Trees
COMS20010 2020, Video 3-4

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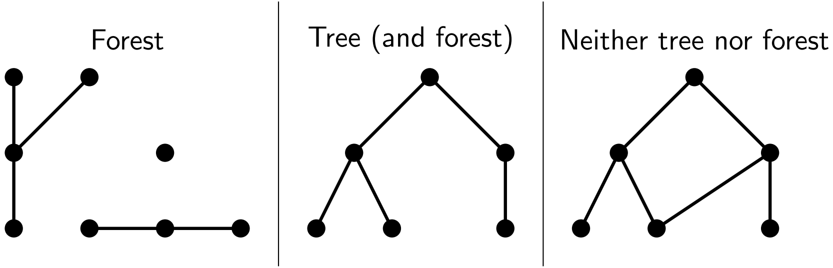

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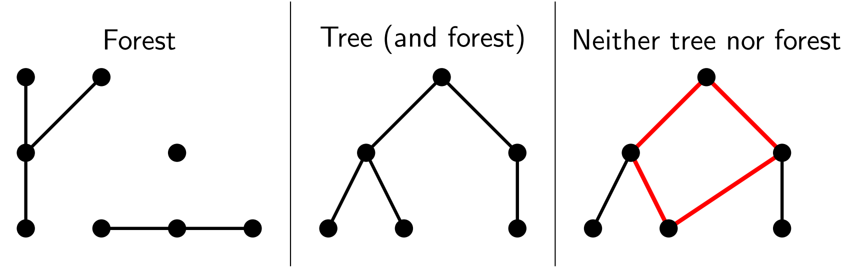

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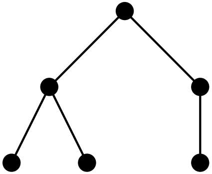

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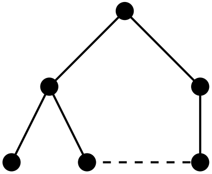

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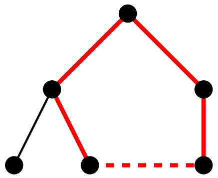

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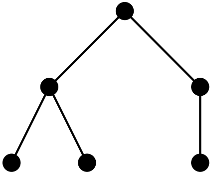

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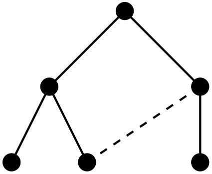

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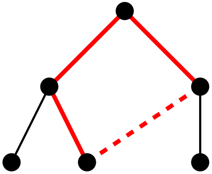

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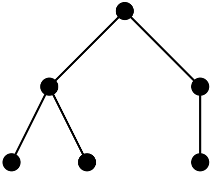

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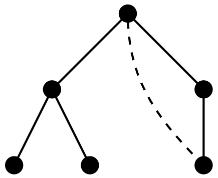

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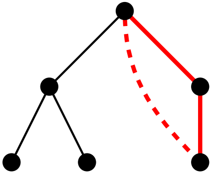

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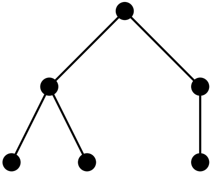

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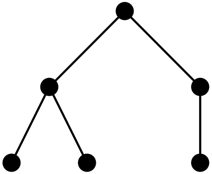

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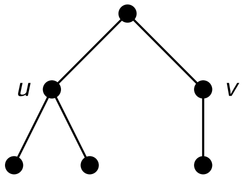

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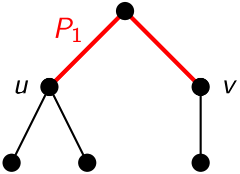

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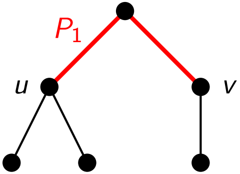

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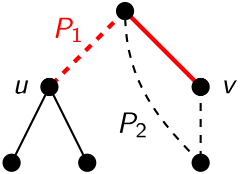

