Linear Algebra

Subspaces

It is often the case that some vector space of interest is contained within a larger vector space whose properties are known.

> **DEFINITION 1** A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

- In general, to show that a nonempty set W with two operations is a vector space one must verify the ten vector space axioms.
- However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are "inherited" from V.

  For example,

  it is *not* necessary to verify that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ holds in W because it holds for all vectors in V including those in W.

On the other hand, it *is* necessary to verify that W is closed under addition and scalar multiplication since it is possible that adding two vectors in W or multiplying a vector in W by a scalar produces a vector in V that is outside of W

Those axioms that are *not* inherited by W are

Axiom 1—Closure of W under addition

Axiom 3—Existence of a zero vector in W

Axiom 4—Existence of a negative in W for every vector in W

Axiom 6—Closure of W under scalar multiplication

so these axioms must be verified to prove that it is a subspace of V.

However, the next theorem shows that if Axiom 1 and Axiom 6 hold in W, then Axioms 3 and 4 hold in W as a consequence and hence need not be verified.

THEOREM 4.2.1 *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*

(*a*) *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.*

(*b*) *If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.*

Proof

- If W is a subspace of V, then all the vector space axioms hold in W, including Axioms 1 and 6, which are precisely conditions (*a*) and (*b*).

- Conversely, assume that conditions (*a*) and (*b*) hold.

Since these are Axioms 1 and 6, and since Axioms 2, 5, 7, 8, 9, and 10 are inherited from V, we only need to show that Axioms 4 and 3 hold in W.

For this purpose, let $\mathbf{u}$ be any vector in W. It follows from condition (*b*) that $k\mathbf{u}$ is a vector in W for every scalar k. In particular, $0\mathbf{u}=\mathbf{0}$ and $(-1)\mathbf{u}=-\mathbf{u}$ are in W, which shows that Axioms 3 and 4 hold in W.

EXAMPLE 1 The Zero Subspace

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\mathbf{0} + \mathbf{0} = \mathbf{0}$$

