Linear Algebra

DEFINITION 2 If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

Examples:

(1) If $u, v, w \in V = \mathrm{R}^2$ and $u = (1,0), v = (0,1)$ & $w = (2,3)$ then the linear combination of w is

$$\boxed{(2,3) = 2(1,0) + 3(0,1)}$$

(2) If $u, v, w \in V = \mathrm{M}_{2\times 2}$ and $u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ & $w = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$ then the linear combination of w is

$$\boxed{\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

(3) If $u, v, w \in V = \mathrm{P}_2(\mathrm{t})$ and $u = x^2 + 3, v = x - 2$ & $w = 2x^2 + 3x$ then the linear combination of w is

$$\boxed{2x^2 + 3x = 2(x^2 + 3) + 3(x - 2)}$$

EXAMPLE 14 Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3.

- Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$
- Show that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that

$$\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$$

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

$$(9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$k_1 + 6k_2 = 9$$

$$2k_1 + 4k_2 = 2$$

$$-k_1 + 2k_2 = 7$$

Solving the above system of equations for k_1 & k_2 using Gauss elimination method

The augmented matrix is given by

Pivot element

$$\tilde{A} = \begin{array}{c} \begin{pmatrix} 1 & 6 & \vdots & 9 \\ 2 & 4 & \vdots & 2 \\ -1 & 2 & \vdots & 7 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Pivot Row

$$= \begin{pmatrix} 1 & 6 & \vdots & 9 \\ 0 & -8 & \vdots & -16 \\ 0 & 8 & \vdots & 16 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\begin{array}{rrrr} R_2 = & 2 & 4 & 2 \\ -2R_1 = & -2 & -12 & -18 \\ \hline R_4 - 2R_1 = & 0 & -8 & -16 \end{array}$$

$$\tilde{A} = \begin{pmatrix} 1 & 6 & \vdots & 9 \\ 0 & -8 & \vdots & -16 \\ 0 & 0 & \vdots & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

Rewriting as system of equations, we get

$$k_1 + 6k_2 = 9$$

$$-8k_2 = -16$$

Using back substitution, we get

$$\boxed{k_2 = 2}$$

$$\boxed{k_1 = -3}$$

Solving this system using Gaussian elimination yields

$$k_1 = -3,$$

$$k_2 = 2,$$

so $$\boxed{\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}}$$

Similarly, for $\mathbf{w}'$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that

$$\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v};$$

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

$$(4, -1, 8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

Solving the above system of equations for k_1, k_2 & k_3 using Gauss elimination method

The augmented matrix is given by

Pivot element

$$\tilde{A} = \begin{array}{c} \begin{pmatrix} 1 & 6 & \vdots & 4 \\ 2 & 4 & \vdots & -1 \\ -1 & 2 & \vdots & 8 \end{pmatrix} \end{array} \begin{array}{l} R_1 \quad \text{Pivot Row} \\ R_2 \\ R_3 \end{array}$$

$$= \begin{pmatrix} 1 & 6 & \vdots & 9 \\ 0 & -8 & \vdots & -9 \\ 0 & 8 & \vdots & 12 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\begin{array}{rrrr} R_2 = & 2 & 4 & -1 \\ -2R_1 = & -2 & -12 & -8 \\ \hline R_4 - 2R_1 = & 0 & -8 & -9 \end{array}$$

$$\tilde{A} = \begin{pmatrix} 1 & 6 & \vdots & 9 \\ 0 & -8 & \vdots & -16 \\ 0 & 0 & \vdots & 3 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

The false statement $0 = 3$ states that the system is inconsistent, so there is no Such k_1 & k_2 exists

Consequently, $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

In each part express the vector as a linear combination of

$\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

(a) $-9 - 7x - 15x^2$ (b) $6 + 11x + 6x^2$

(c) 0 (d) $7 + 8x + 9x^2$

Solution Let $\boldsymbol{p}(\boldsymbol{x}) = -9 - 7x - 15x^2$.

In order for $\boldsymbol{p}(\boldsymbol{x})$ to be linear combination of $\boldsymbol{p_1}(\boldsymbol{x}), \boldsymbol{p_2}(\boldsymbol{x})$ & $\boldsymbol{p_3}(\boldsymbol{x})$, there must be the scalars $k_1, k_2 \& k_3$ such that $\boldsymbol{p}(\boldsymbol{x})$ can be expressed as

$$\boldsymbol{p}(\boldsymbol{x}) = k_1\boldsymbol{p_1}(\boldsymbol{x}) + k_2\boldsymbol{p_2}(\boldsymbol{x}) + k_3\boldsymbol{p_3}(\boldsymbol{x})$$

$$-9 - 7x - 15x^2 = k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2)$$

Equating the corresponding coefficients on both sides, we get

$$-9 = 2k_1 + k_2 + 3k_3$$

$$-7 = k_1 - k_2 + 2k_3$$

$$-15 = 4k_1 + 3k_2 + 5k_3$$

Solving the above system of equations for k_1, k_2 & using Gauss elimination method

The augmented matrix is given by

Pivot element

$$\tilde{A} = \begin{pmatrix} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{pmatrix} \begin{matrix} R_1 \text{ Pivot Row} \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{array}{rcrrrr} R_2 & = & 1 & -1 & 2 & -7 \\ -(\frac{1}{2})R_1 & = & -1 & -\frac{1}{2} & -\frac{3}{2} & \frac{9}{2} \\ \hline R_4 - 2R_1 & = & 0 & -\frac{3}{2} & \frac{1}{2} & \frac{-5}{2} \\ \hline \end{array}$$

$$= \begin{pmatrix} 2 & 1 & 3 & -9 \\ 0 & -3/2 & 1/2 & -5/2 \\ 0 & 1 & -1 & 3 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - (\frac{1}{2})R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\tilde{A} = \begin{pmatrix} 2 & 1 & 3 & -9 \\ 0 & -3/2 & 1/2 & -5/2 \\ 0 & 0 & -2/3 & 4/3 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 + (\frac{2}{3})R_2 \end{matrix}$$

