Comparative Analysis: Direct Enumeration vs. Principle of Inclusion-Exclusion

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1 The Problem

A bag contains chips of four different colors:

- Red (R): 5 chips
- Green (G): 6 chips
- Blue (B): 7 chips
- Yellow (Y): 8 chips
- **Total Chips:** $5 + 6 + 7 + 8 = 26$

We draw 8 chips at random. What is the probability that we get at least one of each color?

2 Total Possible Outcomes

The total number of ways to draw 8 chips from 26 is:

$$N = \binom{26}{8} = \frac{26!}{8!(26-8)!} = 1,562,275$$

3 Method 1: The Direct Method (Listing All Cases)

This method requires us to find every combination of 8 chips that includes at least one of each of the 4 colors. This is done by partitioning the number 8 into 4 positive integers. The five possible partitions for the counts of the four colors are:

- $(5, 1, 1, 1)$
- $(4, 2, 1, 1)$
- $(3, 3, 1, 1)$
- $(3, 2, 2, 1)$
- $(2, 2, 2, 2)$

We must calculate the number of outcomes for all 35 sub-cases derived from these partitions.

3.1 Case 1: Partition (5, 1, 1, 1) - 4 sub-cases

One color is represented by 5 chips and the other three by 1 chip each.

- 5R, 1G, 1B, 1Y: $\binom{5}{5}\binom{6}{1}\binom{7}{1}\binom{8}{1} = 1 \cdot 6 \cdot 7 \cdot 8 = 336$
- 1R, 5G, 1B, 1Y: $\binom{5}{1}\binom{6}{5}\binom{7}{1}\binom{8}{1} = 5 \cdot 6 \cdot 7 \cdot 8 = 1,680$
- 1R, 1G, 5B, 1Y: $\binom{5}{1}\binom{6}{1}\binom{7}{5}\binom{8}{1} = 5 \cdot 6 \cdot 21 \cdot 8 = 5,040$
- 1R, 1G, 1B, 5Y: $\binom{5}{1}\binom{6}{1}\binom{7}{1}\binom{8}{5} = 5 \cdot 6 \cdot 7 \cdot 56 = 11,760$

Subtotal: $336 + 1,680 + 5,040 + 11,760 = 18,816$

3.2 Case 2: Partition (4, 2, 1, 1) - 12 sub-cases

One color is represented by 4 chips, another by 2 chips, and the other two by 1 chip each.

- 4R, 2G, 1B, 1Y: $\binom{5}{4}\binom{6}{2}\binom{7}{1}\binom{8}{1} = 5 \cdot 15 \cdot 7 \cdot 8 = 4,200$
- 4R, 1G, 2B, 1Y: $\binom{5}{4}\binom{6}{1}\binom{7}{2}\binom{8}{1} = 5 \cdot 6 \cdot 21 \cdot 8 = 5,040$
- 4R, 1G, 1B, 2Y: $\binom{5}{4}\binom{6}{1}\binom{7}{1}\binom{8}{2} = 5 \cdot 6 \cdot 7 \cdot 28 = 5,880$
- 2R, 4G, 1B, 1Y: $\binom{5}{2}\binom{6}{4}\binom{7}{1}\binom{8}{1} = 10 \cdot 15 \cdot 7 \cdot 8 = 8,400$
- 1R, 4G, 2B, 1Y: $\binom{5}{1}\binom{6}{4}\binom{7}{2}\binom{8}{1} = 5 \cdot 15 \cdot 21 \cdot 8 = 12,600$
- 1R, 4G, 1B, 2Y: $\binom{5}{1}\binom{6}{4}\binom{7}{1}\binom{8}{2} = 5 \cdot 15 \cdot 7 \cdot 28 = 14,700$
- 2R, 1G, 4B, 1Y: $\binom{5}{2}\binom{6}{1}\binom{7}{4}\binom{8}{1} = 10 \cdot 6 \cdot 35 \cdot 8 = 16,800$
- 1R, 2G, 4B, 1Y: $\binom{5}{1}\binom{6}{2}\binom{7}{4}\binom{8}{1} = 5 \cdot 15 \cdot 35 \cdot 8 = 21,000$
- 1R, 1G, 4B, 2Y: $\binom{5}{1}\binom{6}{1}\binom{7}{4}\binom{8}{2} = 5 \cdot 6 \cdot 35 \cdot 28 = 29,400$
- 2R, 1G, 1B, 4Y: $\binom{5}{2}\binom{6}{1}\binom{7}{1}\binom{8}{4} = 10 \cdot 6 \cdot 7 \cdot 70 = 29,400$
- 1R, 2G, 1B, 4Y: $\binom{5}{1}\binom{6}{2}\binom{7}{1}\binom{8}{4} = 5 \cdot 15 \cdot 7 \cdot 70 = 36,750$
- 1R, 1G, 2B, 4Y: $\binom{5}{1}\binom{6}{1}\binom{7}{2}\binom{8}{4} = 5 \cdot 6 \cdot 21 \cdot 70 = 44,100$

