Linear Transformation

V - vectorspace , W - vectorspace

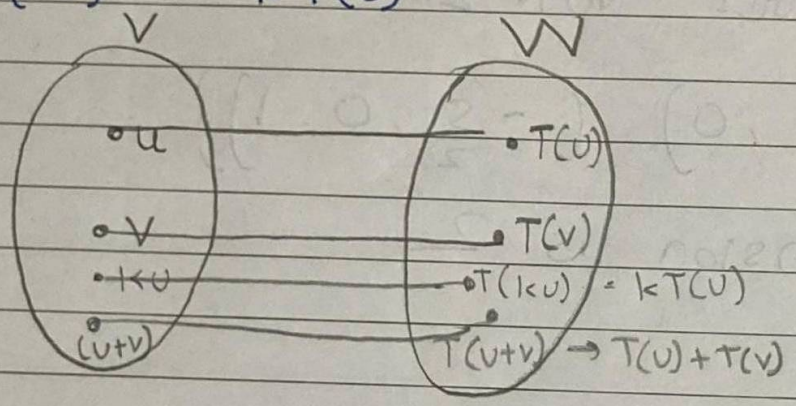
$T: V \to W$

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = \sin x$
$f(x) = x^2$

T is called linear transformation (or) linear map if it satisfies :-

(i) $T(u+v) = T(u) + T(v)$

(ii) $T(ku) = kT(u)$



Is $T: \mathbb{R} \to \mathbb{R}$

$T(x) = \sin x \to$ linear?

$T(x+y) = T(x) + T(y)$

$\sin(x+y) = \sin x + \sin y$ ✗

$\sin x \cos y + \cos x \sin y$

$T: \mathbb{R} \to \mathbb{R}$

$T(x) = 4x$

Is T linear? let x & $y \in \mathbb{R}$; To check :-

$T(x+y) = T(x) + T(y)$; $T(ku) = kT(u)$

$4(x+y) = 4x + 4y$ ✓

$4(kx) = k4x$ ✓

$(4(2x) = 2 \times 4x)$

$(8x = 8x)$

$\therefore$ (T) It is a linear map

Examples:

1) $T: \mathbb{R}^2 \to \mathbb{R}$

$T(x,y) = 2x+y$

Is T linear

To check : $T(u+v) = T(u) + T(v)$

: $T(ku) = kT(u)$

let $u = (x_1, y_1)$ & $v = (x_2, y_2)$

$$u+v = (x_1, y_1) + (x_2, y_2)$$
$$= (x_1 + x_2, y_1 + y_2)$$

$$T(u+v) = T(x_1 + x_2, y_1 + y_2)$$
$$= 2(x_1 + x_2) + (y_1 + y_2)$$
$$= 2x_1 + y_1 + 2x_2 + y_2$$
$$= T(u) + T(v)$$

$$ku = k_1(x_1, y_1) = (kx_1, ky_1)$$
$$T(ku) = T(kx_1, ky_1)$$
$$= 2kx_1 + ky_1$$
$$= k(2x_1 + y_1)$$
$$= kT(u)$$

$\therefore$ T is linear

2. $T: \mathbb{R} \to \mathbb{R}$
$T(x) = x+2$
Is T linear?
let $x, y \in \mathbb{R}$
$T(x,y) = x+y+2$
$T(x)+T(y) = x+2+y+2$ ✗
$= x+y+4$

$T(x+y) \neq T(x)+T(y)$
$\therefore$ T is not linear

Note: Non-zero scalar added can't be linear
Single power is linear
Some examples of
linear maps: ✓ : Linear

$T: \mathbb{R} \to \mathbb{R}$
$T(x) = kx$ ✓

$T: \mathbb{R}^2 \to \mathbb{R}$
$T(x,y) = 2x+3y$ ✓

$T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(x,y) = (2x+y, 3x-4y)$ ✓

$T(x,y,z) = 2x+3y+4z$ ✓

$T: \mathbb{R}^3 \to \mathbb{R}^2$ 2 terms 3 variables
$T(x,y,z) = (x+y+z, 2x-3y+4z)$ ✓

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x, y) = (x+2y, x-y, 2x-y)$

$T: \mathbb{R}^3 \to \mathbb{R}3$
$T(x, y, z) = (2x-3y, x+y-3z, y+3z)$

$T: \mathbb{R} \to \mathbb{R}$
$T(1) = 5$
$T(2) = T(2 \cdot 1)$
$= 2T(1) = 2 \times 5$
$= 10$

$(T(ku) = kT(u))$

$T(3) = T5$

$T(4) = 20$

$T(x) = T(x \cdot 1)$
$= xT(1)$
$= x(5)$
$= 5x$

(Not true for trigo, expo, etc)

$T: \mathbb{R}^2 \to \mathbb{R}$
$T(x, y)$
$T(1, 0) = 4$
$T(0, 1) = 3$
T is linear

Find $T(10, 5)$

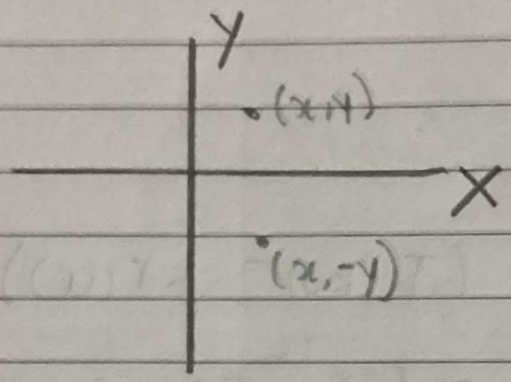
$T(ku) =$
$(10, 5) = 10(1, 0) + 5(0, 1)$
$T(10, 5) = T(10(1, 0) + 5(0, 1))$

$$= T(10(1,0)) + T(5(0,1))$$
$$= 10 \times 4 + 5 \times 3$$
$$= 55$$

12/2/24

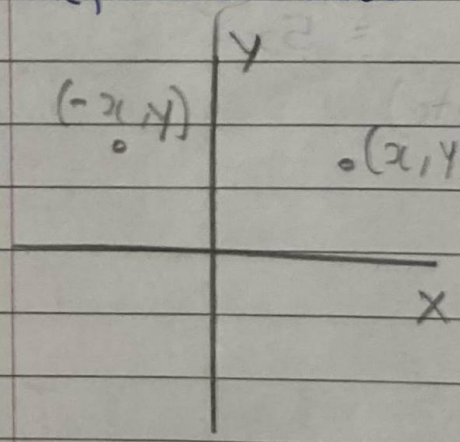
1. Some standard linear transformation on $\mathbb{R}^2$ and $\mathbb{R}^3$

1. Reflection about the x-axis



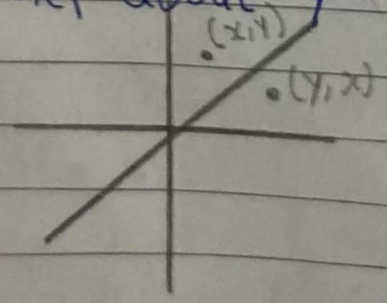
$T(x,y) = (x,-y)$

2. Reflection about the y-axis



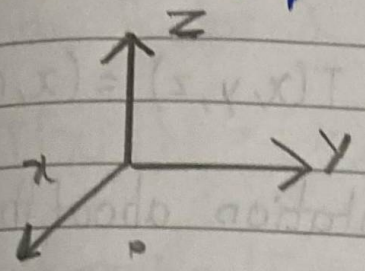
$T(x,y) = (-x,y)$

3. Ref about $y = x$



$T(x,y) = (y,x)$

Ref about xy-plane



$T(x,y,z) = (x,y,-z)$

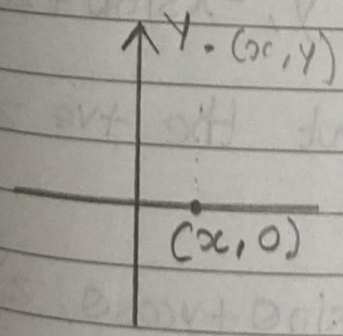
5. Ref about yz plane

$T(x,y,z) = (-x,y,z)$

6. Ref about xz plane

$T(x,y,z) = (x,-y,z)$

7. Orthogonal progression onto the x-axis



$T(x,y) = (x,0)$

8. Orthogonal progression onto the y-axis

$T(x,y) = (0,y)$

9. Orthogonal progression on xy plane

$T(x,y,z) = (x,y,0)$

10. Orthogonal P yz plane $T(x,y,z) = (0, y, z)$

11. Orthogonal P xz plane $T(x,y,z) = (x, 0, z)$

12. Counterclockwise (anti) rotation about the origin through an angle θ

$$t(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

13. Counterclockwise rotation about the +ve x-axis through an angle θ

3D

$$T(x,y,z) = (x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta)$$

14. Counterclockwise rotation about the +ve y-axis through an angle θ

3D

$$T(x,y,z) = (x\cos\theta + z\sin\theta, y, -x\sin\theta + z\cos\theta)$$

15. Counterclockwise rotation about the +ve z-axis through an angle θ

3D

$$T(x,y,z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$$

16. Contraction / Dilation with factor k

$$T(x,y) = (kx, ky) \text{ in } R^2$$

17. Contraction / Dilation with factor k

$$T(x,y,z) = (kx, ky, kz) \text{ in } R3$$

18. Compression / Expansion in the x-direction with factor k in $\mathbb{R}^2$.

$$T(x, y) = (kx, y)$$

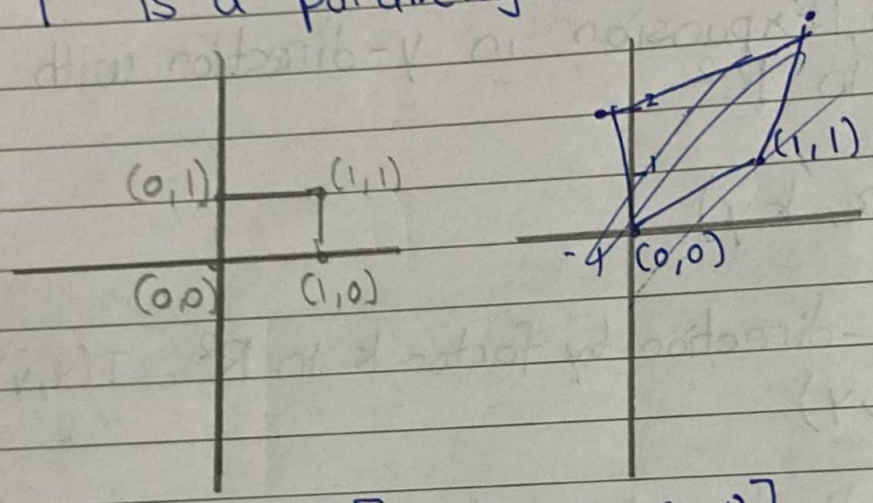
19. Compression / Expansion in y-direction with factor k in $\mathbb{R}^2$

$$T(x, y) = (x, ky)$$

20. Shear in x-direction by factor k in $\mathbb{R}^2$, $T(x,y) = T(x+ky, y)$

21. Shear in y-direction by factor k in $\mathbb{R}^2$, $T(x,y) = T(x, y+kx)$

1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(1,0) = (1,1)$, $T(0,1) = (-1,2)$ let S be a square whose corners are $(0,0), (1,0), (1,1), (0,1)$. Show that the image of S under T is a parallelogram

