Hierarchical Bayes

for Generalized Linear Mixed Effect Model

30 July, 2020

The First Example

NHTS data

The data source

```
NHTS2017 <- (read.csv("~/trippub.csv"))[,c(1,30,62,64,69,72,85)]
# NHTS2017 <- NHTS2017[complete.cases(NHTS2017),]
NHTS2017 <- NHTS2017[NHTS2017$VMT_MILE!=-1&NHTS2017$HHFAMINC>=0&NHTS2017$HH_CBSA!="XXXXX", ]
nhts2017 <- NHTS2017[sample(nrow(NHTS2017), 10000,replace =F), ]
save(nhts2017, file="nhts2017.RData")
```

Select "HOUSEID", "VMT_MILE", and five regressors

excluded the zero-miles VMT, negative household income, and unknown CBSA id (XXXXX)

Sample 10000 observations from the original data

```
load("nhts2017.RData")
str(nhts2017)
```

```
## 'data.frame':    10000 obs. of  7 variables:
##  $ HOUSEID : int  40005335 40743760 30193252 40755224 30073043 40701486 30383843 40307923 40691921 4
##  $ VMT_MILE: num  2.156 3.824 70.617 0.286 3.024 ...
##  $ HHSIZE  : int  2 2 2 1 1 2 2 5 3 1 ...
##  $ HHFAMINC: int  5 9 8 6 5 5 5 9 10 8 ...
##  $ WRKCOUNT: int  2 2 2 1 1 0 1 1 2 0 ...
##  $ LIF_CYC : int  2 2 2 1 9 10 2 4 6 9 ...
##  $ HH_CBSA : Factor w/ 53 levels "12060","12420",..: 17 30 26 17 6 26 48 8 52 42 ...
```

```
summary(nhts2017)
```

```
##     HOUSEID           VMT_MILE          HHSIZE         HHFAMINC        WRKCOUNT        LIF_CYC
##  Min.   :30000008   Min.   :   0.0   Min.   : 1.00   Min.   : 1.0   Min.   :0.00   Min.   : 1.00   1
##  1st Qu.:30262488   1st Qu.:   1.9   1st Qu.: 2.00   1st Qu.: 6.0   1st Qu.:1.00   1st Qu.: 2.00   2
##  Median :30531032   Median :   4.2   Median : 2.00   Median : 7.0   Median :1.00   Median : 5.00   3
##  Mean   :35058745   Mean   :  10.0   Mean   : 2.56   Mean   : 7.1   Mean   :1.35   Mean   : 5.32   4
##  3rd Qu.:40360806   3rd Qu.:  10.3   3rd Qu.: 3.00   3rd Qu.: 9.0   3rd Qu.:2.00   3rd Qu.: 9.00   3
##  Max.   :40794179   Max.   :1861.6   Max.   :11.00   Max.   :11.0   Max.   :7.00   Max.   :10.00   3
##                                                                                                    (
```

```r
table(nhts2017$HH_CBSA)
```

```
##
## 12060 12420 12580 13820 14460 15380 16740 16980 17140 17460 18140 19100 19740 19820 24340 25540 2642
##   480   422    77    17    68   130   171   157    32    39    34  1790    41    41    19    17    97
```

There are $m = 52$ levels of CBSA.

$\mathbf{Y}_j$ is a n_j Vector.

$\mathbf{X}_j$ is a $n_j \times p$ Matrix

```r
ids<-sort(unique(nhts2017$HH_CBSA))
m<-length(ids)
Y<-list() ; X<-list() ; N<-NULL
for(j in 1:m)
{
  Y[[j]]<-nhts2017[nhts2017$HH_CBSA==ids[j],2]
  N[j]<- sum(nhts2017$HH_CBSA==ids[j])
  xj<-nhts2017[nhts2017$HH_CBSA==ids[j], 4]
  xj<-(xj-mean(xj))
  X[[j]]<-cbind( rep(1,N[j]), xj  )
}
```

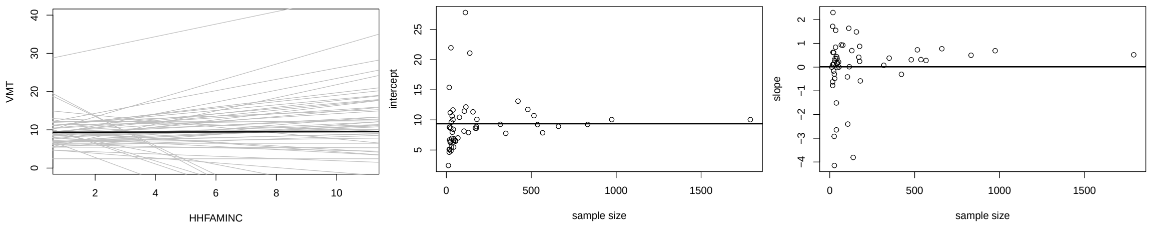
OLS fits

```r
S2.LS<-BETA.LS<-NULL
for(j in 1:m) {
  fit<-lm(Y[[j]]~-1+X[[j]] )
  BETA.LS<-rbind(BETA.LS,c(fit$coef))
  S2.LS<-c(S2.LS, summary(fit)$sigma^2)
}
```

The first panel plots least squares estimates of the regression lines for the 52 CBSA, along with an average of these lines in black. A large majority show an slight increase in expected VMT with increasing household income, although a few show a negative relationship.

