# AUGMENTATION VARIETIES AND POLYNOMIALS

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Abstract. Augmentations arise from a differential graded algebra associated to the conormal bundle of a knot (or a link). The augmentation variety and the augmentation polynomial characterize some properties of the space of augmentations. The goal of the talk is to introduce these concepts, and explain their connections to other knot invariants such as the A-polynomial.

Let $K = K_1 \cup \cdots \cup K_N$ be an $n$-component link in $S^3$. Let $N_K^* S^3$ be the link conormal in $T^*S^3$, and let $\Lambda_K = N_K^* S^3 \cap ST^*S^3$ be the associated Legendrian in the cosphere bundle. Topologically, $ST^*S^3$ is a trivial $S^2$ bundle over $S^3$ (because $S^3$ is a Lie group), and $\Lambda_K$ is a disjoint union of tori.

Let $\mathcal{A}_K$ be the Chekanov-Eliashberg dga associated to $\Lambda_K$,

$$\mathcal{A}_K = R\langle \text{Reeb chords} \rangle.$$

The Reeb chords consist of matrices of chords $A, B, C, D, E, F$ explained the previous talks. We focus on the coefficient ring $R = \mathbb{Z}[H_2(ST^*S^3, \Lambda_K)] = \mathbb{Z}[H_2(S^2)] \oplus H_1(\Lambda_K)$. Let $x_i, p_i \in H_1(\Lambda_K)$ be the longitude and the meridian, and let $t \in H_2(S^2)$ be the fundamental class. Taking $\lambda_i = e^{x_i}, \mu_i = e^{p_i}$ and $Q = e^t$, we have

$$R = [Q^{\pm 1}, \lambda_i^{\pm 1}, \mu_i^{\pm 1}], \quad i = 1, \cdots, n.$$

Moreover, these classes has degree 0 in the dga.

With this dga $\mathcal{A}_K$ in hand, we can proceed to define its augmentations, which consists of dga morphisms to a trivial dga:

$$\epsilon : (\mathcal{A}_K, \partial) \to (k, 0).$$

Here $k$ is any field and by $(k, 0)$ we mean the complex with only a dimension 1 vector space at degree 0, and all the differentials are zero. Being an dga morphism yields that $\epsilon$ satisfies:

$$(1)\, \epsilon(a) = 0, \text{ if } \deg(a) \neq 0, \quad (2)\, \epsilon \circ \partial = 0, \quad (3)\, \epsilon(1) = 1.$$

Note that this definition is purely algebraic. Sometimes augmentations come from Lagrangian fillings, see Day 2 Notes for this geometric interpretation.

Let's see an example:

**Example 1** (Unknot)**.** Let $K$ be an unknot. The coefficient ring is $R = \mathbb{Z}[Q^{\pm 1}, \lambda^{\pm 1}, \mu^{\pm 1}]$. The dga $\mathcal{A}_K$ is generated over two Reeb chords:

$$\begin{aligned} \partial c &= Q - \lambda - \mu + \lambda\mu, & |c| &= 1, \\ \partial e &= 0, & |e| &= 2. \end{aligned}$$

By the condition (1) in the definition of an augmentation, only elements in degree 0 are non-trivially augmented. Set $Q_0 = \epsilon(Q), \lambda_0 = \epsilon(\lambda), \mu_0 = \epsilon(\mu)$, which are invertible numbers in $k^*$ (because the homology classes are invertible). We compute

$$\epsilon \circ \partial(c) = \epsilon(Q - \lambda - \mu + \lambda\mu) = Q_0 - \lambda_0 - \mu_0 + \lambda_0\mu_0. \tag{0.1}$$

By condition (2), an augmentation is an assignment $(Q_0, \lambda_0, \mu_0) \in (k^*)^3$ such that

$$Q_0 - \lambda_0 - \mu_0 + \lambda_0\mu_0 = 0.$$

Now we see that the augmentations of the unknot dga form a space, in this case a variety in $(k^*)^3$. Following the idea we define the augmentation variety.

**Definition 2.** Let $(\mathcal{A}_K, \partial)$ be the dga of a link $K$ with $n$-components, with the usual coefficient ring $R = \mathbb{Z}[Q^{\pm 1}, \lambda_i^{\pm 1}, \mu_i^{\pm 1}]$. The augmentation variety of $K$ is

$$V_K = \Big\{ (\epsilon(Q), \epsilon(\lambda_i), \epsilon(\mu_i)) \,|\, \epsilon : (\mathcal{A}_K, \partial) \to (\mathbb{C}, 0) \text{ augmentation } \Big\} \subset (\mathbb{C}^*)^{2n+1}.$$

*Remark* 3. We quickly make two remarks about calculating the space of augmentations.

(1) First, the space is given by the intersection of hypersurfaces derived from the differentials of degree 1 generators.
(2) Second, we can drop the subscript 0 indicating the augmented value for simplicity, as we have seen in (0.1).