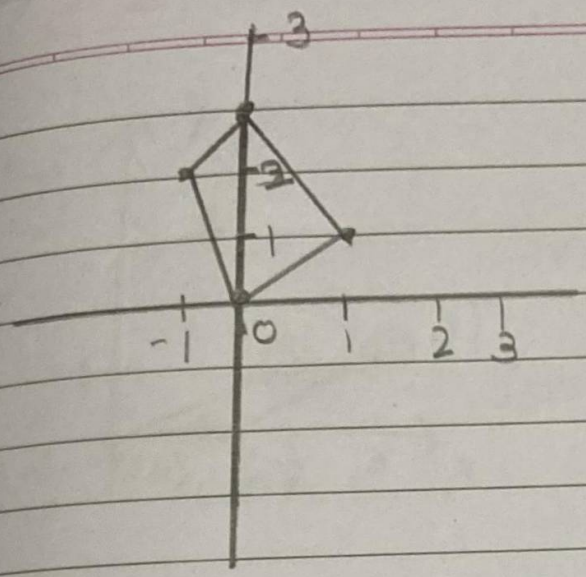


$$T(0,0) = T[0(1,0) + 0(0,1)]$$
$$= 0\,T(1,0) + 0\,T(0,1)$$
$$= 0\,(+1,1) + 0(-1,2)$$
$$= (0,0)$$

$$T(1,0) = (1,1)$$

$$T(0,1) = (-1,2)$$

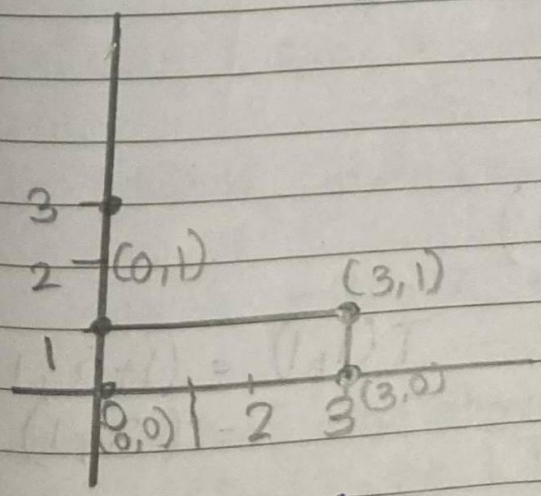
$$T(1,1) = T[1(0,1) + 1(1,0)]$$
$$= T(0,0) + T(1,0)$$
$$= (-1,2) + (1,1)$$
$$= (0,3)$$



2. Draw the image of the unit square with corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Under the following linear maps and explain the geometric of T on the rectangle mentioned above

i) $T(x,y) = (3x, y)$

$T(0,0) = (0,0)$
$T(1,0) = (3\times1, 0) = (3,0)$
$T(1,1) = (3,1)$
$T(0,1) = (0,1)$



Expansion in the x-direction with factor 3

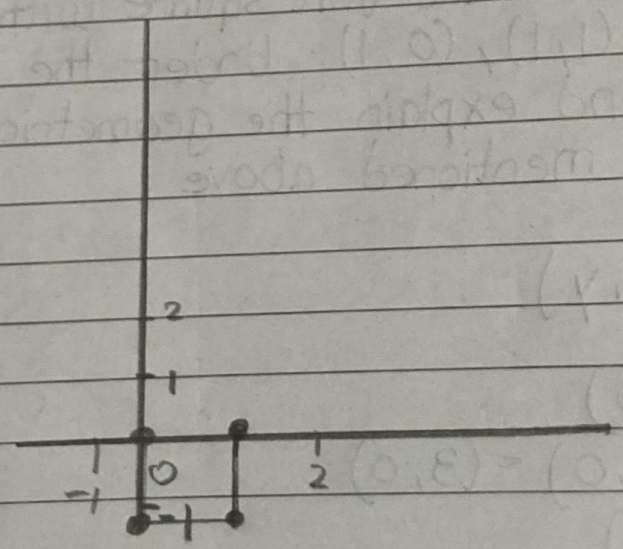
ii) $T(x, y) = (x, -y)$

$T(0,0) = (0,0)$

$T(1,0) = (01,0)$

$T(1,1) = (1,-1)$

$T(0,1) = (0,-1)$



Reflection ~~around~~ along X-axis

HW

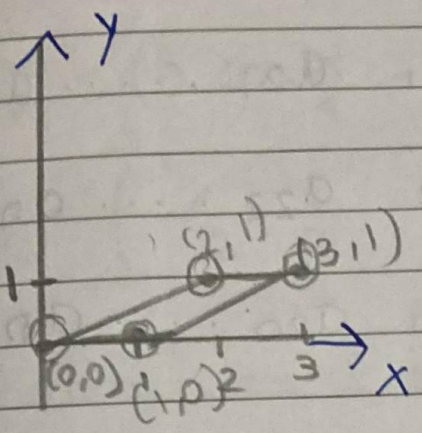
iii $T(x,y) = (x+2y, y)$

$T(0,0) = (0,0)$

$T(1,0) = (1+0, 0)$
$= (1,0)$

$T(1,1) = (1+2, 1)$
$= (3,1)$

$T(0,1) = (0+2, 1)$
$= (2,1)$



Shear in X direction by k factor (2)

Matrix Associated with linear map

Let T be a linear ~~operator~~ transformation from a vectorspace V into itself i.e. $T: V \to V$ and supports $S = \{u_1, u_2, \ldots, u_n\}$ is a basis of V
Now, $\{T(u_1), T(u_2) \ldots T(u_n)\}$ are vectors in V and so each is the linear combination of the vectors in the basis S say

$$T(u_1) = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$T(u_2) = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$

$$\vdots$$

$$T(u_n) = \cancel{a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n}$$

$$a_{n1}u_1 + a_{n2}u_2 + \cdots a_{nn}u_n$$

The transpose of the above matrix of coefficients denoted by $m_S(T)$ or $[T]_S$ is called the matrix represented by T relative to basis S or simply matrix T in basis S.

$$M_S(T) = [T]_S = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+y, x-y)$. Find the matrix associated with T wrt to the Standard basis

Standard basis of $\mathbb{R}^2$ is $B = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$

$$T(e_1) = T(1,0) = (1,1) = (1,0)+(0,1) = 1e_1 + 1e_2$$

$$T(e_2) = T(0,1) = (1,-1) = (1,0)-(0,1) = 1e_1 - 1e_2$$

(Write ~~normal~~ transpose form of matrix)

$$M_B(T) = M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

2. Find the matrix associated with T, where T is rotation by $\frac{\pi}{4}$ (Take origin)

We know that counterclockwise rotation about the origin through an angle θ

$T(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Take $\theta = \pi/4$

$$T(x,y) = \left(x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4},\ x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4}\right)$$

$$T(x,y) = \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}},\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) \qquad T: \mathbb{R}^2 \to \mathbb{R}^2$$

Standard basis of $\mathbb{R}^2$: $\{(1,0), (0,1)\}$

$$T(1,0) = T(e_1) = \left(\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(1,0) + \frac{1}{\sqrt{2}}(0,1)$$

$$= \frac{e_1}{\sqrt{2}} + \frac{e_2}{\sqrt{2}}$$

$$T(0,1) = T(e_2) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}(1,0) + \frac{1}{\sqrt{2}}(0,1)$$

$$= -\frac{e_1}{\sqrt{2}} + \frac{e_2}{\sqrt{2}}$$

(look at coefficients) (Transpose)

Ans:- $M = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ +1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Change of Basis

let T from $[T: V \to U]$ be the linear transformatio

$S' = \{u_1, u_2, \dots, u_m\}$ are basis of U

$S = \{v_1, v_2, \dots v_n\}$ are basis of V

$$T(v_1) = a_{11}u_1 + a_{12}u_2 + \dots + a_{m1}u_m$$
$$T(v_2) = a_{21}u_1 + a_{22}u_2 + \dots + a_{m2}u_m$$
$$\vdots$$
$$T(v_m) = a_{m1}u_1 + a_{m2}u_2 + \dots + a_{mn}u_n$$

$$M_S^{S'} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{n \times m}$$

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$. Defined as $T(x,y,z) = (x+2z, 3x-z)$
Find the matrix associated with T wrt standa
basis. Also verify that $M\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T(x,y,z)$

Standard basis of $\mathbb{R}^3 \to B = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,}$

Standard basis of $\mathbb{R}^2 \to B = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$

$T(1,0,0) = (1,3)$ (See Q)
$= (1,0) + 3(0,1) = 1e_1 + 3e_2$

$T(0,1,0) = (0,0) = 0(1,0) + 0(0,1)$
$= 0e_1 + 0e_2$

$T(0,0,1) = (2,-1) = 2(1,0) - 1(0,1)$
$= 2e_1 - 1e_2$

$$M = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

LHS =

$$\underset{2\times 3}{\begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} x+2z \\ 3x-z \end{bmatrix}$$

$= T(x,y,z)$

= RHS

4. Let $T: \overset{\text{from}}{\mathbb{R}^3} \to \overset{\text{to}}{\mathbb{R}^2}$ $T(x,y,z) = (x-2y, x+y-3z)$
Find the matrix associated with T wrt standard basis and verify that $M\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = T(1,3,-2)$

Standard basis of $\mathbb{R}^3 \to B = \{(1,0,0), (0,1,0), (0,0,1)\}$
Standard basis of $\mathbb{R}^2$ $B' = \{(1,0), (0,1)\}$

$T(1,0,0) = (1,1) = e_1 + e_2$
$T(0,1,0) = (-2,1) = -2e_1 + e_2$
$T(0,0,1) = (0,-3) = 0e_1 - 3e_2$

$$M = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix}$$

$$M \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = LHS$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 \\ 1+3+6 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

$$= T(x, y, z) = (\cancel{x}\, 1-6,\ 1+3+6)$$
$$T(1, 3, -2) = (-5, 10)$$
$$= RHS$$

5 let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, x+y, y)$
Find the matrix associated with T wrt basis $B = \{(1,1), (0,1)\}$ in $\mathbb{R}^2$ and standard basis in $\mathbb{R}^3$

Standard basis in $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$
Basis in $\mathbb{R}^2 = \{\underset{e_1}{(1,1)}, \underset{e_2}{(0,1)}\}$

$$T(\cancel{x,y}\ T(1,1) = e_1 = (1, 2, 1) = e_1 + 2e_2 + e_3$$
$$T(0,1) = e_2 = (0, 1, 1) = 0e_1 + e_2 + e_3$$

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left[(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1) \quad x e_1 + y e_2 + z e_3 \right]$$

6. For the following matrices given below Write the corresponding linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ also draw image of the rectangle with corner points $(0,0)$, $(2,0)$, $(2,1)$ and $(0,1)$. In each case explain the geometric effect of T on this rectangle

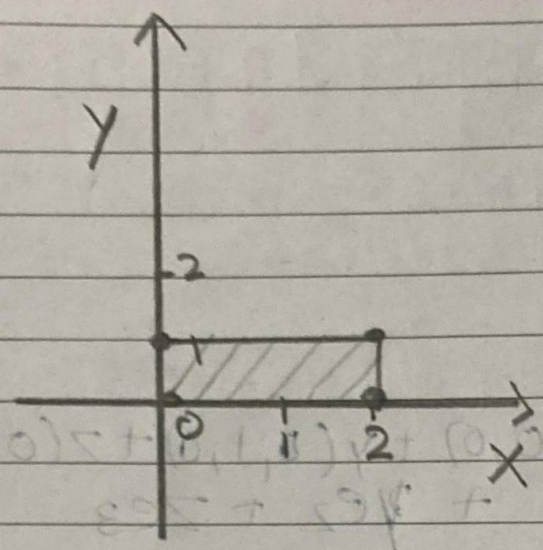
a) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \underline{M}$

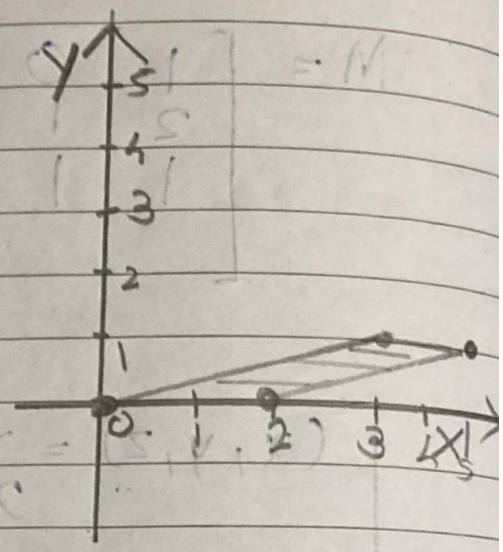
$$T(x,y) = m \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 3y \\ y \end{bmatrix}$$

$$= (x + 3y, y)$$

$T(0,0) = (0,0)$
$T(2,0) = (2,0)$
$T(2,1) = (5,1)$
$T(0,1) = (3,1)$





Shear in x-direction with factor 3

b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = M$

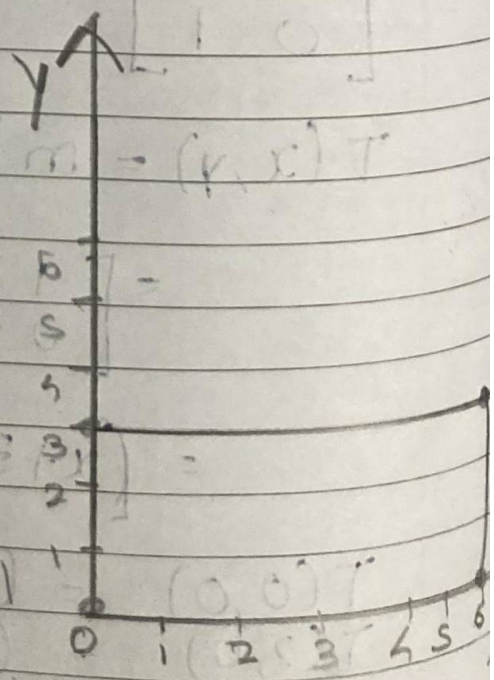
$M \begin{bmatrix} x \\ y \end{bmatrix} = T(x, y)$

