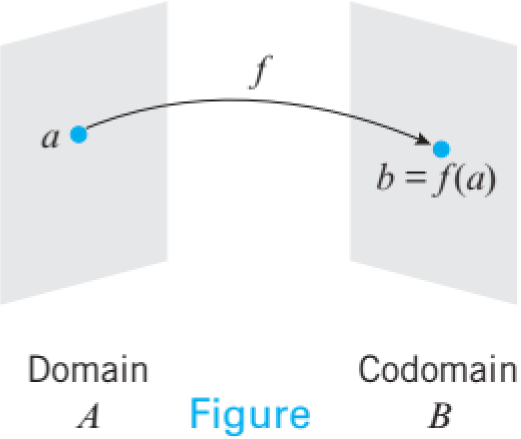
Linear Algebra

Functions

- Recall that a ***function*** is a rule that associates with each element of a set A one and only one element in a set B.



Figure

- If f associates the element b with the element a, then we write

$$b = f(a)$$

and we say that b is the ***image*** of a under f or that $f(a)$ is the ***value*** of f at a.

- The set A is called the ***domain*** of f and the set B the ***codomain*** of f.

- The subset of the codomain that consists of all images of elements in the domain is called the ***range*** of f.

Transformations

DEFINITION 1 If f is a function with domain R^n and codomain R^m, then we say that f is a ***transformation*** from R^n to R^m or that f ***maps*** from R^n to R^m, which we denote by writing

$$f: R^n \to R^m$$

In the special case where $m = n$, a transformation is sometimes called an ***operator*** on R^n.

Matrix Transformations

In this section we will be concerned with the class of transformations from R^n to R^m that arise from linear systems.

$$\begin{aligned}
w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\;\;\vdots \\
w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}$$

which we can write in matrix notation as

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or more briefly as $\mathbf{w} = A\mathbf{x}$

we will view it as a transformation that maps a vector $\mathbf{x}$ in R^n into thevector $\mathbf{w}$ in R^m by multiplying $\mathbf{x}$ on the left by A.

- We call this a ***matrix transformation*** (or ***matrix operator*** in the special case where $m = n$).

  We denote it by

$$T_A \colon R^n \to R^m$$

  or more briefly as $\mathbf{w} = T_A(\mathbf{x})$. We call the transformation T_A ***multiplication by A***.

- On occasion we will find it convenient to express in the schematic form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

  which is read "T_A maps $\mathbf{x}$ into $\mathbf{w}$."

▶ EXAMPLE 1 A Matrix Transformation from R^4 to R^3

The transformation from R^4 to R^3 defined by the equations

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned}$$

can be expressed in matrix form as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from which we see that the transformation can be interpreted as multiplication by

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

Although the image under the transformation T_A of any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

in R^4 could be computed directly from the defining equations, we will find it preferable to use the matrix A. For example, if

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

then it follows that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

► EXAMPLE 2 Zero Transformations

If 0 is the $m \times n$ zero matrix, then

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

so multiplication by zero maps every vector in R^n into the zero vector in R^m.

We call T_0 the ***zero transformation*** from R^n to R^m.

► EXAMPLE 3 Identity Operators

If I is the $n \times n$ identity matrix, then

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

so multiplication by I maps every vector in R^n to itself.

We call T_I the ***identity operator*** on R^n.

Properties of Matrix Transformations

THEOREM 1.8.1 *For every matrix A the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *and for every scalar k:*

(*a*) $T_A(\mathbf{0}) = \mathbf{0}$

(*b*) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ **[Homogeneity property]**

(*c*) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

(*d*) $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

THEOREM 1.8.2 *$T: R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in R^n and for every scalar k:*

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

The additivity and homogeneity properties in Theorem 1.8.2 are called ***linearity conditions***, and a transformation that satisfies these conditions is called a ***linear transformation***.

Matrix transformations are not the only kinds of transformations. For example, if

$$\begin{aligned} w_1 &= x_1^2 + x_2^2 \\ w_2 &= x_1 x_2 \end{aligned} \tag{1}$$

then there are no constants a, b, c, and d for which

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1 x_2 \end{bmatrix}$$

so that the equations in (1) do not define a matrix transformation from R^2 to R^2.

