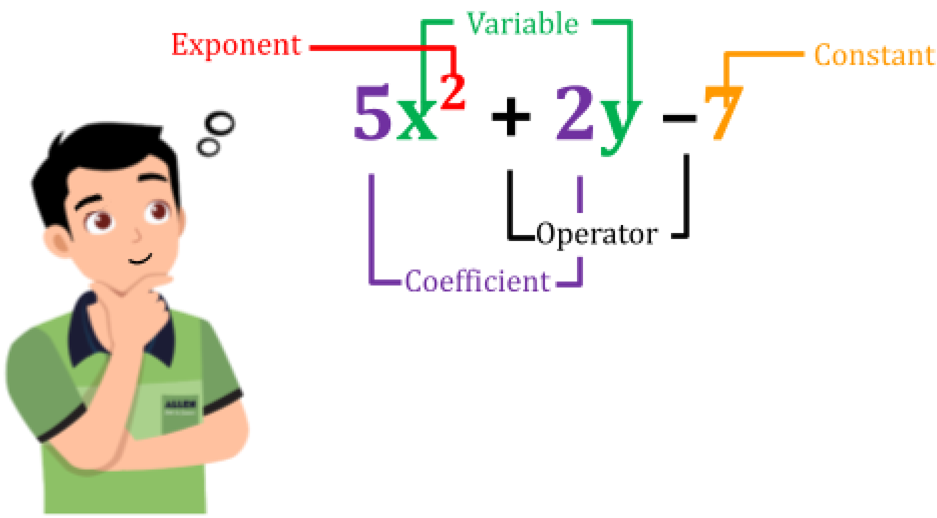
2 Polynomials



Polynomials in one variable

An algebraic expression of the form

$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0x^0$, where

(i) $a_n \neq 0$.

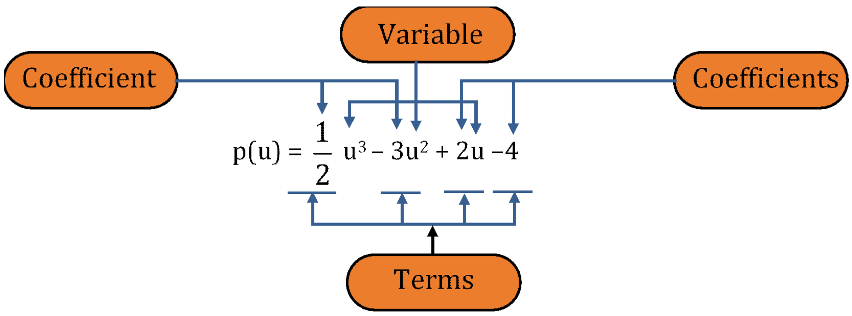
(ii) $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

(iii) Power of x is a non-negative integer, is called a polynomial in one variable.

Hence, $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are coefficients of $x^n, x^{n-1}, \ldots,$ x^0 respectively and $a_nx^n, a_{n-1}x_{n-1}, a_{n-2}x^{n-2}, \ldots$ are terms of the polynomial. Here, the term a_nx^n is called the **leading term** and its coefficient a_n is known as **leading coefficient**.

A polynomial in x is said to be a polynomial in standard form, if the powers of x are either in ascending order or in descending order.

SPOT LIGHT



Do You Remember ?

- In previous classes we have studied about algebraic expressions. Algebraic expressions are combination of constants and variables (their combination called terms) which are separated by mathematical operators (+, –, ×, ÷). In this chapter we will learn about special category of algebraic expressions called polynomials.

  For e.g. $\frac{3x^2y}{z} + 2xy - 1$ is an algebraic expression but not a polynomial.

Check your Concepts 1

Complete the following table :

S. No.	Algebraic expression	Polynomial (Yes/No)	Reason
(i)	$6x^{-2}$	No	Variable has a negative exponent
(ii)	$\frac{1}{x^2}$		
(iii)	$\sqrt{x}$		
(iv)	$4x^2$		

Constant polynomial

Constants 2, – 2, $\sqrt{2}$, $\frac{3}{2}$ and a can be written as $2x^0$, $-2x^0$, $\sqrt{2}\,x^0$, $\frac{3}{2}x^0$ and ax^0 respectively as we know that $x^0 = 1$. Therefore, these constants are expressed as polynomials which contain single term in variable x and the exponent of the variable is 0. Thus, we can define a constant as a constant polynomial.

Note: In particular, the constant number 0 is termed as the zero polynomial. The value of zero polynomial is always zero and its degree is not defined.

Degree of polynomials

Degree of the polynomial in one variable is the largest exponent of the variable.

For e.g., The degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5x^6 - 4x^2 - 6$ is 6.

Be Alert !

- While deciding if an algebraic expression is a polynomial or not, we check the exponent on variable and not constant.

  For e.g. $f(x) = 2\sqrt{x}+1$ is not a polynomial.

  but $f(x) = \sqrt{2}x+1$ is a polynomial.

- To decide the degree of a polynomial in one variable, we check the exponent on variable and not on constant.

