Defining O-notation (recap)
COMS20010 2020, Video lecture 1-3

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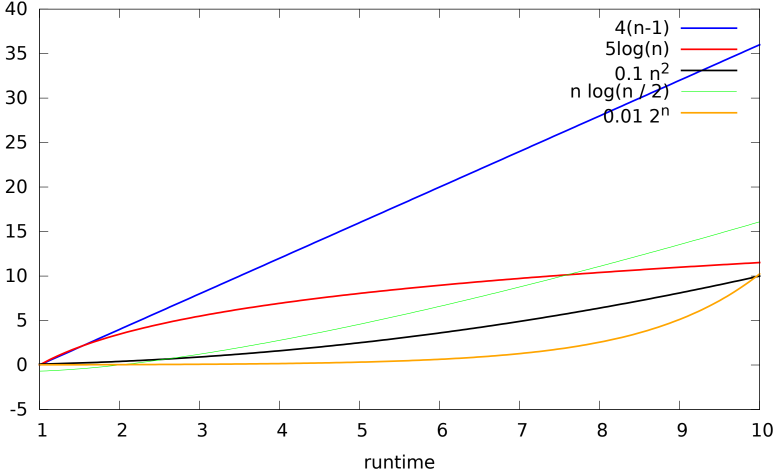
Why O-notation?

Intuition: As input sizes get large, asymptotic growth rate matters more than constant factors. Also, constant factors are implementation-dependent. So we focus on growth rate.



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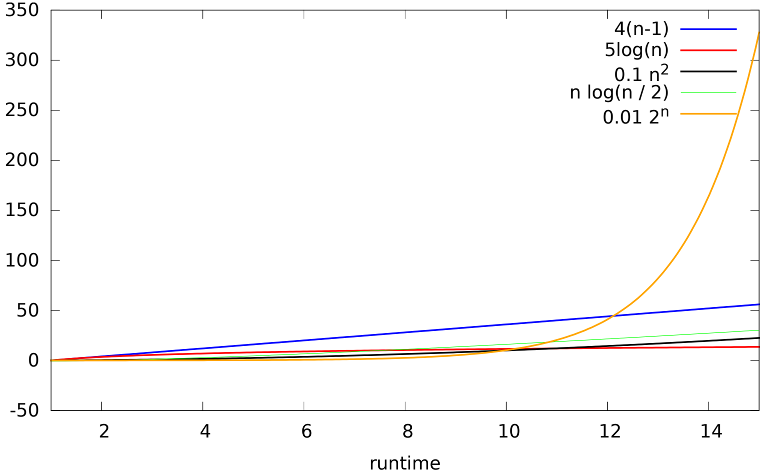

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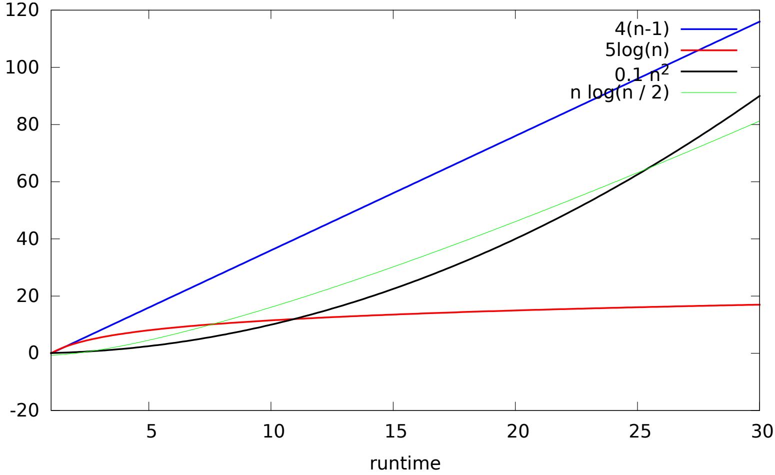

