Linear Algebra

Angle and Orthogonality in Inner Product Spaces

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

- The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\boxed{\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}}$$

- Recall that the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$\boxed{\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)}$$

In this section we will extend the notion of "angle" between vectors to general vector spaces.

We were assured that the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

is valid because it followed from the Cauchy–Schwarz inequality (Theorem 3.2.4) that

$$\boxed{-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \leq 1}$$

as required for the inverse cosine to be defined.

THEOREM 3.2.4 Cauchy–Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\| \tag{22}$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

The following generalization of the Cauchy–Schwarz inequality will enable us to define the angle between two vectors in *any* real inner product space.

THEOREM 6.2.1 Cauchy–Schwarz Inequality

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a real inner product space* V, *then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\| \tag{3}$$

We use the Cauchy–Schwarz inequality to show that

$$\boxed{-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1}$$

This being the case, there is a unique angle θ in radian measure for which

$$\boxed{\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad \text{and} \quad 0 \leq \theta \leq \pi}$$

This enables us to *define* the ***angle*** $\boldsymbol{\theta}$ ***between*** $\mathbf{u}$ ***and*** $\mathbf{v}$ to be

$$\boxed{\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)}$$

▶ EXAMPLE 1 Cosine of the Angle Between Vectors in M_{22}

Let M_{22} have the standard inner product. Find the cosine of the angle between the vectors

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Solution

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \operatorname{tr}(U^T V) \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \\ &= 1(-1) + 2(0) + 3(3) + 4(2) \end{aligned}$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 16}$$

and

$$\begin{aligned} \|\mathbf{u}\| &= \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \\ &= \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} \\ &= \sqrt{1^2 + 2^2 + 3^2 + 4^2} \end{aligned}$$

$$\boxed{\|\mathbf{u}\| = \sqrt{30}}$$

and

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\operatorname{tr}(V^T V)} \\ &= \sqrt{(-1)^2 + 0^2 + 3^2 + 2^2} \end{aligned}$$

$$\boxed{\|\mathbf{v}\| = \sqrt{14}}$$

it follows that

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

$$\boxed{\cos\theta = \frac{16}{\sqrt{30}\sqrt{14}}}$$

- A problem of more interest in general vector spaces is ascertaining whether the angle between vectors is $\pi/2$.
- We should be able to see from the Formula $\theta = \cos^{-1}\left(\dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$ that if $\mathbf{u}$ and $\mathbf{v}$ are *nonzero* vectors, then the angle between them is $\theta = \pi/2$ if and only if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

Accordingly, we make the following definition, which is a generalization of "perpendicularity." between vectors in a general vector spaces

DEFINITION 1 Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V called ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

▸ EXAMPLE 3 Orthogonal Vectors in M_{22}

If M_{22} has the ***standard inner product*** $\langle \mathbf{u}, \mathbf{v}\rangle = \operatorname{tr}(U^T V) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$

then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal since

$$\begin{aligned} \langle U, V\rangle &= u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 \\ &= 1(0) + 0(2) + 1(0) + 1(0) \end{aligned}$$

$$\boxed{\langle U, V\rangle = 0}$$

▸ EXAMPLE 4 Orthogonal Vectors in P_2

Let P_2 have the inner product $\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} p(x)q(x)\,dx$ then $\mathbf{p} = x$ and $\mathbf{q} = x^2$

are orthogonal since

$$\boxed{\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

$$= \int_{-1}^{1} xx^2\,dx$$

$$= \int_{-1}^{1} x^3\,dx$$

$$\boxed{\langle \mathbf{p}, \mathbf{q} \rangle = 0}$$

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2}$$

$$= \int_{-1}^{1} p(x)\,p(x)\;dx$$

$$= \left[\int_{-1}^{1} xx\,dx\right]^{1/2}$$

$$= \left[\int_{-1}^{1} x^2\,dx\right]^{1/2}$$

$$\boxed{\|\mathbf{p}\| = \sqrt{\frac{2}{3}}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2}$$

$$= \int_{-1}^{1} q(x)\,q(x)\,dx$$

$$\|\mathbf{q}\| = \left[\int_{-1}^{1} x^2x^2\,dx\right]^{1/2}$$

$$= \left[\int_{-1}^{1} x^4\,dx\right]^{1/2}$$

$$\boxed{\|\mathbf{q}\| = \sqrt{\frac{2}{5}}}$$

Because $\langle \mathbf{p}, \mathbf{q} \rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to the given inner product.

As the following example shows, orthogonality depends on the inner product in the sense that for different inner products two vectors can be orthogonal with respect to one but not the other.

