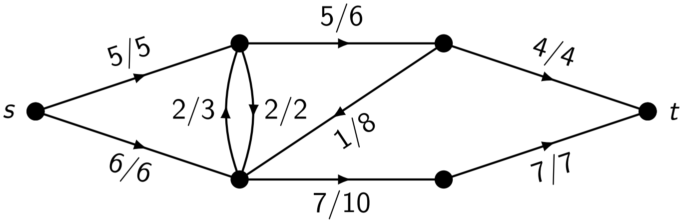
Why the Ford-Fulkerson algorithm looks so familiar
COMS20010 2020, Video 9-1

John Lapinskas, University of Bristol

A **flow network** (G, c, s, t) is a directed graph $G = (V, E)$, a **capacity** $c\colon E \to \mathbb{N}$, a **source** $s \in V$, and a **sink** $t \in V$, with $N^-(s) = N^+(t) = \emptyset$.



A **flow** is a function $f\colon E \to \mathbb{R}$ such that for all $e \in E$ and $v \in V \setminus \{s, t\}$:

- $0 \le f(e) \le c(e)$;
- $f^+(v) := \sum_{u \in N^-(v)} f(u, v) = \sum_{w \in N^+(v)} f(v, w) =: f^-(v)$.

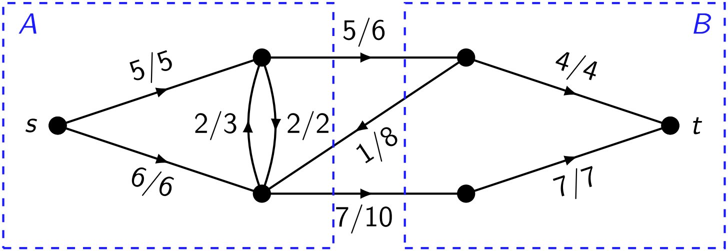
The **value** of f, denoted $v(f)$, is $f^+(s)$.

The problem: Find a **maximum flow**: a flow f maximising $v(f)$.

Theorem: The Ford-Fulkerson algorithm returns a maximum flow. It runs in time $O(v(f^*)|E|)$, where f^* is a maximum flow.

Theorem: There is always a maximum flow with integer values.

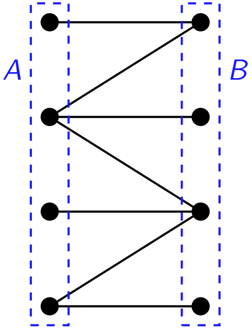
A **cut** is any pair of disjoint sets $A, B \subseteq V$ with $A \cup B = V$, $s \in A$ and $t \in V$. (So A and B partition V, the source is in A and the sink is in B.)



Max-flow min-cut theorem: The value of a maximum flow is equal to the minimum possible flow across a cut.

Matchings in bipartite graphs

Recall that a **matching** in a graph is a collection of disjoint edges.



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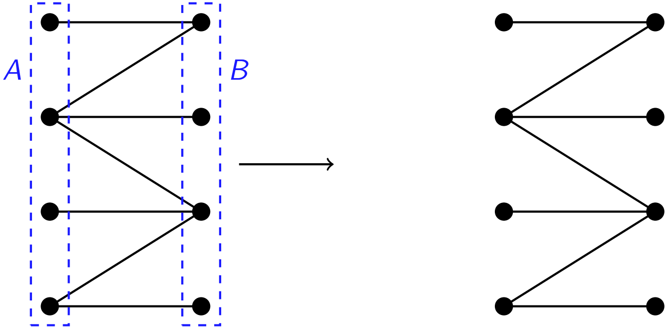

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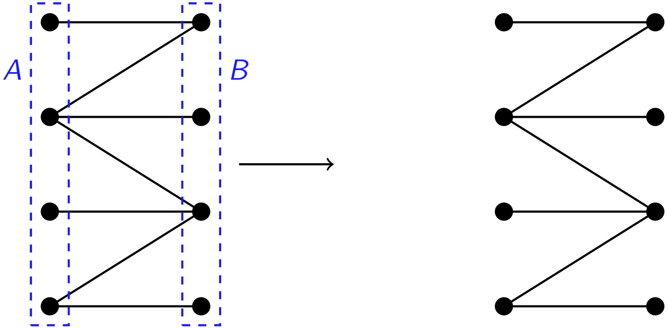

