Linear Algebra

Kernel and Range

- Recall that if A is an $m \times n$ matrix, then the null space of A consists of all vectors $\mathbf{x}$ in R^n such that $A\mathbf{x} = \mathbf{0}$.

From the viewpoint of matrix transformations, the null space of A consists of all vectors in R^n that multiplication by A maps into $\mathbf{0}$

- The column space of A consists of all vectors $\mathbf{b}$ in R^m for which there is at least one vector $\mathbf{x}$ in R^n such that $A\mathbf{x} = \mathbf{b}$.

From the viewpoint of matrix transformations, the column space of A consists of all vectors in R^m that are images of at least one vector in R^n under multiplication by A.

The following definition extends these ideas to general linear transformations.

DEFINITION 2 If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the ***kernel*** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the ***range*** of T and is denoted by $R(T)$.

▶ EXAMPLE 13 Kernel and Range of a Matrix Transformation

If $T_A: R^n \to R^m$ is multiplication by the $m \times n$ matrix A, then, as discussed above, the kernel of T_A is the null space of A, and the range of T_A is the column space of A.

▶ EXAMPLE 14 Kernel and Range of the Zero Transformation

Let $T: V \to W$ be the zero transformation. Since T maps *every* vector in V into $\mathbf{0}$, it follows that $\ker(T) = V$. Moreover, since $\mathbf{0}$ is the *only* image under T of vectors in V, it follows that $R(T) = \{\mathbf{0}\}$.

▶ EXAMPLE 15 Kernel and Range of the Identity Operator

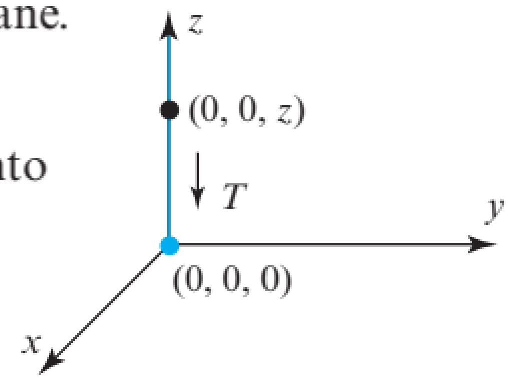
Let $I: V \to V$ be the identity operator. Since $I(\mathbf{v}) = \mathbf{v}$ for all vectors in V, *every* vector in V is the image of some vector (namely, itself); thus $R(I) = V$. Since the *only* vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, it follows that $\ker(I) = \{\mathbf{0}\}$.

▶ EXAMPLE 16 Kernel and Range of an Orthogonal Projection

Let $T : R^3 \to R^3$ be the orthogonal projection onto the xy-plane.

As illustrated in Figure 8.1.2a, the points that T maps into $\mathbf{0} = (0, 0, 0)$ are precisely those on the z-axis, so

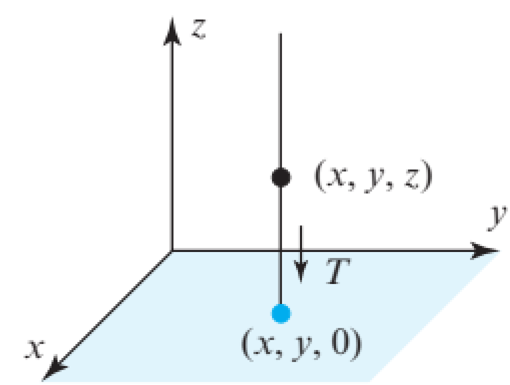
$$\boxed{\ker(T) \text{ is the set of points of the form } (0, 0, z).}$$



(a) $\ker(T)$ is the z-axis.

