Solution to problem number 1.5.11

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Question:
Obtain p, q, r in terms of a, b, c, the sides of the triangle using a matrix equation. Obtain the numerical values.

Solution:
Given in the question:
$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

Now, the side lengths a, b and c can be calculated as:

$$a = \sqrt{(\mathbf{C}-\mathbf{B})^\top.(\mathbf{C}-\mathbf{B})} \tag{1}$$
$$= \sqrt{\begin{pmatrix} 1 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ -11 \end{pmatrix}} \tag{2}$$
$$= \sqrt{1+121} \tag{3}$$
$$= \sqrt{122} \tag{4}$$
$$\tag{5}$$

$$b = \sqrt{(\mathbf{A}-\mathbf{C})^\top.(\mathbf{A}-\mathbf{C})} \tag{6}$$
$$= \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{7}$$
$$= \sqrt{16+16} \tag{8}$$
$$= \sqrt{32} \tag{9}$$
$$\tag{10}$$

$$c = \sqrt{(\mathbf{B}-\mathbf{A})^\top.(\mathbf{B}-\mathbf{A})} \tag{11}$$
$$= \sqrt{\begin{pmatrix} -5 & 7 \end{pmatrix}\begin{pmatrix} -5 \\ 7 \end{pmatrix}} \tag{12}$$
$$= \sqrt{25+49} \tag{13}$$
$$= \sqrt{74} \tag{14}$$

AB being a straight line with F_3 a point on it,

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it can be said that

$$AB = AF_3 + BF_3 \tag{15}$$
$$BC = BD_3 + CD_3 \tag{16}$$
$$CA = AE_3 + BE_3 \tag{17}$$
$$\tag{18}$$
$$\therefore c = m + n, \tag{19}$$
$$a = n + p, \tag{20}$$
$$b = m + p \tag{21}$$

these 3 equations can be written as:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{22}$$
$$\implies \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{23}$$

solving by Guassian elimination method,

$$R_1 \to R_1 + R_2 + R_3, \tag{24}$$
$$\implies \begin{pmatrix} 2 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a+b+c \\ b \\ c \end{pmatrix} \tag{25}$$
$$R_1 \to \frac{R_1}{2}, \tag{26}$$
$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} \frac{a+b+c}{2} \\ b \\ c \end{pmatrix} \tag{27}$$
$$R_2 \to R_1 - R_2, \tag{28}$$
$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} \frac{a+b+c}{2} \\ \frac{a-b+c}{2} \\ c \end{pmatrix} \tag{29}$$
$$R_3 \to R_1 - R_3, \tag{30}$$
$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} \frac{a+b+c}{2} \\ \frac{a-b+c}{2} \\ \frac{a+b-c}{2} \end{pmatrix} \tag{31}$$

$$R_1 \to R_1 - R_2 - R_3, \tag{32}$$

$$\implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} \frac{-a+b+c}{2} \\ \frac{a-b+c}{2} \\ \frac{a+b-c}{2} \end{pmatrix} \tag{33}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} \frac{-a+b+c}{2} \\ \frac{a-b+c}{2} \\ \frac{a+b-c}{2} \end{pmatrix} \tag{34}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{74}+\sqrt{32}-\sqrt{122}}{2} \\ \frac{\sqrt{74}+\sqrt{122}-\sqrt{32}}{2} \\ \frac{\sqrt{122}+\sqrt{32}-\sqrt{74}}{2} \end{pmatrix} \tag{35}$$

$$\therefore m = \frac{\sqrt{74}+\sqrt{32}-\sqrt{122}}{2}, \tag{36}$$

$$n = \frac{\sqrt{74}+\sqrt{122}-\sqrt{32}}{2}, \tag{37}$$

$$p = \frac{\sqrt{122}+\sqrt{32}-\sqrt{74}}{2} \tag{38}$$