Q-10.13.3.10

Yash Patil - EE22BTECH11058

Question: Eight coins are tossed together. The probability of getting exactly 3 heads is
1) $\frac{1}{256}$
2) $\frac{7}{32}$
3) $\frac{5}{32}$
4) $\frac{3}{32}$

Solution: Defining variables:

Parameter	Value	Description
n	8	Number of coins tossed
p	0.5	probability of getting heads
$\mu = np$	4	mean of the distribution
$\sigma^2 = np(1-p)$	2	variance of the distribution
Y	0-8	denotes number of heads obtained

1) **Binomial distribution:** the probability of getting exactly 3 heads is

$$= \binom{8}{3} \times 0.5^3 \times 0.5^5 \tag{1}$$
$$= 0.21875 \tag{2}$$

$\therefore$ option 2 is correct.

2) **Gaussian Distribution:**
The gaussian distribution for Y is

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{3}$$

For getting 3 exactly heads

$$Y = 3 \tag{4}$$

Substituting in equation (3), probability for getting exactly 3 heads is

$$Y = 3 \tag{5}$$
$$p_Y(3) = \frac{1}{\sqrt{2\pi \times 2}} e^{\frac{-(3-4)^2}{2\times 2}} \tag{6}$$
$$= 0.35206 \tag{7}$$

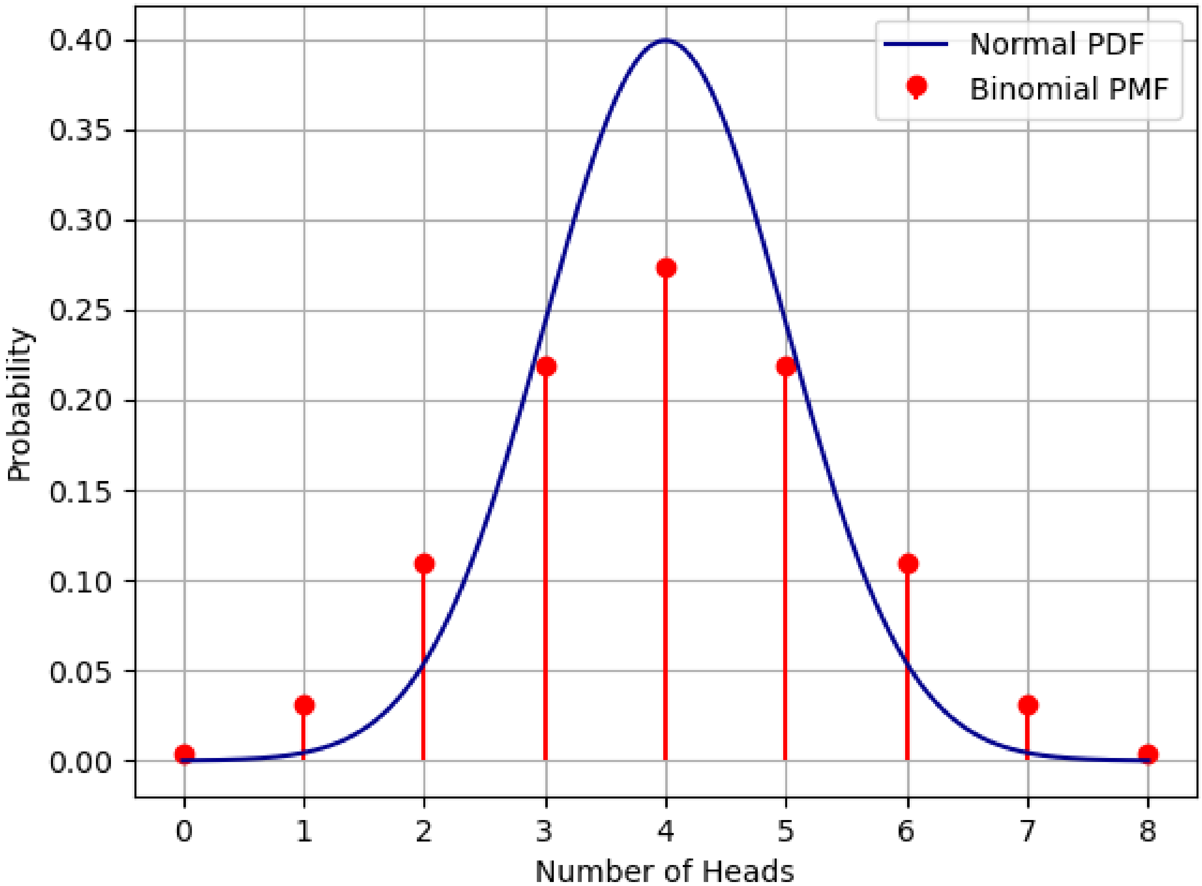

Fig. 1. Binomial distribution vs Gaussian distribution

3) **Using Q function:** Defining a gaussian random variable Z such that

$$Z \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8}$$

Due to continuity correction, $\Pr(Z = x)$ can be approximated as

$$\begin{aligned} p_Z(x) &\approx \Pr(x - 0.5 \leq Z < x + 0.5) && (9)\\ &\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) && (10)\\ &\approx F_Z(x + 0.5) - F_Z(x - 0.5) && (11) \end{aligned}$$

CDF of Z is defined as

$$\begin{aligned} F_Z(x) &= \Pr(Z < x) && (12)\\ &= \Pr\left(\frac{Z - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) && (13) \end{aligned}$$

As

$$\frac{Z-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{14}$$

$$\implies F_Z(x) = 1 - \Pr\left(\frac{Z-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{15}$$

$$= \begin{cases} 1 - Q(\frac{x-\mu}{\sigma}) & x \geq \mu \\ Q(\frac{\mu-x}{\sigma}) & x < \mu \end{cases} \tag{16}$$

Probability in terms of Q function is

$$p_Z(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{17}$$

$\therefore$ Gaussian approximation for $\Pr(Z=3)$ is

$$p_Z(3) \approx Q(0.3536) - Q(1.0608) \tag{18}$$

$$= 0.2174 \tag{19}$$

4) **Comparing all three techniques:**

Event	**Binomial**	**Gaussian**	**Q function**
Getting exactly 3 heads	0.21875	0.35206	0.2174