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$

$T(x, y) = (3x, 3y)$

$T(0,0) = (0,0)$
$T(2,0) = = (6,0)$
$T(2,1) = = (6,3)$
$T(0,1) \quad (0,3)$

Draw initial



Dilation with factor 3
(Expansion)

Composition of Mapping

Consider two mapping

$f: A \to B$ & $g: B \to C$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

The composition of f & g denoted by gof is the mapping ~~gof~~ ~~is~~
$g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$

Ex-

let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$. Find formulas for defining the composition mapping : fog, gof, fof, gog

$(f \circ g)(x) = f[g(x)]$

$= f[2x - 3]$

$= (2x-3)^2 + 3(2x-3) + 1$

$= \cancel{4x^2 - 12x}\ 4x^2 - 12x + \cancel{9} + 6x - \cancel{9} + 1$

$(g \circ f)(x) = 4x^2 - 6x + 1$

$g[x^2 + 3x + 1] = 2[x^2 + 3x + 1] - 3$

$= 2x^2 + 6x + 2 - 3$

$(fof)(x) = f[x^2+3x+1]$

$= (x^2+3x+1)^2 + 3(x^2+3x+1) + 1$

$\cancel{(a^2+b+c)^2 = a^2+2a}$

$= (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1$

$= x^4 + 6x^3 + 14x^2 + 15x + 5$

$gog(x) = g[2x-3]$

$= 2(2x-3)-3$

$= 4x-6-3$

$= 4x-9$

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(x,y) = (x+y, x-y)$ and S (linear map): $\mathbb{R}^2 \to \mathbb{R}$ B defined by $S(a,b) = (a+b)$. Find $S \circ T$ (x,y)

$T: \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{S} \mathbb{R}$

$S \circ T \to R^2 \to R$

$T(x,y) = S[T(x,y)]$
$= S[(x+y),(x-y)]$

$= x+y+x-y$
$= 2x$

2. Define linear transformation $S: \mathbb{R}^2 \to M_{22}$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $S(a,b) = \begin{bmatrix} a+b & b \\ 0 & a-b \end{bmatrix}$ and $T(c,d)$

Find $S \circ T$
$= (2c+d, -d)$

$R_2 \to R_2 \to M_{22}$
$T \qquad S$
ST

$T \circ S = T[S](x,y)]$

$T[M_1]$

$S \circ T = S[T]$

$R^2 \xrightarrow{T} R^2 \xrightarrow{S} M_{2\times2}$

$S \circ T$ (left to right)

$S \circ T: R^2 \to M_{2\times2}$
$S[T] = (S \circ T)(x,y) = S[T(x,y)]$
$= S(2x+y, -y)$

$$= \begin{bmatrix} 2x+y-y & -y \\ 0 & 2x+y+y \end{bmatrix}$$

$$= \begin{bmatrix} 2x & -y \\ 0 & 2x+2y \end{bmatrix}$$

3. Let $T: R^3 \to R^2$, $T(x,y,z) = (\underbrace{x+2z}_{x}, \underbrace{3x-z}_{y})$
and $S: R^2 \to R^3$, $S(x,y) = (x, x+y, y)$

$$R^3 \xrightarrow{T} R^2 \xrightarrow{S} R^3 \qquad S \circ T \to R^3 \to R^3$$

Find $S \circ T$ and $T \circ S$. Write the matrices of T, S, $S \circ T$, $T \circ S$ wrt standard basis and verify the result using matrix method.

$$R^2 \to R^3 \to R^2$$

$$T \circ S : R^2 \to R^2$$

Find $S \circ T$

$\mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{aligned}(S \circ T)(x,y,z) &= S(T(x,y,z)) \\ &= S(\underbrace{x+2z}_{x}, \underbrace{3x-z}_{y}) \\ &= \\ &= S(x+2z, x+2z+3x-z, 3x-z) \\ &= S(x+2z, 4x+z, 3x-z)\end{aligned}$$

$T \circ S(x, y) = T(S(x, y))$

$= T\left(\frac{x}{x}, \frac{x+y}{y}, \frac{y}{z}\right)$

$= (x + 2y, 3x - y)$

For T $R^3 \to R^2$

Standard basis of $R^3 = \{ \underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)} \}$

$T(x, y, z) = (x + 2z; 3x - z)$

$T(\overset{(1,0,0)}{e_1}) = (1, 3) = 1e_1 + 3e_2$ (Write $e_1 e_2 \ldots$

$T(0,1,0) = (0, 0) = 0e_1 + 0e_2$ for after $\to$)

$T(0,0,1) = (2, -1) = 2e_1 - e_2$

$$M_T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

For S $R^2 \to R^3$

$S(x, y) = (x, x+y, y)$

Standard basis of $R^2 = \{ (1,0), (0,1) \}$

$S(1,0) = (1, 1, 0) = e_1 + e_2 + 0e_3$

$S(0,1) = (0, 1, 1) = 0e_1 + 1e_2 + 1e_3$

$$M_S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$R^3 \to R^3$

$S \circ T(x,y,z) = (x+2z, 4x+z, 3x-z)$

Standard basis of $R^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (1,4,3) = 1e_1 + 4e_2 + 3e_3$
$T(0,1,0) = (0,0,0) = 0e_1 + 0e_2 + 0e_3$
$T(0,0,1) = (2,1,-1) = 2e_1 + e_2 - e_3$

$$M_{S \circ T} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 0 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

$T \circ S(x,y) = (x+2y, 3x-y)$

$T(1,0) = (1,3) = e_1 + 3e_2$
$T(0,1) = (2,-1) = 2e_1 - e_2$

$$M_{T \circ S} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$\underline{M_S \cdot M_T}$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1\times1 + 0\times3 & 0 & 2 \\ 1\times1 + 1\times3 & 0 & 1\times2 - 1\times1 \\ 0\times1 + 1\times3 & 0 & 0-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 0 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

$M_T \cdot M_S$

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = M_{T \circ S}$$

~~Theorem~~
For any linear transformation T and S
$m(T \circ S] = m(T) \cdot m(S)$

~~$m(T \circ S$~~

$$M_{T \circ S} = M_T \cdot M_S \qquad (T_1 \text{ OF } T_2)$$

4. Let T_1 & T_2 represent linear maps on $\mathbb{R}^2$ If T_1 represents reflection about Y-axis and T_2 represents expansion in the direction of Y axis by a factor 2. Write the linear transformation ~~T_1~~ $T_1 \circ T_2$ & $T_2 \circ T_1$

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T_2 : R^2 \to R^2$
$T_1(x, y) = (-x, y)$
$T_2(x, 2y) = (x, 2y)$

$T_1 \circ T_2 \qquad R^2 \to R^2$

$T_1(T_2(x, y)) \qquad T_2(T_1(x, y))$

$T_1[\underset{x}{x}, \underset{y}{2y}] \qquad T_2[-x, y]$

$= (-x, 2y) \qquad = [-x, 2y]$

$(T_1 \circ T_2)(x,y) = T_1(T_2(x,y)) = T_1(x, 2y) = (-x, 2y)$

$T_2 \circ T_1 : R^2 \to R^2$

$(T_2 \circ T_1)(x,y) = T_2(T_1(x,y) = T_2(-x,y) = (-x, 2y)$

HW

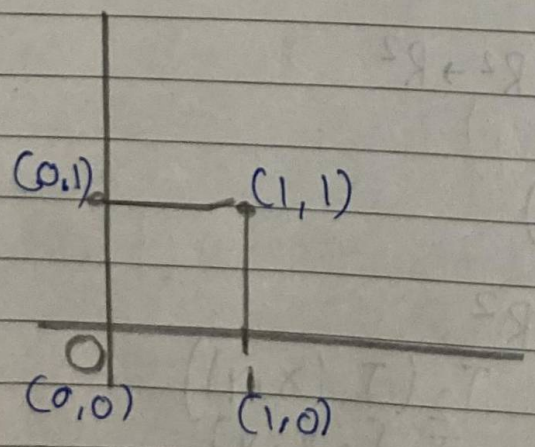
5. Draw the image of the unit square with corner points $(0,0), (1,0), (1,1), (0,1)$ under the map $T = T_1 \circ T_2$
where, T_1 is reflection about the line $x=y$
T_2 is expansion in the direction of y axis by factor of 3

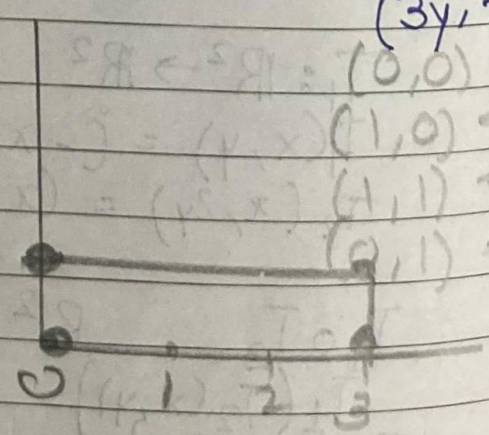
$T_1(x,y) = (y, x)$
$T_2(x,y) = (x, 3y)$

$T = T_1 \circ T_2 : R^2 \to R^2$

$(T_1 \circ T_2)(x,y) = T_1(T_2(x,y)) = T_1(x, 3y) = \cancel{(x, 3y)}$
$(3y, x)$

$(0,0) = (0,0)$
$(1,0) = (0,1)$
$(1,1) = (3,1)$
$(0,1) = (3,0)$





Use matrix method to verify $M_S M_T = M_{SOT}$ in example 1 of today's lecture

$T(1,0) = (1,1)$
$T(0,1) = (1,-1)$

$$\begin{bmatrix} x+y \\ x-y \end{bmatrix} \quad \begin{bmatrix} a+b \end{bmatrix}$$

$2 \times 1 \qquad 1 \times 1$

$$\begin{bmatrix} \cancel{ax+by+ax-by} \\ \cancel{2ax} \end{bmatrix} = \begin{bmatrix} \cancel{2ax} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a+b \end{bmatrix}$$

$$\begin{bmatrix} x+y+x-y \end{bmatrix} = \begin{bmatrix} 2x \end{bmatrix}$$

Tutorial 7

1. $T: R^2 \to R^2$, $T(3,1) = (1,2)$, $T(0,1) = (1,1)$, $T(1,0) = ?$

$\{(3,1),(0,1)\}$ basis of R^2

$(1,0) = a(3,1) + b(0,1)$

$(1,0) = (3a, a+b)$

$3a = 1$

$a = 1/3 \qquad a+b=0$

$b = -1/3$

$(1,0) = \left(\frac{1}{3}(3,1) - \frac{1}{3}(0,1)\right)$

$T(1,0) = T\left(\frac{1}{3}(3,1) - \frac{1}{3}(0,1)\right)$

$= \frac{1}{3}T(3,1) - \frac{1}{3}T(0,1)$

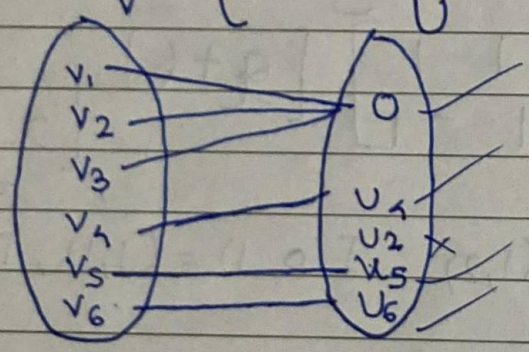
$= \frac{1}{3}(1,2) - \frac{1}{3}(1,1) = (0, 1/3)$

Kernel & Image of Linear Map

Def^n : Let $T : V \to U$ be a linear mapping

The Kernel of T ~~return~~ written as kerT is the set of elements in V that Map x Zero Vector (0) in U i.e.

$$\text{Ker}\cdot T = \{ v \in V \mid T(v) = 0 \} \subseteq V$$



The image (range) of T written as ImT is the set of image points in U i.e.

$$\text{Im}\, T = \{ u \in U \mid \exists v \in V \text{ st } T(v) = u \} \subseteq U$$

$\exists$: There is
st : such that

Theorem

Let $T : V \to U$ be a linear mapping then kernel T is subspace of V and image of T is subspace of U

Important

Rank and Nullity of a linear mapping
Def^n: let $T: V \to U$ be a linear mapping
The Rank of T is defined as the dimension of its image i.e. $\operatorname{Im} T$.