Subtotal: $4,200+5,040+5,880+8,400+12,600+14,700+16,800+21,000+29,400+29,400+36,750+44,100 = 228,270$

3.3 Case 3: Partition (3, 3, 1, 1) - 6 sub-cases

Two colors are represented by 3 chips each, and the other two by 1 chip each.

- 3R, 3G, 1B, 1Y: $\binom{5}{3}\binom{6}{3}\binom{7}{1}\binom{8}{1} = 10 \cdot 20 \cdot 7 \cdot 8 = 11,200$
- 3R, 1G, 3B, 1Y: $\binom{5}{3}\binom{6}{1}\binom{7}{3}\binom{8}{1} = 10 \cdot 6 \cdot 35 \cdot 8 = 16,800$
- 3R, 1G, 1B, 3Y: $\binom{5}{3}\binom{6}{1}\binom{7}{1}\binom{8}{3} = 10 \cdot 6 \cdot 7 \cdot 56 = 23,520$
- 1R, 3G, 3B, 1Y: $\binom{5}{1}\binom{6}{3}\binom{7}{3}\binom{8}{1} = 5 \cdot 20 \cdot 35 \cdot 8 = 28,000$
- 1R, 3G, 1B, 3Y: $\binom{5}{1}\binom{6}{3}\binom{7}{1}\binom{8}{3} = 5 \cdot 20 \cdot 7 \cdot 56 = 39,200$

- 1R, 1G, 3B, 3Y: $\binom{5}{1}\binom{6}{1}\binom{7}{3}\binom{8}{3} = 5 \cdot 6 \cdot 35 \cdot 56 = 58,800$

Subtotal: $11,200 + 16,800 + 23,520 + 28,000 + 39,200 + 58,800 = 177,520$

3.4 Case 4: Partition (3, 2, 2, 1) - 12 sub-cases

One color is represented by 3 chips, two by 2 chips each, and one by 1 chip.

- 3R, 2G, 2B, 1Y: $\binom{5}{3}\binom{6}{2}\binom{7}{2}\binom{8}{1} = 10 \cdot 15 \cdot 21 \cdot 8 = 25,200$
- 3R, 2G, 1B, 2Y: $\binom{5}{3}\binom{6}{2}\binom{7}{1}\binom{8}{2} = 10 \cdot 15 \cdot 7 \cdot 28 = 29,400$
- 3R, 1G, 2B, 2Y: $\binom{5}{3}\binom{6}{1}\binom{7}{2}\binom{8}{2} = 10 \cdot 6 \cdot 21 \cdot 28 = 35,280$
- 2R, 3G, 2B, 1Y: $\binom{5}{2}\binom{6}{3}\binom{7}{2}\binom{8}{1} = 10 \cdot 20 \cdot 21 \cdot 8 = 33,600$
- 2R, 3G, 1B, 2Y: $\binom{5}{2}\binom{6}{3}\binom{7}{1}\binom{8}{2} = 10 \cdot 20 \cdot 7 \cdot 28 = 39,200$
- 1R, 3G, 2B, 2Y: $\binom{5}{1}\binom{6}{3}\binom{7}{2}\binom{8}{2} = 5 \cdot 20 \cdot 21 \cdot 28 = 58,800$
- 2R, 2G, 3B, 1Y: $\binom{5}{2}\binom{6}{2}\binom{7}{3}\binom{8}{1} = 10 \cdot 15 \cdot 35 \cdot 8 = 42,000$
- 2R, 1G, 3B, 2Y: $\binom{5}{2}\binom{6}{1}\binom{7}{3}\binom{8}{2} = 10 \cdot 6 \cdot 35 \cdot 28 = 58,800$
- 1R, 2G, 3B, 2Y: $\binom{5}{1}\binom{6}{2}\binom{7}{3}\binom{8}{2} = 5 \cdot 15 \cdot 35 \cdot 28 = 73,500$
- 2R, 2G, 1B, 3Y: $\binom{5}{2}\binom{6}{2}\binom{7}{1}\binom{8}{3} = 10 \cdot 15 \cdot 7 \cdot 56 = 58,800$
- 2R, 1G, 2B, 3Y: $\binom{5}{2}\binom{6}{1}\binom{7}{2}\binom{8}{3} = 10 \cdot 6 \cdot 21 \cdot 56 = 70,560$
- 1R, 2G, 2B, 3Y: $\binom{5}{1}\binom{6}{2}\binom{7}{2}\binom{8}{3} = 5 \cdot 15 \cdot 21 \cdot 56 = 88,200$

