Linear Algebra

Inner Product Spaces

In this section we will use the most important properties of the dot product on R^n as axioms, which, if satisfied by the vectors in a vector space V, will enable us to extend the notions of length, distance, angle, and perpendicularity to general vector spaces.

Norm, Dot Product, and Distance in R^n

Norm of a Vector

- we will denote the length of a vector $\mathbf{v}$ by the symbol $\|\mathbf{v}\|$, which is read as the *norm* of $\mathbf{v}$, the *length* of $\mathbf{v}$, or the *magnitude* of $\mathbf{v}$ (the term "norm" being a common mathematical synonym for length).

- It follows from the Theorem of Pythagoras that the norm of a vector (v_1, v_2) in R^2 is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

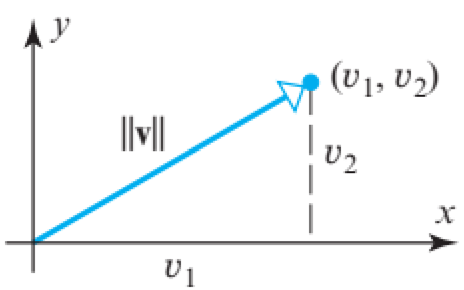


Figure (a)

Similarly, for a vector (v_1, v_2, v_3) in R^3, it follows from Figure and two applications of the Theorem of Pythagoras that

$$\|\mathbf{v}\|^2 = (OR)^2 + (RP)^2$$

$$= (OQ)^2 + (QR)^2 + (RP)^2$$

$$= v_1^2 + v_2^2 + v_3^2$$

and hence that $\boxed{\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$

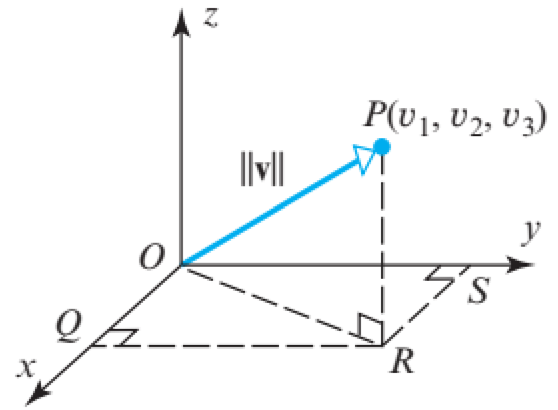


Figure (b)

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{3}$$

▸ EXAMPLE 1 Calculating Norms

The norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

follows from

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2}$$

$$\boxed{\|\mathbf{v}\| = \sqrt{14}}$$

The norm of the vector $\mathbf{v} = (2, -1, 3, -5)$

in R^4 is follows from

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2}$$

$$\boxed{\|\mathbf{v}\| = \sqrt{39}}$$

Our first theorem in this section will generalize to R^n the following three familiar facts about vectors in R^2 and R^3:

- Distances are nonnegative.
- The zero vector is the only vector of length zero.
- Multiplying a vector by a scalar multiplies its length by the absolute value of that scalar.

THEOREM 3.2.1 *If* $\mathbf{v}$ *is a vector in* R^n, *and if* k *is any scalar, then:*

(*a*) $\|\mathbf{v}\| \geq 0$

(*b*) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(*c*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

It is important to recognize that just because these results hold in R^2 and R^3 does not guarantee that they hold in R^n—their validity in R^n must be *proved* using algebraic properties of n-tuples.

We will prove part (c) and leave (a) and (b) as exercises.

Proof (c) If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$, so

$$\begin{aligned}
\|k\mathbf{v}\| &= \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2} \\
&= \sqrt{(k^2)(v_1^2 + v_2^2 + \cdots + v_n^2)} \\
&= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\
&= |k|\|\mathbf{v}\| \quad \blacktriangleleft
\end{aligned}$$

Unit Vectors A vector of norm 1 is called a ***unit vector***.

- Such vectors are useful for specifying a direction
- We can obtain a unit vector in a desired direction by choosing any *nonzero* vector $\mathbf{v}$ in that direction and multiplying $\mathbf{v}$ by the reciprocal of its length.

$$\boxed{\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}}$$

The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called ***normalizing*** $\mathbf{v}$.

EXAMPLE 2 Normalizing a Vector

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$

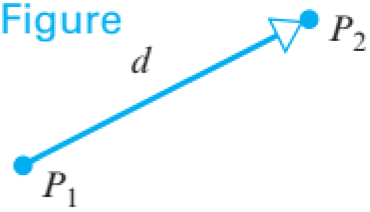
$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1)$$

$$\boxed{\mathbf{u} = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)}$$

Distance in R^n

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in R^2, then the length of the vector $\overrightarrow{P_1P_2}$ is equal to the distance d between the two points

Figure



$d = \|\overrightarrow{P_1P_2}\|$

$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similarly, the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-space is

$$d(\mathbf{u}, \mathbf{v}) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

DEFINITION 2 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

EXAMPLE 4 Calculating Distance in R^n

If $\mathbf{u} = (1, 3, -2, 7)$ and $\mathbf{v} = (0, 7, 2, 2)$ then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

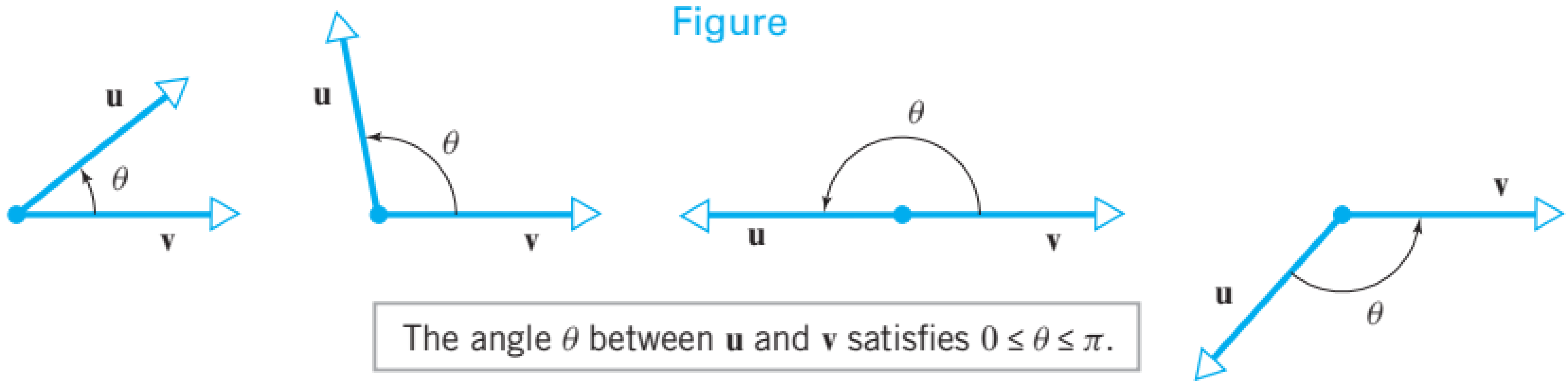
Dot Product

> Our next objective is to define a useful multiplication operation on vectors in R^2 and R^3 and then extend that operation to R^n.

- To do this we will first need to define exactly what we mean by the "angle" between two vectors in R^2 or R^3.
- For this purpose, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^2 or R^3 that have been positioned so that their initial points coincide.

We define the ***angle between* u *and* v** to be the angle θ determined by $\mathbf{u}$ and $\mathbf{v}$ that satisfies the inequalities $0 \le \theta \le \pi$

Figure



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \le \theta \le \pi$.

Dot Product

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

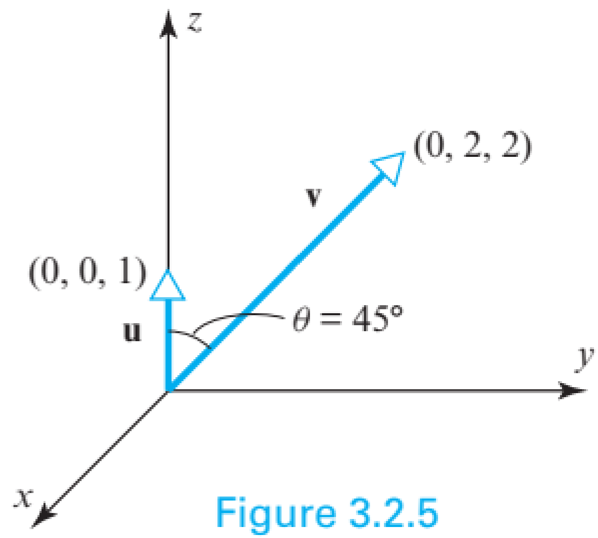


Figure 3.2.5

Solution

The lengths of the vectors are $\|\mathbf{u}\| = 1$

and $\|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$

Thus, it follows from $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$

$$= (1)(2\sqrt{2})(1/\sqrt{2})$$

$$\boxed{\mathbf{u} \cdot \mathbf{v} = 2}$$

Component Form of the Dot Product

- For computational purposes it is desirable to have a formula that expresses the dot product of two vectors in terms of components.
- We will derive such a formula for vectors in 3-space; the derivation for vectors in 2-space is similar.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two nonzero vectors as shown in Figure 3.2.6, then the law of cosines yields

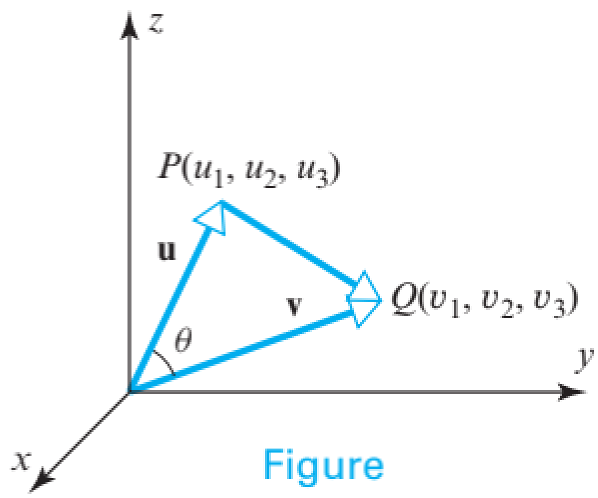
$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Since $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$, we can rewrite as

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

or

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$



Figure

$$\mathbf{u}\cdot\mathbf{v}=\tfrac{1}{2}(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-\|\mathbf{v}-\mathbf{u}\|^2)$$

Substituting

$$\|\mathbf{u}\|^2=u_1^2+u_2^2+u_3^2,$$

$$\|\mathbf{v}\|^2=v_1^2+v_2^2+v_3^2$$

and

$$\|\mathbf{v}-\mathbf{u}\|^2=(v_1-u_1)^2+(v_2-u_2)^2+(v_3-u_3)^2$$

we obtain, after simplifying,

$$\boxed{\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3}$$

The companion formula for vectors in 2-space is

$$\boxed{\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2}$$

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{17}$$

Algebraic Properties of the Dot Product

In the special case where $\mathbf{u} = \mathbf{v}$ in $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2$$

$$\boxed{\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2}$$

Dot products have many of the same algebraic properties as products of real numbers.

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

- Similarity between dot product and matrix multiplication:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be represented as an $n \times 1$ column matrix)

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = [u_1 \quad u_2 \quad \cdots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n]$$

(The result of the dot product of $\mathbf{u}$ and $\mathbf{v}$ is the same as the result of the matrix multiplication of $\mathbf{u}^T$ and $\mathbf{v}$)

Inner Products

In this section we will use the most important properties of the dot product on R^n as axioms, which, if satisfied by the vectors in a vector space V, will enable us to extend the notions of length, distance, angle, and perpendicularity to general vector spaces.

