3.6

3.6 Particle Drift in a Periodic Flow Field

Question 1

We have

$$\frac{dX}{dt} = \alpha \cos k(X(t) - ct) \tag{1}$$

Consider the change of variable $X = \beta X'$, $t = \gamma t'$ with X', T' dimensionless (i.e. β and γ represent our new units of distance and time respectively), so Equation 1 transforms to

$$\frac{dX'}{dt'} = \frac{\alpha\gamma}{\beta} \cos k\big(\beta X'\big(t'\big) - \gamma c t'\big) \tag{2}$$

Now let $\beta = \frac{2\pi}{k}$, $\gamma = \frac{2\pi}{ck}$, then Equation 2 becomes

$$\frac{dX'}{dt'} = a \cos 2\pi\big(X'\big(t'\big) - t'\big) \tag{3}$$

where $a = \frac{\alpha}{c}$, thus by choosing appropriate units for distance and time we can let $k = 2\pi$, $c = 1$ in Equation 1.

Question 2

From Figure 1, we see that For $|a| < 1$ the particle drifts away from the origin and performs an oscillatory motion along the drifting path. For $|a| \geq 1$ we see that the particle tends towards a line of constant gradient (i.e the particle travels at a constant velocity) For Figure 1 a relative tolerance of 10^{-7} and absolute tolerance 10^{-8} was used, we can compare this result to that with relative tolerance of 10^{-14} and absolute tolerance 10^{-16} shown in Figure 2 and find they are very similar thus I am confident the results I obtained before are accurate.

In Figure 3 we see that for a given a, the solutions to Equation (1) with $X(0) \neq 0$ all have similar behaviour.