  For e.g. Degree of polynomial

  $f(x) = 2^5x^3 + 2^4x^2 + x + 1$ is 3 and not 5.

- In your previous class, you have seen the algebraic expression involving more than one variable. We use to find degree of the expression by adding exponents on variables

  For e.g. Degree of $xy^2 + 2xy^3 + 5xyz$ is 4.

  But here we are dealing with polynomials in one variable only so no such procedure will be followed.

Building Concepts 1

Write whether the following statements are True or False. Justify your answer.

(i) $\frac{1}{\sqrt{5}}x^{1/2} + 1$ **is a polynomial**

(ii) $\frac{6\sqrt{x}+x^{3/2}}{\sqrt{x}}$ **is a polynomial, $x \neq 0$**

Explanation

(i) False, because the exponent of the variable is not a whole number.

(ii) True, because $\frac{6\sqrt{x}+x^{3/2}}{\sqrt{x}} = 6 + x$, which is a polynomial.

Building Concepts 2

For the polynomial $\frac{x^3+2x+1}{5} - \frac{7}{2}x^2 - x^6$, write

(i) The degree of the polynomial

(ii) The coefficient of x^3

(iii) The coefficient of x^6

(iv) The constant term

Explanation

Polynomial $\frac{x^3+2x+1}{5} - \frac{7}{2}x^2 - x^6$ can be written as $\frac{x^3}{5}+\frac{2}{5}x+\frac{1}{5} - \frac{7}{2}x^2 - x^6$

or $\quad -x^6 + \frac{x^3}{5} - \frac{7}{2}x^2 + \frac{2}{5}x + \frac{1}{5}$

or $\quad (-1)\,x^6 + (0)\,x^5 + (0)\,x^4 + \left(\frac{1}{5}\right)x^3 + \left(\frac{-7}{2}\right)x^2 + \left(\frac{2}{5}\right)x + \frac{1}{5}$

So,

(i) The degree of the polynomial = 6

(ii) The coefficient of $x^3 = \frac{1}{5}$

(iii) The coefficient of $x^6 = -1$

(iv) The constant term = $\frac{1}{5}$

Building Concepts 3

Which of the following expressions are polynomials? Justify your answer.

(i) 8 **(ii) $\sqrt{3}x^2 - 2x$** **(iii) $1-\sqrt{5x}$**

(iv) $x^2 + 2$ **(v) $\frac{(x-2)(x-4)}{x}$** **(vi) $\frac{1}{x+1}$**

(vii) $\frac{1}{7}a^3 - \frac{2}{\sqrt{3}}a^2 + 4a - 7$ (viii) $\frac{1}{2x}$

Explanation

(i), (ii), (iv), (vii) are polynomials because the exponent of the variable after simplification in each of these is a whole number.

Check your Answers 1

(ii) No, variable has negative exponent in numerals.

(iii) No, variable has 1/2 exponent which is not a whole number.

(iv) Yes, variable has a whole number exponent.

Check your Concepts 2

1. Write 3 algebraic expressions which are not polynomials. Justify your answer.

2. Write the degree of the following polynomials:

(i) $(2a + 1)(5a - 7)$ (ii) $\frac{7}{3}(b)^3 + \frac{3}{7}(b)^7$

Classification of Polynomials

Polynomials classified by degree

Degree	Name	General form	Example
(undefined)	Zero polynomial	0	0
0	(Non-zero) constant polynomial	$a; (a \neq 0)$	1
1	Linear polynomial	$ax + b; (a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c; (a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d; (a \neq 0)$	$x^3 + 1$

Usually, a polynomial of degree n, for n greater than 3, is called a polynomial of degree n, although the phrases quartic polynomial (for n = 4) and quintic polynomial (for n = 5) are sometimes used.

Polynomials classified by number of terms:

Monomials : Polynomials having only one term are called monomials.

E.g. $p(x) = 2x$, $q(y) = 7y^5$, $r(t) = 12t^7$ etc.

Binomials: Polynomials having exactly two unlike terms are called binomials.

E.g. $p(x) = 2x + 1$, $r(y) = 2y^7 + 5y^6$ etc.

Trinomials : Polynomials having exactly three unlike terms are called trinomials.

E.g. $p(x) = 2x^2 + x + 6$, $q(y) = 9y^6 + 4y^2 + 1$ etc.

Building Concepts 4

Classify the following as linear, quadratic & cubic polynomials.

(i) $y + y^3 + 4$ **(ii)** $5x - \frac{3}{2}$ **(iii)** $x^2 - 2x + 3$

(iv) $3x^3 + 2x^2 + 1$

Explanation

(i) It is a cubic polynomial because its degree is 3.

(ii) It is a linear polynomial because its degree is 1.

(iii) It is quadratic polynomial because its degree is 2.

(iv) It is a cubic polynomial because its degree is 3.

Check your Answers 2

1. $x + \frac{1}{x^2}, x^2 + \frac{1}{x^2}, 5x + \frac{1}{5x}$ are not polynomials as some powers on x are negative integer.

2. (i) 2 (ii) 7

SPOT LIGHT

A non zero constant polynomial has no zero.

