Solutions to *Introduction to Commutative Algebra* by Atiyah and Macdonald

Aditya Gudibanda

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Chapter 1

1.1

Problem

Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution

Note the identity $\left(\sum_{i=0}^{k} x_i\right)(1-x) = 1 - x^{k+1}$. Since x is nilpotent, $x^l = 0$ for some $l \geq 1$. Thus, if we set $k = l-1$ in the identity above, we see that $1-x$ has a multiplicative inverse, and is therefore a unit.

To prove the original statement, given that x is a nilpotent unit, it is clear that $-x$ is nilpotent as well, and by the logic above, $1-(-x) = 1+x$ is a unit, as desired.

Suppose u is an arbitrary unit. Since x is nilpotent, $u^{-1}x$ is nilpotent as well, so by the above, $u^{-1}x+1$ is a unit. Since the product of units is a unit, $u(u^{-1}x+1) = x+u$ is a unit, as desired.

1.2

Problem

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.
iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.
iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution

i)

1.3

Problem

Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution

1.4

Problem

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution

1.5

Problem

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that
i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A.
ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?
iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution

1.6

Problem

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution

1.7

Problem

Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution

1.8

Problem

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution

1.9

Problem

Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Solution

1.10

Problem

Let A be a ring, $\mathcal{R}$ its nilradical. Show that the following are equivalent:
i) A has exactly one prime ideal
ii) every element of A is either a unit or nilpotent
iii) $A/\mathcal{R}$ is a field

Solution

1.11

Problem

A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that
i) $2x = 0$ for all $x \in A$
ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements
iii) every finitely generated ideal in A is principal.

Solution

1.12

Problem

A local ring contains no idempotent $\neq 0, 1$.

Solution

1.13

Problem

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be the maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \cup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\bar{K}$ be the set of all elements of L which are algebraic over K. Then $\bar{K}$ is an algebraic closure of K.

Solution

1.14

Problem

In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors of A is a union of prime ideals.

Solution

1.15

Problem

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
ii) $V(0) = X, V(1) = \emptyset$.
iii) If $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V(\cup_{i\in I} E_i) = \cap_{i\in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

Solution

1.16

Problem

Draw pictures of Spec $\mathbb{Z}$, Spec $\mathbb{R}$, Spec $\mathbb{C}[x]$, Spec $\mathbb{R}[x]$, Spec $\mathbb{Z}[x]$.

Solution

1.17

Problem

Solution

1.18

Problem

Solution

1.19

Problem

A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec A is irreducible if and only if the nilradical of A is a prime ideal.

Solution

1.20

Problem

Solution

Chapter 2

2.1

Problem

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution

The tensor product is generated by all possible tensor products of pairs of generators from the two rings we are tensoring. Both rings are cyclic and are generated by 1. So the tensor product is generated by $1 \otimes 1$. Now, note that $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$, and $n(1 \otimes 1) = 1 \otimes n = 1 \otimes 0 = 0$. Thus, the order of $1 \otimes 1$ must divide both m and n. However, since m and n are coprime, the only such positive number is 1. Thus, $1 \otimes 1$ has order 1, so $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is a ring that contains only one element, which is therefore 0, as desired.

2.2

Problem

Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Solution

Consider the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$. We now tensor this with M to get $0 \to \mathfrak{a} \otimes_A M \to A \otimes_A M \to A/\mathfrak{a} \otimes_A M \to 0$. By the first isomorphism theorem, we therefore have that $A/\mathfrak{a} \otimes_A M \cong A \otimes_A M/\mathfrak{a} \otimes_A M$. Clearly, $A \otimes_A M \cong M$ and $\mathfrak{a} \otimes_A M \cong \mathfrak{a}M$, so we have $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$, as desired.

2.3

Problem

Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

Solution

Let $\mathfrak{m}$ be the maximal ideal of A and let $k = A/\mathfrak{m}$ be the residue field. Then $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by 3.2. Since A is a local ring, $\mathfrak{m}$ is the only maximal ideal of A, so by Nakayama's lemma, if $M_k = 0$, then $M = 0$.

Since $M \otimes_A N = 0$, $(M \otimes_A N)_k = M_k \otimes_A N_k = 0$. Since M_k and N_k are both vector spaces over a field, this implies that either $M_k = 0$ or $N_k = 0$, which implies that either $M = 0$ or $N = 0$, as desired.

