Exam

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18.11 Confluent hypergeometric functions

The confluent hypergeometric equation has the form

$$xy" + (c - x)y' - ay = 0; \tag{18.147}$$

it has a regular singularity at $x = 0$ and essential singularity at $x = \infty$.
This equation can be obtained by merging two of the singularities of the ordinary hypergeometric equation (18.136). The parameters a and c are given real numbers.

► Show that setting $x = z/b$ in the hypergeometric equation, and letting $b \to \infty$, yields the confluent hypergeometric equation

Substituting $x = z/b$ into (18.136), with $d/dx = bh/dz$, and letting $u(z) = y(x)$, we obtain

$$bz(1 - \frac{z}{b})\frac{d^2u}{dz^2} + [bc - (a + b + 1)z]\frac{du}{dz} - abu = 0,$$

which clearly has regular singular points at $z = 0$, b and ∞. If we now merge the last two singularities by letting $b \to \infty$, we obtain

$$zu" + (c - z)u' - au = 0,$$

where the primes denote d/dz. Hence $u(z)$ must satisfy the confluent hypergeometric equation. ◄

In our discussion of Bessel, Laguerre and associated Laguerre functions, it was noted that the corresponding second-order differential equation in each case had a single regular singular point at $x = 0$ and an essential singularity at $x = \infty$. From table 16.1, we see that this is also true for the confluent hypergeometric equation. Indeed, this equation can be considered as the 'canonical form' for second-order differential equations with this pattern of singularities. Consequently, as we mention below, the Bessel, Laguerre and associated Laguerre functions can all be written in terms of the *confluent hypergeometric functions*, which are the solutions of (18.147).

The solutions of the confluent hypergeometric equation are obtained from those of the ordinary hypergeomctric equation by again letting $x \to x/b$ and carrying out the limiting process $b \to \infty$. Thus, from (18.141) and (18.143), two linearly independent solutions of (18.147) are (when c is not an integer)

$$y_1(x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + ... \equiv M(a, c; x), \tag{18.148}$$

$$y_2(x) = x^{1-c}M(a - c + 1, 2 - c; x), \tag{18.149}$$

where $M(a, c; x)$ is called the *confluent hypergeometric funcrion* (or *Kummer function*).[§] It is worth nothing, however, that $y_2(x)$ is singular when $c = 2, 3, 4....$ Thus, it is conventional to take the...

[§]We note that alternative notation for the confluent hypergeometric function is ${}_1F_1(a, c; x)$.

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    \documentclass[10pt]{extarticle}
\usepackage[utf8x]{inputenc}
\usepackage{fancyhdr}
\usepackage{amsmath,amssymb}
\usepackage{tikz,lipsum,lmodern}
\usepackage[most]{tcolorbox}
\usepackage[pass,paperwidth=174mm,paperheight=247mm]{geometry}
\usepackage[bottom]{footmisc}
\renewcommand{\thefootnote}{\fnsymbol{footnote}}
\renewcommand*\footnoterule{}

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\section*{\centering\large{18.11 Confluent hypergeometric functions}}

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