Q-10.13.3.10

Yash Patil - EE22BTECH11058

Question: Let X_1, X_2, ..., X_{10} be a random sample of size 10 from a $N_3(\mu, \Sigma)$ distribution, where μ and a non-singular Σ are unknown parameters. If

$$\overline{X_1} = \frac{1}{5}\sum_{i=1}^{5} X_i \tag{1}$$

$$\overline{X_2} = \frac{1}{5}\sum_{i=6}^{10} X_i \tag{2}$$

$$S_1 = \frac{1}{4}\sum_{i=1}^{5}(X_i - \overline{X_1})(X_i - \overline{X_1})^\top \tag{3}$$

$$S_2 = \frac{1}{4}\sum_{i=6}^{10}(X_i - \overline{X_2})(X_i - \overline{X_2})^\top \tag{4}$$

Then which one of the following statements is not true?

1) $\frac{5}{6}(\overline{X_1} - \mu)^\top S_1^- 1(\overline{X_1} - \mu)$ follows a F-distribution with 3 and 2 degrees of freedom
2) $\frac{6}{5(\overline{X_1}-\mu)^\top S_1^- 1(\overline{X_1}-\mu)}$ follows a F-distribution with 3 and 2 degrees of freedom
3) $4(S_1 + S_2)$ follows a Wishart distribution of order 3 with 8 degrees of freedom
4) $5(S_1 + S_2)$ follows a Wishart distribution of order 3 with 10 degrees of freedom

Solution:
Covariance matrix, Σ is defined as

$$\Sigma = E[(X - E[X])(X - E[X])^\top] \tag{5}$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \overline{X})(X_i - \overline{X})^\top \tag{6}$$

$$\text{and,} \tag{7}$$

$$S_1 = \frac{1}{4}\sum_{i=1}^{5}(X_i - \overline{X_1})(X_i - \overline{X_1})^\top \tag{8}$$

$$S_2 = \frac{1}{4}\sum_{i=6}^{10}(X_i - \overline{X_2})(X_i - \overline{X_2})^\top \tag{9}$$

$\therefore$ S_1 and S_2 represent covariance matrix for their respective samples

1) **For option 1 and 2:**
Given,

$$X_i \sim N_3(\mu, \Sigma) \tag{10}$$

$$\implies \overline{X} \sim N_3(\mu, \Sigma/5) \tag{11}$$

$$\therefore \overline{X_1} \sim N_3(\mu, S_1/5) \tag{12}$$

converting to chi-squared distribution,

$$\implies \frac{(\overline{X_1} - \mu)^2}{S_1/5} \sim \chi^2_3 \tag{13}$$

As $\overline{X_1} - \mu$ represents a trivariate distribution,

$$(\overline{X_1} - \mu)^2 = (\overline{X_1} - \mu)^\top(\overline{X_1} - \mu), \tag{14}$$

and bringing S_1 in numerator,

$$\implies 5(\overline{X_1} - \mu)^\top S_1^{-1}(\overline{X_1} - \mu) \sim \chi^2_3 \tag{15}$$

F distribution is defined as

$$\frac{X/d_1}{Y/d_2} \sim F(d_1, d_2) \qquad \text{where} \tag{16}$$

$$X \sim \chi^2_{d_1} \text{ and } Y \sim \chi^2_{d_2} \tag{17}$$

are independant chi-square distributions
Let X represent be a χ^2_3 variable, then

$$\therefore \frac{X/3}{Y/d} \sim F(3, d) \quad \text{and} \tag{18}$$

$$\frac{Y/d}{X/3} \sim F(d, 3) \quad \forall \quad d \in \mathbb{N} \tag{19}$$

Hence for $d = 2$,

$$\frac{5}{3}(\overline{X_1} - \mu)^\top\Sigma^{-1}(\overline{X_1} - \mu) \sim F(3, 2) \tag{20}$$

$$\therefore \frac{5}{6}(\overline{X_1} - \mu)^\top\Sigma^{-1}(\overline{X_1} - \mu) \sim F(3, 2), \text{and similarly} \tag{21}$$

$$\frac{6}{5(\overline{X_1} - \mu)^\top\Sigma^{-1}(\overline{X_1} - \mu)} \sim F(2, 3) \tag{22}$$

Hence, **option 1 and 2 are true**

2) **For option 3 and 4**

By defination, Wishart distribution is given by:

$$x_i \sim N_p(\mu, \Sigma) \quad \forall \quad 1 \le i \le n \tag{23}$$

$$M = \sum_{i=1}^{n} x_i x_i^\top \sim W_p(\Sigma, n) \tag{24}$$

where p denotes order and n denotes the degree of freedom

$$X_i \sim N_3(\mu, \Sigma) \;\; \forall \;\; 1 \le i \le 10 \tag{25}$$

$$\implies Y_i = X_i - \overline{X_1} \sim N_3(0, \Sigma) \;\; \forall \;\; 1 \le i \le 5, \text{and} \tag{26}$$

$$Y_i = X_i - \overline{X_2} \sim N_3(0, \Sigma) \;\; \forall \;\; 6 \le i \le 10 \tag{27}$$

$$\therefore M_1 = S_1 = \sum_{i=1}^{5} Y_i Y_i^\top \sim W_3(\Sigma, 4), \text{and} \tag{28}$$

$$M_2 = S_2 = \sum_{i=6}^{10} Y_i Y_i^\top \sim W_3(\Sigma, 4) \tag{29}$$

$$\therefore M_1 + M_2 \sim W_3(\Sigma, 4 + 4) \tag{30}$$

$$\implies S_1 + S_2 \sim W_3(\Sigma, 8) \tag{31}$$

$\therefore \lambda(S_1 + S_2)$ follow Wishart distribution with order 3 and 8 degrees of freedom where $\lambda \in \mathbb{R}$

Hence, **option 3 is True and option 4 is False**