Linear Algebra

Examples:

Check whether the set of all ordered pairs of real numbers (x, y) form a vector space over $\mathbb{R}$ with vector addition and scalar multiplication given by

$$(i)(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(ii)\ k(x_1, y_1) = (2kx_1, 2ky_1)$$

Proof: **I. Abelian group under addition**

1. Closure property: Let $u, v \in V = \mathbb{R}^2$, where $u = (x_1, y_1), \& \ v = (x_2, y_2)$,

we need to prove that $u + v \in \mathbb{R}^2$

Now

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$u + v = (x_1 + x_2, y_1 + y_2) \quad \in \mathbb{R}^2$$

Therefore

$$\boxed{u, v \in \mathbb{R}^2 \text{ implies } u + v \in \mathbb{R}^2}$$

2. Associativity property:

Let $u, v, w \in V = \mathbb{R}^2$, where $u = (x_1, y_1)$, $v = (x_2, y_2)$ & $w = (x_3, y_3)$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in \mathbb{R}^2$, such that

$$u + e = e + u = u \text{ for all } u \in \mathbb{R}^2$$

Let $e = (0,0)$ such that

$$u + e = (x_1, y_1) + (0,0)$$

$$= (x_1 + 0, y_1 + 0)$$

$$= (x_1, y_1)$$

$$\boxed{u + e = u}$$

$$e + u = (0,0) + (x_1, y_1)$$

$$= (0 + x_1, 0 + y_1)$$

$$= (x_1, y_1)$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in \mathbb{R}^2}$$

4. Inverse Property:

We need to prove that, for every element $u \in \mathbb{R}^2$, there exists $-u \in \mathbb{R}^2$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = (-x_1, -y_1)$ such that

$$u + (-u) = (x_1, y_1) + (-x_1, -y_1)$$

$$= (0,0)$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = (-x_1, -y_1) + (x_1, y_1)$$

$$= (0,0)$$

$$\boxed{(-u) + u = e}$$

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in \mathbb{R}^2$$

Now,

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$= (x_2 + x_1, y_2 + y_1)$$

$$= (x_2, y_2) + (x_1, y_1)$$

$$\boxed{u + v = v + u}$$

Therefore $\boxed{u + v = v + u \text{ for all } u, v \in \mathbb{R}^2}$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in V = \mathbb{R}^2 \ \&\ \alpha \in K \text{ such that } \alpha u \in V = \mathbb{R}^2$$

Now,

$$\boxed{(ii)\ k(x_1, y_1) = (2kx_1, 2ky_1)}$$

$$\alpha u = \alpha(x_1, y_1)$$

$$\alpha u = (2\alpha x_1, 2\alpha y_1) \ \in \mathbb{R}^2$$

Therefore

for any $u \in \mathbb{R}^2$ & $\alpha \in K$ we have $\alpha u \in \mathbb{R}^2$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = \mathbb{R}^2$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,

$(i)\,(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$(ii)\; k(x_1, y_1) = (2kx_1, 2ky_1)$

$$\alpha(u+v) = \alpha\{(x_1, y_1) + (x_2, y_2)\}$$

$$= \alpha(x_1 + x_2, y_1 + y_2)$$

$$= (2\alpha x_1 + 2\alpha x_2, 2\alpha y_1 + 2\alpha y_2)$$

$$= (2\alpha x_1, 2\alpha y_1) + (2\alpha x_2, 2\alpha y_2)$$

$$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V = \mathbb{R}^2 \;\&\; \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

$(ii)\ k(x_1, y_1) = (2kx_1, 2ky_1)$

Now, $$(\alpha + \beta)u = (\alpha + \beta)(x_1, y_1)$$

$$= (2(\alpha + \beta)x_1, 2(\alpha + \beta)y_1)$$

$$= (2\alpha x_1 + 2\beta x_1, 2\alpha y_1 + 2\beta y_1)$$

$$= (2\alpha x_1, 2\alpha y_1) + (2\beta x_1, 2\beta y_1)$$

$$= \alpha(x_1, y_1) + \beta(x_1, y_1)$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

