# Linear Algebra

## Row Space, Column Space, and Null Space

Recall that vectors can be written in comma-delimited form or in matrix form as either row vectors or column vectors.

**DEFINITION 1** For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\ \ \vdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in $R^n$ that are formed from the rows of $A$ are called the ***row vectors*** of $A$, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $R^m$ formed from the columns of $A$ are called the ***column vectors*** of $A$.

## ▶ EXAMPLE 1 Row and Column Vectors of a 2 × 3 Matrix

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of $A$ are

$$\mathbf{r}_1 = [2 \quad 1 \quad 0]$$

$$\text{and} \quad \mathbf{r}_2 = [3 \quad -1 \quad 4]$$

and the column vectors of $A$ are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

$$\mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

# Row Space, Column Space, and Null Space

- In this section we will study some important vector spaces that are associated with matrices.

- Our work here will provide us with a deeper understanding of the relationships between the solutions of a linear system and properties of its coefficient matrix.

**DEFINITION 2** If $A$ is an $m \times n$ matrix, then the subspace of $R^n$ spanned by the row vectors of $A$ is called the ***row space*** of $A$, and the subspace of $R^m$ spanned by the column vectors of $A$ is called the ***column space*** of $A$. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $R^n$, is called the ***null space*** of $A$.

We will sometimes denote the row space of $A$, the column space of $A$, and the null space of $A$ by $\mathrm{row}(A)$, $\mathrm{col}(A)$, and $\mathrm{null}(A)$, respectively.

**Question 1.** What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix $A$?

**Question 2.** What relationships exist among the row space, column space, and null space of a matrix?

Starting with the first question, suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ denote the column vectors of $A$, then the product $A\mathbf{x}$ can be expressed as a linear combination of these vectors with coefficients from $\mathbf{x}$; that is,

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

that is, $\boxed{A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n}$

Thus, a linear system, $A\mathbf{x} = \mathbf{b}$, of $m$ equations in $n$ unknowns can be written as

$$\boxed{x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}}$$

from which we conclude that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is expressible as a linear combination of the column vectors of $A$. This yields the following theorem.

**THEOREM 4.7.1** *A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $A$.*

## ▶ EXAMPLE 2 A Vector b in the Column Space of $A$

Let $A\mathbf{x} = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of $A$ by expressing it as a linear combination of the column vectors of $A$.

**Solution**

Solving the system by Gaussian elimination yields (verify)

$$\boxed{x_1 = 2, \quad x_2 = -1, \quad x_3 = 3}$$

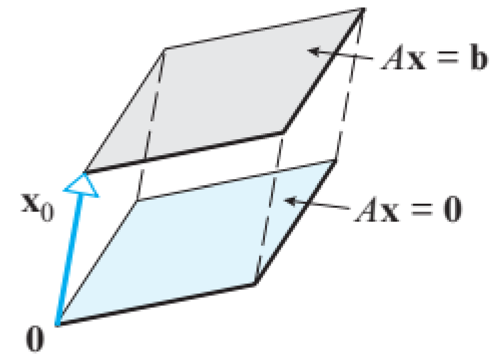
It follows from $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$ that

$$\boxed{2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}}$$

**THEOREM 3.4.4** *The general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.*

***Proof***

- Let $\mathbf{x}_0$ be any specific solution of $A\mathbf{x} = \mathbf{b}$.
- Let $W$ denote the solution set of $A\mathbf{x} = \mathbf{0}$.
- Let $\mathbf{x}_0 + W$ denote the set of all vectors that result by adding $\mathbf{x}_0$ to each vector in $W$.




▲ Figure 3.4.7 The solution set of $A\mathbf{x} = \mathbf{b}$ is a translation of the solution space of $A\mathbf{x} = \mathbf{0}$.

Assume first that $\mathbf{x}$ is a vector in $\mathbf{x}_0 + W$.

This implies that $\mathbf{x}$ is expressible in the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{w}$, where $A\mathbf{x}_0 = \mathbf{b}$ and $A\mathbf{w} = \mathbf{0}$.

Thus,
$$A\mathbf{x} = A(\mathbf{x}_0 + \mathbf{w})$$
$$= A\mathbf{x}_0 + A\mathbf{w}$$
$$\boxed{A\mathbf{x} = \mathbf{b} + \mathbf{0} = \mathbf{b}}$$

which shows that $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Conversely, let $\mathbf{x}$ be any solution of $A\mathbf{x} = \mathbf{b}$.