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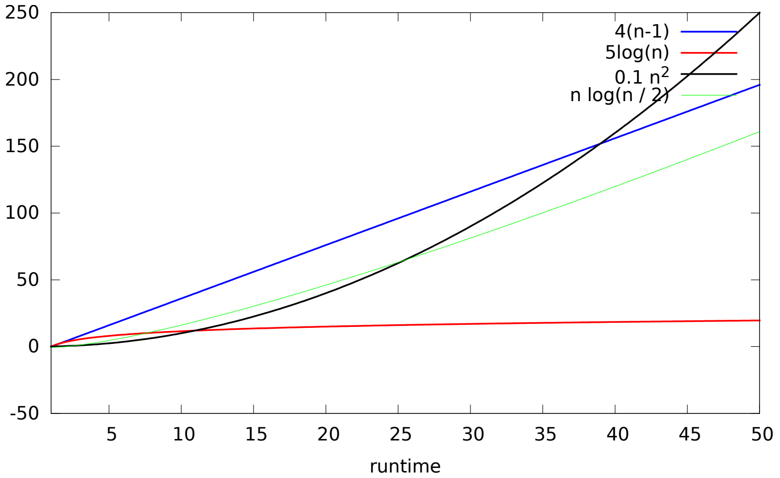

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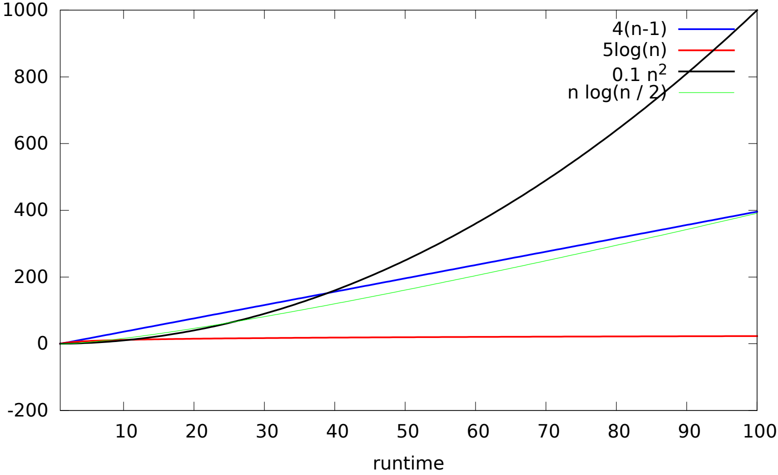

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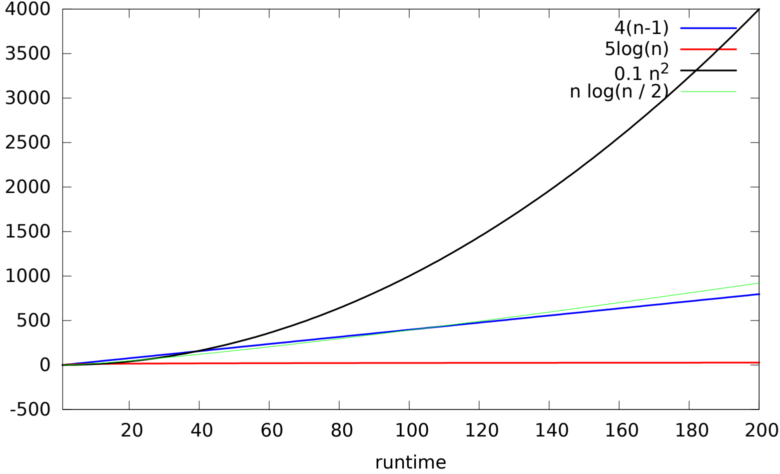

