2.4

2.4 Simulation of Random Samples from Parametric Distributions

The Exponential Distribution

Question 1

By definition of the median

$$\begin{aligned}\int_0^m f\left(x|\theta\left(m\right)\right)dx &= \int_0^m \theta\left(m\right)e^{-\theta(m)x}dx\\ &= \left[e^{-\theta(m)x}\right]_0^m\\ &= 1-e^{-\theta(m)m} = \frac{1}{2}\end{aligned}$$

Therefore

$$\begin{aligned}e^{-\theta(m)m} &= \frac{1}{2}\\ -\theta\left(m\right)m &= \ln\left(\frac{1}{2}\right)\\ \theta\left(m\right) &= \frac{\ln(2)}{m}\end{aligned}$$

So

$$\begin{aligned}g\left(x|m\right) &= f\left(x\left|\frac{\ln\left(2\right)}{m}\right.\right)\\ &= \frac{\ln\left(2\right)}{m}e^{-\frac{\ln(2)}{m}x}\\ &= \frac{\ln\left(2\right)}{m}2^{-\frac{x}{m}}\end{aligned}$$

Question 2

Given

$$u_i = 1-e^{-\theta_0 x_i}$$

We have

$$x_i = -\frac{\ln\left(1-u_i\right)}{\theta_0}$$

The program used to draw figure 1 is shown in the Programs Listings.

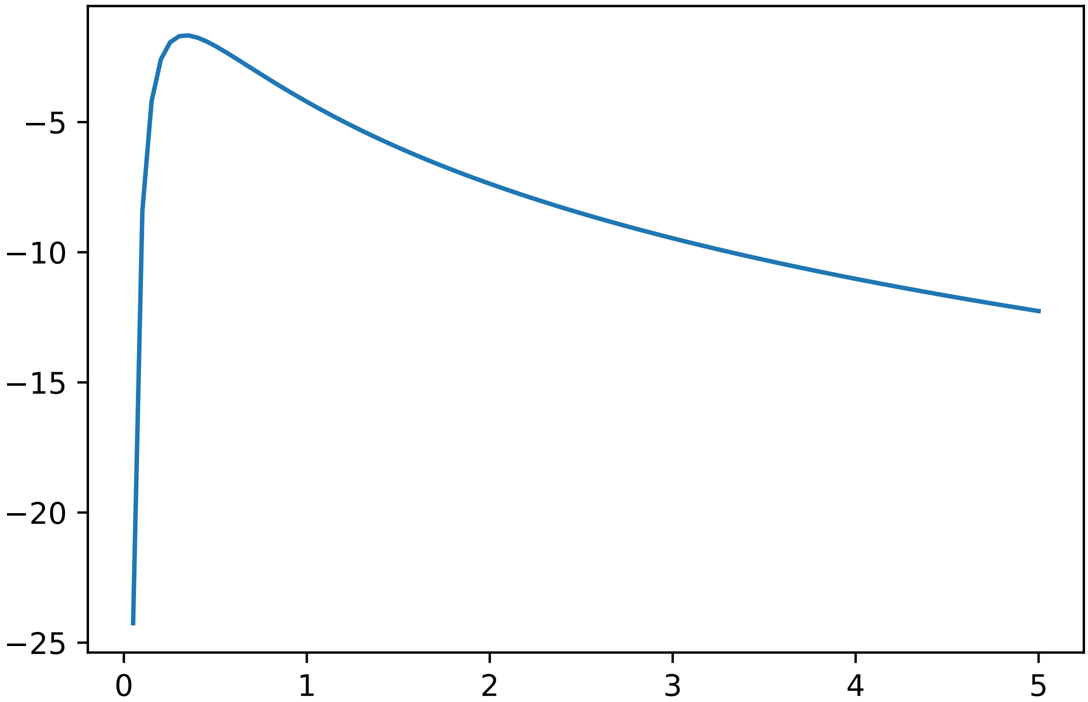

Figure 1: Graph of the log-likelihood of the median, n=6

The log-likelihood of the median is:

$$
\begin{aligned}
l_n(m) &= \sum_{i=1}^{n} \ln(g(x_i|m)) \\
&= n\ln(\ln(2)) - n\ln(m) - \frac{n\bar{X}\ln(2)}{m}\ , \qquad \text{where } \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}
\end{aligned}
$$

therefore

$$
\begin{gathered}
\frac{\partial l_n(m)}{\partial m} = n\left(-\frac{1}{m} + \frac{\bar{X}\ln(2)}{m^2}\right) = 0 \\
\therefore \hat{m}_n = \bar{X}\ln(2)
\end{gathered}
$$

We can see this is indeed a maximum and not a minimum by figure 1 In this case we have that the MLE for the median for this data-set is $\hat{m}_6 = 0.33640786437394693$ whereas the true median for $\theta_0 = 1.2$ is $m = 0.5776226504666211$. We can see that the MLE median is in fact quite far away from the actual median.