Check your Concepts 3

Give an example each of:

(i) A monomial of degree 0

(ii) A binomial of degree 3

(iii) A trinomial of degree 10

Check your Answers 3

(i) 5 (ii) $3x^3 + 1$ (iii) $x^{10} + x^2 + 1$

Value of a polynomial

Consider a polynomial $f(x) = 3x^2 - 4x + 2$. If we replace x by 3 everywhere in the above expression, we get

$f(3) = 3 \times (3)^2 - 4 \times 3 + 2 = 27 - 12 + 2 = 17$

We can say that the value of the polynomial f(x) at x = 3 is 17

Similarly, the value of polynomial $f(x) = 3x^2 - 4x + 2$

at x = – 2 is $f(-2) = 3(-2)^2 - 4 \times (-2) + 2 = 12 + 8 + 2 = 22$

at x = 0 is $f(0) = 3(0)^2 - 4(0) + 2 = 0 - 0 + 2 = 2$

at $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = 3 \times \left(\frac{1}{2}\right)^2 - 4 \times \left(\frac{1}{2}\right) + 2 = \frac{3}{4} - 2 + 2 = \frac{3}{4}$

In general, we can say f(a) is the value of the polynomial f(x) at x = a, where a is a real number.

Zeroes of a polynomial

A real number a is zero of a polynomial f(x) if the value of the polynomial f(x) is zero at x = a i.e. f(a) = 0.

OR

The value of the variable x, for which the polynomial f(x) becomes zero is called zero of the polynomial.

For e.g. Consider a polynomial $p(x) = x^2 - 5x + 6$; replace x by 2 and 3.

$p(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0,$

$p(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

$\therefore$ 2 and 3 are the zeros of the polynomial p(x).

Roots of a polynomial equation

An expression f(x) = 0 is called a polynomial equation if f(x) is a polynomial of degree n ≥ 1.

A real number a is a root of a polynomial f(x) if f(a) = 0 i.e. a is a zero of the polynomial f(x).

For e.g. consider the polynomial f(x) = 3x – 2, then 3x – 2 = 0 is the corresponding polynomial equation.

Here, $f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) - 2 = 0$

i.e. $\frac{2}{3}$ is a zero of the polynomial f(x) = 3x – 2

or $\frac{2}{3}$ is a root of the polynomial equation 3x – 2 = 0

Numerical Ability 1

Find q(0), q(1) and q(2) for each of the following polynomials.

(i) $q(x) = x^2 + 3x$

(ii) $q(y) = 2 + y + 2y^2 - 5y^3$

Solution

(i) $q(x) = x^2 + 3x$

$\therefore$ $q(0) = (0)^2 + 3 \times 0 = 0$

$q(1) = (1)^2 + 3 \times 1 = 4$

$q(2) = (2)^2 + 3 \times 2 = 4 + 6 = 10$

(ii) $q(y) = 2 + y + 2y^2 - 5y^3$

$\therefore$ $q(0) = 2 + 0 + 2(0)^2 - 5(0)^3 = 2$

$q(1) = 2 + 1 + 2(1)^2 - 5(1)^3 = 2 + 1 + 2 - 5 = 0$

$q(2) = 2 + 2 + 2(2)^2 - 5(2)^3 = 2 + 2 + 8 - 40 = -28$

Numerical Ability 2

Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.

Solution

To evaluate $q(a + 1)$, replace x in q(x) with a + 1

$q(x) = x^2 + 3x + 4$

$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4 = a^2 + 2a + 1 + 3a + 3 + 4 = a^2 + 5a + 8$

To evaluate 2q(a), replace x with a in q(x), then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$

$2q(a) = 2(a^2 + 3a + 4) = 2a^2 + 6a + 8$

Now, evaluate $q(a + 1) - 2q(a)$

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8) = a^2 + 5a + 8 - 2a^2 - 6a - 8 = -a^2 - a$

Building Concepts 5

Boating : A motor boat travelling against waves accelerates from a resting position. Suppose the speed of the boat in metre per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.

(i) Find the speed of the boat at 1, 2 and 3 seconds.

(ii) It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Find f(6) and explain what it means.

Explanation

(i) Speed of the boat at 1 second

$= f(1) = -0.04\,(1)^4 + 0.8\,(1)^3 + 0.5\,(1)^2 - 1$

$= -0.04 + 0.8 + 0.5 - 1 = -1.04 + 1.3 = 0.26$ m/sec.

Speed of the boat at 2 second

$= f(2) = -0.04\,(2)^4 + 0.8\,(2)^3 + 0.5\,(2)^2 - 2$

$= -0.04(16) + 0.8(8) + 0.5(4) - 2$

$= -0.64 + 6.4 + 2.0 - 2 = 5.76$ m/sec.

Speed of the boat at 3 second

$= f(3) = -0.04\,(3)^4 + 0.8\,(3)^3 + 0.5\,(3)^2 - 3$

$= -0.04(81) + 0.8(27) + 0.5(9) - 3$

$= -3.24 + 21.6 + 4.5 - 3$

$= -6.24 + 26.1 = 19.86$ m/sec.