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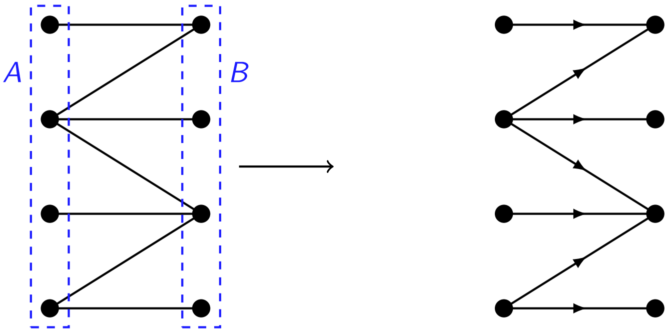

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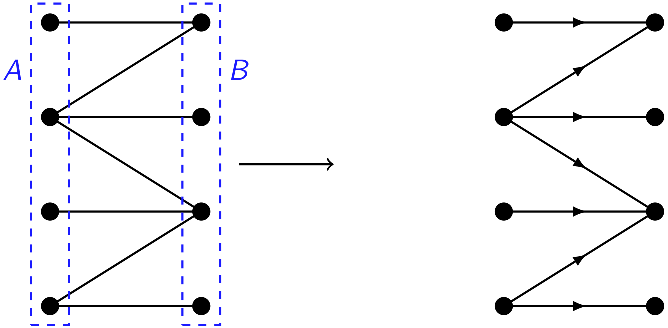

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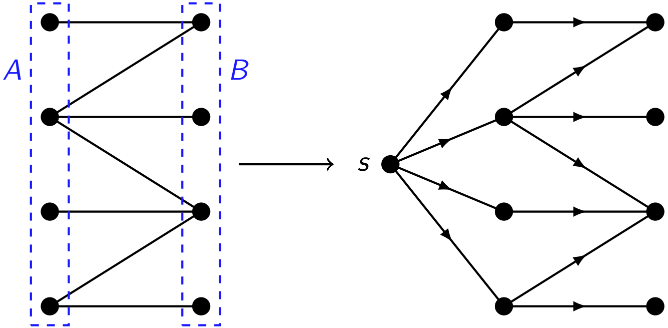

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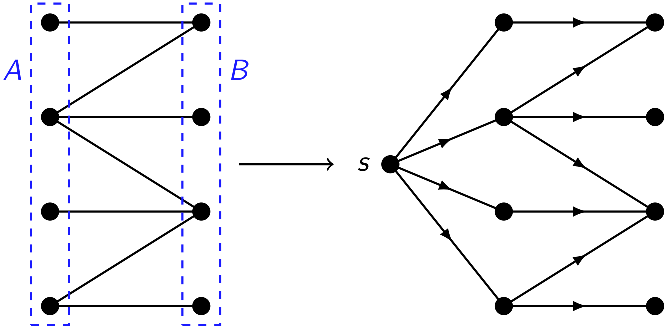

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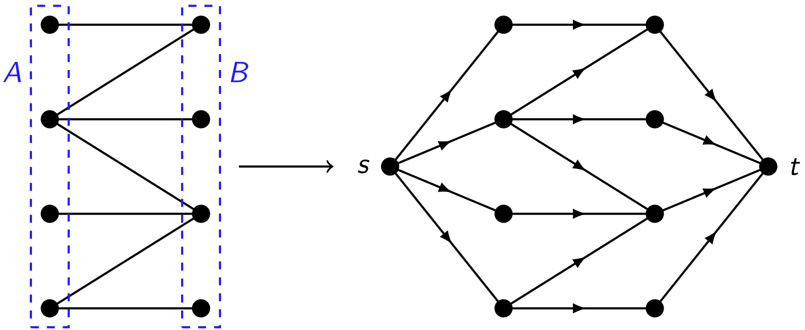

