Solution to problem number 1.5.11

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Question:
Obtain p, q, r in terms of a, b, c, the sides of the triangle using a matrix equation. Obtain the numerical values.

Solution:
Given in the question:
A = $\begin{pmatrix}1\\-1\end{pmatrix}$, B = $\begin{pmatrix}-4\\6\end{pmatrix}$ and C = $\begin{pmatrix}-3\\-5\end{pmatrix}$

Now, the side lengths a, b and c can be calculated as:

$$AB = B - A \tag{1}$$
$$= \begin{pmatrix}-4-1\\6+1\end{pmatrix} = \begin{pmatrix}-5\\7\end{pmatrix} \tag{2}$$
$$BC = C - B \tag{3}$$
$$= \begin{pmatrix}-3+4\\-5-6\end{pmatrix} = \begin{pmatrix}1\\-11\end{pmatrix} \tag{4}$$
$$CA = A - C \tag{5}$$
$$= \begin{pmatrix}1+3\\-1+5\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} \tag{6}$$

Now, the side lengths a, b and c can be calculated as:

$$a = \sqrt{BC^{\top}.BC} \tag{7}$$
$$= \sqrt{\begin{pmatrix}1 & -11\end{pmatrix}\begin{pmatrix}1\\-11\end{pmatrix}} \tag{8}$$
$$= \sqrt{1+121} \tag{9}$$
$$= \sqrt{122} \tag{10}$$

$$b = \sqrt{CA^{\top}.CA} \tag{11}$$
$$= \sqrt{\begin{pmatrix}4 & 4\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}} \tag{12}$$
$$= \sqrt{16+16} \tag{13}$$
$$= \sqrt{32} \tag{14}$$

$$c = \sqrt{AB^{\top}.AB} \tag{15}$$
$$= \sqrt{\begin{pmatrix}-5 & 7\end{pmatrix}\begin{pmatrix}-5\\7\end{pmatrix}} \tag{16}$$
$$= \sqrt{25+49} \tag{17}$$
$$= \sqrt{74} \tag{18}$$

AB being a straight line with F_3 a point on it, it can be said that

$$AB = AF_3 + BF_3 \tag{19}$$
$$BC = BD_3 + CD_3 \tag{20}$$
$$CA = AE_3 + BE_3 \tag{21}$$
$$\tag{22}$$
$$\therefore c = m + n, \tag{23}$$
$$a = n + p, \tag{24}$$
$$b = m + p \tag{25}$$
$$\tag{26}$$

$$\therefore \begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}m\\n\\p\end{pmatrix}, \tag{27}$$
$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0 & 1 & 1\end{pmatrix}\begin{pmatrix}m\\n\\p\end{pmatrix}, \tag{28}$$
$$\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}m\\n\\p\end{pmatrix}, \tag{29}$$

adding these 3 equations (1), (2) and (3) gives:

$$2\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}m\\n\\p\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{30}$$
$$\implies \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}m\\n\\p\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{31}$$
$$= \frac{\sqrt{74} + \sqrt{32} + \sqrt{122}}{2} \tag{32}$$

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subtracting equations (1), (2) and (3) from the above gives us the values of p, m and n respectively

$$\therefore \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{33}$$

$$\implies m = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2}, \tag{34}$$

$$\tag{35}$$

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{36}$$

$$\implies n = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2}, \tag{37}$$

$$\tag{38}$$

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{39}$$

$$\implies p = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \tag{40}$$