Question 3

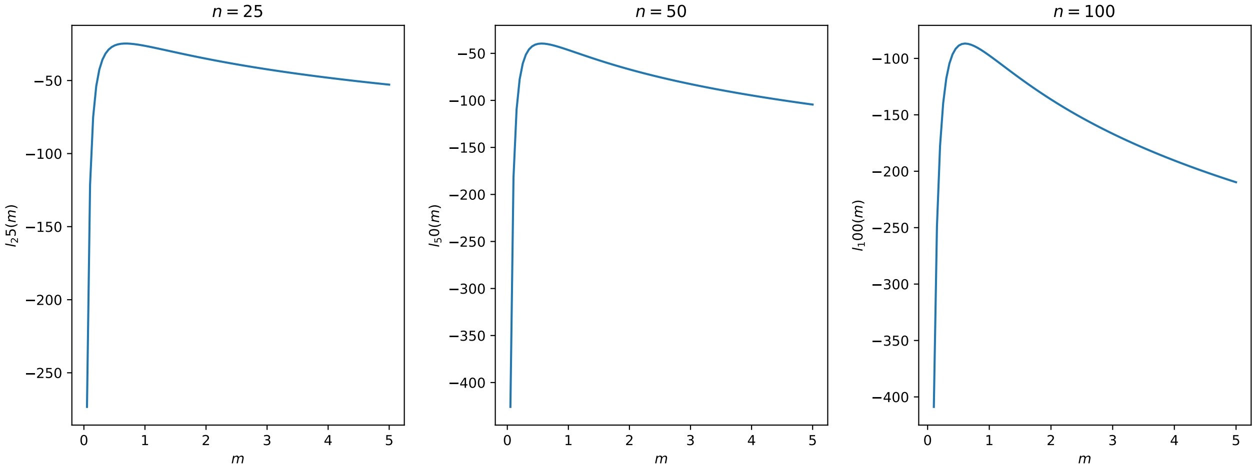

Figure 2: Graph of the log-likelihood of the median, n=25, 50, 100 respectively

Note $\hat{m}_{25} = 0.6845131720181218$, $\hat{m}_{50} = 0.5624737073456703$, $\hat{m}_{100} = 0.6075537305801957$ the MLE for the median are getting closer and closer to the actual median as we use larger and larger sample sizes. We also see that as the sample size becomes larger, the shape of the peak of $l_n(m)$ becomes more pronounced.

Question 4

Given $E \sim \exp(\theta)$, consider its moment generating function

$$\begin{aligned} M_E(\lambda) &= \int_0^\infty e^{\lambda x}\theta e^{-\theta x}dx \\ &= \theta \int_0^\infty e^{-(\theta-\lambda)x}dx \\ &= \frac{\theta}{\theta-\lambda} = \left(1-\frac{\lambda}{\theta}\right)^{-1} \quad \text{for } \lambda < \theta \end{aligned}$$

X, Y independent with $X,\ Y \sim \exp(\kappa)$ and $\mathbb{E}[X] = \mathbb{E}[Y] = \frac{1}{\kappa} = \frac{1}{\theta}$ so $\kappa = \theta$ therefore

$$X,\ Y \sim \exp(\theta)$$

Consider the moment generating function of $X + Y$

$$\begin{aligned} M_{X+Y}(\lambda) &= M_X(\lambda)\, M_Y(\lambda) \quad \text{by independence of } X \text{ and } Y \\ &= \left(1-\frac{\lambda}{\theta}\right)^{-2} \quad \text{for } \lambda < \theta \\ &= M_Z(\lambda) \quad \text{Where } Z \sim \Gamma(2,\theta) \end{aligned}$$

So $X + Y \sim \Gamma(2, \theta)$

Question 5

$$\begin{aligned} F(x) &= \int_0^x \theta^2 t e^{-\theta t}dt \\ &= \left[-\theta t e^{-\theta t}\right]_0^x + \int_0^x \theta e^{-\theta t}dt \\ &= 1 - \theta x e^{-\theta x} - e^{-\theta x} \end{aligned}$$

Therefore $(1+\theta x)e^{-(1+\theta x)} = (1-F(x))/e$ so F^-1 can be expressed in closed form iff the inverse of $(1+\theta x)e^{-(1+\theta x)}$ can be expressed in closed form which in and of itself is true iff the inverse of te^-t has a closed form solution, but it does not, so F^{-1} cannot be expressed as a closed form expression.