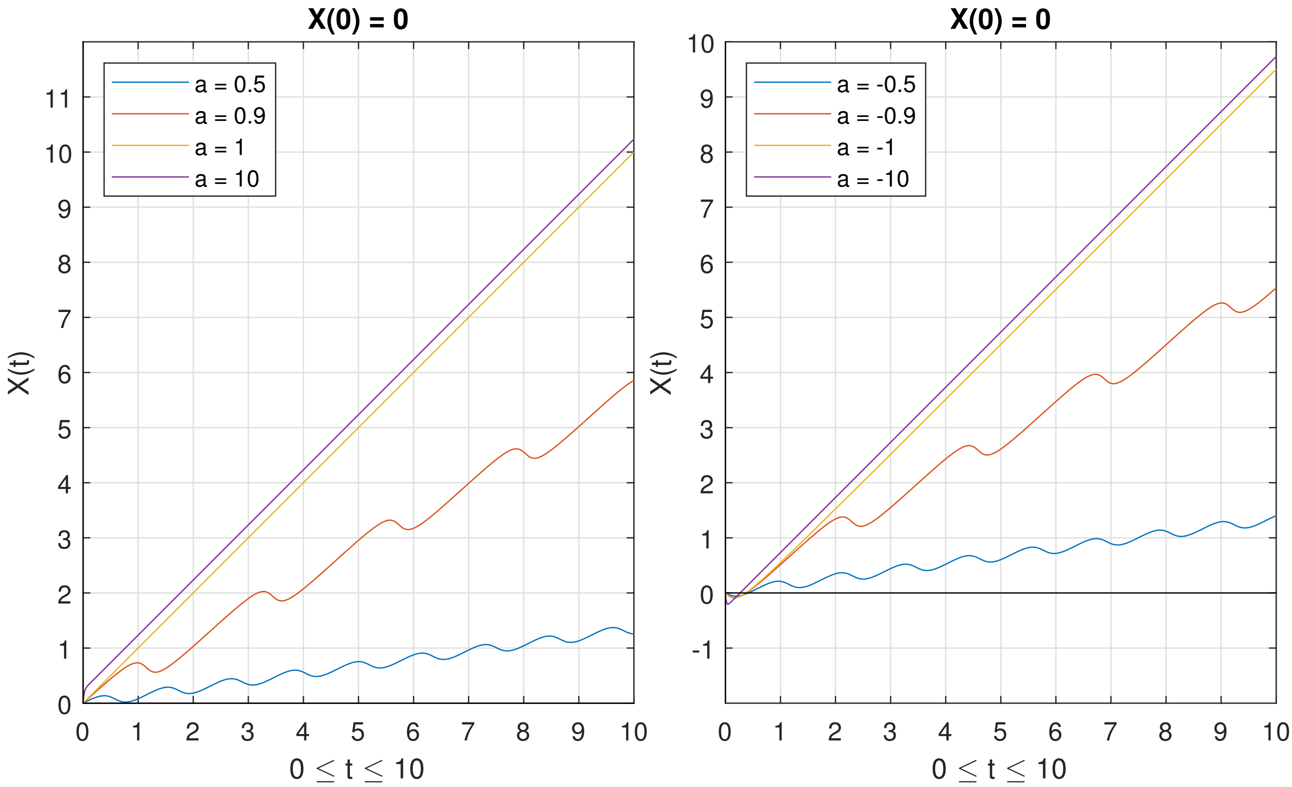


Figure 1: Graphs showing solutions to the ODE for different values of a with a relative tolerance of 10^{-7} and an absolute tolerance 10^{-8}

