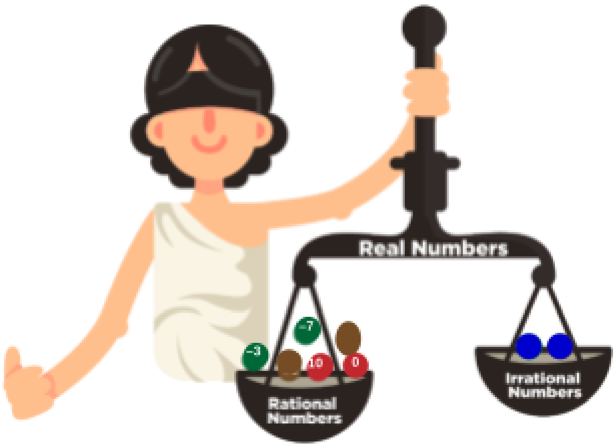
1 Number System



Rational numbers

The rational numbers are the numbers that can be expressed in the form of $\frac{p}{q}$, where p and q are integers and coprime and q ≠ 0. For e.g., – 3, 0, 4.33 etc.

Coprime Numbers : Co-prime numbers or relatively prime numbers are those numbers that have their HCF (Highest Common Factor) as 1. For e.g. (2, 3); (5, 6); (7, 8) etc.

Integers : An integer is a whole number that can be positive, negative, or zero.

Rational numbers between two numbers

One way to find a rational number between two rational numbers is to find their average, called mean.

To find a rational number between x and y, we will find the mean of x and y. i.e. $\frac{x+y}{2}$ which is a rational number lying between x and y. This number will be the mid-value between the given two numbers.

The counting numbers, the whole numbers and the integers are all subsets of set of rational numbers.

SPOT LIGHT

Numerical Ability 1

Find 3 rational numbers between 2 and 5.

Solution

Let a = 2, b = 5

A rational number between 2 and 5 = $\frac{2+5}{2}=\frac{7}{2}$

Second rational number between 2 and $\frac{7}{2}$ = $\frac{2+\frac{7}{2}}{2}=\frac{11}{4}$

Third rational number between $\frac{7}{2}$ and 5 = $\frac{\frac{7}{2}+5}{2}=\frac{17}{4}$

Hence, three rational numbers between 2 and 5 are : $\left[\frac{7}{2},\frac{11}{4},\frac{17}{4}\right]$

Do You Remember ?

(n + 1) Rule

- When denominators of both rational numbers are different then we need to make their denominators same. In case n rational numbers are required between 2 rational numbers, multiply the numerator and denominator of both rational number by n + 1.
  We can write n rational numbers by increasing the numerator by 1 at a time.
- When denominators of both rational numbers are same then we just have to multiply the numerator and denominator of both rational numbers ar required between two given rational numbers.

Numerical Ability 2

Find 4 rational numbers between 4 and 5.

Solution

Let a = 4, b = 5 and n = 4

$$\frac{a(n+1)}{(n+1)}=\frac{4(4+1)}{(4+1)} \text{ and } \frac{b(n+1)}{(n+1)}=\frac{5(4+1)}{(4+1)}$$

$$=\frac{4(5)}{5}=\frac{20}{5} \text{ and } \frac{5(5)}{5}=\frac{25}{5} \Rightarrow \frac{20}{5},\left[\frac{21}{5},\frac{22}{5},\frac{23}{5},\frac{24}{5}\right],\frac{25}{5}$$

Hence, 4 rational numbers between 4 and 5 are $\left[\frac{21}{5},\frac{22}{5},\frac{23}{5},\frac{24}{5}\right]$ or [4.2, 4.4, 4.6, 4.4]

Quick Tips

- We can directly write 4 rational numbers between 4 and 5 as 4.1, 4.2, 4.3, 4.4. But it is not advisable to use this method in subjective test format.

Be Alert !

- While using (n + 1) rule to find rational numbers, make sure to make denominator of unlike fractions same.

Decimal expansion of rational numbers

Every rational number can be expressed as terminating decimal or non-terminating but repeating decimals.

Terminating decimal (The remainder becomes zero)

The word "terminate" means "end". A decimal that ends is a terminating decimal.

OR

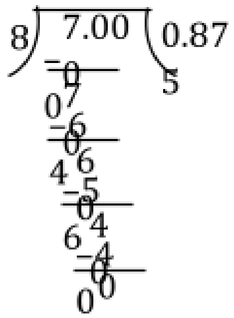
A terminating decimal doesn't keep going. A terminating decimal will have finite number of digits after the decimal point. For e.g.

$\frac{3}{4}$ = 0.75, $\frac{8}{10}$ = 0.8, $\frac{5}{4}$ = 1.25, $\frac{25}{16}$ = 1.5625

Numerical Ability 3

Express $\frac{7}{8}$ in the decimal form.

Solution



We have,

$\therefore \quad \frac{7}{8} = 0.875$

> The set of all rational numbers is usually denoted by a boldface **Q**, which stands for quotient.
>
> **SPOT LIGHT**

Non-terminating & repeating (recurring decimal)

(The remainder never becomes zero)

A decimal in which a digit or a set of finite number of digits repeats periodically is called non-terminating repeating (recurring) decimals. For e.g.

$\frac{5}{3}$=1.6666.........=$1.\overline{6}$

$\frac{7}{11} = 0.636363......... = 0.\overline{63}$

$\frac{1}{999} = 0.001001001......... = 0.\overline{001}$

> Between any two distinct rational numbers x and y, there exists infinitely many rational numbers. This is called **"Denseness property"** of rational numbers.
>
> **SPOT LIGHT**

Numerical Ability 4

Express $\frac{2}{11}$ as a decimal fraction.

Solution

we have

$$\begin{array}{r|l} 11 & 2.000\ (0.1818 \\ & \underline{-11} \\ & 090 \\ & \underline{-88} \\ & 020 \\ & \underline{-11} \\ & 090 \\ & \underline{-88} \\ & 02 \end{array}$$

$\therefore \frac{2}{11} = 0.181818 \ldots\ldots = 0.\overline{18}$

Building Concepts 1

If $\frac{1}{7} = 0.\overline{142857}$ write the decimal expansion of $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}$, and $\frac{5}{7}$ without actually doing the long division.