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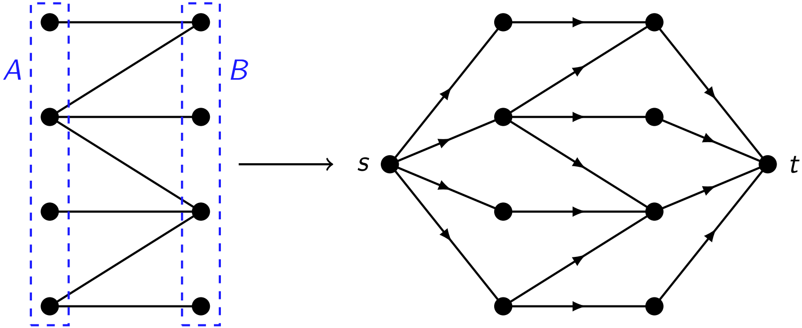

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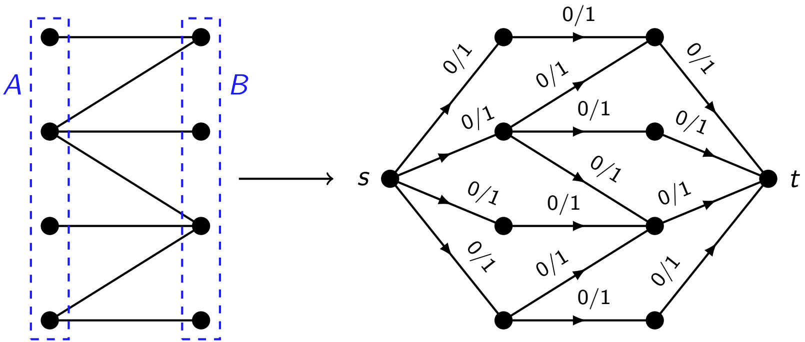

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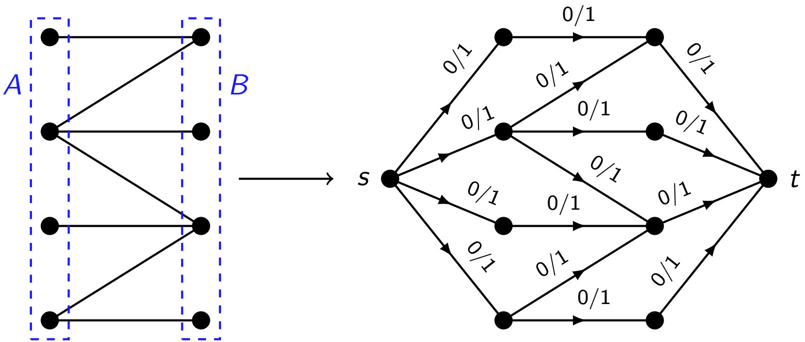

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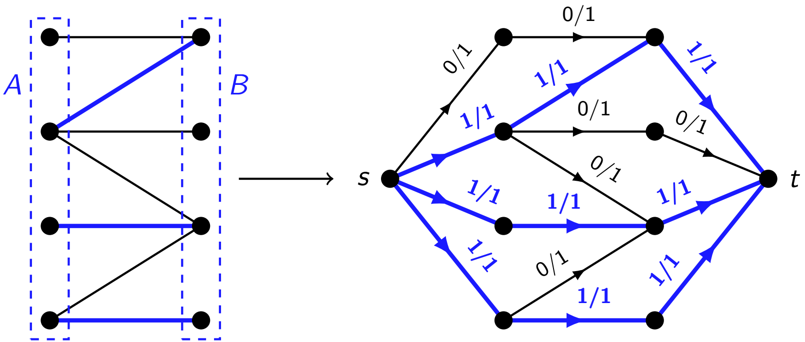

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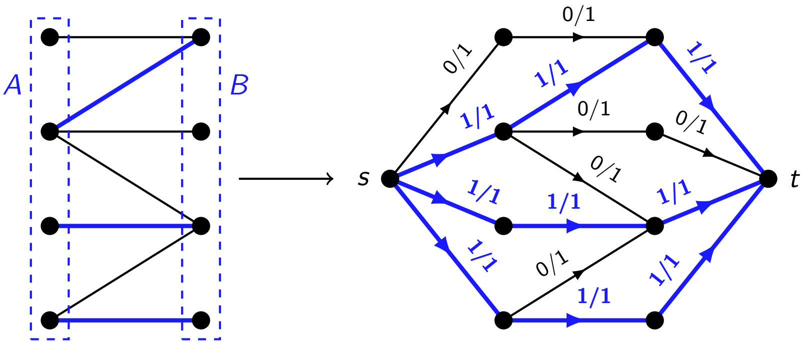

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And Ford-Fulkerson corresponds to our maximum matching algorithm!