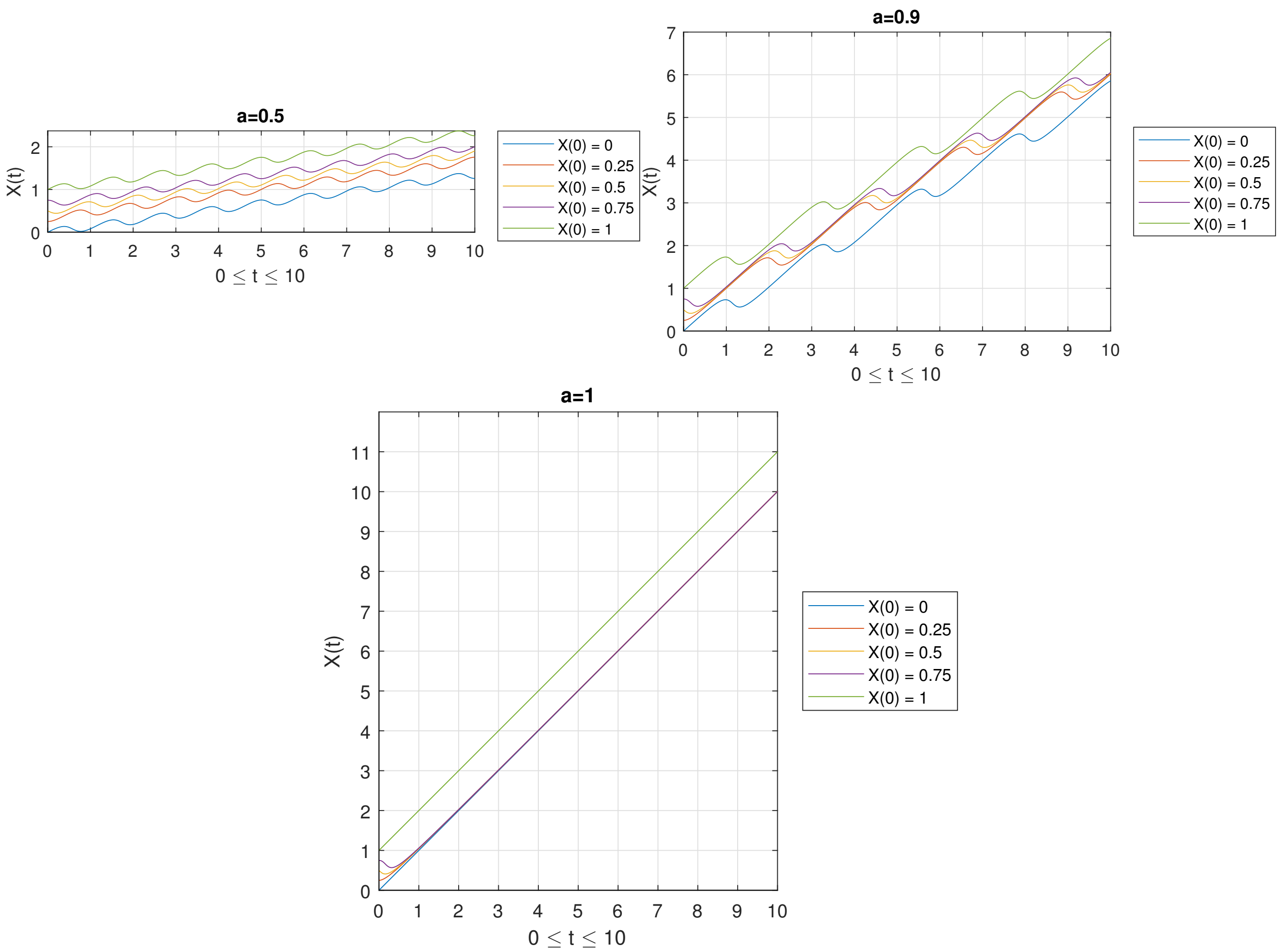


Figure 3: Graphs showing solutions to the ODE for different values of $X(0)$, for 3 values of a