Therefore

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u \text{ for all } u \in V = \mathbb{R}^2 \ \& \ \alpha, \beta \in K}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(2\beta x_1, 2\beta y_1)$$

$$= (4\alpha\beta x_1, 4\alpha\beta y_1)$$

$$\neq \alpha\beta(x_1, y_1)$$

$$\boxed{\alpha(\beta u) \neq (\alpha\beta)u}$$

Axiom (9) fails

10. Property 10:

$$1(u) = 1\,(x_1, y_1)$$

$$1(u) = (2x_1, 2y_1)$$

$$1(u) \neq u$$

Axiom (10) fails

The set of all order pairs $V = \mathbb{R}^2$ with the vector addition and scalar multiplication

$$(i)(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$(ii)\ k(x_1, y_1) = (2kx_1, 2ky_1)$$

is not a vector space, since Axioms (9) and (10) are failed.

3. Examples:

Is the set $V = \mathbb{R}^3$ with vector addition and scalar multiplication given below form a vector space

$$(i)\ (x_1, y_1, z_1) + (x_2, y_2 + z_3) = (x_1 + x_2, y_1 + y_2, z_1 + z_3)$$

$$(ii)\ k(x_1, y_1, z_1) = (kx_1, y_1, z_1)$$

Proof: **I. Abelian group under addition**

1. Closure property:

Let $u, v \in V = \mathbb{R}^3$, where $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$,

we need to prove that $u + v \in \mathbb{R}^3$

Now

$$u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$

$$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \quad \in \mathbb{R}^3$$

Therefore

$$\boxed{u, v \in \mathbb{R}^3 \text{ implies } u + v \in \mathbb{R}^3}$$

2. Associativity property:

Let $u, v, w \in V = \mathbb{R}^3$, where $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ & $w = (x_3, y_3, z_3)$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = (x_1, y_1, z_1) + \{(x_2, y_2, z_2) + (x_3, y_3, z_3)\}$$

$$= (x_1, y_1, z_1) + (x_2 + x_3, y_2 + y_3, z_2 + z_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + z_2 + z_3)$$

$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_3, y_3, z_3)$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in \mathbb{R}^3$, such that

$$u + e = e + u = u \text{ for all } u \in \mathbb{R}^3$$

Let $e = (0,0,0)$ such that

$$u + e = (x_1, y_1, z_1) + (0,0,0)$$

$$= (x_1 + 0, y_1 + 0, z_1 + 0)$$

$$= (x_1, y_1, z_1)$$

$$\boxed{u + e = u}$$

$$e + u = (0,0,0) + (x_1, y_1, z_1)$$

$$= (0 + x_1, 0 + y_1, 0 + z_1)$$

$$= (x_1, y_1, z_1)$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in \mathbb{R}^3}$$

4. Inverse Property:

We need to prove that, for every element $u \in \mathbb{R}^3$, there exists $-u \in \mathbb{R}^3$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = (-x_1, -y_1, -z_1)$ such that

$$u + (-u) = (x_1, y_1, z_1) + (-x_1, -y_1, -z_1)$$

$$= (0,0,0)$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = (-x_1, -y_1, -z_1) + (x_1, y_1, z_1)$$

$$= (0,0,0)$$

$$\boxed{(-u) + u = e}$$

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in \mathbb{R}^3$$

Now,

$$u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$

$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (x_2 + x_1, y_2 + y_1, z_2 + z_1)$$

$$= (x_2, y_2, z_2) + (x_1, y_1, z_1)$$

$$\boxed{u + v = v + u}$$

Therefore $\boxed{u + v = v + u \text{ for all } u, v \in \mathbb{R}^3}$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in V = \mathbb{R}^3 \ \& \ \alpha \in K \text{ such that } \alpha u \in V = \mathbb{R}^3$$