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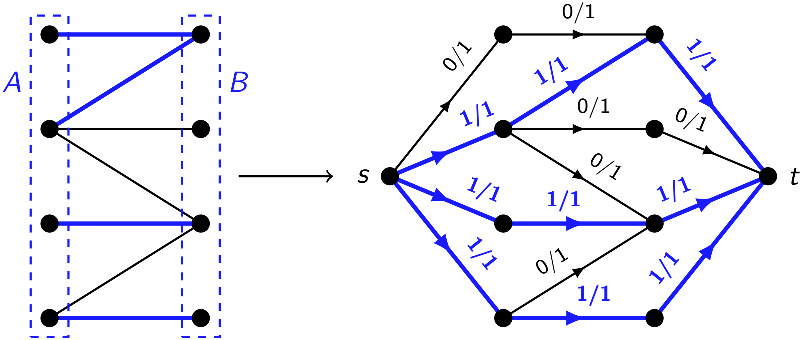

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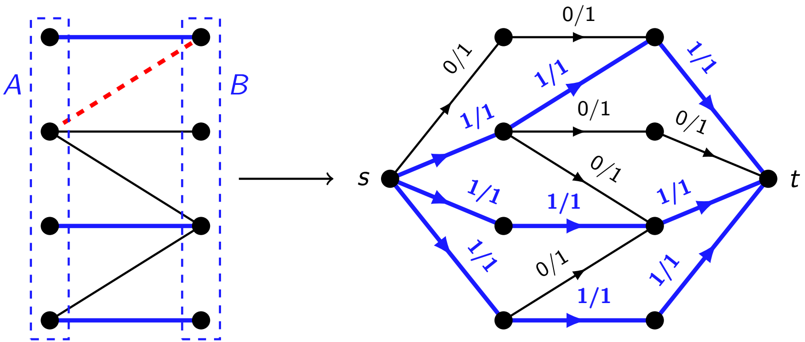

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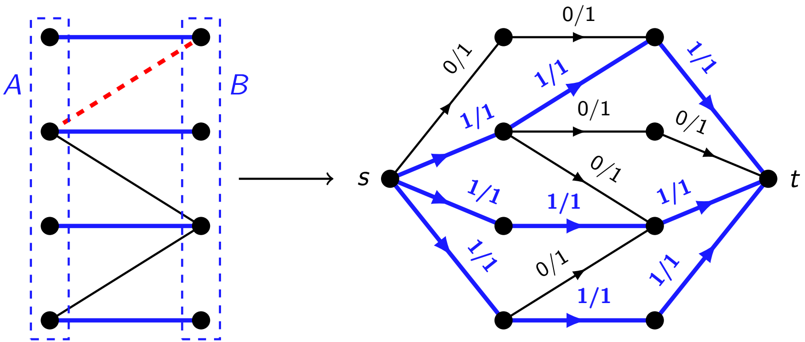

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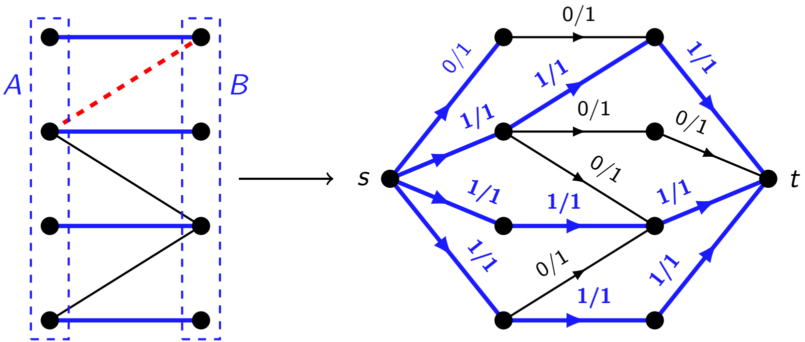