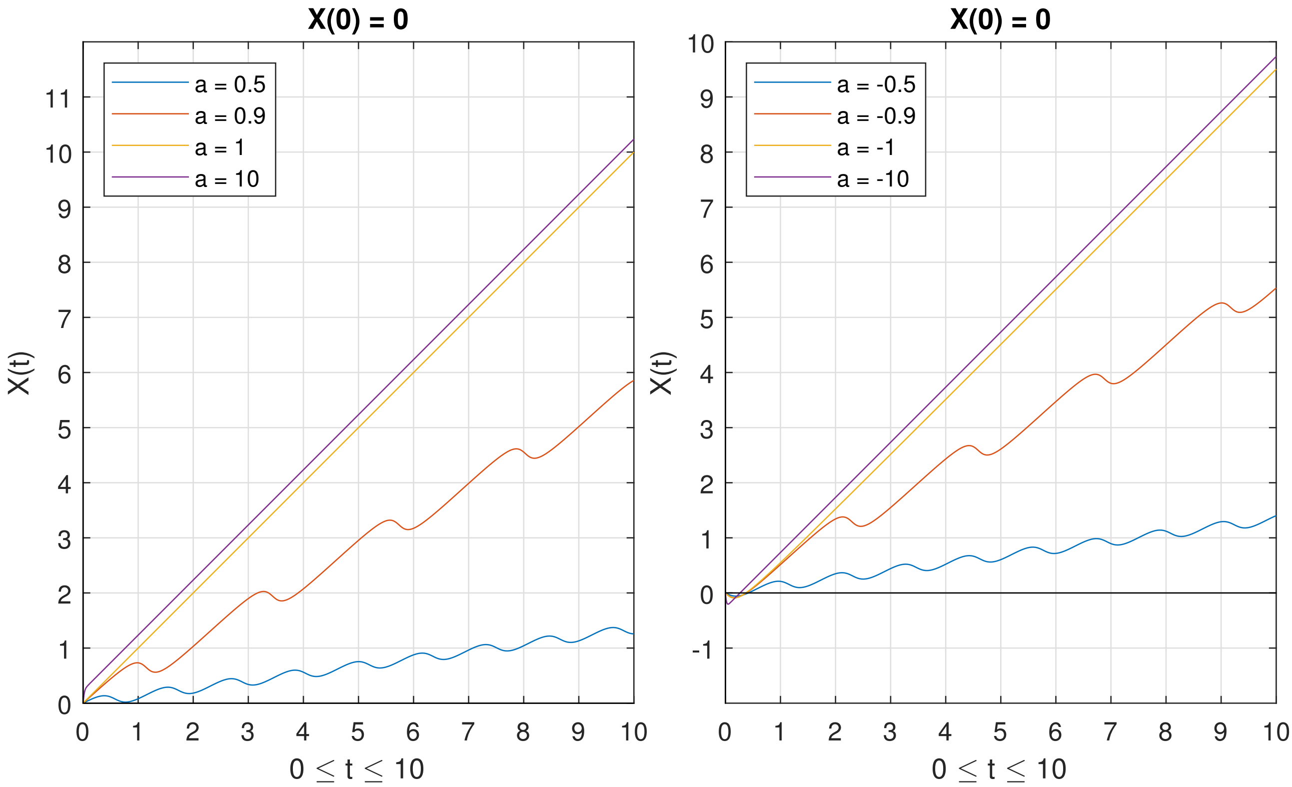


Figure 2: Graphs showing solutions to the ODE for different values of a with a relative tolerance of 10^{-14} and an absolute tolerance 10^{-16}

Question 3

First note that if $X(t)$ is a solution to the ODE then so is $X'(t) = X(t) + n$ where $n \in \mathbb{Z}$ as

$$\begin{aligned}\frac{dX'}{dt} &= \frac{d}{dt}(X(t)+n) = \frac{dX}{dt}\\ &= a\cos 2\pi(X(t)-t)\\ &= a\cos[2\pi(X(t)-t)+2\pi n]\\ &= a\cos 2\pi(X(t)+n-t)\\ &= a\cos 2\pi\big(X'(t)-t\big)\end{aligned}$$

This means that in general we only have to consider solutions with $X(0) = x_0$ where $x_0 \in [0,1)$. We can make this claim stronger and say that time averaged drift velocity is determined by the drift velocity when $X(0) = 0$. This is because the average drift velocity of $X+1$ is equal to drift velocity of X, so given a second solution X' such that $X'(0) = x_0$ where $x_0 \in (0,1)$ we must have $X(t) \leq X'(t) \leq X(t)+1$ (as the trajectory of particles cannot intercept) and as mean drift velocity is given is $lim_{t\to\infty}\frac{X(t)-X(0)}{t} = lim_{t\to\infty}\frac{X(t)}{t}$ the above inequality shows the average drift velocity of X' is the same as the average drift velocity of X.

Now consider solely the solution X such that $X(0) = 0$ and assume X has some drift velocity m, then we can write $X(t) = mt + f(t)$ where m is the average time drift velocity of the particle and $f(t)$ is some oscillatory function, if we assume a small then the magnitude of f is small (by looking at the ODE) so for large t we can ignore this so $X(t) \approx mt$ therefore $\log X(t) \approx log(m) + log(t)$ so if we plot $log(X)$ against $log(t)$ and find the y intercept we can find the mean drift velocity. We are also be able to check that the average time drift velocity exists by checking if the graph of $log(X)$ against $log(t)$ has a gradient which tends to 1.

We see in Figure 4 that for multiple small values of a the graph of $log(X)$ against $log(t)$ has a gradient which tends to 1, suggesting the time average drift velocity does exist for these solutions. We can

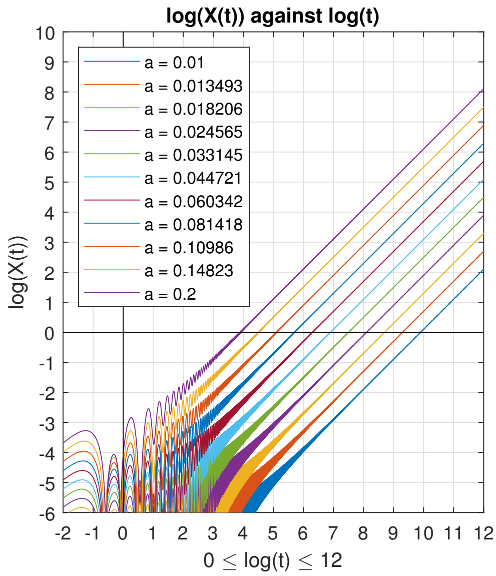


Figure 4: Graphs showing $log(X)$ against $log(t)$

then continue to calculate an estimate for the time average drift velocity and compare to $\frac{1}{2}a^2$ we find by looking at the table below that the approximation is very close to $\frac{1}{2}a^2$