Question 6

The log-likelihood function for θ is

$$\begin{aligned} l_n(\theta) &= \sum_{i=1}^{n} \ln(f(x_i|\theta)) \\ &= 2n\ln(\theta) - n\bar{X}\theta + \sum_{i=1}^{n}\ln(x) \end{aligned}$$

so the MLE satisfies

$$\begin{aligned} \frac{\partial l_n(\theta)}{\partial \theta} &= n\left(\frac{1}{\hat{\theta}} - \bar{X}\right) = 0 \\ \therefore \hat{\theta}_n &= \frac{2}{\bar{X}} \end{aligned}$$

And we can see that this is a maximum as shown in figure 3.

Question 7

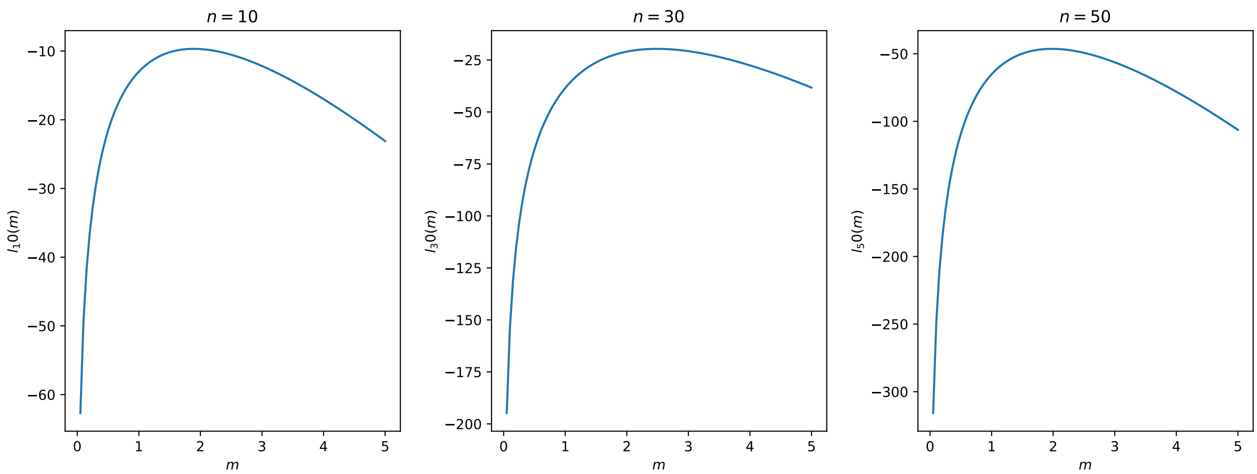

Figure 3: Graph of the log-likelihood of the theta, n=10, 30, 50 respectively

$f(x|\theta) = \theta^2 x e^{-\theta x}$ for $x \geq 0$ is the PDF for a random variable $X \sim \Gamma(2,\theta)$ so from question 5 we know that this distribution is equivalent to the sum of two independent exponential variables, so can use this to generate a random sample of size n from $f(x|\theta_0)$.

Note $\hat{\theta}_{10} = 1.8916039065976777$, $\hat{\theta}_{30} = 2.4902490614918222$, $\hat{\theta}_{50} = 1.9795473338385616$ is slowly converging to the real value of $\theta_0 = 2.2$ but is converging slower than in question 3, and note that the peaks of $l_n(m)$ aren't as well defined as in question 3. Also the MLE in question 3 is unbiased as

$$\begin{aligned} \mathbb{E}[\hat{m}_n] &= \mathbb{E}\left[\bar{X}\ln(2)\right] \\ &= \frac{\ln(2)}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] \\ &= \frac{\ln(2)}{n}\frac{n}{\theta} = \frac{\ln(2)}{\theta} = m \end{aligned}$$

Where $X \sim \exp(\theta)$. Whereas for $\hat{\theta}_n$ we have

$$\begin{aligned}
\mathbb{E}\left[\hat{\theta}_n\right] &= \mathbb{E}\left[\frac{2}{\bar{X}}\right] \\
&= \mathbb{E}\left[\frac{2n}{\sum_{i=1}^{n} X_i}\right] \\
&= \mathbb{E}\left[\frac{2n}{T}\right], \qquad \text{where } T = \sum_{i=1}^{n} X_i \sim \Gamma(2n, \theta) \\
&= 2n \int_0^\infty \frac{1}{t} \frac{\theta^{2n} t^{2n-1} e^{-\theta t}}{(2n-1)!} \\
&= \frac{2n\theta}{2n-1} \int_0^\infty \frac{\theta^{2n-1} t^{2n-2} e^{-\theta t}}{(2n-2)!} \\
&= \frac{2n\theta}{2n-1}
\end{aligned}$$

as $\frac{\theta^{2n-1} t^{2n-2} e^{-\theta t}}{(2n-2)!}$ is the PDF for $\Gamma(2n-1, \theta)$, so $\hat{\theta}_n$ is asymptotically unbiased.

