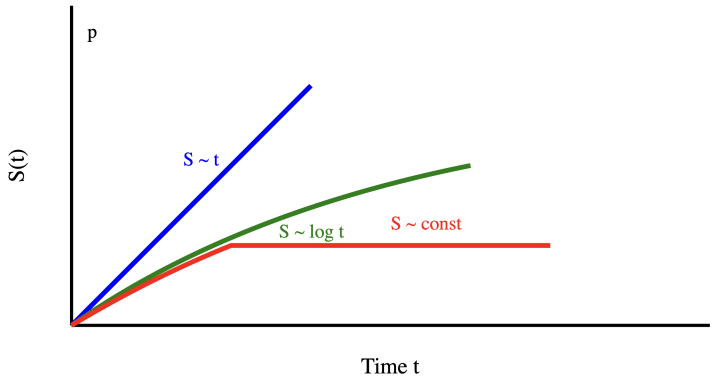
A random unitary circuit consists of:

- Discrete time steps
- Local unitary gates applied to neighboring qubits
- Gates chosen randomly from the Haar measure

You can also construct constrained dynamics using gates that preserve some quantity like PSWAP Gates that preserve excitation number.
While random unitaries provide maximal scrambling, constrained dynamics like PSWAP (partial-swap) gates offer insight into how conservation laws affect entanglement growth. The PSWAP gate acts on two qubits as:

$$PSWAP_{ij} = c|01\rangle\langle 10| + c^*|10\rangle\langle 01| \tag{7}$$

This preserves particle number while allowing controlled exchange. This would mean that while the entanglement grows, the amount of growth is less than that for a random unitary circuit.



3 Entanglement Growth as a Function of Time

3.0.1 Entanglement Entropy

For these notes we will consider that the system starts in a pure state. This is because if the system is in pure state then the von Neumann entropy is a measure of the correlation between the subsystem and complement. In mixed state this interpretation is not direct.
For a bipartite system divided into regions A and B, the entanglement entropy is:

$$S_n(A) = \frac{1}{1n} log_{2[Tr(\rho_A^n)]}(8)$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$is the reduced density matrix. The von Neumann entropy (n→1) is:

$$S_1(A) = Tr(\rho_A log_2 \rho_A) \tag{9}$$

In random unitary circuits without measurements, we expect that the correlations grow extensively. That is the bigger the subsystem size the more the correlation. If a system is of a size L then this growth is not endless and there is in fact a maximal bound upto which the entanglement increases and then saturates. Think a two qubit example with the maximal entanglement being 1 in the case of Bell state. There the temporal behaviour can be categorized as:

- Short times (tL): Linear growth $S(t) \sim v_E t$
- Long times ($\gg L$): Volume law saturation $S \sim L$

The entanglement velocity v_E depends on the specific dynamics eg whether the gates are constrained or not.
Now lets introduce measurements to a 1D chain.

4 1+1 Dimensions...because its easy(?)

Consider a 1D chain of L qubits evolving under:

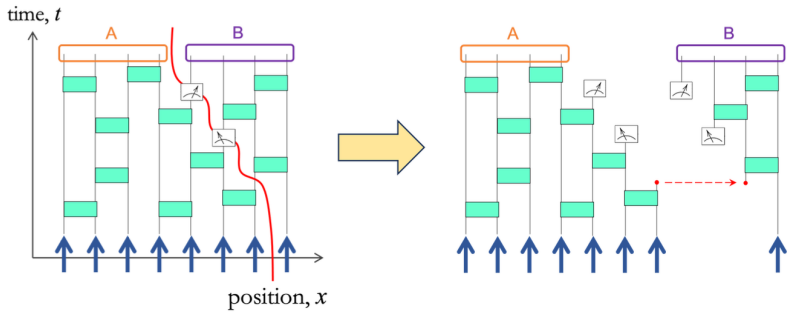


Figure 3: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

- Unitary dynamics: Random two-qubit gates applied in a brickwork pattern
- Measurements: Each qubit measured with probability p after each unitary layer

Then we have a case where there is a competition between the unitary evolution that generates entanglement and projective measurements that destroy entanglement. This leads to a phase transition at critical measurement rate p_c.

