Integration

Integration in $\mathbb{R}$

Definition. ***Partition*** *A finite partition P of $[a,b]$ is an ordered collection of points $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. The* ***order of*** *$P$ is defined to be $|P| = n$ (i.e. the number of subintervals) and the* ***length of*** *P is*

$$l(P) = \max_{i=i,\cdots,|P|} [x_i - x_{i-1}]$$

that is, the length of P is the length of the longest interval whose endpoints are in P. In other words, if $\mathcal{P}_{[a,b]}$ is the set of all finite partitions of $[a,b]$ then $l : \mathcal{P}_{[a,b]} \to \mathbb{R}_+$ gives the worst case scenario for width of subintervals.

Definition. ***Refinement*** *If P and Q are two partitions of $[a,b]$ then Q is the refinement of P if $P \subseteq Q$*

Remark. Any two partition $P, Q \in \mathcal{P}_{[a,b]}$ admits a common partition R, where $R = P \cup Q$

Definition. ***Riemann sum*** *Given a function $f : [a,b] \to \mathbb{R}$, a Riemann sum of f with respect to the partition $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ is any sum of the form*

$$S(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \qquad t_i \in [x_{i-1}, x_i]$$

By how we pick t_i we have

1. ***Left- and Right-endpoint Riemann sums***

$$L(f,P) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \qquad R(f,P) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

2. ***Lower and Upper Riemann sums*** *Fix partition $P \in \mathcal{P}_{[a,b]}$ and $f : [a,b] \to \mathbb{R}$. Define*

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

so m_i is the smallest value f takes on $[x_{i-1}, x_i]$ while M_i is the largest

$$u(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \qquad U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Proposition. *If Q is a refinement of P then we have*

$$u(f,P) \le u(f,Q) \qquad U(f,P) \ge U(f,Q)$$

Intuitively, u is increasing function over refinement of P while U is decreasing for refinement of P

Lemma. *Let A, B be sets such that $A \subseteq B$ then if infimum and supremum exists, we have*

$$\inf A \geq \inf B \qquad \sup A \leq \sup B$$

Definition. *The **Lower and Upper Integral** is defined to be*

$$u(f) = \sup_P [u(f, P)] \qquad U(f) = \inf_P [U(f, P)]$$

In other words, the lower integral is the lower Riemann sum for sufficiently fine P; while the upper integral is the upper Riemann sum for sufficiently fine P.

Definition. ***Riemann Integrable*** *We say that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ with integral I if for every $\epsilon > 0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $P \in \mathcal{P}_{[a,b]}$ satisfies $l(P) < \delta$, then*

$$|S(f, P) - I| < \epsilon$$

where I is denoted as $I = \displaystyle\int_a^b f(x)dx$.

Remark. Roughly, a function is Riemann integrable with integral I if we can approximate I arbitrarily well by taking a sufficiently fine partition P.

The following definition are equivalent

1. *f is Riemann integrable*
2. *$\displaystyle\sup_{P \in \mathcal{P}_{[a,b]}} u(f, P) = \inf_{P \in \mathcal{P}_{[a,b]}} U(f, P)$. In other word, the lower and upper integral are equal*
3. *For every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}_{[a,b]}$ such that $U(f, P) - u(f, P) < \epsilon$*
4. *For every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $P, Q \in \mathcal{P}_{[a,b]}$ satisfy $l(P) < \delta$ and $l(Q) < \delta$ then $|S(f, P) - S(f, Q)| < \epsilon$*

Remark. To prove that a function is Riemann integrable, we use the third definition. Specifically, we pick a partition P with $|P| = n$ and show that the difference between upper and lower Riemann sums converges. To prove that a function is *not* Riemann integrable, the characteristic function of rationals on $[0, 1]$ is not integrable

$$\chi_Q(x) = \begin{cases} 1 & x \in Q \cap [0, 1] \\ 0 & otherwise \end{cases}$$

Since Q is dense in $[0, 1]$, then $M_i = 1$ and $m_i = 0$ and so

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = x_1 - x_0 = 1 \neq 0 = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = u(f, P)$$

holds for any partition P, any $\epsilon < 1$ fails the definition of integrability

Definition. ***Properties of Integral***

1. ***Additivity of Domain*** *If f is integrable on $[a,b]$ and $[b,c]$ then f is integrable on $[a,b]$ and*
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. ***Additivity of Integral*** *If f,g are integralon $[a,b]$ then $f+g$ also integrable on $[a,b]$ and*
$$\int_a^b f(x) + g(x)dx = \int_a^b g(x)dx + \int_a^b f(x)dx$$

3. ***Scalar Multiplication*** *If f integrable on $[a,b]$ and $c \in \mathbb{R}$ then cf is integrable on $[a,b]$*
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