To show that $\mathbf{x}$ is in the set $\mathbf{x}_0 + W$ we must show that $\mathbf{x}$ is expressible in the form. Taking $\mathbf{w} = \mathbf{x} - \mathbf{x}_0$ gives
$$A\mathbf{w} = A(\mathbf{x} - \mathbf{x}_0)$$
$$= A\mathbf{x} - A\mathbf{x}_0$$
$$\boxed{A\mathbf{w} = \mathbf{b} - \mathbf{b} = \mathbf{0}}$$

- Recall that the general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of the system to the general solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

- Keeping in mind that the null space of $A$ is the same as the solution space of $A\mathbf{x} = \mathbf{0}$, we can rephrase the above theorem in the following vector form.

**THEOREM 4.7.2** *If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of $A$, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form*

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{3}$$

*Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.*

The vector $\mathbf{x}_0$ in Formula (3) is called a ***particular solution of*** $A\mathbf{x} = \mathbf{b}$, and the remaining part of the formula is called the ***general solution of*** $A\mathbf{x} = \mathbf{0}$. With this terminology Theorem 4.7.2 can be rephrased as:

*The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.*

Geometrically, the solution set of $A\mathbf{x} = \mathbf{b}$ can be viewed as the translation by $\mathbf{x}_0$ of the solution space of $A\mathbf{x} = \mathbf{0}$ (Figure 4.7.1).

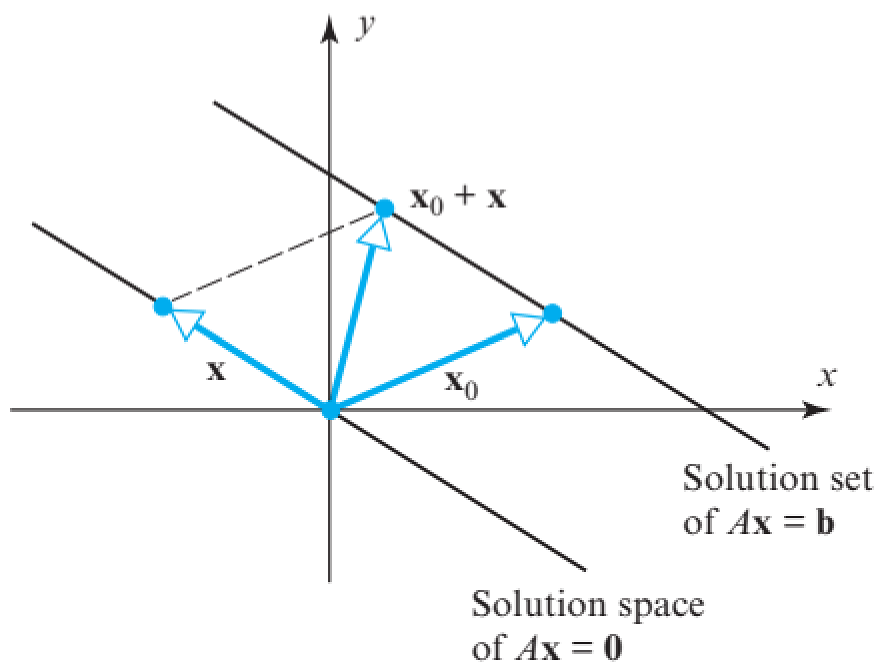



Figure 4.7.1

## ▸ EXAMPLE 3 General Solution of a Linear System $A\mathbf{x} = \mathbf{b}$

we compare the solutions of the linear systems

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

the general solution $\mathbf{x}$ of the nonhomogeneous system and the general solution $\mathbf{x}_h$ of the corresponding homogeneous system are related by

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}}_{\mathbf{x}_0} + \underbrace{r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_h}$$

**THEOREM 4.7.3** *Elementary row operations do not change the null space of a matrix.*

**THEOREM 4.7.4** *Elementary row operations do not change the row space of a matrix.*

But the elementary row operations change the column space of a matrix.

To see why this, compare the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

The matrix $B$ can be obtained from $A$ by adding $-2$ times the first row to the second.

However, this operation has changed the column space of $A$, since that column space consists of all scalar multiples of

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

whereas the column space of $B$ consists of all scalar multiples of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the two are different spaces.

## Echelon Forms

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.

2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***.

(Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

## ► EXAMPLE 1 Row Echelon and Reduced Row Echelon Form

- The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**EXAMPLE** We illustrate the idea by reducing the following matrix row echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

***Step 1.*** Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

**Leftmost nonzero column**

***Step 2.*** Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first and second rows in the preceding matrix were interchanged.

***Step 3.*** If the entry that is now at the top of the column found in Step 1 is $a$, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} \underline{1} & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first row of the preceding matrix was multiplied by $\frac{1}{2}$.

***Step 4.*** Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} \underline{1} & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$-2$ times the first row of the preceding matrix was added to the third row.

***Step 5.*** Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row echelon form.

$$\begin{bmatrix} \underline{1} & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & \underline{-2} & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

**Leftmost nonzero column in the submatrix**

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & \underline{1} & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & \underline{1} & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

–5 times the first row of the submatrix was added to the second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \underline{\frac{1}{2}} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again to Step 1.