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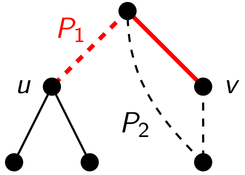

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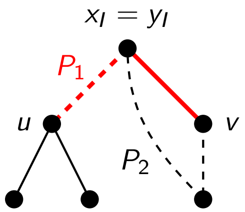

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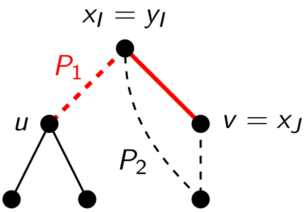

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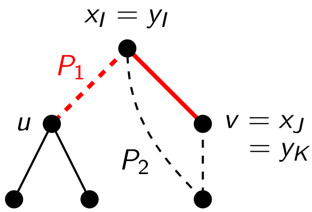

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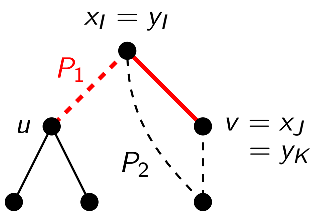

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Then $x_I x_{I+1} \dots x_J y_{K-1} y_{K-2} \dots y_I$ is a cycle, so T is not a tree. □

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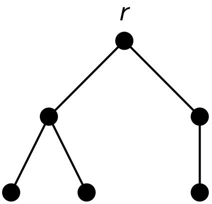

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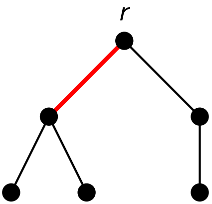

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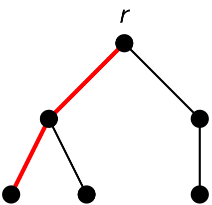

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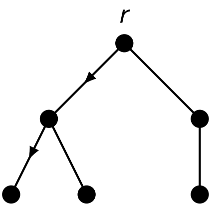

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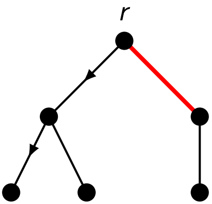

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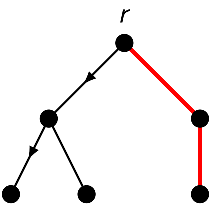

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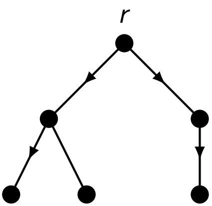

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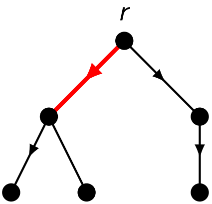

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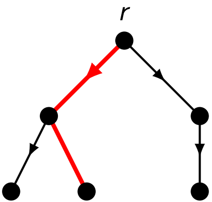

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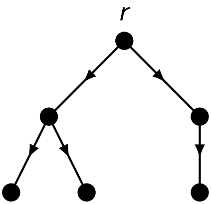

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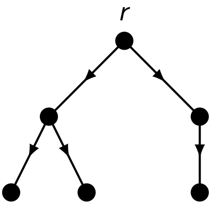

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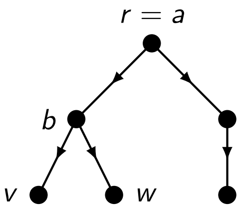

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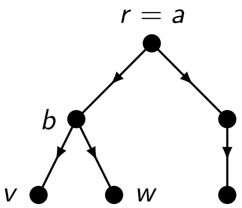

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Suppose some path P_v directs $a \to b$.
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Then both P_v and P_w must start with P_b, since P_b is the **unique** path from r to b.
So P_w also directs $a \to b$. ✓

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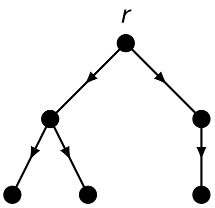
Proof idea: Take an arbitrary root $r \in V$. For all vertices v, let P_v be the unique path from r to v. Direct T's edges along these paths. ✓



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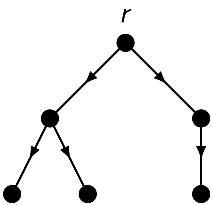

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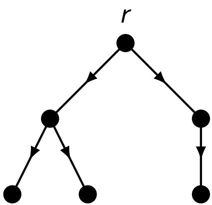

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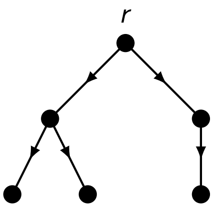

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Bonus: We also just defined rooted trees in terms of graphs.