4. ***Inherited Integrability*** *f is integrable on $[a,b]$ then f is integrable on any subinterval $[c,d] \subseteq [a,b]$*

5. ***Monotonicity of Integral*** *If f,g are integrable on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a,b]$ then*
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

6. ***Subnormality*** *If f is integrable on $[a,b]$ then $|f|$ is integrable on $[a,b]$ and satisfies*
$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

Definition. ***Fundamental Theorem of Calculus***

1. *If f is integrable on $[a,b]$ and $x \in [a,b]$ define $F(x) = \int_a^x f(t)dt$. The function F is continuous on $[a,b]$ and moreover, $F'(x)$ exists and equals $f(x)$ at every point x at which f is continuous*

2. *Let F be continuous function on $[a,b]$ that is differentiable except possibly at finitely many points in $[a,b]$ and take $f = F'$ at all such points. If f is integrable on $[a,b]$, then*
$$\int_a^b f(x)dx = F(b) - F(a)$$

Theorem. ***If f is bounded and monotone on $[a,b]$ then f is integrable***

Theorem. ***Every continous function on $[a,b]$ is integrable***

Now we consider the integral over finitely and possible infinitely many points

Definition. ***Jordan Meansure*** *If $I = [a, b]$ let the length of I be $l(I) = b - a$. If $\mathcal{P}(\mathbb{R})$ is the power-set of $\mathbb{R}$, we define* ***Jordan outer measure*** *as the function $m : \mathcal{P} \to \mathbb{R}_{\geq 0}$ given by*

$$m(S) = \inf \left\{ \sum_{k=1}^{n} l(I_k) : S \subseteq \cup_{k=1}^{n} I_k \text{ where } I \text{ is an interval} \right\}$$

If $m(S)$ exists and $m(\partial S) = 0$, we say that S is ***Jordan Measurable****.*
If $m(S) = 0$ we say that S has ***Jordan Measure Zero****.*

Remark. For proofs of Jordan Measure zero, it suffices to show that for all $\epsilon > 0$, $\displaystyle\sum_{k=1}^{n} l(I_k) < \epsilon$ by definition of infimum. Some examples below,

1. Jordan measure of any finite set is 0
2. Jordan measure of any interval $[a, b]$ is $m([a, b]) = b - a$
3. $m(\mathbb{R})$ does not exist because no finite cover for $\mathbb{R}$
4. If $S = \mathbb{Q} \cap [0, 1]$, then $\partial S = [0, 1]$ and $m(\partial S) = 1 \neq 0$, hence not Jordan measurable

In essense, Jordan measure is an extension of the notion of size (length, area, volume) to shape more complicated than say triangle, rectangles...

Theorem. *If $S \subseteq [a, b]$ is a Jordan measure zero set, and $f : [a, b] \to \mathbb{R}$ is bounded and continuous everywhere except possibly at S, then f is integrable.*

Corollary. *If f, g are integrable on $[a, b]$ and $f = g$ up to a set of Jordan measure zero, then $\displaystyle\int_a^b f(x)dx = \int_a^b g(x)dx$*

4.2 Integration in $\mathbb{R}^n$

Definition.
A ***Rectangle*** *$R \in \mathbb{R}^2$ is any set which can be written as $[a, b] \times [c, d]$. A* ***Partition*** *$P = P_x \times P_y$ is a partition of R where $P_x = \{a = x_0 < \cdots < x_n = b\}$ and $P_y = \{c = y_0 < \cdots < y_m = d\}$ are partitions of their respective intervals $[a, b]$ and $[c, d]$ with* ***subrectangles***

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \quad x = 1, \cdots, n \quad y = 1, \cdots, m$$

The ***Area of rectangle*** *R_{ij} is given by*

$$A(R_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$$

*in which case the **Riemann Sum** for $f : \mathbb{R}^2 \to \mathbb{R}$ over partition P is given by*

$$S(f,P) = \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} f(t_{ij})A(R_{ij}) \qquad t_{ij} \in R_{ij}$$

*and the **upper and lower Riemann sum** are defined as*

$$U(f,P) = \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} \sup_{x \in R_{ij}} f(x)A(R_{ij}) \qquad u(f,P) = \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} \inf_{x \in R_{ij}} f(x)A(R_{ij})$$

*$f : \mathbb{R}^2 \to \mathbb{R}$ is **Riemann Integrable** if for any $\epsilon > 0$ there exists a partition P (i.e. exists δ such that $A(P) < \delta$ where $A(P) = \max\{A(R_{ij})\}$ for all subrectangles R_{ij} of P) such that*

$$U(f,P) - u(f,P) < \epsilon$$

*The **Integral** is given by*

$$\int\int_R f dA \quad or \quad \int\int f(x,y)dxdy$$