Sampling: Probability

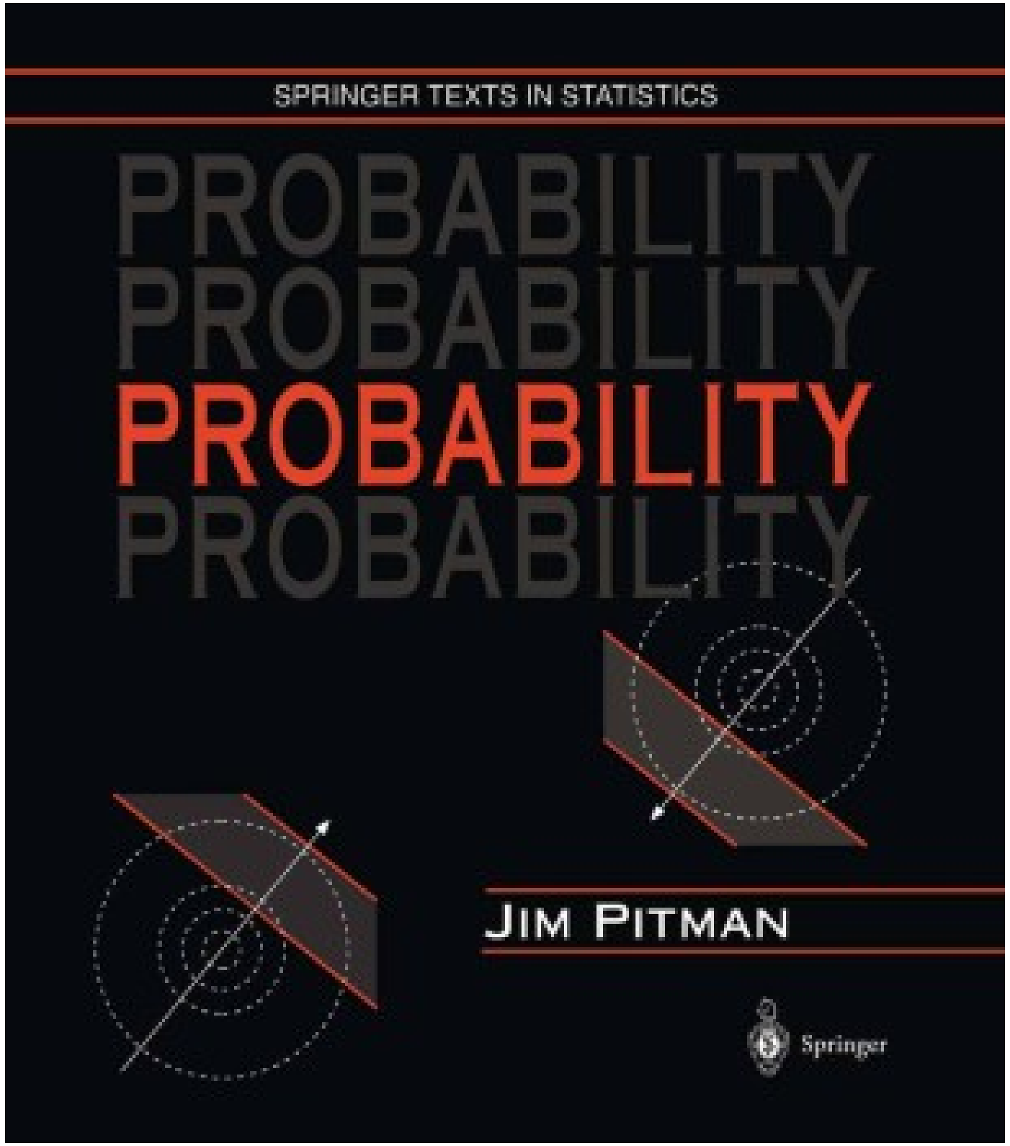
```
$ echo "Data Science Institute"
```

Learning Goal

How do we calculate and interpret probabilities? What is a statistical distribution?

Key Texts

- Pitman, 1993, *Probability* , Springer, Chapters 1-3
- Image source: [Springer]



Introduction

Definitions

- An **outcome space** is a set of all possible outcomes of some kind, often represented by Ω.
  - For example, Ω = {A, B, C,..., Z} is an outcome space containing all letters of the alphabet
- An **event** is a subset of an outcome space.
  - There are often many possible events for a specific outcome space
  - Possible events for Ω above could be vowels {A, E, I, O, U} or letters before E {A, B, C, D}
- A **probability** is a function of an event describing how likely it is to occur

Equally Likely Outcomes

- If all outcomes in a set Ω are equally likely, the probability of event *A* is the number of outcomes in *A* divided by the total number of outcomes,

> $$P(A) = \frac{\#A}{\#\Omega}$$

- $P(A)$ can be read as “the probability of A”.