Question 8

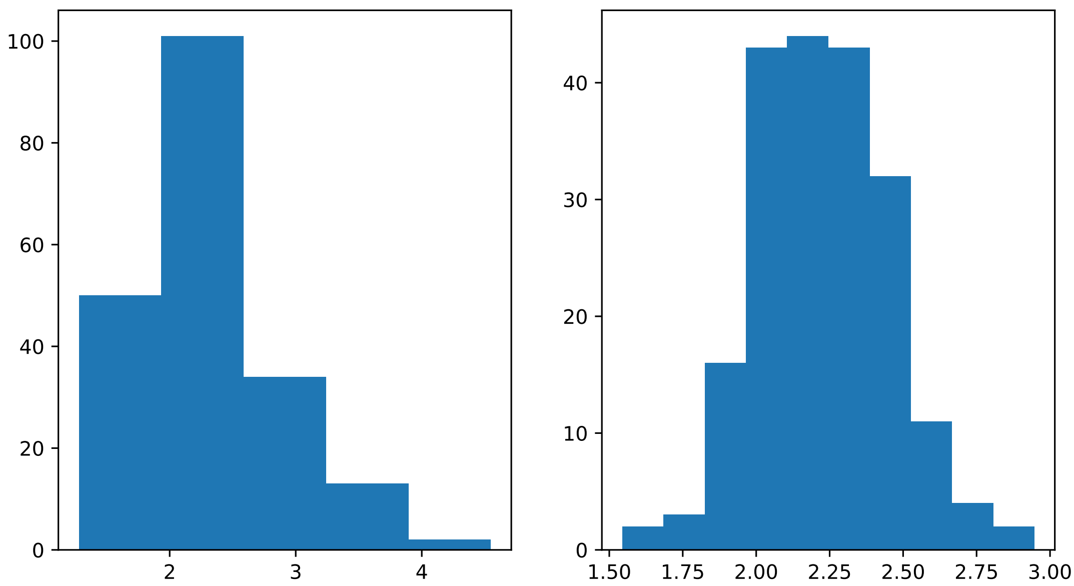

Figure 4: histogram of the MLE of theta, n=10, 50 respectively

We can see that for the histogram when $n = 10$ there's a peak for $2 < \theta < 2.5$ and outside of this peak the bars diminish in size, the same peak appears in the histogram for $n = 50$ however the variance of this data-set seems to be smaller as the bars outside of this peak are much smaller than their equivalent for $n = 10$, this implies that a larger sample will lead to a better estimate of θ_0.

The Normal Distribution

Question 9

Given

$$X = \mu_1 + \sigma\sqrt{V}\cos\Phi, \; Y = \mu_2 + \sigma\sqrt{V}\sin\Phi$$

We Have

$$V = \frac{(X-\mu_1)^2 + (Y-\mu_2)^2}{\sigma^2}, \ \Phi = \arctan\frac{Y-\mu_2}{X-\mu_1}$$

so

$$\begin{aligned} \frac{\partial V}{\partial X} &= \frac{2(X-\mu_1)}{\sigma^2} & \frac{\partial V}{\partial Y} &= \frac{2(Y-\mu_2)}{\sigma^2} \\ \frac{\partial \Phi}{\partial X} &= \frac{-(Y-\mu_2)}{(X-\mu_1)^2+(Y-\mu_2)^2} & \frac{\partial \Phi}{\partial Y} &= \frac{X-\mu_1}{(X-\mu_1)^2+(Y-\mu_2)^2} \end{aligned}$$

Thus the Jacobian is

$$\begin{aligned} \left|\frac{\partial(\Phi, V)}{\partial(X,Y)}\right| &= \left|\frac{\partial V}{\partial X}\frac{\partial \Phi}{\partial Y} - \frac{\partial V}{\partial Y}\frac{\partial \Phi}{\partial X}\right| \\ &= \left|\frac{2\left((X-\mu_1)^2+(Y-\mu_2)^2\right)}{\sigma^2\left((X-\mu_1)^2+(Y-\mu_2)^2\right)}\right| \\ &= \frac{2}{\sigma^2} \end{aligned}$$

Therefore

$$\begin{aligned} g(x,y) &= f\left(\phi(x,y), v(x,y)\right) \\ &= \frac{2}{\sigma^2}\frac{1}{4\pi}e^{-\frac{1}{2}\left(\frac{(x-\mu_1)^2+(y-\mu_2)^2}{\sigma^2}\right)} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu_1)^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\mu_2)^2}{2\sigma^2}} \end{aligned}$$

so by factorisation X, Y independent with $X \sim N(\mu_1, \sigma^2)$ and $Y \sim N(\mu_2, \sigma^2)$