**Leftmost nonzero column in the new submatrix**

$$\begin{bmatrix} \underline{1} & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & \underline{1} & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \underline{1} & 2 \end{bmatrix}$$

The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

$\boxed{\text{The } \textit{entire} \text{ matrix is now in row echelon form.}}$

**THEOREM 4.7.5** *If a matrix $R$ is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of $R$, and the column vectors with the leading 1's of the row vectors form a basis for the column space of $R$.*

## EXAMPLE Bases for the Row and Column Spaces of a Matrix in Row Echelon Form

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

***Solution***

Since the matrix $R$ is in row echelon form, it follows from Theorem 4.7.5 that the vectors

$$\mathbf{r}_1 = [1 \quad -2 \quad 5 \quad 0 \quad 3]$$
$$\mathbf{r}_2 = [0 \quad 1 \quad 3 \quad 0 \quad 0]$$
$$\mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 1 \quad 0]$$

form a basis for the row space of $R$, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of $R$.

### ▸ EXAMPLE 6 Basis for a Row Space by Row Reduction

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

***Solution***

Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of $A$ by finding a basis for the row space of any row echelon form of $A$.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$$

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \end{matrix}$$

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \end{matrix}$$

Reducing $A$ to row echelon form, we obtain (verify)

$$R = \begin{bmatrix} \underline{1} & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & \underline{1} & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & \underline{1} & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4.7.5, the nonzero row vectors of $R$ form a basis for the row space of $R$ and hence form a basis for the row space of $A$. These basis vectors are

$$\boxed{\begin{aligned} \mathbf{r}_1 &= [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4] \\ \mathbf{r}_2 &= [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5] \end{aligned}}$$

## Basis for the Column Space of a Matrix

Although elementary row operations can change the column space of a matrix, it follows from Theorem 4.7.6($b$) that they do not change the *dimension* of its column space.

**THEOREM 4.7.6** *If $A$ and $B$ are row equivalent matrices, then:*

($a$) *A given set of column vectors of $A$ is linearly independent if and only if the corresponding column vectors of $B$ are linearly independent.*

($b$) *A given set of column vectors of $A$ forms a basis for the column space of $A$ if and only if the corresponding column vectors of $B$ form a basis for the column space of $B$.*

## ▸ EXAMPLE 7 Basis for a Column Space by Row Reduction

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that consists of column vectors of $A$.

**Solution** We observed in Example 6 that the matrix

$$R = \begin{bmatrix} \underline{1} & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & \underline{1} & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & \underline{1} & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is a row echelon form of $A$.

- Keeping in mind that $A$ and $R$ can have different column spaces, we cannot find a basis for the column space of $A$ directly from the column vectors of $R$.

- However, it follows from Theorem 4.7.6(*b*) that if we can find a set of column vectors of $R$ that forms a basis for the column space of $R$, then the *corresponding* column vectors of $A$ will form a basis for the column space of $A$.

- Since the first, third, and fifth columns of $R$ contain the leading 1's of the row vectors, the vectors

$$\mathbf{c}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}'_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}'_5 = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of $R$.

- Thus, the corresponding column vectors of $A$, which are

$$\boxed{\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}}$$

form a basis for the column space of $A$.

## Bases Formed from Row and Column Vectors of a Matrix

- In Example 6, we found a basis for the row space of a matrix by reducing that matrix to row echelon form.
- However, the basis vectors produced by that method were not all row vectors of the original matrix.

The following adaptation of the technique used in Example 7 shows how to find a basis for the row space of a matrix that consists entirely of row vectors of that matrix.

**EXAMPLE 9 Basis for the Row Space of a Matrix**

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from $A$.

## Solution

We will transpose $A$, thereby converting the row space of $A$ into the column space of $A^T$; then we will use the method of Example 7 to find a basis for the column space of $A^T$; and then we will transpose again to convert column vectors back to row vectors.

Transposing $A$ yields

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

and then reducing this matrix to row echelon form we obtain

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \underline{1} & 2 & 0 & 2 \\ 0 & \underline{1} & -5 & -10 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns contain the leading 1's, so the corresponding column vectors in $A^T$ form a basis for the column space of $A^T$; these are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

$$\boxed{\mathbf{r}_1 = [1 \quad -2 \quad 0 \quad 0 \quad 3],}$$

$$\boxed{\mathbf{r}_2 = [2 \quad -5 \quad -3 \quad -2 \quad 6],}$$

$$\boxed{\mathbf{r}_4 = [2 \quad 6 \quad 18 \quad 8 \quad 6]}$$

for the row space of $A$.

THANK YOU