Example: Rolling a die

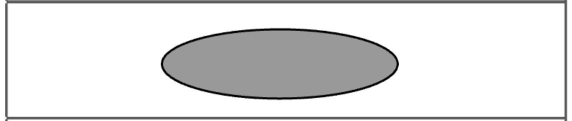
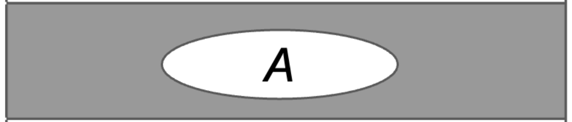
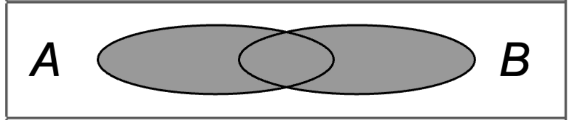
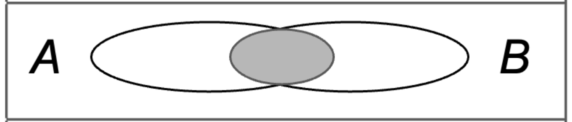
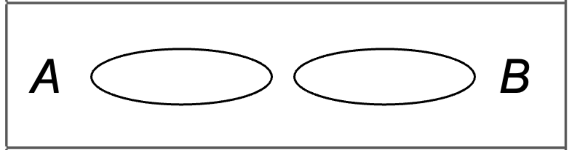
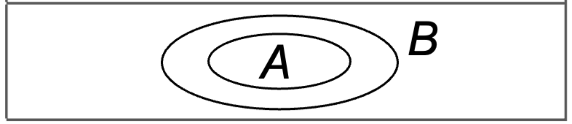
- For a six-sided die, the outcome space is,

> $$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Some possible events and their probabilities are,

Description	Event	Probability
An even number is rolled	$A = \{2, 4, 6\}$	$3/6 = 1/2 = 0.5$
A number less than 6 is rolled	$B = \{1, 2, 3, 4, 5\}$	$5/6 = 0.833...$
A 6 is rolled	$C = \{6\}$	$1/6 = 0.166...$

Sets and Events

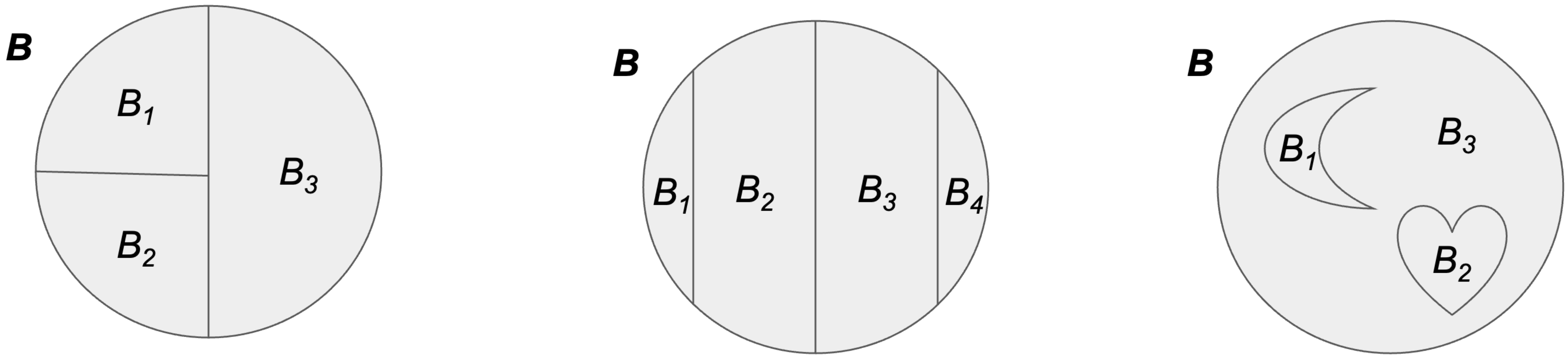
Event language	Set language	Set notation	Venn diagram
Outcome space	Universal set	Ω	
Event	Subset of Ω	A, B, C, etc.	
Impossible event	Empty set	$\emptyset$	
Not A, or the opposite of A	Complement of A	A^C	
Either A or B or both	Union of A and B	$A \cup B$	
Both A and B	Intersection of A and B	$A \cap B$	
A and B are mutually exclusive	A and B are disjoint	$A \cap B = \emptyset$	
If A, then B	A is a subset of B	$A \subseteq B$	

Partitions

- An event B is **partitioned** into n events $B_1, B_2, \ldots, B_n$ if

1. $B = B_1 \cup B_2 \cup \ldots \cup B_n$ – every outcome in *B* belongs to some event B_i , none are left out
2. $B_1, B_2, \ldots, B_n$ are **mutually exclusive** – if an outcome is in event B_i , it is not in any other event

Adapted from Pitman (1993), Figure 1



Rules of Probability

- For an event *B* over an outcome space Ω,
- **Non-negativity**: $P(B) \geq 0$
- **Addition**: If $B_1, B_2, \dots, B_n$ is a partition of *B* , then

> $$P(B) = P(B_1) + P(B_2) + \ldots + P(B_n)$$

- **Total one**: $P(\Omega) = 1$

Example: Drawing cards

- Suppose you have a regular deck of cards. Let B represent the event "drawing a heart". Let B_1 and B_2 be a partition of B , with B_1 = "drawing non-numeric heart card (J, Q, K, A)" and B_2 = "drawing an ace or numeric heart card (2,...,10)"
- B_1 and B_2 is a valid partition, since all heart cards are either numeric or non numeric, and a card cannot be both a numeric card and a non-numeric card (mutual exclusivity)

> $$P(B) = 13/52 = 1/4 \geq 0$$
> $$P(B1) + P(B2) = 4/52 + 9/52 = 13/52 = 1/4 = P(B)$$
> $$P(\Omega) = P(\text{“draw any card in the deck”}) = 1$$

Conditional Probability

Conditional Probability

- ⚠️ **Conditional probability** can be described as the probability that event *A* will happen **given that event *B* has already happened** ⚠️
- The notation for conditional probability is $P(A|B)$
- The formula for conditional probability is,

> $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- As long as $P(B) \neq 0$

Example: Rolling a die

- Suppose you roll a regular die, but haven't yet looked at the result. Let event A be "rolling a 4", and let event B be "rolling an even number".
- The probability that number you rolled is a 4 is,

> $$P(A) = \frac{1}{6}$$

- Now suppose I look at the die and tell you that the number you rolled is **even** . Given this new information, the probability that the number you rolled is a 4 is,

> $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\text{"rolling a 4" and "rolling an even number"})}{P(\text{"rolling an even number"})} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Independence

- If events A and B are **independent** , event A is not affected by the occurrence of event B.
- This can be described mathematically as,

$$P(A|B) = P(A)$$

- From the formula for conditional probability, we can then derive the following formula for independent events,

$$P(A \cap B) = P(A)P(B)$$

- The same applies for any number of disjoint events.