2.4

Problem

Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution

$\Rightarrow$: Suppose that M is flat. The maximal ideals of M are $\{\times_{i \neq j} M_i \times \mathfrak{m}_j | j \in I, \mathfrak{m}_\mathrm{j} \in M_j\}$, where $\mathfrak{m}_j$ are the maximal ideals of M_j. The quotients of all these maximal ideals must also be flat. These are all $M_j/\mathfrak{m}_j$, over all $j \in I$ and over all $\mathfrak{m}_j$ which are maximal in M_j. For a specific j, this implies that all the maximal ideals are flat. This implies that M_j is flat. So this direction is done.

$\Leftarrow$: Just go the other direction.

2.5

Problem

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Solution

2.6

Problem

For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r (m_t \in M)$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x] \cong A[x] \otimes_A M$.

Solution

From Proposition 2.14, we know that $(M+N) \otimes P \cong (M \otimes P) + (N \otimes P)$. We can consider $M[x]$ to be an infinite direct sum of M, and $A[x]$ to be an infinite direct sum of A. From Proposition 2.14, we know that when M is a module of A, $A \otimes M \cong M$. These two facts immediately imply the desired conclusion, $M[x] \cong A[x] \otimes_A M$.

2.7

Problem

Let $\mathfrak{p}$ be a prime ideal of A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

Solution

Let $f, g \in A[x] - \mathfrak{p}[x]$. Let

$$f = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$$

and let

$$g = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$$

Suppose a_i is the coefficient with smallest index that is not in $\mathfrak{p}$, and let b_j be defined likewise. These must both exist because $f, g \notin \mathfrak{p}[x]$. Then consider the coefficient of x^{i+j} in fg. WLOG, let $i < j$. Then the coefficient of x^{i+j} in fg is

$$\sum_{r=0}^{i-1} a_r b_{i+j-r} + a_i b_j + \sum_{s=0}^{j-1} a_{i+j-s} b_s$$

Both the sums are in $\mathfrak{p}$ because $a_r \in \mathfrak{p}$ for $r < i$ and $b_s \in \mathfrak{p}$ for $s < j$, by the definitions of i and j. But a_i and b_j are not in $\mathfrak{p}$, and $\mathfrak{p}$ is prime, so $a_i b_j$ is not in $\mathfrak{p}$. Thus, fg is not in $\mathfrak{p}[x]$. Thus, $\mathfrak{p}[x]$ is prime in $A[x]$, as desired.

2.8

Problem

i) If M and N are flat A-modules, then so is $M \otimes_A N$.

ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution

i) Let $0 \to B \to C \to D \to 0$ be an exact sequence of A-modules. Since M is flat,

$$0 \to B \otimes_A M \to C \otimes_A M \to D \otimes_A M \to 0$$

is exact. Since N is flat,

$$0 \to (B \otimes_A M) \otimes_A N \to (C \otimes_A M) \otimes_A N \to (D \otimes_A M) \otimes_A N \to 0$$

is also exact. By Proposition 2.14, $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$, so therefore we have

$$0 \to B \otimes_A (M \otimes_A N) \to C \otimes_A (M \otimes_A N) \to D \otimes_A (M \otimes_A N) \to 0$$

is exact. Thus, $M \otimes_A N$ is flat, as desired.

ii)

2.9

Problem

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Solution

Let $u_1, u_2, \cdots, u_m$ be the generators of M' and let $v_1, v_2, \cdots v_n$ be the generators of M''. Let $w_i = f(u_i), 1 \leq i \leq m$. Since M surjects onto M'', for each of the v_i, there is an x_i such that $f(x_i) = v_i$.

Let f be the map from M to M'' in the exact sequence. For each $m \in M$, either m goes to 0 or $f(m)$ is a finite sum $s_1 v_1 + s_2 v_2 + \cdots + s_n v_n$. In the first case, it is in the submodule generated by $w_1, w_2, \cdots, w_m$. In the second case, it is in the submodule generated by $x_1, x_2, \cdots, x_n$. In all cases, m is in the submodule generated by $w_1, w_2, \cdots, w_m, x_1, x_2, \cdots, x_n$. Thus M is finitely generated as desired.

2.10

Problem

Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution

2.11

Problem

Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.

Solution

Let $\mathfrak{m}$ be a maximal ideal of A, let $k = A/\mathfrak{m}$, and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes_k \phi : k \otimes_k A^m \to k \otimes_k A^n$ is an isomorphism of vector spaces of dimensions m and n. Thus, $m = n$.

2.12

Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that Ker (ϕ) is finitely generated.

2.13 Solution

Ker $(\phi) \subset M$ and M is finitely generated, so Ker (ϕ) must be finitely generated.

2.14

Problem

Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution