Breadth-first search
COMS20010 2020, Video 4-3

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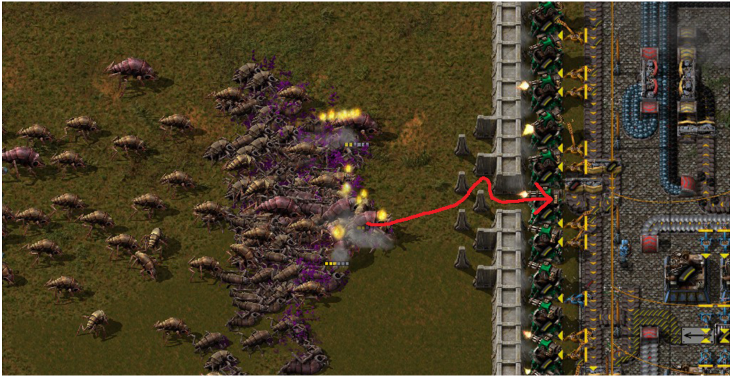
Shortest path-finding

Last time: Given a graph G and two vertices $x, y \in V(G)$, is there a path from x to y?

E.g. can an enemy attack the base without breaking down a wall?



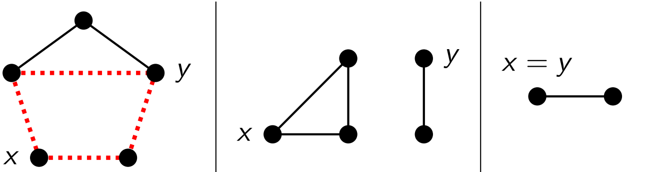
This time: What is the **shortest** path from x to y?

Graph distance

This time: What is the **shortest** path from x to y?

What do we mean by “shortest”?

The **distance** between x and y, $\boldsymbol{d(x, y)}$, is the length in edges of a shortest path between x and y, or ∞ if no such path exists.

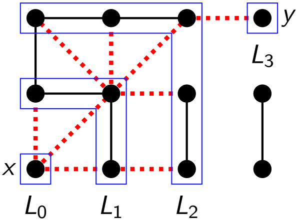


$d(x, y) = 2$ | $d(x, y) = \infty$ | $d(x, y) = 0$

In directed graphs, it’s the same except that the path is **from** x **to** y. So... we might not have $d(x, y) = d(y, x)$!

Breadth-first search: The idea

Input: A graph G and two vertices x and y.
Output: A shortest path from x to y.



Let L_i be the set of vertices at distance i from x. So $L_0 = \{x\}$.

L_1 is everything adjacent to x.

L_2 is everything adjacent to L_1, but **not** in L_0 or L_1.

In general, L_{i+1} is everything adjacent to L_i and not in $L_0 \cup \cdots \cup L_i$.

By continuing this until we find y, keeping track of which edges we use, we get a shortest path to y.

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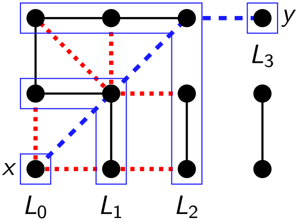

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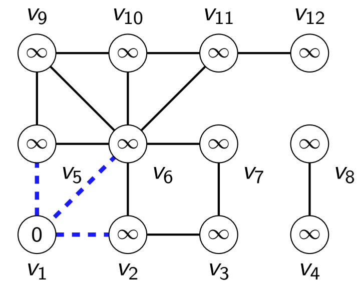

$\texttt{queue}: (1,5), (1,6), (1,2)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
Output : $d(v, y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \dots, v_n$.
Let $\texttt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\texttt{L}[1] \leftarrow 0$, $\texttt{pred}[1] \leftarrow \texttt{None}$.
Let $\texttt{queue}$ be a queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$.
while $\texttt{queue}$ *is not empty* **do**
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    if $\texttt{L}[j] = \infty$ **then**
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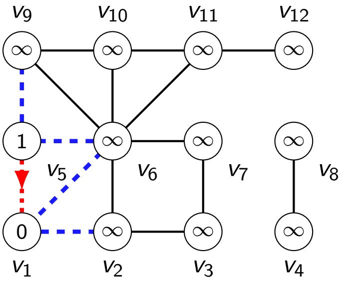
        Set $\texttt{L}[j] \leftarrow \texttt{L}[i] + 1$, $\texttt{pred}[j] = i$.
Return $\texttt{L}$ and $\texttt{pred}$.