Example: Selecting a high school student

- Pitman (1993), Exercises 1.4, Exercise 9
- Three high schools have senior classes of size 100, 400, and 500, respectively. Here are two schemes for selecting a student from among the three senior classes:
  i. Make a list of all 1000 seniors, and choose a student at random from this list.
  ii. Pick one school at random, then pick a student at random from the senior class in that school.
- Are these two strategies equivalent?

Random Variables

Random Variables

- **Random variables** are a way to describe a set possible outcomes with a distribution of probabilities over the set of outcomes.
- Usually denoted with capital letters: X, Y, Z, etc.
- Random variables are similar to events
  - Events are a specific outcome or set of outcomes, while random variables describe possible outcomes and their various probabilities
  - An event “Number on the die is a five” or {5} or $X = 5$ is one possible outcome of the random variable X, “the number obtained by rolling a die”

Random Variables

- The **range** of a random variable is all of the values it could possibly take. This can be continuous ($0 \leq X \leq 10$) or discrete ($X \in \{1, 2, 3\}$).
- The distribution of random variable is determined by the probabilities of values within its range,

> $$P(X = x) \text{ for x } \in \text{ range of } X$$

Example: Rolling two dice

- Let X represent the sum of the faces showing on two rolled dice.

x	2	3	4	5	6	7	8	9	10	11	12
Possible outcomes	1+1	1+2, 2+1	1+3, 2+2, 3+1	1+4, 2+3, 3+2, 4+1	1+5, 2+4, 3+3, 4+2, 5+1	1+6, 2+5, 3+4, 4+3, 5+2, 6+1	2+6, 3+5, 4+4, 5+3, 6+2	3+6, 4+5, 5+4, 6+3	4+6, 5+5, 6+4	5+6, 6+5	6+6
P(X = x)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

Indicator Variables

- **Indicator variables** , denoted I_A, are specific type of random variable that take the value 0 or 1 to indicate a the occurrence of a given event A .
- Some examples of indicator variables may be votes in a two-party election (with event A being a vote for a particular candidate), votes for or against a bill, satisfied versus not satisfied reviews for a product, etc.

Distributions

Distributions

- ⚠️ A **probability distribution** is a statistical function that describes the probabilities of all possible events in an outcome space. ⚠️
- Distributions can be **discrete** (if the outcome space is distinct events, like rolling a die) or **continuous** (if the outcome space is a range of values, like choosing any real number between 1 and 10).
- Some features of interests for distributions may be their mean, variance, mode, skew, etc.

Binomial Distribution

Binomial Distribution

- The **binomial distribution** concerns sequences of events with two possible outcomes: **success** and **failure**.
- Success occurs with probability ***p*** , and failure occurs with probability ***q = 1 - p*** *. * Trials defined this way are called **Bernoulli trials** .
- ⚠️ The binomial distribution helps determine the probability of getting ***k* successes in *n* independent trials (with replacement)** ⚠️

Binomial Distribution Formula

- For *n* independent trials with probability *p* of success and probability *q* = 1 - *p* of failure, the probability of ***k* success in *n* independent trials** (with replacement) is,

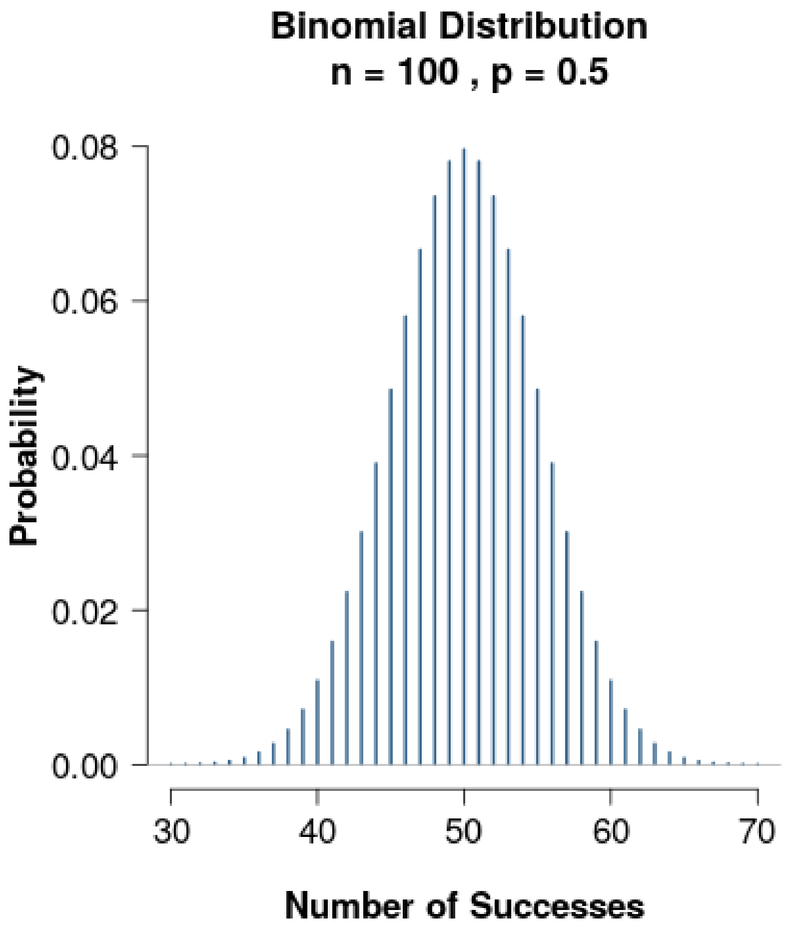
> $$P(\text{k success in n trials}) = \binom{n}{k} p^k q^{n-k}$$

