Solution to problem number 1.5.11

G V V Sharma*

Question:
Obtain p, q, r in terms of a, b, c, the sides of the triangle using a matrix equation. Obtain the numerical values.

Solution:
Given in the question:
$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

Now, the side lengths a, b and c can be calculated as:

$$a = \sqrt{(\mathbf{C}-\mathbf{B})^{\top}.(\mathbf{C}-\mathbf{B})} \tag{1}$$
$$= \sqrt{\begin{pmatrix} 1 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ -11 \end{pmatrix}} \tag{2}$$
$$= \sqrt{1+121} \tag{3}$$
$$= \sqrt{122} \tag{4}$$
$$\tag{5}$$
$$b = \sqrt{(\mathbf{A}-\mathbf{C})^{\top}.(\mathbf{A}-\mathbf{C})} \tag{6}$$
$$= \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{7}$$
$$= \sqrt{16+16} \tag{8}$$
$$= \sqrt{32} \tag{9}$$
$$\tag{10}$$
$$c = \sqrt{(\mathbf{B}-\mathbf{A})^{\top}.(\mathbf{B}-\mathbf{A})} \tag{11}$$
$$= \sqrt{\begin{pmatrix} -5 & 7 \end{pmatrix}\begin{pmatrix} -5 \\ 7 \end{pmatrix}} \tag{12}$$
$$= \sqrt{25+49} \tag{13}$$
$$= \sqrt{74} \tag{14}$$

AB being a straight line with F_3 a point on it, it can be said that

$$AB = AF_3 + BF_3 \tag{15}$$
$$BC = BD_3 + CD_3 \tag{16}$$
$$CA = AE_3 + BE_3 \tag{17}$$
$$\tag{18}$$
$$\therefore c = m + n, \tag{19}$$
$$a = n + p, \tag{20}$$
$$b = m + p \tag{21}$$
$$\tag{22}$$
$$\therefore \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix}, \tag{23}$$
$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix}, \tag{24}$$
$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix}, \tag{25}$$

adding these 3 equations (1), (2) and (3) gives:

$$2\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{26}$$
$$\implies \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{27}$$
$$= \frac{\sqrt{74}+\sqrt{32}+\sqrt{122}}{2} \tag{28}$$

subtracting equations (1), (2) and (3) from the above gives us the values of p, m and n respectively

$$\therefore \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{29}$$
$$\implies m = \frac{\sqrt{74}+\sqrt{32}-\sqrt{122}}{2}, \tag{30}$$
$$\tag{31}$$
$$\tag{32}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

$$\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}m\\ n\\ p\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} \tag{33}$$

$$\implies n = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2}, \tag{34}$$

$$\tag{35}$$

$$\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}m\\ n\\ p\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} \tag{36}$$

$$\implies p = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \tag{37}$$