As illustrated in Figure 8.1.2b, T maps the points in R^3 to the xy-plane, where each point in that plane is the image of each point on the vertical line above it.

$$\boxed{\text{Thus, } R(T) \text{ is the set of points of the form } (x, y, 0).}$$

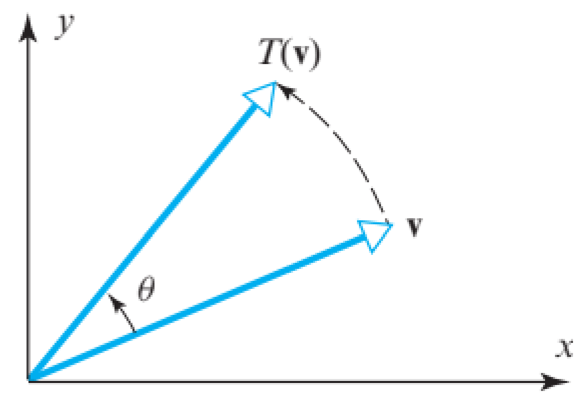


(b) $R(T)$ is the entire xy-plane

▶ Figure 8.1.2

▶ EXAMPLE 17 Kernel and Range of a Rotation

Let $T: R^2 \to R^2$ be the linear operator that rotates each vector in the xy-plane through the angle θ (Figure 8.1.3).



▲ Figure 8.1.3

Since *every* vector in the xy-plane can be obtained by rotating some vector through the angle θ, it follows that

$$\boxed{R(T) = R^2.}$$

Moreover, the only vector that rotates into $\mathbf{0}$ is $\mathbf{0}$, so

$$\boxed{\ker(T) = \{\mathbf{0}\}.}$$

▶ EXAMPLE 18 Kernel of a Differentiation Transformation

Let $V = C^1(-\infty, \infty)$ be the vector space of functions with continuous first derivatives on $(-\infty, \infty)$, let $W = F(-\infty, \infty)$ be the vector space of all real-valued functions defined on $(-\infty, \infty)$, and let $D: V \to W$ be the differentiation transformation $D(\mathbf{f}) = f'(x)$. The kernel of D is the set of functions in V with derivative zero. From calculus, this is the set of constant functions on $(-\infty, \infty)$. ◀

Properties of Kernel and Range

THEOREM 8.1.3 *If $T: V \rightarrow W$ is a linear transformation, then:*

(a) *The kernel of T is a subspace of V.*

(b) *The range of T is a subspace of W.*

Rank and Nullity of Linear Transformations

DEFINITION 3 Let $T: V \rightarrow W$ be a linear transformation. If the range of T is finite-dimensional, then its dimension is called the ***rank of T***; and if the kernel of T is finite-dimensional, then its dimension is called the ***nullity of T***. The rank of T is denoted by $\text{rank}(T)$ and the nullity of T by $\text{nullity}(T)$.

THEOREM 8.1.4 Dimension Theorem for Linear Transformations

If $T: V \rightarrow W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and

$$\mathit{rank}(T) + \mathit{nullity}(T) = \dim(V) \tag{7}$$

Example

Find $\ker T$, where $T\colon\ E^3 \to E^2$ is defined by

$$T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 - x_3).$$

Solution

Since $\ker T = \{\mathbf{x} | T(\mathbf{x}) = \mathbf{0}\}$, we must solve $T((x_1, x_2, x_3)) = (0, 0)$, that is,

$$(x_1 + x_2, x_2 - x_3) = (0, 0)$$

The resulting equations are

$$x_1 + x_2 = 0$$
$$x_2 - x_3 = 0$$

which have solution $(-k, k, k)$. Therefore

$$\boxed{\ker T = \{\mathbf{v} \in E^3 | \mathbf{v} = k(-1, 1, 1)\} = \operatorname{span}\{(-1, 1, 1)\}}$$

Example

Define T from E^3 to E^3 by

$$T((a, b, c)) = (a - b + c, 2a + b - c, -a - 2b + 2c).$$

Determine range T and dim(range T).

Solution

- Let $\mathbf{y} = (y_1, y_2, y_3)$ be in range T.
- Thus $\mathbf{y} = T((a, b, c))$ for some vector (a, b, c) in E^3.
- That is, the equation $\mathbf{y} = T((a, b, c))$ **must be consistent**.

Solving $T(\mathbf{x}) = \mathbf{y}$, we find

$$(a - b + c, 2a + b - c, -a - 2b + 2c) = (y_1, y_2, y_3)$$

The equations are

$$\begin{aligned} a - b + c &= y_1 \\ 2a + b - c &= y_2 \\ -a - 2b + 2c &= y_3 \end{aligned}$$

We reduce the equations and see what conditions the consistency forces.

