Solutions to *Introduction to Commutative Algebra* by Atiyah and Macdonald

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Chapter 1

1.1

Problem

Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution

Note the identity $\left(\sum_{i=0}^{k} x_i\right)(1-x) = 1 - x^{k+1}$. Since x is nilpotent, $x^l = 0$ for some $l \geq 1$. Thus, if we set $k = l-1$ in the identity above, we see that $1-x$ has a multiplicative inverse, and is therefore a unit.

To prove the original statement, given that x is a nilpotent unit, it is clear that $-x$ is nilpotent as well, and by the logic above, $1-(-x) = 1+x$ is a unit, as desired.

Suppose u is an arbitrary unit. Since x is nilpotent, $u^{-1}x$ is nilpotent as well, so by the above, $u^{-1}x+1$ is a unit. Since the product of units is a unit, $u(u^{-1}x+1) = x+u$ is a unit, as desired.

1.2

Problem

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.

iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution

TODO

1.3

Problem

Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution

TODO

1.4

Problem

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution

One inclusion holds for any ring. Suppose $1 - fg$ is always a unit for all g. Letting $g = x$, all coefficients of f must be nilpotent by exercise 2.

1.5

Problem

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that

i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?

iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution

TODO

1.6

Problem

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution

The Jacobson radical is contained in the nilradical in any ring.

Suppose $\mathfrak{R}$ contains a nonzero idempotent x. Then $1 - xx = 1 - x$ is a unit. But $x(1 - x) = 0$ so $1 - x$ is also a zero divisor.

1.7

Problem

Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution

Let $\mathfrak{a}$ be a prime ideal. Then $A/\mathfrak{a}$ is an integral domain. Suppose that $x \in A$. Then $x^n = x$ for some $n > 1$. Let $\bar{x}$ be the image of x under the projection to the quotient by $\mathfrak{a}$. Then $\bar{x}^n - \bar{x} = \bar{x}(\bar{x}^{n-1} - 1) = 0$. Since $A/\mathfrak{a}$ is an integral domain, this implies that either $\bar{x} = 0$ or $\bar{x}^{n-1} = 1$. If $\bar{x}^{n-1} = 1$, then $\bar{x} \cdot \bar{x}^{n-2} = 1$, so $\bar{x}$ has an inverse. This implies that for every $\bar{x} \in A/\mathfrak{a}$, either $\bar{x} = 0$ or $\bar{x}$ has an inverse. Thus, $A/\mathfrak{a}$ is a field, so $\mathfrak{a}$ is maximal, as desired.

1.8

Problem

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution

For any prime $\mathfrak{p}$, $A/\mathfrak{p}$ is an integral domain. For any $x \in A$, let $\bar{x} = x + \mathfrak{p}$. $\bar{x}(\bar{x}^{n-1} - 1) = \bar{x}^n - \bar{x} = 0$. For $x \notin \mathfrak{p}$, $\bar{x}^{n-1} = 1$, so $\bar{x}$ is a unit. As $A/\mathfrak{p}$ is a field, $\mathfrak{p}$ is maximal.

1.9

Problem

Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Solution

$\Leftarrow$: Suppose $\mathfrak{a}$ is an intersection of prime ideals. Clearly, $\mathfrak{a} \subset r(\mathfrak{a})$. Suppose $x \notin \mathfrak{a}$. Then there is some prime ideal $\mathfrak{p}$ which x is not in, but in which $\mathfrak{a}$ is contained. Thus, no power of x will ever be in $\mathfrak{p}$, so no power of x will ever be in $\mathfrak{a}$. Thus, $\mathfrak{a} = r(\mathfrak{a})$, as desired.

$\Rightarrow$: Simply apply Proposition 1.14 - the radical of an ideal $\mathfrak{a}$, in this case $\mathfrak{a}$, is the intersection of the prime ideals which contain $\mathfrak{a}$.

1.10

Problem

Let A be a ring, $\mathcal{R}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal

ii) every element of A is either a unit or nilpotent

iii) $A/\mathcal{R}$ is a field

Solution

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i $\Rightarrow$ iii: Suppose A has exactly one prime ideal. Let $\mathfrak{m}$ be the maximal ideal of A. Then this must be the unique prime ideal, so it is also the nilradical $\mathcal{R}$. Thus, $\mathcal{R}$ is maximal, so $A/\mathcal{R}$ is a field.

iii $\Rightarrow$ ii: Suppose $A/\mathcal{R}$ is a field. Then R is maximal, so all the elements in $A - \mathcal{R}$ are units. All the elements within $\mathcal{R}$ are obviously nilpotent. Thus, every element of A is either a unit or nilpotent, as desired.

ii $\Rightarrow$ i: Let $\mathfrak{p}$ be a prime ideal. Clearly it must contain all the nilpotent units and can contain none of the units. Thus, every prime ideal must contain exactly all of the nilpotents. Thus, A has exactly one prime ideal.

1.11

Problem

A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that

i) $2x = 0$ for all $x \in A$

ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements

iii) every finitely generated ideal in A is principal.