(ii) $f(6) = -0.04\,(6)^4 + 0.8\,(6)^3 + 0.5\,(6)^2 - 6$

$= -51.84 + 172.8 + 18 - 6$

$= -57.84 + 190.8 = 132.96$ m/sec.

This means, the speed of the boat at 6th second = 132.96 m/sec.

Every linear polynomial has one and only one real zero.

SPOT LIGHT

Numerical Ability 3

Verify whether the indicated values of variables are zeros of the polynomials corresponding to them.

(i) $p(y) = 4y - 4\pi,\ y = 4, \pi$

(ii) $q(u) = (u + 1)(u + 2),\ u = -1, 2$

Solution

(i) $p(y) = 4y - 4\pi$

$p(4) = 4(4) - 4\pi = 16 - 4\pi \neq 0$

$p(\pi) = 4\pi - 4\pi = 0$

$\Rightarrow$ π is a zero of the polynomial and 4 is not a zero of the polynomial.

(ii) $q(u) = (u + 1)(u + 2)$

$q(-1) = (-1 + 1)(-1 + 2) = (0)(1) = 0$

$q(2) = (2 + 1)(2 + 2) = (3)(4) = 12 \neq 0$

$\Rightarrow$ – 1 is a zero of the polynomial and 2 is not a zero of the polynomial.

Numerical Ability 4

Find the zero of the polynomials given below

(i) $p(x) = 3x + \pi$

(ii) $q(u) = (u + 3)(u - 4)$

Solution

(i) $p(x) = 3x + \pi$

It's corresponding polynomial equation is $3x + \pi = 0$

$\Rightarrow$ $3x = -\pi$

$\therefore$ $x = -\dfrac{\pi}{3}$ is a zero of the polynomial.

(ii) $q(u) = (u + 3)(u - 4)$

It's corresponding polynomial equation is $(u + 3)(u - 4) = 0$

$\Rightarrow$ either $u + 3 = 0$ or $u - 4 = 0$

$\Rightarrow$ $u = -3$ or $u = 4$

$\therefore$ – 3 and 4 are the zeros of the polynomial.

Building Concepts 6

Show that $p(x) = x^2 + 4x + 7$ has no zero.

Explanation

$p(x) = x^2 + 4x + 7 = x^2 + 4x + 4 + 3$

$= (x + 2)^2 + 3 \geq 3$

Hence, $p(x) \neq 0$ for any x

$\therefore$ p(x) has no zero.

Remainder theorem

Statement : Let p(x) be a polynomial of degree ≥ 1 and 'a' is any real number. If p(x) is divided by $(x - a)$, then the remainder is p(a).

For e.g. Let p(x) be $x^3 - 7x^2 + 6x + 4$

Divide p(x) with $(x - 6)$ and to find the remainder, put $x = 6$ in p(x) i.e. p(6) will be the remainder.

$\therefore$ required remainder will be

$p(6) = (6)^3 - 7.6^2 + 6.6 + 4 = 216 - 252 + 36 + 4 = 256 - 252 = 4$

$x - 6\,)\,x^3 - 7x^2 + 6x + 4\,(\,x^2 - x$

$\quad x^3 - 6x^2$

$\quad - \quad +$

$\quad - x^2 + 6x + 4$

$\quad - x^2 + 6x$

$\quad + \quad -$

Remainder = 4

Thus, p(a) is remainder on dividing p(x) by $(x - a)$.

Remark

(i) $p(-a)$ is remainder on dividing p(x) by $(x + a)$ $[x + a = 0 \Rightarrow x = -a]$

(ii) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by $(ax - b)$ $[ax - b = 0 \Rightarrow x = b/a]$

(iii) $p\left(\frac{-b}{a}\right)$ is remainder on dividing p(x) by $(ax + b)$ $[ax + b = 0 \Rightarrow x = -b/a]$

(iv) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by $(b - ax)$ $[b - ax = 0 \Rightarrow x = b/a]$

Numerical Ability 5

Find the remainder when

(i) $x^3 - ax^2 + 6x - a$ is divided by $x - a$

(ii) $2x^4 + x^3 - 2x^2 + x + 1$ by $2x - 1$

Solution

(i) Let $p(x) = x^3 - ax^2 + 6x - a$

zero of $x - a$ is a

$p(a) = a^3 - a(a)^2 + 6(a) - a$

$= a^3 - a^3 + 6a - a = 5a$

So, by the Remainder theorem, remainder = $5a$

(ii) Let $p(x) = 2x^4 + x^3 - 2x^2 + x + 1$

zero of $2x - 1$ is $\frac{1}{2}$

So, $p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) + 1$

$= \frac{1}{8} + \frac{1}{8} - \frac{1}{2} + \frac{1}{2} + 1$

$= \frac{1}{4} + 1 = \frac{5}{4}$

So, by the Remainder theorem, remainder = $\frac{5}{4}$

Numerical Ability 6

If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a.

Solution

Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when $p(x)$ and $q(x)$ are divided by $(x - 3)$ are $p(3)$ and $q(3)$ respectively.