Question 10

Assume $X_i \sim N(\mu, 1)$ then $\bar{X} \sim N(\mu, \frac{1}{n})$

$$\begin{aligned} \mathbb{P}\left(\mu + a \le \bar{X} \le \mu + b\right) &= \mathbb{P}\left(a\sqrt{n} \le \sqrt{n}\left(\bar{X}-\mu\right) \le b\sqrt{n}\right) \\ &= \Phi\left(b\sqrt{n}\right) - \Phi\left(a\sqrt{n}\right) = 0.8 \end{aligned}$$

where $\Phi(x)$ is the CDF of the standard Normal distribution assume $b = -a$ then $\Phi\left(b\sqrt{n}\right) - \Phi\left(a\sqrt{n}\right) = 2\Phi\left(b\sqrt{n}\right) - 1 = 0.8$, therefore $\Phi\left(b\sqrt{n}\right) = 0.9$

$$b = \frac{\Phi^{-1}(0.9)}{\sqrt{n}}$$

Therefore

$$\mathbb{P}\left(\bar{X} - b \le \mu \le \bar{X} + b\right) = \mathbb{P}\left(\bar{X} - \frac{\Phi^{-1}(0.9)}{\sqrt{n}} \le \mu \le \bar{X} + \frac{\Phi^{-1}(0.9)}{\sqrt{n}}\right) = 0.8$$

So

$$\left[\bar{X} - \frac{\Phi^{-1}(0.9)}{\sqrt{n}}, \bar{X} + \frac{\Phi^{-1}(0.9)}{\sqrt{n}}\right]$$

is an 80% confidence interval Note $\phi^{-1}(0.9) \approx 1.2815516$

Question 11

sample mean	lower bound	upper bound	in confidence interval?
-0.008505	-0.136660	0.119650	1
0.057399	-0.070756	0.185554	1
-0.176074	-0.304229	-0.047919	0
-0.033177	-0.161332	0.094979	1
0.120678	-0.007477	0.248834	1
-0.070341	-0.198496	0.057815	1
0.111881	-0.016274	0.240036	1
0.026286	-0.101869	0.154441	1
-0.040425	-0.168580	0.087730	1
-0.271566	-0.399721	-0.143411	0
0.080570	-0.047585	0.208726	1
-0.048870	-0.177026	0.079285	1
0.090336	-0.037819	0.218491	1
0.023836	-0.104319	0.151991	1
0.053248	-0.074907	0.181403	1
0.008922	-0.119233	0.137077	1
0.109602	-0.018553	0.237757	1
-0.168518	-0.296673	-0.040363	0
-0.004521	-0.132677	0.123634	1
0.062790	-0.065365	0.190945	1
-0.016768	-0.144923	0.111387	1
0.116252	-0.011903	0.244407	1
-0.080835	-0.208990	0.047320	1
-0.095345	-0.223500	0.032810	1
-0.169443	-0.297598	-0.041288	0

Figure 5: Table of the sample mean,the lower and upper bound of the confidence interval and whether the mean lies within the confidence interval

There were 4 occurrences where confidence interval didn't contain $\mu = 0$.

Question 12

If the previous questions were to be repeated with a different value of μ, then by definition of a $100\alpha\%$ confidence interval, there would be an 80% chance that the mean lay within the CI, and thus a 20% chance that the CI not contain the mean