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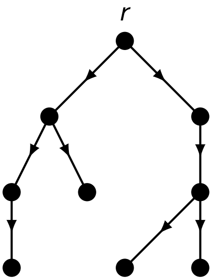

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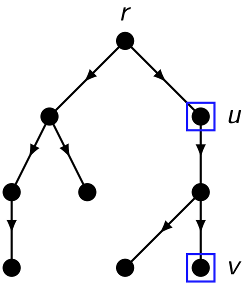

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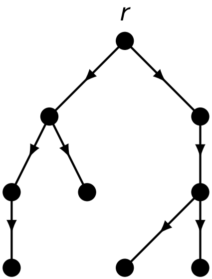

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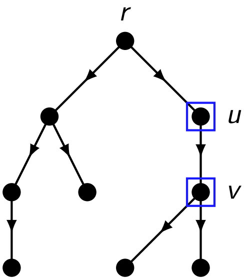

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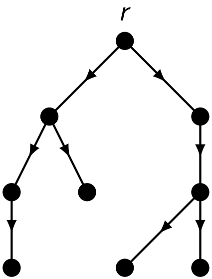

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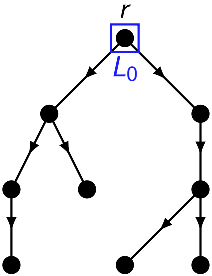

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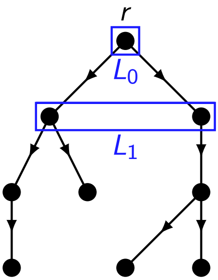

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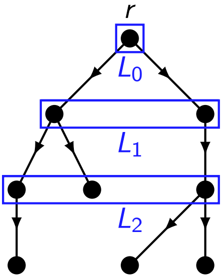

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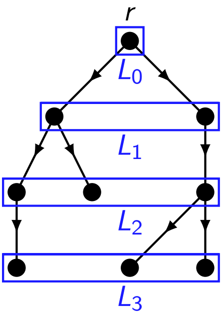

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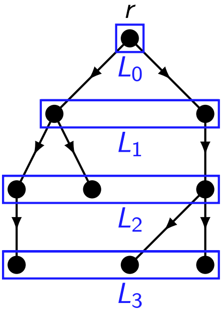

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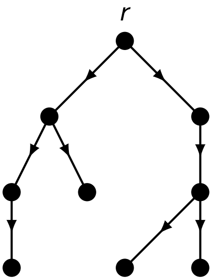

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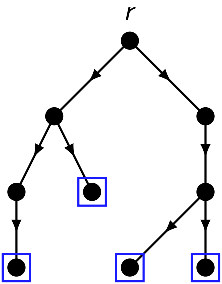

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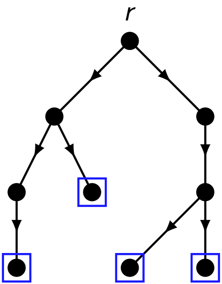

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Solving for x gives $x \geq 2$, so we're done! □

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When you're actually working with trees, it's good to have one single result that tells you that all the "obvious" things are true. This is that result.