Solution

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i) $(x+1)^2 = x^2 + 2x + 1 = x + 2x + 1 = x + 1 \Rightarrow 2x = 0$, as desired.

ii) Let $\mathfrak{p}$ be a prime ideal. Then every element in $A/\mathfrak{p}$ satisfies $\bar{x}^2 = \bar{x}$. Since $A/\mathfrak{p}$ is an integral domain, this means that $\bar{x} = 0$ or $\bar{x} = 1$. Thus, $A/\mathfrak{p}$ has only two elements and is hence a field, so $\mathfrak{p}$ is maximal, as desired.

iii) It suffices to be able to reduce the ideal (a, b) and show that it is equivalent to another ideal (c). To this end, let $c = ab + a + b$. It is clear that $c \in (a, b)$. Next, $ca = a^2b + a^2 + ab = ab + a + ab = a$. Next, $cb = ab^2 + ab + b^2 = ab + ab + b = b$. Thus, $(a, b) = (c)$. So given any finite set of generators, we can reduce these to one generator, by induction, by reducing the number of generators one by one.

1.12

Problem

A local ring contains no idempotent $\neq 0, 1$.

Solution

Let $x \in A$, the local ring. Let $\mathfrak{m}$ be the maximal ideal of A. Suppose x is idempotent. Then in $A/\mathfrak{m}$, $\bar{x}^2 = \bar{x} \Rightarrow \bar{x}(\bar{x} - 1) = 0$. Since $A/\mathfrak{m}$ is a field, this implies that either $\bar{x} = 0$ or $\bar{x} = 1$.

If $\bar{x} = 1$, then x is a unit, so if $x^2 = x$, then $x(x - 1) = 0$, so x is either 0 or 1. 0 is not a unit, so $x = 1$.

If $\bar{x} = 0$, then $x \in \mathfrak{m}$, so $x - 1$ is a unit. Since $x(x - 1) = 0$, this means that $x = 0$.

Thus, x is either 0 or 1, as desired.

1.13

Problem

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be the maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \cup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\bar{K}$ be the set of all elements of L which are algebraic over K. Then $\bar{K}$ is an algebraic closure of K.

Solution

TODO

1.14

Problem

In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors of A is a union of prime ideals.

Solution

If $(x) + \mathfrak{m}$ contains a non-zero-divisor z_1 and $(y) + \mathfrak{m}$ contains a non-zer-divisor z_2 then $z_1 z_2$ is not a zero divisor and is in $(xy) + \mathfrak{m}$.

1.15

Problem

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

ii) $V(0) = X, V(1) = \emptyset$.

iii) If $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V(\cup_{i \in I} E_i) = \cap_{i \in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

Solution

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i) If $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{a} \subset \mathfrak{p}$, so $E \subset \mathfrak{p}$. Thus, $\mathfrak{p} \in V(E)$. So $V(\mathfrak{a}) \subset V(E)$. For the other direction, if $\mathfrak{p} \in V(E)$, then $E \subset \mathfrak{p}$. So $\mathfrak{p}$ must contain all the elements in the ideal generated by E, which is $\mathfrak{a}$. Thus, $\mathfrak{a} \subset \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a})$. Thus, $V(E) = V(\mathfrak{a})$. Next, let $\mathfrak{p} \in V(\mathfrak{a})$. Let $x \in r(\mathfrak{a}) - \mathfrak{a}$, with $n \in \mathbb{N}$ being the minimal exponent such that $x^n \in \mathfrak{a}$. Clearly, $n > 1$. Then consider the fact that $x \cdot x^{n-1} \in \mathfrak{a}$. By the definition of n, both x and x^{n-1} are not in $\mathfrak{a}$. But $x^n \in \mathfrak{a} \Rightarrow x^n \in \mathfrak{p}$, so either x or x^{n-1} must be in $\mathfrak{p}$. If it is x^{n-1}, then we perform the same argument with x and x^{n-2}, etc. until we reach the conclusion that $x \in \mathfrak{p}$. Thus, $r(\mathfrak{a}) \subset \mathfrak{p}$, so $\mathfrak{p} \in V(r(\mathfrak{a}))$. For the other direction, $V(r(\mathfrak{a})) \subset V(\mathfrak{a})$ is obvious because $\mathfrak{a} \subset r(\mathfrak{a})$. Thus, $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$, as desired.

ii) $V(0) = X$, because every prime ideal contains 0. $V(1) = \emptyset$, because if an ideal contains 1, then it contains all of A and it cannot be prime.

iii) If $\mathfrak{p} \in V(\cup_{i\in I} E_i)$, then $\mathfrak{p} \in V(E_i)$, $i \in I$, so $p \in \cap_{i\in I} V(E_i)$. If $\mathfrak{p} \in \cap_{i\in I} V(E_i)$, then $\cap_{i\in I} E_i \in \mathfrak{p}$, so $E_i \in \mathfrak{p}$, $i \in I$, so $\cup_{i\in I} E_i \subset \mathfrak{p}$, so $\mathfrak{p} \in V(\cup_{i\in I} E_i)$. Thus, $V(\cup_{i\in I} E_i) = \cap_{i\in I} V(E_i)$, as desired.