- $\binom{n}{k}$ is called " *n* choose *k* " and describes the number of possible combinations of *k* successes and *n* - *k* failures:

> $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Distribution

- For a large number of trials *n* , we expect the number of successes to be about *np* .
- For n = 100 and p = 0.5, the distribution of number of successes is centered around 50 (the most likely) and the total number of successes gets less likely as the numbers get farther from 50.



Example: Drawing cards

- Suppose you draw n = 5 cards from a standard deck, and your desired outcome is drawing a club. Then p = 13/52 = ¼ and q = 1 - p = 1 - ¼ = ¾.
- For k = 1 success, the possible combinations of cards drawn are:

> {➡️♣️⬅️♣️♣️♣️♣️},
> {♣️➡️♣️⬅️♣️♣️♣️},
> {♣️♣️➡️♣️⬅️♣️♣️},
> {♣️♣️♣️➡️♣️⬅️♣️},
> {♣️♣️♣️♣️➡️♣️⬅️}

- Mathematically this can be represented as,

$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1(4 \cdot 3 \cdot 2 \cdot 1)} = 5$$

Example: Drawing cards

- Since the trials are independent, the probability of getting 1 success in 5 trials is the product of the probability of getting a club on one trial and the probability of getting non-clubs on four trials.

> $$P(\text{Club})P(\text{Non-club})P(\text{Non-club})P(\text{Non-club})P(\text{Non-club}) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{1^1}{4}\cdot\frac{3^4}{4} = 0.0791$$

- Putting the two calculations together, we have,

> $$P(1 \text{ club in } 5 \text{ draws}) = \binom{5}{1}\frac{1^1}{4}\cdot\frac{3^4}{4} = 5\cdot 0.0791 = 0.396$$

Uniform Distribution

Uniform Distribution

- The **uniform distribution** describes a situation in which every outcome on a certain set or interval is **equally likely** .
- This can be represented mathematically as,

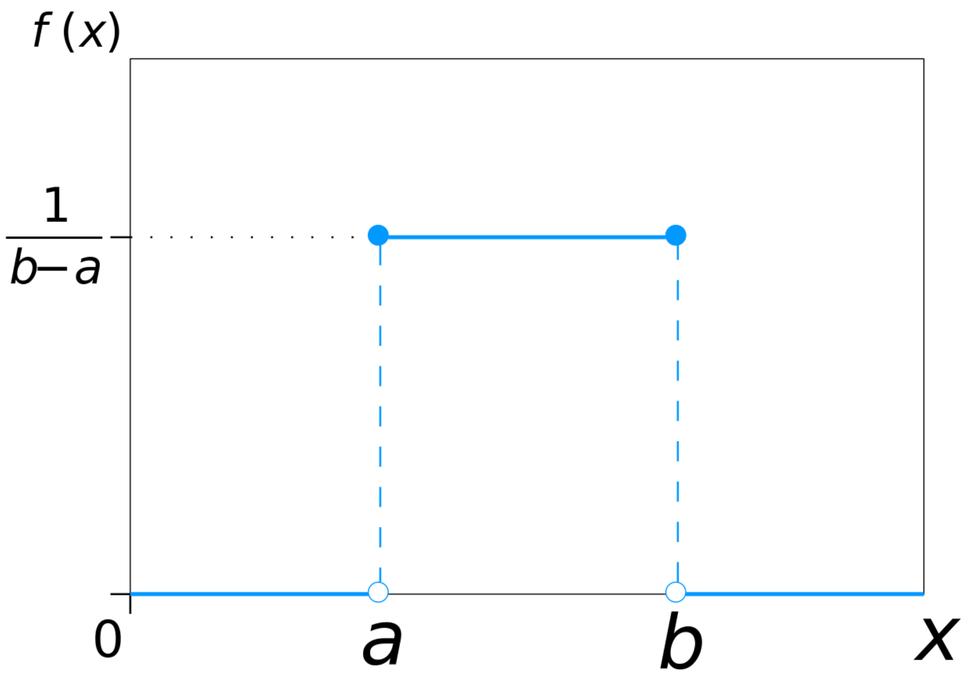
> $$P(X = x) = \begin{cases} \frac{1}{|\Omega|} & \text{if } x \in \Omega, \\ 0 & \text{otherwise} \end{cases}$$

or

> $$P(X = x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise} \end{cases}$$

Uniform Distribution

- Examples
  - Rolling a die
  - Drawing any card from a normal deck
  - Choosing a random number between 1 and 100
  - Choosing a random student in a classroom



