

# Trees

## COMS20010 (Algorithms II)

John Lapinskas, University of Bristol

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In this course, we will think of trees as examples of graphs.

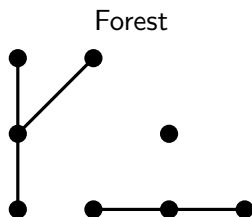
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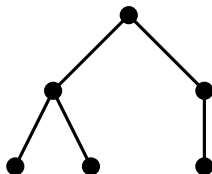
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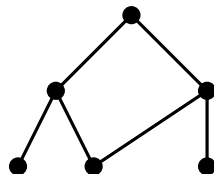
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Tree (and forest)



Neither tree nor forest



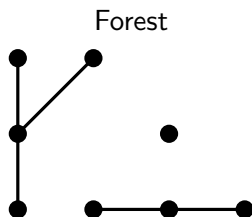
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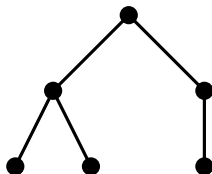
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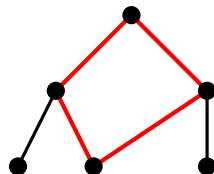
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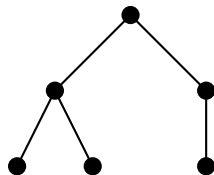
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A **tree** is a connected graph with no cycles.

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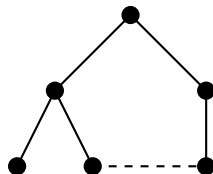
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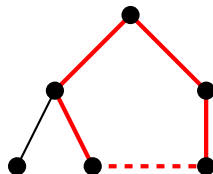
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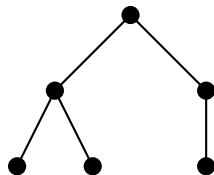
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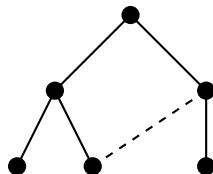




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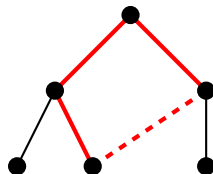
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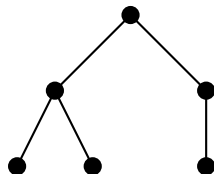
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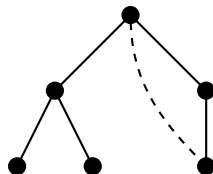
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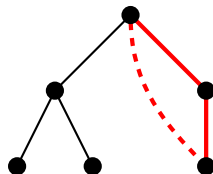
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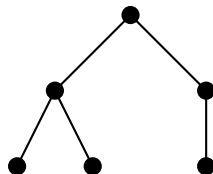


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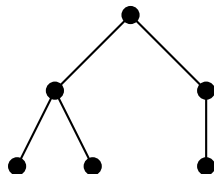


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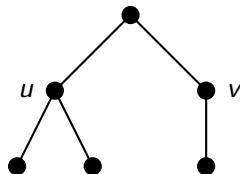
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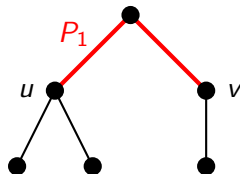


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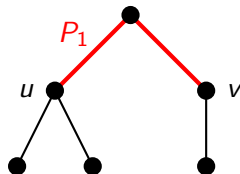
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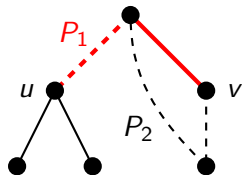
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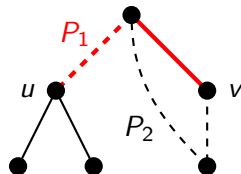
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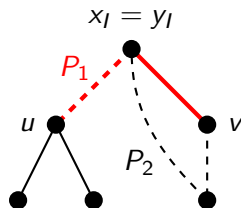
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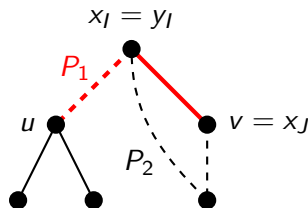
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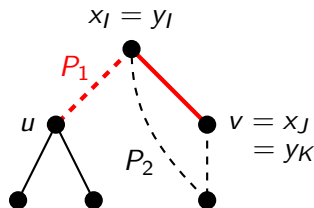
Let  $J = \min\{i > I: x_i \in \{y_I, \dots, y_k\}\}$  be the point of remerging.

