Project 2. Solving the wave equation

Consider the wave equation in 2D [8, 10]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{in } \Omega \,, \\ u = g & \text{on } \partial\Omega \,, \\ u(t=0) = u_0 & \text{in } \Omega \,, \\ \dfrac{\partial u}{\partial t} = u_1 & \text{in } \Omega \,. \end{cases} \tag{3}$$

Implement a finite element solver for problem (3). Discuss the choice of the time and space discretization methods, the properties of the chosen method (especially in terms of numerical dissipation and dispersion, see [8, 10]) and the computational and algorithmic aspects of the solver.

da 9.6.2
con c = 1

$$\int_\Omega u_{tt}(\underline{x},t)\, v(\underline{x},t)\, d\underline{x}\, dt - \int_\Omega \Delta u(\underline{x},t)\, v(\underline{x},t)\, d\underline{x}\, dt = \int_\Omega f(\underline{x},t)\, v(\underline{x},t)$$

WP $$\int_\Omega u_{tt}(\underline{x},t)\, v(\underline{x},t)\, d\underline{x}\, dt + \overbrace{\int_\Omega \nabla u(\underline{x},t)\, \nabla v(\underline{x},t)\, d\underline{x}\, dt}^{a(u,v)} = \underbrace{\int_\Omega f(\underline{x},t)\, v(\underline{x},t)\, d\underline{x}\, dt}_{F(v)}$$

Gal $$\int_\Omega \frac{\partial^2 u_h}{\partial t}(\underline{x},t)\, v_h(\underline{x},t)\, d\underline{x}\, dt + a(u_h, v_h) = F(v_h)$$

...

$$\sum_j \frac{\partial^2}{\partial t^2} u_j(t) \underbrace{\int \varphi_j \varphi_i \, dx}_{M_{ij}} + \sum_j u_j(t) \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \underbrace{F(\varphi_i)}_{\vec{F}} \quad \forall i$$

$$M \frac{\partial^2 u}{\partial t^2} + A\vec{u} = \vec{F}$$

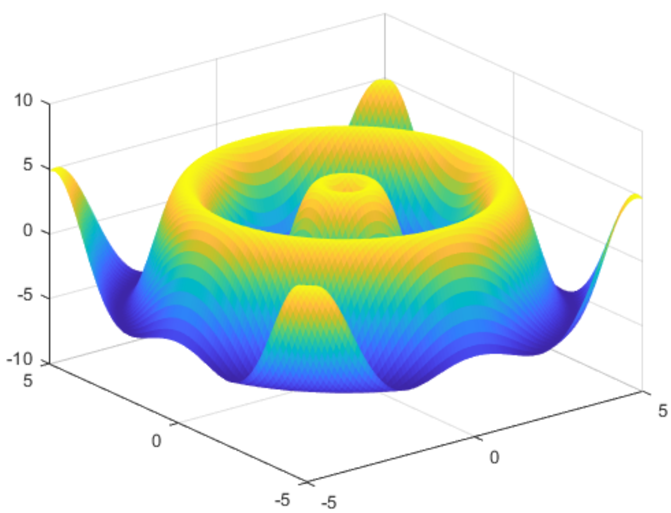

Figure 1: Example solution to the wave equation.

13.2.1 The wave equation

1D

Let us consider the following second order hyperbolic equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = f,} \quad x \in (\alpha, \beta), \quad t > 0. \tag{13.9}$$

Let

$$u(x,0) = u_0(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = v_0(x), \quad x \in (\alpha, \beta),$$

be the initial data and let us suppose, moreover, that u is identically null at the boundary

$$u(\alpha, t) = 0 \quad \text{and} \quad u(\beta, t) = 0, \quad t > 0. \tag{13.10}$$

In this case, u can represent the vertical displacement of a vibrating elastic chord with lenght $\beta - \alpha$, fixed at the endpoints, and γ is a coefficient that depends on the specific mass of the chord and on its tension. The chord is subject to a vertical force whose density is f. The functions $u_0(x)$ and $v_0(x)$ describe the initial displacement and the velocity of the chord.