Nullity of T is defined as the dimension of its kernel i.e. $\ker T$

$\therefore$ Nullity of $T = \dim(\ker T)$

Rank of $T = \dim(\operatorname{Im} T)$

Theorem
Rank-Nullity Theorem

Let V be a finite dimension vectorspace and $T: V \to U$ be a linear mapping then
$\dim V = \dim(\operatorname{Im} T) + \dim(\ker T)$
OR
$\dim V$ = Rank of T + Nullity of T

(No Proof)

1. Let $T: \overset{V}{\mathbb{R}^3} \to \overset{U}{\mathbb{R}^3}$ be a projection of vector V into the xy plane. Verify Rank-Nullity Theorem

For any $v = (x,y,z) \in \mathbb{R}^3$, $T(x,y,z) = (x,y,0)$

$$\ker T = \{ (x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0) \}$$

$$= \{ (x,y,z) \mid (x,y,0) = (0,0,0) \}$$

$$= \{ (x,y,z) \mid x=0, y=0, z \in \mathbb{R} \}$$

$$= \{ (0,0,z) \mid z \in \mathbb{R} \}$$

$$= \{ z(0,0,1) \mid z \in \mathbb{R} \}$$

$B = \{(0,0,1)\}$ is linearly independent & linear span of $\ker T$

$\therefore B = \{(0,0,1)\}$ is Basis of $\ker T$

$\dim \ker T = 1$
$= \text{Nullity of } T$

$$\operatorname{Im} T = \{ T(x,y,z) \mid x,y,z \in \mathbb{R} \}$$

$$= \{ (x,y,0) \mid x,y \in \mathbb{R} \}$$

$\{x(1,0,0) + y(0,1,0) \mid x, y \in \mathbb{R}\}$

$C = \{(1,0,0), (0,1,0)\}$ are linearly independent & linear span of $\operatorname{Im} T$

C is Basis of $\operatorname{Im} T$

$\dim(\operatorname{Im} T) = 2 =$ Rank of T

Here $V = \mathbb{R}^3$, $\dim(\mathbb{R}^3) = 3$

$\therefore \dim(\operatorname{Im} T) + \dim(\ker T) =$ Rank of T + Nullity of T

$= 2 + 1$

$= 3$

$= \dim(\mathbb{R}^3)$

Hence, Rank-Nullity Theorem is verified

2. Verify Rank-Nullity Theorem for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}$, $T(x,y,z) = 3x - 2y + z$

$\ker T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid 3x - 2y + z = 0\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid z = -3x + 2y\}$

(ker T)

$= \{ (x, y, -3x + 2y) \mid x, y \in \mathbb{R} \}$

$= \{ (x, 0, -3x) + (0, y, 2y) \mid x, y \in \mathbb{R} \}$

$= \{ x(1, 0, -3) + y(0, 1, 2) \mid x, y \in \mathbb{R} \}$

$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix}$ (Row-Echelon)

Rank = 2
Dim = 2

$B = \{ (1, 0, -3), (0, 1, 2) \}$ are linearly independent & linearly span of ker T

B is basis of ker T

dim (ker T) = 2 = Nullity of T

$\text{Im } T = \{ T(x, y, z) \mid x, y, z \in \mathbb{R} \}$

$= \{ 3x - 2y + z \mid x, y, z \in \mathbb{R} \}$

$\text{Im } T = \mathbb{R}$

dim (Im T) = dim ($\mathbb{R}$) = 1 = Rank of T

$V = \mathbb{R}^3$

dim ($\mathbb{R}^3$) = 3

= dim (ker T) + dim (Im T)

= Nullity of T + Rank of T

$= 2+1$
$= \dim V$

Hence, Rank-Nullity Theorem is verified

3 Find the range and kernel of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (x, x+y, y)$
Also, verify Rank-Nullity Theorem

$V = \mathbb{R}^2$
$U = \mathbb{R}^3$

$\ker T = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0,0)\}$
$= \{(x,y) \in \mathbb{R}^2 \mid (x, x+y, y) = (0,0,0)\}$
$= \{(x,y) \in \mathbb{R}^2 \mid x=0, y=0\}$

$\ker T = \{(0,0)\}$

$\dim(\ker T) = 0$
$=$ Nullity of T

$\operatorname{Im} T = \{T(x,y) \mid x,y \in \mathbb{R}\}$
$= \{(x, x+y, y) \mid x,y \in \mathbb{R}\}$
$=$ Take $x = u$, $y = v$
$= \{(u, u+v, v) \mid u,v \in \mathbb{R}\}$

$= \{ x(1,1,0) + y(0,1,1) \mid x, y \in \mathbb{R} \}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$B = \{ (1,1,0), (0,1,1) \}$ is linearly independent and linear span of $\operatorname{Im} T$

$\dim(\operatorname{Im} T) = 2 =$ Rank of T

$V = \mathbb{R}^2$, $\dim V = \dim(\mathbb{R}^2) = 2$

Rank of T + Nullity of $T = 2 + 0 = 2$
$= \dim V$

Hence, verified

4. Find the range and kernel of given linear transformation. Also find Rank and Nullity of given transformation. Hence, verify rank nullity theorem.

$T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x+z, x+y+2z, 2x+y$

$V \quad U$

$\operatorname{Ker} T = \{ (x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = (0, 0$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid (x+z, x+y+2z, 2x+y+3z) = (0,0,0) \}$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid x+z=0,\ x+y+2z=0,\ 2x+y+3z=0 \}$

$A|B$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right)$$

$= R_2 \to R_2 - R_1 ;\ R_3 \to R_3 - 2R_1$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$= R_3 \to R_3 - R_2$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$f = n - r$
$= 3 - 2$
$= 1$

$A(r) = 2 = A|B(r)$

$x + z = 0$
$y + z = 0$

$x = -z$
$y = -z$

$\ker T =$
$= \{ (x,y,z) \in \mathbb{R}^3 \mid x = -z,\ y = -z,\ z \in \mathbb{R} \}$
$= \{ (-z,-z,z) \mid z \in \mathbb{R} \}$

Stop for Kernel T

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Date

$$\left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right)$$

$R_2 \to R_2 - \frac{3}{5} R_1 \qquad R_3 \to R_3 - \frac{7}{5} R_1$

$$= \left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & -11/5 & 1/5 & -3/5 \end{array}\right)$$

$R_3 \to R_3 + \frac{1}{11} R_2 \Rightarrow \left(\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$5x + \frac{3z}{11} + \frac{9}{11} + 7z = 4$

$5x + \frac{80}{11} z = 4 - \frac{9}{11}$

$5x + \frac{80}{11} z = \frac{35}{11}$

$5x = -\frac{80z}{11} + \frac{35}{11}$

$x = -\frac{16z}{11} + \frac{7}{11}$

$11y = z + 3$

$\frac{121}{5} y = \frac{11}{5} z + \frac{33}{5}$

$y = \frac{z}{11} + \frac{3}{11}$

$5x + \frac{3z}{11} + \frac{9}{11} + 7$

$5x = \frac{-42}{11} - 3$

$y = -5z$

$5z = -y \qquad x - 4z$

$x - 5 \qquad x$

$x + 6z = 0$

$x = -6z$

$= \{ z(-1,-1,1) \mid z \in \mathbb{R} \}$

$B = \{ (-1,-1,1) \}$

Linearly independent, linear span of kerT

$\therefore$ B is basis of kerT

$\text{Dim}(\ker T) = 1$

$= \text{Nullity of } T$

$\text{Im}\, T = \{ T(x,y,z) \mid x,y,z \in \mathbb{R} \}$

$= \{ (x+z,\ x+y+2z,\ 2x+y+3z) \mid x,y,z \in \mathbb{R} \}$

let's see 2 ways

$= \{ x(1,1,2) + y(0,1,1) + z(1,2,3) \mid x,y,z \in \mathbb{R} \}$

$$\cancel{A/B_2} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$= R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$R_3 - R_2$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore B = \{ (1,1,2), (0,1,1) \}$

B is linearly independent & linear span of Im T

$\dim(\text{Im}T) = 2 = \text{Rank of } T$

$V = \mathbb{R}^3$, $\dim V = \dim(\mathbb{R}^3) = 3$

Rank of T + Nullity of T $= 2+1 = 3$
$= \dim V$

Hence, verified

2ND WAY

$\text{Im}T = \{(x+z, x+y+2z, 2x+y+3z) \mid x, y, z \in \mathbb{R}\}$

$x + z = r$

$x + y + 2z = s$

$2x + y + 3z = x + z + x + y + 2z$
$= r + s$

$= \{(r, s, r+s) \mid r, s \in \mathbb{R}\}$

$= \{r(1,0,1) + s(0,1,1)\}$

$B = \{(1,0,1), (0,1,1)\}$

B is linearly independent & L Span of Im T

$\dim(\text{Im}T) = 2 = \text{Rank of } T$

$V = \mathbb{R}3$, $\dim V = \dim(\mathbb{R}^3) = 3$

Rank of T + Nullity of T $= 2+1 = 3$
$= \dim V$

5. Verify Rank Nullity Theorem $T: \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(e_1) = e_1 + e_3$, $T(e_2) = e_2 + e_3$, $T(e_3) = -e_3$

$(1,0,1)$ $(0,1,1)$ $(0,0,-1)$

For any $(x,y,z) \in \mathbb{R}^3$

$(x,y,z) = xe_1 + ye_2 + ze_3$

$$T(x,y,z) = T[xe_1 + ye_2 + ze_3]$$
$$= T(xe_1) + T(ye_2) + T(ze_3)$$
$$= xT(e_1) + yT(e_2) + zT(e_3)$$
$$= x(1,0,1) + y(0,1,1) + z(0,0,-1)$$

$T(x,y,z) = (x, y, x+y-z)$

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$$
$$= \{(x,y,z \in \mathbb{R}^3 \mid (x,y,x+y-z) = (0,0,0)\}$$
$$= \{(x,y,z \in \mathbb{R}^3 \mid x=0, y=0, x+y=z\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x=0, y=0, z=0\}$$
$$= \{(0,0,0)\}$$

$\ker T = \{(0,0,0)\}$

$\dim(\ker T) = 0$

$= \text{Nullity of } T$

$$\operatorname{Im} T = \{T(x,y,z) \mid x,y,z \in \mathbb{R}\}$$
$$= \{(x, y, x+y-z) \mid x,y,z \in \mathbb{R}\}$$
$$= \{x(1,0,1) + y(0,1,1) + z(0,0,-1) \mid x,y,z \in \mathbb{R}\}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$B = \{(1,0,1), (0,1,1), (0,0,-1)\}$ are linearly independent & linear span of $\operatorname{Im} T$

where B is basis of $\operatorname{Im} T$

$\therefore$ Rank of $T = 3$

~~Im T~~

$V = \mathbb{R}^3$, $\dim V = \dim(\mathbb{R}^3) = 3$

Rank of T + Nullity of $T = 3 + 0 = 3$
$= \dim V$

Hence, verified Rank-Nullity Theorem

$x = r$, $y = s$, $x + y - z = t$

$\operatorname{Im} T = \{(r, s, t) \mid r, s, t \in \mathbb{R}\}$
$= \mathbb{R}^3$

$\dim(\operatorname{Im} T) =$

HW

5. Verify Rank-Nullity Theorem for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x,y,z) = (x-y+z, y-z, 2x-5y+5z)$

$V = \mathbb{R}^3$
$U = \mathbb{R}^3$

$\text{Ker } T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$
$= \{(x,y,z) \in \mathbb{R}^3 \mid (x-y+z, y-z, 2x-5y+5z) = (0,0,0)\}$

$$A|B = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & -5 & 5 & 0 \end{array}\right)$$

$R_3 \to R_3 - 2R_1$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$R_3 \to R_3 + 3R_1$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$f = n - r$
$= 3 - 2 = 1$

$\{z(0,1,1) \mid z \in \mathbb{R}\}$

$R(A) = 2$
$R(A/B) = 2$

$x - y + z = 0$
$y - z = 0 \Rightarrow y = z$

$x - z + z = 0 \quad x = 0$

$B = \{(0,1,1)\}$ is linearly independent & linear span of $\ker T$

B is basis of $\ker T$

$\dim(\ker T) = 1$

$= \text{Nullity of } T$

$\operatorname{Im} T = \{T(x,y,z) \mid x,y,z \in R\}$

$= \{(x-y+z, y-z, 2x-5y+5z) \mid x,y,z \in R\}$

$= x(1,0,2) + y(-1,1,-5) + z(1,-1,5) \mid x,y,z \in R\}$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 2 & -5 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & -5 \\ 1 & -5 & 5 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 + R_2 \to R_3$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_2 + R_1 \to R_2$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$V = R^3$