By the given condition, we have $p(3) = q(3)$

$\Rightarrow$ $a \times (3)^3 + 4 \times (3)^2 + 3 \times 3 - 4 = (3)^3 - 4 \times 3 + a$

$\Rightarrow$ $27a + 36 + 9 - 4 = 27 - 12 + a$

$\Rightarrow$ $26a + 26 = 0$

$\Rightarrow$ 26a = – 26

$\Rightarrow$ a = – 1

Factor theorem

Statement : Let f(x) be a polynomial of degree ≥ 1 and a be any real constant such that

f(a) = 0, then (x – a) is a factor of f(x). Conversely, if (x – a) is a factor of f(x), then f(a) = 0.

Proof : By Remainder theorem, if f(x) is divided by (x – a), the remainder will be f(a). Let q(x) be the quotient. Then, we can write

f(x) = (x – a) × q(x) + f(a) (Dividend = Divisor × Quotient + Remainder)

If f(a) = 0, then f(x) = (x – a) × q(x)

Thus, (x – a) is a factor of f(x).

Converse : Let (x – a) is a factor of f(x).

Then, we have a polynomial q(x) such that f(x) = (x – a) × q(x)

Replacing x by a, we get f(a) = 0.

Hence proved.

Numerical Ability 7

Use the factor theorem to determine whether (x – 1) is a factor of

f(x) = $2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

Solution

By using factor theorem, (x – 1) is a factor of f(x), only when f(1) = 0

f(1) = $2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2}$

= $2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2} = 0$

Hence, (x – 1) is a factor of f(x).

Numerical Ability 8

For what value of k, (x – 1) is a factor of p(x) = kx^2 – 3x + k ?

Solution

Here, p(x) = kx^2 – 3x + k

$\because$ x – 1 is a factor of p(x)

$\therefore \quad p(1) = 0$ (zero of x – 1

is 1)

or $\quad k(1)^2 - 3(1) + k = 0$

$\Rightarrow \quad k - 3 + k = 0$ or $2k - 3 = 0$

$\therefore \quad k = \frac{3}{2}$

Factorisation of Quadratic polynomials

Factorisation of a quadratic polynomial by splitting the middle term

Let $(px + q)$ and $(rx + s)$ be the 2 linear factors of the quadratic polynomial

$ax^2 + bx + c$, $(a \neq 0)$.

Then, $ax^2 + bx + c = (px + q)(rx + s) = prx^2 + (ps + qr)x + qs$

On comparing the coefficients, we get

$a = pr$; $b = ps + qr$ and $c = qs$

Clearly, b is the sum of two numbers ps and qr, whose product $(ps \times qr)$ is equal to ac.

This means, for factorising a quadratic polynomial $ax^2 + bx + c$ into two linear factors, we need to write b as sum of two numbers whose product is ac.

Numerical Ability 9

Factorise the quadratic polynomial $3x^2 + 7x + 2$ by splitting the middle term.

Solution

$p(x) = 3x^2 + 7x + 2$

The coefficient of the middle term is 7. Now, we find two numbers u and v such that $u + v = 7$ and $u \times v = 3 \times 2 = 6$. By inspection, we find $u = 1$ and $v = 6$. Then, we have

$3x^2 + 7x + 2 = 3x^2 + (1 + 6)x + 2$ {By splitting the middle term}

$= 3x^2 + x + 6x + 2$

$= (3x^2 + x) + (6x + 2)$

$= x(3x + 1) + 2(3x + 1)$

$= (3x + 1)(x + 2)$

Active Maths 1

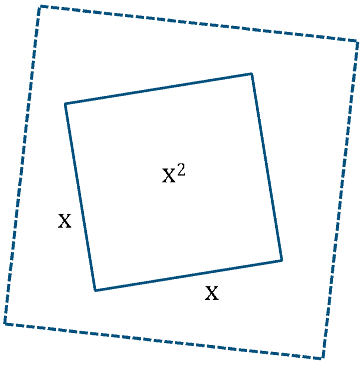
How to represent the factorisation of the quadratic polynomial geometrically?

Exploring the concept.

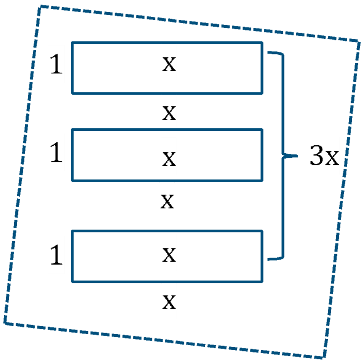
Consider the quadratic polynomial $x^2 + 5x + 6$.

1. The polynomial $x^2 + 5x + 6 \Rightarrow x^2 + 3x + 2x + 6$ can be factorised as $(x + 3)(x + 2)$.

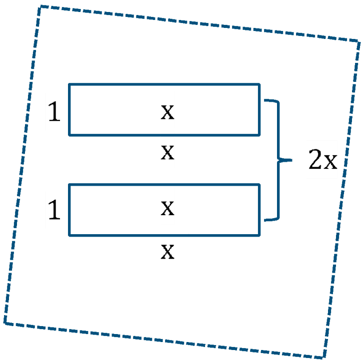
2. To represent x^2, draw a square of x units.



