

这是标题

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1 李代数的有用公式

以下公式对于左扰动和右扰动均成立。通过级数展开：

$$\exp([\boldsymbol{\theta}]_{\times}) = \mathbf{I} + [\boldsymbol{\theta}]_{\times} + \frac{1}{2!} [\boldsymbol{\theta}]_{\times}^2 + \frac{1}{3!} [\boldsymbol{\theta}]_{\times}^3 + \dots$$

可以得到：

$$\exp([\delta\boldsymbol{\theta}]_{\times}) = \mathbf{I} + [\delta\boldsymbol{\theta}]_{\times} \quad (1)$$

$$\exp(t [\boldsymbol{\theta}]_{\times}) = \exp([\boldsymbol{\theta}]_{\times})^t \quad (2)$$

$$\mathbf{R} \exp([\delta\boldsymbol{\theta}]_{\times}) \mathbf{R}^{-1} = \exp(\mathbf{R} [\delta\boldsymbol{\theta}]_{\times} \mathbf{R}^{-1}) \quad (3)$$

$$\mathbf{R}^{-1} \exp([\delta\boldsymbol{\theta}]_{\times}) \mathbf{R} = \exp(\mathbf{R}^{-1} [\delta\boldsymbol{\theta}]_{\times} \mathbf{R}) \quad (4)$$

$$\mathbf{R}^{-1} = \mathbf{R}^T \quad (5)$$

$$\mathbf{Ad}_R \delta\boldsymbol{\theta} = (\mathbf{R} [\delta\boldsymbol{\theta}]_{\times} \mathbf{R}^{-1})^{\vee} = \mathbf{R} \delta\boldsymbol{\theta} \quad (6)$$

$$\mathbf{Ad}_R^{-1} \delta\boldsymbol{\theta} = (\mathbf{R}^{-1} [\delta\boldsymbol{\theta}]_{\times} \mathbf{R})^{\vee} = \mathbf{R}^{-1} \delta\boldsymbol{\theta} \quad (7)$$

以下公式以右扰动模型为依据：

$$\delta^R \boldsymbol{\theta} = \mathbf{Q} \ominus \mathbf{R} = \text{Log}(\mathbf{R}^{-1} \mathbf{Q}) \quad (8)$$

以下公式以左扰动模型为依据：

$$\delta^I \boldsymbol{\theta} = \mathbf{Q} \ominus \mathbf{R} = \text{Log}(\mathbf{Q} \mathbf{R}^{-1}) \quad (9)$$

将右扰动（局部扰动）化为左扰动（全局扰动）的公：

$$\begin{cases} \mathbf{R} \exp([\delta^R \boldsymbol{\theta}]_{\times}) = \exp(\mathbf{R} [\delta^R \boldsymbol{\theta}]_{\times} \mathbf{R}^{-1}) \mathbf{R} \\ [\delta^I \boldsymbol{\theta}]_{\times} = \mathbf{R} [\delta^R \boldsymbol{\theta}]_{\times} \mathbf{R}^{-1} \end{cases} \quad (10)$$

将左扰动化为右扰动的公式：

$$\begin{cases} \exp([\delta^I \boldsymbol{\theta}]_{\times}) \mathbf{R} = \mathbf{R} \exp(\mathbf{R}^{-1} [\delta^I \boldsymbol{\theta}]_{\times} \mathbf{R}) \\ [\delta^R \boldsymbol{\theta}]_{\times} = \mathbf{R}^{-1} [\delta^I \boldsymbol{\theta}]_{\times} \mathbf{R} \end{cases} \quad (11)$$

注意：大写 LOG 直接将旋转矩阵变成三维向量，小写 log 将旋转矩阵变成李代数，Exp 和 exp 有同样意韵，但是恰好相反：

$$\begin{cases} \exp([\boldsymbol{\theta}]_{\times}) = \text{Exp}(\boldsymbol{\theta}) \\ \log(\mathbf{R})^{\vee} = \text{Log}(\mathbf{R}) \end{cases}$$