Poisson Distribution

Poisson Distribution

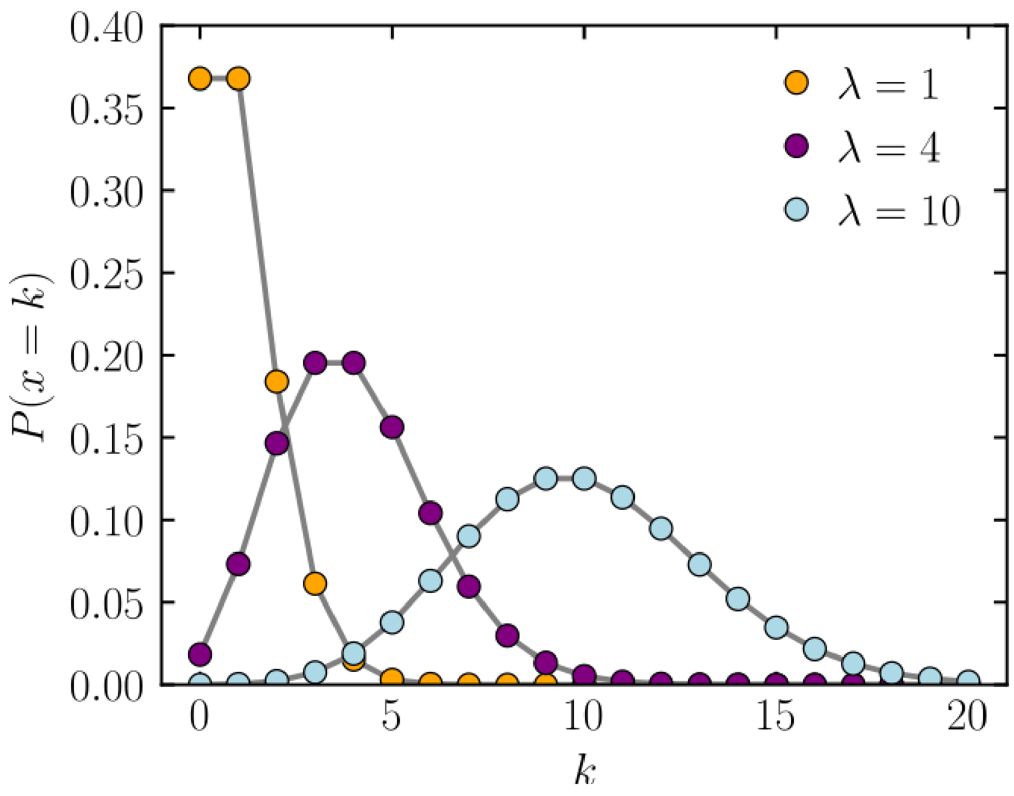
- The **Poisson distribution** is an approximation of the distribution of the **number *N* of occurrences of events of some kind**, when the events all have small probabilities and are independent.

> $$P(N = k) \approx \frac{e^{-\mu}\mu^k}{k!} \text{ for k} = 1, 2, \ldots$$

- The Poisson distribution is a discrete probability distribution.

Poisson Distribution

- Examples
  - Number of wins in n games of roulette for a gambler who bets on a single number each game
  - Number of rain drops that land on a particular area of a roof during a set time interval
  - Number of people who enter a store in a certain time interval



Normal Distribution

Normal Distribution

- The **normal distribution** is one of the most common and important distributions
- It is represented by the equation,

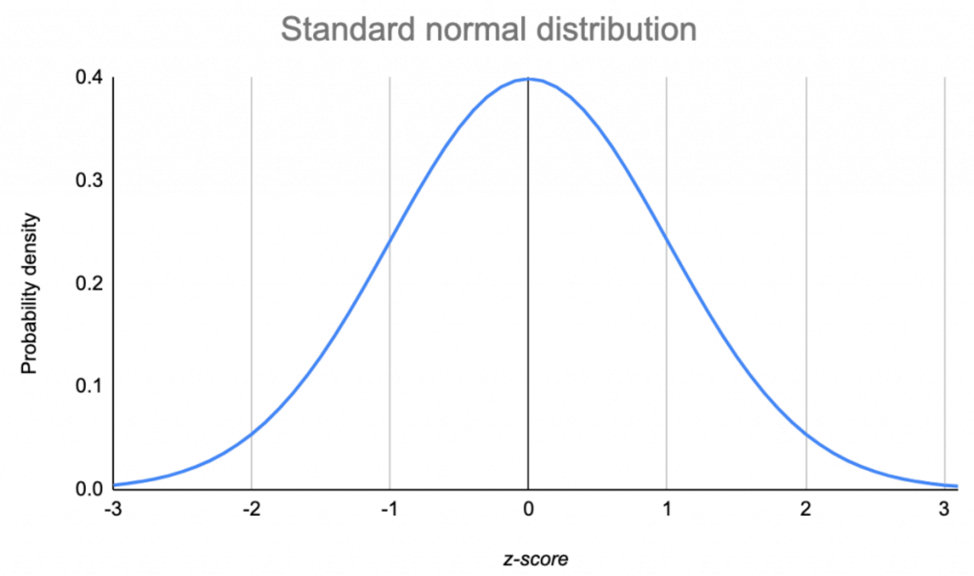
> $$P(X = x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- where μ is the mean of X and σ is the standard deviation.
- A random variable X following a normal distribution is often denoted,

