这是标题

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1 李代数的有用公式

以下公式对于左扰动和右扰动均成立。通过级数展开：

$$\exp(\lfloor\boldsymbol{\theta}\rfloor_\times)=\boldsymbol{I}+\lfloor\boldsymbol{\theta}\rfloor_\times+\frac{1}{2!}\lfloor\boldsymbol{\theta}\rfloor_\times^2+\frac{1}{3!}\lfloor\boldsymbol{\theta}\rfloor_\times^3+\dots$$

可以得到：

$$\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)=\boldsymbol{I}+\lfloor\delta\boldsymbol{\theta}\rfloor_\times \tag{1}$$

$$\exp(t\lfloor\boldsymbol{\theta}\rfloor_\times)=\exp(\lfloor t\boldsymbol{\theta}\rfloor_\times)=\exp(\lfloor\boldsymbol{\theta}\rfloor_\times)^t \tag{2}$$

$$\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}^{-1}=\exp(\boldsymbol{R}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1}) \tag{3}$$

$$\boldsymbol{R}^{-1}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}=\exp(\boldsymbol{R}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}) \tag{4}$$

$$\boldsymbol{R}^{-1}=\boldsymbol{R}^T \tag{5}$$

$$\mathbf{Ad}_{\boldsymbol{R}}\delta\boldsymbol{\theta}=(\boldsymbol{R}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1})^\vee=\boldsymbol{R}\delta\boldsymbol{\theta} \tag{6}$$

$$\mathbf{Ad}_{\boldsymbol{R}}{}^{-1}\delta\boldsymbol{\theta}=(\boldsymbol{R}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R})^\vee=\boldsymbol{R}^{-1}\delta\boldsymbol{\theta} \tag{7}$$

以下公式以右扰动模型为依据：

$$\delta^{\boldsymbol{R}}\boldsymbol{\theta}=\boldsymbol{Q}\ominus\boldsymbol{R}=\mathrm{Log}(\boldsymbol{R}^{-1}\boldsymbol{Q}) \tag{8}$$

以下公式以左扰动模型为依据：

$$\delta^{\boldsymbol{I}}\boldsymbol{\theta}=\boldsymbol{Q}\ominus\boldsymbol{R}=\mathrm{Log}(\boldsymbol{Q}\boldsymbol{R}^{-1}) \tag{9}$$

将右扰动 (局部扰动) 化为左扰动 (全局扰动) 的公：

$$\begin{cases}\boldsymbol{R}\exp(\lfloor\delta^{\boldsymbol{R}}\boldsymbol{\theta}\rfloor_\times)=\exp(\boldsymbol{R}\lfloor\delta^{\boldsymbol{R}}\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1})\boldsymbol{R}\\ \lfloor\delta^{\boldsymbol{I}}\boldsymbol{\theta}\rfloor_\times=\boldsymbol{R}\lfloor\delta^{\boldsymbol{R}}\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1}\end{cases} \tag{10}$$

将左扰动化为右扰动的公式：

$$\begin{cases}\exp(\lfloor\delta^{\boldsymbol{I}}\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}=\boldsymbol{R}\exp(\boldsymbol{R}^{-1}\lfloor\delta^{\boldsymbol{I}}\boldsymbol{\theta}\rfloor_\times\boldsymbol{R})\\ \lfloor\delta^{\boldsymbol{R}}\boldsymbol{\theta}\rfloor_\times=\boldsymbol{R}^{-1}\lfloor\delta^{\boldsymbol{I}}\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}\end{cases} \tag{11}$$

注意：大写 LOG 直接将旋转矩阵变成三维向量，小写 log 将旋转矩阵变成李代数，Exp 和 exp 有同样意韵，但是恰好相反：

$$\begin{cases}\exp(\lfloor\boldsymbol{\theta}\rfloor_\times)=\mathrm{Exp}(\boldsymbol{\theta})\\ \log(\boldsymbol{R})^\vee=\mathrm{Log}(\boldsymbol{R})\end{cases}$$

2 线性化

记非线性函数为 $f(\boldsymbol{R})$，则对于右扰动模型：

$$f(\boldsymbol{R}\oplus\delta^{\boldsymbol{R}}\boldsymbol{\theta})=f(\boldsymbol{R})\oplus\frac{{}^{\boldsymbol{R}}\partial f(\boldsymbol{R})}{\partial\delta^{\boldsymbol{R}}\boldsymbol{\theta}}\delta^{\boldsymbol{R}}\boldsymbol{\theta}$$

对于左扰动模型：

$$f(\delta^{\boldsymbol{I}}\boldsymbol{\theta}\oplus\boldsymbol{R})=\frac{{}^{\boldsymbol{I}}\partial f(\boldsymbol{R})}{\partial\delta^{\boldsymbol{I}}\boldsymbol{\theta}}\delta^{\boldsymbol{I}}\boldsymbol{\theta}\oplus f(\boldsymbol{R})$$