4.1 How Measurements Reduce Entanglement

To understand this we have to understand what projective measurements do.
A projective measurement in the computational basis: Projects the qubit into $|0\rangle$ or $|1\rangle$. The outcome probabilities for a single qubit would be given by: $p_0 = |\langle 0|\psi\rangle|^2$ and $p_1 = |\langle 1|\psi\rangle|^2$. After measurement the post-measurement state

$$|\psi'\rangle = P_i|\psi\rangle/\sqrt{p_i} \tag{10}$$

So what does this do to the entanglement? Lets take an example.
Consider a Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Before measurement the entanglement between the qubits is maximal and the state of each individual qubit is maximally mixed.
After measuring qubit 1: if outcome is 0:$|\psi'\rangle = |00\rangle$ and the entanglement is now 0.
Hence, the measurement destroys entanglement by collapsing superpositions. This is actually a crucial ingredient of MIPT. After every measurement you pick on the the pure state options and work with that.

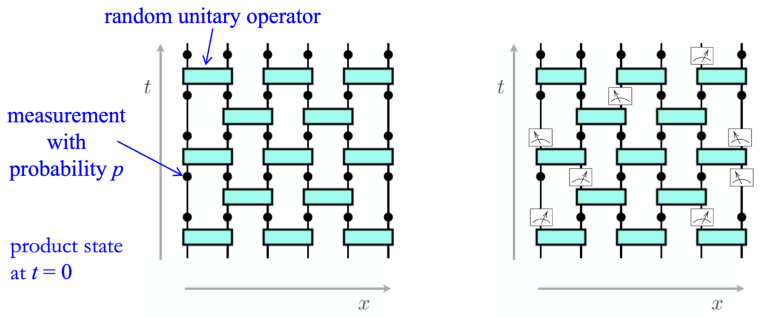


Figure 4: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

4.2 Measurement Protocol: Pure State Evolution vs. Mixed States

Track individual quantum trajectories, not ensemble averages. For a specific measurement record $\{m_i\}; |\psi_{\{m_i\}}\rangle$ = (normalized state after measurements). The density matrix averaged over outcomes:

$$\rho_{\text{avg}} = \sum_{\{m_i\}} p(\{m_i\}) |\psi_{\{m_i\}}\rangle\langle\psi_{\{m_i\}} \tag{11}$$

MIPT and the phase transition appears in individual trajectories, not in ρ_{avg}! This is crucial because this ensures that at every time step after making measurements we still have a pure state that we are evolving rather than a mixed state. This is also a necessary feature for MIPT. Had you taken the average state after measurement, you will not see MIPT.

5 Phase transition

At the critical point p_c: Entanglement grows logarithmically: $S(t) \sim A \log t$. We expect that the there are scale-invariant correlations, which means that the entanglement entropy of 2 qubit subsytems, 3 qubit subsytems, 4, 5...and so on look the same. This would also correspond to a decay of mutual information with distance. The expected correlation length is $\xi \sim |p - p_c|^{-\nu}$

6 Fun math trick: Mapping to percolation problem and minimal cut

6.1 The Minimal Cut Picture

The nth Rényi entropy S_n can be computed via minimal cut (MC). MC says that compute the minimum number of bonds that must be cut to separate