Breadth-first search: Implementation



$\mathtt{queue}: (1,6), (1,2), (5,9), (5,6)$

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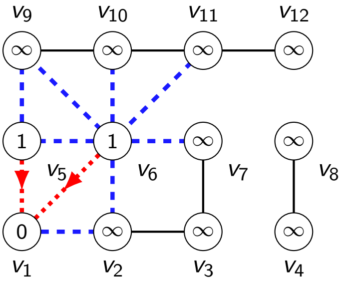

$\texttt{queue}: (1,2), (5,9), (5,6), (6,5), (6,9), (6,10), (6,11), (6,7), (6,2)$

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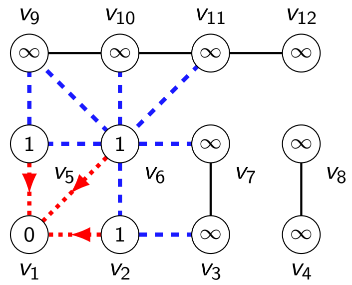

queue : $(5,9),(5,6),(6,5),(6,9),$
$(6,10),(6,11),(6,7),(6,2),$
$(2,6),(2,3)$

Algorithm: BFS

Input : Graph $G=(V,E)$, vertex $v \in V$.
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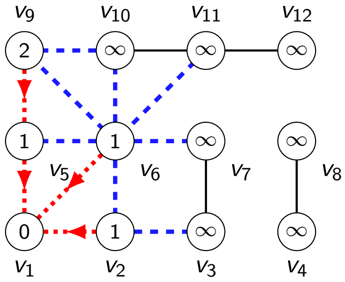

$\texttt{queue}: (5,6),(6,5),(6,9),(6,10),$
$(6,11),(6,7),(6,2),(2,6),$
$(2,3),(9,10),(9,6)$

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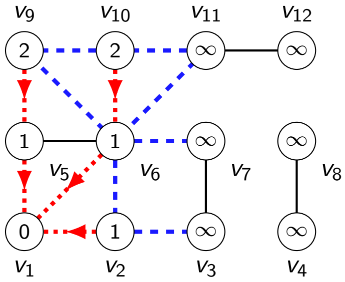

queue : $(6, 11), (6, 7), (6, 2), (2, 6),$
$(2, 3), (9, 10), (9, 6), (10, 11),$
$(10, 9)$

Algorithm: BFS

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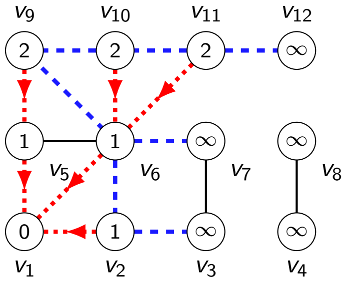

queue : $(6,7),(6,2),(2,6),(2,3),$
$(9,10),(9,6),(10,11),(10,9),$
$(11,10),(11,12)$

Algorithm: BFS

Input : Graph $G=(V,E)$, vertex $v \in V$.
Output : $d(v,y)$ for all $y \in V$ and "a way of finding shortest paths".

