Linear Algebra

Basis for a Vector Space

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a ***basis*** for V if:

(a) S spans V.

(b) S is linearly independent.

A vector space V is said to be ***finite-dimensional*** if there is a finite set of vectors in S that spans V and is said to be ***infinite-dimensional*** if no such set exists.

▶ EXAMPLE 1 The Standard Basis for R^n

Recall that the standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),$$
$$\mathbf{e}_2 = (0, 1, 0, \ldots, 0),$$
$$, \ldots ,$$
$$\mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

span R^n and they are linearly independent. Thus, they form a basis for R^n that we call the ***standard basis for R^n***.

In particular, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ is the standard basis for R^3.

▶ EXAMPLE 2 The Standard Basis for P_n

The set $S = \{1, x, x^2, \dots, x^n\}$ is a basis for the vector space P_n of polynomials of degree n or less. Since the vectors span P_n and are linearly independent

Thus, they form a basis for P_n that we call the ***standard basis for P_n***.

▶ EXAMPLE 3 Another Basis for R^3

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3.

Solution We must show that these vectors are linearly independent and span R^3.

To prove linear independence we must show that the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has only the trivial solution; and to prove that the vectors span R^3 we must show that every vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

By equating corresponding components on the two sides, these two equations can be expressed as the linear systems

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad\quad + 4c_3 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad\quad + 4c_3 &= b_3 \end{aligned} \tag{1}$$

Thus, we have reduced the problem to showing that in (1) the homogeneous system has only the trivial solution and that the nonhomogeneous system is consistent for all values of b_1, b_2, and b_3.

But the two systems have the same coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

so : we can prove both results at the same time by showing that $\det(A) \neq 0$.

where $\det(A) = -1$, which proves that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for R^3.

▶ EXAMPLE 4 The Standard Basis for M_{mn}

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Solution We must show that the matrices are linearly independent and span M_{22}.

- To prove linear independence we must show that the equation

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \mathbf{0} \tag{4}$$

has only the trivial solution, where $\mathbf{0}$ is the 2×2 zero matrix;

- To that the matrices span M_{22} we must show that every 2×2 matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B \tag{5}$$

The matrix forms of Equations (4) and (5) are

$$c_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+c_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+c_3\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+c_4\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix}c_1 & c_2\\ c_3 & c_4\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

it has only the trivial solution

$$\boxed{c_1=c_2=c_3=c_4=0}$$

This proves that the matrices are linearly independent.

and

$$c_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+c_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}+c_3\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}+c_4\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix}c_1 & c_2\\ c_3 & c_4\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

it has the solution $\boxed{c_1=a,\quad c_2=b,\quad c_3=c,\quad c_4=d}$

This proves that every 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This proves that the matrices M_1, M_2, M_3, M_4 span M_{22} and form a basis for M_{22}.

More generally, the mn different matrices whose entries are zero except for a single entry of 1 form a basis for M_{mn} called the ***standard basis for*** $\boldsymbol{M_{mn}}$.

THEOREM 4.4.1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

Proof To see that there is only *one* way to express a vector as a linear combination of the vectors in S, suppose that some vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

and also as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Subtracting the second equation from the first gives

$$\mathbf{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$$

Since the right side of this equation is a linear combination of vectors in S, the linear independence of S implies that

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \ldots, \quad c_n - k_n = 0$$

that is,

$$\boxed{c_1 = k_1, \quad c_2 = k_2, \ldots, \quad c_n = k_n}$$

Thus, the two expressions for $\mathbf{v}$ are the same.

Dimension

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

THEOREM 4.5.2 *Let V be an n-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.*

(*a*) *If a set in V has more than n vectors, then it is linearly dependent.*

(*b*) *If a set in V has fewer than n vectors, then it does not span V.*

▶ EXAMPLE 1 Dimensions of Some Familiar Vector Spaces

$$\dim(R^n) = n \qquad \textbf{[The standard basis has } \boldsymbol{n} \textbf{ vectors.]}$$

$$\dim(P_n) = n + 1 \qquad \textbf{[The standard basis has } \boldsymbol{n+1} \textbf{ vectors.]}$$

$$\dim(M_{mn}) = mn \qquad \textbf{[The standard basis has } \boldsymbol{mn} \textbf{ vectors.]}$$

