Linear Algebra

7. Example: Show that the set of all points in R^2 lying on a line, which passes through the origin, is a vector space under usual vector addition and scalar multiplication operations defined on R^2.

Proof: The equation of a straight line in R^2 (xy-plane) is of the form $ax + by = c$ where not both a and c equal 0 . When $c = 0$, it passes through the origin.

Let V be the set consisting of all ordered pairs in R^2 lying on a line which passes through the origin. (i.e) V consisting of all points , which are satisfying the equation $ax + by = 0$.

1. Closure property: **I. Abelian group under addition**

Let $u, v \in V$, where $u = (x_1, y_1)$ & $v = (x_2, y_2)$ and satisfies the equations

$$ax_1 + by_1 = 0 \text{ ------- } (1)$$

$$ax_2 + by_2 = 0 \text{ ------- } (2)$$

we need to prove that $u + v \in V$

(1)+(2) $a(x_1+x_2) + b(y_1 + y_2) = 0$ $\Rightarrow$ $u + v = (x_1 + x_2, y_1 + y_2)$ is also satisfy the equation $ax + by = 0$.

Therefore

$$\boxed{u, v \in V \text{ implies } u + v \in V}$$

2. Associativity property:

Let $u, v, w \in V$, where $u = (x_1, y_1)$, $v = (x_2, y_2)$ & $w = (x_3, y_3)$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in V$, such that

$$u + e = e + u = u \text{ for all } u \in V$$

Let $e = (0,0)$, (here $e = (0,0)$ satisfies the equation $ax + by = 0$)

$$u + e = (x_1, y_1) + (0,0)$$

$$= (x_1 + 0, y_1 + 0)$$

$$= (x_1, y_1)$$

$$\boxed{u + e = u}$$

$$e + u = (0,0) + (x_1, y_1)$$

$$= (0 + x_1, 0 + y_1)$$

$$= (x_1, y_1)$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in V}$$

4. Inverse Property:

We need to prove that, for every element $u \in V$, there exists $-u \in V$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = (-x_1, -y_1)$, (here $-u$ satisfies the equation $a(-x_1) + b(-y_1) = 0$)

$$u + (-u) = (x_1, y_1) + (-x_1, -y_1)$$

$$= (0,0)$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = (-x_1, -y_1) + (x_1, y_1)$$

$$= (0,0)$$

$$\boxed{(-u) + u = e}$$

Therefore $$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in V$$

Now,

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$= (x_2 + x_1, y_2 + y_1)$$

$$= (x_2, y_2) + (x_1, y_1)$$

$$\boxed{u + v = v + u}$$

Therefore $\boxed{u + v = v + u \text{ for all } u, v \in V}$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in V \;\&\; \alpha \in K \text{ such that } \alpha u \in V = \mathbb{R}^2$$

Now,

$u \in V$, this is $u = (x_1, y_1)$ satisfies

$$ax_1 + by_1 = 0$$

$$\Rightarrow \quad a(\alpha x_1) + b(\alpha y_1) = 0$$

(i.e) $\alpha u = (\alpha x_1, \alpha y_1)$ also lies on the same line

$$\alpha u = (\alpha x_1, \alpha y_1) \in V$$

Therefore

for any $u \in V$ & $\alpha \in K$ we have $\alpha u \in V$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,
$$\alpha(u+v) = \alpha\{(x_1, y_1) + (x_2, y_2)\}$$

$$= \alpha(x_1 + x_2, y_1 + y_2)$$

$$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$$

$$= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$$

$$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V \ \& \ \alpha \in K}$$

8. Distributive property of vector over scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = (\alpha + \beta)(x_1, y_1)$$

$$(\alpha + \beta)u = \big((\alpha + \beta)x_1, (\alpha + \beta)y_1\big)$$

$$= (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1, \alpha y_1) + (\beta x_1, \beta y_1)$$

$$= \alpha(x_1, y_1) + \beta(x_1, y_1)$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

