Linear Algebra

Examples:

Show that the set of all 2×2 matrices with real entries is a vector space, if the vector addition is defined as matrix addition and scalar multiplication is defined as scalar matrix multiplication

$$(i)\ \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{pmatrix}$$

$$(ii)\ k\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{pmatrix}$$

Solution: **I. Abelian group under addition**

1. Closure property: Let $u, v \in M_{2\times 2}$

we need to prove that $u+v \in M_{2\times 2}$

Now

$$u+v = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$$u+v = \begin{pmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{pmatrix} \quad \in M_{2\times 2}$$

Therefore

$$\boxed{u, v \in M_{2\times 2} \text{implies } u+v \in M_{2\times 2}}$$

2. Associativity property:

Let $u, v, w \in M_{2\times 2}$, where $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$, $v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ & $w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \left\{ \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \right\}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} + w_{11} & v_{12} + w_{12} \\ v_{21} + w_{21} & v_{22} + +w_{22} \end{pmatrix}$$

$$u + (v + w) = \begin{pmatrix} u_{11} + v_{11} + w_{11} & u_{12} + v_{12} + w_{12} \\ u_{21} + v_{21} + w_{21} & u_{22} + v_{22} + +w_{22} \end{pmatrix} \quad \text{------}(1)$$

Similarly, we can prove

$$(u + v) + w = \begin{pmatrix} u_{11} + v_{11} + w_{11} & u_{12} + v_{12} + w_{12} \\ u_{21} + v_{21} + w_{21} & u_{22} + v_{22} + +w_{22} \end{pmatrix} \text{------}(2)$$

Therefore, we have

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in M_{2\times 2}$, such that

$$u + e = e + u = u \text{ for all } u \in M_{2\times 2}$$

Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ such that

$$u + e = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$\boxed{u + e = u}$$

$$e + u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in M_{2\times 2}}$$

4. Inverse Property:

We need to prove that, for every element $u \in M_{2\times 2}$, there exists $-u \in M_{2\times 2}$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$ such that

$$u + (-u) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{(-u) + u = e}$$

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in M_{2\times 2}$$

Now,

$$u + v = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} + u_{11} & v_{12} + u_{12} \\ v_{21} + u_{21} & v_{22} + u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$\boxed{u + v = v + u}$$

Therefore $\boxed{u + v = v + u \text{ for all } u, v \in M_{2\times 2}}$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in M_{2\times 2} \,\&\, \alpha \in K \text{ such that } \alpha u \in M_{2\times 2}$$

Now,

$$\alpha u = \alpha \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$\alpha u = \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix} \quad \in M_{2\times 2}$$

Therefore

for any $u \in M_{2\times 2} \,\&\, \alpha \in K$ we have $\alpha u \in M_{2\times 2}$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = M_{2\times2}$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,

$$\alpha(u+v) = \alpha\left\{\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\right\}$$

$$= \alpha\begin{pmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u_{11}+\alpha v_{11} & \alpha u_{12}+\alpha v_{12} \\ \alpha u_{21}+\alpha v_{21} & \alpha u_{22}+v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix} + \begin{pmatrix} \alpha v_{11} & \alpha v_{12} \\ \alpha v_{21} & \alpha v_{22} \end{pmatrix}$$

$$= \alpha\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \alpha\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

Therefore $\boxed{\alpha(u+v) = \alpha u + \alpha v}$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in M_{2\times2} \text{ \& } \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in M_{2\times 2}$ & $\alpha, \beta \in K$

$$(\alpha+\beta)u = \alpha u + \beta u$$

Now,

$$(\alpha+\beta)u = (\alpha+\beta)\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha+\beta)u_{11} & (\alpha+\beta)u_{12} \\ (\alpha+\beta)u_{21} & (\alpha+\beta)u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u_{11}+\beta u_{11} & \alpha u_{12}+\beta u_{12} \\ \alpha u_{21}+\beta u_{21} & \alpha u_{22}+\beta u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u_{11} & \alpha u_{12} \\ \alpha u_{21} & \alpha u_{22} \end{pmatrix} + \begin{pmatrix} \beta u_{11} & \beta u_{12} \\ \beta u_{21} & \beta u_{22} \end{pmatrix}$$