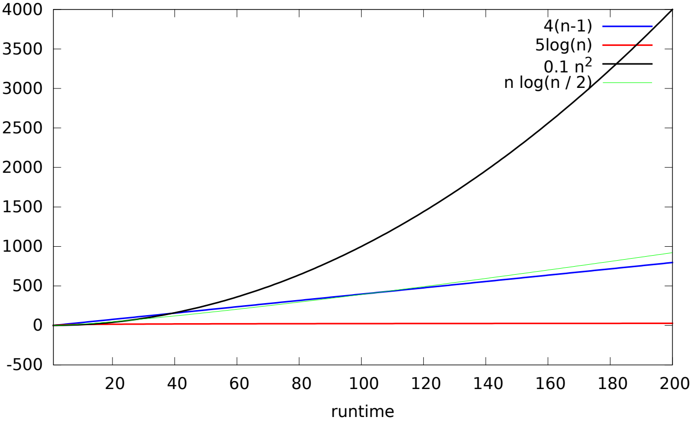

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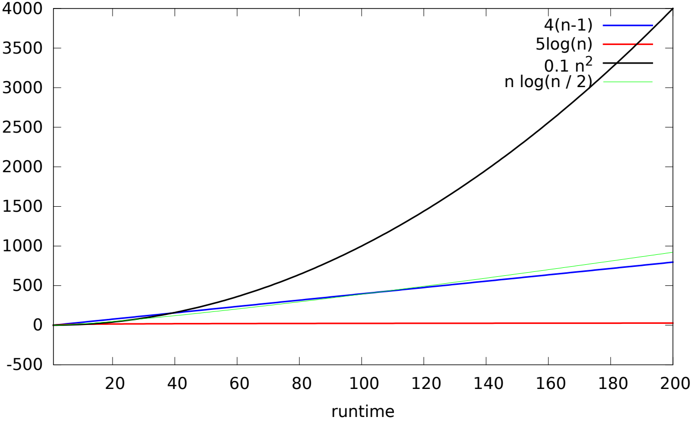

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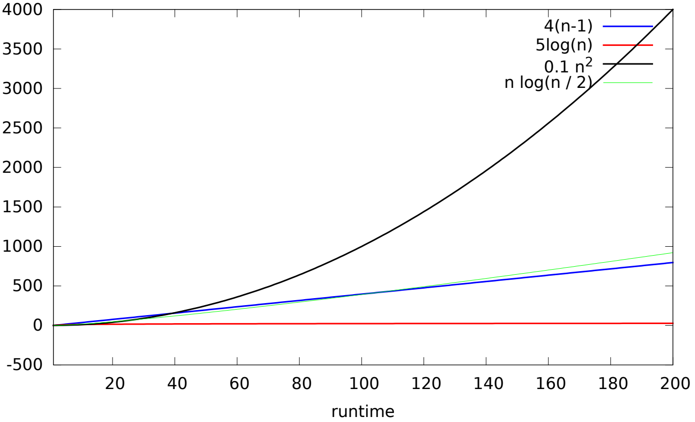

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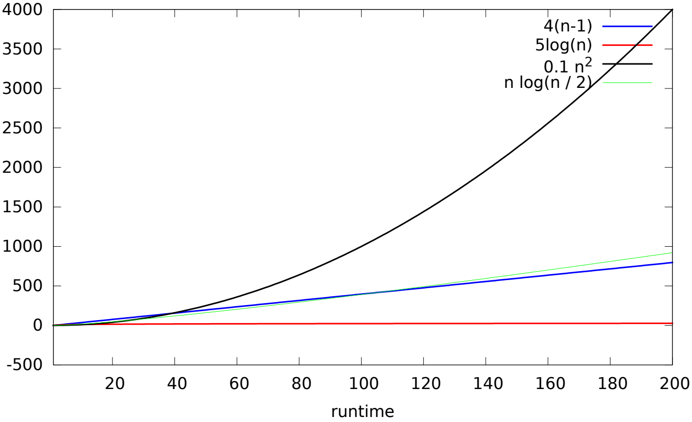

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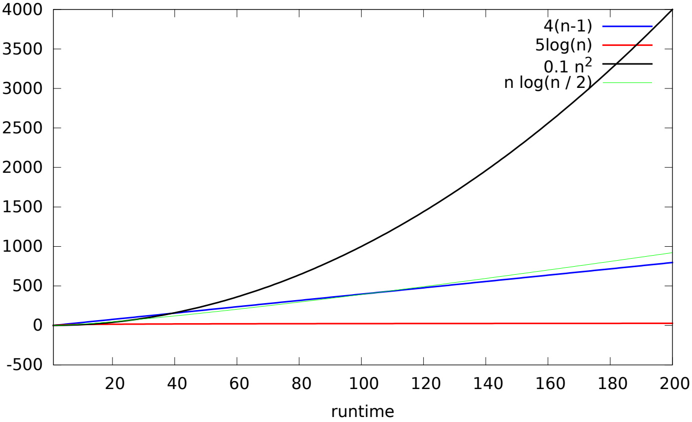