2 线性化

记非线性函数为 $f(\mathbf{R})$ ，则对于右扰动模型：

$$f(\mathbf{R} \oplus \delta^R \boldsymbol{\theta}) = f(\mathbf{R}) \oplus \frac{{}^R \partial f(\mathbf{R})}{\partial \delta^R \boldsymbol{\theta}} \delta^R \boldsymbol{\theta}$$

对于左扰动模型：

$$f(\delta^I \boldsymbol{\theta} \oplus \mathbf{R}) = \frac{{}^I \partial f(\mathbf{R})}{\partial \delta^I \boldsymbol{\theta}} \delta^I \boldsymbol{\theta} \oplus f(\mathbf{R})$$

且二者导数之间应有如下的关系（可以用来验证推导结果）：

$$\begin{aligned} \mathbf{Ad}_{f(\mathbf{R})} \frac{{}^R \partial f(\mathbf{R})}{\partial \delta^R \boldsymbol{\theta}} &= \frac{{}^I \partial f(\mathbf{R})}{\partial \delta^I \boldsymbol{\theta}} \mathbf{Ad}_R \\ &\rightarrow \mathbf{Ad}_R = \mathbf{R} \end{aligned}$$

3 公式应用

注意，以下推导过程中 \mathbf{R} 、 \mathbf{Q} 为旋转矩阵， $\delta\boldsymbol{\theta}$ 为对旋转矩阵 \mathbf{R} 施加的扰动量， \mathbf{p} 为点向量。

3.1 左扰动

$$\begin{aligned} \frac{{}^I \partial \mathbf{R}}{\partial \delta \boldsymbol{\theta}} &= \frac{\exp([\delta \boldsymbol{\theta}]_{\times}) \mathbf{R} \ominus \mathbf{R}}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}(\exp([\delta \boldsymbol{\theta}]_{\times}) \mathbf{R} \mathbf{R}^{-1})}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}(\exp([\delta \boldsymbol{\theta}]_{\times}))}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}(\text{Exp}(\delta \boldsymbol{\theta}))}{\delta \boldsymbol{\theta}} = \mathbf{I} \\ \frac{{}^I \partial \mathbf{R}^{-1}}{\partial \delta \boldsymbol{\theta}} &= \frac{(\exp([\delta \boldsymbol{\theta}]_{\times}) \mathbf{R})^{-1} \ominus \mathbf{R}^{-1}}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}((\exp([\delta \boldsymbol{\theta}]_{\times}) \mathbf{R})^{-1} \mathbf{R})}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}(\mathbf{R}^{-1} \exp(-[\delta \boldsymbol{\theta}]_{\times}) \mathbf{R})}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}(\exp(-\mathbf{R}^{-1} [\delta \boldsymbol{\theta}]_{\times} \mathbf{R}))}{\delta \boldsymbol{\theta}} \\ &= \frac{\text{Log}(\text{Exp}((- \mathbf{R}^{-1} [\delta \boldsymbol{\theta}]_{\times} \mathbf{R})^{\vee}))}{\delta \boldsymbol{\theta}} \\ &= \frac{(- \mathbf{R}^{-1} [\delta \boldsymbol{\theta}]_{\times} \mathbf{R})^{\vee}}{\delta \boldsymbol{\theta}} \\ &= \frac{- \mathbf{R}^{-1} \delta \boldsymbol{\theta}}{\delta \boldsymbol{\theta}} = - \mathbf{R}^{-1} \end{aligned}$$

$$\begin{aligned}
\frac{{}^I\partial R p}{\partial \delta \theta} &= \frac{\exp([\delta \theta]_{\times}) R p - R p}{\delta \theta} \\
&= \frac{(I + [\delta \theta]_{\times}) R p - R p}{\delta \theta} \\
&= \frac{[\delta \theta]_{\times} R p}{\delta \theta} \\
&= \frac{-[R p]_{\times} \delta \theta}{\delta \theta} = -[R p]_{\times}
\end{aligned}$$