iv) Since $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b}$, it is clear that $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{ab})$. Now suppose $\mathfrak{p} \in V(\mathfrak{ab})$. Now suppose $x \in \mathfrak{a} \cap \mathfrak{b}$. Then clearly $x^2 \in \mathfrak{ab}$. So since $\mathfrak{p}$ is prime, $x \in \mathfrak{p}$. Thus, $V(\mathfrak{ab}) \in V(\mathfrak{a} \cap \mathfrak{b})$. Thus, $V(\mathfrak{ab}) = V(\mathfrak{a} \cap \mathfrak{b})$. This could also have been proven by noting that $r(\mathfrak{ab}) = r(\mathfrak{a} \cap \mathfrak{b})$ and using part i. Next, let $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. Suppose $\mathfrak{p}$ was a prime ideal such that $\mathfrak{a} - \mathfrak{p} \neq \emptyset, \mathfrak{b} - \mathfrak{p} \neq \emptyset$. Let $x \in \mathfrak{a} - \mathfrak{p}$ and $y \in \mathfrak{b} - \mathfrak{p}$. Then $xy \in \mathfrak{ab} \subset \mathfrak{p}$, but $x, y \notin \mathfrak{p}$, a contradiction. Thus, $\mathfrak{p}$ must contain either $\mathfrak{a}$ or $\mathfrak{b}$, so $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. Next, suppose $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. WLOG, suppose $\mathfrak{p} \in V(\mathfrak{a})$. Then clearly $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. Thus, $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$, as desired.

1.16

Problem

Draw pictures of Spec $\mathbb{Z}$, Spec $\mathbb{R}$, Spec $\mathbb{C}[x]$, Spec $\mathbb{R}[x]$, Spec $\mathbb{Z}[x]$.

Solution

TODO

1.17

Problem

For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$

ii) $X_f = \varnothing \iff f$ is nilpotent

iii) $X_f = X \iff f$ is a unit

iv) $X_f = X_g \iff r((f)) = r((g))$

v) X is quasi-compact.

vi) each X_f is quasi-compact.

vii) An open subset of X is quasi-compact if and only if it is the finite union of sets X_f.

Solution

The complement of $V(E)$ is $\bigcup_{f \in E} X_f$ by De Morgan's laws. $X_f \cap X_g = V((f)) \cup V((g)) = V((fg))$.

The nilpotent elements are precisely those contained in every prime ideal.

The units are precisely those that generate (1) and $V(1) = \varnothing$.

If $X_f = X_g$ then $V(r((f))) = V(r((g)))$. Any prime ideal containing $r((f))$ contains $r((g))$ and vice versa so the intersection of these is $r((f)) = r((g))$.

Without loss of generality, X is covered by X_f for $f \in S$. $V(S) = \varnothing$ so S is not contained in any maximal ideal and must generate (1). So $1 = \sum_{i \in J} g_i f_i$ for some finite subset J. and so $(f_i)_{i \in J} = (1)$ and X_{f_i} cover X.

The intersection of prime ideals containing S is the radical $r(S)$. $V(S) = V((f))$ so $(f) \subseteq \mathfrak{p}$ for all $S \subseteq \mathfrak{p}$ which implies $(f) \subseteq r(S)$. So $f^n = \sum_{i \in J} g_i f_i$ for a finite subset J and some $n > 0$. So $\{X_{f_i} : i \in J\}$ covers X_f.

If an open subset of X is not a finite union of X_f then any cover using X_f must be the infinite union of X_f. If it is a finite union of X_f then since each X_f is individually compact we must have that every open cover has a finite subcover for each X_f in the union, which is a finite subcover of the open subset.

1.18

Problem

Show that

i) The set $\{x\}$ is closed.

ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$

iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$

iv) X is a T_0-space.

Solution

If $\mathfrak{p}_x$ is maximal then there is an open set containing just $\{x\}$. If there is an open set containing just $\{x\}$ then x is not a subset of any other prime ideal and is therefore maximal.

The smallest closed set containing $\{x\}$ consists of the prime ideals containing x.

y contains x if and only if y is in the set of all prime ideals satisfying this property.

Every distinct pair of points satisfy $x \not\subseteq y$ without loss of generality. In this case $y \notin \overline{\{x\}}$.

1.19

Problem

A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec A is irreducible if and only if the nilradical of A is a prime ideal.

Solution

If the nilradical is prime then every open set contains the nilradical, as it is a point in our space.

For $fg \in \mathfrak{N}$ with $f, g \notin \mathfrak{N}$, $X_{fg} = \varnothing$ but X_f and X_g are nonempty.

1.20

Problem

Let X be a topological space.

i) If Y is an irreducible subspace of X then the closure $\overline{Y}$ in X is irreducible.

ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.

iii) The maximal irreducible subspaces are closed and cover X. What are the irreducible components of a Hausdorff space?

iv) If A is a ring and $X = \mathrm{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

Solution

For A, B in the closure of Y, $A \cap Y$ and $B \cap Y$ are open. The nonempty case would have nonempty intersection, so without loss of generality $A \cap Y$ is empty. Then $\overline{Y} \setminus Y$ has nonempty interior, which is impossible.

The union of a chain of irreducible subspaces is irreducible. To see this, every point in the union of a chain is contained in some element of that chain. Let X_i be such that A is nonempty and open and X_j be such that B is nonempty and open. Without loss of generality, $X_i \subseteq X_j$, so A and B are nonempty and open in X_j and have nonempty intersection there, and therefore nonempty intersection in the union. By Zorn's lemma, there is a maximal irreducible subspace.

Every singleton is an irreducible subspace. Every point's singleton is contained in some maximal irreducible subspace, which proves covering. Moreover, the maximal irreducible subspaces must be equal to their closure and therefore are closed.

The irreducible components of a Hausdorff space are singletons, as every two distinct points have disjoint open sets containing them. This irreducibility in fact is equivalent to the Hausdorff condition.

Every two open sets in $V(\mathfrak{p})$ must both contain the point $\mathfrak{p}$. Every nonzero ideal contains a minimal prime ideal, so these closed irreducible subspaces are covering. If $V(\mathfrak{p}) \cup V(\mathfrak{p}')$ is irreducible then $V(\mathfrak{p}\mathfrak{p}')$ is irreducible, but for distinct minimal prime ideals, this product is 0.

1.21

Problem

Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$ then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A. Consider the induced mapping $\phi^* : Y \to X$.

i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$.

ii) If $\mathfrak{a}$ is an ideal of A then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

iii) If $\mathfrak{b}$ is an ideal of B then $\overline{\phi^*(V_b)} = V(b^c)$.

iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker(\phi))$ of X.

v) $\phi^*(Y)$ is dense in $X \iff \ker(\phi) \subseteq \mathfrak{N}$.

vi) Let $\psi : B \to C$ be another ring homomorphism. $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions in A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

Solution

$f \in \phi^{-1}(\mathfrak{q}) \iff \phi(f) \in \mathfrak{q}$. So $\phi^{-1}(\mathfrak{q}) \in V(f) \iff \mathfrak{q} \in V(\phi(f))$.

$\phi^{*-1}(V(\mathfrak{a})) = \bigcap_{f \in \mathfrak{a}} \phi^{*-1}(V(f)) = \bigcap_{f \in \mathfrak{a}} V(\phi(f))$ So this maps to $V(\mathfrak{a}^e)$, since it is the smallest prime set containing $\phi(f)$ for $f \in \mathfrak{a}$.

A prime ideal contains $r(\mathfrak{b})^c$ iff it contains the contraction of every prime ideal containing $\mathfrak{b}$, which is the image of $V(\mathfrak{b})$ under ϕ^*. The result follows by $V(\bigcap A) = \bar{A}$.

In the map $\mathfrak{b} \mapsto b^c$ and inverse map $\mathfrak{a} \mapsto \mathfrak{a}^e$, Continuity of both maps follows by the second and third parts. The set of contracted ideals is the set of prime ideals containing $\ker(\phi)$. So the composition of the maps is the identity.

$\ker(\phi) \subseteq \mathfrak{N} \iff V(\ker(\phi)) = X$ which implies ϕ^* is a homeomorphism. Density implies that no prime ideal is contained in $X \setminus V(\ker(\phi))$.

$$\begin{aligned}
(\psi \circ \phi)^*(\mathfrak{q}) &= (\psi \circ \phi)^{-1}(\mathfrak{q}) \\
&= \phi^{-1}(\psi^{-1}(\mathfrak{q})) \\
&= (\phi^* \circ \psi^*)(\mathfrak{q})
\end{aligned}$$

There are two prime ideals in B which are $(1) \times (0)$ and $(0) \times (1)$ since $(0) \times (0)$ is the product of these two prime ideals and $(1) \times (1)$ is the entire space. $\phi^*((0) \times (1))$ is the prime ideal $\mathfrak{p}$. $\phi^*((1) \times (0))$ is the prime ideal 0. Since there are only two elements each, ϕ^* is a bijection. $\phi^*(V((1) \times (0)))$ contains just 0 and not $\mathfrak{p}$, which is not closed. So ϕ^* is not an open mapping.

1.22

Problem

Let A be the direct product of rings A_i. Show that $\mathrm{Spec}(A)$ is the disjoint union of open subspaces X_i where X_i is canonically homeomorphic with $\mathrm{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following are equivalent:

i) $X = \mathrm{Spec}(A)$ is disconnected.

ii) $A \simeq A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.

iii) A contains an idempotent $\neq 0, 1$.