Our first goal in this section is to extend the notion of a dot product to general real vector spaces by using those four properties as axioms.

DEFINITION 1 An ***inner product*** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a ***real inner product space***.

- Because the axioms for a real inner product space are based on properties of the dot product, these inner product space axioms will be satisfied automatically if we define the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n to be

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- This inner product is commonly called the ***Euclidean inner product*** (or the ***standard inner product***) on R^n to distinguish it from other possible inner products that might be defined on R^n.

We call R^n with the Euclidean inner product ***Euclidean n-space***.

- Inner products can be used to define notions of norm and distance in a general inner product space just with dot products in R^n.

Recall that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in Euclidean n-space, then norm and distance can be expressed in terms of the dot product as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\boxed{\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}}$$

and

$$\boxed{d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}}$$

Note:

$$\mathbf{u}\cdot\mathbf{v} = \text{dot product (Euclidean inner product for } R^n\text{)}$$

$$<\mathbf{u}\,,\,\mathbf{v}> = \text{general inner product for a vector space } V$$

DEFINITION 2 If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle}$$

A vector of norm 1 is called a ***unit vector***.

THEOREM 6.1.1 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a real inner product space* V, *and if* k *is a scalar, then*:

(*a*) $\|\mathbf{v}\| \geq 0$ *with equality if and only if* $\mathbf{v} = \mathbf{0}$.

(*b*) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$.

(*c*) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(*d*) $d(\mathbf{u}, \mathbf{v}) \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{v}$.

Weighted Euclidean Inner Product

- Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which it is desirable to modify it by *weighting* each term differently.

- More precisely, if $w_1, w_2, \ldots, w_n$ are *positive* real numbers, which we will call ***weights***, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n}$$

defines an inner product on R^n that we call the ***weighted Euclidean inner product with weights*** $\boldsymbol{w_1, w_2, \ldots, w_n}$.

▶ EXAMPLE 1 Weighted Euclidean Inner Product

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

satisfies the four inner product axioms.

Solution

$$\boxed{\text{1.} \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle}$$

Axiom 1: Interchanging **u** and **v** in Formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

does not change the sum on the right side, so

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle.}$$

$$\boxed{\text{2.} \quad \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle}$$

Axiom 2: If $\mathbf{w} = (w_1, w_2)$, then

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2}$$

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2$$

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2$$

$$= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2)$$

$$= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2)$$

$$\boxed{\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle}$$

$$\boxed{3. \quad \langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle}$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2}$$

Axiom 3:

$$\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2$$

$$= k(3u_1 v_1 + 2u_2 v_2)$$

$$\boxed{\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle}$$

Axiom 4: $\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1 v_1) + 2(v_2 v_2) = 3v_1^2 + 2v_2^2 \geq 0$

with equality if and only if $v_1 = v_2 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$.

EXAMPLE 2 Calculating with a Weighted Euclidean Inner Product

- It is important to keep in mind that norm and distance depend on the inner product being used.
- If the inner product is changed, then the norms and distances between vectors also change.

For example, for the vectors $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$ in R^2 with the Euclidean inner product we have

$$\boxed{\|\mathbf{u}\| = \sqrt{1^2 + 0^2} = 1}$$

and

$$\boxed{d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(1, -1)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}}$$

but if we change to the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

we have

$$\boxed{\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = [3(1)(1) + 2(0)(0)]^{1/2} = \sqrt{3}}$$

and

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle (1, -1), (1, -1) \rangle^{1/2}$$

$$= [3(1)(1) + 2(-1)(-1)]^{1/2}$$

$$\boxed{d(\mathbf{u}, \mathbf{v}) = \sqrt{5}}$$

DEFINITION 3 If V is an inner product space, then the set of points in V that satisfy

$$\|\mathbf{u}\| = 1$$

is called the ***unit sphere*** or sometimes the ***unit circle*** in V.