Therefore

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u \text{ for all } u \in V \text{ \& } \alpha, \beta \in K}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta x_1, \beta y_1)$$

$$= (\alpha\beta x_1, \alpha\beta y_1)$$

$$= \alpha\beta(x_1, y_1)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

Therefore $\boxed{\alpha(\beta u) = (\alpha\beta)u \text{ for all } u \in V \text{ \& } \alpha, \beta \in K}$

10. Property 10:

$$1(u) = 1(x_1, y_1)$$

$$= ((1)x_1, (1)y_1)$$

$$= (x_1, y_1)$$

$$\boxed{1(u) = u}$$

The set of all points in R^2 lying on a line, which passes through the origin, is a vector space under usual vector addition and scalar multiplication defined on R^2.

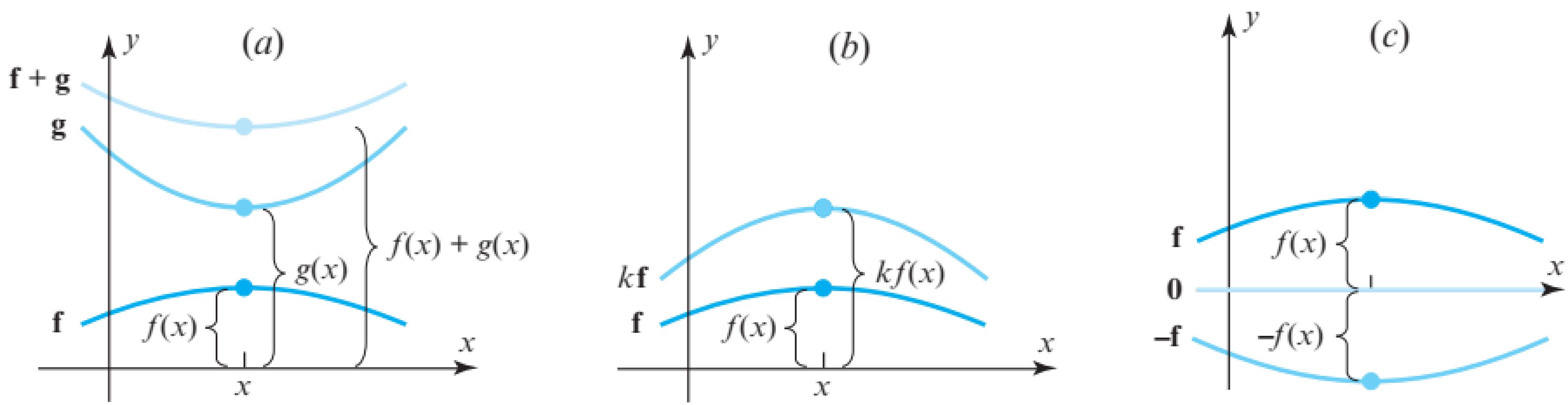
EXAMPLE The Vector Space of Real-Valued Functions $F(-\infty, \infty)$

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$.

If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$$

$$(k\mathbf{f})(x) = kf(x)$$



Figure

1. Closure property: **Axioms 1 and 6:**

Let $u, v \in V$, where $u = f(x)$ & $v = g(x)$, we need to prove that $u + v \in V$

and we need to prove that, $u \in V$ & $\alpha \in K$ such that $\alpha u \in V$.

These closure axioms require that if we add two functions that are defined at each x in the interval $(-\infty, \infty)$, then sums and scalar multiples of those functions must also be defined at each x in the interval $(-\infty, \infty)$.