$$\begin{aligned}
\frac{{}^I\partial R^{-1} p}{\partial \delta \theta} &= \frac{(\exp([\delta \theta]_{\times}) R)^{-1} p - R^{-1} p}{\delta \theta} \\
&= \frac{R^{-1} \exp(-[\delta \theta]_{\times}) p - R^{-1} p}{\delta \theta} \\
&= \frac{R^{-1} (I - [\delta \theta]_{\times}) p - R^{-1} p}{\delta \theta} \\
&= \frac{-R^{-1} [\delta \theta]_{\times} p}{\delta \theta} \\
&= \frac{R^{-1} [p]_{\times} \delta \theta}{\delta \theta} = R^{-1} [p]_{\times}
\end{aligned}$$

$$\begin{aligned}
\frac{{}^I\partial R Q}{\partial \delta \theta} &= \frac{\exp([\delta \theta]_{\times}) R Q \ominus R Q}{\delta \theta} \\
&= \frac{\text{Log}(\exp([\delta \theta]_{\times}) R Q Q^{-1} R^{-1})}{\delta \theta} \\
&= \frac{\text{Log}(\exp([\delta \theta]_{\times}))}{\delta \theta} = I
\end{aligned}$$

$$\begin{aligned}
\frac{{}^I\partial R^{-1} Q}{\partial \delta \theta} &= \frac{(\exp([\delta \theta]_{\times}) R)^{-1} Q \ominus R^{-1} Q}{\delta \theta} \\
&= \frac{\text{Log}(R^{-1} \exp(-[\delta \theta]_{\times}) Q Q^{-1} R)}{\delta \theta} \\
&= \frac{\text{Log}(\exp(R^{-1} [-\delta \theta]_{\times} R))}{\delta \theta} \\
&= \frac{(R^{-1} [-\delta \theta]_{\times} R)^{\vee}}{\delta \theta} = -R^{-1}
\end{aligned}$$

$$\begin{aligned}
\frac{{}^I\partial Q R}{\partial \delta \theta} &= \frac{Q \exp([\delta \theta]_{\times}) R \ominus Q R}{\delta \theta} \\
&= \frac{\text{Log}(Q \exp([\delta \theta]_{\times}) R R^{-1} Q^{-1})}{\delta \theta} \\
&= \frac{\text{Log}(Q \exp([\delta \theta]_{\times}) Q^{-1})}{\delta \theta} \\
&= \frac{\text{Log}(\exp(Q [\delta \theta]_{\times} Q^{-1}))}{\delta \theta} \\
&= \frac{(Q [\delta \theta]_{\times} Q^{-1})^{\vee}}{\delta \theta} = Q
\end{aligned}$$

3.2 右扰动

$$\begin{aligned}
\frac{{}^R\partial R}{\partial \delta \theta} &= \frac{R \exp([\delta \theta]_{\times}) \ominus R}{\delta \theta} \\
&= \frac{\text{Log}(R^{-1} R \exp([\delta \theta]_{\times}))}{\delta \theta} \\
&= \frac{\text{Log}(\exp([\delta \theta]_{\times}))}{\delta \theta} \\
&= \frac{\text{Log}(\text{Exp}(\delta \theta))}{\delta \theta} = I
\end{aligned}$$

$$\begin{aligned}
\frac{{}^R\partial R^{-1}}{\partial \delta \theta} &= \frac{(R \exp([\delta \theta]_{\times}))^{-1} \ominus R^{-1}}{\delta \theta} \\
&= \frac{\text{Log}(R \exp(-[\delta \theta]_{\times}) R^{-1})}{\delta \theta} \\
&= \frac{\text{Log}(\exp(-R [\delta \theta]_{\times} R^{-1}))}{\delta \theta} \\
&= \frac{\text{Log}(\text{Exp}((-R [\delta \theta]_{\times} R^{-1})^{\vee}))}{\delta \theta} \\
&= \frac{(-R [\delta \theta]_{\times} R^{-1})^{\vee}}{\delta \theta} \\
&= \frac{-R \delta \theta}{\delta \theta} = -R
\end{aligned}$$