EXAMPLE 3 Unusual Unit Circles in R^2

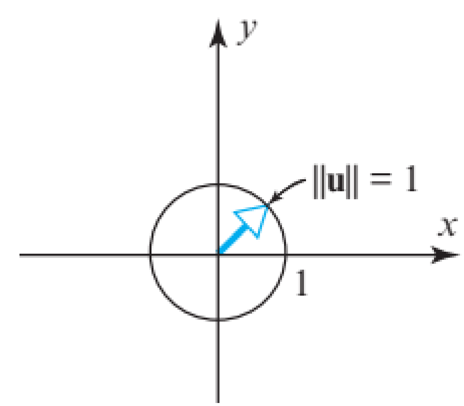
(a) Sketch the unit circle in an xy-coordinate system in R^2 using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$.

(b) Sketch the unit circle in an xy-coordinate system in R^2 using the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9} u_1 v_1 + \frac{1}{4} u_2 v_2$.

Solution (a)

If $\mathbf{u} = (x, y)$, then

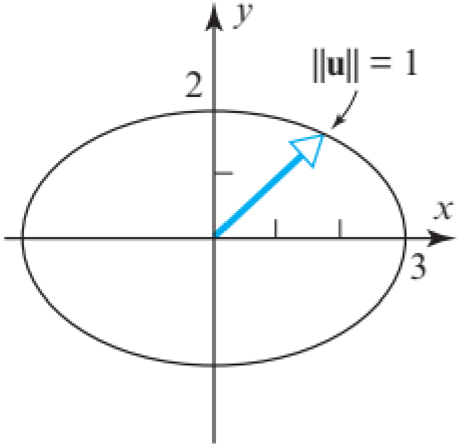
$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{x^2 + y^2},$$



(a) The unit circle using the standard Euclidean inner product.

Solution (b) If $\mathbf{u} = (x, y)$, then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\frac{1}{9}x^2 + \frac{1}{4}y^2}$, so the equation of the unit circle is $\sqrt{\frac{1}{9}x^2 + \frac{1}{4}y^2} = 1$, or on squaring both sides,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



(*b*) The unit circle using a weighted Euclidean inner product.

EXAMPLE

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Show that the expression

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 - 2u_2v_2$$

does *not* define an inner product on R^2 and list all inner product axioms that fail to hold.

Solution

$\boxed{1.\quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle}$

Axiom 1: Interchanging **u** and **v** in Formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 - 2u_2v_2$$

does not change the sum on the right side, so

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle.}$$

$\boxed{2.\quad \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle}$

Axiom 2: If $\mathbf{w} = (w_1, w_2)$, then

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 - 2u_2v_2}$$

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 3(u_1 + v_1)w_1 - 2(u_2 + v_2)w_2$$

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = 3(u_1+v_1)w_1 - 2(u_2+v_2)w_2$$

$$= 3(u_1w_1 + v_1w_1) - 2(u_2w_2 + v_2w_2)$$

$$= (3u_1w_1 - 2u_2w_2) + (3v_1w_1 - 2v_2w_2)$$

$$\boxed{\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle}$$

$$\boxed{3.\quad \langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle}$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 - 2u_2v_2}$$

Axiom 3: $\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 - 2(ku_2)v_2$

$$= k(3u_1v_1 - 2u_2v_2)$$

$$\boxed{\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle}$$

Axiom 4: $\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1v_1) - 2(v_2v_2)$

$$\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 - 2v_2^2$$

$3v_1^2 - 2v_2^2 < 0$ for some values of v_1 & v_2, for instance for $v = (0,1), 3v_1^2 - 2v_2^2$ is less than zero.

$Therefore$

Axiom 4 fails to hold.

In Exercises 33–34, let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Show that the expression

33. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$

34. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$

does *not* define an inner product on R^3, and list all inner product axioms that fail to hold.