Explanation

As we have,

$\frac{1}{7} = 0.\overline{142857}$

$\frac{2}{7} = 2 \times \frac{1}{7} = 0.\overline{285714}$; $\frac{3}{7} = 3 \times \frac{1}{7} = 0.\overline{428571}$

$\frac{4}{7} = 4 \times \frac{1}{7} = 0.\overline{571428}$; $\frac{5}{7} = 5 \times \frac{1}{7} = 0.\overline{714285}$.

Method to convert non-terminating decimal to the form $\frac{p}{q}$.

In a non-terminating repeating decimals, we have two types of decimal representations

(a) Pure recurring decimal

(b) Mixed recurring decimal

(a) Pure recurring decimal

It is a decimal representation in which all the digits after the decimal point are repeated.

Following are the steps to convert it in the form of $\frac{p}{q}$.

Step-1 : Denote pure recurring decimal as x.

Step-2 : Write the number in decimal form by removing bar from top of repeating digits.

Step-3 : Count the number of digits having bar on their heads.

Step-4 : Multiply the repeating decimal by 10, 100, 1000, ... depending upon 1 place repetition, 2 place repetition, 3 place repetition and so on present in decimal number.

Step-5 : Subtract the number obtained in step 2 from a number obtained in step 4.

Step-6 : Find the value of x in the form $\frac{p}{q}$.

(b) Mixed recurring decimal

It is a decimal representation in which there are one or more digits present before the repeating digits after decimal point. Following are the steps to convert it in the form of $\frac{p}{q}$.

Step-1 : Denote mixed recurring decimal as x.

Step-2 : Count the number of digits after the decimal point which do not have bar on them. Let it be 'n'.

Step-3 : Multiply both sides of x by 10^n to get only repeating digits on the right side of the decimal point.

Step-4 : Further use the method of converting pure recurring decimal in the form of $\frac{p}{q}$ and get the value of x.

Numerical Ability 5

Express each of the following pure recurring decimals in the form $\frac{p}{q}$.

(i) $0.\overline{6}$ **(ii)** $0.\overline{585}$ **(iii)** $23.\overline{43}$

Solution

(i) Let $x = 0.\overline{6} \Rightarrow x = 0.666\ldots$ ... (1)

Here, we have only one repeating digit, so we multiply both sides of eq. (1) by 10 to get

$\Rightarrow$ $10x = 6.66\ldots$ ... (2)

On subtracting (1) from (2), we get;

$\Rightarrow$ $10x - x = (6.66 \ldots\ldots) - (0.66\ldots\ldots)$

$\Rightarrow$ $9x = 6 \Rightarrow x = \frac{6}{9}$

$\Rightarrow$ $x = \frac{2}{3}$

Hence, $0.\overline{6} = \frac{2}{3}$

(ii) Let $x = 0.\overline{585} \Rightarrow x = 0.585585585\ldots$ ... (1)

Here, we have three repeating digits, so we multiply both sides of eq. (1) by 1000 to get

$\Rightarrow$ $1000x = 585.585585\ldots$ ... (2)

On subtracting eq. (1) from eq. (2), we get

$1000x - x = (585.585585\ldots\ldots) - (0.585585 \ldots\ldots)$

$\Rightarrow$ $999x = 585$

$\Rightarrow$ $x = \frac{585}{999} = \frac{195}{333} = \frac{65}{111}$

Hence, $0.\overline{585} = \frac{65}{111}$

(iii) Let $x = 23.\overline{43} \Rightarrow x = 23.434343$ ... (1)

Multiplying both sides of eq. (1) by 100, we get

$100\,x = 2343.4343$ ... (2)

Subtracting (1) from (2) we get

$100\,x - x = (2343.4343\ldots\ldots) - (23.4343\ldots\ldots)$

$\Rightarrow \quad 99x = 2320 \Rightarrow x = \dfrac{2320}{99}$

Hence, $23.\overline{43} = \dfrac{2320}{99}$

Alternate method

We have, $23.\overline{43} = 23 + 0.\overline{43} = 23 + \dfrac{43}{99}$

Using the above rule, we have $0.\overline{43} = \dfrac{43}{99}$

$\Rightarrow \quad 23.\overline{43} = \dfrac{23 \times 99 + 43}{99}$

$= \dfrac{2277 + 43}{99} = \dfrac{2320}{99}$

The decimal representation of π upto 50 decimal places is 3.141592 6535897932384626 4338327950288419 716939937510....

SPOT LIGHT

Numerical Ability 6

Express the following mixed recurring decimals in the form $\dfrac{p}{q}$.

(i) $0.3\overline{2}$ **(ii) $0.12\overline{3}$** **(iii) $15.7\overline{12}$**

Solution

(i) Let $x = 0.3\overline{2}$

Clearly, there is just one digit on the right side of the decimal point which is without bar. So, we multiply both sides by 10 so that only the repeating digit is left on the right side of the decimal point.

$\therefore \quad 10x = 3.\overline{2}$

$\Rightarrow \quad 10x = 3 + 0.\overline{2}$

$\Rightarrow \quad 10x = 3 + \dfrac{2}{9}$

$\Rightarrow \quad 10\,x = \dfrac{9\times3+2}{9}$

$\Rightarrow \quad 10\,x = \dfrac{29}{9}$

$\Rightarrow \quad x = \dfrac{29}{90}$

(ii) Let $x = 0.12\overline{3}$

Clearly, there are two digits on the right side of the decimal point which is without bar. Now, we multiply both sides of equation by $10^2 = 100$ so that only the repeating digit is left on the right side of the decimal point.

$\therefore \quad 100\,x = 12.\overline{3}$

$\Rightarrow \quad 100\,x = 12 + 0.\overline{3}$

$\Rightarrow \quad 100\,x = 12 + \dfrac{3}{9}$

$\Rightarrow \quad 100\,x = \dfrac{12\times9+3}{9}$

$\Rightarrow \quad 100\,x = \dfrac{108+3}{9}$

$\Rightarrow \quad 100\,x = \dfrac{111}{9}$

$\Rightarrow \quad x = \dfrac{111}{900} = \dfrac{37}{300}$

(iii) Let $x = 15.7\overline{12}$

Clearly, there is just one digit on the right side of the decimal point which is without bar. Now, we multiply both sides by 10 so that only the repeating digit is left on the right side of the decimal point.