The χ^2 Distribution

Question 13

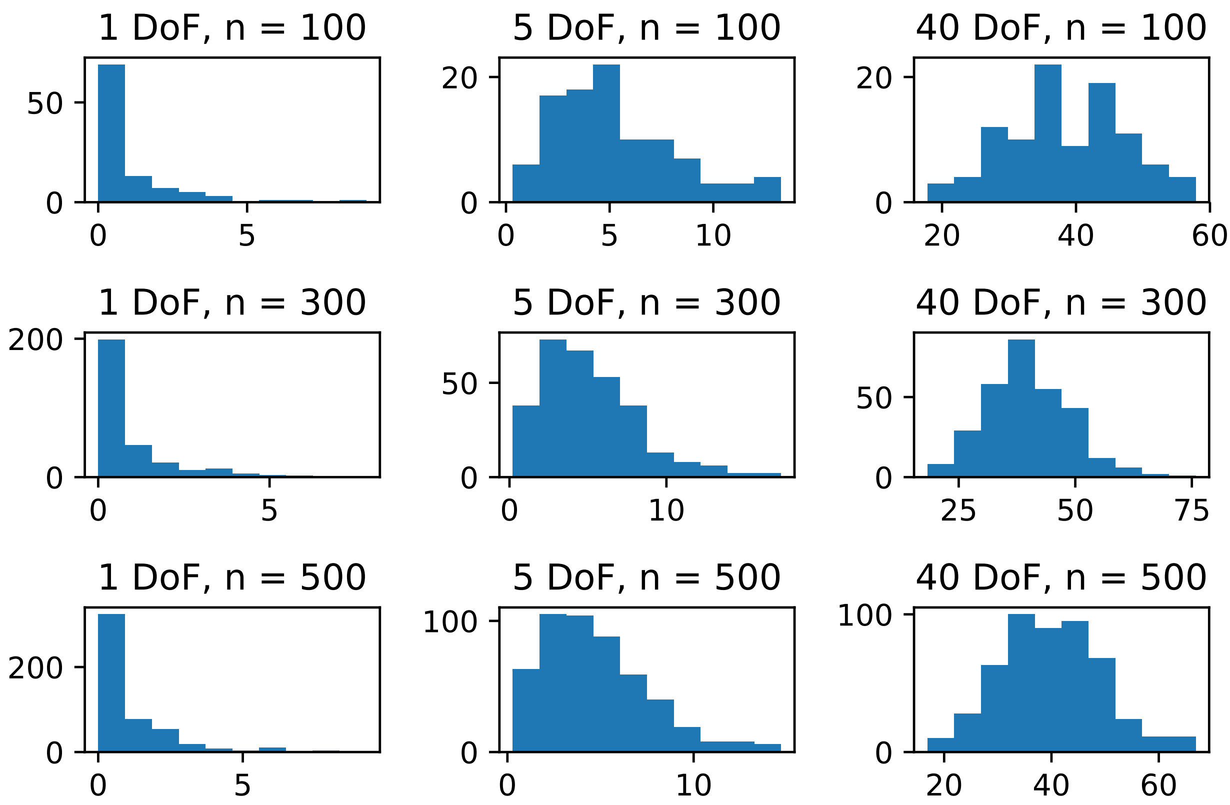

Figure 6: Histograms of χ^2 distribution for various degrees of freedom and sample sizes

As you increase the degrees of freedom the variance of the data increases as well as the location of the mode. The shape of the graph becomes more rounded as the degrees of freedom increases. We also may note that for 1 degree of freedom changing n from 100 to 500 has little to no effect on the shape on the graph whereas for 5 and (especially) 40 degrees of freedom, increasing the number of samples gives a much more accurate graph of the distribution.

Program Listings

Helper functions

These were functions that were defined to help make the project easier but aren't directly to do with the project
These were the modules I imported

```
from random import uniform
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

This function was used to draw tables

```
def draw_table(table, file_name, column_format):
    template = r'''\documentclass[preview]{{standalone}}
    \usepackage{{booktabs}}
    \begin{{document}}
    {}
    \end{{document}}
    '''
    dataframe = pd.DataFrame.from_dict(table)
    with open(file_name+".tex", 'w') as file:
        file.write(template.format(dataframe.to_latex(
                escape=False,
                column_format = column_format,
                index = False)))
```

The Exponential Distribution

The following function was used to generate a random sample from the distribution $\exp(\theta)$

```
def exp_sample(n, t0):
    samples = []
    for _ in range(n):
        samples.append(-np.log(1-uniform(0,1))/t0)
    return samples
```

The following was used to find the log-likelihood function for the median of a given sample

```
def log_like_median(samples):
    def l(m):
        result = 1
        for xi in samples:
            result *= np.log(2)/m*2**(-xi/m)
        return np.log(result)
    return l
```

The following was used to draw the figures and provide the data shown in question 2

```
samples = exp_sample(6, 1.2)
samples.sort()
l6 = log_like_median(samples)
x_values = np.linspace(0,5,100)
```

```
y_values_6 = l6(x_values)

median_MLE = np.log(2)*sum(samples)/6
with open('data.txt', 'w') as file:
    file.write("Question 2\n")
    file.write("The true median is: " + str(np.log(2)/1.2) +"\n")
    file.write("The median MLE is: " + str(median_MLE)+"\n")
    file.write("The samples:" + str(samples)+"\n\n")

fig = plt.figure()
ax = fig.add_subplot(1, 1, 1)

ax.plot(x_values, y_values_6)
plt.show()

fig.savefig("Q2.pdf")
```

The following was used to draw the figures and provide the data shown in question 3

```
# n=25

samples_25 = exp_sample(25, 1.2)
samples_25.sort()
l25 = log_like_median(samples_25)
x_values = np.linspace(0,5,100)
y_values_25 = l25(x_values)

median_MLE_25 = np.log(2)*sum(samples_25)/25
with open('data.txt', 'a') as file:
    file.write("Question 3\n")
    file.write("n = 25\n")
    file.write("The median MLE is: " + str(median_MLE_25) + "\n")
    file.write("The samples:" + str(samples_25) + "\n\n")

# n=50

samples_50 = exp_sample(50, 1.2)
samples_50.sort()
l50 = log_like_median(samples_50)
x_values = np.linspace(0,5,100)
y_values_50 = l50(x_values)

median_MLE_50 = np.log(2)*sum(samples_50)/50
with open('data.txt', 'a') as file:
    file.write("n = 50\n")
    file.write("The median MLE is: " + str(median_MLE_50) + "\n")
    file.write("The samples:" + str(samples_50) + "\n\n")