Solution

Let $V(\mathfrak{a})$ and $V(\mathfrak{b})$ be closed and open. Then $\mathfrak{a}+\mathfrak{b}=1$. So $a+b=1$ with $a\in\mathfrak{a}$ and $b\in\mathfrak{b}$. $V(\mathfrak{ab})$ is empty so is contained in the nilradical. So $(ab)^n=0$ and $(a^n)+(b^n)=1$ so we can take $e\in(a^n)$ and $1-e\in(b^n)$. $e-e^2=0$ so e is idempotent.

$(1,0)^2=(1,0)$ is a nonzero idempotent.

If x is idempotent, then $(1-x)^2=1-x$ is also idempotent. $(x)\cap(1-x)=(0)$ because the product is (0). Nonzero $\mathfrak{a}$ has nonzero intersection with exactly one of (x) or $(1-x)$. $\phi : A\to A/(e)\times A/(1-e)$ is a bijective homomorphism.

$e^2=e$ and $(1-e)^2=1-e$. (e) and $(1-e)$ are coprime with (0) intersection. $V((e))\cup V((1-e))=V((0))=X$. $V((e))\cap V((1-e))=V((1))=\varnothing$.

1.23

Problem

Let A be a Boolean ring and let $X=\operatorname{Spec}(A)$.

i) For $f\in A$, the lset X_f is both open and closed in X.

ii) Let $f_1,\ldots,f_n\in A$. $X_{f_1}\cup\cdots\cup X_{f_n}=X_f$ for some f.

iii) The sets X_f are the only subsets of X which are both open and closed.

iv) X is a compact Hausdorff space.

Solution

X_f and X_{1-f} are disjoint since every maximal ideal contains one or the other.

Every finitely generated ideal is principal. $(f_1,\ldots,f_n)=(f)$ for some f so $\bigcup_i X_{f_i}=X_f$.

X is quasi-compact, so $Y\subset X$ is quasi-compact. Y is open, so it can be written as the union of finitely many X_f. So Y is equal to X_g for some g.

If for every f, $(f)\subset g$ iff $(f)\subset h$ then g and h are equal. Otherwise, $g\in X_f$ and $h\in X_{1-f}$ for some f, disjoint open sets.

1.24

Problem

Let L be a Boolean lattice. Define $a+b=(a\wedge b')\vee(a'\wedge b)$ and $ab=a\wedge b$. L is a Boolean ring. Conversely, show every Boolean ring induces a Boolean lattice.

Solution

$\wedge$ is associative, idempotent, commutative, and unital.

Verification of commutativity and associativity is a matter of computation.

$a+0=(a\wedge 1)\vee 0=a$.

Distributivity follows from distributivity in the lattice and de Morgan's laws.

(the converse) 1 is the unit, 0 is the additive identity. Reflexivity of $\leq$ is idempotency of $\cdot$. Transitivity of $\leq$ is simply multiplication. $a=ab$ and $b=ab$ implies $a=b$. The complement of a is $(1-a)$. $\wedge$ distributes over $\vee$ and vice versa.

1.25

Problem

From the last two exercises deduce Stone's theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.

Solution

The open-and-closed subsets of the spectrum of the corresponding Boolean ring form a Hausdorff space such that the open and closed subsets are precisely the sets X_f for $f\in A$.

$X_f\leq X_g$ if and only if $f=fg$ which induces the Boolean lattice.

1.26

Problem

Let $f \in C(X)$. Let $U_f = \{x \in X : f(x) \neq 0\}$. Let $\tilde{X}$ be the set of all maximal ideals. Let $\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$. Let $\mathfrak{m}_x$ be the set of all functions f such that $f(x) = 0$. The map $\mu : X \to \tilde{X}$ sends $x \mapsto \mathfrak{m}_x$. Show that $\mu(U_f) = \tilde{U}_f$.

Solution

$x \in U_f \iff f(x) \neq 0 \iff f \notin m_x \iff \mu(x) \in \tilde{U}_f$.

1.27

Problem

Regular mappings acting on k-algebra homomorphisms by right composition put regular mappings in one-to-one correspondence with k-algebra homomorphisms.

Solution

1.28

Problem

Solution

Chapter 2

2.1

Problem

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution

The tensor product is generated by all possible tensor products of pairs of generators from the two rings we are tensoring. Both rings are cyclic and are generated by 1. So the tensor product is generated by $1 \otimes 1$. Now, note that $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$, and $n(1 \otimes 1) = 1 \otimes n = 1 \otimes 0 = 0$. Thus, the order of $1 \otimes 1$ must divide both m and n. However, since m and n are coprime, the only such positive number is 1. Thus, $1 \otimes 1$ has order 1, so $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is a ring that contains only one element, which is therefore 0, as desired.

2.2

Problem

Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Solution

Consider the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$. We now tensor this with M to get $0 \to \mathfrak{a} \otimes_A M \to A \otimes_A M \to A/\mathfrak{a} \otimes_A M \to 0$. By the first isomorphism theorem, we therefore have that $A/\mathfrak{a} \otimes_A M \cong A \otimes_A M / \mathfrak{a} \otimes_A M$. Clearly, $A \otimes_A M \cong M$ and $\mathfrak{a} \otimes_A M \cong \mathfrak{a}M$, so we have $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$, as desired.