$\therefore \quad 10\,x = 157.\overline{12}$

$\Rightarrow \quad 10\,x = 157 + 0.\overline{12}$

$\Rightarrow \quad 10\,x = 157 + \dfrac{12}{99}$

$\Rightarrow \quad 10\,x = 157 + \dfrac{4}{33}$

$\Rightarrow \quad 10\,x = \dfrac{157 \times 33 + 4}{33}$

$\Rightarrow \quad 10\,x = \dfrac{5181 + 4}{33}$

$\Rightarrow \quad 10\,x = \dfrac{5185}{33}$

$\Rightarrow \quad x = \dfrac{5185}{330}$

$= \dfrac{1037}{66}$

Quick Tips

- We can directly write the pure recurring decimal in $\dfrac{p}{q}$ form:
- If whole number part is zero.

$0.\overline{6}=\frac{6}{9}$ $\quad$ $0.\overline{23}=\frac{23}{99}$ $\quad$ $0.\overline{539}=\frac{539}{999}$

• We can directly write the mixed recurring decimal to $\frac{p}{q}$ form:

$$\left(\frac{p}{q}\right)\text{form}=\frac{(\text{Complete number})-(\text{number formed by Non}-\text{repeating digit})}{\text{No.of 9 as no.of repeating digits after that write no.of 0 as no.of non repeating digits after decimal.}}$$

(i) $12.7\overline{62}=\frac{12762-127}{990}$

$=\frac{12635}{990}=\frac{2527}{198}$

(ii) $25.63\overline{2}=\frac{25632-2563}{900}=\frac{23069}{900}$

Irrational numbers

Around 400 BC, followers of the famous mathematician and philosopher, Pythagoras, were the first to discover the numbers which were not rationals such as the length of the diagonal of a square with side one unit long and the ratio of circumference to the diameter of a circle. These and other needs led to the introduction of the Irrational Numbers.

A number is called an irrational number, if it cannot be written in the form p/q, where p & q are integers, coprime and q ≠ 0. All Non-terminating & Non-repeating decimal numbers are Irrational numbers.

For e.g. $\sqrt{2}, \sqrt{3}, 3\sqrt{2}, 2+\sqrt{3}, \sqrt{2+\sqrt{3}}, \pi$, etc....

Decimal expansion of irrational numbers

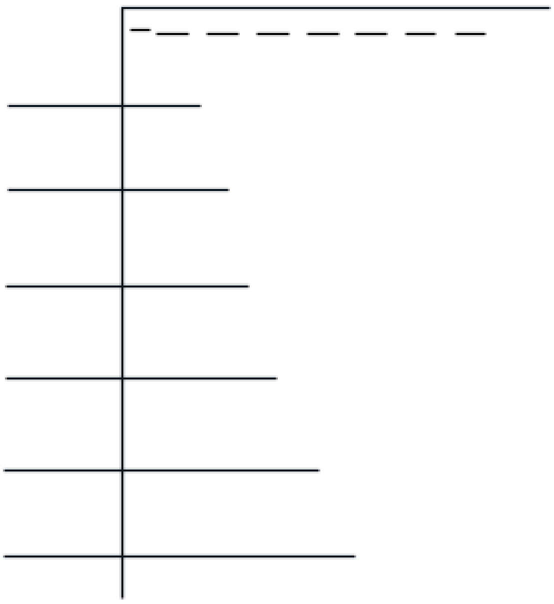
Every irrational number can be expressed as non-terminating and non-repeating decimal.

For e.g. $\sqrt{2}$ = 1.4142135........

Note : An irrational number between two numbers a and b is $\sqrt{ab}$, where ab cannot be a perfect square.

Do You Remember ?

Decimal expansion of $\sqrt{2}$ by division method:



1.4 1 4 2 1 3 5 ……

2.00 00 00 00 00 00 00……

1

100

96

400

281

11900

11296

60400

56564

383600

282841

100759….

1

24

281

2824

282841

1

+4

+1

+4

28282

+2

+1

282842

So, $\sqrt{2}$ = 1.41142135…………,

Building Concepts 2

(i) Find two irrational numbers between 2 and 2.5

(ii) Find one rational and one irrational number between 0.101001000100001……. and 0.1001000100001……

(iii) Find two rational numbers between $\sqrt{2}$ and $\sqrt{3}$

(iv) Find two irrational numbers between $\sqrt{2}$ and $\sqrt{3}$

Explanation

(i) Irrational number between 2 and 2.5 = $\sqrt{(2)(2.5)} = \sqrt{(2)\left(\frac{5}{2}\right)} = \sqrt{5}$

Again, irrational number between 2 and $\sqrt{5} = \sqrt{(2)(\sqrt{5})} = \sqrt{2\sqrt{5}}$.

So, required irrational numbers between 2 and 2.5 are $\sqrt{5}$ and $\sqrt{2\sqrt{5}}$.

(ii) One rational number between given numbers = 0.101

(Terminating decimal)

One irrational number between given numbers = 0.1002000100001...

(Non-Terminating Non-repeating).

(iii) $\sqrt{2}$ = 1.4142... and $\sqrt{3}$ = 1.732... Now $\sqrt{2} < 1.5 < 1.6 < \sqrt{3}$

$\Rightarrow$ 1.5 and 1.6 i.e. $\frac{3}{2}$ and $\frac{8}{5}$ are two rational between $\sqrt{2}$ and $\sqrt{3}$.

(iv) $2 < 2.1 < 2.2 < 3$

$\Rightarrow$ $\sqrt{2} < \sqrt{2.1} < \sqrt{2.2} < \sqrt{3}$

$\Rightarrow$ $\sqrt{2.1}$ and $\sqrt{2.2}$ are two irrationals between $\sqrt{2}$ and $\sqrt{3}$.

Be Alert !