THEOREM 1.8.3 *Every linear transformation from R^n to R^m is a matrix transformation, and conversely, every matrix transformation from R^n to R^m is a linear transformation.*

THEOREM 1.8.4 *If $T_A: R^n \rightarrow R^m$ and $T_B: R^n \rightarrow R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.*

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

▶ EXAMPLE 4 Finding a Standard Matrix

Find the standard matrix A for the linear transformation $T: R^2 \rightarrow R^2$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

Solution

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Thus, it follows that the standard matrix is

$$\boxed{A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}}$$

▶ EXAMPLE 5 Computing with Standard Matrices

For the linear transformation in Example 4, use the standard matrix A obtained in that example to find

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

Solution The transformation is multiplication by A, so

$$\boxed{T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix}}$$

▸ EXAMPLE 6 Finding a Standard Matrix

Rewrite the transformation $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$ in column-vector form and find its standard matrix.

Solution

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, the standard matrix is

$$\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$

General Linear Transformations

Up to now our study of linear transformations has focused on transformations from R^n to R^m.

In this section we will turn our attention to linear transformations involving general vector spaces.

Definitions and Terminology

- We defined a *matrix transformation* $T_A: R^n \to R^m$ to be a mapping of the form

$$T_A(\mathbf{x}) = A\mathbf{x}$$

  in which A is an $m \times n$ matrix.

- We subsequently established in Theorem that the matrix transformations are precisely the *linear transformations* from R^n to R^m, that is,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$\text{and} \quad T(k\mathbf{u}) = kT(\mathbf{u})$$

We will use these two properties as the starting point for defining more general linear transformations.

DEFINITION 1 If $T : V \rightarrow W$ is a mapping from a vector space V to a vector space W, then T is called a ***linear transformation*** from V to W if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars k:

(i) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

(ii) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

In the special case where $V = W$, the linear transformation T is called a ***linear operator*** on the vector space V.

More generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in V and $k_1, k_2, \ldots, k_r$ are any scalars, then

$$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_rT(\mathbf{v}_r) \tag{1}$$

THEOREM 8.1.1 *If $T : V \rightarrow W$ is a linear transformation, then:*

(*a*) $T(\mathbf{0}) = \mathbf{0}$.

(*b*) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ *for all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* V.

▶ EXAMPLE 1 Matrix Transformations

Because we have based the definition of a general linear transformation on the homogeneity and additivity properties of *matrix transformations*, it follows that every matrix transformation $T_A\colon R^n \to R^m$ is also a linear transformation in this more general sense with $V = R^n$ and $W = R^m$.

▶ EXAMPLE 2 The Zero Transformation

Let V and W be any two vector spaces. The mapping $T\colon V \to W$ such that $T(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v}$ in V is a linear transformation called the ***zero transformation***.

To see that T is linear, observe that

$$T(\mathbf{u}+\mathbf{v}) = \mathbf{0},$$

$$T(\mathbf{u}) = \mathbf{0},$$

$$T(\mathbf{v}) = \mathbf{0},$$

$$\text{and} \quad T(k\mathbf{u}) = \mathbf{0}$$

Therefore,

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$\text{and} \quad T(k\mathbf{u}) = kT(\mathbf{u})$$

▶ EXAMPLE 3 The Identity Operator

Let V be any vector space. The mapping $I: V \to V$ defined by $I(\mathbf{v}) = \mathbf{v}$ is called the ***identity operator*** on V. We will leave it for you to verify that I is linear.