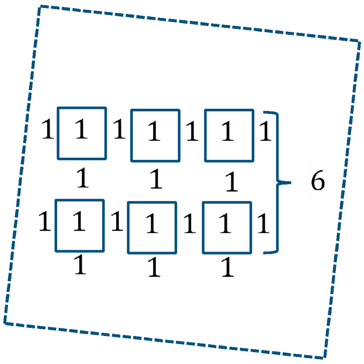
3. To represent 3x, draw three rectangular strips of dimension $(1 \times x)$.



4. To represent 2x, draw two rectangular strips of dimensions $(1 \times x)$.

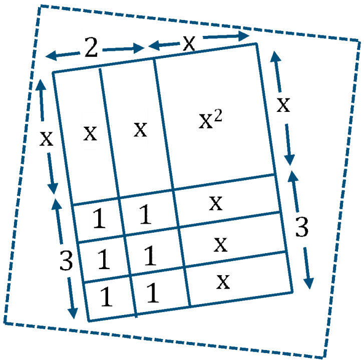


5. To represent 6, draw 6 unit squares.



6. Cut all the strips.

7. Now, paste all the strips together on the white sheet of paper as shown in figure.



Drawing conclusions

$x^2 + 5x + 6$

area of 5 rectangular strips = $5x = 2x + 3x$

area of square = x^2

area of 6 unit squares = 6

total area of rectangle obtained = $x^2 + 3x + 2x + 6 = x^2 + 5x + 6 = (x + 3)(x + 2)$

Thus, we verified the factors of a quadratic polynomial geometrically by paper cutting and pasting.

By using paper cutting and pasting method, represent the factors of following quadratic expressions :

(i) $x^2 - x - 6$ (ii) $2x^2 + 5x + 2$

Factorisation of quadratic polynomials by factor theorem

In this method for factorising a polynomial into linear factors, obtain the constant term by making the coefficient of x^2 unity.

For e.g. consider a quadratic polynomial $ax^2 + bx + c$ $(a \neq 0)$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Let $(x + p)$ and $(x + q)$ be the two linear factors of $\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

Then, $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\,(x + p)\,(x + q) = a\,[x^2 + (p + q)\,x + pq]$

On comparing the coefficients, we get $\frac{c}{a} = pq$

Clearly, the two zeros $(-p)$ and $(-q)$ of the quadratic polynomial $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$ are factors of the constant term $\frac{c}{a}$.

This is the concept behind using the factor theorem for factorisation.

Algorithm to factorise a quadratic polynomials

Step-1 : Obtain the constant term by making the coefficient of x^2 unity.

Step-2 : Take one of the factors of the constant term say a and replace x by it in the given example. If the polynomial reduces to zero, then $(x - a)$ is a factor of polynomial.

Step-3 : In case of quadratic polynomial, divide the given polynomial by $(x - a)$ to get the other linear factor.

Numerical Ability 10

Factorise $x^2 - 5x - 24$ by using the factor theorem.

Solution

$p(x) = x^2 - 5x - 24$

Here, coefficient of the leading term is 1 and the constant term is -24. A zero of the polynomial p(x) will be a factor of the number -24.

By inspection, we find that the number 8 is a divisor of -24 and also we have

$p(8) = (8)^2 - 5(8) - 24 = 64 - 40 - 24 = 0$

i.e. 8 is a zero of the polynomial p(x).

Then $(x - 8)$ is a factor of the polynomial.

We can express $x^2 - 5x - 24 = (x^2 - 8x) + (3x - 24)$

$= x\,(x - 8) + 3\,(x - 8) = (x - 8)\,(x + 3)$

We can also find the second factor of the polynomial by dividing $x^2 - 5x - 24$ by $(x - 8)$.

If the sum of the coefficients of odd powers in a polynomial is equal to sum of coefficients of even powers, then $(x + 1)$ is a factor of the polynomial.

SPOT LIGHT

Quick Tips

- Do you commit mistakes of sign while doing factorisation of quadratic polynomials (i.e middle term splitting)?

It to easy to decide the sign of middle terms. Let us take few examples :

$x^2 - 7x + 12$

Here product of 1st & last term is positive so both the factors will be of "same sign" (they will be either +ve or –ve)

$\Rightarrow$ $x^2 - 4x - 3x + 12$

Also, in the polynomial $x^2 + x - 12$

Here product of 1st and last term is –ve so both the factors will be of opposite sign.

$\Rightarrow$ $x^2 + 4x - 3x - 12$

Factorisation of cubic polynomials

Step-1: Make the coefficient of the leading term as 1 and factorise the constant term and find a suitable factor of constant term, so that it makes value of polynomial zero. We get one linear factor as $(x - a)$. For example factors of –120 are many but we find a suitable factor of – 120 which is a zero of the given polynomial (as in Numerical Ability 1).

Step-2: Divide the given polynomial by $(x - a)$, to get the other factor (quadratic). Factorise the quadratic factor so obtained by the appropriate method to get the 2 linear factors (Already studied in factorisation of quadratic polynomials).

Numerical Ability 11

Factorise $x^3 - 23x^2 + 142x - 120$.