> $$X \sim \mathcal{N}(\mu, \sigma^2)$$

Normal Distribution

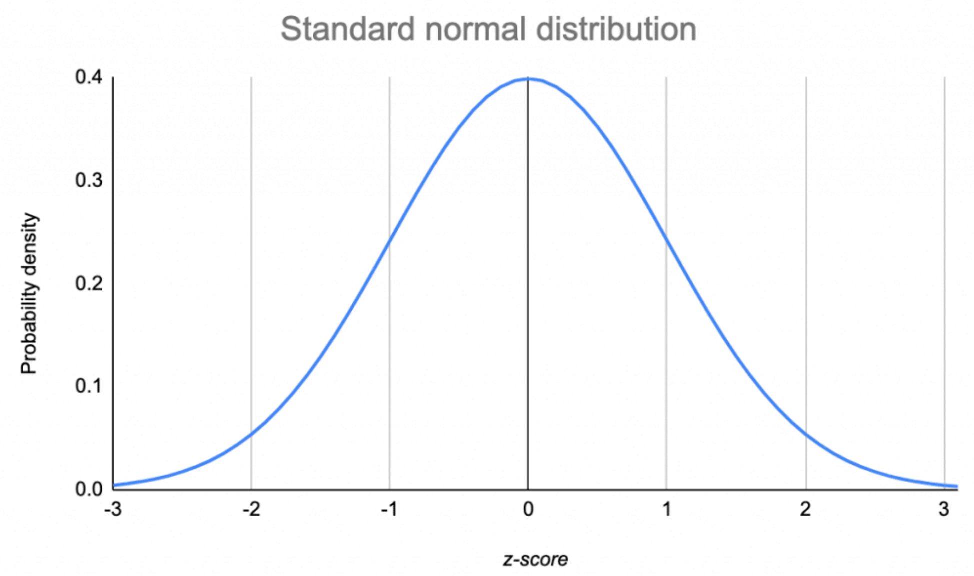
- The normal distribution is centered and symmetric about μ. σ describes the horizontal spread (how wide the distribution is).
- The normal distribution can be used to be use to approximate other distributions for easy calculations of probabilities



Standard Normal Distribution

- The standard normal distribution is a normal distribution with mean 0 and standard deviation 1.
- In general, a random variable X with a normal distribution can be standardized using the following formula,

$$Z = \frac{X - \mu}{\sigma}$$



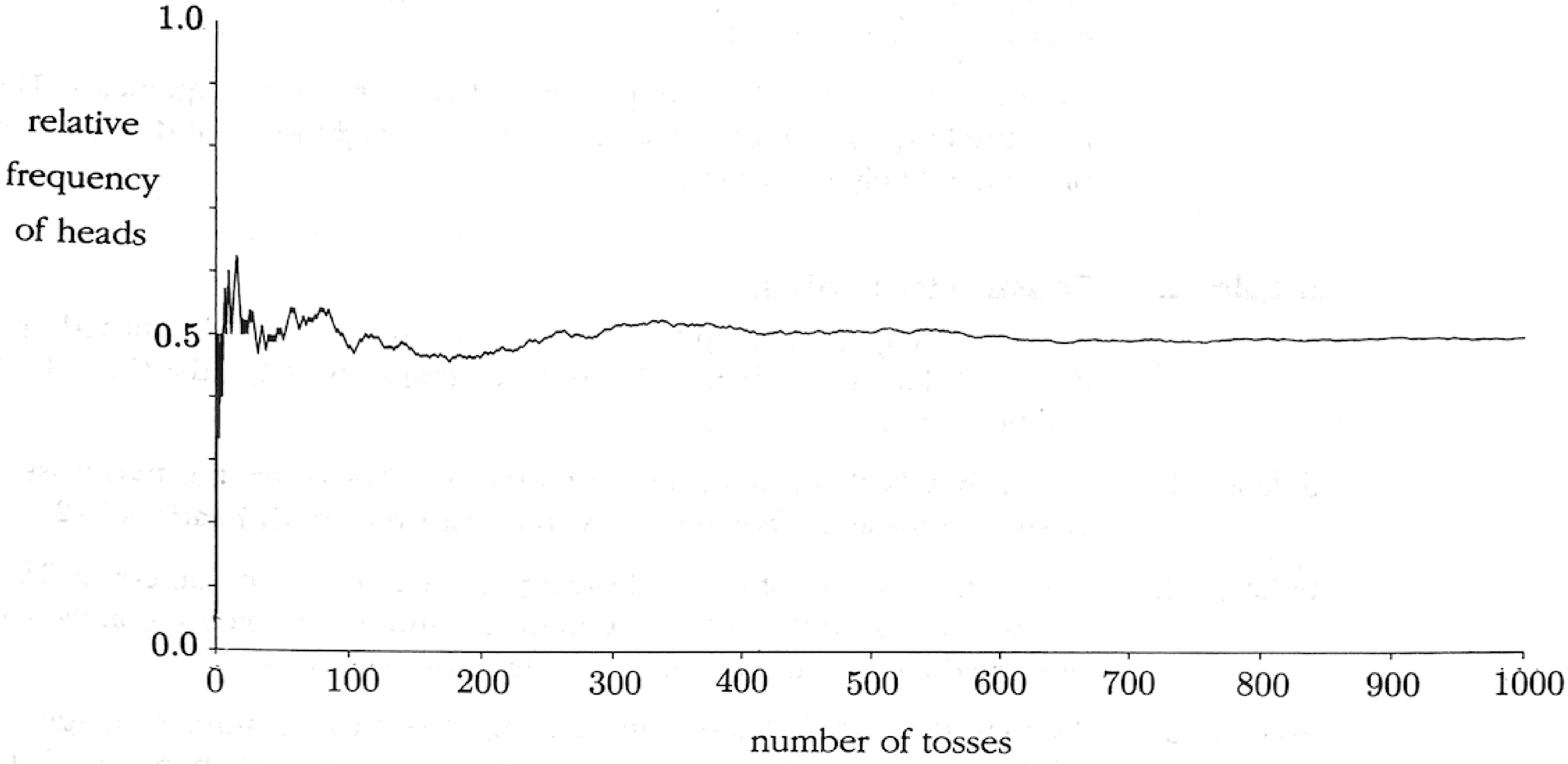
Example: Z-Scores

- Z-Scores represent the probability that a value is less than or equal to the value of a given standardized random variable.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Law of Large Numbers

