QR- decomposition

Suppose A is an $m \times n$ matrix with linearly independent column vectors;

$$A = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

$u_1 = (a_{11}, a_{21}, a_{31})$ $u_2 = (a_{12}, a_{22}, a_{32})$,

$u_3 = (a_{13}, a_{23}, a_{33})$

since column's are linearly independent

$\{u_1, u_2, u_3\}$ forms a basis for $\mathbb{R}^3$

(Note: suppose $A_{4\times 3}$ matrix.

Then $\{u_1, u_2, u_3\}$ is just linearly independent subset of $\mathbb{R}^4$ & It spans a 3-dimensional subspace)

We are going to discuss our results only for $n\times n$ matrix. and we will generalize for $m\times n$ matrix

$\{u_1, u_2, u_3\}$ forms a basis for $\mathbb{R}^3$

By Gram-schmidt process, we get

$\{q_1, q_2, q_3\}$ - orthonormal basis.

what is the relation between

$$A = [u_1 | u_2 | u_3] , \quad Q = [q_1 | q_2 | q_3]$$

since $\{q_1, q_2, q_3\}$ is a basis for $\mathbb{R}^3$

Recall

$$u_1 = \langle u_1, q_1 \rangle q_1 + \langle u_1, q_2 \rangle q_2 + \langle u_1, q_3 \rangle q_3$$

$$u_2 = \langle u_2, q_1 \rangle q_1 + \langle u_2, q_2 \rangle q_2 + \langle u_2, q_3 \rangle q_3$$

$$u_3 = \langle u_3, q_1 \rangle q_1 + \langle u_3, q_2 \rangle q_2 + \langle u_3, q_3 \rangle q_3$$

$$[u_1 | u_2 | u_3] = [q_1 | q_2 | q_3] \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ \langle u_1, q_2 \rangle & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ \langle u_1, q_3 \rangle & \langle u_2, q_3 \rangle & \langle u_3, q_3 \rangle \end{bmatrix}$$

[From Gram-Schmidt process, we can observe that for $j \geq 2$, q_j is orthogonal to $u_1, \ldots, u_{j-1}$]

$$\therefore \langle u_1, q_2 \rangle = 0 , \quad \langle u_1, q_3 \rangle = 0$$

$$\langle u_2, q_3 \rangle = 0$$

$$\Rightarrow \quad A = Q \underbrace{\begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}}_{= R}$$

$$A = QR$$

Generalizing for $m \times n$ matrix

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

$$[u_1 | u_2 | \cdots | u_n] = [q_1 | q_2 | \cdots | q_n] \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \cdots & \langle u_n, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \cdots & \langle u_n, q_2 \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle u_n, q_n \rangle \end{bmatrix}$$

where Q is an $m \times n$ matrix with orthonormal column vectors (ie) $QQ^T = I$) & R is an $n \times n$ invertible upper triangular matrix.

Example:

1) Find a QR decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Sol:

$u_1 = (1, 2, 1)$, $u_2 = (1, 0, 1)$, $u_3 = (1, 1, 2)$

After Gram-Schmidt process

we get, $q_1 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $q_2 = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$q_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$\langle u_1, q_1 \rangle = 1 \cdot \frac{1}{\sqrt{6}} + 2 \cdot \frac{2}{\sqrt{6}} + 1 \cdot \frac{1}{\sqrt{6}} = \frac{6}{\sqrt{6}}$

$\langle u_2, q_1 \rangle = 1 \cdot \frac{1}{\sqrt{6}} + 1 \cdot \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}} = 2/\sqrt{6}$

$\langle u_3, q_1 \rangle = \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} = 5/\sqrt{6}$

$\langle u_2, q_2 \rangle = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

$\langle u_3, q_2 \rangle = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = 2/\sqrt{3}$

$\langle u_3, q_3 \rangle = -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$2/6 + \frac{2}{3}$

$\frac{2+4}{6}$

$$\therefore \quad A = \underbrace{\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 6/\sqrt{6} & 2/\sqrt{6} & 5/\sqrt{6} \\ 0 & 2/\sqrt{3} & 2/\sqrt{3} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}}_{R}$$

$\frac{5}{6} + \frac{2}{3} - \frac{1}{2}$

$\frac{5+4-3}{6}$

$$QR = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{(Check)}$$

$\frac{10}{6} - \frac{2}{3}$

2) Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

sol

$u_1 = (1, 1, 1)$ $u_2 = (0, 1, 1)$, $u_3 = (0, 0, 1)$

Apply Gram-Schmidt process, we get

$$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad q_2 = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

Check.

$A = QR$.