2.3

Problem

Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

Solution

Let $\mathfrak{m}$ be the maximal ideal of A and let $k = A/\mathfrak{m}$ be the residue field. Then $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by 3.2. Since A is a local ring, $\mathfrak{m}$ is the only maximal ideal of A, so by Nakayama's lemma, if $M_k = 0$, then $M = 0$.

Since $M \otimes_A N = 0$, $(M \otimes_A N)_k = M_k \otimes_A N_k = 0$. Since M_k and N_k are both vector spaces over a field, this implies that either $M_k = 0$ or $N_k = 0$, which implies that either $M = 0$ or $N = 0$, as desired.

2.4

Problem

Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution

$\Rightarrow$: Suppose that M is flat. The maximal ideals of M are $\{\times_{i \neq j} M_i \times \mathfrak{m}_j | j \in I, \mathfrak{m}_{\mathrm{j}} \in M_j\}$, where $\mathfrak{m}_j$ are the maximal ideals of M_j. The quotients of all these maximal ideals must also be flat. These are all $M_j/\mathfrak{m}_j$, over all $j \in I$ and over all $\mathfrak{m}_j$ which are maximal in M_j. For a specific j, this implies that all the maximal ideals are flat. This implies that M_j is flat. So this direction is done.

$\Leftarrow$: Just go the other direction.

2.5

Problem

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Solution

$A[x]$ is just the infinite direct sum of A, considered as an A-module, so $A[x]$ is flat if A is flat as an A-algebra, by problem 2.4. But for any A-module M, $M \otimes_A A \cong M$, so clearly A is flat. Thus, $A[x]$ is a flat A-algebra.

2.6

Problem

For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r (m_t \in M)$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x] \cong A[x] \otimes_A M$.

Solution

From Proposition 2.14, we know that $(M + N) \otimes P \cong (M \otimes P) + (N \otimes P)$. We can consider $M[x]$ to be an infinite direct sum of M, and $A[x]$ to be an infinite direct sum of A. From Proposition 2.14, we know that when M is a module of A, $A \otimes M \cong M$. These two facts immediately imply the desired conclusion, $M[x] \cong A[x] \otimes_A M$.

2.7

Problem

Let $\mathfrak{p}$ be a prime ideal of A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

Solution

Let $f, g \in A[x] - \mathfrak{p}[x]$. Let

$$f = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$$

and let

$$g = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$$

Suppose a_i is the coefficient with smallest index that is not in $\mathfrak{p}$, and let b_j be defined likewise. These must both exist because $f, g \notin \mathfrak{p}[x]$. Then consider the coefficient of x^{i+j} in fg. WLOG, let $i < j$. Then the coefficient of x^{i+j} in fg is

$$\sum_{r=0}^{i-1} a_r b_{i+j-r} + a_i b_j + \sum_{s=0}^{j-1} a_{i+j-s} b_s$$

Both the sums are in $\mathfrak{p}$ because $a_r \in \mathfrak{p}$ for $r < i$ and $b_s \in \mathfrak{p}$ for $s < j$, by the definitions of i and j. But a_i and b_j are not in $\mathfrak{p}$, and $\mathfrak{p}$ is prime, so a_ib_j is not in $\mathfrak{p}$. Thus, fg is not in $\mathfrak{p}[x]$. Thus, $\mathfrak{p}[x]$ is prime in $A[x]$, as desired.

2.8

Problem

i) If M and N are flat A-modules, then so is $M \otimes_A N$.

ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution

i) Let $0 \to B \to C \to D \to 0$ be an exact sequence of A-modules. Since M is flat,

$$0 \to B \otimes_A M \to C \otimes_A M \to D \otimes_A M \to 0$$

is exact. Since N is flat,

$$0 \to (B \otimes_A M) \otimes_A N \to (C \otimes_A M) \otimes_A N \to (D \otimes_A M) \otimes_A N \to 0$$

is also exact. By Proposition 2.14, $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$, so therefore we have

$$0 \to B \otimes_A (M \otimes_A N) \to C \otimes_A (M \otimes_A N) \to D \otimes_A (M \otimes_A N) \to 0$$

is exact. Thus, $M \otimes_A N$ is flat, as desired.

ii) Let $0 \to C \to D \to E \to 0$ be an exact sequence of A-modules. Since B is a flat A-algebra,

$$0 \to C \otimes_A B \to D \otimes_A B \to E \otimes_A B \to 0$$

is exact. Since N is a flat B-module,

$$0 \to C \otimes_A B \otimes_B N \to D \otimes_A B \otimes_B N \to E \otimes_A B \otimes_B N \to 0$$

But $B \otimes_B N \cong N$, so we get

$$0 \to C \otimes_A N \to D \otimes_A N \to E \otimes_A N \to 0$$

which implies that N is flat as an A-module, as desired.