Solution

$p(x) = x^3 - 23x^2 + 142x - 120$

The coefficient of the leading term is 1 and the constant term is – 120. Factors of – 120 are many but we find a suitable factor of – 120 which is a zero of the polynomial.

By inspection, we find that

$p(1) = (1)^3 - 23(1)^2 + 142(1) - 120$

$= 1 - 23 + 142 - 120 = 0$

$\Rightarrow$ $(x - 1)$ is a factor of the polynomial.

Now, we can express the given polynomial as below :

$= x^3 - 23x^2 + 142x - 120$

$= (x^3 - x^2) + (-22x^2 + 22x) + (120x - 120)$

$= x^2(x - 1) - 22x(x - 1) + 120(x - 1)$

$= (x - 1)(x^2 - 22x + 120)$

$= (x - 1)\{x^2 + (-12 - 10)x + 120\}$

$= (x - 1)\{(x^2 - 12x) + (-10x + 120)\}$

$= (x - 1)\{x(x - 12) - 10(x - 12)\}$

$= (x - 1)\{(x - 12)(x - 10)\}$

$= (x - 1)(x - 10)(x - 12)$

SPOT LIGHT

If the sum of the coefficients of a polynomial is zero, then $(x - 1)$ is a factor of the polynomial.

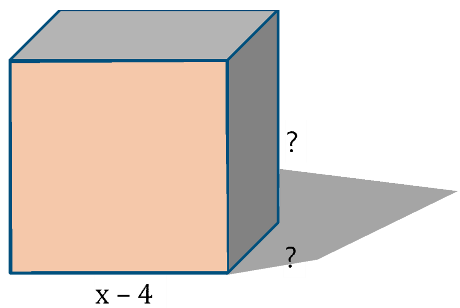
Building Concepts 7

The volume of the rectangular prism is given by
$v(x) = x^3 + 3x^2 - 36x + 32$. Find the missing measures, if one measure is $x - 4$.

Explanation

The volume of a rectangular prism is $l \times b \times h$. One measure is $x - 4$, so, $x - 4$ is a factor of $v(x)$.

$$
\begin{array}{l}
x - 4\ \big)\ x^3 + 3x^2 - 36x + 32\ \big(\ x^2 + 7x - 8 \\
\qquad\quad x^3 - 4x^2 \\
\qquad\ \ - \quad + \\
\hline
\qquad\qquad 7x^2 - 36x + 32 \\
\qquad\qquad 7x^2 - 28x \\
\qquad\qquad - \qquad + \\
\hline
\qquad\qquad\qquad -8x + 32 \\
\qquad\qquad\qquad -8x + 32 \\
\qquad\qquad\qquad + \qquad - \\
\hline
\qquad\qquad\qquad\qquad 0
\end{array}
$$



So, $x^3 + 3x^2 - 36x + 32$

$= (x - 4)(x^2 + 7x - 8)$

$= (x - 4)(x^2 + 8x - x - 8)$

$= (x - 4)\{x(x + 8) - 1(x + 8)\}$

$= (x - 4)(x + 8)(x - 1)$

So, the missing measures are $(x + 8)$ and $(x - 1)$.

Quick Tips

- While factorising cubic and bi-quadratic polynomials, we should start with small values of variables to check for factors. As a matter of fact many questions will become zero for the values ±1, ±2, ±3.

Algebraic identities

An algebraic identity is an algebraic equation that is true for all values of the variables present in the equation.

I. (i) $(x + y)^2 = x^2 + 2xy + y^2$ (ii) $(x - y)^2 = x^2 - 2xy + y^2$

II. $x^2 - y^2 = (x + y)(x - y)$

III. $(x + a)(x + b) = x^2 + (a + b)x + ab$

IV. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

V. (i) $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$ (ii) $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

VI. (i) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

VII. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Quick Tips

- While using an algebraic identity, it should be clear to us that every identity has 2 forms.

$$\underbrace{(a+b)(a-b)}_{\text{factorised form}} = \underbrace{a^2 - b^2}_{\substack{\text{Product form} \\ \text{or} \\ \text{expanded form}}}$$

$$\underbrace{(a+b)^2}_{\text{factorised form}} = \underbrace{a^2 + 2ab + b^2}_{\text{expanded form}}$$

- The only way of remembering the algebraic identities is to do more and more questions related to them.

Numerical Ability 12

Find the product using appropriate identities.