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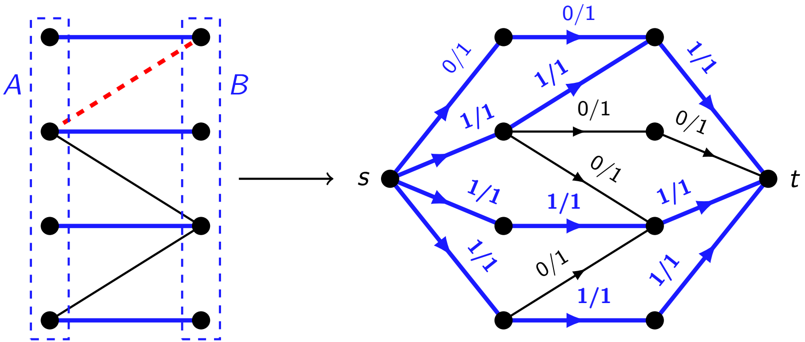

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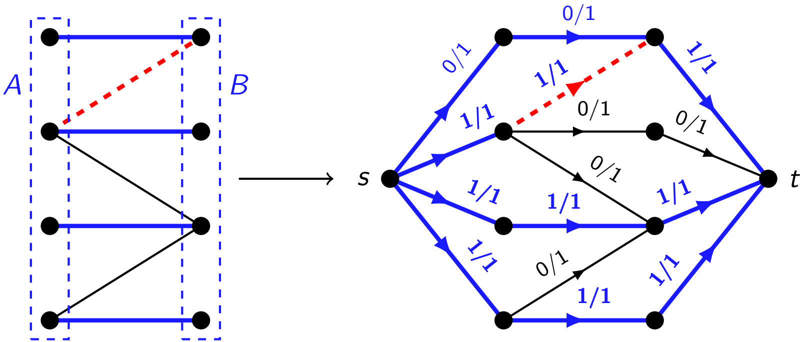

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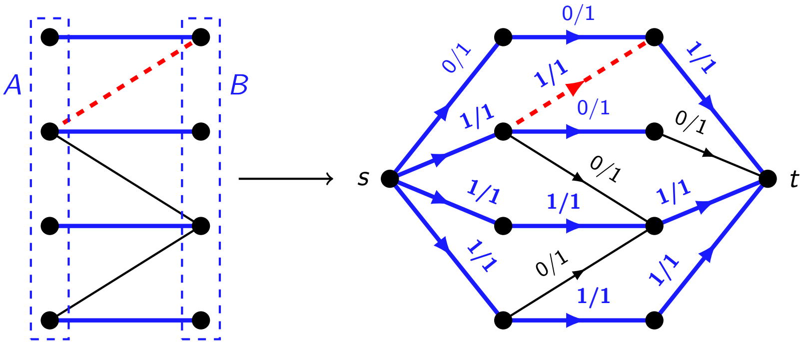

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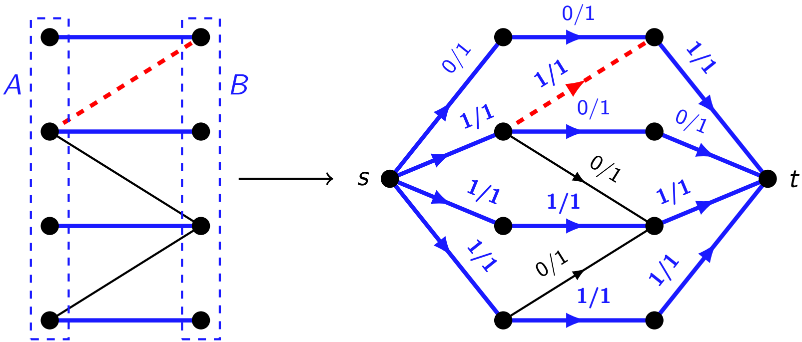