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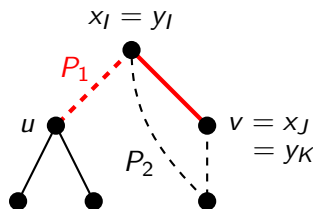
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Then  $x_I x_{I+1} \dots x_J y_{K-1} y_{K-2} \dots y_I$  is a cycle, so  $T$  is not a tree.

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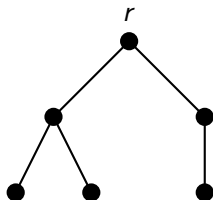
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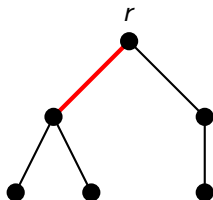
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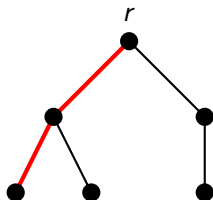
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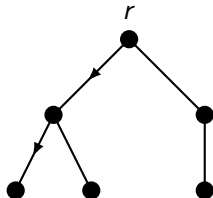
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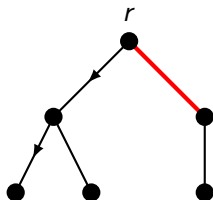
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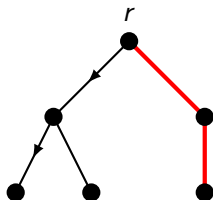
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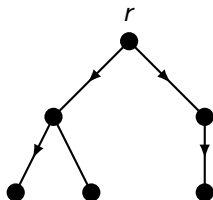
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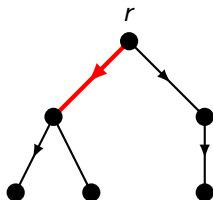
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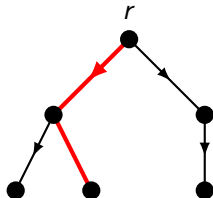
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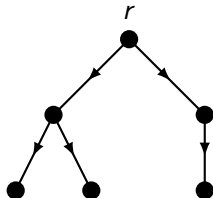
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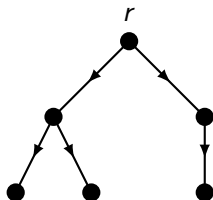
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Why are the directions consistent?



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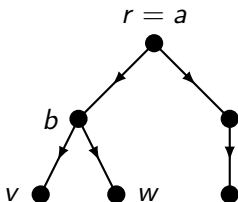
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Let  $r \in V$  be arbitrary — this will be the **root**. Every vertex  $v \neq r$  has a unique path  $P_v$  from  $r$  to  $v$  by the lemma. Direct its edges from  $r$  to  $v$ .

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Suppose some path  $P_v$  directs  $a \rightarrow b$ .

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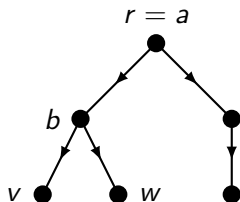
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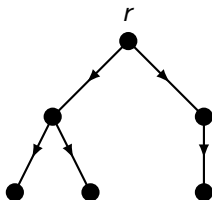
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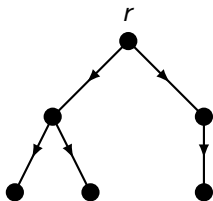
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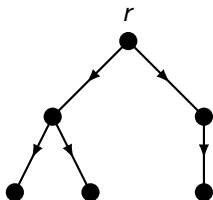
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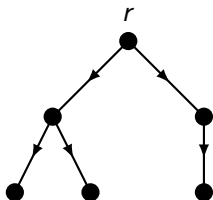
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**Bonus:** We also just defined rooted trees in terms of graphs.

