

Linear programming COMS20010 (Algorithms II)

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These two videos are a very basic overview of a deep and rich theory.

As an example problem: which Warhammer models should Games Workshop produce in order to make as much money as possible?

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- Games Workshop makes a profit of £4 per noise marine and £10 per doomwheel, so...**they wish to maximise $4N + 10D$.**
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- Similarly, their metal plant can turn out 4kg of finished parts per day. One noise marine contains 60g of metal, and one doomwheel contains 10g, so...

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More succinctly, the problem is:

$$\begin{aligned}4N + 10D &\rightarrow \max, \text{ subject to} \\5N + 100D &\leq 5000; \\60N + 10D &\leq 4000; \\N &\leq 100; \\D &\leq 50; \\N, D &\geq 0.\end{aligned}$$

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We can write this in matrix form:

$$\begin{aligned}4N + 10D &\rightarrow \max, \text{ subject to} \\ \begin{pmatrix} 5 & 100 \\ 60 & 10 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N \\ D \end{pmatrix} &\leq \begin{pmatrix} 5000 \\ 4000 \\ 100 \\ 50 \end{pmatrix}; \\ N, D &\geq 0.\end{aligned}$$

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The desired output is a vector $\vec{x} \in \mathbb{R}^n$ maximising $f(\vec{x})$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$.

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We say a $\vec{x} \in \mathbb{R}^n$ is a **feasible** solution to a linear program if $\vec{x} \geq \vec{0}$ and $A\vec{x} \leq \vec{b}$, and an **optimal** solution if $f(\vec{y}) \leq f(\vec{x})$ for all feasible $\vec{y} \in \mathbb{R}^n$.

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But these are the only two things that can go wrong — any bounded linear program with at least one feasible solution has an optimal solution.

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All of these can be implemented in the above framework, which is known as **standard form**.

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As an example, let's turn the following LP into standard form:

$$4x - 5y + z \rightarrow \min \text{ subject to}$$

$$x + y + z = 5;$$

$$x + 2y \geq 2;$$

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Removing non-negativity: If y doesn't have to be non-negative, we can replace it by $y_1 - y_2$ where $y_1, y_2 \geq 0$. We think of y_1 as the positive part and y_2 as the negative part.

There will be feasible solutions with both $y_1 > 0$ and $y_2 > 0$, but this doesn't matter — any optimal solution of the old problem will be an optimal solution of the new one and vice versa.

Standard form: We are given a linear objective function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, an $m \times n$ matrix A , and an m -dimensional vector $\vec{b} \in \mathbb{R}^m$. The desired output is a vector $\vec{x} \in \mathbb{R}^n$ maximising $f(\vec{x})$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$.

As an example, let's turn the following LP into standard form:

$$\begin{aligned} -4x + 5(y_1 - y_2) - z &\rightarrow \max \text{ subject to} \\ x + (y_1 - y_2) + z &\leq 5; \\ -x - (y_1 - y_2) - z &\leq -5; \\ -x - 2(y_1 - y_2) &\leq -2; \\ x, y_1, y_2, z &\geq 0. \end{aligned}$$

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As an example, let's turn the following LP into standard form:

$$\begin{aligned} & -4x + 5y_1 - 5y_2 - z \rightarrow \max \text{ subject to} \\ & \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \\ z \end{pmatrix} \leq \begin{pmatrix} 5 \\ -5 \\ -2 \end{pmatrix}; \\ & x, y_1, y_2, z \geq 0. \end{aligned}$$

The problem is now in standard form! And these techniques are fully general.

So we have **reduced** the problem of solving a general linear program, which might have a minimisation goal, $=$ or \leq constraints, and/or negative variables, to that of solving a linear program in standard form.

That makes it easier to find an algorithm!