▶ EXAMPLE 2 Dimension of Span(S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then every vector in span(S) is expressible as a linear combination of the vectors in S. Thus, if the vectors in S are *linearly independent*, they automatically form a basis for span(S), from which we can conclude that

$$\boxed{\dim[\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}] = r}$$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

▶ EXAMPLE 3 Dimension of a Solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0\\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution The augmented matrix for the given homogeneous system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & 0\\ 0 & 0 & 5 & 10 & 0 & 15 & 0\\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

the reduced row echelon form is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + 4x_4 + 2x_5 \qquad\quad &= 0 \\ x_3 + 2x_4 \qquad\qquad\qquad &= 0 \\ x_6 &= 0 \end{aligned}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we found the solution of this system to be

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

or, alternatively, as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space.

Thus, the solution space has dimension 3.

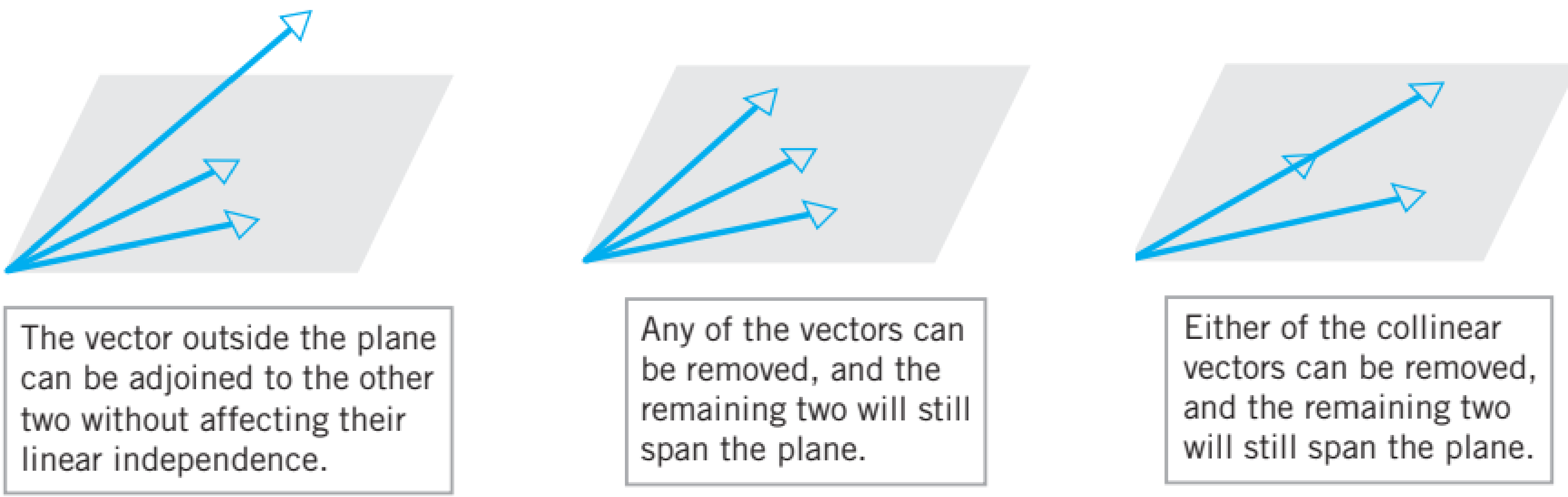
THEOREM 4.5.3 Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(*a*) *If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.*

(*b*) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,*

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$$



Figure

EXAMPLE 4 Applying the Plus/Minus Theorem

Show that $\mathbf{p}_1 = 1 - x^2$, $\mathbf{p}_2 = 2 - x^2$, and $\mathbf{p}_3 = x^3$ are linearly independent vectors.

Solution

- The set $S = \{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent since neither vector in S is a scalar multiple of the other.
- Since the vector $\mathbf{p}_3$ cannot be expressed as a linear combination of the vectors in S it can be adjoined to S to produce a linearly independent set $S \cup \{\mathbf{p}_3\} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

THEOREM 4.5.4 *Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.*

Proof Assume that S has exactly n vectors and spans V.