Number the vertices of G as $v = v_1, \ldots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
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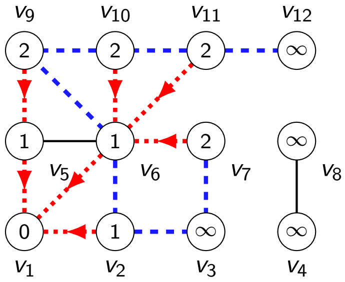
Breadth-first search: Implementation



queue : $(6,2),(2,6),(2,3),(9,10),$
$(9,6),(10,11),(10,9),(11,10),$
$(11,12),(7,3)$

Algorithm: BFS

Input : Graph $G = (V, E)$, vertex $v \in V$.
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Number the vertices of G as $v = v_1, \dots, v_n$.
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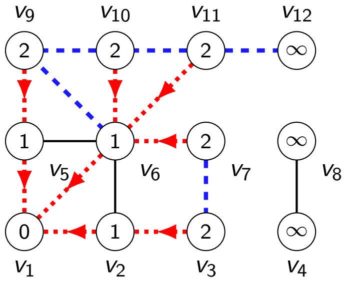

queue : $(9, 10), (9, 6), (10, 11), (10, 9),$
$(11, 10), (11, 12), (7, 3), (3, 7)$

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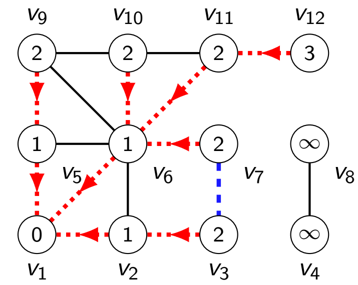

queue : $(7, 3), (3, 7)$

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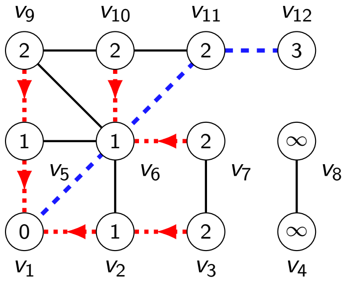

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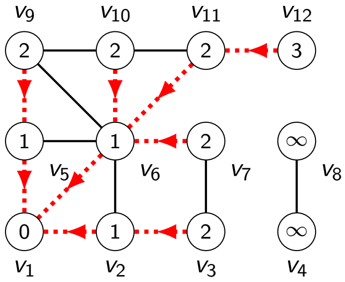
Return L and pred.
```

In the output, $\mathtt{L}[i] = d(v, v_i)$. By following edges back from v_i via `pred`, we can also quickly reconstruct a shortest path from v to v_i.

E.g. $v_1 v_5 v_{11} v_{12}$ is a shortest path from v_1 to v_{12}.

Breadth-first search: Implementation



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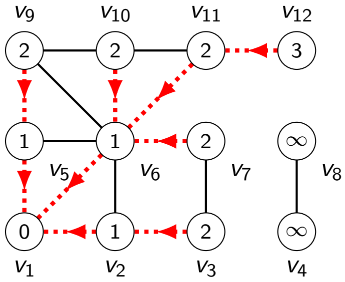
Return `L` and `pred`.

Time analysis: If G is in adjacency list form, each edge is added to `queue` at most twice, incurring $O(1)$ overhead each time, so the running time is $O(|V| + |E|)$.

Breadth-first search: Implementation



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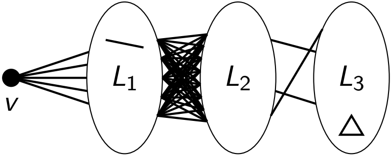
Return L and pred.

Important: There is a significant **space** inefficiency in this version of breadth-first search! See example sheet.

BFS trees

Definition: A **BFS tree** T of G is a rooted tree (call its root x) with:

1. $V(T)$ is the vertex set of a component of G;
2. The i'th layer of T is $\{x\colon d_G(x,v) = i\}$;
3. If $\{x,y\} \in E(G)$, then $|d_G(v,x) - d_G(v,y)| \leq 1$, i.e. x and y must be in the same or adjacent layers of T.



Theorem: The tree of edges from `pred` is always a BFS tree.

Proof: We already proved (1) and (2), so suppose $\{x,y\} \in E(G)$.

If P is a shortest path from v to x, then Pxy is a path from v to y, so $d(v,y) \leq d(v,x) + 1$. Likewise $d(v,x) \leq d(v,y) + 1$. ✓