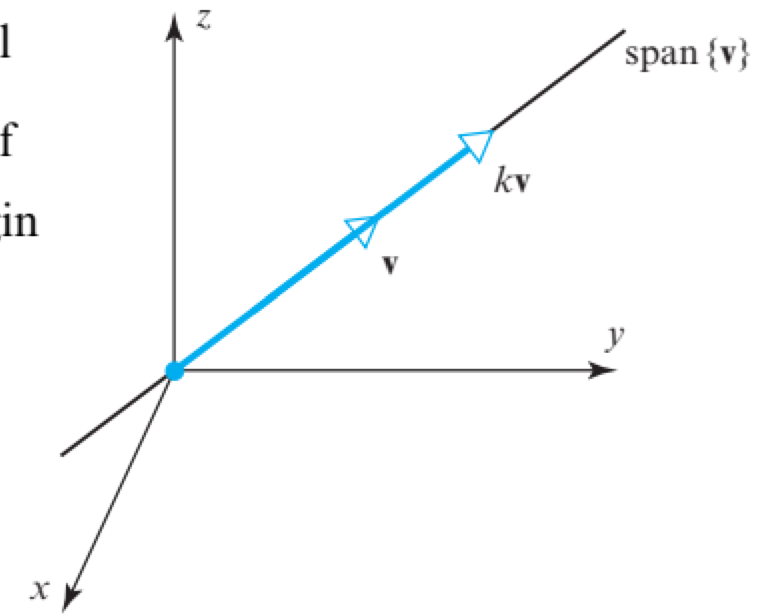
Linear Algebra

DEFINITION 3 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V ***generated*** by S, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ ***span*** W. We denote this subspace as

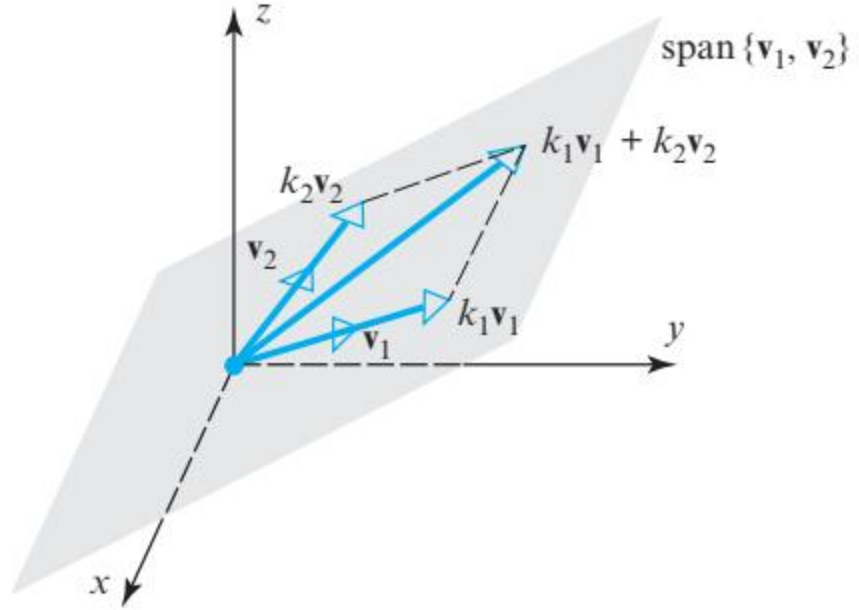
$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S)$$

EXAMPLE 12 A Geometric View of Spanning in R^2 and R^3

(a) If $\mathbf{v}$ is a nonzero vector in R^2 or R^3 that has its initial point at the origin, then span$\{\mathbf{v}\}$, which is the set of all scalar multiples of $\mathbf{v}$, is the line through the origin determined by $\mathbf{v}$.



(b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors in R^3 that have their initial points at the origin, then span$\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, is the plane through the origin determined by these two vectors.



(b) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

EXAMPLE 11 The Standard Unit Vectors Span R^n

The standard unit vectors in R^n are

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),$$

$$\mathbf{e}_2 = (0, 1, 0, \ldots, 0),$$

$$, \ldots ,$$

$$\mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\boxed{\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n}$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

Examples: The vectors $e_1 = (1,0)$ & $e_2 = (0,1)$ span R^2, since any vector (a, b) can be expressed as a linear combination of e_1 and e_2 as follows

$(a, b) = a(1,0) + b(0,1)$	$(2,3) = 2(1,0) + 3(0,1)$
$(a, b) = ae_1 + be_2$	$(2,3) = 2e_1 + 3e_2$

EXAMPLE Testing for Spanning

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.

Solution

We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

Thus, our problem reduces to ascertaining whether this system is consistent for all values of b_1, b_2, and b_3.

One way of doing this is to use parts (*e*) and (*g*) of Theorem 2.3.8, which state that

the system is consistent if and only if its coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \text{ has a nonzero determinant.}$$

But this is *not* the case here since $\det(A) = 0$, so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

THEOREM 2.3.8 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A can be expressed as a product of elementary matrices.*

(*e*) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*

(*f*) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*

(*g*) $\det(A) \neq 0$.