Lemma: The following are equivalent for an n-vertex graph $T = (V, E)$:

(A) T is connected and has no cycles, i.e. is a tree;

(B) T has $n-1$ edges and is connected;

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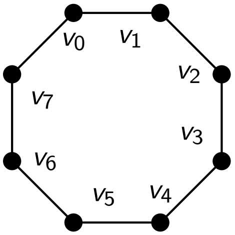

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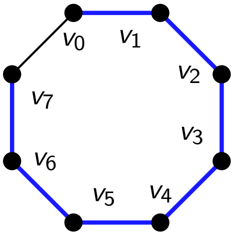

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(A) $\Rightarrow$ (B), (C) and (D): ✓

(D) $\Rightarrow$ (A): T has a path between any pair of vertices, so it's connected.

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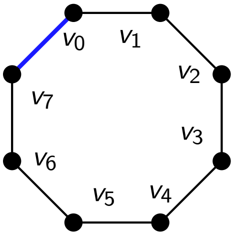

- the path $v_0 \dots v_k$; and
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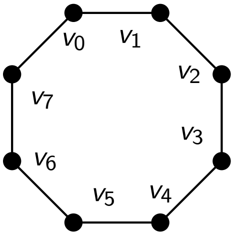

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Every edge of T is in some C_i, so $|E| = \sum_i (|V(C_i)| - 1) = n - r$. But we know $|E| = n - 1$, so we must have $r = 1$. ✓

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We form a new graph T' by repeatedly removing edges from cycles in T (in arbitrary order) until no more cycles remain.

Then T' has no cycles, and the Claim implies it's connected, so it's a tree. So by (A) $\Rightarrow$ (B) (or Lemma 2), T' has $n - 1$ edges.

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So T must have had **more than** $n-1$ edges — a contradiction. $\square$

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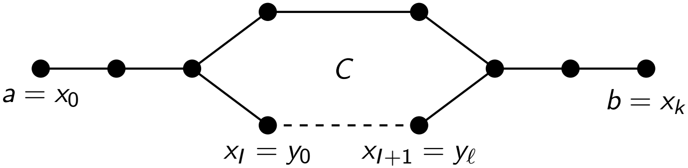

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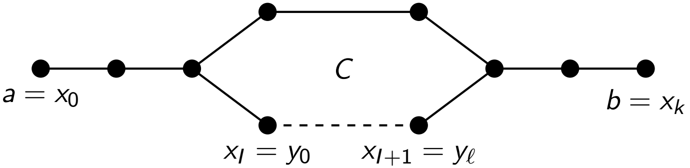

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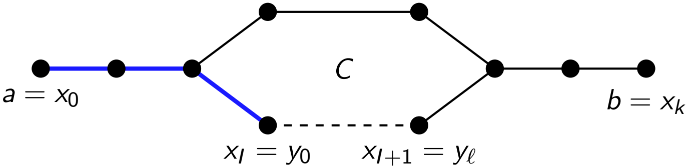

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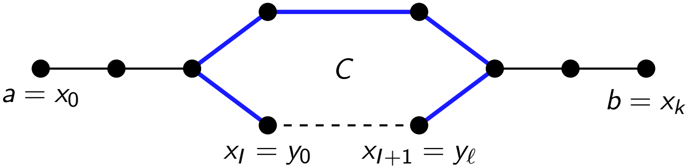

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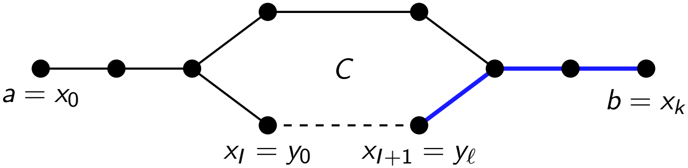
If e is in P: Write $e = \{x_l, x_{l+1}\}$. Let $C = y_0 \dots y_\ell$ be a cycle in T containing e — without loss of generality we can take $y_0 = x_l$ and $y_\ell = x_{l+1}$.



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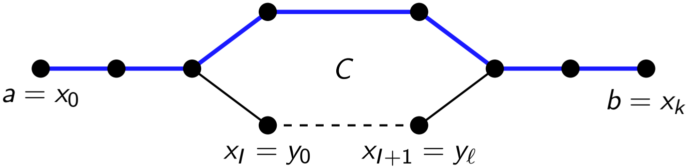

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And there was much rejoicing.