**Example 4** (Hopf link)**.** The coefficient ring is $R = \mathbb{Z}[Q^{\pm 1}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \mu_1^{\pm 1}, \mu_2^{\pm 1}]$. Degree 0 and 1 Reeb chords are

$$\begin{aligned} \deg = 0 : &\quad a_{12}, a_{21}; \\ \deg = 1 : &\quad b_{12}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}. \end{aligned}$$

There are other Reeb chords in higher degrees (see Day 2 Notes), but are irrelevant (see (1) in Remark 3). The differentials of the degree 1 terms are:

$$\partial(b_{12}) = (\lambda_1\mu_1\lambda_2^{-1}\mu_2^{-1} - 1)a_{12}; \tag{0.2}$$
$$\partial(b_{21}) = a_{21}(1 - \lambda_1^{-1}\mu_1^{-1}\lambda_2\mu_2); \tag{0.3}$$
$$\partial(c_{11}) = Q - \lambda_1 - \mu_1 + \lambda_1\mu_1 + \lambda_1\mu_1\mu_2^{-1}a_{12}a_{21}; \tag{0.4}$$
$$\partial(c_{12}) = Qa_{12} + \lambda_1\mu_2^{-1}a_{12}(\mu_1\mu_2 - \mu_1 - \mu_2 + \mu_1 a_{12}a_{21}); \tag{0.5}$$
$$\partial(c_{21}) = \lambda_2 a_{21} - \mu_1 a_{21}; \tag{0.6}$$
$$\partial(c_{22}) = Q - \lambda_2 - \mu_2 + \lambda_2\mu_2 + \lambda_2 a_{21}a_{12}; \tag{0.7}$$

Considering (0.3), we study the following two cases:

**A**: Suppose $a_{21} \neq 0$. Then (0.6) implies $\lambda_2 = \mu_1$, and together with (0.3) there is $\lambda_1 = \mu_2$. Equation (0.7) yields

$$Q = \lambda_2 + \mu_2 - \lambda_2\mu_2 - \lambda_2 a_{12}a_{21}.$$

All the other equations are redundant now. To summarize, we get a component of the augmentation variety

$$V_A = \{\lambda_1 = \mu_2, \lambda_2 = \mu_1\} \subset (\mathbb{C}^*)^5.$$

The (augmented) value of $Q$ depends on the values of $a_{12}$ and $a_{21}$. Moreover, $Q$ can be taken over the entire $\mathbb{C}^*$, and there are many choices of $a_{12}$ and $a_{21}$ for each value of $Q$.

**B**: Suppose $1-\lambda_1^{-1}\mu_1^{-1}\lambda_2\mu_2 \neq 0$. Then (0.3) implies $a_{21}=0$, and (0.2) implies $a_{12}=0$. From (0.4) and (0.7) we get another component of the augmentation variety:

$$V_B = \{Q-\lambda_1-\mu_1+\lambda_1\mu_1 = 0, Q-\lambda_2-\mu_2+\lambda_2\mu_2=0\} \subset (\mathbb{C}^*)^5.$$

Combing the two cases, the augmentation variety of the Hopf link is

$$V_{Hopf} = V_A \cup V_B.$$

*Remark* 5. We have some further remarks regarding to the calculation.

(1) A point in the augmentation variety may corresponds to many augmentations. See case A of the Hopf link calculation.
(2) Adjoining the augmentations of each component gives rise to an augmentation of the link, and that is $V_B$ in our calculation. There are other augmentations do not come in this way. Such an augmentation evaluates non-trivially on the mixed Reeb chords (which begins and ends on different components of $\Lambda_K$). In our case, $a_{12}$ and $a_{21}$ are degree 0 mixed Reeb chords, and they cannot be both zero in case A.
(3) If a link $K$ is the union of two sublinks $K_1$ and $K_2$, which are unlinked. Then any mixed Reeb chords between $K_1$ and $K_2$ is trivially augmented. One can think of moving $K_1$ and $K_2$ far apart from each other in the ambient $S^3$. As an example, unlink has its augmentation variety

$$V_{unlink} = V_B.$$

(4) In general, the augmentation variety has codimension $n$ in $(\mathbb{C}^*)^{2n+1}$. The variety associated to a knot has codimension 1.

Following the last remark, we make the definition of the augmentation polynomial

**Definition 6.** Let $K$ be a knot (single component). Let

$$V_K = \Big\{(\epsilon(Q),\epsilon(\lambda),\epsilon(\mu)) \,|\, \epsilon : (\mathcal{A}_K,\partial) \to (\mathbb{C},0) \text{ augmentation } \Big\} \subset (\mathbb{C}^*)^3.$$

When the maximal-dimension of the Zariski closure of $V_K$ is a codimension 1 subvariety of $(\mathbb{C}^*)^3$, this variety is the vanishing set of a reduced (no repeating factors) polynomial $Aug(Q,\lambda,\mu)$, called the augmentation polynomial.

The augmentation polynomial is well defined up to units in $\mathbb{Z}[Q^{\pm1},\lambda^{\pm1},\mu^{\pm1}]$. We can choose the polynomial such that

- the coefficients are integers,
- the coefficients have no common factors of prime numbers,
- there are no negative powers.

Then we obtain $Aug_K(Q,\lambda,\mu) \in \mathbb{Z}[Q,\lambda,\mu]$, which is well-defined up to $\pm1$.