$$\boxed{(\alpha+\beta)u = \alpha u + \beta u}$$

Therefore

$$\boxed{(\alpha+\beta)u = \alpha u + \beta u \text{ for all } u \in M_{2\times 2} \text{ \& } \alpha, \beta \in K}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in M_{2\times 2}$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now, $$\alpha(\beta u) = \alpha\begin{pmatrix} \beta u_{11} & \beta u_{12} \\ \beta u_{21} & \beta u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\beta u_{11} & \alpha\beta u_{12} \\ \alpha\beta u_{21} & \alpha\beta u_{22} \end{pmatrix}$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

10. Property 10:

$$1(u) = 1\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$1(u) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$1(u) = u$$

The set of all 2×2 matrices $M_{2\times 2}$ with real entries is a vector space, under vector addition is defined as matrix addition and scalar multiplication is defined as scalar matrix multiplication

Examples:

Check, whether the set of all 2×2 matrices of the form $\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$ (where a & b are real numbers) is a vector space, under standard matrix addition and scalar multiplication.

Solution:

1. Closure property: Let $u, v \in V$, where u $= \begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix}$ & $v = \begin{pmatrix} a_2 & 1 \\ 1 & b_2 \end{pmatrix}$

we need to prove that $u + v \in V$

$$u + v = \begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ 1 & b_2 \end{pmatrix}$$

$$u + v = \begin{pmatrix} a_1 + a_2 & 2 \\ 2 & b_1 + b_2 \end{pmatrix} \notin V$$

Therefore $\boxed{u, v \in V \text{ implies } u + v \notin V}$

Therefore V is not a vector space.

Examples:

Check, whether the set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ (where a & b are real numbers) is a vector space, under standard matrix addition and scalar multiplication.

Solution: Let V be the set, consisting of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

I. Abelian group under addition

1. Closure property: Let $u, v \in V$

we need to prove that $u + v \in V$

Now
$$u + v = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

$$u + v = \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix} \in V$$

Therefore
$$\boxed{u, v \in V \text{ implies } u + v \in V}$$

2. Associativity property:

Let $u, v, w \in V$, where $u = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$, $v = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$ & $w = \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix}$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \left\{ \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 + a_3 & 0 \\ 0 & b_2 + b_3 \end{pmatrix}$$

$$u + (v + w) = \begin{pmatrix} a_1 + a_2 + a_3 & 0 \\ 0 & b_1 + b_2 + b_3 \end{pmatrix}$$

Similarly, we can prove

$$(u + v) + w = \begin{pmatrix} a_1 + a_2 + a_3 & 0 \\ 0 & b_1 + b_2 + b_3 \end{pmatrix}$$

Therefore, we have

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in V$, such that

$$u + e = e + u = u \text{ for all } u \in V$$

Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ such that

$$u + e = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$\boxed{u + e = u}$$

$$e + u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in V}$$

4. Inverse Property:

We need to prove that, for every element $u \in V$, there exists $-u \in V$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = \begin{pmatrix} -a_1 & 0 \\ 0 & -b_1 \end{pmatrix}$ such that

$$u + (-u) = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} -a_1 & 0 \\ 0 & -b_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = \begin{pmatrix} -a_1 & 0 \\ 0 & -b_1 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{(-u) + u = e}$$

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in V$$

Now,

$$u + v = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 + a_1 & 0 \\ 0 & b_2 + b_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$\boxed{u + v = v + u}$$

Therefore $\boxed{u + v = v + u \text{ for all } u, v \in V}$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in V \; \& \; \alpha \in K \text{ such that } \alpha u \in V$$

Now,

$$\alpha u = \alpha \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$\alpha u = \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix} \quad \in V$$