$\dim(R^3) = 3$

$\dim(\ker T) + \dim(\operatorname{Im} T) =$

$= \text{Nullity of } T + \text{Rank of } T$

$= 1 + 2$

$= 3$

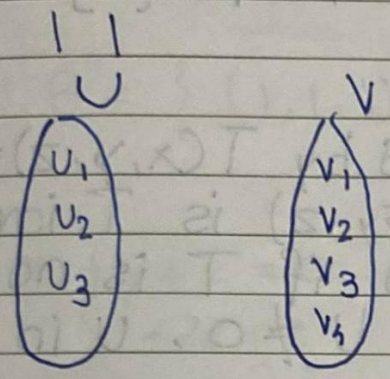
$\therefore$ Rank Nullity Theorem is verified

$\{(1,0,2), (0,1,-3)\}$ is linearly independent & linear span of $\operatorname{Im} T$

B is basis of $\operatorname{Im} T$

$\dim(\operatorname{Im} T) = 2 = \text{Rank of } T$

Inverse Linear Transformation



— One One / Injective

→ Onto / Surjective

Pre image not there

Both / Bijective

Let $T: V \to V$ be a linear transformation. ~~It~~ If T is ~~set~~ said to be Invertible. If it has an inverse

$$A \cdot A^{-1} = I$$
$$2 \times \tfrac{1}{2} = 1$$

There exists T^{-1} from $V \to V$ such that $T \circ T^{-1} = I = T^{-1} \circ T$ on otherhand T is invertible as a mapping if ~~in~~ T is both one-to-one and onto

Theorem

Let T be a linear transformation on finite dimensional vector space V then the following 4 conditions are equivalent

i) T is nonsingular i.e. $\ker T = \{0\}$
ii) T is one-to-one
iii) T is onto mapping
iv) T is invertible

The linear transformation T is said to be nonsingular if and only if $\ker T = \{0\}$ otherwise T is singular

1. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y+z, x+2y-z, 3x+5y-z)$ is T invertible? Justify your ans / Find if T is non singular. If not find $U \neq 0$, U in $\mathbb{R}^3$ such that $T(U) = 0$

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 / (x+y+z, x+2y-z, 3x+5y-z) = (0,0,0)\}$$

$x+y+z=0$
$x+2y-z=0$
$3x+5y-z=0$

$A|B =$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 3 & 5 & -1 & 0 \end{array}\right)$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 3R_1$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right)$$

$R_3 \to R_3 - 2R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

X $\therefore B = \{(1,1,1), (0,1,-2)\}$ is linearly independent and linear span for $\ker T$

X $B = \{(1,1,1), (0,1,-2)\}$ is a basis of $\ker T$

$$x + y + z = 0$$
$$y - 2z = 0$$
$$y = 2z$$
$$x = -3z$$

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 \mid x = -3z,\ y = 2z,\ z \in \mathbb{R}\}$$
$$= \{(-3z, 2z, z) \mid z \in \mathbb{R} \neq \{0\}$$

$\therefore$ $\ker T$ is singular

$\therefore$ T is $\boxed{\text{not}}$ invertible
T^{-1} does not exist

Take $u = (-3, 2, 1) \neq 0$

$$T(-3,2,1) = (-3+2+1, -3+4-1, -9+10-1)$$
$$= (0, 0, 0)$$

(See Same $T: V \to V$)

2. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x,y,z) = (x+y-2z,\ x+2y+z,\ 2x+2y-3z)$. Find if T is non singular

$$\ker T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid (x+y-2z,\ x+2y+z,\ 2x+2y-3z) = (0,0,0)\}$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{r} x+y-2z=0 \\ x+2y+z=0 \\ 2x+2y-3z=0 \end{array}\right\}$$

$A|B =$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right)$$

$R_2 \to R_2 - R_1$; $R_2 \to R_2 - 2R_1$

$$= \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Rank of A = Rank of $A|B = 2$

$A(r)$ = no of var

$\therefore$ Trivial sol^n

$x + y - 2z = 0$ — ①

$y + 3z = 0$ — ②

$z = 0$ — ③

from ③ to ②

$y + 3 \times 0 = 0$

$y = 0$

From ② to ①

$x = 0$

$\ker T = \{(0,0,0)\}$

T is non singular

T is invertible

We got to find invertible

3. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined By $T(x,y) = (2x+y, 3x-5y)$ is invertible. Find T inverse

$$\ker T = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0)\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid (2x+y, 3x-5y) = (0,0)\}$$
$$= \{(x,y \in \mathbb{R}^2 \mid \begin{matrix} 2x+y=0 \\ 3x-5y=0 \end{matrix}\}$$

$A|B =$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & -5 & 0 \end{array}\right)$$

$R_2 \to R_2 - \frac{3}{2} R_1$

$$= \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -13/2 & 0 \end{array}\right)$$

$x = 0$
$y = 0$
Rank is 2
$A(r) = A|B(r) = 2 =$ no of var
$\therefore$ We get trivial sol^n
T is non-singular
T is invertible $\Rightarrow T^{-1}$ Exists

$T(x, y) = (2x+y, 3x-5y)$

Consider $T(x,y) = (u,v)$

$(x, y) = T^{-1}(u,v)$ — X

(x,y) in terms of (u,v)

We have to find (x,y) in terms of (u,v)

$T(x,y) = (2x+y, 3x-5y) = (u,v)$

$2x+y = u$ — ①

$3x-5y = v$ — ②

Multiply eq ① by 5

Add ② & ③

$10x + \cancel{5y} = 5u$ - ③

$3x - \cancel{5y} = v$

$13x = 5u+v$

$x = \frac{5u+v}{13}$

Sub value of x in ①

$2 \times \left(\frac{5u+v}{13}\right) + y = u$

$y = \frac{13u}{13} - \frac{10u-2v}{13}$

$y = \frac{3u-2v}{13}$

From X

$T^{-1}(u,v) = (x,y)$

$T^{-1}(u,v) = \left(\frac{5u+v}{13}, \frac{3u-2v}{13}\right)$

(sub nos for ref)

$\left(T^{-1}(1,1) = \frac{6}{13}, \frac{1}{13}\right)$

$T(x,y) = 2\times\frac{6}{13}+1, \frac{3\times 6}{13} - \frac{5\times 1}{13})$

$= (1,1)$

HW ~~Show that~~

4. Let F be a linear operator on $\mathbb{R}^2$ defined by $F(x,y) = (2x+y, 3x+2y)$ Show F is invertible Also Find F^{-1}

$\ker F = \{(x,y,z) \mid F(x,y,z) = (0,0,0)\}$

$= \{(x,y) \mid (2x+y, 3x+2y) = (0,0)\}$

$A|B = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 2 & 0 \end{array}\right)$

$R_2 \to R_2 - \frac{3}{2}R_1$

$= \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1/2 & 0 \end{array}\right)$

$y = 0$
$x = 0$ from ①
Trivial sol^n

$\therefore$ B is $\{(0,0)\}$ is non-singular

It is invertible

F^{-1} exists

$$F(x,y) = (2x+y, 3x+2y)$$
$$F(x,y) = (u,v)$$
$$(x,y) = F^{-1}(u,v)$$

$$(2x+y, 3x+2y) = (u,v)$$

$$2x+y = u \quad -①$$
$$3x+2y = v \quad -②$$

$\times 2$ in ①

$$2x + 2y = 2u$$
$$3x + 2y = v$$
$$(-) \quad (-) \quad (-)$$
$$x = 2u - v$$

$$4u - 2v + y = u$$
$$y = 2v - 3u$$
$$y = -3u + 2v$$

Sub (x,y) in ①

$$4u - 2v - 3u + 2v = u$$

$$F^{-1}(u,v) = (2u-v, -3u+2v)$$
$$F^{-1}(1,1) = (2-1, -3+2)$$
$$= (1, -1)$$

$(2\times 1+1,\ 3\times 1-2y)$
$(3,1)$

5. Find the linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ for which $T(1,2)=(2,3)$ and $T(0,1)=(1,4)$. Find the formula for $T(x,y)$. Is T invertible? If so, find T^{-1}

$T(0,1)=(1,4)$
$T(1,2)=(2,3)$

$B=\{(1,2),(0,1)\}$ basis of $\mathbb{R}^2$

For $(a,y)\in\mathbb{R}^2$

$(x,y) = a(1,2)+b(0,1)$
$(x,y) = (a,\ 2a+b)$

$a=x \qquad b=y-2a$
$\qquad\qquad b=y-2x$

$(x,y) = (x)(1,2)+(y-2x)(0,1)$
$= x$

$T(x,y) = T[x(1,2)+(y-2x)(0,1)]$
$= T[x(1,2)]+T[(y-2x)(0,1)]$
$= xT(1,2)+(y-2x)T(0,1)$
$= x(2,3)+(y-2x)(1,4)$
$= (2x+y-2x,\ 3x+4y-8x)$

$T(x,y) = (y, 4y-5x)$

$\ker T = \{ (x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0) \}$

$= \{ (x,y) \in \mathbb{R}^2 \mid (y, 4y-5x) = (0,0) \}$

$y = 0 \quad 4y - 5x = 0$

$= \{ (x,y) \in \mathbb{R}^2 \mid x = 0, y = 0 \}$

$= \{ (0,0) \in \mathbb{R}^2 \}$

$= \{ (0,0) \}$

$B = \{ (0,0) \}^T$ is non singular

So it T is invertible

T^{-1} is existing

$T(x,y) = (u,v)$

$(x,y) = T^{-1}(u,v) \longrightarrow (*)$

Write x, y in terms u, v

$(y, 4y-5x) = (u,v)$

$y = u \quad - (1)$

$4y - 5x = v \quad - (2)$

Sub y in (2)

$4u - 5x = v$

$4u - v = 5x$

$\frac{4u - v}{5} = x$

From (*)

$T^{-1}(u, v) = \left(\frac{4u - v}{5}, u\right)$

Imp

6. Show that $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$ is invertible. Find the matrix associated with T^{-1} wrt to the basis $B = \{(1,1,1), (1,1,0), (1,0,0)\}$

$\text{Ker}T = \{(x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = (0, 0, 0)\}$

$= \{(x, y, z) \in \mathbb{R}^3 \mid (2y + z, x - 4y, 3x) = (0, 0, 0)\}$

$= \{(x, y, z) \in \mathbb{R}^3 \mid 2y + z = 0, x - 4y = 0, 3x = 0\}$

$= \{(x, y, z) \in \mathbb{R}^3 \mid 2y + z = 0, x - 4y = 0, x = 0\}$

$= \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, y = 0, z = 0\}$

$\text{Ker}T = \{(0, 0, 0)\}$

T is non-singular
T is invertible
T^{-1} exists

$T(x, y, z) = (u, v, w)$

$(x, y, z) = T^{-1}(u, v, w)$ — (*)

Write x, y, z in terms u, v, w

$(2y+z,\ x-4y,\ 3x) = T^{-1}(u, v, w)$

$2y + z = u$ — ①
$x - 4y = v$ — ②
$3x = w$ — ③

$$\boxed{x = \frac{w}{3}}$$

From ③ sub in ②

$\frac{w}{3} - 4y = v$

$\frac{w - v}{3} = 4y$

$\frac{w - 3v}{12} = y$

$2\left[\frac{w-3v}{12}\right] + z = u$

$\frac{w - 3v}{6} + z = u$

$z = u - \frac{w + 3v}{6}$

$z = \frac{6u - w + 3v}{6}$

From ⊛

$$T^{-1}(u, v, w) = \left(\frac{w}{3}, \frac{w-3v}{12}, \frac{6u-w+3v}{6} \right)$$

$$B = \{\underset{r}{(1,1,1)}, \underset{s}{(1,1,0)}, \underset{t}{(1,0,0)}\}$$

$$T^{-1}(r) = T^{-1}(1,1,1)$$

$$= \left(\frac{1}{3}, \frac{-1}{6}, \frac{4}{3} \right)$$

$$T^{-1}(s) = T^{-1}(1,1,0)$$

$$= \left(0, \frac{-1}{4}, \frac{3}{2} \right)$$

$$T^{-1}(t) = T^{-1}(1,0,0)$$

$$= (0, 0, 1)$$

$$(x, y, z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$$

$$(x, y, z) = (a+b+c, a+b, a)$$

$$a+b+c = x$$
$$a+b = y$$
$$a = z$$

$$\left(\tfrac{1}{3}, \tfrac{-1}{6}, \tfrac{4}{3} \right) = \frac{4}{3} r - \frac{3}{2} s + \frac{1}{2} t$$

$$\left(0, \frac{-1}{4}, \frac{3}{2} \right) = \frac{3}{2} r - \frac{7}{4} s + \frac{1}{4} t$$