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 2 & 1 & -1 & y_2 \\ -1 & -2 & 2 & y_3 \end{array}\right)$$

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 3 & -3 & y_2 - 2y_1 \\ 0 & -3 & 3 & y_1 + y_3 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 1 & -1 & \left(\frac{1}{3}\right)(y_2 - 2y_1) \\ 0 & -3 & 3 & y_1 + y_3 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \to R_2/3 \\ R_3 \end{array}$$

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 1 & -1 & \left(\frac{1}{3}\right)(y_2 - 2y_1) \\ 0 & 0 & 0 & -y_1 + y_2 + y_3 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

The above system will be consistent if

$$-y_1 + y_2 + y_3 = 0$$

That is,

$$\boxed{\text{range } T = \{(y_1, y_2, y_3) | y_1 = y_2 + y_3\}}$$

The dimension of range T is 2, since the equation $-y_1 + y_2 + y_3 = 0$ allows the assignment of arbitrary values to any **two** of the values of y_k.

Compositions and Inverse Transformations

Compositions of Matrix Transformations

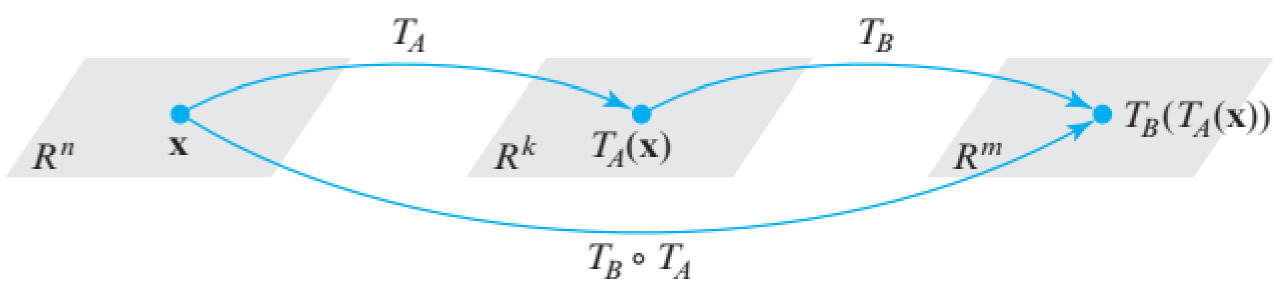
- Suppose that T_A is a matrix transformation from R^n to R^k and T_B is a matrix transformation from R^k to R^m.
- If $\mathbf{x}$ is a vector in R^n, then T_A maps this vector into a vector $T_A(\mathbf{x})$ in R^k, and T_B, in turn, maps that vector into the vector $T_B(T_A(\mathbf{x}))$ in R^m.
- This process creates a transformation from R^n to R^m that we call the ***composition of T_B with T_A*** and

$$T_B \circ T_A$$

which is read "T_B circle T_A."

Figure

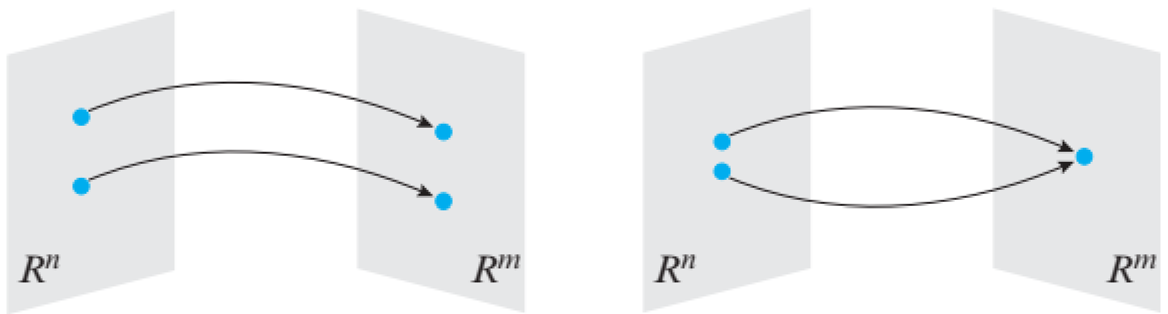
$$\boxed{T_B \circ T_A = T_{BA}}$$



One-to-One Matrix Transformations

DEFINITION 1 A matrix transformation $T_A: R^n \to R^m$ is said to be ***one-to-one*** if T_A maps distinct vectors (points) in R^n into distinct vectors (points) in R^m.