(i) $(x + 8)(x + 8)$

(ii) $(3x - 2y)(3x - 2y)$

(iii) $(x + 0.1)(x - 0.1)$

Solution

(i) $(x + 8)(x + 8) = (x + 8)^2 = x^2 + 2(x)(8) + (8)^2 = x^2 + 16x + 64.$

(ii) $(3x - 2y)(3x - 2y) = (3x - 2y)^2$

$= (3x)^2 - 2(3x)(2y) + (2y)^2 = 9x^2 - 12xy + 4y^2$

(iii) $(x + 0.1)(x - 0.1) = (x)^2 - (0.1)^2$

$= x^2 - 0.01$

Numerical Ability 13

Expand each of the following using suitable identities

(i) $(3x - 4y)^2$ (ii) $(3x - y)^3$ (iii) $(3x + 4y + 5z)^2$

Solution

(i) $(3x - 4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$

$= 9x^2 - 24xy + 16y^2$

(ii) $(3x - y)^3 = (3x)^3 - (y)^3 - 3(3x)(y)(3x - y)$

$= 27x^3 - y^3 - 9xy\,(3x - y)$

$= 27x^3 - y^3 - (9xy)\,(3x) + (9xy)(y)$

$= 27x^3 - y^3 - 27x^2y + 9xy^2$

(iii) $(3x + 4y + 5z)^2 = (3x)^2 + (4y)^2 + (5z)^2 + 2(3x)(4y) + 2(4y)(5z) + 2(5z)(3x)$

$= 9x^2 + 16y^2 + 25z^2 + 24xy + 40yz + 30zx$

Numerical Ability 14

Evaluate the following without directly multiplying

(i) 101 × 103

(ii) 185 × 185 – 15 × 15

(iii) $(103)^3$

(iv) $(198)^3$

Solution

(i) $101 \times 103 = (100 + 1) \times (100 + 3)$

$= (100)^2 + (1 + 3)(100) + (1)(3) = 10000 + 400 + 3$

$= 10403$

(ii) $185 \times 185 - 15 \times 15 = (185)^2 - (15)^2 = (185 + 15)(185 - 15)$

$= (200)(170) = 34000$

(iii) $(103)^3 = (100 + 3)^3 = (100)^3 + (3)^3 + 3\,(100)(3)(100 + 3)$

$= 1000000 + 27 + (900)(103) = 1000027 + 92700$

$= 1092727$

(iv) $(198)^3 = (200 - 2)^3 = (200)^3 - (2)^3 - 3\,(200)(2)(200 - 2)$

$= 8000000 - 8 - 1200(200 - 2)$

$= 8000000 - 8 - 240000 + 2400$

$= 7762392$

Numerical Ability 15

Factorise the following :

(i) $4x^2 + 20xy + 25y^2$

(ii) $25x^2y^2z^2 - 36u^2$

(iii) $125x^3y^3 + 27z^3$

(iv) $125x^3 + 225x^2y + 135xy^2 + 27y^3$

(v) $2x^2 + y^2 + 9z^2 + 2\sqrt{2}xy - 6yz - 6\sqrt{2}zx$

Solution

(i) $4x^2 + 20xy + 25y^2 = (2x)^2 + 2(2x)(5y) + (5y)^2 = (2x + 5y)^2$

(ii) $25x^2y^2z^2 - 36u^2 = (5xyz)^2 - (6u)^2$

$= (5xyz + 6u)(5xyz - 6u)$

(iii) $125x^3y^3 + 27z^3 = (5xy)^3 + (3z)^3$

$= (5xy + 3z)\{(5xy)^2 - (5xy)(3z) + (3z)^2\}$

$= (5xy + 3z)(25x^2y^2 - 15xyz + 9z^2)$

(iv) $125x^3 + 225x^2y + 135xy^2 + 27y^3$

$= (5x)^3 + 45xy(5x + 3y) + (3y)^3$

$= (5x)^3 + 3(5x)(3y)(5x + 3y) + (3y)^3$

$= (5x + 3y)^3$

(v) $2x^2 + y^2 + 9z^2 + 2\sqrt{2}xy - 6yz - 6\sqrt{2}zx$

$= (\sqrt{2}x)^2 + (y)^2 + (3z)^2 + 2(\sqrt{2}x)(y) - 2(y)(3z) - 2(\sqrt{2}x)(3z)$

$= (\sqrt{2}x + y - 3z)^2$

Memory Map

Polynomials

is an algebraic expressions f(x) of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_1x^1 + a_0$ Where $a_0, a_1, \ldots a_n$ are real numbers and all indices of variable x are non-negative integers.

$a_0, a_1x, \ldots\ldots a_nx^n$ are terms

$a_0, a_1, \ldots\ldots a_n$ are coefficients.

Degree of Polynomial

Highest power of variable is called degree.

Value of Polynomial

The value of a polynomial f(x) at x = a is obtained by substituting x = a in given polynomial

Zero or Root of polynomial

x = r is a zero of a polynomial p(x) if p(r) = 0

Types of Polynomials

On basis of terms

- Monomial
- Binomial
- Trinomial
- Quadrinomial

On basis of degree

- Zero
- Constant
- Linear
- Quadratic
- Cubic
- Bi-Quadratic

Remainder Theorem

If p(x) is divided by (x - a) then remainder is given by p(a).

Factorisation of Polynomials

- By taking common factor
- By making perfect square
- By factor theorem
- By splitting the middle term

Factor Theorem

If (x - a) is a factor of p(x) then p(a) = 0 and vice versa.

Algebraic Identities

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)^2 = a^2 - 2ab + b^2$
- $a^2 - b^2 = (a + b)(a - b)$
- $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
- $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
- $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Special case : If a + b + c = 0 then $a^3 + b^3 + c^3 = 3abc$