Therefore

for any $u \in V$ & $\alpha \in K$ we have $\alpha u \in V$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,

$$\alpha(u+v) = \alpha\left\{\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}\right\}$$

$$= \alpha\begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 + \alpha a_2 & 0 \\ 0 & \alpha b_1 + \alpha b_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix} + \begin{pmatrix} \alpha a_2 & 0 \\ 0 & \alpha b_2 \end{pmatrix}$$

$$= \alpha\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \alpha\begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V \ \& \ \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = (\alpha + \beta)\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha + \beta)a_1 & 0 \\ 0 & (\alpha + \beta)b_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\, a_1 + \beta a_1 & 0 \\ 0 & \alpha b_1 + \beta b_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & 0 \\ 0 & \alpha b_1 \end{pmatrix} + \begin{pmatrix} \beta a_1 & 0 \\ 0 & \beta b_1 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} + \beta \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

Therefore $\boxed{(\alpha + \beta)u = \alpha u + \beta u \text{ for all } u \in V \ \& \ \alpha, \beta \in K}$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now, $$\alpha(\beta u) = \alpha\begin{pmatrix}\beta a_1 & 0 \\ 0 & \beta b_1\end{pmatrix}$$

$$= \begin{pmatrix}\alpha\beta a_1 & 0 \\ 0 & \alpha\beta b_1\end{pmatrix}$$

$$= \alpha\beta\begin{pmatrix}a_1 & 0 \\ 0 & b_1\end{pmatrix}$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

10. Property 10:

$$1(u) = 1\begin{pmatrix}a_1 & 0 \\ 0 & b_1\end{pmatrix}$$

$$1(u) = \begin{pmatrix}a_1 & 0 \\ 0 & b_1\end{pmatrix}$$

$$1(u) = u$$

Therefore, the set of all 2×2 matrices of the form $\begin{pmatrix}a & 0 \\ 0 & b\end{pmatrix}$ (where a & b are real numbers) is a vector space, under standard matrix addition and scalar multiplication.

6. Examples:

Show that the set of all polynomials of degree less than or equal to 2, $P_2(t)$ is a vector space under the operations

$$(i)\ (a_0 + a_1t + a_2t^2) + (b_0 + b_1t + b_2t^2) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$$

$$(ii)\ k(a_0 + a_1t + a_2t^2) = ka_0 + ka_1t + ka_2t^2$$

Proof: **I. Abelian group under addition**

1. Closure property:

Let $u, v \in P_2(t)$, where $u = a_0 + a_1t + a_2t^2$ & $v = b_0 + b_1t + b_2t^2$,

we need to prove that $u + v \in P_2(t)$

Now

$$u + v = (a_0 + a_1t + a_2t^2) + (b_0 + b_1t + b_2t^2)$$

$$u + v = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 \quad \in P_2(t)$$

Therefore

$$\boxed{u, v \in P_2(t) \text{ implies } u + v \in P_2(t)}$$

2. Associativity property:

Let $u, v, w \in P_2(t)$,

where $u = a_0 + a_1 t + a_2 t^2, v = b_0 + b_1 t + b_2 t^2$ & $w = c_0 + c_1 t + c_2 t^2$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = (a_0 + a_1 t + a_2 t^2) + \{(b_0 + b_1 t + b_2 t^2) + c_0 + c_1 t + c_2 t^2\}$$

$$= (a_0 + a_1 t + a_2 t^2) + (b_0 + c_0) + (b_1 + c_1)t + (b_2 + c_2)t^2$$

$$= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)t + (a_2 + b_2 + c_2)t^2$$

$$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (c_0 + c_1 t + c_2 t^2)$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in P_2(t)$, such that

$$u + e = e + u = u \text{ for all } u \in P_2(t)$$

Let $e = 0 + 0(t) + 0(t^2)$ such that

$$u + e = [a_0 + a_1 t + a_2 t^2] + [0 + 0(t) + 0(t^2)]$$

$$= (a_0 + 0) + (a_1 + 0)t + (a_2 + 0)t^2$$

$$= a_0 + a_1 t + a_2 t^2$$

$$\boxed{u + e = u}$$

Similarly, we can prove that

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in V}$$

4. Inverse Property:

We need to prove that, for every element $u \in P_2(t)$, there exists $v \in P_2(t)$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = -a_0 - a_1 t - a_2 t^2$ such that

$$u + v = [a_0 + a_1 t + a_2 t^2] + [-a_0 - a_1 t - a_2 t^2]$$

$$= (a_0 + (-a_0)) + (a_1 + (-a_1))t + (a_2 + (-a_2))t^2$$

$$= 0 + 0(t) + 0(t^2)$$

$$\boxed{u + v = e}$$

Similarly, we can prove that $\boxed{v + u = e}$

Therefore $\boxed{u + v = v + u = e}$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in P_2(t)$$