- To prove that S is a basis, we must show that S is a linearly independent set.
- But if this is not so, then some vector $\mathbf{v}$ in S is a linear combination of the remaining vectors.
- If we remove this vector from S, then it follows from Theorem 4.5.3(b) that the remaining set of $n-1$ vectors still spans V.
- But this is impossible since no set with fewer than n vectors can span an n-dimensional vector space.

$\boxed{\text{Thus } S \text{ is linearly independent.}}$

Assume that S has exactly n vectors and is a linearly independent set.

- To prove that S is a basis, we must show that S spans V.
- But if this is not so, then there is some vector $\mathbf{v}$ in V that is not in span(S).
- If we insert this vector into S, then it follows from Theorem 4.5.3(a) that this set of $n+1$ vectors is still linearly independent.
- But this is impossible, since no set with more than n vectors in an n-dimensional vector space can be linearly independent.

$\boxed{\text{Thus } S \text{ spans } V.}$

EXAMPLE 5 Bases by Inspection

(a) Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2.

(b) Explain why the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3.

Solution (a)

- Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2, and hence they form a basis for R^2.

Solution (b)

- The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the xz-plane (why?).
- The vector $\mathbf{v}_3$ is outside of the xz-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.
- Since R^3 is three-dimensional, Theorem 4.5.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space R^3.

THEOREM 4.5.5 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(*a*) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

THEOREM 4.5.6 *If W is a subspace of a finite-dimensional vector space V, then:*

(*a*) *W is finite-dimensional.*

(*b*) $\dim(W) \leq \dim(V)$.

(*c*) $W = V$ *if and only if* $\dim(W) = \dim(V)$.

7. In each part, find a basis for the given subspace of R^3, and state its dimension.

(a) The plane $3x - 2y + 5z = 0$.

(b) The plane $x - y = 0$.

(c) The line $x = 2t,\ y = -t,\ z = 4t$.

(d) All vectors of the form (a, b, c), where $b = a + c$.

Solution (a) Find a basis for the solution space to the equation $3x - 2y + 5z = 0$.

- First find a parametric description for the solutions.

We can do this by putting $y = s, z = t$, and solving for x.

The solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3}s - \frac{5}{3}t \\ s \\ t \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{pmatrix} t$$

The vectors $\left(\frac{2}{3}, 1, 0\right)$ & $(-\frac{5}{3}, 0,)$ span the solution space.

Therefore, the dimension of the solution space is 2.

Solution (b) Find a basis for the solution space to the equation $x - y = 0$

- First find a parametric description for the solutions.

  We can do this by putting $y = s, z = t$, and solving for x.

  The solutions are
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t$$

  The vectors $(1,1,0)$& $(0,0,1)$ span the solution space.

  Therefore, the dimension of the solution space is 2.

Solution

(d) All vectors of the form (a, b, c), where $b = a + c$.

The vectors in question can also be written in the form

$$\begin{pmatrix} a \\ b \\ a+b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} a + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} b$$

The vectors $(1, 1, 0)$, $(0, 1, 1)$ span the space in question, hence has dimension 2.

9. Find the dimension of each of the following vector spaces.

(a) The vector space of all diagonal $n \times n$ matrices.

(b) The vector space of all symmetric $n \times n$ matrices.

(c) The vector space of all upper triangular $n \times n$ matrices.

Solution (a) Let D_{nn} denote the specified vector space of diagonal matrices.

- One basis for D_{nn} consists of the n different $n \times n$ matrices, each of which has exactly one of the diagonal entries equal to 1 and all other entries in the matrix equal to 0.

$$\boxed{\text{Thus } D_{nn} \text{ has dimension } n.}$$

For instance for D_{33}, we could use the 3 matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as basis vectors.

$$\boxed{\text{Thus } D_{33} \text{ has dimension } 3.}$$

10. Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.

(a) $\mathbf{v}_1 = (-1, 2, 3), \ \mathbf{v}_2 = (1, -2, -2)$

(b) $\mathbf{v}_1 = (1, -1, 0), \ \mathbf{v}_2 = (3, 1, -2)$

Solution (c) Let S_{nn} denote the specified vector space of symmetric matrices.

A typical matrix in S_{33} would look like

$$\begin{bmatrix} a & x & y \\ x & b & z \\ y & z & c \end{bmatrix}$$

the main diagonal can be anything

but the entries off the main diagonal must

be paired up to match symmetrically.

THANK YOU