EXAMPLE 2 Orthogonality Depends on the Inner Product

The vectors $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$ are orthogonal with respect to the Euclidean inner product on R^2 since

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1)(1) + (1)(-1)$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 0}$$

However, they are not orthogonal with respect to the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$$

since

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(1) + 2(1)(-1)$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 1 \neq 0}$$

In Exercises 13–14, show that the vectors are not orthogonal with respect to the Euclidean inner product on R^2, and then find a value of k for which the vectors are orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + ku_2v_2$.

13. $\mathbf{u} = (1, 3)$, $\mathbf{v} = (2, -1)$ **14.** $\mathbf{u} = (2, -4)$, $\mathbf{v} = (0, 3)$

19. Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad x_2 = 2$$

Show that the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to this inner product.

THEOREM 6.2.3 Generalized Theorem of Pythagoras

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in a real inner product space, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2\end{aligned}$$

The orthogonality of $\mathbf{u}$ and $\mathbf{v}$ implies that $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, so

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Generalized Theorem of Pythagoras:

If **u** and **v** are orthogonal vectors in an inner product space, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Example: Consider the inner product space P_n with inner product

$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$ and the vectors $\mathrm{p}(x) = x$ and $\mathrm{q}(x) = x^2$.

$$\boxed{\|u\|^2 = \langle u, u\rangle}$$

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \int_{-1}^{1} (x+x^2)(x+x^2)dx$$

$$\|u+v\|^2 = \int_{-1}^{1} (x^2+2x^3+x^4)dx$$

$$\|u+v\|^2 = \left[\frac{x^3}{3}+\frac{2x^4}{4}+\frac{x^5}{5}\right]_{-1}^{1} = \left(\frac{1}{3}+\frac{2}{4}+\frac{1}{5}\right) - \left(-\frac{1}{3}+\frac{2}{4}-\frac{1}{5}\right) = \frac{16}{15}$$

$$\boxed{\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx}$$

$$\|u\|^2 = \langle u, u\rangle = \int_{-1}^{1} u(x)u(x)dx$$

$$\|u\|^2 = \int_{-1}^{1} x\, x\, dx$$

$$\|u\|^2 = \int_{-1}^{1} x^2\, dx$$

$$\|u\|^2 = \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$\boxed{\|u\|^2 = \frac{2}{3}}$$

$$\|v\|^2 = \langle v, v\rangle = \int_{-1}^{1} v(x)v(x)dx$$

$$\|v\|^2 = \int_{-1}^{1} x^2\, x^2\, dx$$

$$\|v\|^2 = \int_{-1}^{1} x^4\, dx$$

$$\|v\|^2 = \left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$\boxed{\|v\|^2 = \frac{2}{5}}$$

Now

$$\boxed{\|u\|^2 + \|v\|^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}}$$

Therefore

$$\boxed{\|u + v\|^2 = \|u\|^2 + \|v\|^2}$$

DEFINITION 2 If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$.

Problem : 1

(a) Let W be the line $y = x$ in an xy-coordinate system in R^2. Describe the subspace $W^{\perp}$.

(b) Let W be the y-axis in an xyz-coordinate system in R^3. Describe the subspace $W^{\perp}$.

(c) Let W be the yz-plane of an xyz-coordinate system in R^3. Describe the subspace $W^{\perp}$.

Answer:

(a) The line $y = -x$

(b) The xz-plane

(c) The x-axis

THEOREM 3.4.3 *If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.*

To take geometric observation, consider the homogeneous system

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \quad &\quad\; \vdots \qquad\qquad\quad\; \vdots \qquad\; \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{aligned}$$

If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$, then we can rewrite this system in dot product form as

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{x} &= \mathbf{0} \\ \mathbf{r}_2 \cdot \mathbf{x} &= \mathbf{0} \\ \vdots \quad &\quad \vdots \\ \mathbf{r}_m \cdot \mathbf{x} &= \mathbf{0} \end{aligned}$$

from which we see that every solution vector $\mathbf{x}$ is orthogonal to every row vector of the coefficient matrix.

THEOREM

If A is an $m \times n$ matrix, then

(a) *The nullspace of A and the row space of A are orthogonal complements in R^n with respect to the Euclidean inner product.*

(b) *The nullspace of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product.*

Proof (b)

Since the column space of A is the row space of A^T, the proof follows by applying the result

▶ EXAMPLE 6 Basis for an Orthogonal Complement

Let W be the subspace of R^6 spanned by the vectors

$$\mathbf{w}_1 = (1, 3, -2, 0, 2, 0), \qquad \mathbf{w}_2 = (2, 6, -5, -2, 4, -3),$$
$$\mathbf{w}_3 = (0, 0, 5, 10, 0, 15), \qquad \mathbf{w}_4 = (2, 6, 0, 8, 4, 18)$$

Find a basis for the orthogonal complement of W.

Solution. The subspace W is the same as the row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Since the row space and null space of A are orthogonal complements, our problem reduces to finding a basis for the null space of this matrix.