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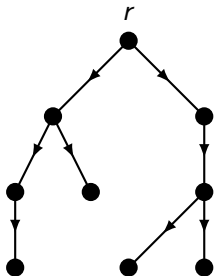
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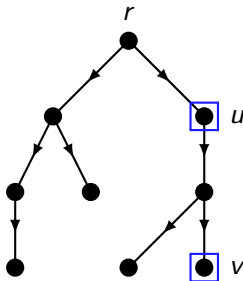
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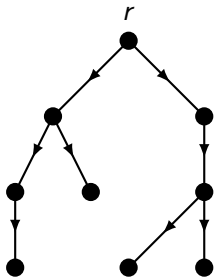
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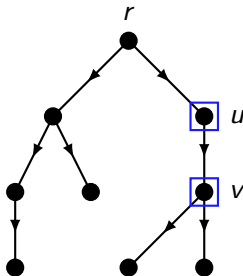
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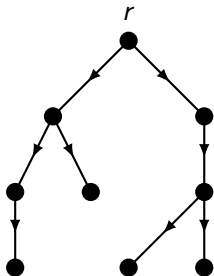
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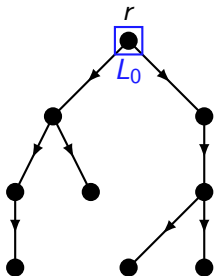
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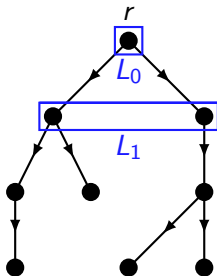
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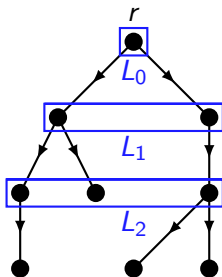
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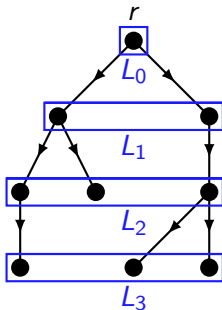
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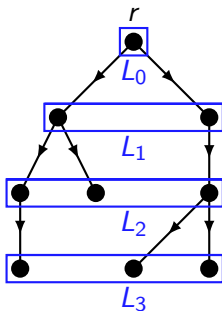
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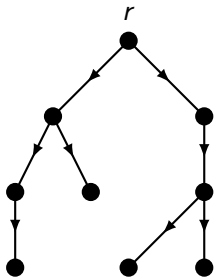
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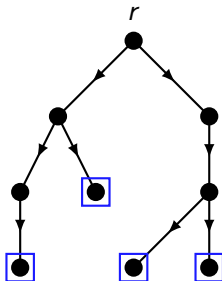
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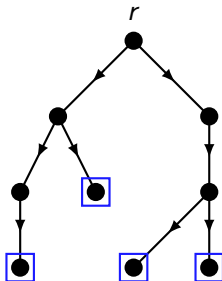
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**Proof:** Let  $x$  be the number of leaves in  $T$ .

By the handshaking lemma,  $|E| = \frac{1}{2} \sum_{v \in V} d(v)$ . Also,  $|E| = n - 1$ .

Since  $T$  is connected and  $n \geq 2$ , every vertex has degree at least 1.

So all non-leaves have degree at least 2, and  $\sum_{v \in V} d(v) \geq 2(n - x) + x$ .

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**Lemma 1:** Any pair of vertices in a tree is joined by a **unique** path. □

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A **leaf** is a degree-1 vertex.

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Solving for  $x$  gives  $x \geq 2$ , so we're done! □

# The Fundamental Lemma of Trees

A **tree** is a connected graph with no cycles.

