Linear Algebra

Linear Independence

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***.

THEOREM 4.3.1 *A nonempty set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *in a vector space* V *is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

EXAMPLE Linear Independence of the Standard Unit Vectors in R^3

consider the standard unit vectors $\mathbf{v}_1 = (1, 0, 0),$

$$\mathbf{v}_2 = (0, 1, 0),$$

$$\mathbf{v}_3 = (0, 0, 1)$$

To prove linear independence we must show that the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

are $\boxed{k_1 = 0, k_2 = 0, k_3 = 0.}$

This becomes evident by writing this equation in its component form

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$$

$$(k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow \quad \boxed{k_1 = 0, k_2 = 0, k_3 = 0.}$$

The most basic linearly independent set in R^n is the set of standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

▶ EXAMPLE 2 Linear Independence in R^3

Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \tag{1}$$

can be satisfied with coefficients that are not all zero.

To see whether this is so, let us rewrite (1) in the component form

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

Thus, our problem reduces to determining whether this system has nontrivial solutions.

Solving the above system of equations for k_1, k_2 & k_3 using Gauss elimination method

The augmented matrix is given by

Pivot element

$$\tilde{A} = \begin{array}{c} \begin{pmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{l} R_1 \text{ Pivot Row} \\ R_2 \\ R_3 \end{array}$$

$$= \begin{pmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$R_2 =$	-2	6	2	0
$2R_1 =$	2	10	6	0
$R_2 + 2R_1 =$	0	16	8	0

$$\tilde{A} = \begin{pmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$R_3 =$	3	-1	1	0
$-3R_1 =$	-3	-15	-9	0
$R_3 - 3R_1 =$	0	-16	-8	0

Rewriting as system of equations, we get

$$k_1 + 5k_2 + 3k_3 = 0$$

$$16\,k_2 + 8k_3 = 0$$

Rewriting as system of equations, we get

$$k_1 + 5k_2 + 3k_3 = 0 \text{ ------------- } (1)$$

$$16\,k_2 + 8k_3 = 0 \text{ ------------- } (2)$$

Let $k_3 = t$ (since k_3 is arbitrary), from (1) and (2) we get

$$\boxed{k_3 = t}$$

$$\boxed{k_2 = -\frac{1}{2}t}$$

$$\boxed{k_1 = -\frac{1}{2}t}$$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

Method 2:

A second method for establishing the linear dependence is to take advantage of the fact that the coefficient matrix

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

is square and compute its determinant.

The $\det(A) = 0$, it follows that the above system of equations has nontrivial solution. Therefore, the given vectors are linearly dependent.

Note:

Because we have established that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent, at least one of them is a linear combination of the others, for example,

$$\boxed{\mathbf{v}_3 = \tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2}$$

▸ EXAMPLE Linear Independence of Polynomials

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0}$$

can be satisfied with coefficients that are not all zero.

Now, rewriting above equation in its polynomial form gives

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0$$

or, equivalently,

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

Since this equation must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero

Thus, the linear dependence or independence of the given polynomials hinges on whether the following linear system has a nontrivial solution:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned}$$

We leave it for you to show that this linear system has nontrivial solutions either by solving it directly or by showing that the coefficient matrix has determinant zero.

$$\boxed{\text{Thus, the set } \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \text{ is linearly dependent.}}$$

THANK YOU