Solution 33

$\boxed{1.\ \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle}$

Axiom 1: Interchanging $\mathbf{u}$ and $\mathbf{v}$ in Formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$$

does not change the sum on the right side, so

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle.}$$

$\boxed{2.\ \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle}$

Axiom 2: if $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = (u_1 + v_1)^2 w_1^2 + (u_2 + v_2)^2 w_2^2 + (u_3 + v_3)^2 w_3^2$$

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \underline{2u_1 v_1 w_1^2 + 2u_2 v_2 w_2^2 + 2u_3 v_3 w_3^2}$$

$\boxed{\text{Axiom 2 fails.}}$

$\boxed{3. \quad \langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle}$

$\boxed{\langle \mathbf{u}, \mathbf{v}\rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2}$

Axiom 3:

$$\langle k\mathbf{u}, \mathbf{v}\rangle = k^2 u_1^2 v_1^2 + k^2 u_2^2 v_2^2 + k^2 u_3^2 v_3^2$$

$$\langle k\mathbf{u}, \mathbf{v}\rangle = k^2\langle \mathbf{u}, \mathbf{v}\rangle.$$

$\Rightarrow$ $\langle k\mathbf{u}, \mathbf{v}\rangle \neq k\langle \mathbf{u}, \mathbf{v}\rangle$ unless $k = 0$ or $k = 1$, so

$\boxed{\text{Axiom 3 fails.}}$

$\boxed{\langle \mathbf{u}, \mathbf{v}\rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2}$

Axiom 4:

$$\langle \mathbf{v}, \mathbf{v}\rangle = v_1^2\, v_1^2 + v_2^2\, v_2^2 + v_3^2\, v_3^2 \geq 0$$

with equality if and only if $v_1 = v_2 = v_3 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$.

Inner Products Generated by Matrices

The Euclidean inner product and the weighted Euclidean inner products are special cases of a general class of inner products on R^n called ***matrix inner products***.

- To define this class of inner products, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n that are expressed in *column form*, and let A be an *invertible* $n \times n$ matrix.

- It can be shown that if $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

also defines an inner product; it is called the ***inner product on R^n generated by A***.

- Recall that if $\mathbf{u}$ and $\mathbf{v}$ are in column form, then $\mathbf{u} \cdot \mathbf{v}$ can be written as $\mathbf{v}^T\mathbf{u}$ from which it follows that $\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$ can be expressed as

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u}} \quad \text{or equivalently as} \quad \boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A\mathbf{u}}$$

EXAMPLE 4 Matrices Generating Weighted Euclidean Inner Products

- The standard Euclidean and weighted Euclidean inner products are special cases of matrix inner products.
- The standard Euclidean inner product on R^n is generated by the $n \times n$ identity matrix, since setting $A = I$ in Formula $\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$ yields

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = I\mathbf{u} \cdot I\mathbf{v} = \mathbf{u} \cdot \mathbf{v}}$$

- The weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

is generated by the matrix

$$A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{w_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{w_n} \end{bmatrix}$$

This can be seen by observing that $A^T A$ is the $n \times n$ diagonal matrix whose diagonal entries are the weights $w_1, w_2, \ldots, w_n$.

▶ EXAMPLE 5 Example 1 Revisited

The weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ discussed in Example 1 is the inner product on R^2 generated by

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

EXAMPLE 6 The Standard Inner Product on M_{nn}

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V)} \tag{8}$$

defines an inner product on M_{nn} called the ***standard inner product*** on that space.

- This can be proved by confirming that the four inner product space axioms are satisfied, but we can see why this is so by computing (8) for the 2×2 matrices

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

This yields $\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V)$

$$= tr\left(\begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix} . \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}\right) = tr\begin{pmatrix} u_1 v_1 + u_3 v_3 & u_1 v_2 + u_3 v_4 \\ u_2 v_1 + u_4 v_3 & u_2 v_2 + u_4 v_4 \end{pmatrix}$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4}$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4}$$

which is just the dot product of the corresponding entries in the two matrices. And it

follows from this that

$$\boxed{\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}\langle U^T U \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}}$$

