Q-10.13.3.10

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Question: Let X_1, X_2, ..., X_{10} be a random sample of size 10 from a $N_3(\mu, \Sigma)$ distribution, where μ and a non-singular Σ are unknown parameters. If

$$\overline{X_1} = \frac{1}{5}\sum_{i=1}^{5} X_i \tag{1}$$

$$\overline{X_2} = \frac{1}{5}\sum_{i=6}^{10} X_i \tag{2}$$

$$S_1 = \frac{1}{4}\sum_{i=1}^{5}(X_i - \overline{X_1})(X_i - \overline{X_1})^\top \tag{3}$$

$$S_2 = \frac{1}{4}\sum_{i=6}^{10}(X_i - \overline{X_2})(X_i - \overline{X_2})^\top \tag{4}$$

Then which one of the following statements is not true?

1) $\frac{5}{6}(\overline{X_1} - \mu)^\top S_1^- 1(\overline{X_1} - \mu)$ follows a F-distribution with 3 and 2 degrees of freedom
2) $\frac{6}{5(\overline{X_1}-\mu)^\top S_1^- 1(\overline{X_1}-\mu)}$ follows a F-distribution with 3 and 2 degrees of freedom
3) $4(S_1 + S_2)$ follows a Wishart distribution of order 3 with 8 degrees of freedom
4) $5(S_1 + S_2)$ follows a Wishart distribution of order 3 with 10 degrees of freedom

Solution:

Covariance matrix, Σ is defined as

$$\Sigma = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^\top] \tag{5}$$

$$\text{and,} \tag{6}$$

$$S_1 = \frac{1}{4}\sum_{i=1}^{5}(\mathbf{X_i} - \overline{\mathbf{X_1}})(\mathbf{X_i} - \overline{\mathbf{X_1}})^\top \tag{7}$$

$$S_2 = \frac{1}{4}\sum_{i=6}^{10}(\mathbf{X_i} - \overline{\mathbf{X_2}})(\mathbf{X_i} - \overline{\mathbf{X_2}})^\top \tag{8}$$

$\therefore$ S_1 and S_2 represent sample covariance matrix for their respective samples

1) **For option 1 and 2:**
2) **For option 3 and 4**

By defination, Wishart distribution is given by:

$$\mathbf{x_i} \sim N_p(\mu, \Sigma) \quad \forall \quad 1 \leq i \leq n \tag{9}$$

$$M = \sum_{i=1}^{n} x_i x_i^\top \sim W_p(\Sigma, n) \tag{10}$$

where p denotes order and n denotes the degree of freedom

Pdf of Wishart distribution is given by:

$$f(M) = \frac{1}{2^{(\frac{np}{2})}\Gamma_p(\frac{n}{2})|\Sigma|^{\frac{n}{2}}}|M|^{\frac{n-p-1}{2}} e^{\frac{-1}{2}tr(\Sigma^{-1}M)} \tag{11}$$

where

$$\Gamma_p\left(\frac{n}{2}\right) = \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma\left(\frac{n}{2} - \frac{j-1}{2}\right) \tag{12}$$

$$\mathbf{X_i} \sim N_3(\mu, \Sigma) \ \ \forall \ \ 1 \leq i \leq 10 \tag{13}$$

$$\implies \mathbf{Y_i} = \mathbf{X_i} - \overline{\mathbf{X_1}} \sim N_3(0, \Sigma) \ \ \forall \ \ 1 \leq i \leq 5, \text{and} \tag{14}$$

$$\mathbf{Y_i} = \mathbf{X_i} - \overline{\mathbf{X_2}} \sim N_3(0, \Sigma) \ \ \forall \ \ 6 \leq i \leq 10 \tag{15}$$

$$\therefore M_1 = 4\mathbf{S_1} = \sum_{i=1}^{5} \mathbf{Y_i}\mathbf{Y_i}^\top \sim W_3(\Sigma, 4), \text{and} \tag{16}$$

$$M_2 = 4\mathbf{S_2} = \sum_{i=6}^{10} \mathbf{Y_i}\mathbf{Y_i}^\top \sim W_3(\Sigma, 4) \tag{17}$$

$$\therefore M_1 + M_2 \sim W_3(\Sigma, 4 + 4) \tag{18}$$

$$\implies 4\mathbf{S_1} + 4\mathbf{S_2} \sim W_3(\Sigma, 8) \tag{19}$$

$\therefore$ $4(\mathbf{S_1} + \mathbf{S_2})$ follows a Wishart distribution with order 3 and 8 degrees of freedom

and,

$5(\mathbf{S_1} + \mathbf{S_2}) = \frac{5}{4}(M_1 + M_2)$ also follows a Wishart distribution with order 3 and 8 degrees of freedom

Hence, **option 3 is True and option 4 is False**