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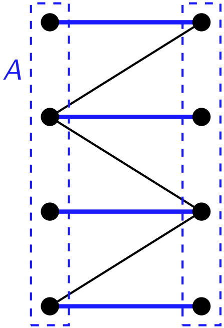

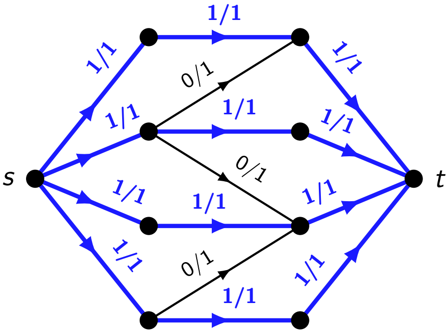

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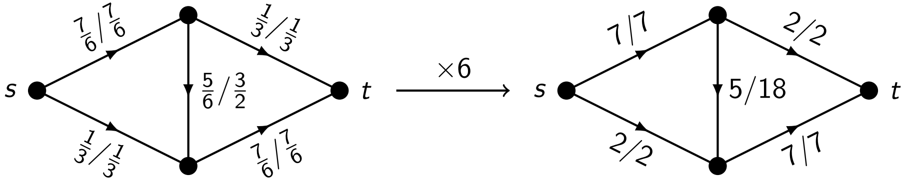

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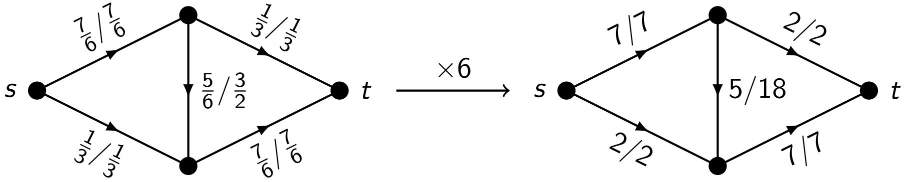

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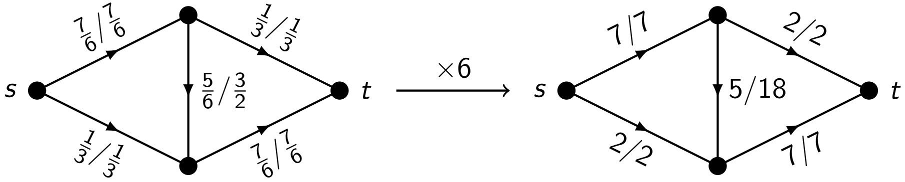

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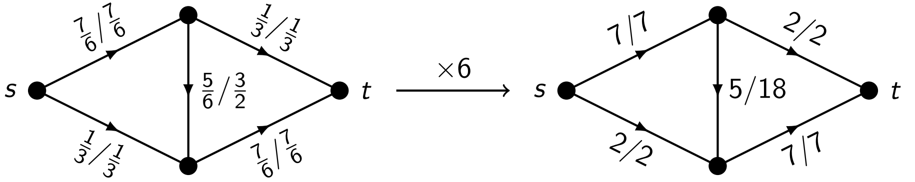

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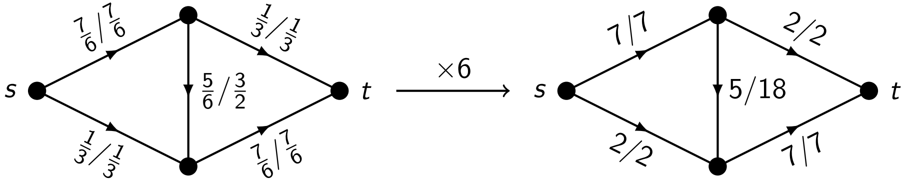

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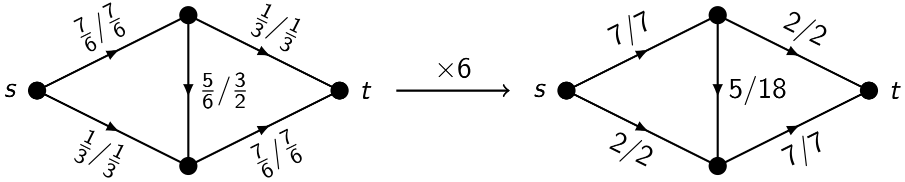