Figure 4.10.5



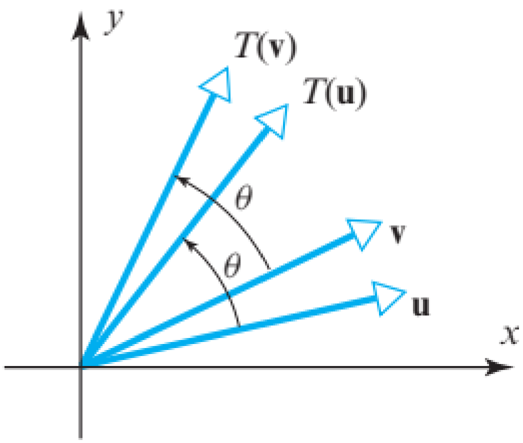
One-to-one   Not one-to-one

This idea can be expressed in various ways.

1. T_A is one-to-one if for each vector $\mathbf{b}$ in the range of A there is exactly one vector $\mathbf{x}$ in R^n such that $T_A\mathbf{x} = \mathbf{b}$.

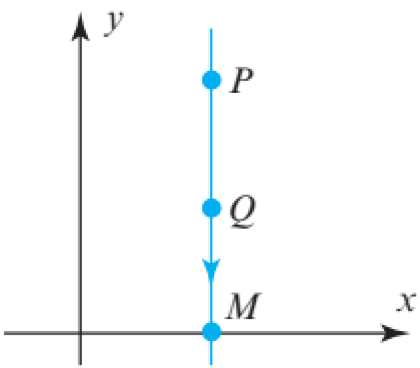
2. T_A is one-to-one if the equality $T_A(\mathbf{u}) = T_A(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.

- Rotation operators on R^2 are one-to-one since distinct vectors that are rotated through the same angle have distinct images



▲ Figure 4.10.6 Distinct vectors **u** and **v** are rotated into distinct vectors $T(\mathbf{u})$ and $T(\mathbf{v})$.

- In contrast, the orthogonal projection of R^2 onto the x-axis is not one-to-one because it maps distinct points on the same vertical line into the same point



▲ Figure 4.10.7 The distinct points P and Q are mapped into the same point M.

THEOREM 4.10.1 *If A is an $n \times n$ matrix and $T_A: R^n \to R^n$ is the corresponding matrix operator, then the following statements are equivalent.*

(*a*) *A is invertible.*

(*b*) *The kernel of T_A is $\{\mathbf{0}\}$.*

(*c*) *The range of T_A is R^n.*

(*d*) *T_A is one-to-one.*

▶ EXAMPLE 5 The Rotation Operator on R^2 Is One-to-One

As was illustrated in Figure 4.10.6, the operator $T: R^2 \to R^2$ that rotates vectors through an angle θ is one-to-one. In accordance with parts (*a*) and (*d*) of Theorem 4.10.1, show that the standard matrix for T is invertible.

Solution

We will show that the standard matrix for T is invertible by showing that its determinant is nonzero.

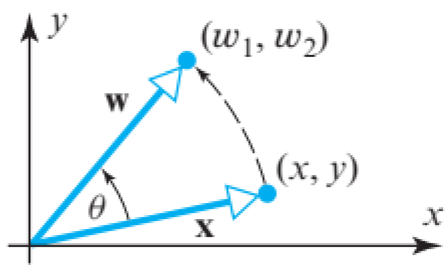
From Table 5 of Section 4.9 the standard matrix for T is

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is invertible because

$$\det[T] = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Table 5

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

▶ EXAMPLE 6 Projection Operators Are Not One-to-One

As illustrated in Figure 4.10.7, the operator $T: R^2 \to R^2$ that projects onto the x-axis in the xy-plane is not one-to-one. In accordance with parts (a) and (d) of Theorem 4.10.1, show that the standard matrix for T is not invertible.

Solution

We will show that the standard matrix for T is not invertible by showing that its determinant is zero. From Table 3 of Section 4.9 the standard matrix for T is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ Since $\det[T] = 0$, the operator T is not one-to-one.