Subtotal: $25,200 + 29,400 + 35,280 + 33,600 + 39,200 + 58,800 + 42,000 + 58,800 + 73,500 + 58,800 + 70,560 + 88,200 = 613,340$

3.5 Case 5: Partition (2, 2, 2, 2) - 1 sub-case

Each color is represented by 2 chips.

- 2R, 2G, 2B, 2Y: $\binom{5}{2}\binom{6}{2}\binom{7}{2}\binom{8}{2} = 10 \cdot 15 \cdot 21 \cdot 28 = 88,200$

Subtotal: $88,200$

4 Total Successful Outcomes and Final Probability

Summing the outcomes from all five partitions gives the total number of ways to draw 8 chips with at least one of each color.

$$\text{Total Favorable Outcomes} = 18,816 + 228,270 + 177,520 + 613,340 + 88,200 = 1,126,146$$

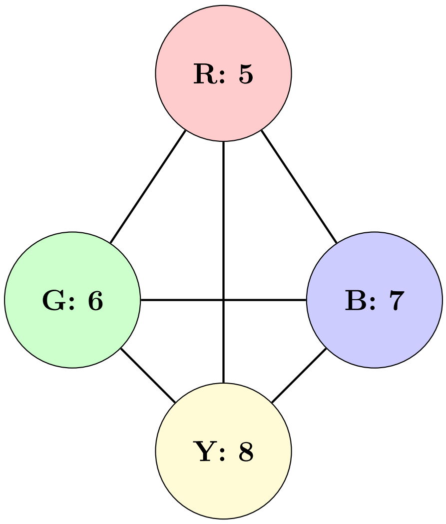
The probability is the ratio of successful outcomes to total possible outcomes:

$$P(\text{at least one of each color}) = \frac{1,126,146}{1,562,275} \approx 0.720837240562641 \approx 72\%$$

5 Method 2: The Principle of Inclusion-Exclusion

Here's a visual representation of the problem. The way this will be used is by masking nodes to determine the terms to use in the inclusion-exclusion formula.

Complete Graph: All Four Colortexts Required



By the complement rule, $P(\text{at least one of each color}) = 1 - P(\text{missing at least one color})$

Using the inclusion-exclusion principle (Condensed notation to save room):

$$\begin{aligned}
P(\text{missing } \geq 1 \text{ color}) = {} & P(\neg R) + P(\neg G) + P(\neg B) + P(\neg Y) && (1)\\
& - P(\neg R \& \neg G) - P(\neg R \& \neg B) - P(\neg R \& \neg Y) && (2)\\
& - P(\neg G \& \neg B) - P(\neg G \& \neg Y) - P(\neg B \& \neg Y) && (3)\\
& + P(\neg R \& \neg G \& \neg B) + P(\neg R \& \neg G \& \neg Y) && (4)\\
& + P(\neg R \& \neg B \& \neg Y) + P(\neg G \& \neg B \& \neg Y) && (5)\\
& - P(\neg R \& \neg G \& \neg B \& \neg Y) && (6)
\end{aligned}$$