且二者导数之间应有如下的关系 (可以用来验证推导结果)：

$$\mathbf{Ad}_{f(\boldsymbol{R})}\frac{{}^{\boldsymbol{R}}\partial f(\boldsymbol{R})}{\partial\delta^{\boldsymbol{R}}\boldsymbol{\theta}}=\frac{{}^{\boldsymbol{I}}\partial f(\boldsymbol{R})}{\partial\delta^{\boldsymbol{I}}\boldsymbol{\theta}}\mathbf{Ad}_{\boldsymbol{R}}$$
$$\rightarrow\mathbf{Ad}_{\boldsymbol{R}}=\boldsymbol{R}$$

3 公式应用

注意，以下推导过程中 $\boldsymbol{R}$、$\boldsymbol{Q}$ 为旋转矩阵，$\delta\boldsymbol{\theta}$ 为对旋转矩阵 $\boldsymbol{R}$ 施加的扰动量，$\boldsymbol{p}$ 为点向量。

3.1 左扰动

$$\begin{aligned}\frac{{}^{\boldsymbol{I}}\partial\boldsymbol{R}}{\partial\delta\boldsymbol{\theta}}&=\frac{\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\ominus\boldsymbol{R}}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{Log}(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\boldsymbol{R}^{-1})}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{Log}(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{LOG}(\mathrm{Exp}(\delta\boldsymbol{\theta}))}{\delta\boldsymbol{\theta}}=\boldsymbol{I}\end{aligned}$$

$$\begin{aligned}\frac{{}^{\boldsymbol{I}}\partial\boldsymbol{R}^{-1}}{\partial\delta\boldsymbol{\theta}}&=\frac{(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R})^{-1}\ominus\boldsymbol{R}^{-1}}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{Log}((\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R})^{-1}\boldsymbol{R})}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{Log}(\boldsymbol{R}^{-1}\exp(-\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R})}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{Log}(\exp(-\boldsymbol{R}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}))}{\delta\boldsymbol{\theta}}\\&=\frac{\mathrm{Log}(\mathrm{Exp}((-\boldsymbol{R}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R})^\vee))}{\delta\boldsymbol{\theta}}\\&=\frac{(-\boldsymbol{R}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R})^\vee}{\delta\boldsymbol{\theta}}\\&=\frac{-\boldsymbol{R}^{-1}\delta\boldsymbol{\theta}}{\delta\boldsymbol{\theta}}=-\boldsymbol{R}^{-1}\end{aligned}$$

$$
\begin{aligned}
\frac{{}^{I}\partial \boldsymbol{R}\boldsymbol{p}}{\partial\delta\boldsymbol{\theta}} &= \frac{\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\boldsymbol{p}-\boldsymbol{R}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{(\boldsymbol{I}+\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\boldsymbol{p}-\boldsymbol{R}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\lfloor\delta\boldsymbol{\theta}\rfloor_\times \boldsymbol{R}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{-\lfloor\boldsymbol{R}\boldsymbol{p}\rfloor_\times \delta\boldsymbol{\theta}}{\delta\boldsymbol{\theta}} = -\lfloor\boldsymbol{R}\boldsymbol{p}\rfloor_\times
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{I}\partial \boldsymbol{R}^{-1}\boldsymbol{p}}{\partial\delta\boldsymbol{\theta}} &= \frac{(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R})^{-1}\boldsymbol{p}-\boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\boldsymbol{R}^{-1}\exp(-\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{p}-\boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\boldsymbol{R}^{-1}(\boldsymbol{I}-\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{p}-\boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{-\boldsymbol{R}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times \boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\boldsymbol{R}^{-1}\lfloor\boldsymbol{p}\rfloor_\times \delta\boldsymbol{\theta}}{\delta\boldsymbol{\theta}} = \boldsymbol{R}^{-1}\lfloor\boldsymbol{p}\rfloor_\times
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{I}\partial \boldsymbol{R}\boldsymbol{Q}}{\partial\delta\boldsymbol{\theta}} &= \frac{\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\boldsymbol{Q}\ominus\boldsymbol{R}\boldsymbol{Q}}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\boldsymbol{Q}\boldsymbol{Q}^{-1}\boldsymbol{R}^{-1})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))}{\delta\boldsymbol{\theta}} = \boldsymbol{I}
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{I}\partial \boldsymbol{Q}\boldsymbol{R}}{\partial\delta\boldsymbol{\theta}} &= \frac{\boldsymbol{Q}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\ominus\boldsymbol{Q}\boldsymbol{R}}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{Q}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{Q}^{-1})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{Q}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{Q}^{-1})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(\boldsymbol{Q}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{Q}^{-1}))}{\delta\boldsymbol{\theta}} \\
&= \frac{(\boldsymbol{Q}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{Q}^{-1})^\vee}{\delta\boldsymbol{\theta}} = \boldsymbol{Q}
\end{aligned}
$$

