1AC Autumn 2015: Combinatorics I

Overview

Dr Richard Mycroft

A question to start...

- How many anagrams are there of the word ‘CAT’?
- What about ‘CHEESE’?

(The anagrams don’t have to be English words – the question is simply how many ways are there to rearrange the given letters)

$$\mathrm{CAT}, \mathrm{CTA}, \mathrm{ATC}, \ldots$$

What is combinatorics?

Mirriam-Webster defines:

COMBINATORICS

"Branch of mathematics concerned with the selection, arrangement, and combination of objects chosen from a finite set. The number of possible bridge hands is a simple example; more complex problems include scheduling classes in classrooms at a large university and designing a routing system for telephone signals. No standard algebraic procedures apply to all combinatorial problems; a separate logical analysis may be required for each problem. Combinatorics has its roots in antiquity, but new uses in computer science and systems management have increased its importance in recent years. See also PERMUTATIONS *and* COMBINATIONS.*"*

Discrete mathematics

Essentially synonymous is discrete mathematics, the mathematics of distinct 'separated' objects, such as integers, as opposed to 'continuous' objects such as the real numbers.

In particular, discrete mathematics has essentially no calculus, since this this is defined in terms of 'continuous' sets.

Discrete mathematics

Essentially synonymous is discrete mathematics, the mathematics of distinct ‘separated’ objects, such as integers, as opposed to ‘continuous’ objects such as the real numbers.

In particular, discrete mathematics has essentially no calculus, since this this is defined in terms of ‘continuous’ sets.

Discrete mathematics has grown hugely over the last 60-70 years, as

- scientific developments suggest that much of the world around us is discrete in nature, and
- discrete mathematics is crucial to many technological developments, such as in computing.

Course sections

- I. Fundamentals
- II. Finite Counting
- III. Infinite Counting
- IV. Graph Theory

Fundamentals

This introductory section introduces some of the fundamental concepts which will frequently appear throughout your degree programme. These include:

- A set is a collection of numbers/vectors/objects/anything, e.g.

$$\{2, 6, 12, 17\}$$

- A relation is a 'property' of two objects which may or may not be true. For example, $=$ and $<$ are relations on $\mathbb{N}$:

$$3 = 5 \text{ is false, } 7 = 7 \text{ is true}$$

$$1 < 9 \text{ is true, } 6 < 6 \text{ is false}$$

This section also includes important results relating to set sizes, such as the inclusion-exclusion formulae, product rule and pigeonhole principle.

Finite counting

Typical question: how many _______ are there?

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \; \ldots \; ?$$

Finite counting

Typical question: how many _______ are there?

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \; \ldots \; ?$$

e.g. How many integers between 1 and 10 are not divisible by either 2 or 3?

- Count them!

Finite counting

Typical question: how many _______ are there?

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \ \ldots\ ?$$

e.g. How many integers between 1 and 10 are not divisible by either 2 or 3?

- Count them!

But how many integers between 1 and 10,000 are not divisible by either 2 or 3?

- Could take a while unless we *find a better way to count*!

Counting and probability

Counting is important in its own right, but also for calculating probabilities: if every outcome of a random experiment is equally likely, then the probability that a certain event occurs is

$$\frac{\text{number of outcomes in which event occurs}}{\text{total number of outcomes}}.$$

Counting and probability

Counting is important in its own right, but also for calculating probabilities: if every outcome of a random experiment is equally likely, then the probability that a certain event occurs is

$$\frac{\text{number of outcomes in which event occurs}}{\text{total number of outcomes}}.$$

For example: In Texas Hold'em poker, each player is dealt a hand of two cards. What is the probability that a given player is dealt 'pocket rockets' (two aces, the best hand)?

Infinite counting

We all learn as young children what we mean by the the number of objects in a finite collection, and what it means for there to be 'more' of one thing than another. But how do these definitions extend to infinite sets?

Are there more multiples of 2 than multiples of 3? Or are there equally many of each?

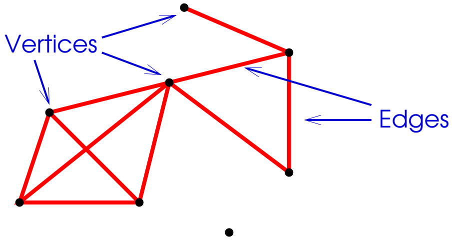
$$2, \quad 4, \quad 6, \quad 8, \quad 10, \quad 12, \quad 14, \quad 16, \quad 18, \quad 20, \ \ldots \ ?$$

$$3, \quad 6, \quad 9, \quad 12, \quad 15, \quad 18, \quad 21, \quad 24, \quad 27, \quad 30, \ \ldots \ ?$$

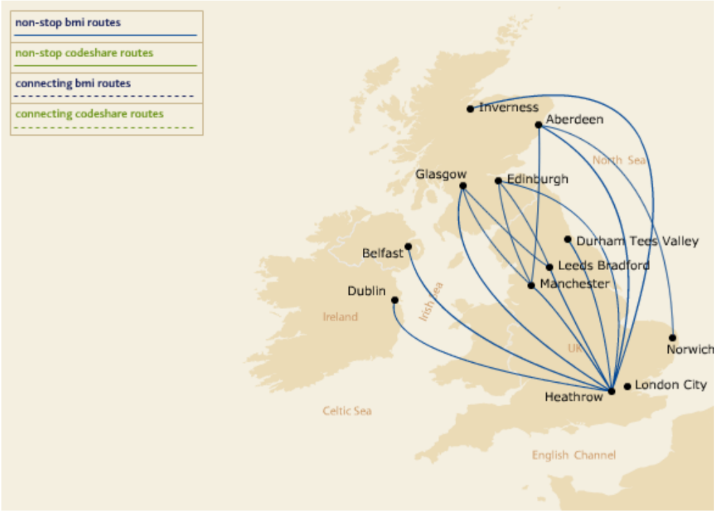
Do all infinite sets have the same size (infinity?)

Graph theory

A graph consists of a number of points, called vertices, and connections between these points called edges.



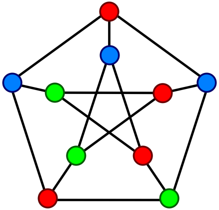
Graphs are useful for representing any kind of 'network' or 'relation', where it is the connections between objects/places/things that matter most.



The 'BMI flight network' graph: vertices are cities and an edge between two cities indicates a connecting flight.

Some typical questions in graph theory

- Is there a path between two given vertices? If so, what is the shortest route?
- Does a given graph G contain a copy of a fixed smaller graph H? How many such copies are there? What conditions guarantee the existence of such a copy?
- Can we visit every vertex once without visiting any vertex twice? Or every edge once without traversing any edge twice?
- How many colours do you need to colour each vertex so that no edge goes between vertices of the same colour?



Precision in writing

- In general, solutions and proofs in combinatorics must be written clearly and precisely using standard English words – specific notation is much less extensive than for e.g. Calculus.
- It is essential to be precise and to avoid ambiguity. However, English is frequently ambiguous or context-dependent:

Accurate and precise mathematical writing will be a major theme of this course.