Now,

$$\alpha u = \alpha(x_1, y_1, z_1)$$

$$\alpha u = (\alpha x_1, y_1, z_1) \quad \in \mathbb{R}^3$$

Therefore

$$\text{for any } u \in \mathbb{R}^3 \ \& \ \alpha \in K \text{ we have } \alpha u \in \mathbb{R}^3$$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = \mathbb{R}^3$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,

$$\alpha(u+v) = \alpha u + \alpha v$$

$$\alpha\{(x_1, y_1, z_1) + (x_2, y_2, z_2)\} = \alpha(x_1, y_1, z_1) + \alpha(x_2, y_2, z_2)$$

$$\alpha(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (\alpha x_1, y_1, z_1) + (\alpha x_2, y_2, z_2)$$

$$(\alpha x_1 + \alpha x_2, y_1 + y_2, z_1 + z_2) = (\alpha x_1 + \alpha x_2, y_1 + y_2, z_1 + z_2)$$

$$\boxed{LHS = RHS}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V = \mathbb{R}^3 \text{ \& } \alpha \in K}$$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = \mathbb{R}^3$ & $\alpha \in K$

$$\alpha(u + v) = \alpha u + \alpha v$$

Now,

$$\alpha(u + v) = \alpha\{(x_1, y_1, z_1) + (x_2, y_2, z_2)\}$$

$$= \alpha(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (\alpha x_1 + \alpha x_2, y_1 + y_2, z_1 + z_2)$$

$$= (\alpha x_1, y_1, z_1) + (\alpha x_2, y_2, z_2)$$

$$= \alpha(x_1, y_1, z_1) + \alpha(x_2, y_2, z_2)$$

$$\boxed{\alpha(u + v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u + v) = \alpha u + \alpha v \text{ for all } u, v \in V = \mathbb{R}^3 \text{ \& } \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^3$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = \big((\alpha + \beta)x_1, y_1, z_1\big)$$

$$(\alpha + \beta)u = (\alpha x_1 + \beta x_1, y_1, z_1)$$

$$\alpha u + \beta u = \alpha(x_1, y_1, z_1) + \beta(x_1, y_1, z_1)$$

$$= (\alpha x_1, y_1, z_1) + (\beta x_1, y_1, z_1)$$

$$\alpha u + \beta u = (\alpha x_1 + \beta x_1, 2y_1, 2z_1)$$

$$\boxed{(\alpha + \beta)u \neq \alpha u + \beta u}$$

Axiom 8 fails

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^3$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta x_1, y_1, z_1)$$

$$= (\alpha\beta x_1, y_1, z_1)$$

$$= \alpha\beta(x_1, y_1, z_1)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

Therefore $\boxed{\alpha(\beta u) = (\alpha\beta)u \text{for all } u \in V = \mathbb{R}^3 \text{ \& } \alpha, \beta \in K}$

9. Associative property of vector addition and scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^3$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta x_1, y_1, z_1)$$
$$= (\alpha\beta x_1, y_1, z_1)$$

$$(\alpha\beta)u = (\alpha\beta x_1, y_1, z_1)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

10. Property 10:

$$1(u) = \big((1)x_1, y_1, z_1\big)$$
$$= (x_1, y_1, z_1)$$

$$\boxed{1(u) = u}$$

The set $V = \mathbb{R}^3$ with the vector addition and scalar multiplication

$(i)\ (x_1, y_1, z_1) + (x_2, y_2 + z_3) = (x_1 + x_2, y_1 + y_2, z_1 + z_3)$

$(ii)\ k(x_1, y_1, z_1) = (kx_1, y_1, z_1)$

is not a vector space, since Axiom 8 fails.

4. Examples:

Is the set of all positive real numbers V with vector addition and scalar multiplication given below form a vector space ?

$$(i)\ x+y=xy$$

$$(ii)\ kx=x^k$$

Proof: **I. Abelian group under addition**

1. Closure property:

Let $u, v \in V$, where $u=x$ & $v=y$,

we need to prove that $u+v \in V$

$$\boxed{(i)\ x+y=xy}$$

Now

$$u+v=x+y$$

$$u+v=xy \quad \in V$$

Therefore

$$\boxed{u, v \in V \text{ implies } u+v \in V}$$

2. Associativity property:

Let $u, v, w \in V$, where $u = x, \quad v = y$ & $w = z$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$\boxed{(i)\ x + y = xy}$$