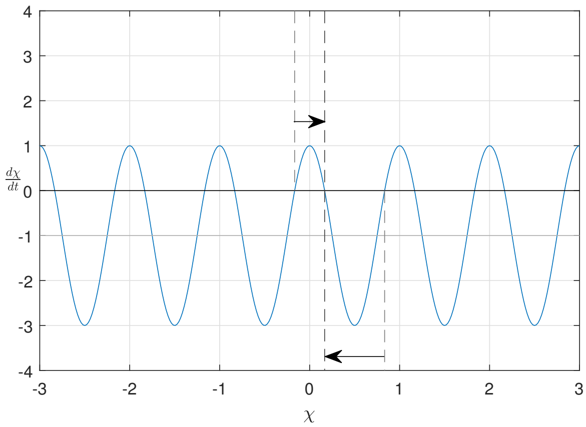
a	y_int	calc_drift_vel	half_a_squared	abs_error	rel_error
0.01	-9.9466	4.789e-05	5e-05	2.1102e-06	0.042204
0.013493	-9.3208	8.9544e-05	9.1028e-05	1.4841e-06	0.016303
0.018206	-8.6996	0.00016666	0.00016572	9.3703e-07	0.0056542
0.024565	-8.0804	0.00030955	0.00030171	7.8453e-06	0.026003
0.033145	-7.5013	0.00055234	0.00054928	3.0643e-06	0.0055788
0.044721	-6.9131	0.00099463	0.001	5.375e-06	0.005375
0.060342	-6.3173	0.0018048	0.0018206	1.5738e-05	0.0086448
0.081418	-5.7046	0.0033307	0.0033145	1.6242e-05	0.0049002
0.10986	-5.1023	0.0060826	0.0060342	4.847e-05	0.0080325
0.14823	-4.508	0.01102	0.010986	3.4856e-05	0.0031729
0.2	-3.9134	0.019972	0.02	2.8367e-05	0.0014183

Question 4

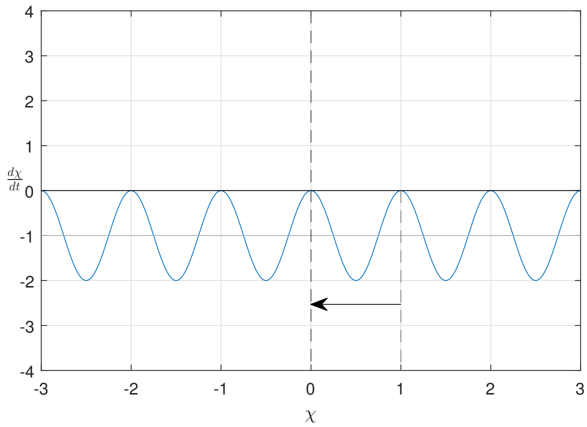
If we were to try and plot $\frac{dX}{dt}$ against X we would face a problem with the t dependence in $\frac{dX}{dt}$ so consider the substitution $\chi(t) = X(t) - t$, then if X satisfies Equation (1), χ satisfies

$$\frac{d\chi}{dt} = a\cos 2\pi\chi - 1 \tag{4}$$

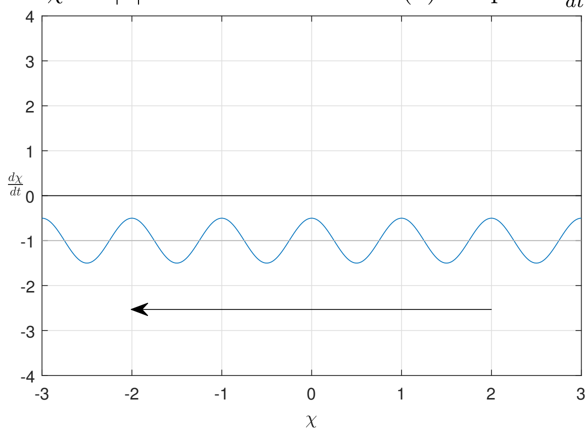
Now plot $\frac{d\chi}{dt}$ against χ for various values of a