Computing each term:

$$P(\text{no red}) = \frac{\binom{21}{8}}{\binom{26}{8}} = \frac{203,490}{1,562,275} \tag{7}$$

$$P(\text{no green}) = \frac{\binom{20}{8}}{\binom{26}{8}} = \frac{125,970}{1,562,275} \tag{8}$$

$$P(\text{no blue}) = \frac{\binom{19}{8}}{\binom{26}{8}} = \frac{75,582}{1,562,275} \tag{9}$$

$$P(\text{no yellow}) = \frac{\binom{18}{8}}{\binom{26}{8}} = \frac{43,758}{1,562,275} \tag{10}$$

And for missing two colors:

$$P(\text{no red \& no green}) = \frac{\binom{15}{8}}{\binom{26}{8}} = \frac{6,435}{1,562,275} \tag{11}$$

$$P(\text{no red \& no blue}) = \frac{\binom{14}{8}}{\binom{26}{8}} = \frac{3,003}{1,562,275} \tag{12}$$

$$P(\text{no red \& no yellow}) = \frac{\binom{13}{8}}{\binom{26}{8}} = \frac{1,287}{1,562,275} \tag{13}$$

$$P(\text{no green \& no blue}) = \frac{\binom{13}{8}}{\binom{26}{8}} = \frac{1,287}{1,562,275} \tag{14}$$

$$P(\text{no green \& no yellow}) = \frac{\binom{12}{8}}{\binom{26}{8}} = \frac{495}{1,562,275} \tag{15}$$

$$P(\text{no blue \& no yellow}) = \frac{\binom{11}{8}}{\binom{26}{8}} = \frac{165}{1,562,275} \tag{16}$$

For missing three colors:

$$P(\text{no red \& no green \& no blue}) = \frac{\binom{8}{8}}{\binom{26}{8}} = \frac{1}{1,562,275} \tag{17}$$

$$P(\text{no red \& no green \& no yellow}) = \frac{\binom{7}{8}}{\binom{26}{8}} = 0 \text{ (impossible)} \tag{18}$$

$$P(\text{no red \& no blue \& no yellow}) = \frac{\binom{6}{8}}{\binom{26}{8}} = 0 \text{ (impossible)} \tag{19}$$

$$P(\text{no green \& no blue \& no yellow}) = \frac{\binom{5}{8}}{\binom{26}{8}} = 0 \text{ (impossible)} \tag{20}$$

For missing all four colors:

$$P(\text{no red \& no green \& no blue \& no yellow}) = \frac{\binom{0}{8}}{\binom{26}{8}} = 0 \text{ (impossible)} \tag{21}$$

Combining all the numerator terms:

$$\begin{aligned}
&(203,490 + 125,970 + 75,582 + 43,758) && (22)\\
&\quad - (6,435 + 3,003 + 1,287 + 1,287 + 495 + 165) && (23)\\
&\quad + 1 && (24)\\
&= 448,800 - 12,672 + 1 && (25)\\
&= 436,129 && (26)
\end{aligned}$$

Therefore:

$$P(\text{missing } \geq \text{ 1 color}) = \frac{436,129}{1,562,275} \approx 0.27916275943735896$$

$$P(\text{at least one of each color}) = 1 - \frac{436,129}{1,562,275} \approx 0.7208372405626411 \approx 72\%$$

6 Comparison

In most cases, the direct enumeration method is straightforward and easy to understand if you think about it as partitioning the problem into cases and sub-cases. However, this example showed that it required 35 terms!

We can significantly reduce the number of terms to 11 by applying method 2, where we take the complement and apply inclusion-exclusion. The individual terms can be easily found by masking nodes corresponding to the colors we want to exclude.

For example, if you wanted to compute $P(\text{no red \& no yellow})$, you would mask the red and yellow nodes and compute $\binom{13}{8}$. This is because there are 6 green and 7 blue chips left, and we want to choose 8 of them $\binom{6+7}{8}$.