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V)$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

$$= 1(-1) + 2(0) + 3(3) + 4(2)$$

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = 16}$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

$$= \sqrt{\operatorname{tr}(U^T U)}$$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

$$= \sqrt{1^2 + 2^2 + 3^2 + 4^2}$$

$$\boxed{\|\mathbf{u}\| = \sqrt{30}}$$

and

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\operatorname{tr}(V^T V)}$$

$$= \sqrt{(-1)^2 + 0^2 + 3^2 + 2^2}$$

$$\boxed{\|\mathbf{v}\| = \sqrt{14}}$$

$$u = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\mathrm{tr}(U^T U)}$$

$$= \left\| \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} \right\|$$

$$= \sqrt{tr\left(\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}\right)}$$

$$= \sqrt{tr\left(\begin{bmatrix} 17 & 1 \\ 1 & 1 \end{bmatrix}\right)}$$

$$\boxed{d(\mathbf{u}, \mathbf{v}) = \sqrt{18}}$$

▸ EXAMPLE 7 The Standard Inner Product on P_n

If $\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n$ and $\mathbf{q} = b_0 + b_1x + \cdots + b_nx^n$ are polynomials in P_n, then the following formula defines an inner product on P_n (verify)

that we will call the ***standard inner product*** on this space:

$$\boxed{\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n}$$

- The norm of a polynomial $\mathbf{p}$ relative to this inner product is

$$\boxed{\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}}$$

- The inner product of

$$p(x) = 2 + x$$

and $q(x) = x + x^2$

is

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$$

$$= (2)(0) + (1)(1) + (0)(1)$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = 1$$

$$\boxed{p(x) = 2 + x}$$

$$\boxed{q(x) = x + x^2}$$

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

$$= \sqrt{(2)(2) + (1)(1)}$$

$$\boxed{\|\mathbf{p}\| = \sqrt{5}}$$

and

$$\boxed{\|\mathbf{q}\| = \sqrt{(1)(1) + (1)(1)} = \sqrt{2}}$$

and

$$d(p(x), q(x)) = \|p(x) - q(x)\|$$

$$= \|2 - x^2\|$$

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

$$= \sqrt{(2)(2) + (0)(0) + (-1)(-1)}$$

$$\boxed{d(p(x), q(x)) = \sqrt{5}}$$

EXAMPLE 8 The Evaluation Inner Product on P_n

- If $\mathbf{p} = p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $\mathbf{q} = q(x) = b_0 + b_1x + \cdots + b_nx^n$ are polynomials in P_n, and if $x_0, x_1, \ldots, x_n$ are distinct real numbers (called ***sample points***), then the formula

$$\boxed{\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \cdots + p(x_n)q(x_n)}$$

defines an inner product on P_n called the ***evaluation inner product*** at $x_0, x_1, \ldots, x_n$.

- The norm of a polynomial $\mathbf{p}$ relative to the evaluation inner product is

$$\boxed{\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{[p(x_0)]^2 + [p(x_1)]^2 + \cdots + [p(x_n)]^2}}$$

EXAMPLE 9 Working with the Evaluation Inner Product

Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad \text{and} \quad x_2 = 2$$

Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ and $\|\mathbf{p}\|$ for the polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$.

Solution

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \cdots + p(x_n)q(x_n)$$

$$= p(-2)q(-2) + p(0)q(0) + p(2)q(2)$$

$$= (4)(-1) + (0)(1) + (4)(3)$$

$$\boxed{\langle \mathbf{p}, \mathbf{q} \rangle = 8}$$

and

$$\|\mathbf{p}\| = \sqrt{[p(x_0)]^2 + [p(x_1)]^2 + [p(x_2)]^2}$$

$$= \sqrt{[p(-2)]^2 + [p(0)]^2 + [p(2)]^2}$$

$$= \sqrt{4^2 + 0^2 + 4^2}$$

$$\boxed{\|\mathbf{p}\| = \sqrt{32}}$$

▶ EXAMPLE 10 An Integral Inner Product on $C[a, b]$

Let $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ be two functions in $C[a, b]$ and define