# n=100

samples_100 = exp_sample(100, 1.2)
samples_100.sort()
l100 = log_like_median(samples_100)
x_values = np.linspace(0,5,100)
```

```
y_values_100 = l100(x_values)

median_MLE_100 = np.log(2)*sum(samples_100)/100
with open('data.txt', 'a') as file:
    file.write("n = 100\n")
    file.write("The median MLE is: " + str(median_MLE_100) + "\n")
    file.write("The samples:" + str(samples_100) + "\n\n")

# The Graph

fig = plt.figure(figsize=(13,5))
ax = fig.add_subplot(1, 3, 1)
ax.plot(x_values, y_values_25)
ax.set_xlabel("$m$")
ax.set_ylabel("$l_25(m)$")
ax.set_title("$n=25$")

ax = fig.add_subplot(1, 3, 2)
ax.plot(x_values, y_values_50)
ax.set_xlabel("$m$")
ax.set_ylabel("$l_50(m)$")
ax.set_title("$n=50$")


ax = fig.add_subplot(1, 3, 3)
ax.plot(x_values, y_values_100)
ax.set_xlabel("$m$")
ax.set_ylabel("$l_100(m)$")
ax.set_title("$n=100$")
plt.tight_layout()

fig.savefig("Q3.pdf")
```

The following function was used to generate a random sample from the distribution $\Gamma(2, \theta)$

```
def gamma_sample(n, t0):
    samples = []
    for _ in range(n):
        samples.append(sum(exp_sample(2, t0)))
    return samples
```

The following was used to find the log-likelihood function for theta of a given sample

```
def log_like_theta(samples):
    def l(t):
        result = 1
        for xi in samples:
            result *= t*t*xi*np.e**(-t*xi)
        return np.log(result)
    return l
```

The following was used to draw the figure and provide the data shown in question 7

```
samples_dist_10 = gamma_sample(10, 2.2)
samples_dist_10.sort()
l_dist_10 = log_like_theta(samples_dist_10)
x_values = np.linspace(0,5,100)
y_values_dist_10 = l_dist_10(x_values)

theta_MLE_dist_10 = 2/(sum(samples_dist_10)/10)
with open('data.txt', 'a') as file:
    file.write("Question 7\n")
    file.write("n = 10\n")
    file.write("The Theta MLE is: " + str(theta_MLE_dist_10) + "\n")
    file.write("The samples:" + str(samples_dist_10) + "\n\n")

# n = 30

samples_dist_30 = gamma_sample(30, 2.2)
samples_dist_30.sort()
l_dist_30 = log_like_theta(samples_dist_30)
x_values = np.linspace(0,5,100)
y_values_dist_30 = l_dist_30(x_values)

theta_MLE_dist_30 = 2/(sum(samples_dist_30)/30)
with open('data.txt', 'a') as file:
    file.write("n = 30\n")
    file.write("The Theta MLE is: " + str(theta_MLE_dist_30) + "\n")
    file.write("The samples:" + str(samples_dist_30) + "\n\n")



# n = 50

samples_dist_50 = gamma_sample(50, 2.2)
samples_dist_50.sort()
l_dist_50 = log_like_theta(samples_dist_50)
x_values = np.linspace(0,5,100)
y_values_dist_50 = l_dist_50(x_values)

theta_MLE_dist_50 = 2/(sum(samples_dist_50)/50)
with open('data.txt', 'a') as file:
    file.write("n = 50\n")
    file.write("The Theta MLE is: " + str(theta_MLE_dist_50) + "\n")
    file.write("The samples:" + str(samples_dist_50) + "\n\n")

# The Graph

fig = plt.figure(figsize = (13,5))
ax = fig.add_subplot(1, 3, 1)
ax.plot(x_values, y_values_dist_10)
ax.set_xlabel("$m$")
ax.set_ylabel("$l_10(m)$")
ax.set_title("$n=10$")

ax = fig.add_subplot(1, 3, 2)
```

```
ax.plot(x_values, y_values_dist_30)
ax.set_xlabel("$m$")
ax.set_ylabel("$l_30(m)$")
ax.set_title("$n=30$")

ax = fig.add_subplot(1, 3, 3)
ax.plot(x_values, y_values_dist_50)
ax.set_xlabel("$m$")
ax.set_ylabel("$l_50(m)$")
ax.set_title("$n=50$")
plt.tight_layout()

fig.savefig("Q7.pdf")
```

The following was used to draw the figure and provide the data shown in question 8