This follows from Formulas

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$$

$$(k\mathbf{f})(x) = kf(x)$$

Therefore

$$\boxed{u, v \in V \text{ implies } u + v \in V}$$

and

for any $u \in V$ & $\alpha \in K$ we have $\alpha u \in V$

3. Identity Property:

We need to prove that, there exists an element $e \in V$, such that

$$u + e = e + u = u \text{ for all } u \in V$$

Let $e = \mathbf{0}$ (zero function)

This axiom requires that there exists a function in $F(-\infty, \infty)$, which when added to any other function $\mathbf{f}$ in $F(-\infty, \infty)$ produces $\mathbf{f}$ back again as the result.

The function whose value at every point x in the interval $(-\infty, \infty)$ is zero has this property.

Geometrically, the graph of the function $\mathbf{0}$ is the line that coincides with the x-axis.

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in V}$$

4. Inverse Property:

We need to prove that, for every element $u = f(x) \in V$, there exists $-u \in V$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = -f(x)$.

This axiom requires that for each function **f** in $F(-\infty, \infty)$ there exists a function $-\mathbf{f}$ in $F(-\infty, \infty)$, which when added to **f** produces the function **0**.

The function defined by $-\mathbf{f}(x) = -f(x)$ has this property.

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

2. Associativity property:

5. Commutative Property:

7. Distributive property of scalar multiplication over vector addition:

8. Distributive property of vector over scalar multiplication

9. Associative property of vector with scalar multiplication

10. Property 10: $\mathbf{1}(\boldsymbol{u}) = \boldsymbol{u}$

The validity of each of these axioms follows from properties of real numbers.

For example, if $\mathbf{f}$ and $\mathbf{g}$ are functions in $F(-\infty, \infty)$, then Axiom **5** requires that

$$\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f}.$$

This follows from the computation

$$(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$$

$$= \mathbf{g}(x) + \mathbf{f}(x)$$

$$\boxed{(\mathbf{f} + \mathbf{g})(x) = (\mathbf{g} + \mathbf{f})(x)}$$

The set of all real valued functions F$(-\infty, \infty)$ that **are defined at each** $\boldsymbol{x}$ **in** $(-\infty, \infty)$**is a vector space under the operations**

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$$

$$(k\mathbf{f})(x) = kf(x)$$

In Example 6 the functions were defined on the entire interval $(-\infty, \infty)$. However, the arguments used in that example apply as well on all subintervals of $(-\infty, \infty)$, such as a closed interval $[a, b]$ or an open interval (a, b). We will denote the vector spaces of functions on these intervals by $F[a, b]$ and $F(a, b)$, respectively.

EXAMPLE The Vector Space of Infinite Sequences of Real Numbers R^∞

Let V consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \ldots, u_n, \ldots)$$

in which $u_1, u_2, \ldots, u_n, \ldots$ is an infinite sequence of real numbers.

We define two addition and scalar multiplication componentwise by

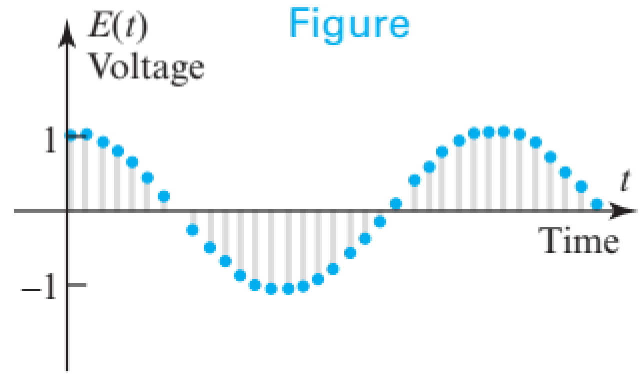
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots)$$

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n, \ldots)$$

We will denote this vector space by the symbol R^∞.

R^∞ is a vector space.

Vector spaces of this type arise when a transmitted signal of indefinite duration is digitized by sampling its values at discrete time intervals



Figure

THEOREM 4.1.1 *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:*

(*a*) $0\mathbf{u} = \mathbf{0}$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{u} = -\mathbf{u}$

(*d*) *If* $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = \mathbf{0}$.

THANK YOU