$$\begin{aligned}
\frac{{}^R\partial R p}{\partial \delta \theta} &= \frac{R \exp([\delta \theta]_{\times}) p - R p}{\delta \theta} \\
&= \frac{R (I + [\delta \theta]_{\times}) p - R p}{\delta \theta} \\
&= \frac{R [\delta \theta]_{\times} p}{\delta \theta} \\
&= \frac{-R [p]_{\times} \delta \theta}{\delta \theta} = -R [p]_{\times}
\end{aligned}$$

$$\begin{aligned}
\frac{{}^R\partial R^{-1} p}{\partial \delta \theta} &= \frac{(R \exp([\delta \theta]_{\times}))^{-1} p - R^{-1} p}{\delta \theta} \\
&= \frac{\exp(-[\delta \theta]_{\times}) R^{-1} p - R^{-1} p}{\delta \theta} \\
&= \frac{(I - [\delta \theta]_{\times}) R^{-1} p - R^{-1} p}{\delta \theta} \\
&= \frac{-[\delta \theta]_{\times} R^{-1} p}{\delta \theta} \\
&= \frac{[R^{-1} p]_{\times} \delta \theta}{\delta \theta} = [R^{-1} p]_{\times}
\end{aligned}$$

$$\begin{aligned}
\frac{{}^R\partial RQ}{\partial \delta\theta} &= \frac{R \exp([\delta\theta]_{\times})Q \ominus RQ}{\delta\theta} \\
&= \frac{\text{Log}(Q^{-1}R^{-1}R \exp([\delta\theta]_{\times})Q)}{\delta\theta} \\
&= \frac{\text{Log}(Q^{-1} \exp([\delta\theta]_{\times})Q)}{\delta\theta} \\
&= \frac{\text{Log}(\exp(Q^{-1}[\delta\theta]_{\times}Q))}{\delta\theta} \\
&= \frac{\text{Log}(\text{Exp}((Q^{-1}[\delta\theta]_{\times}Q)^{\vee}))}{\delta\theta} \\
&= \frac{(Q^{-1}[\delta\theta]_{\times}Q)^{\vee}}{\delta\theta} = Q^{-1} \\
\frac{{}^I\partial R^{-1}Q}{\partial \delta\theta} &= \frac{(R \exp([\delta\theta]_{\times}))^{-1}Q \ominus R^{-1}Q}{\delta\theta} \\
&= \frac{\text{Log}(Q^{-1}R \exp(-[\delta\theta]_{\times})R^{-1}Q)}{\delta\theta} \\
&= \frac{\text{Log}(\exp(-Q^{-1}R[\delta\theta]_{\times}R^{-1}Q))}{\delta\theta} \\
&= \frac{(-Q^{-1}R[\delta\theta]_{\times}R^{-1}Q)^{\vee}}{\delta\theta} \\
&= \frac{-Q^{-1}R\delta\theta}{\delta\theta} = -Q^{-1}R \\
\frac{{}^R\partial QR}{\partial \delta\theta} &= \frac{QR \exp([\delta\theta]_{\times}) \ominus QR}{\delta\theta} \\
&= \frac{\text{Log}(R^{-1}Q^{-1}QR \exp([\delta\theta]_{\times}))}{\delta\theta} \\
&= \frac{\text{Log}(\exp([\delta\theta]_{\times}))}{\delta\theta} = I
\end{aligned}$$

4 符号说明

符号 $[\delta\mathbf{v}]_{\times}$ 表示将向量 $\delta\mathbf{v}$ 变换成反对称矩阵 (从向量空间到李代数空间)。比如对于三维空间旋转而言：

$$\begin{cases} \delta\theta = \begin{pmatrix} \delta\theta_x & \delta\theta_y & \delta\theta_z \end{pmatrix}^T \\ [\delta\theta]_{\times} = \begin{pmatrix} 0 & -\delta\theta_z & \delta\theta_y \\ \delta\theta_z & 0 & -\delta\theta_x \\ -\delta\theta_y & \delta\theta_x & 0 \end{pmatrix} \end{cases}$$

事实上：

$$\exp([\delta\theta]_{\times}) = I + [\delta\theta]_{\times} = R(\delta\theta)$$

正好对应了小角度旋转情况下的旋转矩阵。