Table 3

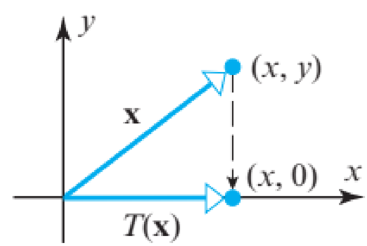
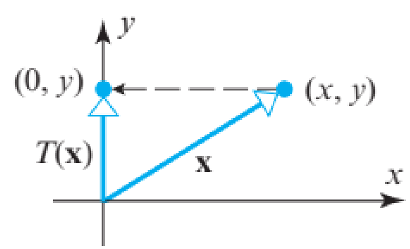
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Table 4

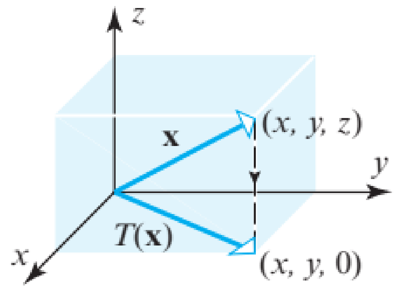
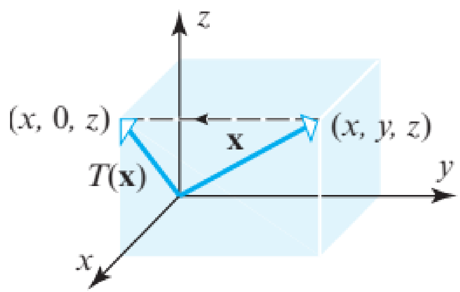
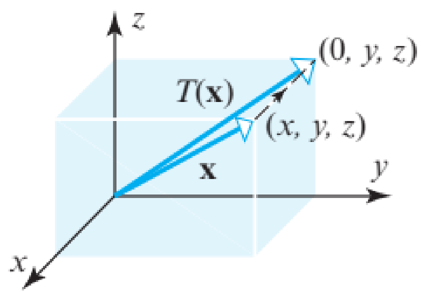
Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table 6

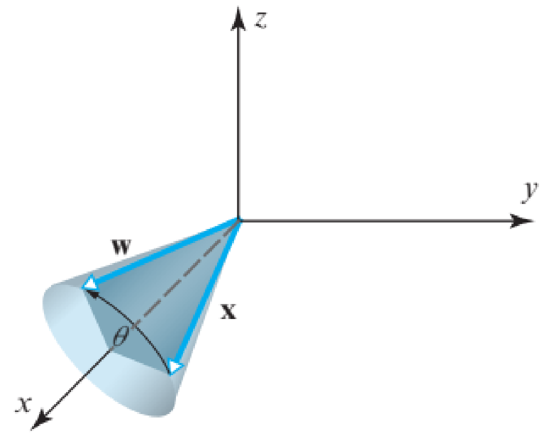
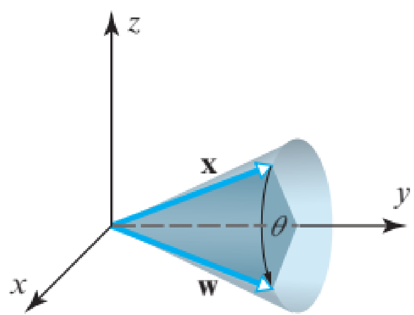
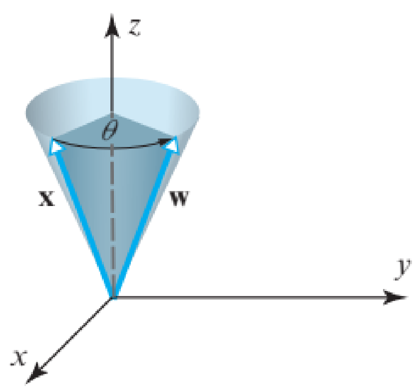
Operator	**Illustration**	**Rotation Equations**	**Standard Matrix**
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table 7