- While writing the irrational numbers of the form 2.010010001… etc. please check that the answer should lie between the given two numbers.

Real numbers

Rational numbers together with irrational numbers are said to be real numbers. That is, a real number is either rational or irrational.

For e.g. 2, $-\frac{3}{2}$, 0, 1.5, $\sqrt{2}, \sqrt[3]{5}, \sqrt[5]{11}, \pi$ etc. are real numbers.

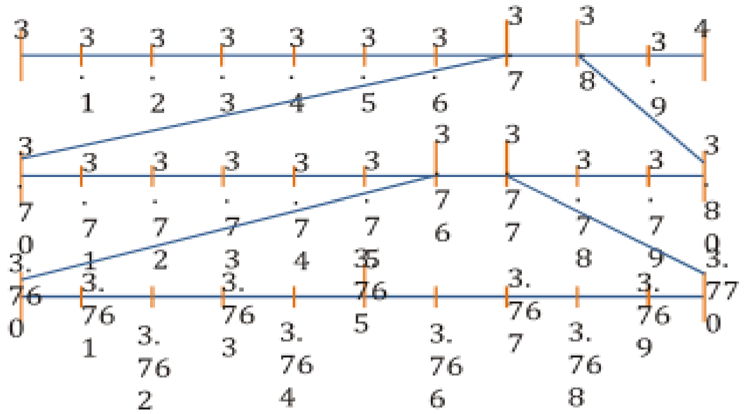
Representation of real numbers on the number line by means of magnifying glass

Representation of rational numbers on the number line

The process of visualization of numbers on the number line through a magnifying glass is known as successive magnification. Sometimes, we are unable to check the numbers like 3.765 and $4.\overline{26}$ on the number line we seek the help of magnifying glass by dividing the part into subparts and subparts again into equal subparts to ensure the accuracy of the given number.

For e.g. represent 3.765 on the number line. This number lies between 3 and 4. The distance 3 and 4 is divided into 10 equal parts. Then the first mark to the right of 3 will represent 3.1 and second 3.2 and so on. Now, 3.765 lies between 3.7 and 3.8. We divide the distance between 3.7 and 3.8 into 10 equal parts.

3.76 will be on the right of 3.7 at the 6^{th} mark, and 3.77 will be on the right of 3.7 at the 7^{th} mark and 3.765 will lie between 3.76 and 3.77 and so on.



Building Concepts 3

Represent 3.728 on the number line through successive magnification.

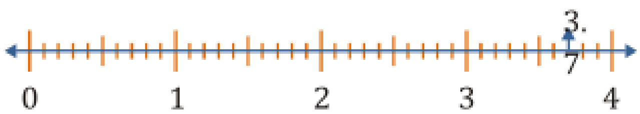
Explanation

We have to locate the point 3.728 on the number line. This number lies between 3 and 4.

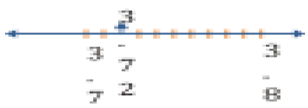
First go to 3.7. You divide the portion of the number line between 3 and 4 in 10 equal parts. Now first mark from the left will give you 3.1, the 2nd mark will give you 3.2 and so on.

To get 3.7 you reach at 7th mark.

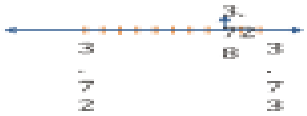
Again to get 3.72, you divide the portion of the number line between 3.7 and 3.8 in 10 equal parts, to get 3.72, you reach 2nd mark from the left.



Again to reach 3.728 you further divide the portion of the number line between 3.72 to 3.73 in 10 equal parts.



To get the point 3.728 on the number line you reach 8th point from right to 3.72 on this subdivision.



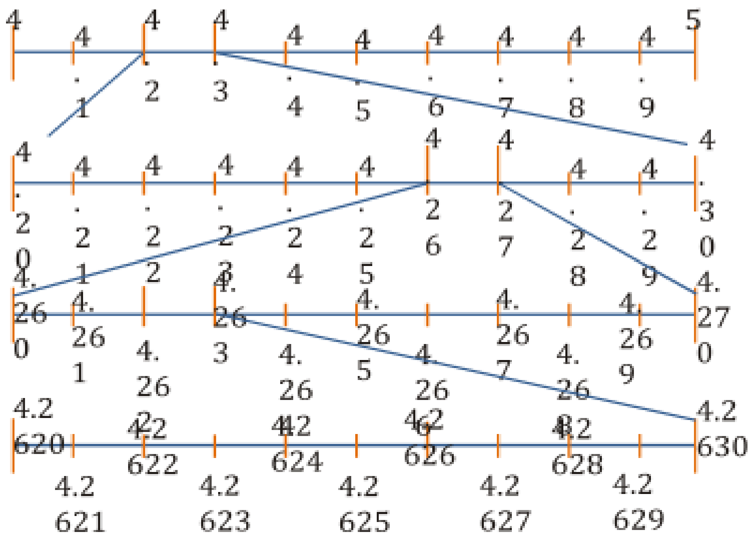
Building Concepts 4

Visualize $4.\overline{26}$ on the number line, up to 4 decimal places.

Explanation

We can locate the point 4.2626 on the number line.

$4.\overline{26}$ = 4.262626.......= 4.2626 (up to 4 decimal places)



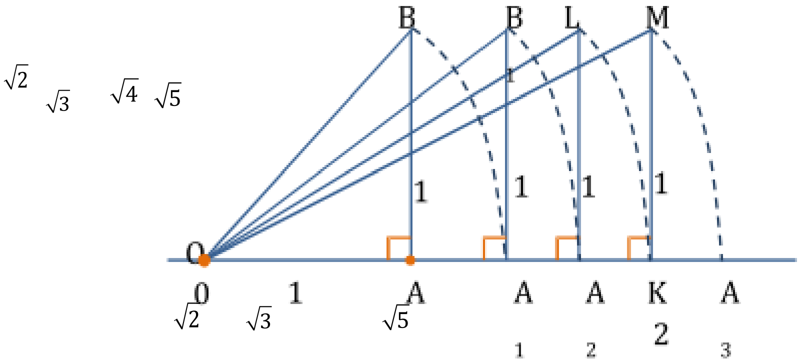
> π is an irrational number, it's decimal representation is non terminating and non-repeating.
>
> **SPOT LIGHT**