▶ EXAMPLE 4 Dilation and Contraction Operators

If V is a vector space and k is any scalar, then the mapping $T: V \to V$ given by $T(\mathbf{x}) = k\mathbf{x}$ is a linear operator on V, for if c is any scalar and if $\mathbf{u}$ and $\mathbf{v}$ are any vectors in V, then

$$\begin{aligned} T(c\mathbf{u}) &= k(c\mathbf{u}) \\ &= c(k\mathbf{u}) \\ \boxed{T(c\mathbf{u}) = cT(\mathbf{u})} \end{aligned} \qquad \begin{aligned} T(\mathbf{u}+\mathbf{v}) &= k(\mathbf{u}+\mathbf{v}) \\ &= k\mathbf{u} + k\mathbf{v} \\ \boxed{T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})} \end{aligned}$$

If $0 < k < 1$, then T is called the ***contraction*** of V with factor k, and if $k > 1$, it is called the ***dilation*** of V with factor k.

▶ EXAMPLE 5 A Linear Transformation from P_n to P_{n+1}

Let $\mathbf{p} = p(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial in P_n, and define the transformation $T: P_n \to P_{n+1}$ by

$$T(\mathbf{p}) = T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \cdots + c_n x^{n+1}$$

This transformation is linear because for any scalar k and any polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in P_n we have

$$\begin{aligned} T(k\mathbf{p}) &= T(kp(x)) \\ &= x(kp(x)) \\ &= k(xp(x)) \\ \end{aligned}$$

$$\boxed{T(k\mathbf{p}) = kT(\mathbf{p})}$$

$$\begin{aligned} T(\mathbf{p}_1 + \mathbf{p}_2) &= T(p_1(x) + p_2(x)) \\ &= x(p_1(x) + p_2(x)) \\ &= xp_1(x) + xp_2(x) \end{aligned}$$

$$\boxed{T(\mathbf{p}_1 + \mathbf{p}_2) = T(\mathbf{p}_1) + T(\mathbf{p}_2)}$$

▶ EXAMPLE 7 Transformations on Matrix Spaces

Let M_{nn} be the vector space of $n \times n$ matrices. In each part determine whether the transformation is linear.

(a) $T_1(A) = A^T$ (b) $T_2(A) = \det(A)$

Solution (a)

$$T_1(kA) = (kA)^T = kA^T$$
$$T_1(kA) = kT_1(A)$$

$$T_1(A+B) = (A+B)^T = A^T + B^T$$
$$T_1(A+B) = T_1(A) + T_1(B)$$

so T_1 is linear.

Solution (b)

$$T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A)$$

Note that additivity also fails because

$\det(A+B)$ and $\det(A) + \det(B)$

are not generally equal.

Thus, T_2 is not homogeneous and hence not linear

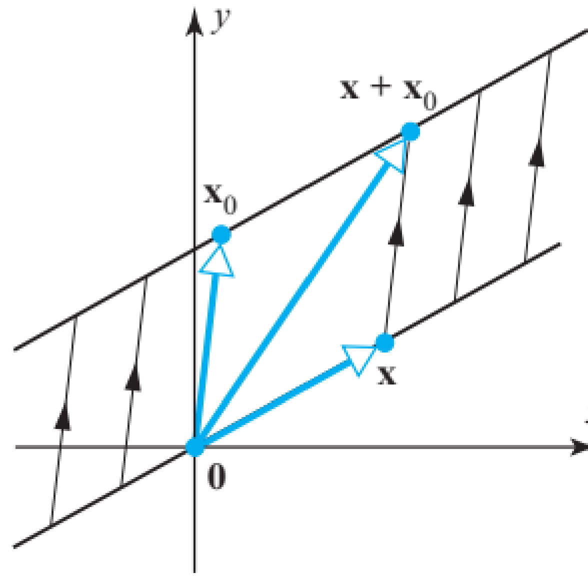
EXAMPLE 8 Translation Is Not Linear

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$$

Part (*a*) of Theorem 8.1.1 states that a linear transformation maps $\mathbf{0}$ to $\mathbf{0}$. This property is useful for identifying transformations that are *not* linear.

For example, if $\mathbf{x}_0$ is a fixed nonzero vector in R^2, then the transformation

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$$



▲ Figure 8.1.1 $T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$ translates each point $\mathbf{x}$ along a line parallel to $\mathbf{x}_0$ through a distance $\|\mathbf{x}_0\|$.

has the geometric effect of translating each point $\mathbf{x}$ in a direction parallel to $\mathbf{x}_0$ through a distance of $\|\mathbf{x}_0\|$ (Figure 8.1.1). This cannot be a linear transformation since $T(\mathbf{0}) = \mathbf{x}_0$, so T does not map $\mathbf{0}$ to $\mathbf{0}$.