$$u + (v + w) = x + \{y + z\}$$

$$= x + yz$$

$$= xyz$$

$$= xy + z$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in V$, such that

$$u + e = e + u = u \text{ for all } u \in V$$

Let $e = 1$ such that

$$u + e = x + 1$$

$$= x.1$$

$$= x$$

$$\boxed{u + e = u}$$

$$e + u = 1 + x$$

$$= 1.x$$

$$= x$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in V}$$

4. Inverse Property:

We need to prove that, for every element $u \in V$, there exists $v \in V$ such that

$$u + v = v + u = e$$

Let $v = \frac{1}{u}$ such that

$$u + v = x + \frac{1}{x}$$

$$= x \, \frac{1}{x}$$

$$= 1$$

$$\boxed{u + v = e}$$

$$v + u = \frac{1}{x} + x$$

$$= \frac{1}{x} \, x$$

$$= 1$$

$$\boxed{v + u = e}$$

Therefore

$$\boxed{u + v = v + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in V$$

Now,

$$u + v = x + y$$

$$= xy$$

$$= yx$$

$$= y + x$$

$$\boxed{u + v = v + u}$$

Therefore $\boxed{u + v = v + u \text{ for all } u, v \in V}$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$u \in V$ & $\alpha \in K$ such that $\alpha u \in V$

Now,

$$\boxed{(ii)\ kx = x^k}$$

$$\alpha u = \alpha\ x$$

$$\alpha u = x^{\alpha} \quad \in V$$

Therefore

for any $u \in V$ & $\alpha \in K$ we have $\alpha u \in V$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V$ & $\alpha \in K$

$$\alpha(u + v) = \alpha u + \alpha v$$

Now, $$\alpha(u + v) = \alpha u + \alpha v$$

$$\alpha\{x + y\} = \alpha\, x + \alpha y$$

$$\alpha(x\, y) = x^{\alpha} + y^{\alpha}$$

$$(xy\,)^{\alpha} = x^{\alpha}\, y^{\alpha}$$

$$x^{\alpha}\, y^{\alpha} = x^{\alpha}\, y^{\alpha}$$

$$\boxed{LHS = RHS}$$

Therefore

$$\boxed{\alpha(u + v) = \alpha u + \alpha v \text{ for all } u, v \in V \ \& \ \alpha \in K}$$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now, $\alpha(u+v) = \alpha\{x+y\}$

$$= \alpha(x\,y)$$

$$= (x\,y)^{\alpha}$$

$$= x^{\alpha} y^{\alpha}$$

$$= x^{\alpha} + y^{\alpha}$$

$$= \alpha x + \alpha\,y$$

Therefore $\boxed{\alpha(u+v) = \alpha u + \alpha v}$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V \,\&\, \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = (\alpha + \beta)\, x$$

$$(\alpha + \beta)u = x^{(\alpha+\beta)}$$

$$\alpha u + \beta u = \alpha\, x + \beta\, x$$

$$= x^{\alpha} + x^{\beta}$$

$$\alpha u + \beta u = x^{\alpha}\, x^{\beta}$$

$$\alpha u + \beta u = x^{\alpha+\beta}$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta\ x\ )$$

$$= \alpha\ x^{\beta}$$

$$= x^{\alpha\beta}$$

$$= (\alpha\beta)\ x$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

Therefore $\boxed{\alpha(\beta u) = (\alpha\beta)u \text{ for all } u \in V \ \&\ \alpha, \beta \in K}$

9. Associative property of vector addition and scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta\, x)$$
$$= \alpha\, x^{\beta}$$
$$= x^{\alpha\beta}$$

$$(\alpha\beta)u = (\alpha\beta)x$$
$$(\alpha\beta)u = x^{\alpha\beta}$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

10. Property 10:

$$1(u) = 1\, x$$
$$= x^1$$
$$= x$$

$$\boxed{1(u) = u}$$

The set $V = \mathbb{R}^+$ with the vector addition and scalar multiplication defined as follows

$$(i)\ x + y = xy$$
$$(ii)\ kx = x^k$$

is a vector space.

THANK YOU