Representation of irrational numbers on the number line

Consider the number line and mark a point O on it and let it represent zero. Let A represent 1 unit on the number line. So, OA = 1. At A draw AB perpendicular to OA.
Let AB = OA = 1 unit

$\therefore$ By Pythagoras Theorem,

$$OB = \sqrt{(OA)^2+(AB)^2} = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$$



Taking O as centre and radius = OB = $\sqrt{2}$, draw a circle cutting the number line at A_1, where

$OA_1 = OB = \sqrt{2}$

$\Rightarrow$ A_1 represents $\sqrt{2}$ on number line. Now draw A_1B_1 perpendicular to number line at A_1 and let $A_1B_1 = 1$

$\therefore$ $OB_1 = \sqrt{(OA_1)^2 + (A_1B_1)^2} = \sqrt{(\sqrt{2})^2 + (1)^2} = \sqrt{2+1} = \sqrt{3}$

Taking O as centre and $OB_1 = \sqrt{3}$ as radius,

draw a circle cutting the number line at A_2 where $OA_2 = OB_1 = \sqrt{3}$.

$\Rightarrow$ A_2 represents $\sqrt{3}$ on number line

Continue this process and get the point K on number line where

$OK = OL = \sqrt{(OA_2)^2 + (A_2L)^2} = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2.$

$\Rightarrow$ K represents 2 on number line.

Again, get a point A_3 on number line where

$OA_3 = OM = \sqrt{(OK)^2 + (KM)^2} = \sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}$.

$\Rightarrow$ A_3 represents $\sqrt{5}$ on number line.

In this way, we can show that there exist points on number line representing $\sqrt{6}, \sqrt{7}, \sqrt{8}$ etc. which are irrational numbers.

In fact, for every irrational number, there exists a unique point on the number line.

Active Maths 1

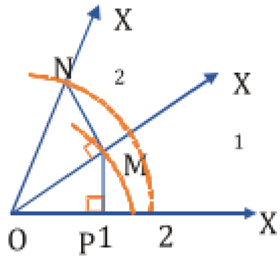
How to make a square root spiral by using paper folding?

Exploring the concept

1. Draw a line OX on the tracing paper. Mark point O on one end and mark points 0, 1, 2, 3, ... at equal distance of 1 unit by paper folding.

2. Fold the paper along the line that passes through the point marked '1' and perpendicular to the line OX. (Fold the paper in such a way that point 'O' coincides with point '2') Make a crease and unfold it.

$\sqrt{3}$ $\sqrt{2}$

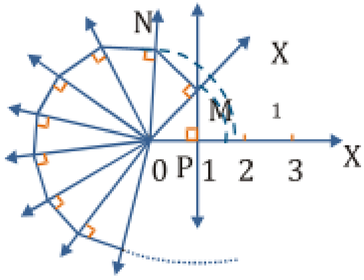


3. From the point marked '1' draw a line of length 1 unit moving along the crease. Mark the point as M such that PM = 1 unit. Join OM, clearly OM = $\sqrt{2}$ units

(∴ OM = $\sqrt{OP^2 + PM^2} = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$ units)

4. Fold the paper along the line (fold on point M in such a way that point O joined with any point lie on OX1) that passes through point M and perpendicular to OM at M. Make a crease and unfold it. From the point M draw a line of 1 unit moving upward, along the crease. Mark the point as N such that MN = 1 unit. Join ON where ON = $\sqrt{3}$.

$\sqrt{4}$ $\sqrt{3}$ $\sqrt{2}$ $\sqrt{5}$ $\sqrt{6}$ $\sqrt{7}$ $\sqrt{8}$ $\sqrt{9}$ $\sqrt{10}$



5. Keep this process continuously to get $\sqrt{4}, \sqrt{5}, \sqrt{6}, ..., \sqrt{10}$

Drawing conclusions

This is the way we get square root spiral pattern by using paper folding. In the same way, you can locate $\sqrt{n}$ for any positive integer n, after $\sqrt{n-1}$ has been located.

Method to find $\sqrt{x}$ units for any given positive real number x geometrically.

(i) Let x be a positive real number. Take AB = x units and BC = 1 unit on the real line.

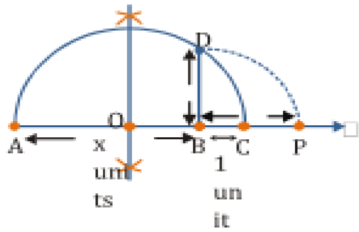
(ii) Find the midpoint O of AC and draw a semicircle with centre O and radius OA or OC.

(iii) At B, draw a line BD ⊥ AC, where D is a point on the semicircle.

(iv) Further, with centre B and radius BD, draw an arc intersecting the real line ✦ at P.

Therefore, BP = BD = $\sqrt{x}$.

Justification : We have, In right triangle OBD,



OD = OA = OC = $\dfrac{x+1}{2}$ units (radius of the semicircle)

OB = OC – BC = $\left(\dfrac{x+1}{2}-1\right)$ units = $\left(\dfrac{x-1}{2}\right)$ units.

In right ΔOBD, we have $OD^2 = OB^2 + BD^2$

$BD^2 = OD^2 - OB^2$ and BD = $\sqrt{OD^2 - OB^2}$ [By Pythagoras theorem]

$= \sqrt{\left(\dfrac{x+1}{2}\right)^2 - \left(\dfrac{x-1}{2}\right)^2}$ units

$= \sqrt{\left(\dfrac{x+1}{2}+\dfrac{x-1}{2}\right)\left(\dfrac{x+1}{2}-\dfrac{x-1}{2}\right)}$ units [∴ $A^2 - B^2 = (A + B)(A - B)$]

$= \sqrt{x \times 1}$ units = $\sqrt{x}$ units.

So, BD = $\sqrt{x}$ units.

Thus, $\sqrt{x}$ exists for all positive real numbers.