```
n=10
N=200
set_of_samples = [gamma_sample(n, 2.2) for _ in range(N)]
set_of_MLEs_10 = [2/(sum(x)/n) for x in set_of_samples]

n=50

set_of_samples = [gamma_sample(n, 2.2) for _ in range(N)]
set_of_MLEs_50 = [2/(sum(x)/n) for x in set_of_samples]

fig = plt.figure(figsize=(9,5))

ax = fig.add_subplot(1, 2, 1)
ax.hist(set_of_MLEs_10, bins=5)


ax = fig.add_subplot(1, 2, 2)
ax.hist(set_of_MLEs_50, bins=10)

fig.savefig("Q8.pdf")
```

The Normal Distribution

The following generates a random sample of size n from distribution $N(\mu, 1)$

```
def norm_sample(mu, n):
    samples = []
    while len(samples)<n:
        A = uniform(0,1)
        B = uniform(0,1)
        Phi = 2*np.pi*A
        V = -2*np.log(1-B)
        X = mu + V**(0.5)*np.cos(Phi)
        Y = mu + V**(0.5)*np.sin(Phi)
        samples.append(X)
        samples.append(Y)
    return samples[0:n]
```

The following was used to draw the figure and provide the data shown in question 11

```
sample = norm_sample(0,100)
x_bar = sum(sample)/100
CI = [x_bar - 1.2815516/10, x_bar + 1.2815516/10]
if CI[0]<0 and CI[1]>0:
    print("Sample Mean within CI")
else:
    print("Sample Mean not within CI")
with open('data.txt', 'a') as file:
    file.write("Question 11\n")
    file.write("sample mean = " + str(x_bar) + "\n")
    file.write("The CI is: " + str(CI) + "\n")
    file.write("Does lie in CI:" + str(CI[0]<0 and CI[1]>0) + "\n\n")

table_25 = {"sample mean":[],
         "lower bound":[],
         "upper bound":[],
         "in confidence interval?":[]}

for _ in range(25):
    s = norm_sample(0,100)
    m = sum(s)/100
    lb = m - 1.2815516/10
    ub = m + 1.2815516/10
    in_ci = int(lb<0 and ub>0)
    table_25["sample mean"].append(m)
    table_25["lower bound"].append(lb)
    table_25["upper bound"].append(ub)
    table_25["in confidence interval?"].append(in_ci)


draw_table(table_25, "table_25", "|c||l|l|l|c|")

with open('data.txt', 'a') as file:
    file.write("n = 25\n")
    file.write("mu = 0\n")
    file.write("# mu not in CI: " + str(25-sum(table_25["in"])) + "\n\n")
```

The χ^2 Distribution

The following generates a random sample of size n from a χ^2_k distribution

```
def chi_sample(DoF, n):
    samples = []
    for _ in range(n):
        norm_samples = norm_sample(0, DoF)
        sqr = [x*x for x in norm_samples]
        samples.append(sum(sqr))
    return samples
```

The following was used to draw the figure and provide the data shown in question 11

```
chi_samples_1_100 = chi_sample(1, 100)
chi_samples_1_300 = chi_sample(1, 300)
```

```
chi_samples_1_500 = chi_sample(1, 500)

chi_samples_5_100 = chi_sample(5, 100)
chi_samples_5_300 = chi_sample(5, 300)
chi_samples_5_500 = chi_sample(5, 500)

chi_samples_40_100 = chi_sample(40, 100)
chi_samples_40_300 = chi_sample(40, 300)
chi_samples_40_500 = chi_sample(40, 500)



fig = plt.figure()
ax11 = fig.add_subplot(3, 3, 1)
ax11.hist(chi_samples_1_100, bins=10)
ax11.set_title("1 DoF, n = 100")

ax12 = fig.add_subplot(3, 3, 2)
ax12.hist(chi_samples_5_100, bins=10)
ax12.set_title("5 DoF, n = 100")

ax13 = fig.add_subplot(3, 3, 3)
ax13.hist(chi_samples_40_100, bins=10)
ax13.set_title("40 DoF, n = 100")

ax21 = fig.add_subplot(3, 3, 4)
ax21.hist(chi_samples_1_300, bins=10)
ax21.set_title("1 DoF, n = 300")

ax22 = fig.add_subplot(3, 3, 5)
ax22.hist(chi_samples_5_300, bins=10)
ax22.set_title("5 DoF, n = 300")

ax23 = fig.add_subplot(3, 3, 6)
ax23.hist(chi_samples_40_300, bins=10)
ax23.set_title("40 DoF, n = 300")

ax31 = fig.add_subplot(3, 3, 7)
ax31.hist(chi_samples_1_500, bins=10)
ax31.set_title("1 DoF, n = 500")

ax32 = fig.add_subplot(3, 3, 8)
ax32.hist(chi_samples_5_500, bins=10)
ax32.set_title("5 DoF, n = 500")

ax33 = fig.add_subplot(3, 3, 9)
ax33.hist(chi_samples_40_500, bins=10)
ax33.set_title("40 DoF, n = 500")

plt.tight_layout()
plt.show()

fig.savefig("Q13.pdf")
```