2.9

Problem

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Solution

Let $u_1, u_2, \cdots, u_m$ be the generators of M' and let $v_1, v_2, \cdots v_n$ be the generators of M''. Let $w_i = f(u_i), 1 \leq i \leq m$. Since M surjects onto M'', for each of the v_i, there is an x_i such that $f(x_i) = v_i$.

Let f be the map from M to M'' in the exact sequence. For each $m \in M$, either m goes to 0 or $f(m)$ is a finite sum $s_1 v_1 + s_2 v_2 + \cdots + s_n v_n$. In the first case, it is in the submodule generated by $w_1, w_2, \cdots, w_m$. In the second case, it is in the submodule generated by $x_1, x_2, \cdots, x_n$. In all cases, m is in the submodule generated by $w_1, w_2, \cdots, w_m, x_1, x_2, \cdots, x_n$. Thus M is finitely generated as desired.

2.10

Problem

Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution

The induce homomorphism is surjective, so $u(x) + \mathfrak{a}N$ passes through all values of $N/\mathfrak{a}N$. $\Im(u) + \mathfrak{a}N = N$ so $\Im(u) = N$ by Nakayama's lemma.

2.11

Problem

Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.

Solution

Let $\mathfrak{m}$ be a maximal ideal of A, let $k = A/\mathfrak{m}$, and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes_k \phi : k \otimes_k A^m \to k \otimes_k A^n$ is an isomorphism of vector spaces of dimensions m and n. Thus, $m = n$.

2.12

Problem

Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that Ker (ϕ) is finitely generated.

Solution

Ker $(\phi) \subset M$ and M is finitely generated, so Ker (ϕ) must be finitely generated.

2.13

Problem

Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution

Let $p : N_B \to N$ be $p(b \otimes y) = by$ which is induced uniquely by the universal property of the tensor product, as $\tilde{p}(b, y) = by$ is bilinear.

$r(a, n) = p(f(a) \otimes n)$ is bilinear in A and N. If $r(a, n) = 0$ then $a \otimes n = 0$. $r(1, n) = n$, so $1 \otimes n$ is bijective,

Considering $g \circ p$, we have $\ker(g \circ p) \oplus \Im(g \circ p) = N_B$. p is surjective so $\Im(g \circ p) = \Im(g)$. By the injectivity of g, $\ker(g \circ p) = \ker(p)$.

Chapter 3

3.1

Problem

Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution

$\Leftarrow$: Let $m/t \in S^{-1}M$. Now note that $m/t = 0/1 \Leftrightarrow mu = 0$ for some $u \in S$. But since $sM = 0$, $ms = 0$ for all $m \in M$, so setting $u = s$ establishes that $m/t = 0$, so $S^{-1}M = 0$.

$\Rightarrow$: Suppose that $S^{-1}M = 0$. Then for all $m/1 \in S^{-1}M$, we have $m/1 = 0/1$, so there exists a $u \in S$ such that $mu = 0$. Let $e_1, e_2, \cdots, e_n$ be the generators of M. Let their corresponding annihilators in S be $u_1, u_2, \cdots, u_n$. Then $u_1 u_2 \cdots u_n$ annihilates every element of M. Thus, we are done.

3.2

Problem

Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1+\mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Solution

I will show that $\mathfrak{a}$ is contained in every maximal ideal of A. Suppose not. Then there would exist $a \in \mathfrak{a}$, $m \in M$ such that $a+m = 1$. This implies that a is a unit. Thus, $\mathfrak{a}$ is contained in every maximal ideal of A. It is also clear that $\mathfrak{a}$ and S are disjoint. By the one-to-one correspondence between prime ideals of A and prime ideals of $S^{-1}A$, this means that $S^{-1}\mathfrak{a}$ is contained in all the maximal ideals of $S^{-1}A$. The one-to-one correspondence applies to maximal ideals because they are disjoint from $S = 1 + \mathfrak{a}$. If $m = 1 + a$, this implies that a is a unit. Therefore, $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, as desired.

3.3

Problem

Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Solution

Let $f : (ST)^{-1}A \to U^{-1}(S^{-1}A)$ be defined by $f(a/st) = (a/s)/(t/1)$. This map is clearly surjective. Now suppose that $f(a_1/s_1t_1) = f(a_2/s_2t_2)$. Then $(a_1/s_1)/(t_1/1) = (a_2/s_2)/(t_2/1)$. Then there is some $a_3/s_3 \in S^{-1}A$ such that $((a_1/s_1)(t_2/1) - (a_2/s_2)(t_1/1))a_3/s_3 = 0 \Rightarrow (a_1t_2/s_1 - a_2t_1/s_2)a_3/s_3 = 0 \Rightarrow (a_1t_2s_2 - a_2t_1s_1)/s_1s_2 \cdot (a_3/s_3) = 0 \Rightarrow a_1t_2s_2a_3 - a_2t_1s_1a_3 = 0$ This means that $a_1/(s_1t_1) = a_2/(s_2t_2)$, so f is injective as well. Thus, f is an isomorphism, as desired.