Law of Large Numbers



Law of Large Numbers

- ⚠️ If the number of trials n is large, the proportion of successes in n independent trials will, with overwhelming probability, be very close to p , the probability of success on each trial ⚠️
- Intuition for sampling
  - As the number of units sampled increases, the proportion of units that exhibits a certain trait will grow closer and closer to the true proportion of individuals in the population with that trait

Expected Value

Expected Value

- The **expected value** or **expectation** of a random variable X is the mean of the distribution of X, denoted E(X) or μ. This is represented mathematically as,

> $$E(X) = \sum_{\text{Every x}} xP(X = x)$$

- The expected value is the average of all possible values of X weighted by their probabilities.
- The expected value of indicator variable I_A is the probability of event A,

> $$E(I_A) = P(A)$$

Example: Sampling a student

- Suppose you are randomly sampling a student from a school. There are 200 students each of ages 16, Let random variable *X* represent the age of the student sampled.
- The expected age of the student selected is,

> $$E(X) = \sum_{\text{Every x}} xP(X = x) = 16(\frac{200}{600}) + 17(\frac{200}{600}) + 18(\frac{200}{600}) = 17$$

- Now suppose there are 100 students age 16, 200 students age 17, and 300 students age 18. The new expected age is,

> $$E(X) = \sum_{\text{Every x}} xP(X = x) = 16(\frac{100}{600}) + 17(\frac{200}{600}) + 18(\frac{300}{600}) = 17.333$$

Properties of Expectation

- Constants : The expectation of a constant random variable is its constant value

> $$E(c) = c$$

- Scalar multiplication : For random variable X multiplied by constant c,

> $$E(cX) = cE(X)$$

- Addition : The expectation of a sum of random variables is the sum of the expectations

> $$E(X + Y) = E(X) + E(Y)$$

Variance and Standard Deviation

Variance and Standard Deviation

- The **variance** of X, denoted Var(X), is the mean squared deviation of X from its expected value E(X),

> $$Var(x) = E([X - E(X)]^2) = E(X^2) - [E(X)]^2$$

- The **standard deviation** of X, denoted SD(X), is the square root of the variance of X:

> $$SD(X) = \sqrt{Var(X)}$$

Variance Properties

- **Addition**: for independent random variables X_1, X_2,..., X_n, the variance of their sum is,

> $$Var(X_1 + ... + X_n) = Var(X_1) + ... + Var(X_n)$$

- **Scalar multiplication**: for random variable X and scalar c,

> $$Var(cX) = c^2 Var(X)$$

Variance and Standard Deviation

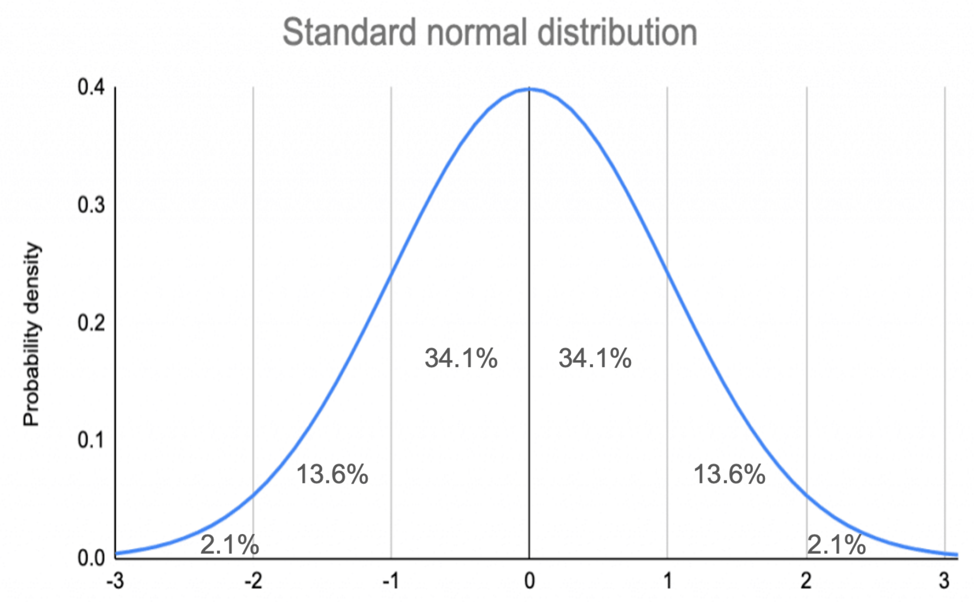
- Variance is often denoted σ^2, with SD denoted σ
- Variance and SD describe how spread out the distribution of a variable is
- SD is often easier to interpret since its units are the same as the mean
- ⚠️ **In general** ⚠️
  - For a random variable X with some distribution, you should expect the value of X to be around the expected value E(X), plus or minus a few times the standard deviation SD(X)

Example: Normal Distribution

- On a normal distribution, ~68% of the probability density lies within one SD of the mean:

> $$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$$

- For the standard normal *N(0, 1)* , this means *X* is fairly likely to be between -1 and 1, and that 2.5 would be a very unlikely value of *X*.