Now,

$$u + v = (a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t + b_2 t^2)$$

$$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$$

$$= (b_0 + a_0) + (b_1 + a_1)t + (b_2 + a_2)t^2$$

$$= (b_0 + b_1 t + b_2 t^2) + (a_0 + a_1 t + a_2 t^2)$$

$$\boxed{u + v = v + u}$$

Therefore $$\boxed{u + v = v + u \text{ for all } u, v \in P_2(t)}$$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in P_2(t) \,\&\, \alpha \in K \text{ such that } \alpha u \in P_2(t)$$

Now,

$$\alpha u = \alpha(a_0 + a_1 t + a_2 t^2)$$

$$\alpha u = \alpha a_0 + (\alpha a_1)t + (\alpha a_2)t^2 \in P_2(t)$$

Therefore

for any $u \in P_2(t) \,\&\, \alpha \in K$ we have $\alpha u \in P_2(t)$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in P_2(t)$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,

$$\alpha(u+v) = \alpha[(a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t + b_2 t^2)]$$

$$= \alpha[(a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2]$$

$$= (\alpha a_0 + \alpha b_0) + (\alpha a_1 + \alpha b_1)t + (\alpha a_2 + \alpha b_2)t^2$$

$$= (\alpha a_0 + \alpha a_1 t + \alpha a_2 t^2) + (\alpha b_0 + \alpha b_1 t + \alpha b_2 t^2)$$

$$= \alpha(a_0 + a_1 t + a_2 t^2) + \alpha(b_0 + b_1 t + b_2 t^2)$$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in P_2(t) \,\&\, \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in P_2(t)$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = (\alpha + \beta)(a_0 + a_1 t + a_2 t^2)$$

$$= (\alpha + \beta)a_0 + (\alpha + \beta)a_1 t + (\alpha + \beta)a_2 t^2$$

$$= (\alpha a_0 + \beta a_0) + (\alpha a_1 + \beta a_1)t + (\alpha a_2 + \beta a_2)t^2$$

$$= (\alpha a_0 + \alpha a_1 t + \alpha a_2 t^2) + (\beta a_0 + \beta a_1 t + \beta a_2 t^2)$$

$$= \alpha(a_0 + a_1 t + a_2 t^2) + \beta(a_0 + a_1 t + a_2 t^2)$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in P_2(t)$& $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta a_0 + \beta a_1 t + \beta a_2 t^2)$$

$$= \alpha\beta a_0 + \alpha\beta a_1 t + \alpha\beta a_2 t^2$$

$$= (\alpha\beta)(a_0 + a_1 t + a_2 t^2)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

Therefore

$$\boxed{\alpha(\beta u) = (\alpha\beta)u \text{ for all } u \in P_2(t) \;\&\; \alpha, \beta \in K}$$

10. Property 10:

$$1(u) = 1(a_0 + a_1t + a_2t^2)$$

$$= a_0 + a_1t + a_2t^2$$

$$= u$$

$$\boxed{1(u) = u}$$

the set of all polynomials of degree less than or equal to 2, P_2 (t) is a vector space under the operations

(i) $(a_0 + a_1t + a_2t^2) + (b_0 + b_1t + b_2t^2) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$

(ii) $k(a_0 + a_1t + a_2t^2) = ka_0 + ka_1t + ka_2t^2$

is a vector space.

Note:

The set of all polynomials of degree equal to n is not a vector space, since all polynomials are degree n so the identity element will not exists more over closure under addition will fail.

Example:

Let V be set consisting of all the polynomials of degree equal to 2.

Let $u, v \in V$, where $u = 3x^2 + 2x + 1$ & $v = -3x^2 + 5x + 1$

Then

$$u + v = (3x^2 + 2x + 1) + (-3x^2 + 5x + 1)$$

$$u + v = 7x + 2 \quad \notin V$$

That is, $u + v$ is not a polynomials of degree equal to 2.

THANK YOU