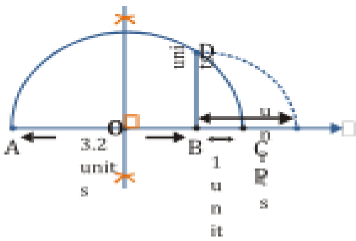
Hence, the point P represents $\sqrt{x}$ on the real number line.

Building Concepts 5

Represent $\sqrt{3.2}$ units geometrically on the number line.

Explanation

$\sqrt{3.2}\sqrt{3.2}$



Let ✦ be the number line.

Draw a line segment AB = 3.2 units

and BC = 1 unit. Find the midpoint O of AC.

Draw a semicircle with centre O and radius OA or OC.

Draw BD ⊥ AC intersecting the semicircle at D. Then BD = $\sqrt{3.2}$ units. Now, with centre B and radius BD, draw an arc intersecting the number line at P.

Hence, BD = BP = $\sqrt{3.2}$ units

Operations on real number

Following are some useful results on real numbers.

(i) Negative of an irrational number is an irrational number.

For e.g. $\sqrt{2}$ is an irrational number because its value i.e 1.414… is a non terminating non

repeating decimal. If we consider the value of $-\sqrt{2}$ i.e 1.414…, it is still a non terminating non repeating decimal so $-\sqrt{2}$ is also an irrational number.

(ii) The sum or difference of a rational number and an irrational number is an irrational number.

For e.g. let us consider the value of $2+\sqrt{2}$ where 2 is a rational number and $\sqrt{2}$ is an irrational number, their sum will be

$$\begin{array}{r} 1.414\ldots\ldots \\ +2.000\ldots\ldots \\ \hline 3.414\ldots\ldots \end{array}$$

which is a non terminating non repeating decimal Similarly difference of a rational & an irrational number will be an irrational number.

(iii) The product of a non-zero rational number and an irrational number is an irrational number.

For e.g. Let us consider the value of 2 × 1.01001000100001… which is equals to 2.020020002… which is an irrational number.

(iv) The sum, difference, product and quotient of two irrational numbers need not be an irrational number.

For e.g. $\sqrt{2}$ and $-\sqrt{2}$ are two irrational numbers and $\sqrt{2} + (-\sqrt{2}) = \sqrt{2} - \sqrt{2} = 0$ which is rational.

Similarly if we multiply $\sqrt{2}$ by $2\sqrt{2}$, $\sqrt{2} \times 2\sqrt{2} = 4$ which is rational.

If we divide $\sqrt{12}$ by $\sqrt{3}$, $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2$ which is rational.

Also on the other hand, $\sqrt{2}+\sqrt{3}$ is irrational.

$\sqrt{5}-\sqrt{2}$ is irrational, $\sqrt{2}\times\sqrt{3}=\sqrt{6}$ is irrational and so on.

> An irrational number of the type $\sqrt[n]{x}$, where it is not possible to find exactly the n^{th} root of x, where x is a positive rational number is called a surd.
>
> SPOT LIGHT

Some identities using radical sign

Let a and b be positive real numbers, then

(i) $\sqrt{ab}=\sqrt{a}\sqrt{b}$

(ii) $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

(iii) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$

(iv) $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$

(v) $(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d})=\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(vi) $(\sqrt{a}+\sqrt{b})^2=a+2\sqrt{ab}+b$

(vii) Conjugate of $a+\sqrt{b}$ is $a-\sqrt{b}$

Numerical Ability 7

(i) Add 7 + $\sqrt{5}$ and – 7 + $\sqrt{5}$

(ii) Multiply $7\sqrt{3}$ by $2\sqrt{3}$

(iii) Divide $6\sqrt{27}$ by $3\sqrt{12}$

Solution

(i) $(7+\sqrt{5})+(-7+\sqrt{5})=(7-7)+(\sqrt{5}+\sqrt{5})=0+2\sqrt{5}=2\sqrt{5}$

(ii) $7\sqrt{3}\times 2\sqrt{3}=7\times 2\times 3=42$

(iii) $\frac{6\sqrt{27}}{3\sqrt{12}}=\frac{\sqrt{36\times 27}}{\sqrt{9\times 12}}=\frac{\sqrt{972}}{\sqrt{108}}=\sqrt{9}=3$

Numerical Ability 8

Simplify:

(i) $(\sqrt{5}+3)(\sqrt{5}+2)$ **(ii)** $(\sqrt{7}+\sqrt{2})^2$ **(iii)** $(6+\sqrt{2})(6-\sqrt{2})$

Solution

(i) $(\sqrt{5}+3)(\sqrt{5}+2)=5+2\sqrt{5}+3\sqrt{5}+6=11+5\sqrt{5}$

(ii) $(\sqrt{7}+\sqrt{2})^2=(\sqrt{7})^2+(\sqrt{2})^2+2\sqrt{7}\sqrt{2}=7+2+2\sqrt{14}=9+2\sqrt{14}$

(iii) $(6+\sqrt{2})(6-\sqrt{2})=(6)^2-(\sqrt{2})^2=36-2=34$

Rationalisation

The process of converting the irrational terms to rational (mostly in denominator) in a mathematical expression is known as rationalisation.

Why do we need rationalisation?

Let us take an example. Suppose your teacher wants you to calculate the value of $\frac{1}{\sqrt{2}}$ and value of $\sqrt{2}$ is 1.414 (approx.)

To calculate this you need to divide 1 by 1.414 which will be a tedious task to do. But if we do some manipulation in the given expression as $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{(\sqrt{2})^2}=\frac{\sqrt{2}}{2}$

$\Rightarrow \frac{1.414}{2} = 0.707$

As you can see multiplying $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$ both in numerator and denominator converted the irrational term i.e. $\sqrt{2}$ into a rational term i.e. 2. This has converted the calculation into an easier one.

- Rationalisation converts irrational terms to rational in the denominator which can make the calculation easier.

Rationalising factor

In the above example, we have multiplied $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$ in both numerator and denominator to convert $\sqrt{2}$ into 2 in the denominator.

So, the factor by which we multiply numerator and denominator to rationalise the denominator is called rationalising factor.