EXAMPLE 9 The Evaluation Transformation

Let V be a subspace of $F(-\infty, \infty)$, let

$$x_1, x_2, \ldots, x_n$$

be a sequence of distinct real numbers, and let $T: V \rightarrow R^n$ be the transformation

$$T(f) = \big(f(x_1), f(x_2), \ldots, f(x_n)\big) \tag{2}$$

that associates with f the n-tuple of function values at $x_1, x_2, \ldots, x_n$.

We call this the ***evaluation transformation*** on V at $x_1, x_2, \ldots, x_n$. Thus, for example, if

$$x_1 = -1, \quad x_2 = 2, \quad x_3 = 4$$

and if $f(x) = x^2 - 1$, then

$$T(f) = \big(f(x_1), f(x_2), f(x_3)\big) = (0, 3, 15)$$

The evaluation transformation in (2) is linear, for if k is any scalar, and if f and g are any functions in V, then

$$T(kf) = \big((kf)(x_1), (kf)(x_2), \ldots, (kf)(x_n)\big)$$

$$= \big(kf(x_1), kf(x_2), \ldots, kf(x_n)\big)$$

$$= k\big(f(x_1), f(x_2), \ldots, f(x_n)\big)$$

$$T(kf) = kT(f)$$

and

$$T(f+g) = \big((f+g)(x_1), (f+g)(x_2), \ldots, (f+g)(x_n)\big)$$

$$= \big(f(x_1) + g(x_1), f(x_2) + g(x_2), \ldots, f(x_n) + g(x_n)\big)$$

$$= \big(f(x_1), f(x_2), \ldots, f(x_n)\big) + \big(g(x_1), g(x_2), \ldots, g(x_n)\big)$$

$$T(f+g) = T(f) + T(g)$$

Finding Linear Transformations from Images of Basis Vectors

If $T_A: R^n \to R^m$ is multiplication by A, and if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n, then A can be expressed as

$$\boxed{A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)]}$$

It follows from this that the image of any vector $\mathbf{v} = (c_1, c_2, \ldots, c_n)$ in R^n under multiplication by A can be expressed as

$$\boxed{T_A(\mathbf{v}) = c_1 T_A(\mathbf{e}_1) + c_2 T_A(\mathbf{e}_2) + \cdots + c_n T_A(\mathbf{e}_n)}$$

This formula tells us that for a matrix transformation the image of any vector is expressible as a linear combination of the images of the standard basis vectors. This is a special case of the following more general result.

THEOREM 8.1.2 *Let $T: V \to W$ be a linear transformation, where V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then the image of any vector $\mathbf{v}$ in V can be expressed as*

$$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) \tag{3}$$

where $c_1, c_2, \ldots, c_n$ are the coefficients required to express $\mathbf{v}$ as a linear combination of the vectors in the basis S.

Proof Express $\mathbf{v}$ as $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ and use the linearity of T.

▶ EXAMPLE 10 Computing with Images of Basis Vectors

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0)$$

Let $T: R^3 \to R^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (2, -1), \quad T(\mathbf{v}_3) = (4, 3)$$

Find a formula for $T(x_1, x_2, x_3)$, and then use that formula to compute $T(2, -3, 5)$.

Solution

We first need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_1 + c_2 \phantom{{}+c_3} &= x_2 \\ c_1 \phantom{{}+c_2+c_3} &= x_3 \end{aligned}$$

which yields $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - x_2$, so

$$\begin{aligned} (x_1, x_2, x_3) &= x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) \\ &= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3 \end{aligned}$$

Thus

$$\begin{aligned} T(x_1, x_2, x_3) &= x_3 T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \end{aligned}$$

$$\boxed{T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)}$$

From this formula we obtain

$$T(2, -3, 5) = (9, 23)$$

THANK YOU