In Example 4 of Section 4.7 we showed that

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for this null space.

Expressing these vectors in comma-delimited form (to match that of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$),

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

You may want to check that these vectors are orthogonal to $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ by computing the necessary dot products. ◀

Exercise

Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

(a) Find bases for the column space of A and nullspace of A^T.

(b) Verify that every vector in the column space of A is orthogonal to every vector in the nullspace of A^T

Problem : 2

(a) Let W be the plane in R^3 with equation $x-2y-3z=0$. Find parametric equations for $W^{\perp}$.

(b) Let W be the line in R^3 with parametric equations

$$x=2t,\quad y=-5t,\quad z=4t$$

Find an equation for $W^{\perp}$.

(c) Let W be the intersection of the two planes

$$x+y+z=0 \quad \text{and} \quad x-y+z=0$$

in R^3. Find an equation for $W^{\perp}$.

Solution (a):

Given that W is a plane given by $x-2y-3z=0$ ---- (1)

The basis vectors, which span W can be obtained by letting $y=t$ & $z=s$

Thus, the set of all (x, y, z) lines on W will be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t+3s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad => \quad \boxed{\text{Span } (W) = \left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

The set of all vectors on $W^{\perp}$ will be orthogonal to the basis of W. Therefore, we are looking for the null space of the matrix A with basis of W as rows.

$$\tilde{A} = \begin{array}{c} \begin{pmatrix} 2 & 1 & 0 & \vdots & 0 \\ 3 & 0 & 1 & \vdots & 0 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\tilde{A} = \begin{pmatrix} 1 & 1/2 & 0 & \vdots & 0 \\ 3 & 0 & 1 & \vdots & 0 \end{pmatrix} \begin{array}{l} R_1 \to \frac{R_1}{2} \\ R_2 \end{array}$$

$$\tilde{A} = \begin{pmatrix} 1 & 1/2 & 0 & \vdots & 0 \\ 0 & -3/2 & 1 & \vdots & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \end{array}$$

$$\tilde{A} = \begin{pmatrix} 1 & 1/2 & 0 & \vdots & 0 \\ 0 & 1 & -2/3 & \vdots & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \to (-\frac{2}{3})R_2 \end{array}$$

Leading 1's

From which, we have

$$x + \frac{1}{2}y = 0$$

$$y - \frac{2}{3}z = 0$$

z is arbitrary, let $z = t$

Consequently basis for $W^{\perp}$ is

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} t}$$

Solution (c):

Let W be the intersection of the planes $x+y+z=0$ and $x-y+z=0$. The intersection these plane will be a straight line, which can be obtained by solving these two plane

$$\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

From which, we have

$$x+y+z=0$$

$$2y=0$$

which implies $y=0$ and

z is arbitrary, let $z=t$

thus $x=-z$.

W is the line in R^3 given by $\begin{pmatrix}-t\\ 0\\ t\end{pmatrix}$.

The basis for W is $\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$.

Now, to find $W^{\perp}$:

$W^{\perp}$ is the set of vectors (x,y,z) orthogonal to $\mathrm{v}=(-1,0,1)$. Let $u=(x,y,z)\in W^{\perp}$

$$\langle u,v\rangle=0$$

$$(-1)x+0(y)+1(z)=0$$

$$\boxed{x-z=0}$$

y & z are arbitrary and $x = z$, we get

$$\boxed{\text{The basis for } W^{\perp} \text{ is } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

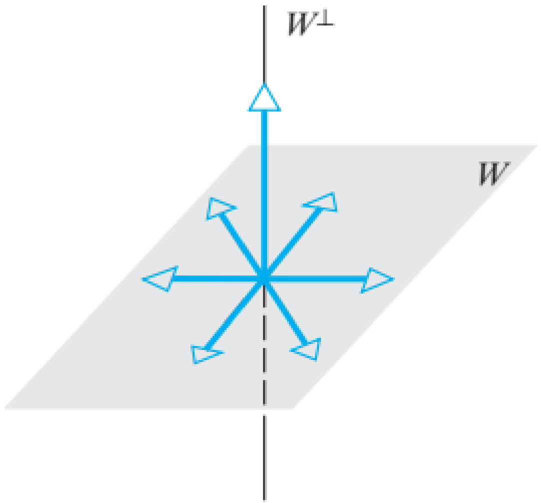
THEOREM 6.2.4 *If W is a subspace of a real inner product space V, then:*

(*a*) $W^{\perp}$ *is a subspace of* V.

(*b*) $W \cap W^{\perp} = \{\mathbf{0}\}$.

THEOREM 6.2.5 *If W is a subspace of a real finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is,*

$$(W^{\perp})^{\perp} = W$$



▲ Figure 6.2.2 Each vector in W is orthogonal to each vector in $W^{\perp}$ and conversely.

THANK YOU