13. Determine whether the following polynomials span P_2.

$$\mathbf{p}_1 = 1 - x + 2x^2, \quad \mathbf{p}_2 = 3 + x, \quad \mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

Solution We must examine, whether an arbitrary polynomial $p(x)$ in $P_2(x)$ can be expressed as a linear combination of $p_1(x), p_2(x), p_3(x)$ & $p_4(x)$

$$p(x) = k_1 p_1(x) + k_2 p_2(x) + k_3\, p_3(x) + k_4\, p_4(x)$$

$$a_2x^2 + a_1x + a_0 = k_1(1 - x + 2x^2) + k_2(3 + x) + k_3(5 - x + 4x^2) + k_4(-2 - 2x + 2x^2)$$

$$a_2x^2 + a_1x + a_0 = (k_1 + 3k_2 + 5k_3 - 2k_4) + (-k_1 + k_2 - k_3 - 2k_4)x + (2k_1 + 4k_3 + 2k_4)x^2$$

Equating the coefficients on both sides, we get

$$k_1 + 3k_2 + 5k_3 - 2k_4 = a_0$$

$$-k_1 + k_2 - k_3 - 2k_4 = a_1$$

$$2k_1 + 4k_3 + 2k_4 = a_2$$

If the above system of equations is consistent for arbitrary a_0, a_1 & a_2 then any polynomial $p(x)$ in $P_2(x)$ can be expressed as a linear combination of $p_1(x), p_2(x), p_3(x)$ & $p_4(x)$

THEOREM 4.2.4 *The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.*

Proof Let W be the solution set of the system. The set W is not empty because it contains at least the trivial solution $\mathbf{x} = \mathbf{0}$.

To show that W is a subspace of R^n, we must show that it is closed under addition and scalar multiplication. To do this, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be vectors in W. Since these vectors are solutions of $A\mathbf{x} = \mathbf{0}$, we have

$$A\mathbf{x}_1 = \mathbf{0} \quad \text{and} \quad A\mathbf{x}_2 = \mathbf{0}$$

It follows from these equations and the distributive property of matrix multiplication that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so W is closed under addition. Similarly, if k is any scalar then

$$A(k\mathbf{x}_1) = kA\mathbf{x}_1 = k\mathbf{0} = \mathbf{0}$$

so W is also closed under scalar multiplication. ◀

▶ EXAMPLE 16 Solution Spaces of Homogeneous Systems

In each part, solve the system by any method and then give a geometric description of the solution set.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution (b)

The augmented matrix is given by

Pivot element

$$\tilde{A} = \begin{array}{c} \left(\begin{array}{cccc} 1 & -2 & 3 & 0 \\ -3 & 7 & -8 & 0 \\ -2 & 4 & -6 & 0 \end{array}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Pivot Row

$$= \left(\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \qquad\quad R_1 \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

Pivot Row

$$\begin{array}{rcccc} R_2 = & -3 & 7 & -8 & 0 \\ 3R_1 = & 3 & -6 & 9 & 0 \\ \hline R_4 + 3R_1 = & 0 & 1 & 1 & 0 \end{array}$$

$$\begin{array}{rcccc} R_3 = & -2 & 4 & -6 & 0 \\ 2R_1 = & 2 & -4 & 6 & 0 \\ \hline R_3 + 2R_1 = & 0 & 0 & 0 & 0 \end{array}$$

Rewriting as system of equations, we get

$$k_1 - 2k_2 + 3k_3 = 0$$

$$k_2 + k_3 = 0$$

Rewriting as system of equations, we get

$$k_1 - 2k_2 + 3k_3 = 0 \text{ ------------- } (1)$$

$$k_2 + k_3 = 0 \text{ ------------- } (2)$$

Let $k_3 = t$ (since k_3 is arbitrary), from (1) and (2) we get

$$\boxed{k_3 = t}$$

$$\boxed{k_2 = -t}$$

$$\boxed{k_1 = -5t}$$

(i.e.) $\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix} t$

The solution space of the given homogeneous system (b) is a set of all linear combination of the vector $\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}$, that is the line passes through the origin that is parallel to the vector $\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}$.

Solution (a)

The augmented matrix is given by

Pivot element

Pivot Row

$$\tilde{A} = \begin{array}{c} \left(\begin{matrix} 1 & -2 & 3 & 0 \\ -2 & 4 & -6 & 0 \\ 3 & -6 & 9 & 0 \end{matrix}\right) \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$= \left(\begin{matrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right) \begin{array}{l} R_1 \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

Rewriting as system of equations, we get

$$k_1 - 2k_2 + 3k_3 = 0$$

$R_2 =$	-2	4	-6	0
$2R_1 =$	2	-4	6	0
$R_4 + 2R_1 =$	0	0	0	0

$R_3 =$	3	-6	9	0
$-3R_1 =$	-3	6	-9	0
$R_3 - 3R_1 =$	0	0	0	0

Rewriting as system of equations, we get

$$k_1 - 2k_2 + 3k_3 = 0 \text{ ------------- } (1)$$

Let $k_2 = t$ & $k_3 = s$
(since k_2 & k_3 are arbitrary)

$$\boxed{k_3 = s}$$

$$\boxed{k_2 = t}$$

$$\boxed{k_1 = 2t - 3s}$$

(i.e.) $\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 2t - 3s \\ t \\ s \end{pmatrix}$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} -3s \\ 0 \\ s \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} s$$

The solution space of the given homogeneous system (a) is a set of all linear combination of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$, that is the plane passes through the origin Spanned by the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

THANK YOU