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx$$

We will show that this formula defines an inner product on $C[a, b]$ by verifying the four inner product axioms for functions $\mathbf{f} = f(x)$, $\mathbf{g} = g(x)$, and $\mathbf{h} = h(x)$ in $C[a, b]$:

Axiom 1:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx$$

$$= \int_a^b g(x)f(x)\,dx$$

$$\boxed{\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle}$$

Axiom 2:

$$\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \int_a^b (f(x) + g(x))h(x)\,dx$$

$$= \int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx$$

$$\boxed{\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle = \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle}$$

Axiom 3:

$$\langle k\mathbf{f}, \mathbf{g}\rangle = \int_a^b kf(x)g(x)\,dx$$

$$= k\int_a^b f(x)g(x)\,dx$$

$$\boxed{\langle k\mathbf{f}, \mathbf{g}\rangle = k\langle \mathbf{f}, \mathbf{g}\rangle}$$

Axiom 4:

If $\mathbf{f} = f(x)$ is any function in $C[a, b]$, then

$$\langle \mathbf{f}, \mathbf{f}\rangle = \int_a^b f^2(x)\,dx \geq 0$$

since $f^2(x) \geq 0$ for all x in the interval $[a, b]$. Moreover, because f is continuous on $[a, b]$, the equality holds if and only if the function f is identically zero on $[a, b]$, that is, if and only if $\mathbf{f} = \mathbf{0}$; and this proves that Axiom 4 holds.

Norm of a Vector in $C[a, b]$

If $C[a,b]$ has the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx$, then the norm of a function $\mathbf{f} = f(x)$ relative to this inner product is

$$\boxed{\|\mathbf{f}\| = \langle \mathbf{f}, \mathbf{f} \rangle^{1/2} = \sqrt{\int_a^b f^2(x)\,dx}}$$

- The unit sphere in this space consists of all functions $\mathbf{f}$ in $C[a,b]$ that satisfy the equation

$$\boxed{\int_a^b f^2(x)\,dx = 1}$$

Algebraic Properties of Inner Products

THEOREM 6.1.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in a real inner product space* V, *and if* k *is a scalar, then:*

(*a*) $\langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$

(*b*) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$

(*c*) $\langle \mathbf{u}, \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{w}\rangle$

(*d*) $\langle \mathbf{u}-\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{w}\rangle$

(*e*) $k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, k\mathbf{v}\rangle$

▶ EXAMPLE 12 Calculating with Inner Products

$$\begin{aligned}
\langle \mathbf{u}-2\mathbf{v}, 3\mathbf{u}+4\mathbf{v}\rangle &= \langle \mathbf{u}, 3\mathbf{u}+4\mathbf{v}\rangle - \langle 2\mathbf{v}, 3\mathbf{u}+4\mathbf{v}\rangle \\
&= \langle \mathbf{u}, 3\mathbf{u}\rangle + \langle \mathbf{u}, 4\mathbf{v}\rangle - \langle 2\mathbf{v}, 3\mathbf{u}\rangle - \langle 2\mathbf{v}, 4\mathbf{v}\rangle \\
&= 3\langle \mathbf{u}, \mathbf{u}\rangle + 4\langle \mathbf{u}, \mathbf{v}\rangle - 6\langle \mathbf{v}, \mathbf{u}\rangle - 8\langle \mathbf{v}, \mathbf{v}\rangle \\
&= 3\|\mathbf{u}\|^2 + 4\langle \mathbf{u}, \mathbf{v}\rangle - 6\langle \mathbf{u}, \mathbf{v}\rangle - 8\|\mathbf{v}\|^2
\end{aligned}$$

$$\boxed{\langle \mathbf{u}-2\mathbf{v}, 3\mathbf{u}+4\mathbf{v}\rangle = 3\|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle - 8\|\mathbf{v}\|^2}$$

THANK YOU