(a) Graph of $\frac{d\chi}{dt}$ against χ for $|a| > 1$



(b) Graph of $\frac{d\chi}{dt}$ against χ for $|a| = 1$



(c) Graph of $\frac{d\chi}{dt}$ against χ for $|a| < 1$

Figure 5: Graphs of $\frac{d\chi}{dt}$ against χ for various values of a

In Figure 5, we can see that for $|a| \geq 1$, $\chi(t)$ tends to a steady state (i.e. a point where $\frac{d\chi}{dt} = 0$) which implies that a particle following the path $X(t)$ eventually ends up travelling at an (almost) constant velocity of 1, we also see that in both of these cases, although the flow is periodic and oscillatory, the solutions are not. For $|a| < 1$ we see that χ is monotonically decreasing however the value of $\frac{d\chi}{dt}$ spends more time being greater than -1 than less than so the average of $\frac{d\chi}{dt}$ is greater than -1, so the average drift velocity is greater than 0. Also we can note that χ has an oscillatory nature therefore so does X.

Question 5

Consider the substitution $\chi(t) = X(t) - t$, then if X satisfies Equation (3) then χ satisfies

$$\frac{d\chi}{dt} = a\cos 2\pi\chi - 1 \tag{5}$$

This is a separable ODE equation thus can find χ by doing

$$\int \frac{\frac{d\chi}{dt}}{a\cos 2\pi\chi - 1} dt = \int 1 dt$$

and has the solution

$$\chi(t) = \begin{cases} \frac{1}{\pi}\arctan\left(\frac{1}{2\pi(t-c)}\right) & a=1 \\ -\frac{1}{\pi}\arctan(2\pi(t-c)) & a=-1 \\ \frac{1}{\pi}\arctan\left(\frac{\sqrt{1-a^2}}{a+1}\tan\left(\pi\sqrt{1-a^2}(c-t)\right)\right) & |a|<1 \\ \frac{1}{\pi}\arctan\left(\sqrt{\frac{a-1}{a+1}}\tanh\left(\pi\sqrt{a^2-1}(t-c)\right)\right) & |a|>1 \end{cases}$$

Note that in the case where $a = 1$ there's a discontinuity in $\chi(t)$ when $t = c$ and when $|a| < 1$ there's a discontinuity whenever $t = c - \frac{2n+1}{2\sqrt{1-a^2}}$ where $n \in \mathbb{Z}$. At these point $\chi(t^+) - \chi(t^-) \in \mathbb{Z}$ and note that if χ is a solution to Equation (5), then so is $\chi + n$ where $n \in \mathbb{Z}$ so using this in the cases $a = 1$ and $|a| < 1$ we can find continuous solutions to Equation (5), therefore we can find continuous solutions $X(t) = \chi(t) + t$ to Equation (3). Due to this continuation we should note that in the case where $|a| < 1$ that $\chi\left(t + \frac{1}{\sqrt{1-a^2}}\right) - \chi(t) = -1$.

Using $\chi(t)$ defined above we can easily deduce the solution $X(t)$ as

$$X(t) = t + \chi(t)$$

With regards to the time averaged drift velocity, using question 4 we can see this is 1 for $|a| > 1$. For $|a| < 1$ we will first calculate $\frac{d\chi}{dt}$

$$\begin{aligned} \frac{d\chi}{dt} &= \frac{d}{dt}\left(\frac{1}{\pi}\arctan\left(\frac{\sqrt{1-a^2}}{a+1}\tan\left(\pi\sqrt{1-a^2}(c-t)\right)\right)\right) \\ &= -\frac{1-a^2}{1+a\cos\left(\pi\sqrt{1-a^2}(c-t)\right)} \end{aligned}$$

So $\frac{d\chi}{dt}$ is periodic with period $T = \frac{1}{\sqrt{1-a^2}}$ so $\frac{dX}{dt} = \frac{d\chi}{dt} + 1$ is periodic with period T so the average drift velocity v is

$$\begin{aligned} v &= \frac{1}{T}\int_t^{t+T} \frac{dX}{dt} dt \\ &= \frac{1}{T}\int_t^{t+T} \left(\frac{d\chi}{dt} + 1\right) dt \\ &= \frac{1}{T}\left(T + \chi\left(t + \frac{1}{\sqrt{1-a^2}}\right) - \chi(t)\right) \\ &= 1 - \frac{1}{T} \\ &= 1 - \sqrt{1-a^2} \\ &= 1 - \left(1 - \frac{1}{2}a^2 - \frac{1}{8}a^4 + \dots\right) \\ &\approx \frac{1}{2}a^2 \end{aligned}$$

This holds for $|a|$ sufficiently small.