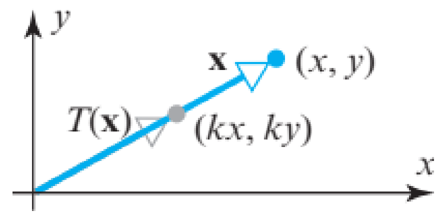
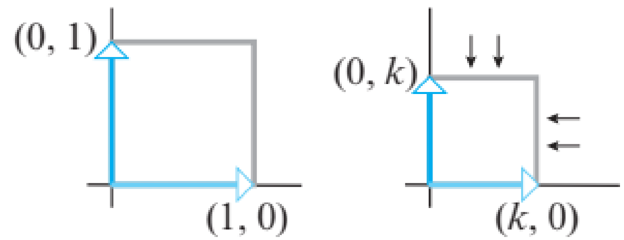
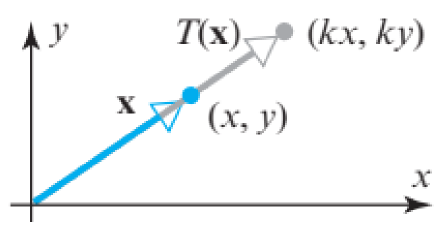
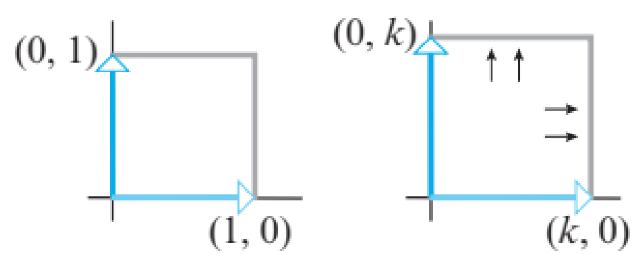
<table>
<tr><th>Operator</th><th>Illustration
$T(x, y) = (kx, ky)$</th><th>Effect on the
Unit Square</th><th>Standard
Matrix</th></tr>
<tr><td>Contraction with factor k in R^2
$(0 \le k < 1)$</td><td>
</td><td>
</td><td rowspan="2">$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$</td></tr>
<tr><td>Dilation with factor k in R^2
$(k > 1)$</td><td>
</td><td>
</td></tr>
</table>

Table 8

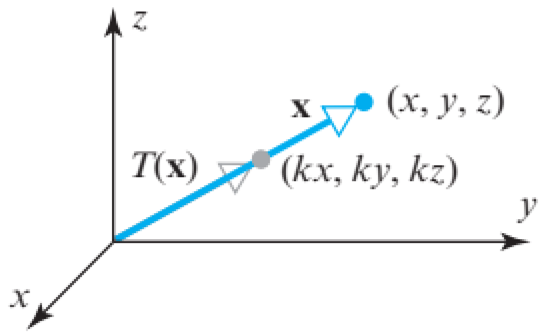
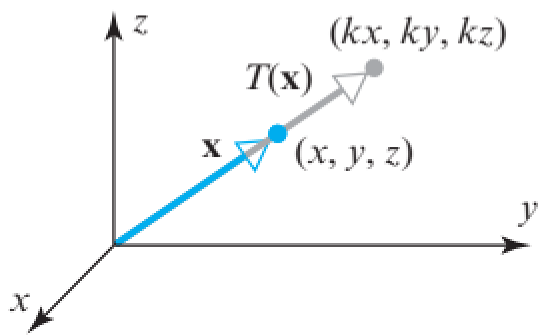
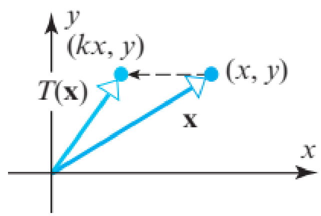
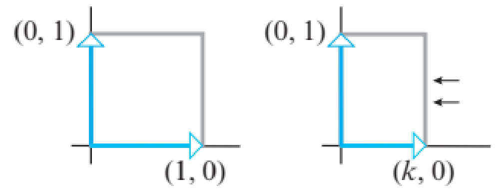
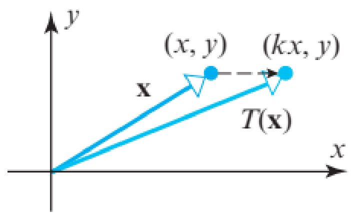
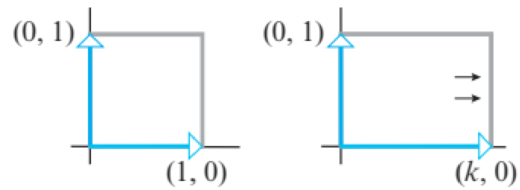
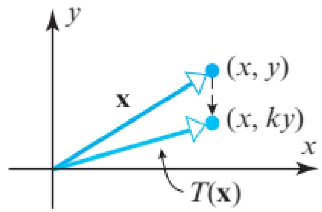
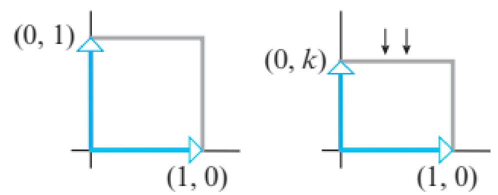
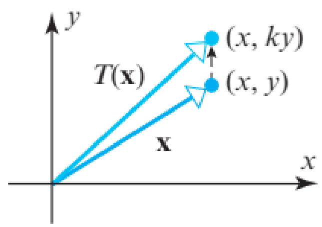
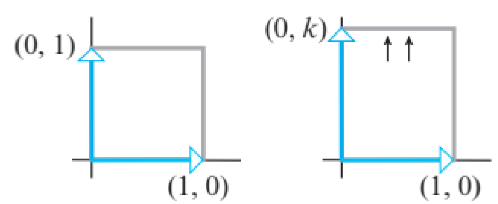
<table>
<tr><th>Operator</th><th>Illustration
$T(x, y, z) = (kx, ky, kz)$</th><th>Standard
Matrix</th></tr>
<tr><td>Contraction with factor k in R^3
$(0 \le k < 1)$</td><td>
</td><td rowspan="2">$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$</td></tr>
<tr><td>Dilation with factor k in R^3
$(k > 1)$</td><td>
</td></tr>
</table>