3.2 右扰动

$$
\begin{aligned}
\frac{{}^{R}\partial \boldsymbol{R}}{\partial\delta\boldsymbol{\theta}} &= \frac{\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\ominus\boldsymbol{R}}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{R}^{-1}\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{LOG}(\mathrm{Exp}(\delta\boldsymbol{\theta}))}{\delta\boldsymbol{\theta}} = \boldsymbol{I}
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{R}\partial \boldsymbol{R}^{-1}}{\partial\delta\boldsymbol{\theta}} &= \frac{(\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))^{-1}\ominus\boldsymbol{R}^{-1}}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{R}\exp(-\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}^{-1})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(-\boldsymbol{R}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1}))}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\mathrm{Exp}((-\boldsymbol{R}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1})^\vee))}{\delta\boldsymbol{\theta}} \\
&= \frac{(-\boldsymbol{R}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{R}^{-1})^\vee}{\delta\boldsymbol{\theta}} \\
&= \frac{-\boldsymbol{R}\delta\boldsymbol{\theta}}{\delta\boldsymbol{\theta}} = -\boldsymbol{R}
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{R}\partial \boldsymbol{R}\boldsymbol{p}}{\partial\delta\boldsymbol{\theta}} &= \frac{\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{p}-\boldsymbol{R}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\boldsymbol{R}(\boldsymbol{I}+\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{p}-\boldsymbol{R}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\boldsymbol{R}\lfloor\delta\boldsymbol{\theta}\rfloor_\times \boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{-\boldsymbol{R}\lfloor\boldsymbol{p}\rfloor_\times \delta\boldsymbol{\theta}}{\delta\boldsymbol{\theta}} = -\boldsymbol{R}\lfloor\boldsymbol{p}\rfloor_\times
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{R}\partial \boldsymbol{R}^{-1}\boldsymbol{p}}{\partial\delta\boldsymbol{\theta}} &= \frac{(\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))^{-1}\boldsymbol{p}-\boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\exp(-\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}^{-1}\boldsymbol{p}-\boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{(\boldsymbol{I}-\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{R}^{-1}\boldsymbol{p}-\boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{-\lfloor\delta\boldsymbol{\theta}\rfloor_\times \boldsymbol{R}^{-1}\boldsymbol{p}}{\delta\boldsymbol{\theta}} \\
&= \frac{\lfloor\boldsymbol{R}^{-1}\boldsymbol{p}\rfloor_\times \delta\boldsymbol{\theta}}{\delta\boldsymbol{\theta}} = \lfloor\boldsymbol{R}^{-1}\boldsymbol{p}\rfloor_\times
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{R}\partial \boldsymbol{R}\boldsymbol{Q}}{\partial\delta\boldsymbol{\theta}} &= \frac{\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{Q}\ominus\boldsymbol{R}\boldsymbol{Q}}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{Q}^{-1}\boldsymbol{R}^{-1}\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{Q})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{Q}^{-1}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times)\boldsymbol{Q})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(\boldsymbol{Q}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{Q})}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\mathrm{Exp}((\boldsymbol{Q}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{Q})^\vee))}{\delta\boldsymbol{\theta}} \\
&= \frac{(\boldsymbol{Q}^{-1}\lfloor\delta\boldsymbol{\theta}\rfloor_\times\boldsymbol{Q})^\vee}{\delta\boldsymbol{\theta}} = \boldsymbol{Q}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{\boldsymbol{R}}\partial \boldsymbol{Q}\boldsymbol{R}}{\partial \delta\boldsymbol{\theta}} &= \frac{\boldsymbol{Q}\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times) \ominus \boldsymbol{Q}\boldsymbol{R}}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\boldsymbol{R}^{-1}\boldsymbol{Q}^{-1}\boldsymbol{Q}\boldsymbol{R}\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))}{\delta\boldsymbol{\theta}} \\
&= \frac{\mathrm{Log}(\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times))}{\delta\boldsymbol{\theta}} = \boldsymbol{I}
\end{aligned}
$$

4 符号说明

符号 $\lfloor\delta\boldsymbol{v}\rfloor_\times$ 表示将向量 $\delta\boldsymbol{v}$ 变换成反对称矩阵 (从向量空间到李代数空间)。比如对于三维维空间旋转而言:

$$
\begin{cases}
\delta\boldsymbol{\theta} = \begin{pmatrix}\delta\theta_x & \delta\theta_y & \delta\theta_z\end{pmatrix}^T \\
\lfloor\delta\boldsymbol{\theta}\rfloor_\times = \begin{pmatrix} 0 & -\delta\theta_z & \delta\theta_y \\ \delta\theta_z & 0 & -\delta\theta_x \\ -\delta\theta_y & \delta\theta_x & 0 \end{pmatrix}
\end{cases}
$$

事实上:

$$
\exp(\lfloor\delta\boldsymbol{\theta}\rfloor_\times) = \boldsymbol{I} + \lfloor\delta\boldsymbol{\theta}\rfloor_\times = \boldsymbol{R}(\delta\boldsymbol{\theta})
$$

正好对应了小角度旋转情况下的旋转矩阵。