Program Listings

Function used to generate Figures 1 and 2

```
function graph_a_values(tspan, reltol, abstol, a_values, x0,...
    graph_title, filename)
% graphs solution to the ODE

    figure('visible','off')
    opts = odeset('RelTol',reltol,'AbsTol',abstol);
    miny = inf;
    maxy = -inf;
    for a = a_values
        [t, x] = ode45(@(t,x) flowODE(t,x,a),...
                       tspan,x0,opts);
        plot(t,x, 'DisplayName',join(['a = ', num2str(a)]));
        grid on;
        miny = min([min(min(x)) miny]);
        maxy = max([max(max(x)) maxy]);
        yticks(floor(miny):1:ceil(maxy));
        xticks(floor(tspan(1)):1:ceil(tspan(2)));
        daspect([1 1 1]);
        hold on;
        legend('show');
    end
    xlabel(join([num2str(tspan(1)), "t", num2str(tspan(2))],...
                " \leq "));
    ylabel('X(t)');
    legend('Location','northwest','AutoUpdate','off');
    line([0 0], ylim, 'Color','black');
    line(xlim, [0 0], 'Color','black');
    hold off;
    title(graph_title);
    print(filename, '-dpng','-r600');
end
```

Function used to generate Figure 3

```
function graph_x_values(tspan, reltol, abstol, a, x0_values,...
    graph_title, filename)
% graphs solution to the ODE

    figure('visible','off')
    opts = odeset('RelTol',reltol,'AbsTol',abstol);
    miny = inf;
    maxy = -inf;
    for x0 = x0_values
        [t, x] = ode45(@(t,x) flowODE(t,x,a),...
                       tspan,x0,opts);
        plot(t,x, 'DisplayName',join(['X(0) = ', num2str(x0)]));
        grid on;
        miny = min([min(min(x)) miny]);
        maxy = max([max(max(x)) maxy]);
        yticks(floor(miny):1:ceil(maxy));
        xticks(floor(tspan(1)):1:ceil(tspan(2)));
```

```
        daspect([1 1 1]);
        hold on;
        legend('show');
    end
    xlabel(join([num2str(tspan(1)), "t", num2str(tspan(2))],...
                " \leq "));
    ylabel('X(t)');
    legend('Location','eastoutside','AutoUpdate','off');
    line([0 0], ylim, 'Color','black');
    line(xlim, [0 0], 'Color','black');
    hold off;
    title(graph_title);
    print(filename, '-dpng','-r600');
end
```