Table 9

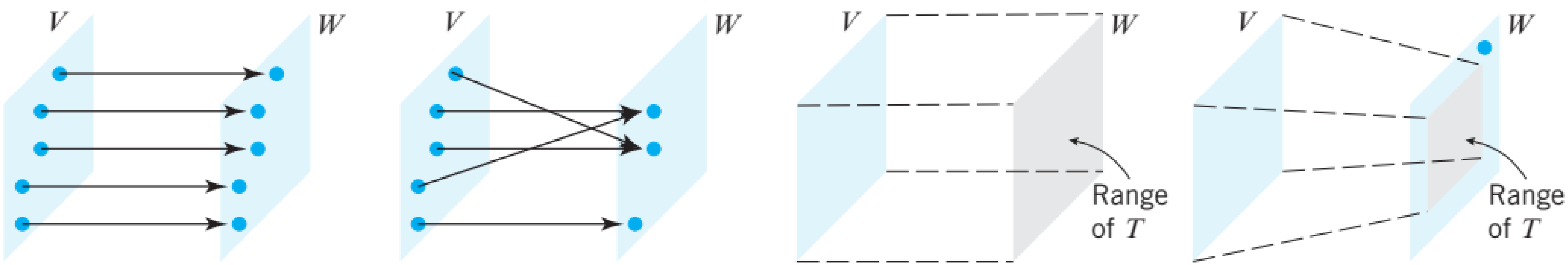
Operator	**Illustration** $T(x, y) = (kx, y)$	**Effect on the Unit Square**	**Standard Matrix**
Compression in the x-direction with factor k in R^2 $(0 \leq k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion in the x-direction with factor k in R^2 $(k > 1)$			
Operator	**Illustration** $T(x, y) = (x, ky)$	**Effect on the Unit Square**	**Standard Matrix**
Compression in the y-direction with factor k in R^2 $(0 \leq k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

Compositions and Inverse Transformations

In this section we will extend some of those ideas to general linear transformations.

DEFINITION 1 If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be ***one-to-one*** if T maps distinct vectors in V into distinct vectors in W.

DEFINITION 2 If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be ***onto*** (or ***onto*** W) if every vector in W is the image of at least one vector in V.



One-to-one. Distinct vectors in V have distinct images in W.

Not one-to-one. There exist distinct vectors in V with the same image.

Onto W. Every vector in W is the image of some vector in V.

Not onto W. Not every vector in W is the image of some vector in V.

THEOREM 8.2.1 *If $T : V \to W$ is a linear transformation, then the following statements are equivalent.*

(a) T *is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

THEOREM 8.2.2 *If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent.*

(a) T *is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

(c) T *is onto* [*i.e.*, $R(T) = W$].