Central Limit Theorem

Law of Averages

- Let $X_1, ..., X_n$ be a sequence of independent random variables with the same distribution as random variable X. Let μ = E(X) denote the expected value for all X_i . Let the following random variable represent the average of $X_1, ..., X_n$

> $$\bar{X}_n = \frac{X_1 + X_2 + ... + X_n}{n}$$

- Then for every ε > 0, no matter how small,

> $$P(|\bar{X}_n - \mu| < \epsilon) \rightarrow \ 1 \text{ as n } \rightarrow \infty$$

Law of Averages

- "The probability that the difference between the calculated mean and the true mean is very small approaches 1 as n approaches infinity"

> $$P(|\bar{X}_n - \mu| < \epsilon) \to \ 1 \text{ as n } \to \infty$$

- ⚠️ As the number of random variables increases, the average of the random variables will be arbitrary close to the expected value of any one random variable ⚠️

Example: Tossing a coin

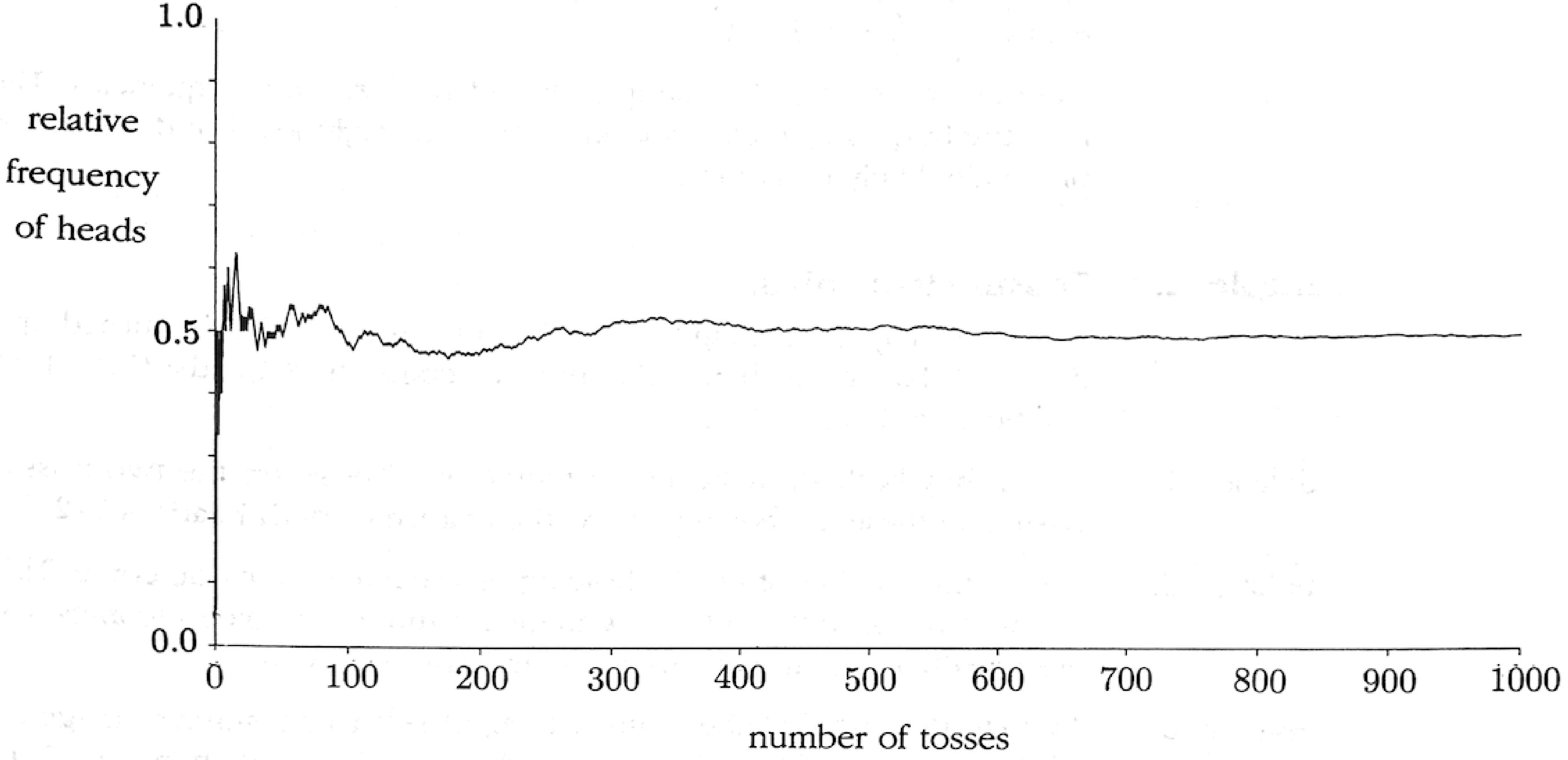
- Suppose we are tossing a fair coin. We assign a value of 0 for heads and 1 for tails. Let indicator variable *I* represent the result of a given going toss. Then,

  > $$E(I_A) = 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5$$

- Observe the following sequences of coin tosses:

n	Results	
1	{h}	$\bar{X}_1 = \frac{0}{1} = 0$
3	{h, h, t}	$\bar{X}_3 = \frac{0+0+1}{3} = 0.333\ldots$
5	{h, h, t, h, t}	$\bar{X}_5 = \frac{0+0+1+0+1}{5} = 0.4$

Example: Tossing a coin



Central Limit Theorem

- ⚠️ Let S_n = X_1 + X_2 + ... + X_n be the sum of n independent random variables each with the same distribution. For large n, the distribution of S_n is approximately normal , with mean E(S_n) = $n\mu$ and variance Var(S_n) = σ 2 n , where μ = E(X_i) and σ 2 = Var(X_i). ⚠️

- This also holds true for,

> $$\bar{X}_n = \frac{S_n}{n}, \text{ with } E(\bar{X}_n) = \mu \text{ and } Var(\bar{X}_n) = \frac{\sigma^2}{n}\$$$

> $$\bar{X}_n \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Next

Populations, censuses, surveys, and observational data