Rationalising factors in different cases:

(i) $\dfrac{1}{\sqrt{a}}$ → The rationalising factor will be $\sqrt{a}$

$$\frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

(ii) $\dfrac{1}{a \pm \sqrt{b}}$ The rationalising factor will be $a \mp \sqrt{b}$

$$\frac{1}{a \pm \sqrt{b}} \times \frac{a \mp \sqrt{b}}{a \mp \sqrt{b}} = \frac{a \mp \sqrt{b}}{(a)^2 - (\sqrt{b})^2} = \frac{a \mp \sqrt{b}}{a^2 - b}$$

(iii) $\dfrac{1}{\sqrt{a} \pm \sqrt{b}}$ → The rationalising factor will be $\sqrt{a} \mp \sqrt{b}$

$$\frac{1}{\sqrt{a} \pm \sqrt{b}} \times \frac{\sqrt{a} \mp \sqrt{b}}{\sqrt{a} \mp \sqrt{b}} = \frac{\sqrt{a} \mp \sqrt{b}}{(\sqrt{a})^2 - (\sqrt{b})^2} = \frac{\sqrt{a} \mp \sqrt{b}}{a - b}$$

Be Alert !

- In all cases, a, b are not perfect squares.

Numerical Ability 9

Simplify the following by rationalising the denominator :

(i) $\dfrac{1}{\sqrt{7} + \sqrt{6}}$ **(ii)** $\dfrac{3 + \sqrt{2}}{3 - \sqrt{2}}$

Solution

(i) We have, $\dfrac{1}{\sqrt{7} + \sqrt{6}} = \dfrac{1}{\sqrt{7} + \sqrt{6}} \times \dfrac{\sqrt{7} - \sqrt{6}}{\sqrt{7} - \sqrt{6}}$ (Multiply and divide by $\sqrt{7} - \sqrt{6}$)

$= \dfrac{\sqrt{7} - \sqrt{6}}{(\sqrt{7} - \sqrt{6})(\sqrt{7} + \sqrt{6})} = \dfrac{\sqrt{7} - \sqrt{6}}{(\sqrt{7})^2 - (\sqrt{6})^2}$ [$\because$ (a + b) (a – b) = $a^2 - b^2$]

$= \dfrac{\sqrt{7} - \sqrt{6}}{7 - 6} = \sqrt{7} - \sqrt{6}$

(ii) Multiplying the numerator and denominator by the conjugate of $3 - \sqrt{2}$, we get

$$\frac{3+\sqrt{2}}{3-\sqrt{2}} = \frac{3+\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{\left(3+\sqrt{2}\right)^2}{(3)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{9+2+2\times3\times\sqrt{2}}{9-2} = \frac{11+6\sqrt{2}}{7}$$

Numerical Ability 10

If $\frac{3+2\sqrt{2}}{3-\sqrt{2}} = a + b\sqrt{2}$, where a and b are rational numbers. Find the values of a and b.

Solution

L.H.S. = $\frac{3+2\sqrt{2}}{3-\sqrt{2}} = \frac{(3+2\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}$ (Use rationalization)

$$= \frac{9+3\sqrt{2}+6\sqrt{2}+4}{9-2} = \frac{13}{7} + \frac{9}{7}\sqrt{2}$$

$\therefore \frac{13}{7} + \frac{9}{7}\sqrt{2} = a + b\sqrt{2}$

equating the rational and irrational parts; We get a = $\frac{13}{7}$, b = $\frac{9}{7}$

Numerical Ability 11

If $x = 3 - 2\sqrt{2}$, find $x^2 + \frac{1}{x^2}$.

Solution

We have, $x = 3 - 2\sqrt{2}$

$$\therefore \quad \frac{1}{x}=\frac{1}{3-2\sqrt{2}}=\frac{1}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{3+2\sqrt{2}}{(3)^2-(2\sqrt{2})^2}=\frac{3+2\sqrt{2}}{9-8}=3+2\sqrt{2}$$

Thus, $x+\frac{1}{x}=3-2\sqrt{2}+3+2\sqrt{2}$

$x+\frac{1}{x}=6$

Squaring both sides $\left(x+\frac{1}{x}\right)^2=6^2$

$x^2+\frac{1}{x^2}+2=36$

$x^2+\frac{1}{x^2}=34$

Laws of exponents

Let a > 0 be a real number and p and q be rational numbers.

Then, we have

(i) $a^p \cdot a^q = a^{p+q}$

(ii) $(a^p)^q = a^{pq}$

(iii) $\frac{a^p}{a^q}=a^{p-q}$

(iv) $a^p b^p = (ab)^p$

(v) $a^{-p} = \frac{1}{a^p}$

(vi) $a^{p/q} = (a^p)^{1/q} = (a^{1/q})^p$

(vii) $\left(\frac{a}{b}\right)^p=\frac{a^p}{b^p}$

Exponent was introduced by the mathematician Michael Stifel in 1544 in his book Arithmetica Integra.

SPOT LIGHT

Do You Remember ?

- Any non zero real number raised to power zero is equal to 1.

e.g. $(x)^0 = 1$, $(1000)^0 = 1$

Numerical Ability 12

Evaluate the following:

(i) $(\sqrt[3]{64})^{\frac{-1}{2}}$ **(ii)** $(\sqrt{25})^{-7} \times (\sqrt{5})^{-5}$

(iii) $\left(\frac{4}{5}\right)^{7} \div \left(\frac{5}{4}\right)^{-5}$ **(iv)** $\left(\frac{121}{169}\right)^{-3/2}$

Solution

(i) $(\sqrt[3]{64})^{\frac{-1}{2}} = \left[(64)^{\frac{1}{3}}\right]^{\frac{-1}{2}} = (64)^{\frac{1}{3}\times\frac{-1}{2}} = (64)^{\frac{-1}{6}}$

$= (2^6)^{\frac{-1}{6}} = 2^{6\times\left(\frac{-1}{6}\right)} = 2^{-1} = \frac{1}{2}$

(ii) $(\sqrt{25})^{-7} \times (\sqrt{5})^{-5}$

$= \left[(25)^{\frac{1}{2}}\right]^{-7} \times \left[(5)^{\frac{1}{2}}\right]^{-5} = \left[(5)^{2\times\frac{1}{2}}\right]^{-7} \times (5)^{-7-\frac{-5}{2}}$