Finally, we point out the following schemes for approximating the wave equation (13.9), again in the case $f = 0$:

$\Delta t / h$

- **Leap-Frog**

$$\boxed{\underbrace{u_j^{n+1} - 2u_j^n + u_j^{n-1}}_{\Delta t} = (\gamma\lambda)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n).} \tag{13.23}$$

- **Newmark**

$$\boxed{u_j^{n+1} - 2u_j^n + u_j^{n-1} = \frac{(\gamma\lambda)^2}{4} \left(w_j^{n-1} + 2w_j^n + w_j^{n+1} \right),} \tag{13.24}$$

where $w_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$.

LAB 4 - heat function

$$M \frac{d\underline{u}}{dt} + A\underline{u} = \underline{f}(t) \quad \text{con } A_{ij} = \int_\Omega \mu \nabla\varphi_i \nabla\varphi_j \, dx \; ; \; M_{ij} = \int_\Omega \varphi_i \varphi_j \, dx \; ; \; \underline{f}_i = \int_\Omega f(t)\varphi_i \, dx$$

$$M \frac{\underline{u}^{n+1} - \underline{u}^n}{\Delta t} + \theta A \underline{u}^{n+1} + (1-\theta) A \underline{u}^n = \theta \underline{f}^{n+1} + (1-\theta)\underline{f}^n \qquad \theta \in (0,1)$$

$$\left(\frac{M}{\Delta t} + \theta A\right) \underline{u}^{n+1} = \left(\frac{M}{\Delta t} - (1-\theta) A\right) \underline{u}^n + \theta \underline{f}^{n+1} + (1-\theta)\underline{f}^n$$

$$M \frac{\partial^2 u}{\partial t^2} + A\vec{u} = \vec{F}$$

$$M \frac{d^2 u}{dt^2} = F - Au$$

$$\downarrow$$

$$M \frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} = F^n - Au^n$$

$$\downarrow$$

$$\left[M + (\Delta t)^2 A\right] u^{(n+1)} = (\Delta t^2) F^{(n)} + 2Mu^{(n)} - Mu^{(n-1)}$$

$v = \frac{\partial u}{\partial t} \quad a = \frac{\partial^2 u}{\partial t}$

Formule del metodo di Newmark:

$$u^{n+1} = u^n + \Delta t\, v^n + (\Delta t)^2 \left[\left(\tfrac{1}{2} - \beta\right) a^n + \beta a^{n+1} \right]$$
$$v^{n+1} = v^n + \Delta t \left[(1-\gamma)\, a^n + \gamma a^{n+1} \right]$$
*

γ e β definiscono una famiglia di metodi

Partendo da $M \frac{\partial^2 u}{\partial t^2} = F - Au$

$$M a^{(n)} = F^{(n)} - A u^{(n)}$$

$$M a^{(n+1)} = F^{(n+1)} - A u^{(n+1)}$$
$$M a^{(n+1)} = F^{(n+1)} - A \left(u^n + \Delta t\, v^n + (\Delta t)^2 \left[\left(\tfrac{1}{2} - \beta\right) a^n + \beta a^{n+1} \right] \right)$$

$$\left[M - (\Delta t)^2 \beta A \right] a^{(n+1)} = F^{(n+1)} - A \left[u^n + \Delta t\, v^n + (\Delta t)^2 \left(\tfrac{1}{2} - \beta\right) a^n \right]$$

si risolve $a^{(n+1)}$, aggiorna $u^{(n+1)}$ e $v^{(n+1)}$ con le formule *

con $A_{ij} = \int \mu \nabla \varphi_i \nabla \varphi_j \, dx$; $M_{ij} = \int \varphi_i \varphi_j \, dx$; $\underline{f}_i = \int_\Omega f(t) \varphi_i \, dx$