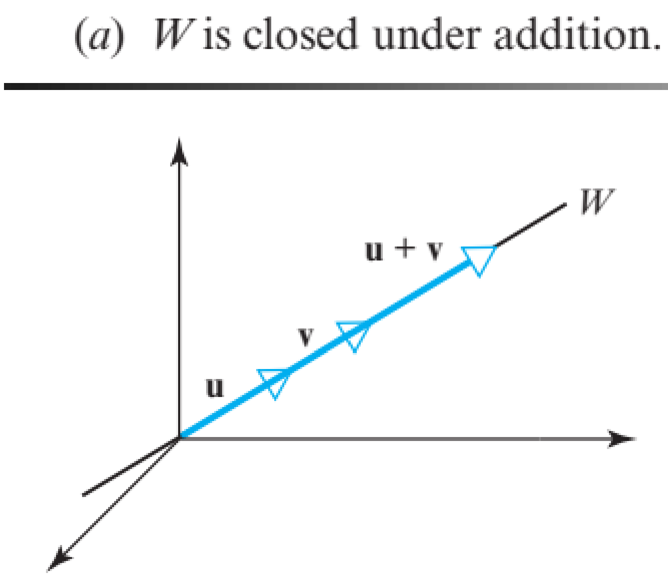
and for any scalar k

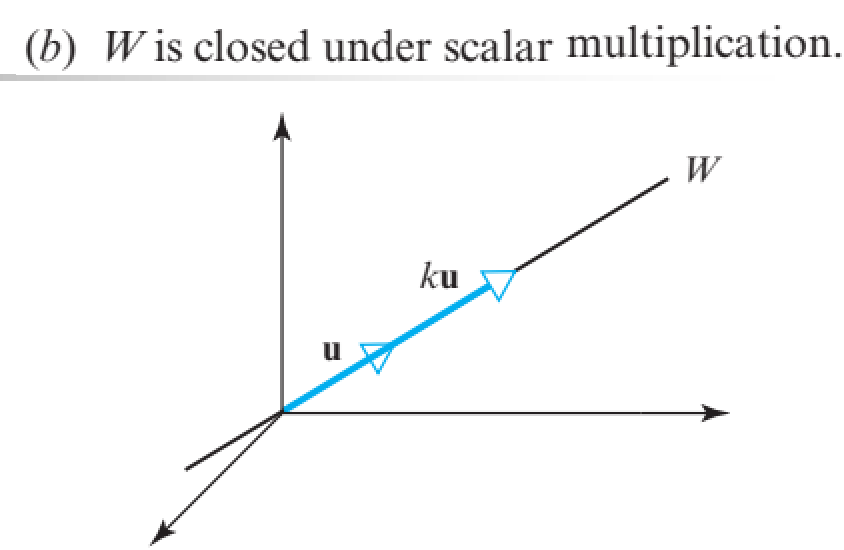
$$k\mathbf{0} = \mathbf{0}$$

We call W the ***zero subspace*** of V.

EXAMPLE 2 Lines Through the Origin Are Subspaces of R^2 and of R^3

- If W is a line through the origin of either R^2 or R^3, then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication





EXAMPLE 3 Planes Through the Origin Are Subspaces of R^3

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a plane W through the origin of R^3, then it is evident geometrically that $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ also lie in same plane W for any scalar k (Figure 4.2.3).

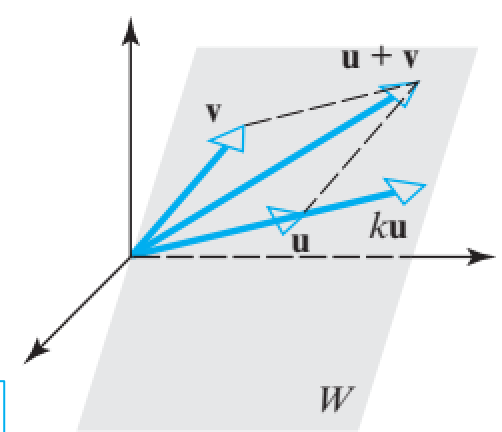


Figure 4.2.3 The vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$.

Thus W is closed under addition and scalar multiplication.

Table

Subspaces of R^2	Subspaces of R^3
• $\{\mathbf{0}\}$ • Lines through the origin • R^2	• $\{\mathbf{0}\}$ • Lines through the origin • Planes through the origin • R^3

EXAMPLE 4 A Subset of R^2 That Is Not a Subspace

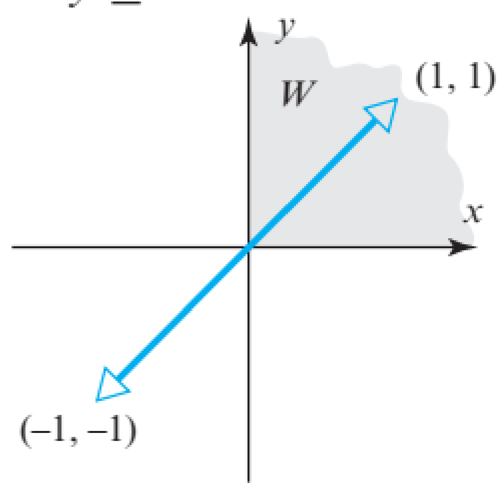
Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$ (the shaded region in Figure 4.2.4).

This set is not a subspace of R^2 because it is not closed under scalar multiplication.

For example,

$$\mathbf{v} = (1, 1) \text{ is a vector in } W,$$

but $(-1)\mathbf{v} = (-1, -1)$ is not.



▲ Figure 4.2.4 W is not closed under scalar multiplication.

EXAMPLE 5 Subspaces of M_{nn}

We know that the sum of two symmetric $n \times n$ matrices is symmetric and that a scalar multiple of a symmetric $n \times n$ matrix is symmetric.

Thus, the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of M_{nn}.

Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn}.

EXAMPLE 6 A Subset of M_{nn} That Is Not a Subspace

The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}, failing on two counts—it is not closed under addition and not closed under scalar multiplication.

We will illustrate this with an example in M_{22} that you can readily adapt to M_{nn}.

Consider the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\text{and} \quad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

$$U + V = \begin{pmatrix} 0 & 4 \\ 0 & 10 \end{pmatrix} \notin W \qquad \text{(Since } U + V \text{ is not invertible)}$$

The matrix $0U$ is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

EXAMPLE 7 The Subspace $C(-\infty, \infty)$

- There is a theorem in calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous.