3.4

Problem

Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Solution

Let $g_1 : B \to S^{-1}B$ and $g_2 : B \to T^{-1}B$ be the canonical maps. g_2 induces a unique map $h_2 \circ g_1 = g_2$. g_1 induces a unique map $h_1 \circ g_2 = g_1$. $h_1 \circ h_2$ is unique by the previous exercise, and is the identity. Similarly, $h_2 \circ h_1$ is the identity.

3.5

Problem

Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Solution

Suppose $x \in A$ was nilpotent and $\neq 0$. It must be contained in every prime ideal, so let $\mathfrak{p}$ be some prime ideal of A. Let $y \in A - \mathfrak{p}$, which is clearly not nilpotent, and consider the element $x/y \in A_\mathfrak{p}$. Suppose that $x^n = 0$. Then $(x/y)^n = x^n/y^n = 0/y^n$, which is equal to $0/1$, the zero element of $A_\mathfrak{p}$. This violates the assumption that $A_\mathfrak{p}$ has no nonzero nilpotent element. Thus we are done.

3.6

Problem

Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.

Solution

Σ has maximal elements by Zorn's lemma, because $A - \{0\}$ is a multiplicatively closed subset that contains all the others, and inclusion is a partial order. Suppose $S \in \Sigma$ is maximal. Consider $A - S$. If this was not prime, then S would not be multiplicatively closed, because there would exist $x, y \in S$ such that $xy \in A - S$. Thus, $A - S$ is prime. Suppose that $A - S$ was not minimal, and suppose $T \subset S$ was a proper subset of S forming a prime ideal. Then $A - T$ would be multiplicatively closed and contain S, which contradicts the fact that S is maximal. Thus, $A - S$ is a minimal prime ideal, as desired.

For the other direction, suppose $A - S$ is a minimal prime ideal. Then clearly S is multiplicatively closed by the definition of prime ideals. If S was not maximal, then suppose $S \subset T$ for some T that is multiplicatively closed. Then $A - T$ would be prime and be contained in $A - S$ which contradicts the fact that $A - S$ is minimal. Thus, we are done.

3.7

Problem

A multiplicatively closed subset S of a ring A is said to be *saturated* if

$$xy \in S \Leftrightarrow x, y \in S.$$

Prove that

i) S is saturated $\Leftrightarrow$ $A - S$ is a union of prime ideals.

ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\bar{S}$ containing S, and that $\bar{S}$ is the complement in A of the union of prime ideals which do not meet S. ($\bar{S}$ is called the *saturation* of S.)

If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\bar{S}$.

Solution

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i) Suppose $A - S$ is a union of prime ideals. Let $x, y \in S$. Then xy cannot be in any of the prime ideals comprising $A - S$, so $xy \in S$. Next, suppose $xy \in S$. WLOG, if $x \in A - S$, then x is in some prime ideal in $A - S$, so xy is also in the same prime ideal, implying $xy \in A - S$, a contradiction. Thus, $x, y \in S$. Thus, S is saturated.

For the other direction, suppose S is saturated. Let T be the union of all prime ideals which do not meet S. Let $x \in (A - S) - T$, and let $y \in S$.

If $xy \in S$, then $x \in S$, a contradiction. If $xy \in T$, then xy is in some prime ideal, but both x and y are not in that prime ideal, a contradiction. Thus, $xy \in (A - S) - T$. One can clearly see that $(A - S) - T$ must be a prime ideal, and it does not meet S, a contradiction because we defined T be the union of all prime ideals that do not meet S. Thus, $(A - S) - T$ must be the empty set, so $A - S = T$, so $A - S$ is the union of prime ideals, as desired.

ii) Let T be the union of all prime ideals that do not meet S, and consider $A - T$. This contains S, and is saturated by part i. Suppose there was a smaller set U containing S that is saturated. Then let $x \in (A-T)-U$. Then applying the same argument from part i, we see that $(A-T)-U$ is a prime ideal which does not meet S, and hence must be in T, a contradiction. Thus, T is in fact the unique smallest saturated subset containing S, as desired.

$1 \bar{+} \mathfrak{a}$ is the complement of the union of all prime ideals that do not meet $1 + \mathfrak{a}$.

3.8

Problem

Let S, T be multiplicatively closed subsets of A, such that $S \subset T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following statements are equivalent:

i) ϕ is bijective.

ii) For each $t \in T, t/1$ is a unit in $S^{-1}A$.

iii) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.

iv) T is contained in the saturation of S (Exercise 7)

v) Every prime ideal which meets T also meets S.

Solution

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i $\Rightarrow$ ii: Since ϕ is surjective, for each $a/t \in T^{-1}A$, there exists $a_1/s \in S^{-1}A$ such that a_1/s, considered as an element of $T^{-1}A$, is equal to a/t. So $\exists u \in A$ such that $u(ta_1 - sa) = 0$.

ii) TODO
iii) TODO
iv) TODO
v) TODO