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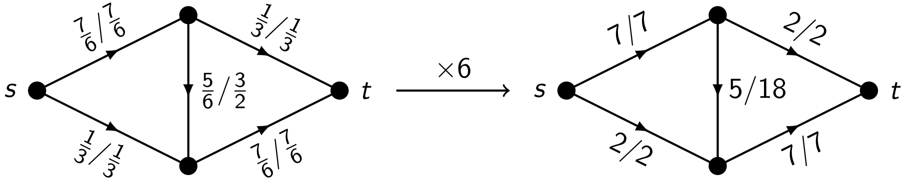

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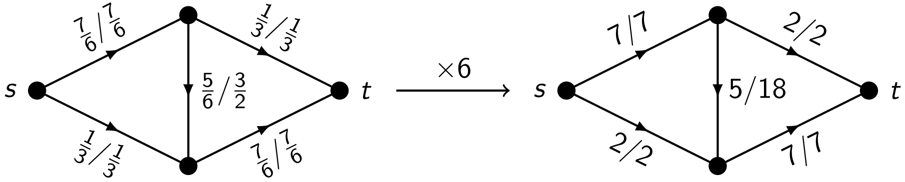

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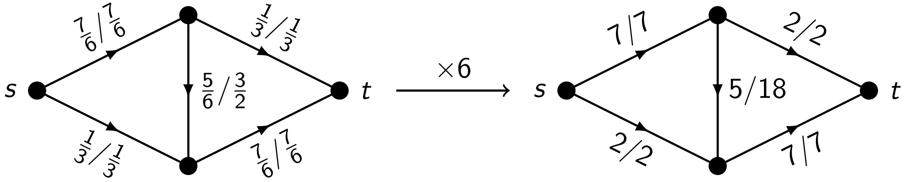

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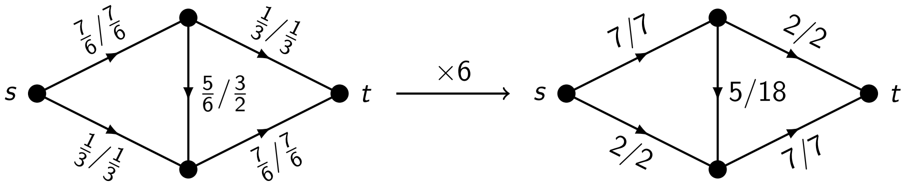

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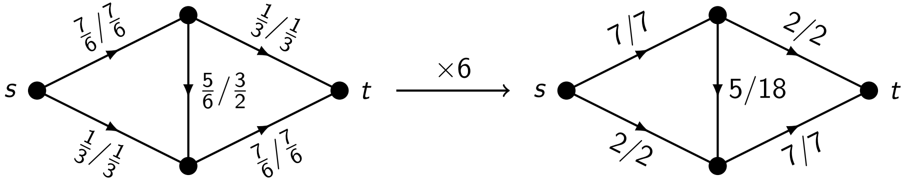

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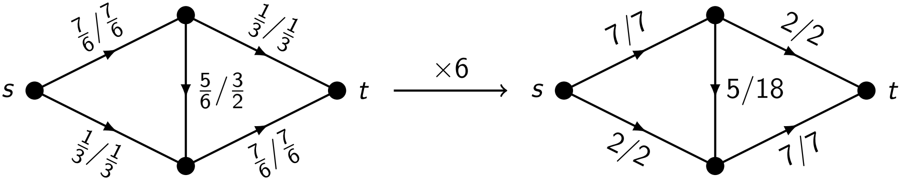

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So if the denominators of capacities in (G, c, s, t) are $b_1, \dots, b_m$, then we find $L = \operatorname{lcm}(b_1, \dots, b_m)$, then find the max flow in (G, Lc, s, t).

A better algorithm: Edmonds-Karp

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In fact, if we allow **irrational** edge capacities, it may never terminate... We prove this on the problem sheet!

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In other words, we just have to use breadth-first search on the residual graph G_f to find augmenting paths, rather than depth-first search! This is the **Edmonds-Karp** algorithm.