- Rephrased in vector language, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$.

We will denote this subspace by $C(-\infty, \infty)$.

EXAMPLE 8 Functions with Continuous Derivatives

A function with a continuous derivative is said to be *continuously differentiable*.

- There is a theorem in calculus which states that the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable.

- Thus, the functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C^1(-\infty, \infty)$

  where the superscript emphasizes that the *first* derivatives are continuous.

- To take this a step further, the set of functions with m continuous derivatives on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$ as is the set of functions with derivatives of all orders on $(-\infty, \infty)$. We will denote these subspaces by $C^m(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$, respectively.

Remark

- In our previous examples we considered functions that were defined at all points of the interval $(-\infty, \infty)$.
- Sometimes we will want to consider functions that are only defined on some subinterval of $(-\infty, \infty)$, say the closed interval $[a, b]$ or the open interval (a, b).
- In such cases we will make an appropriate notation change. For example, $C[a, b]$ is the space of continuous functions on $[a, b]$ and $C(a, b)$ is the space of continuous functions on (a, b).

EXAMPLE 9 The Subspace of All Polynomials

Recall that a ***polynomial*** is a function that can be expressed in the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are constants.

- It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial.
- Thus, the set W of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$.

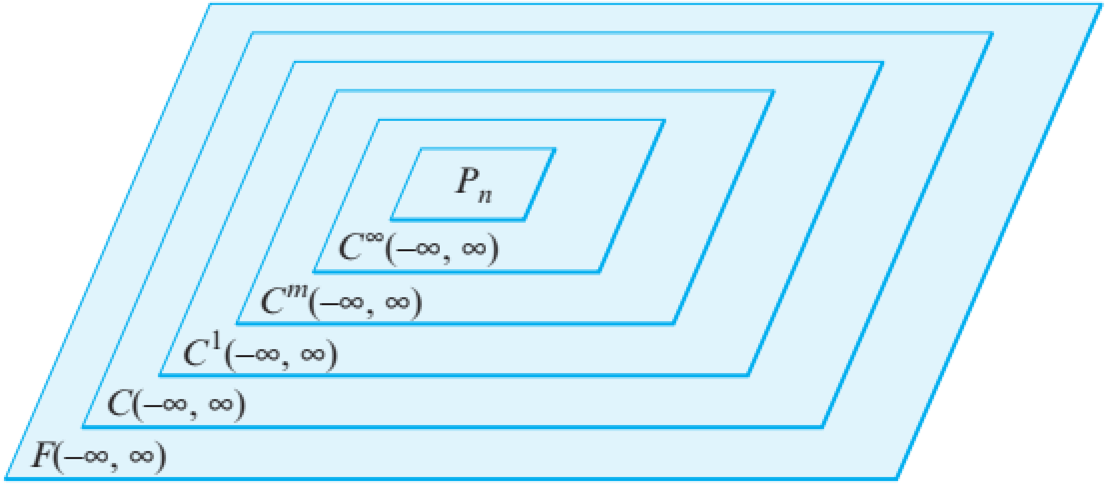
We will denote this space by P_∞.

The set of all polynomials of degree *n or less* than n form a subspace of $F(-\infty, \infty)$.

We will denote this space by P_n.

The Hierarchy of Function Spaces

- It is proved in calculus that polynomials are continuous functions and have continuous derivatives of all orders on $(-\infty, \infty)$.
- Thus, it follows that P_∞ is not only a subspace of $F(-\infty, \infty)$, as previously observed, but is also a subspace of $C^\infty(-\infty, \infty)$.



Figure

Building Subspaces

The following theorem provides a useful way of creating a new subspace from known subspaces.

THEOREM 4.2.2 *If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.*

Proof Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$.

- This set is not empty because each of these subspaces contains the zero vector of V, and hence so does their intersection.
- Thus, it remains to show that W is closed under addition and scalar multiplication.
- To prove closure under addition, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W.

  Since W is the intersection of $W_1, W_2, \ldots, W_r$, it follows that $\mathbf{u}$ and $\mathbf{v}$ also lie in each of these subspaces.
- Moreover, since these subspaces are closed under addition and scalar multiplication, they also all contain the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for every scalar k, and hence so does their intersection W.

  This proves that W is closed under addition and scalar multiplication.

THANK YOU