Q-10.13.3.10

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Question: Let X_1, X_2, ..., X_{10} be a random sample of size 10 from a $N_3(\mu, \Sigma)$ distribution, where μ and a non-singular Σ are unknown parameters. If

$$\overline{X_1} = \frac{1}{5}\sum_{i=1}^{5} X_i \tag{1}$$

$$\overline{X_2} = \frac{1}{5}\sum_{i=6}^{10} X_i \tag{2}$$

$$S_1 = \frac{1}{4}\sum_{i=1}^{5}(X_i - \overline{X_1})(X_i - \overline{X_1})^\top \tag{3}$$

$$S_2 = \frac{1}{4}\sum_{i=6}^{10}(X_i - \overline{X_2})(X_i - \overline{X_2})^\top \tag{4}$$

Then which one of the following statements is not true?

1) $\frac{5}{6}(\overline{X_1} - \mu)^\top S_1^- 1(\overline{X_1} - \mu)$ follows a F-distribution with 3 and 2 degrees of freedom
2) $\frac{6}{5(\overline{X_1} - \mu)^\top S_1^- 1(\overline{X_1} - \mu)}$ follows a F-distribution with 3 and 2 degrees of freedom
3) $4(S_1 + S_2)$ follows a Wishart distribution of order 3 with 8 degrees of freedom
4) $5(S_1 + S_2)$ follows a Wishart distribution of order 3 with 10 degrees of freedom

Solution:
Covariance matrix, Σ is defined as

$$\Sigma = E[(X - E[X])(Y - E[Y])] \tag{5}$$

$$\text{as in this case X = Y} \tag{6}$$

$$\Sigma = E[(X - E[X])(X - E[X])^\top] \tag{7}$$

$$= \sum(X_i - \overline{X})(X_i - \overline{X})^\top \tag{8}$$

$$\text{and} \tag{9}$$

$$S_1 = \frac{1}{4}\sum_{i=1}^{5}(X_i - \overline{X_1})(X_i - \overline{X_1})^\top \tag{10}$$

Given,

$$X_i \sim N_3(\mu, \Sigma) \tag{11}$$

$$\implies \overline{X} \sim N_3(\mu, \Sigma/5) \tag{12}$$

$$\therefore \overline{X_1} \sim N_3(\mu, \Sigma/5) \tag{13}$$

converting to chi-squared distribution,

$$\implies \frac{(\overline{X_1} - \mu)^2}{\Sigma/5} \sim \chi_3^2 \tag{14}$$

As $\overline{X_1} - \mu$ represents a trivariate distribution,

$$(\overline{X_1} - \mu)^2 = (\overline{X_1} - \mu)^\top(\overline{X_1} - \mu), \tag{15}$$

$$\tag{16}$$

and bringing Σ in numerator,

$$\implies 5 \times (\overline{X_1} - \mu)^\top\Sigma^{-1}(\overline{X_1} - \mu) \sim \chi_3^2 \tag{17}$$

$$\tag{18}$$

Converting χ^2 distribution into F-distribution by dividing it with the degree of freedom,

$$\implies \frac{5}{3} \times (\overline{X_1} - \mu)^\top\Sigma^{-1}(\overline{X_1} - \mu) \sim \chi_3^2 \tag{19}$$

$$\tag{20}$$