▸ EXAMPLE 2 Basic Transformations That Are One-to-One and Onto

The linear transformations $T_1: P_3 \to R^4$ and $T_2: M_{22} \to R^4$ defined by

$$T_1(a + bx + cx^2 + dx^3) = (a, b, c, d)$$

$$T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$

are both one-to-one and onto (verify by showing that their kernels contain only the zero vector).

▶ **EXAMPLE 3 A One-to-One Linear Transformation That Is Not Onto**

Let $T: P_n \to P_{n+1}$ be the linear transformation

$$T(\mathbf{p}) = T(p(x)) = xp(x)$$

discussed in Example 5 of Section 8.1. If

$$\mathbf{p} = p(x) = c_0 + c_1x + \cdots + c_nx^n \quad \text{and} \quad \mathbf{q} = q(x) = d_0 + d_1x + \cdots + d_nx^n$$

are distinct polynomials, then they differ in at least one coefficient. Thus,

$$T(\mathbf{p}) = c_0x + c_1x^2 + \cdots + c_nx^{n+1} \quad \text{and} \quad T(\mathbf{q}) = d_0x + d_1x^2 + \cdots + d_nx^{n+1}$$

also differ in at least one coefficient. It follows that T is one-to-one since it maps distinct polynomials $\mathbf{p}$ and $\mathbf{q}$ into distinct polynomials $T(\mathbf{p})$ and $T(\mathbf{q})$. However, it is not onto because all images under T have a zero constant term. Thus, for example, there is no vector in P_n that maps into the constant polynomial 1.

▶ **EXAMPLE 5 Differentiation Is Not One-to-One**

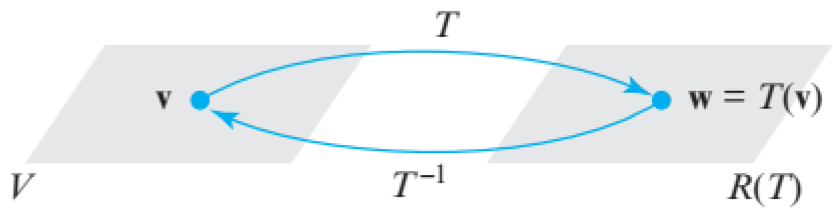
Let

$$D: C^1(-\infty, \infty) \to F(-\infty, \infty)$$

be the differentiation transformation discussed in Example 11 of Section 8.1. This linear transformation is *not* one-to-one because it maps functions that differ by a constant into the same function. For example,

$$D(x^2) = D(x^2 + 1) = 2x \quad ◀$$

If $T: V \to W$ is a one-to-one linear transformation with range $R(T)$, and if $\mathbf{w}$ is any vector in $R(T)$, then the fact that T is one-to-one means that there is *exactly one* vector $\mathbf{v}$ in V for which $T(\mathbf{v}) = \mathbf{w}$. This fact allows us to define a new function, called the ***inverse of T*** (and denoted by T^{-1}), that is defined on the range of T and that maps $\mathbf{w}$ back into $\mathbf{v}$ (Figure 8.2.4).



▶ Figure 8.2.4 The inverse of T maps $T(\mathbf{v})$ back into $\mathbf{v}$.

▶ EXAMPLE 9 An Inverse Transformation

Let $T: R^3 \to R^3$ be the linear operator defined by the formula

$$T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3)$$

Determine whether T is one-to-one; if so, find $T^{-1}(x_1, x_2, x_3)$.

Solution It follows from Formula (15) of Section 1.8 that the standard matrix for T is

$$[T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

(verify). This matrix is invertible, and from Formula (9) of Section 4.10 the standard matrix for T^{-1} is

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

It follows that

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = [T^{-1}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 - 3x_3 \\ -11x_1 + 6x_2 + 9x_3 \\ -12x_1 + 7x_2 + 10x_3 \end{bmatrix}$$

Expressing this result in horizontal notation yields

$$T^{-1}(x_1, x_2, x_3) = (4x_1 - 2x_2 - 3x_3, -11x_1 + 6x_2 + 9x_3, -12x_1 + 7x_2 + 10x_3)$$ ◀

THANK YOU