Solutions to *Introduction to Commutative Algebra* by Atiyah and Macdonald

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Chapter 1

1.1

Problem

Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution

Note the identity $\left(\sum_{i=0}^{k} x_i\right)(1 - x) = 1 - x^{k+1}$. Since x is nilpotent, $x^l = 0$ for some $l \geq 1$. Thus, if we set $k = l - 1$ in the identity above, we see that $1 - x$ has a multiplicative inverse, and is therefore a unit.

To prove the original statement, given that x is a nilpotent unit, it is clear that $-x$ is nilpotent as well, and by the logic above, $1 - (-x) = 1 + x$ is a unit, as desired.

Suppose u is an arbitrary unit. Since x is nilpotent, $u^{-1}x$ is nilpotent as well, so by the above, $u^{-1}x + 1$ is a unit. Since the product of units is a unit, $u(u^{-1}x + 1) = x + u$ is a unit, as desired.

1.2

Problem

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.

iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution

i)

1.3

Problem

Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution

1.4

Problem

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution

1.5

Problem

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?

iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution

1.6

Problem

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution

1.7

Problem

Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution

1.8

Problem

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution

1.9

Problem

Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Solution

1.10

Problem

Let A be a ring, $\mathcal{R}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal
ii) every element of A is either a unit or nilpotent
iii) $A/\mathcal{R}$ is a field

Solution

1.11

Problem

A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$
ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements
iii) every finitely generated ideal in A is principal.

Solution

1.12

Problem

A local ring contains no idempotent $\neq 0, 1$.

Solution

1.13

Problem

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be the maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \cup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\bar{K}$ be the set of all elements of L which are algebraic over K. Then $\bar{K}$ is an algebraic closure of K.

Solution

1.14

Problem

In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors of A is a union of prime ideals.

Solution

1.15

Problem

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
ii) $V(0) = X, V(1) = \emptyset$.
iii) If $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V(\cup_{i\in I} E_i) = \cap_{i\in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

Solution

1.16

Problem

Draw pictures of Spec $\mathbb{Z}$, Spec $\mathbb{R}$, Spec $\mathbb{C}[x]$, Spec $\mathbb{R}[x]$, Spec $\mathbb{Z}[x]$.

Solution

1.17

Problem

Solution

1.18

Problem

Solution

1.19

Problem

A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec A is irreducible if and only if the nilradical of A is a prime ideal.

Solution

1.20

Problem

Solution