$(0,0,1) = 1r - 1s + 0t$

$$M_T^{-1} \begin{bmatrix} 4/3 & 3/2 & 1 \\ -3/2 & -7/4 & -1 \\ 1/2 & 1/4 & 0 \end{bmatrix}$$

Eigen values and Eigenvectors of the matrices

let A be a square matrix of order $n \times n$ lambda is said to be eigenvalue of A if there exists a non-zero vector X such that $AX = \lambda X$ and X is called the eigen vector corresponding to eigenvalue (λ)

$A_{n \times n} X_{n \times 1}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$(AX - \lambda X) = 0$
$(A - \lambda I)X = 0$

$\boxed{\det(A - \lambda I) = 0}$ is called characteristic polynomial

$\det(A - \lambda I) = 0$ is characteristic equation of A
The roots of characteristic eqn are called characteristic roots or eigenvalue of A and X is called characteristic vector corresponding eigen vector

to eigen value λ

Theorem:- Cayley Hamilton Theorem

Every square matrix satisfies its characteristic equation

1. Find the characteristic equation of A given below:-

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \left| \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \right|$$

$$\det = (3-\lambda)(1-\lambda) - 8$$

$$\Rightarrow 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

$$(X) \quad \lambda^2 - 5\lambda + \lambda - 5 = 0$$

$$\lambda(\lambda - 5) + 1(\lambda - 5) = 0$$

$$(\lambda + 1)(\lambda - 5) = 0$$

NOTE:

For 2x2 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Characteristic Eq^n of A is

$\lambda^2 -$ [Sum of diagonal elements] $\lambda +$ Det

$$\lambda^2 - 4\lambda + (-5) = 0$$

2. Find the characteristic eq^n of matrix given below and verify that it satisfies Cayley Hamilton Theorem

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & -3 \\ -2 & 1 & 0 \end{bmatrix}$$

Characteristic eq^n

$$\det (A - \lambda I) = 0$$

$$\det \left[\begin{Bmatrix} 1 & 1 & 3 \\ 1 & 0 & -3 \\ -2 & 1 & 0 \end{Bmatrix} - \lambda \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \right]$$

$$= \left| \begin{Bmatrix} 1-\lambda & 1 & 3 \\ 1 & -\lambda & -3 \\ -2 & 1 & -\lambda \end{Bmatrix} \right|$$

$$= (1-\lambda)(\lambda^2+3) - 1(-\lambda-6) + 3(1-2\lambda)$$

$$= (\lambda^2+3) - (\lambda^3-3\lambda)) - \lambda + 6 + 3 - 6\lambda$$

$$= -\lambda^3 + \lambda^2 + 3 + 3\lambda - \lambda + 6 + 3 - 6\lambda$$

$$= -\lambda^3 + \lambda^2 - 8\lambda + 12$$

For 3x3 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\lambda^3 - [\text{Sum of diagonal elements}]\lambda^2 + \left[\begin{array}{c}\text{Sum of Minors} \\ \text{of diagonal} \\ \text{elements}\end{array}\right]\lambda - \det(A) = 0$$

$$\cancel{\lambda^3 - \lambda^2 +}$$

$$\lambda^3 - [a_{11} + a_{22} + a_{33}]\lambda^2 + \left\{ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right\}\lambda$$

$$- \det A = 0$$

$$\lambda^3 - \lambda^2[1] + ([3+6-1]\lambda) - 12 = 0$$

$$\lambda^3 - \lambda^2 + 8\lambda - 12 = 0$$

$\therefore$ By Cayley Hamilton Theorem, the characteristic eqn A satisfies its characteristic eqn

i.e.

$$A^3 - A^2 + 8A - 12I = 0$$

$$LHS = A^3 - A^2 + 8A - 12I$$

$$= \begin{bmatrix} 0 & -4 & -24 \\ -1 & 10 & 27 \\ 15 & -10 & 3 \end{bmatrix} - \begin{bmatrix} -4 & 4 & 0 \\ 7 & -2 & 3 \\ -1 & -2 & -9 \end{bmatrix}$$

$$+ \begin{bmatrix} 8 & 8 & 24 \\ 8 & 0 & -24 \\ -16 & 8 & 0 \end{bmatrix} - \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= RHS$$

Hence, proved

3. Verify Cayley Hamilton theorem for matrix A and hence find A^{-1} & A^4 for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\cancel{\lambda^3 + \lambda^2}$$

$$\lambda^3 - (-1)\lambda^2 + \left[-3 - 10 - 5\right]\lambda - \det(A) - 40 = 0$$

$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

By Cayley Hamilton Theorem

$$LHS = A^3 + A^2 - 18A - 40I = 0$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - \begin{bmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{bmatrix}$$

$$- \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= RHS$

Hence Verified

To find A^{-1}

We know that $A^3 + A^2 - 18A - 40I = 0$

Multiply both sides by A^{-1}

$$A^2 + A - 18I - 40A^{-1} = 0$$

$$40A^{-1} = A^2 + A - 18I$$

$$A^{-1} = \frac{1}{40}\left[A^2 + A - 18I\right]$$

$$= \frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} -3/40 & 1/8 & 11/40 \\ 7/20 & -1/4 & 1/20 \\ 1/8 & 1/8 & -1/8 \end{bmatrix}$$

We kT $A^3 + A^2 - 18A - 40I = 0$

Multiply by A on bs

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$A^4 = -A^3 + 18A^2 + 40A$$

$$\begin{bmatrix} -44 & -33 & -46 \\ -24 & -13 & -74 \\ -52 & -14 & -8 \end{bmatrix} + \begin{bmatrix} 252 & 54 & \\ 216 & 162 & \\ 36 & 72 & 25 \end{bmatrix}$$

$$+\begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 40 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

4. Verify Cayley Hamilton Theorem for following matrices. Hence, find

A^{-1}, A^4

1. $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ ~~3~~ $\cancel{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}}$

2. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ 3 $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$

1. $\det(A - \lambda I) = 0$

Equation or the eigen equation

$+\lambda^3 -$ (sum of diagonal elements) $\lambda^2 +$ (Minor sum of diag elements) $\lambda - \det(A) = 0$

$\lambda^3 - 16\lambda^2 + (-28 + 50 + 13)\lambda - 50 = 0$

$\lambda^3 - 16\lambda^2 + 35\lambda - 50 = 0$

By CHT

A must satisfy the characteristic eqn

$$A^3 - 16A^2 + 55A - 50I = 0$$

$$\begin{bmatrix} 449 & 119 & 491 \\ 566 & 171 & 644 \\ 754 & 324 & 986 \end{bmatrix} - \begin{bmatrix} 784 & 64 & 656 \\ 896 & 176 & 864 \\ 864 & 864 & 1376 \end{bmatrix}$$

$$+ \begin{bmatrix} 385 & -55 & 165 \\ 330 & 55 & 220 \\ 110 & 220 & 440 \end{bmatrix} - \begin{bmatrix} 273 & -39 & 117 \\ 234 & 39 & 156 \\ 78 & 156 & 312 \end{bmatrix} - \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

$$= \begin{bmatrix} -2562 & 16 & -48 \\ -96 & -2466 & -64 \\ -32 & -64 & -2578 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, CHT verified

On both sides div by A

$$A^2 - 16A + 55I - 50A^{-1} = 0$$

$$\frac{1}{50}\left(A^2 - 16A + 55I\right) = A^{-1}$$

$$\frac{1}{50}\left(\begin{bmatrix} 49 & 4 & 41 \\ 56 & 11 & 54 \\ 54 & 34 & 86 \end{bmatrix} - \begin{bmatrix} 112 & -16 & 48 \\ 96 & 16 & 64 \\ 32 & 64 & 128 \end{bmatrix} + \begin{bmatrix} 55 & 0 & \\ 0 & 55 & \\ 0 & 0 & \end{bmatrix}\right.$$

$$A^{-1} = \begin{bmatrix} -4/25 & 2/15 & -7/50 \\ -4/5 & 1 & -1/5 \\ 11/25 & -3/5 & 13/50 \end{bmatrix}$$

Multiply by A on bs

$$A^4 - 16A^3 + 55A^2 - 50A = 0$$

$$A^4 = 16A^3 - 55A^2 + 50A$$

$$= \begin{bmatrix} 7184 & 1904 & 7856 \\ 9056 & 2736 & 10304 \\ 12064 & 5184 & 15776 \end{bmatrix} - \begin{bmatrix} 28695 & 220\ 2 \\ 3080 & 605\ 2 \\ 2970 & 1870\ 4 \end{bmatrix}$$

$$+ \begin{bmatrix} 350 & -50 & 150 \\ 300 & 50 & 200 \\ 100 & 200 & 400 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 4839 & 1634 & 5751 \\ 6276 & 2181 & 7534 \\ 9194 & 3514 & 11446 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 2 & -2 \\ -1 & +3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$\cancel{R_2 \to R_2 + R_1}$

$$\det(A - \lambda I) = 0$$

$$\lambda^3 - 5\lambda^2 + [3 + 1 + 5]\lambda - 1 = 0$$

To prove CHT,

$= A^3 - 5A^2 + 9A - 1I = 0$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - \begin{bmatrix} -5 & 68 & -26 \\ -20 & 45 & 80 \\ 80 & -30 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & \\ -9 & 27 & \\ 0 & -18 & \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, proved

(div by A on both sides)

$A^2 - 5A + 9I - 1A^{-1} = 0$

$A^{-1} = A^2 - 5A + 9I$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -8 \\ -4 & 12 & 0 \\ 0 & -8 & 4 \end{bmatrix} \begin{bmatrix} 5 & 10 & \\ -5 & 15 & \\ 0 & -10 & \end{bmatrix}$$

$$+ \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 4 & 4 \\ 0 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

(Mul by A on both sides

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$= \begin{bmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} - \begin{bmatrix} -9 & 108 & -36 \\ -36 & 63 & 18 \\ 18 & -72 & 9 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

3. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$

$$\lambda^3 - 2\lambda^2 + (3 - -1 + 2)\lambda - 18 = 0$$

$$A^3 - 2A^2 + 4A - 18I = 0$$

$$\begin{bmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 16 \\ 10 & -10 & 4 \\ 10 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 8 & 4 \\ -4 & 0 & 12 \\ 8 & -4 & 4 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(div by A on bs)

$$A^2 - 2A + 4I - 18A^{-1} = 0$$

$$\frac{1}{18}\left[A^2 - 2A + 4I\right] = A^{-1}$$

$$A^{-1} = \frac{1}{18}\left\{\begin{bmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 2 \\ -2 & 0 & 6 \\ 4 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1/6 & -1/6 & 1/3 \\ 7/18 & -1/18 & -2/9 \\ 1/18 & 5/18 & 1/9 \end{bmatrix}$$

(Mul by A on bs)

$$A^4 - 2A^3 + 4A^2 - 18A = 0$$

$$A^4 = 2A^3 - 4A^2 + 18A$$

$$= \begin{bmatrix} 32 & -12 & 24 \\ 28 & 16 & -16 \\ 4 & 20 & 28 \end{bmatrix} - \begin{bmatrix} 4 & 4 & 32 \\ 20 & -20 & 8 \\ 20 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 18 & 36 & 18 \\ -18 & 0 & 54 \\ 36 & -18 & 18 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 46 & 20 & 10 \\ -10 & 36 & 30 \\ 20 & -10 & 46 \end{bmatrix}$$

1. Find the characteristic equation of matrix A given below and hence find matrix represented $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A where A is

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\lambda^2 - 4\lambda - 5 = 0$$

By CHT,

$$A^2 - 4A - 5I = 0$$

$$\begin{array}{r|l} & x^3 - 2x + 3 \\ \hline x^2 - 4x - 5 & x^5 - 4x^4 - 7x^3 + 11x^2 - x - 10 \\ & x^5 - 4x^4 - 5x^3 \\ & (-) \quad (+) \quad (+) \\ \hline & -2x^3 + 11x^2 - x - 10 \\ & -2x^3 + 8x^2 + 10x \\ & (+) \quad (-) \quad (-) \\ \hline & 3x^2 - 11x - 10 \\ & 3x^2 - 12x - 15 \\ & (-) \quad (+) \quad (+) \\ \hline & x + 5 \end{array}$$

Stop till degree 1

$$d \overline{)\, p \,}\ q \quad r$$

$$p = dq + r$$

$$x^5 - 4x^4 - 7x^3 + 11x^2 - x - 10 = (x^2 - 4x - 5)(x^3 - 2x + 3) + (x + 5)$$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + (A + 5I)$$

$$= 0 + (A + 5I)$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$

2. Find characteristic eqn of a matrix A and hence find the matrix represented by $A^7 - 4A^6 - 20A^5 - 34A^4 - 4A^3 - 20A^2 - 33A + I$

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Characteristic eqn $= \lambda^3 - 4\lambda^2 + (-4 - 6 - 10) - 35$

$$= \lambda^3 - 4\lambda^2 - 20\lambda - 35$$

By CHT,

$$A^3 - 4A^2 - 20A - 35 = 0$$

$$\begin{array}{l}
\qquad\qquad\qquad\qquad x^4 + x \\
x^3 - 4x^2 - 20x - 35 \overline{\big)\, x^7 - 4x^6 - 20x^5 - 34x^4 - 4x^3 - 20x^2 - 33x + 1} \\
\qquad\qquad\qquad\qquad x^7 - 4x^6 - 20x^5 - 35x^4 \\
\qquad\qquad\qquad\qquad (-) \quad (+) \quad (+) \quad (+) \\
\hline
\qquad\qquad\qquad\qquad 0 \quad 0 \quad 0 \quad + x^4 - 4x^3 - 20x^2 - 33x + 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad x^4 - 4x^3 - 20x^2 - 35x \\
\qquad\qquad\qquad\qquad\qquad\qquad (-) \quad (+) \quad (+) \quad (+) \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2x + 1
\end{array}$$

$P = dq + r$

$\begin{bmatrix} 5 & 4 \\ & 8 \end{bmatrix}$

× A and hence
$A^7 - 4A^6 - 20A^5$

$(-4 - 6 - 10)$
35

35

$0x^2 - 33x + 1$

$x^2 - 33x + 1$
$2 - 35x$
$(+)$
$2x + 1$

(directly A)

(A) $(x^7 - 4x^6 - 20x^5 - 34x^4 - 4x^3 - 20x^2 - 33x + 1) = (x^3 - 4x^2 - 20x - 35)(x^4 + x) + (2x + 1)$

$$= 2A + I$$

$$= 2\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 14 \\ 8 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}$$

3. Find characteristic eqn of matrix A and hence find $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Characteristic eqn $= \lambda^3 - 5\lambda^2 + [2 + 3 + 2]\lambda - 3$

$= \lambda^3 - 5\lambda^2 + 7\lambda - 3$

By Cayley H. Theorem
$A^3 - 5A^2 + 7A - 3I = 0$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$\underline{A^8 - 5A^7 + 7A^6 - 3A^5} +.$

$A^5(A$

$$
\begin{array}{r|l}
 & x^5 + x \\
\hline
x^3 - 5x^2 + 7x - 3 & x^8 - 5x^7 + 7x^6 - 3x^5 + x^4 - 5x^3 + 8x^2 - 2x + 1 \\
 & x^8 - 5x^7 + 7x^6 - 3x^5 \\
 & (-) \\
\hline
 & 0 + x^4 - 5x^3 + 8x^2 - 2x + 1 \\
 & x^4 - 5x^3 + 7x^2 - 3x + 0 \\
 & (-) \quad (+) \quad (-) \quad (+) \quad (-) \\
\hline
 & x^2 + x + 1
\end{array}
$$

$dq + r =$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I =$

$(x^3 - 5x^2 + 7x - 3)(x^5 + x)$

$+ (x^2 + x + 1)$

$(x^2 + x + 1)$

$= A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

4. Find eigen values of following matrix

i) $\begin{bmatrix} 0 & -6 & -12 \\ 6 & 0 & -12 \\ 12 & 12 & 0 \end{bmatrix}$ (Skew-sym matrix)

Characteristic eq^n $= \lambda^3 - 0\lambda^2 + [+36 + 144 + 144]\lambda - 0$

$= \lambda^3 - 0\lambda^2 + 324\lambda - 0$

By CHT,

$$\lambda^3 + 324\lambda = 0$$
$$A^3 + 324A = 0$$
$$\lambda = 18i, -18i, 0$$

ii) $\begin{bmatrix} 0 & 2 & -6 \\ -2 & 0 & -9 \\ 6 & 9 & 0 \end{bmatrix}$

Char eq^n $= \lambda^3 - 0\lambda^2 + [4 + 36 + 81]\lambda - 0$

$0 = \lambda^3 + 121\lambda$

$\lambda = 11i, -11i, 0$

iii $\begin{bmatrix} 4/9 & 8/9 & 1/9 \\ -7/9 & 4/9 & -4/9 \\ -4/9 & 1/9 & 8/9 \end{bmatrix}$

Char eqn

$= \lambda^3 - \frac{16}{9}\lambda^2 + \left(\frac{36}{9} + \frac{36}{9} + \frac{72}{9}\right)\lambda - 1$

$= \lambda^3 - \frac{16}{9}\lambda^2 + (16\lambda) - 1 = 0$

By CHT

~~$A^3 - 16$~~

$X_1 = 0.0629$

$X_2 = 0.8575 \pm 3.89318i$

5. Find eigenvalue and eigenvectors of given matrix

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (Symmetric/Orthogonal

~~$\lambda^2 - 0\lambda - 1$~~ $= 0$

$= \lambda^2 - 0\lambda - 1 = 0$

$\lambda^2 - 1 = 0$

$\lambda = 1$ or -1

Note:

If A,

- The eigenvalues of symmetric matrix are always real
- The eigenvalues of skew-symmetric matrix are either 0 or pure imaginary and it comes in pairs
- The eigenvalues of orthogonal matrix are of unit modulus (1 or -1)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda = 1, -1$

$(A - \lambda I)X = 0$

let $\lambda_1 = 1$

$(A - \lambda_1 I)X_1 = 0$

$(A - I)X_1 = 0$

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\cancel{-x_1 + x_2 + x_1 - x_2 = 0}$

$R_2 \to R_2 + R_1$

$$= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} x \\ x_1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1$

$(A - \lambda_2 I) X_2 = 0$

$(A + I) X_2 = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_2 = 0$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = x_2 = -x_1$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$X_2 = X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. Obtain the eigenvalues & eigen ~~functions~~ vectors of the following

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Char eqn $= \lambda^3 - [10\lambda^2] + [15 + 6 + 6]\lambda - 18 = 0$

$= \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

$\lambda = (1, 3, 6)$ Eigen values of A

For $\lambda_1 = 1 \Rightarrow$ value

$$(A - \lambda_1 I) X_1 = 0$$

$$(A - I) X_1 = 0$$

$$\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0$$
$$2x_3 = 0$$
$$x_3 = 0$$

$$x_1 = -2x_2 \quad \Rightarrow \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}$$

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \text{Vector}$$

For $\lambda_2 = 3$

$$(A - \lambda_2 I) X_2 = 0$$

$$(A - 3I) X_2 = 0$$

$$\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$= \begin{bmatrix} -1 & 2 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= -x_1 + 2x_2 = 0$$
$$6x_2 = 0$$
$$x_2 = 0$$
$$-x_1 = 0$$
$$0 = 0$$

$$x_2 = \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \Rightarrow \text{vector}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_3 = 6$

$$(A - \lambda_3 I) X_3 = 0$$

$$(A - 6I)(X_3) = 0$$

$$\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{aligned} -4x_1 + 2x_2 &= 0 \\ +2x_1 - x_2 &= 0 \\ -3x_3 &= 0 \Rightarrow x_3 = 0 \end{aligned}$$

$$\begin{aligned} \cancel{-4x_1 + 2x_2} &= 0 \\ \cancel{4x_1 - 2x_2} &= 0 \\ 0 \quad & 0 \end{aligned}$$

$$x_1 = \frac{x_2}{2}$$

$$\begin{aligned} -4x_1 &= -2x_2 \\ 2x_1 &= x_2 \\ x_1 &= \frac{x_2}{2} \end{aligned}$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{\begin{bmatrix} x_2/2 \\ x_2 \\ 0 \end{bmatrix}} = \cancel{x_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} x_1 \\ 2x_2 \\ 0 \end{bmatrix} = \cancel{x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$$

$$= x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

3. Find eigenvalues & eigen vectors of matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\lambda^3 - [7\lambda^2] + [4+3+4]\lambda - 5 = 0$$
$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

Roots

$$\lambda = 5, 1, 1$$

Trace of A = Sum of diagonal Elements
= Sum of Eigenvalues

det A = Product of Eigenvalues

For $\lambda_1 = 5$

$$(A - \lambda_1 I) X_1 = 0$$

$$\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$= \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

matrix

5 = 0

Elements
lves

values

] = 0

]

$R_2 \to R_1 + 3R_2$

$$= \begin{bmatrix} -3 & 2 & 1 \\ 0 & -4 & 4 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 + 3R_3$

$$= \begin{bmatrix} 0 & 4 & -4 \\ 1 & -2 & +1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_2 - 4x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$x_2 = x_3$

$x_1 = +2x_2 - x_3$

$x_1 = +2x_2 - x_2$
$= +x_2$

$$X_1 = \begin{bmatrix} -x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} +1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 1$

$(A - \lambda_2 I) X_2 = 0$

$$\left\{\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$; $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= x_1 + 2x_2 + x_3 = 0$$

$$2x_2 = -x_1 - x_3$$

$$x_2 = \left[\frac{-x_1 - x_3}{2}\right]$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ \frac{-x_1 - x_3}{2} \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} +1 \\ -1/2 \\ 0 \end{bmatrix} + x_3 = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

OR

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

AM of $\lambda = 1$ is 2

GM of $\lambda = 1$ is 2

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

4 Find eigenvalues and eigen vectors of

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Characteristic eq^n

$$\lambda^3 - [7\lambda^2] + [11 + 6 + -1]\lambda - 12 = 0$$

$$\lambda^3 - 7\lambda^2 + [16\lambda] - 12 = 0$$

$$\lambda = 3, 2, 2 \quad \text{(see product of det)}$$

$$\lambda_1 = 3$$

$$(A - \lambda_1 I) X_1 = 0$$

$$\left(\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{3}{2} R_2$$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + \frac{9}{10} R_7$

$$= \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 0 & 0 & 5/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{5}{2} x_3 = 0$$

$$10x_2 + 5x_3 = 0$$

$$-2x_1 - 6x_2 - 4x_3 = 0$$

$$x_1 + 3x_2 + 2x_3 = 0$$
$$x_1 + 3x_2 - 4x_2 = 0$$

$$10x_2 = 0$$
$$x_2 = 0$$

$$x_1 = \cdot$$

$$\lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$\lambda_2 = 2$$

$$\left(\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$; $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 16 \\ 0 & -10 & -4 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 16 \\ 0 & 5 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 3R_3$

$$= \begin{bmatrix} 1 & 10 & 5 \\ 0 & 0 & -20 \\ 0 & 5 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 12 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 + 12x_3 = 0$

$x_1 + 10x_2 = 0$

$x_2 = -12x_3$ $\quad x_2 = \frac{-x_1}{10}$

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$$x_3 \begin{bmatrix} -1 \\ -2/5 \\ 1 \end{bmatrix}$$

$$\frac{x_3}{5} = \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix}$$

AM of $\lambda = 3$ is 1 AM of $\lambda = 2$ is 2
GM of $\lambda = 3$ is 1 GM of $\lambda = 2$ is 1

5. Find eigenvalues & eigen vectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Char eqn $= \lambda^3 - 6\lambda^2 + [4+4+4]\lambda - 8 = 0$

$$= \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda = (\cancel{4, 1+i, 1-i})$$
$$(2, 2, 2)$$

$\lambda_1 = 2$

$$(A\lambda_1 - I)X = 0$$

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2 is 2
2 is 1

rs of

$+3+4]\lambda$

$-8=0$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$
$x_3 = 0$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Algebraic Multiplicity of $\lambda = 2$ is 3
Geometric Multiplicity of $\lambda = 2$ is 1

Algebraic Multiplicity

Let λ be an eigen value of A repeated r times then r is called Algebraic Multiplicity of λ

Geometric Multiplicity

The number of linearly independent eigen vector corresponding to eigenvalue λ is called its geometric multiplicity

G is less than A

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Matrix	Eigenvalue	Vectors
A	λ	X
A^2	λ^2	X
A^n	λ^n	X
A^T	λ	
(kiss) kA	$k\lambda$	X
I	1	
kI	k	
A^{-1}	$\frac{1}{\lambda}$	
$\text{Adj}A$	$\frac{\text{Product of all eigen values}}{\lambda}$	

$$AX = \lambda X$$

$$A(AX) = A(\lambda X)$$

$$A^2 X = A\lambda X$$

$$A^2 X = \lambda^2 X$$

$$kAX = k\lambda X$$

$$(kA)X = (k\lambda)X$$

$$\frac{(\lambda_1 \cdot \lambda_2 \cdot \lambda_3)}{\lambda_1}$$

(λX)
$+ \lambda X$
$\lambda^2 X$

$= k\lambda X$
$= (k\lambda) X$

NOTE:
The eigenvalues of diagonal matrix or upper triangular matrix or lower triangular matrix are its diagonal elements.

1. If $A = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$ then find eigenvalues of

$6A^{-1} + A^3 + 2I$

~~Char eqn~~

Since A is an upper triangular matrix the eigen values of A are its diagonal elements $\lambda = 2, 3$

Eigen values of $6A^{-1} + A^3 + 2I$

$$\frac{6}{\lambda} + \lambda^3 + 2(1)$$

For $\lambda = 2$

$= \frac{6}{2} + 2^3 + 2$

$= 13$

For $\lambda = 3$

$\frac{6}{3} + 3^3 + 2$

$2 + 27 + 2$

$= 31$

$\therefore$ Eigen values of $6A^{-1} + A^3 + 2I$ are $13, 31$

2. Find eigenvalues of Adj. A where $A = \begin{bmatrix} 1 & 2 \\ \cancel{6} & \cancel{5} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Since A is an upper triangular matrix
The eigen values ~~of~~ are diag. elements

$$\lambda = (1, 2, 4, 6)$$

Eigenvalues of adj A = Prod of

$$\frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{\lambda_1}, \quad \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{\lambda_2}, \quad \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{\lambda_3}, \quad \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4}{\lambda_4}.$$

For $\lambda_1 = \underset{\lambda_1}{1}, \underset{\lambda_2}{2}, \underset{\lambda_3}{4}, \underset{\lambda_4}{6}$

$$\lambda_1 = \frac{48}{1}, \frac{48}{2}, \frac{48}{4}, \frac{48}{6}$$

$$48, 24, 12, 8$$

If A is $\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$. Find eigenvalues and eigenvectors

of $A^2 + 2A + I$

$\lambda \to X \qquad \lambda^2 + 2\lambda + 1, x$

$\lambda^2 - \lambda + (-15) = 0$

$\lambda^2 - \lambda - 15 = 0$

$X = \left(\frac{1 + \sqrt{61}}{2}, \frac{1 - \sqrt{61}}{2} \right)$

$(A - I\lambda) X \qquad (A - \lambda I) X \to$ Eigenvector

*

Find eigenvalues and eigenvectors

$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$* \quad (A - \lambda_1 I) X_1$

$\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - (1 +$

1. $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$

$= \lambda^2 - 2\lambda + (-15)$

$= \lambda^2 - 2\lambda - 15$

$(A - \lambda I) X = 0$

$\lambda = 5$ or -3

For $\lambda_1 = 5$

$\left(\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) X_1 = 0$

$\begin{bmatrix} -4 & 8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= -4x + 8y + 2x - 4y = (0, 0)$

$= -2x + 4y = (0, 0)$

$2x_1 = 4x_2$

$x_1 = 2x_2$

For $\lambda_2 = -3$

$$\left(\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) x_2 = 0$$

$$= \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 + 8x_2 + 2x_1 + 4x_2 = 0$$

$$6x_1 + 12x_2 = 0$$

$$x_1 = -2x_2$$

$$x_1 = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$A^2 + 2A + I$

Value

$= \lambda_1^2 + 2\lambda_1 + 1$ $\qquad$ $\lambda_2^2 + (2\lambda_2) + 1$

$= 5^2 + (10) + 1$ $\qquad$ $= 9 - 6 + 1$

$= 36$ $\qquad$ $= 4$

Eigen vector same as A

FOR EDUCATIONAL USE

2

i $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Char eqn $= \lambda^3 - 6\lambda^2 + (4+5+2)\lambda -$

$= \lambda^3 - 6\lambda^2 + (11\lambda) - 6$

$\lambda = (1, 3, 2)$

For $\lambda = 1$

$(A - \lambda_1 I) X_1 = 0$

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_3 = 0$

$R_2 \to R_2 - 2R_1$

$$\cancel{\begin{bmatrix} 0 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\cancel{x_3 = 0}$$

$$\cancel{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_3 = 0$$

$$x_1 + x_2 + 0 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 = -x_2$$

$$X_1 = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$(A - \lambda_2 I) X_2 = 0$$

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$= \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - x_3 = 0$$

$$-2x_1 = x_3$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 - x_2 - 2x_1 = 0$$

$$-x_2 - x_1 = 0$$

$$-x_1 = x_2$$

$$2x_1 + 2x_2 = 0$$

$$x_1 = -x_2$$

$$X_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 2$

$$\left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -x_3$$

$$2x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2$$

$$= x_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Similarity of Matrices

A matrix 'A' is said to be similar to matrix B if there exists a non-singular matrix M such that $M^{-1}AM = B$

$$M^{-1}AM = B$$

Properties :-

- If A is similar to B and B is similar to C then, A is similar to C
- If A is similar to B then A^2 is similar to B^2 (in general A^n is similar to B^n)
- Similar matrices have same eigenvalues
- The determinant of similar matrix is same

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

FOR EDUCATIONAL USE

- Matrix similar to diagonal matrix. Diagonalization of matrix

A (sq) matrix A is said to be diagonalizable if it is similar to diagonal matrix i.e. A is said to be diagonalizable if there exists a non-singular matrix M. such that

$$M^{-1}AM = D$$

where, D is a diagonal matrix

The necessary and sufficient condition

A square matrix A is diagonalizable if and only if Algabraic multiplicity of each eigenvalue is equal to its Geometric Multiplicity

NOTE: A square matrix with distinct eigen values is always diagonalizable

1. Check whether the following matrix is similar to diagonal matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

Characteristic eqn $= \lambda^3 - 3\lambda^2 + [0 + -3 + 6]\lambda - 1$

$= \lambda^3 - 3\lambda^2 + 3\lambda - 1$

ix · Diagonalization

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$\lambda^2 + [6 + -3$
$+ 6]\lambda - 1$

$+ 3\lambda - 1$

$\lambda = (1, 1, 1)$

AM = 3
of $\lambda = 1$

GM
of $\lambda = 1$

$(A - \lambda_1 I) X_1$

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$x_1 \cancel{+} - x_2 + x_3 = 0$$
$$3x_2 - 3x_3 = 0$$

$$\Rightarrow x_2 = x_3$$

Sub x_2 value in ①

$$x_1 - x_3 + x_3 = 0$$
$$x_1 = 0$$

$$X = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

There exists only one linearly Independent eigen vector $\therefore$ GM of $\lambda = 1$ is $\underline{1}$

$\therefore$ AM $\neq$ GM

$\therefore$ Thus, A is not diagonalizable

2. Is Matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ diagonalizable?

If so find transforming matrix M and the diagonal matrix D

Char eq^n = $\lambda^3 - [2\lambda^2] + [0 + -3 + 4]\lambda - [0]$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda = 0$$

$\lambda = 0, 1, 1$

AM of 1 is 2 (repeated)

GM of 1 is :-

$(A - \lambda I) X_1 = 0$

$(A - I) X_1 = 0$

$$\left(\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$= \begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 1/2 R_1$

$$= \begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

FOR EDUCATIONAL USE

$$x_2 = -\frac{\cancel{6}^{4/6}}{\cancel{2}_{1}} x_3$$

$$x_2 = -\frac{\cancel{2}^{2/3}}{2} x_3$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\cancel{3/2}\ -2/3\, x_3 \\ x_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3x_1 \\ -2x_3 \\ 3x_3 \end{bmatrix}$$

$$= \left\{ \frac{1}{3} x_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3} x_2 \begin{bmatrix} 0 \\ -2 \\ +3 \end{bmatrix} \right\}$$

$$= \frac{1}{3} \left\{ x_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right\}$$

$X_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$ are eigenvectors corr to $\lambda = 1$

GM of $\lambda_1 = 2$

AM = GM

For $\lambda = 1$

$\therefore$ Matrix A is diagonalizable

Eigenvector corr to 0

$$\begin{bmatrix} 3x_1 \\ -2x_3 \\ 3x_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -2 \\ +3 \end{bmatrix}\Big\}$$

$\}$

are eigenvectors

gonalizable

$$(A - \lambda_2 I) X_2 = 0$$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_2 + R_3$$

$$= \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$= \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 + x_3 = 0$$

$$2x_2 = -x_3$$

$$x_1 - 6x_2 - 4x_3 = 0$$

$$x_1 - 6x_2 + 8x_2 = 0$$

$$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2$$

$$-x_1 = 2x_2$$

$$\begin{bmatrix} -2x_2 \\ x_2 \\ -2x_2 \end{bmatrix}$$

$$\Rightarrow x_2 \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$\therefore$ It is eigenvector corr $\lambda = 0$

$$M = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$$

$$MAM^{-1} = D$$

$$M^{-1} \Rightarrow \begin{bmatrix} 1/3 & -2 & -4/3 \\ 0 & -2 & -1 \\ 0 & -3 & -2 \end{bmatrix}$$

$$M^{-1}AM$$

$$\underset{M^{-1}}{\begin{bmatrix} 1/3 & -2 & -4/3 \\ 0 & -2 & -1 \\ 0 & -3 & -2 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}} \underset{M}{\begin{bmatrix} 3 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{bmatrix}$$

HW

Is $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable

If so find M & D

$\lambda^3 - 5\lambda^2 + [2+2+2]\lambda - 4$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(A\cancel{\lambda} - I\lambda)X$

$(\cancel{\lambda}[\ 2$

Eigenvalues are $(\underset{\lambda_1}{1}, \underset{\lambda_2}{2}, \underset{\lambda_3}{2})$

Finding eigen vector for $\lambda_2 = 2$

$(A - \lambda_2 I)X_2 = 0$

$$\left\{\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y = 0$

$-z = 0$

FOR EDUCATIONAL USE

$$X_1 = x\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$X_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

AM for $\lambda = 2$ is $\underline{2}$
GM for $\lambda = 2$ is $\underline{1}$

$\lambda = 2$ Not diagonizable

For $\lambda = 1$

$$(A - \lambda_1 I)X_1 = 0$$

$$\left\{\begin{bmatrix}2 & 3 & 4\\0 & 2 & -1\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 4\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x + 3y + 4z = 0$$
$$y - z = 0$$
$$y = z$$

$$x + 3y + 4y = 0$$
$$x + 7y = 0$$

$x = -7y$

$$\cancel{X_2} \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$ AM is 1
GM is 1

$\therefore$ It is diagonizable

$$M = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M^{-1}$$

It is not invertible

$M^{-1}AM = D$

No, Diagonal exists

1. Show that the matrix A is diagonalizable

$$A = \begin{bmatrix} -9 & 5 & 5 \\ -8 & 3 & 5 \\ -16 & 8 & 7 \end{bmatrix}$$. Find the transformatio matrix M and diagonal

Characteristic eqn $= +\lambda^3 - \lambda^2 + [-1 + 5 + 1] - 3$

$+\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$= \lambda^3 - \lambda^2 - 5\lambda - 3$

$\lambda = \underline{\underline{3, -1, -1}}$

For $\lambda_2 = -1$

$(A - \lambda_2 I)X = 0$

$$\left(\begin{bmatrix} -9 & 5 & 5 \\ -8 & 3 & 5 \\ -16 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 5 & 5 \\ -8 & 5 & 5 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} -8 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-8x_1 + 4x_2 + 4x_3 = 0$$

$$-2x_1 + x_2 + x_3 = 0$$

$$x_2 + x_3 = 2x_1$$

$$x_3 = 2x_1 - x_2$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 - x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \& \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Are linearly independent eigenvector corr to $\lambda = -1$

$\therefore$ AM of -1 is 2

GM of -1 is 2

AM = GM for $\lambda = -1$

$\therefore$ It is diagonalizable

$\therefore$

For $\lambda_2 = 3$

$$(A - \lambda_2 I) X_2 = 0$$

$$\left(\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

divide All R_1, R_2, R_3 by 4.

$$\begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$;

$$= \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{x_1} \cdot R_1 \to R_1 + R_2$

$$= \begin{bmatrix} -1 & 1 & 2 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = x_2$

$x_3 = 2x_2$

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 2x_2 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

FOR EDUCATIONAL USE

$$M^{-1}AM = D$$

$$M^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$