Functions used to generate figure 4, and the Table in Question 3

```
function T = log_graph_a_values(tspan, reltol, abstol, a_values, x0,...
    graph_title, filename)
% graphs solution to the ODE

    figure('visible','off')
    opts = odeset('RelTol',reltol,'AbsTol',abstol);
    miny = inf;
    maxy = -inf;
    yint = [];
    for a = a_values
        [t, x] = ode45(@(t,x) flowODE(t,x,a),...
                       tspan,x0,opts);
        x = arrayfun(@(y) log(y), x);
        t = arrayfun(@(y) log(y), t);
        ind = binary_search(t, 7);
        t1 = t(ind);
        x1 = x(ind);
        t2 = t(end);
        x2 = x(end);
        c = (x1*t2-x2*t1)/(t2-t1);
        yint = [yint; [a, c, exp(c), 0.5*a^2,...
            abs(exp(c)-0.5*a^2), abs(exp(c)-0.5*a^2)/(0.5*a^2)]];
        plot(t,x, 'DisplayName',join(['a = ', num2str(a)]));
        miny = min([min(min(x)), miny]);
        maxy = max([max(max(x)), maxy]);
        daspect([1, 1, 1]);
        hold on;
        legend('show');
    end
    T = array2table(yint,'VariableNames',{'a','y_int','calc_drift_vel',...
        'half_a_squared', 'abs_error', 'rel_error'});
    disp(T)
    xlabel(join(['0', "log(t)", num2str(log(tspan(2)))],...
                " \leq "));
    ylabel('log(X(t))');
    axis([-2, log(tspan(2)), -6, 10])
    legend('Location','northwest','AutoUpdate','off');
```

```
    line([0 0], ylim, 'Color','black');
    line(xlim, [0 0], 'Color','black');
    yticks(-100:1:20);
    xticks(-100:1:ceil(log(tspan(2))));
    grid on;
    hold off;
    title(graph_title);
    print(filename, '-dpng','-r600');
end

function result = binary_search(arr, T)
    n = size(arr);
    L = 1;
    R = n(1);
    result = floor((L+R)/2);
    while L <= R
        m = floor((L+R)/2);
        if arr(m) >= T && arr(m-1) < T
            result = m;
            break
        elseif arr(m) < T
            L = m+1;
        else
            R = m-1;
        end
    end
end
```

Code used to generate Figure 5

```
x = linspace(-3,3, 300);
y1 = 2*cos(2*pi*x)-1;
y2 = 1*cos(2*pi*x)-1;
y3 = 0.5*cos(2*pi*x)-1;

figure('visible','off')
plot(x,y1)
xticks(-10:1:10)
xlabel("\chi")
yticks(-10:1:10)
yl = ylabel("$\frac{d\chi}{dt}$",'Interpreter','latex');
set(yl,'rotation',0,'VerticalAlignment','bottom')
axis([-3, 3, -4, 4])
line([-10, 10], [0, 0], 'color', 'black')
line([-10, 10], [-1, -1], 'color', [0.7, 0.7, 0.7])
line([1/6, 1/6], [10, -10], 'color', [0.25, 0.25, 0.25], 'LineStyle', '--
    ')
line([-1/6, -1/6], [10, 0], 'color', [0.5, 0.5, 0.5], 'LineStyle', '--')
line([5/6, 5/6], [0, -10], 'color', [0.5, 0.5, 0.5], 'LineStyle', '--')
grid on
daspect([1 2 1])
annotation('arrow',[0.495 0.54],[0.65 0.65])
annotation('arrow',[0.625 0.54],[0.2 0.2])
print("large a", '-dpng','-r600');
```

```
figure('visible','off')
plot(x,y2)
xticks(-10:1:10)
xlabel("\chi")
yticks(-10:1:10)
yl = ylabel("$\frac{d\chi}{dt}$",'Interpreter','latex');
set(yl,'rotation',0,'VerticalAlignment','bottom')
axis([-3, 3, -4, 4])
line([-10, 10], [0, 0], 'color', 'black')
line([-10, 10], [-1, -1], 'color', [0.7, 0.7, 0.7])
line([0, 0], [10, -10], 'color', [0.25, 0.25, 0.25], 'LineStyle', '--')
line([1, 1], [0, -10], 'color', [0.5, 0.5, 0.5], 'LineStyle', '--')
grid on
daspect([1 2 1])
annotation('arrow',[0.647 0.5175],[0.3 0.3])
print("mid a", '-dpng','-r600');

figure('visible','off')
plot(x,y3)
xticks(-10:1:10)
xlabel("\chi")
yticks(-10:1:10)
yl = ylabel("$\frac{d\chi}{dt}$",'Interpreter','latex');
set(yl,'rotation',0,'VerticalAlignment','bottom')
axis([-3, 3, -4, 4])
line([-10, 10], [0, 0], 'color', 'black')
line([-10, 10], [-1, -1], 'color', [0.7, 0.7, 0.7])
grid on
daspect([1 2 1])
annotation('arrow',[0.7765 0.2585],[0.3 0.3])
print("small a", '-dpng','-r600');
```