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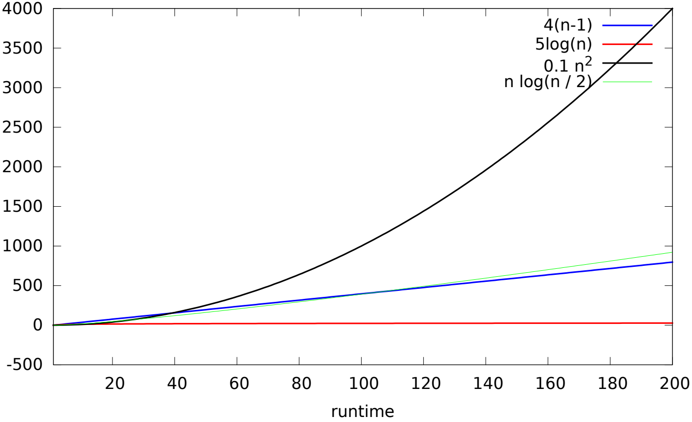

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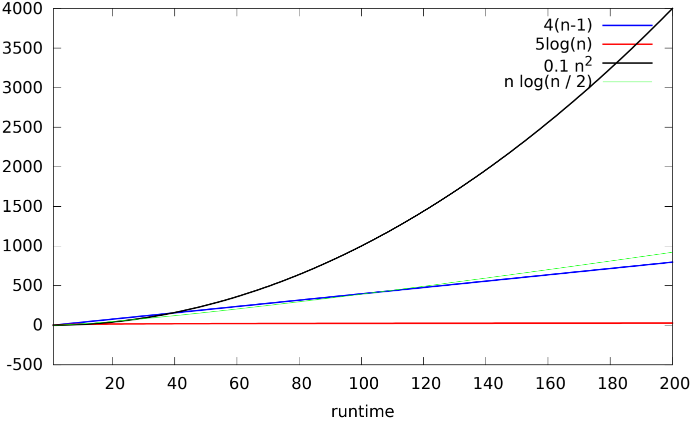

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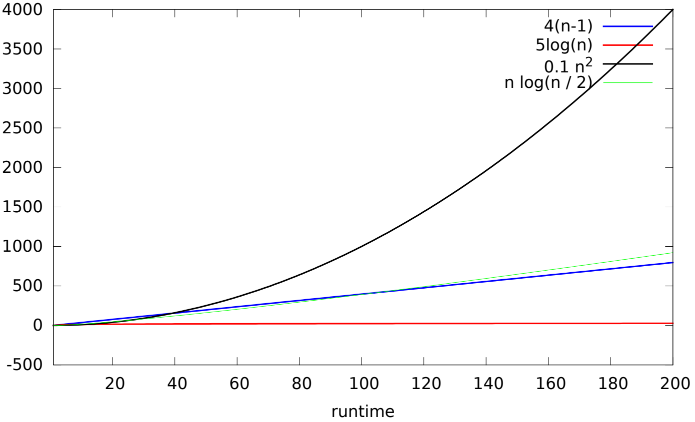

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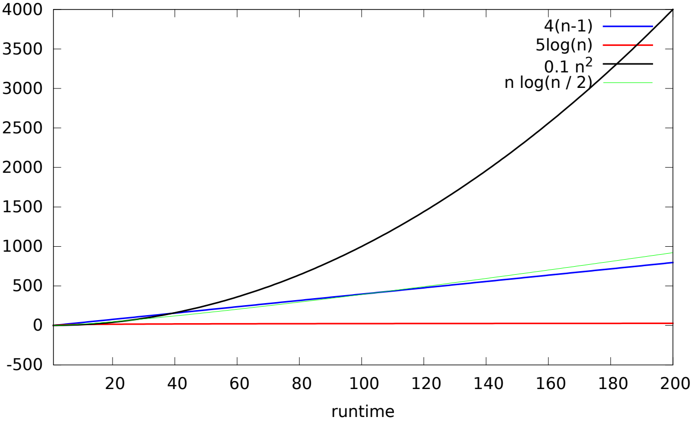