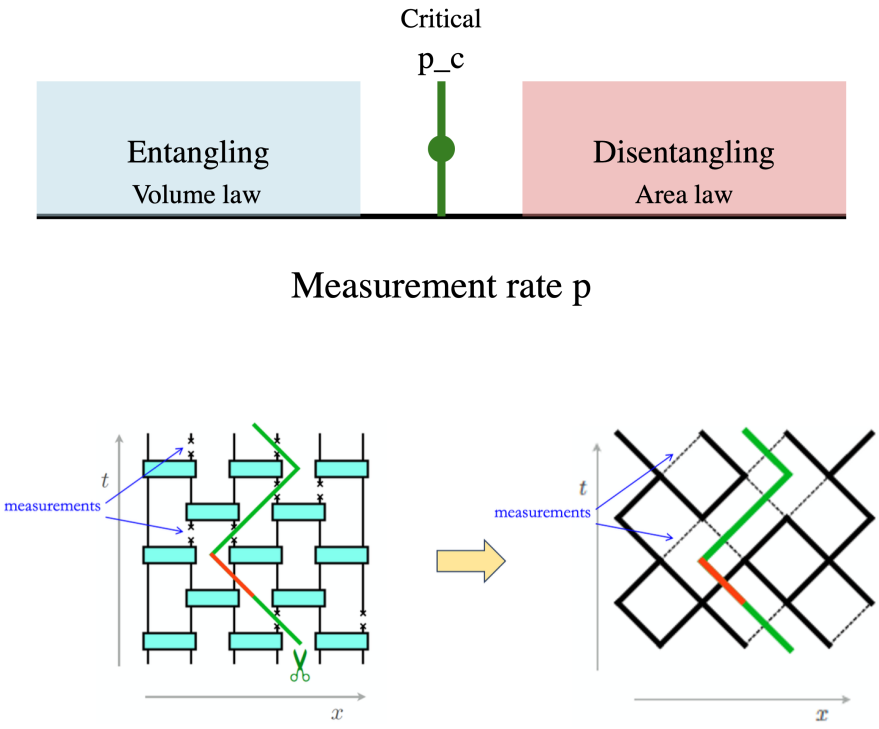


Figure 5: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

regions A and B. This is to say that measurements act as "broken bonds" in the percolation problem. This means that the nth entropy is given by

$$S_n = N_{cut}\log(2) \tag{12}$$

When there are measurements involved, you can traverse the bod freely because there is no entanglement there in time or space. So, essentially minimal cut says that find a line that starts at the top between A and B and exits the circuit while passing across the smallest possible number of circuit legs. Locations of measurements act like pre-broken legs and can be traversed for free.
It turns out that this problem has a lot to do with polymer growth problems. KPZ universality to be specific. And borrowing results from there we can conclude,

- $S(x,t)$ is linear in time for short times, and then saturates to a constant value proportional to the size x.
- Statistical fluctuations in $S(x,t)$ grow with time as $\delta S \propto t^{\frac{1}{3}}$

- Fluctuations in $S(x,t)$ have a correlation length ξ in space that grows as $\xi \sim t^{2/3}$

This means that any systems that obey area law entanglement also have this behaviour for the correlaions.

7 Classical Percolation Mapping

Apparently this work also has connections to classical percolation problems. Here are the key things to keep in mind along with the picture of a fisherperson's net...

- At p ¡ pc, the network resembles a "ripped up fisherman's net", with big holes of size $\xi \sim 1/(pcp)\nu$, which grow larger as one approaches the transition (ν ¿0 is the correlation length critical exponent). Those big holes in general have many dangling loose ends, but these are irrelevant for the minimal cut. The minimal cut therefore requires making roughly one cut per length scale ξ.
- Exactly at p= pc, the correlation length ξ diverges and the network has no characteristic length scale. One can say that the network resembles a scale-free distribution of "empty chambers" of all length scales. The minimal cut involves passing from a small-sized chamber (which is generically where the cut begins) to ever larger length scales until one finds an empty chamber comparable in diameter to the system size.
- At p ¿ pc, the network falls apart into many disconnected pieces with characteristic size $\xi \sim 1/(ppc)\nu$. The minimal cut may begin inside such a piece, but after exiting that piece the cut can proceed freely to the boundary of the network.

This ends up resulting in the following conditions for entanglement entropy.

- Entangling phase ($p < p_c$)→ Minimal cut must traverse O(t) unbroken bonds → $S_0(t) \sim v_0 t$
- Critical point ($p = p_c$)→Scale-invariant percolation clusters→ $S_0(t) \sim A \ln t$
- Disentangling phase ($p > p_c$)→Minimal cut finds paths through broken bonds→ $S_0(t) \sim O(1)$

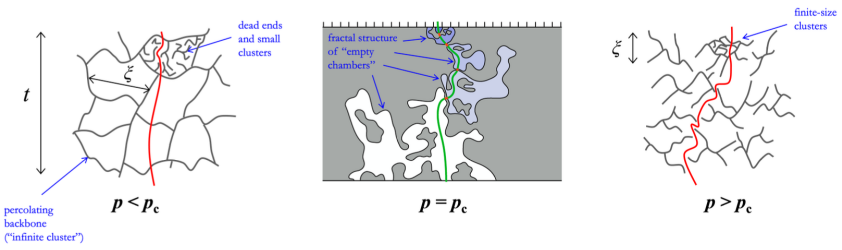


Figure 11: Pictures of the minimal cut in a random network for different measurement regimes.

Figure 6: Figure credits: Brian Skinner `https://arxiv.org/abs/2307.02986`

8 References

- Lecture Notes: Introduction to random unitary circuits and the measurement-induced entanglement phase transition