

NP-completeness of 3-SAT

COMS20010 2020, Video 9-4

John Lapinskas, University of Bristol

An easier problem to reduce from: 3-SAT

A **literal** is either a variable x or its negation $\neg x$. An **OR clause** is an OR of literals. A **CNF** formula is an AND of OR clauses. The **SAT** problem asks: "Is the input CNF formula satisfiable?"

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Certainly $3\text{-SAT} \in \text{NP}$, but our proof that SAT is NP-hard breaks for 3-SAT. So to prove NP-hardness, we will reduce SAT to 3-SAT; the result then follows since SAT is NP-hard.

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C_i has width 2: Say $C_i = x \vee y$. Then we would like to replace C_i with $x \vee y \vee \text{False}$ in F' , since this is True if and only if $x \vee y = \text{True}$.

But False is not a literal... Can we add a new variable which is always False in any satisfying assignment?

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But False is not a literal... Can we add a new variable which is always False in any satisfying assignment? Yes! If we add this CNF to F :

$$F_z = (\neg z_1 \vee z_2 \vee z_3) \wedge (\neg z_1 \vee z_2 \vee \neg z_3) \wedge (\neg z_1 \vee \neg z_2 \vee z_3) \wedge (\neg z_1 \vee \neg z_2 \vee \neg z_3)$$

then z_1 is forced to be False: No matter what value z_2 and z_3 take, their literals must both be False in one of the above OR clauses. ✓

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If C_i has width 3: We can just leave it as it is. ✓

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Width 1 clauses: ✓ **Width 2 clauses:** ✓ **Width 3 clauses:** ✓

If C_i has width $k \geq 4$: Say $C_i = \ell_1 \vee \cdots \vee \ell_k$. We would like to replace

$$C_i \rightarrow (e_1 = \ell_1 \vee \ell_2) \wedge (e_2 = e_1 \vee \ell_3) \wedge \cdots \wedge (e_{k-2} = e_{k-3} \vee \ell_{k-1}) \wedge (e_{k-2} \vee \ell_k),$$

as given the values of ℓ_1, \dots, ℓ_k , this is satisfiable if and only if

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as given the values of ℓ_1, \dots, ℓ_k , this is satisfiable if and only if $\ell_1 \vee \cdots \vee \ell_k = \text{True}$. How do we implement the e_i 's? We have

$$(a = b \vee c) \text{ if and only if } (a \vee \neg b) \wedge (a \vee \neg c) \wedge (\neg a \vee b \vee c);$$

the first two clauses on the right enforce $a = \text{False} \Rightarrow b \vee c = \text{False}$,
and the last enforces $b \vee c = \text{False} \Rightarrow a = \text{False}$. □

Example of $\text{SAT} \leq_c \text{3-SAT}$ reduction

Suppose our original SAT instance is:

$$F = u \wedge (\neg u \vee \neg v) \wedge (v \vee \neg w \vee x \vee \neg y \vee \neg z) \wedge (y \vee z) \wedge (\neg v \vee w \vee z).$$

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$$\begin{aligned} F' = & (u \vee f_1 \vee \text{False}) \wedge (\neg u \vee \neg v \vee f_1) \\ & \wedge (e_1 = v \vee \neg w) \wedge (e_2 = e_1 \vee x) \wedge (e_3 = e_2 \vee \neg y) \\ & \wedge (e_3 \vee \neg z \vee f_1) \wedge (y \vee z \vee f_1) \wedge (\neg v \vee w \vee z) \\ & \wedge (\neg f_1 \vee a_1 \vee a_2) \wedge (\neg f_1 \vee a_1 \vee \neg a_2) \wedge (\neg f_1 \vee \neg a_1 \vee a_2) \\ & \wedge (\neg f_1 \vee \neg a_1 \vee \neg a_2). \end{aligned}$$

We simulate the first instance of False with $f_1 \dots$ and the second instance with f_2 .

Example of $\text{SAT} \leq_c 3\text{-SAT}$ reduction

Suppose our original SAT instance is:

$$F = u \wedge (\neg u \vee \neg v) \wedge (v \vee \neg w \vee x \vee \neg y \vee \neg z) \wedge (y \vee z) \wedge (\neg v \vee w \vee z).$$

We transform this into a 3-SAT instance as follows:

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We simulate the first instance of False with f_1 ... and the second instance with f_2 .

Example of $\text{SAT} \leq_c 3\text{-SAT}$ reduction

Suppose our original SAT instance is:

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Finally, we simulate the equality clauses.

Example of $\text{SAT} \leq_c 3\text{-SAT}$ reduction

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Suppose our original SAT instance is:

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Finally, we simulate the equality clauses.

Example of $\text{SAT} \leq_c 3\text{-SAT}$ reduction

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Phew! We could have done this directly, without the gadgets as intermediate steps, but they made it much easier to think about...