**Example 7.** The subscripts $O$, LT, RT stand for unknot, left trefoil, and right trefoil.

$$\begin{aligned}
Aug_O(\lambda,\mu,Q) &= Q-\lambda-\mu+\lambda\mu,\\
Aug_{\mathrm{LT}}(\lambda,\mu,Q) &= (\mu^3Q^2-\mu^4Q)+(Q^2-\mu Q^2-2\mu^2Q+2\mu^2Q^2-\mu^3Q+\mu^4)\lambda\\
&\qquad\qquad + (-Q^2+\mu Q^2)\lambda^2,\\
Aug_{\mathrm{RT}}(\lambda,\mu,Q) &= (Q^3-\mu Q^2)+(-Q^3+\mu Q^2-2\mu^2Q+2\mu^2Q^2+\mu^3Q-\mu^4Q)\lambda\\
&\qquad\qquad + (-\mu^3+\mu^4)\lambda^2.
\end{aligned}$$

In the rest of the notes, we explain the relation between the two variable augmentation polynomial $Aug_K(\lambda, \mu)$ and the $A$-polynomial $A_K(\lambda, \mu)$:

$$A_K(\lambda, \mu) \mid Aug_K(\lambda, \mu^2). \tag{0.8}$$

The knot group $\pi_K$ is the fundamental group of the knot complement, i.e. $\pi_K = \pi_1(S^3 \backslash K)$. It is finitely generated by the meridians, and finitely presented by the crossing in a 2D diagram. Let $m$ be a preferred meridian and let $\ell$ be the longitude (defined by a Seifert surface). Note that $m$ and $\ell$ commute in $\pi_K$.

Consider a $SL(2, \mathbb{C})$ representation $\rho : \pi_K \to SL(2, \mathbb{C})$. By choosing a suitable basis, we can get

$$\rho(m) = \begin{pmatrix} \mu_0 & * \\ 0 & \mu_0^{-1} \end{pmatrix}, \quad \rho(\ell) = \begin{pmatrix} \lambda_0 & * \\ 0 & \lambda_0^{-1} \end{pmatrix}. \tag{0.9}$$

We can extract the pair $(\lambda_0, \mu_0) \in (\mathbb{C}^*)^2$ from the representation. Define the $A$-polynomial variety to be

$$V_K^A := \{(\lambda_0, \mu_0) \in (\mathbb{C}^*)^2 \mid \text{There exists } \rho \in Rep(\pi_K, SL(2, \mathbb{C}) \text{ with } (0.9))\} \in (\mathbb{C}^*)^2.$$

The maximal-dimensional part of the Zariski closure of $V_K^A$ is the zero set of the $A$-polynomial $A_K(\lambda, \mu)$.

On the other side of the story, we take the conjecture definition for the two variable augmentation polynomial $Aug_K(\lambda, \mu) = Aug_K(Q = 1, \lambda, \mu)$. (See [Ng] for more details.)

Recall that an augmentation is by definition a dga morphism $(\mathcal{A}_K, \partial) \to (k, 0)$. Because $\mathcal{A}_K$ has no negative degree terms (see Day 2 Notes), the definition is equivalent to an algebra morphism $HC_0(K) \to k$, where $HC_0(K)$ is the degree 0 homology. Specialize to $Q = 1$, the homology $HC_0(K)|_{Q=1}$ is isomorphic to $Cord(K)$, a quotient algebra generated over $\pi_K$ (see Day 2 Notes for details).

Consider $GL(n, \mathbb{C})$ representations of $\pi_K$. It is called a KCH representation if after choosing a suitable basis, there is

$$\rho(m) = \begin{pmatrix} \mu_0 & 0 \\ 0 & Id_{n-1} \end{pmatrix}, \quad \rho(\ell) = \begin{pmatrix} \lambda_0 & 0 \\ 0 & diag_{n-1} \end{pmatrix}. \tag{0.10}$$

Given a KCH representation, Ng defines an augmentation $\epsilon : Cord(K) \to \mathbb{C}$ by: $\epsilon(\mu) = \mu_0$, $\epsilon(\lambda) = \lambda_0$, and $\epsilon([\gamma]) = \rho(\gamma)_{11}$, the $(1, 1)$-entry of the matrix. Cornwell shows that each augmentation with $Q = 1$, $\mu \neq 1$ lifts to a KCH representation.

Now we are ready to show (0.8).

*Proof.* The case $\mu = \pm 1$ can be considered separately which we omit here. Consider a $SL(2, \mathbb{C})$ representation $\rho$. Assuming $\mu \neq 1$, $\rho(m)$ has distinct eigenvalues. By choosing a suitable basis, we get

$$\rho(m) = \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix}, \quad \rho(\ell) = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}.$$

Define $\hat{\rho}(\gamma) = \mu^{lk(K,\gamma)}\rho(\gamma)$, where $lk(K, \gamma)$ is the linking number between $K$ and $\gamma$. It is straightforward to check $\hat{\rho}$ is a KCH representation, and gives rise to $(\mu_0^2, \lambda_0)$ in the augmentation variety. □

## References

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