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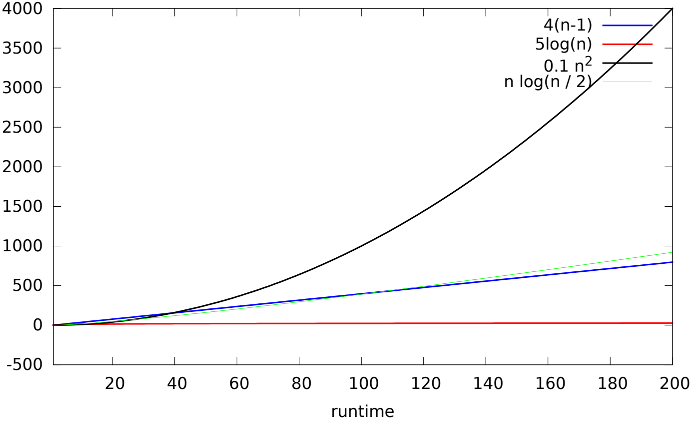

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This rigorous definition is "just" a more precise version of our intuition.

Other O-notation

$f(n) \in O(g(n))$ is good notation for "f grows no faster than g, ignoring constants". But what if we want to say "g grows no slower than f"?

Notation	Intuitive meaning	Analogue
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$f(n) \in \Omega(g(n))$	f grows at least as fast as g	$\geq$
$f(n) \in \Theta(g(n))$	f at the same rate as g	$=$
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Looking at it like this, it's much easier to see that

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So we prove $n^2 - 5n + 12 \in \Theta(n^2)$ by taking $c = \frac{1}{2}$, $C = 13$, and $n_0 = 10$.

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So we're given a constant c, and we need to show $n! \geq c \cdot 2^n$ when n is sufficiently large. Remember we have

$$n! = \underbrace{n \cdot (n-1) \cdot \dots \cdot 1}_{n \text{ terms}}, \qquad 2^n = \underbrace{2 \cdot 2 \cdot \dots \cdot 2}_{n \text{ terms}}.$$

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So we prove $n! = \omega(2^n)$ by taking $n_0 \geq \log c + 6$.

Multi-variable O-notation

We will often need O-notation for functions of more than one variable.

For example, an algorithm running on an n-vertex m-edge graph will often have running time depending on both m and n.

What does it mean to say that e.g. $f(m,n) \in O(mn)$ or $f(m,n) \in \Theta(m^2 \log n)$?

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For example, $f(m,n) \in O(g(m,n))$ when there exist C, $\boldsymbol{m_0}$ **and** $\boldsymbol{n_0}$ such that $f(m,n) \leq C \cdot mn$ whenever $m \geq m_0$ **and** $n \geq n_0$.

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All the useful properties of single-variable O-notation (see next video!) carry over to multi-variable O-notation, so e.g. if $f(m, n) \in O(g(m, n))$ and $f(m, n) \in \Omega(g(m, n))$ then we still have $f(m, n) \in \Theta(g(m, n))$.

An important clarification (added after recording)

O-notation can behave strangely with negative functions.

But we only care about O-notation for running times, which are positive!

So whenever you are asked to prove something general about O-notation in this course, you can assume the functions involved are non-negative.

But logarithms get used to bound running times all the time, and e.g. $n \log(n/100)$ is negative for small n. Since it's positive for large n, we'd still like to be able to say e.g. $n \log(n/100) \in \Theta(n \log n)$.

So the formal requirement is that the functions involved are **eventually non-negative** — that is, before we can say $f(n) \in O(g(n))$ or similar, we require that $f(n), g(n) \geq 0$ for all sufficiently large n.

Any fact that holds about O-notation for non-negative functions will also hold for eventually non-negative functions, by taking n_0 large enough that "eventually non-negative" becomes "non-negative".