MATH 738
HW 2
Due 11/29

(1) Let $\mathcal{C}$ be an additive category. An ideal of $\mathcal{C}$ is a collection of subspaces

$$I(X,Y) \subseteq \mathrm{Hom}(X,Y)$$

for each pair of objects X, Y such that for any $f : X \to X'$ and $g : Y \to Y'$ we have

$$g \circ I(X,Y) \circ f \subseteq I(X',Y').$$

Show that $\overline{\mathcal{C}}$ with the same objects as $\mathcal{C}$ and with

$$\mathrm{Hom}_{\overline{\mathcal{C}}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/I(X,Y)$$

has the structure of an additive category such that

$$X \mapsto X$$

$$f \mapsto \overline{f}$$

is a essentially surjective and full additive functor.

(2) Let P, Q be a bounded above complexes of projectives. Directly show that any quasi-isomorphism

$$Q \to P$$

has an inverse up to homotopy.

(3) Assume we have a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^i & \longrightarrow & B^i & \longrightarrow & C^i & \longrightarrow & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle g^i} & & \downarrow{\scriptstyle h^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \longrightarrow & D^i & \longrightarrow & E^i & \longrightarrow & F^i & \longrightarrow & D^{i+1} & \longrightarrow & \cdots \end{array}$$

in an abelian category $\mathcal{A}$ with exact rows. Show that if any two of $f^\bullet, g^\bullet, h^\bullet$ are all isomorphisms, then so is the third.

(4) Let $\mathcal{T}$ a compactly generated triangulated category and let $\mathcal{P}(X)$ be a predicate depending on objects of $\mathcal{T}$. Assume that

- if X and Y are isomorphic and $\mathcal{P}(X)$ is true, then $\mathcal{P}(Y)$ is true for Y
- $\mathcal{P}$ is true for a set of compact generators,
- if $X \to Y \to Z \to X[1]$ is a triangle and $\mathcal{P}$ is true for two of the three X, Y, and Z, then $\mathcal{P}$ is true for the third.
- if $\mathcal{P}$ is true for X_a for $a \in A$, then $\mathcal{P}$ is true for $\bigoplus_a X_a$.

Show that $\mathcal{P}$ is true for all objects of $\mathcal{T}$.

(5) An *cohomological functor* $H : \mathcal{T}^{op} \to \mathrm{Ab}$ is a functor that takes triangles to long exact sequences and for which the natural map

$$H\left(\bigoplus_{a \in A} X_a\right) \to \prod_{a \in A} H(X_a)$$

is an isomorphism. Show that if $\mathcal{T}$ is compactly generated, then any cohomological functor is representable. (Hint: use the resolution constructed in the notes replacing $\mathrm{Hom}(-,X)$ with H.)