$= (5)^{-19/2}$

(iii) $\left(\frac{4}{5}\right)^{7} \div \left(\frac{5}{4}\right)^{-5} = \left(\frac{4}{5}\right)^{7} \times \left(\frac{4}{5}\right)^{-5}$

$= \left(\frac{4}{5}\right)^{7-5} = \left(\frac{4}{5}\right)^{2} = \frac{16}{25}$

(iv) $\left(\frac{121}{169}\right)^{-3/2} = \left(\frac{11\times 11}{13\times 13}\right)^{-3/2}$

$= \left(\frac{11^2}{13^2}\right)^{-3/2} = \left(\frac{11}{13}\right)^{2\times\frac{-3}{2}}$

$= \left(\frac{11}{13}\right)^{-3} = \left(\frac{13}{11}\right)^{3} = \frac{2197}{1331}$

Numerical Ability 13

Simplify:

(i) $\left(\dfrac{4x^5y}{16xy^4}\right)^3$ **(ii)** $\dfrac{3x^2y^{-3}}{12x^6y^3}$ **(iii)** $5x^2y(2x^4y^{-3})$

(iv) $(2fh^4)^4(fg)^6$ **(v)** $\left(\dfrac{-2a^3b^2x^0}{3a^2b^3c^7}\right)^{-2}$

Solution

(i) $\left(\dfrac{4x^5y}{16xy^4}\right)^3 = \left(\dfrac{1}{4}x^{5-1}.y^{1-4}\right)^3 = \left(\dfrac{1}{4}x^4.y^{-3}\right)^3$

$= \dfrac{1}{64}x^{12}y^{-9} = \dfrac{x^{12}}{64y^9}$

(ii) $\dfrac{3x^2y^{-3}}{12x^6y^3} = \dfrac{1}{4}x^{2-6}y^{-3-3}$

$= \dfrac{1}{4}x^{-4}y^{-6} = \dfrac{1}{4x^4y^6}$

(iii) $5x^2y(2x^4y^{-3}) = 10x^{2+4}.y^{1-3}$

$= 10x^6y^{-2} = \dfrac{10x^6}{y^2}$

(iv) $(2fg^4)^4(fg)^6$

$= 2^4f^4g^{16}f^6g^6$

$= 16f^{4+6}g^{16+6}$

$= 16f^{10}g^{22}$

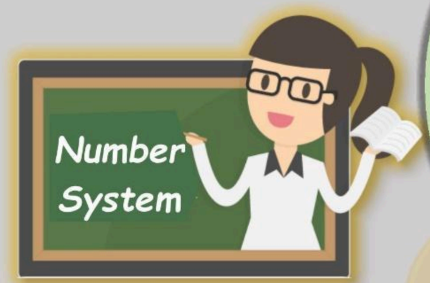
(v) $\left(\dfrac{-2a^3b^2c^0}{3a^2b^3c^7}\right)^{-2}$

$= \left(\dfrac{-2}{3}a^{3-2}b^{2-3}c^{0-7}\right)^{-2} = \left(\dfrac{-2}{3}a^1b^{-1}c^{-7}\right)^{-2}$

$= \left(\frac{-2}{3}\right)^{-2} a^{-2}b^{2}c^{14} = \left(-\frac{3}{2}\right)^{-2} a^{-2}b^{2}c^{14}$

$= \frac{9}{4}\frac{b^{2}c^{14}}{a^{2}}$

Memory Map



Rational Numbers

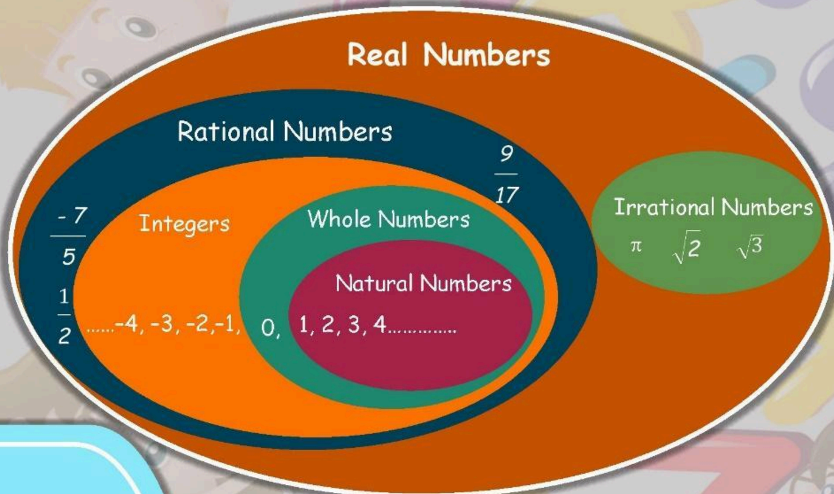
- A rational number is a type of real number, which is in the form of p/q where $q \neq 0$.
- Decimal expansion of a rational number is of two
  - Terminating
  - Non-terminating but recurring
- Rational Numbers b/w two Integral numbers (Average or Mean Method)

Irrational Numbers

- **Irrational numbers** are real numbers that cannot be represented as simple fractions.
- Decimal expansion of irrational number is non-terminating and non-recurring.
- Irrational number between two real numbers a and b is given by $\sqrt{ab}$.

Operation on Real Numbers

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$
- $(\sqrt{a}+\sqrt{b})^2 = a + 2\sqrt{ab} + b$



Representing Real Numbers on the Number line

Process of visualisation of representation of numbers on the number line through a magnifying glass is the process of successive magnification.

Rationalisation

Term	Rationalising factor
$\sqrt{a}$	$\sqrt{a}$
$\sqrt{a}+\sqrt{b}$	$\sqrt{a}-\sqrt{b}$
$\sqrt{a}+b$	$\sqrt{a}-b$
$a+\sqrt{b}$	$a-\sqrt{b}$

Decimal expansion of Numbers

- Terminating
- Non-terminating
  - Repeating
    - Pure
    - Mixed
  - Non-repeating

Laws of Exponents

- $a^m \times a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $a^m \times b^m = (ab)^m$