Rewriting as system of equations, we get

$$2k_1 + k_2 + 3k_3 = -9$$

$$-\frac{3}{2}\,k_2 + \left(\frac{1}{2}\right)k_3 = -\frac{5}{2}$$

$$-\frac{2}{3}k_3 = \frac{4}{3}$$

Using back substitution, we get

$$\boxed{k_3 = -2}$$

$$\boxed{k_2 = 1}$$

$$\boxed{k_1 = -2}$$

Therefore, $\boldsymbol{p}(\boldsymbol{x}) = -2\boldsymbol{p}_{\boldsymbol{1}}(\boldsymbol{x}) + \boldsymbol{p}_{\boldsymbol{2}}(\boldsymbol{x})-2\boldsymbol{p}_{\boldsymbol{3}}(\boldsymbol{x})$

Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

Solution (a)

$$\begin{pmatrix} 6 & -8 \\ -1 & -8 \end{pmatrix} = k_1 \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix} + k_2 \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -8 \\ -1 & -8 \end{pmatrix} = \begin{pmatrix} 4k_1 & 0 \\ -2k_1 & -2k_1 \end{pmatrix} + \begin{pmatrix} k_2 & -k_2 \\ 2k_2 & 3k_2 \end{pmatrix} + \begin{pmatrix} 0 & 2k_3 \\ k_3 & 4k_3 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -8 \\ -1 & -8 \end{pmatrix} = \begin{pmatrix} 4k_1 + k_2 & -k_2 + 2k_3 \\ -2k_1 + 2k_2 + k_3 & -2k_1 + 3k_2 + 4k_3 \end{pmatrix}$$

Equating corresponding components gives

$$6 = 4k_1 + k_2$$

$$-8 = -k_2 + 2k_3$$

$$-1 = -2k_1 + 2k_2 + k_3$$

$$-8 = -2k_1 + 3k_2 + 4k_3$$

Solving the above system of equations for $k_1, \& \, k_2$ using Gauss elimination method

The augmented matrix is given by

Pivot element

$$\tilde{A} = \left(\begin{array}{cccc} 4 & 1 & 0 & \vdots & 6 \\ 0 & -1 & 2 & \vdots & -8 \\ -2 & 2 & 1 & \vdots & -1 \\ -2 & 3 & 4 & \vdots & -8 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

Pivot Row

$$\begin{array}{rcccc} R_3 = & -2 & 2 & 1 & -1 \\ \frac{1}{2}R_1 = & 2 & \frac{1}{2} & 0 & 3 \\ \hline R_3 + \frac{1}{2}R_1 = & 0 & \frac{5}{2} & 1 & 2 \\ \hline \end{array}$$

$$\tilde{A} = \left(\begin{array}{cccc} 4 & 1 & 0 & \vdots & 6 \\ 0 & -1 & 2 & \vdots & -8 \\ 0 & 5/2 & 1 & \vdots & 2 \\ 0 & 5/2 & 4 & \vdots & -5 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \to R_2 \\ R_3 \to R_3 + \frac{1}{2}R_1 \\ R_4 \to R_4 + \frac{1}{2}R_1 \end{array}$$

$$\begin{array}{rcccc} R_3 = & 0 & -1 & 2 & -8 \\ (2/5)R_2 = & 0 & 1 & 2/5 & 4/5 \\ \hline R_3 + (2/5)R_2 = & 0 & 0 & 12/5 & -36/5 \\ \hline \end{array}$$

$$\tilde{A} = \left(\begin{array}{cccc} 4 & 1 & 0 & \vdots & 6 \\ 0 & -1 & 2 & \vdots & -8 \\ 0 & 0 & 12/5 & \vdots & -36/5 \\ 0 & 0 & 24/5 & \vdots & -5 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + 2/5R_2 \\ R_4 \to R_4 + 2/5R_2 \end{array}$$

Pivot Row

$$\tilde{A} = \begin{pmatrix} 4 & 1 & 0 & \vdots & 6 \\ 0 & -1 & 2 & \vdots & -8 \\ 0 & 0 & 12/5 & \vdots & -36/5 \\ 0 & 0 & 0 & & 47/5 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \quad \text{Pivot Row} \\ R_4 \to R_4 - 2R_3 \end{array}$$

The false statement $0 = 47/5$ states that the system is inconsistent, so there is no Such $k_1, k_2 \& k_3$ exists. Consequently, the given matrix $\begin{pmatrix} 6 & -8 \\ -1 & -8 \end{pmatrix}$ can not be Expressed as a liner combination of the given matrices $A, B \& C$.

THEOREM 4.2.3 *If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:*

(a) The set W of all possible linear combinations of the vectors in S is a subspace of V.

(b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

Proof (a) Let W be the set of all possible linear combinations of the vectors in S.

- We must show that W is closed under addition and scalar multiplication.
- To prove closure under addition, let

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r$$

$$\text{and} \quad \mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r$$

be two vectors in W.

- It follows that their sum can be written as

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{w}_1 + (c_2 + k_2)\mathbf{w}_2 + \cdots + (c_r + k_r)\mathbf{w}_r$$

which is a linear combination of the vectors in S.

- Thus, W is closed under addition.

- Similarly, we prove that W is also closed under scalar multiplication and hence is a subspace of V.

Proof (b)

Let W' be any subspace of V that contains all of the vectors in S.

Since W' is closed under addition and scalar multiplication, it contains all linear combinations of the vectors in S and hence contains W.

THANK YOU