**Lemma 1:** Any pair of vertices in a tree is joined by a **unique** path. ☐

**Lemma 2:** Any  $n$ -vertex tree has  $n - 1$  edges. ☐

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When you're actually working with trees, it's good to have one single result that tells you that all the “obvious” things are true. This is that result.

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- (A)  $T$  is connected and has no cycles, i.e. is a tree;
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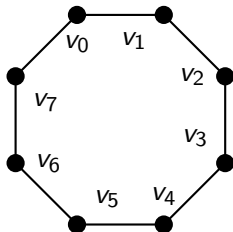
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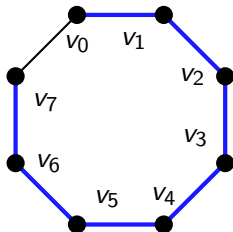
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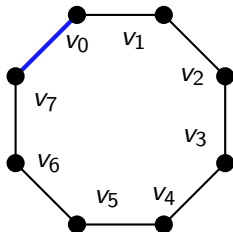
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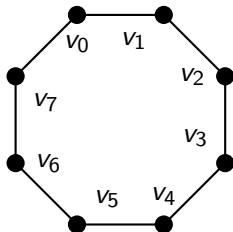
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So  $T$  has no cycles.

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Each of these components has no cycles, and is connected, so it's a tree.  
So by (A)  $\Rightarrow$  (B) (or Lemma 2), each  $C_i$  has  $|V(C_i)| - 1$  edges.

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Every edge of  $T$  is in some  $C_i$ , so  $|E| = \sum_i (|V(C_i)| - 1) = n - r$ .

But we know  $|E| = n - 1$ , so we must have  $r = 1$ .

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So  $T$  must have had **more than**  $n - 1$  edges — a contradiction.      □

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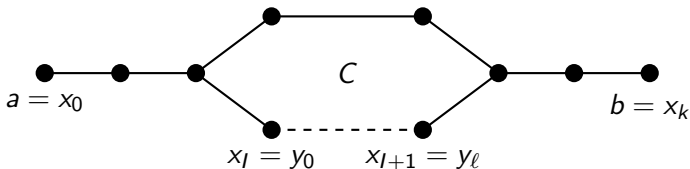
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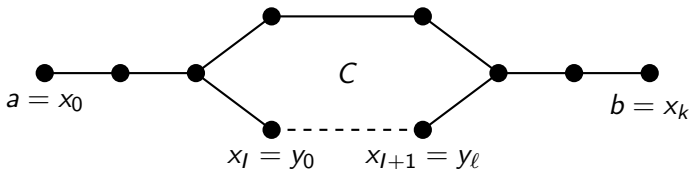
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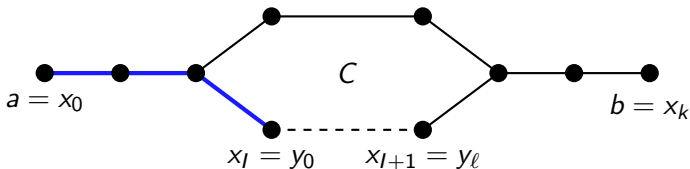
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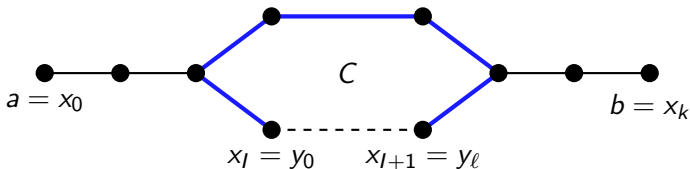
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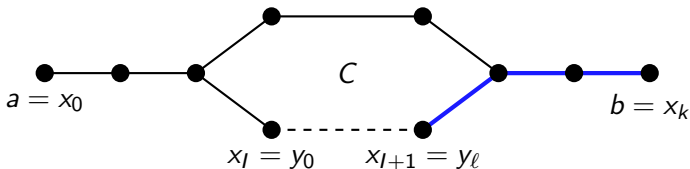
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And there was much rejoicing.