The second and third panels of the figure relate the least squares estimates to sample size. Notice that CBSAs with the higher sample sizes have regression coefficients that are generally closer to the average, whereas CBSAs with extreme coefficients are generally those with low sample sizes. This phenomenon confirms that the smaller the sample size for the group, the more probable that unrepresentative data are sampled and an extreme least squares estimate is produced.

```r
plot( range(nhts2017[,4]),c(0,40),type="n",xlab="HHFAMINC", ylab="VMT")  # range(NHTS2017[,2])
for(j in 1:m) {    abline(BETA.LS[j,1],BETA.LS[j,2],col="gray")  }

BETA.MLS<-apply(BETA.LS,2,mean)
abline(BETA.MLS[1],BETA.MLS[2],lwd=2)

plot(N,BETA.LS[,1],xlab="sample size",ylab="intercept")
abline(h= BETA.MLS[1],col="black",lwd=2)
plot(N,BETA.LS[,2],xlab="sample size",ylab="slope")
abline(h= BETA.MLS[2],col="black",lwd=2)
```



A hierarchical regression model

$$\mathbf{Y}_{i,j} = \boldsymbol{\beta_j^T x_{i,j}} + \varepsilon_{i,j} = \boldsymbol{\theta^T x_{i,j} + \gamma_j^T x_{i,j}} + \varepsilon_{i,j}$$

- mvnormal simulation

$\boldsymbol{\beta}_{1:m} \overset{iid}{\sim} N_p(\boldsymbol{\theta}, \Sigma)$, $\boldsymbol{\gamma}_{1:m} \overset{iid}{\sim} N_p(0, \Sigma)$

$\boldsymbol{\theta} \sim N_p(\boldsymbol{\mu_0}, \boldsymbol{\Lambda_0})$,

```
rmvnorm<-function(n,mu,Sigma)
{
  E<-matrix(rnorm(n*length(mu)),n,length(mu))
  t(  t(E%*%chol(Sigma)) +c(mu))
}
```

- Wishart simulation

$\Sigma \sim Inverse - Wishart(\eta_0, \boldsymbol{S_0^{-1}})$,

```
rwish<-function(n,nu0,S0)
{
  sS0 <- chol(S0)
  S<-array( dim=c( dim(S0),n ) )
  for(i in 1:n)
  {
     Z <- matrix(rnorm(nu0 * dim(S0)[1]), nu0, dim(S0)[1]) %*% sS0
     S[,,i]<- t(Z)%*%Z
  }
  S[,,1:n]
}
```

$\sigma^2 \sim Inverse - Gamma(\frac{1}{2}\nu_0, \frac{1}{2}\nu_0\sigma_0^2)$

MCMC

Full conditional distributions

$$\{\beta|\boldsymbol{y_j},\boldsymbol{X_j},\boldsymbol{\theta},\sigma^2,\Sigma\} \sim N_p\left([\Sigma^{-1}+\frac{1}{\sigma^2}\mathbf{X_j'X_j}]^{-1}[\Sigma^{-1}\boldsymbol{\theta}+\frac{1}{\sigma^2}\mathbf{X_j'y_j}],[\Sigma^{-1}+\frac{1}{\sigma^2}\mathbf{X_j'X_j}]^{-1}\right)$$

$$\{\boldsymbol{\theta}|\boldsymbol{\beta_{1:m}},\Sigma\} \sim N_p(\boldsymbol{\mu_m},\boldsymbol{\Lambda_m}); \quad \Lambda_m=(\Lambda_0^{-1}+m\Sigma^{-1})^{-1}; \quad \boldsymbol{\mu_m}=\Lambda_m(\Lambda_0^{-1}\boldsymbol{\mu_0}+m\Sigma^{-1}\bar{\boldsymbol{\beta}})$$

$$\{\Sigma|\boldsymbol{\theta},\boldsymbol{\beta_{1:m}}\} \sim Inverse-Wishart(\eta_0+m,[\boldsymbol{S_0}+\boldsymbol{S_\theta}]^{-1}); \quad \boldsymbol{S_\theta}=\sum_{j=1}^{m}(\boldsymbol{\beta_j}-\boldsymbol{\theta})(\boldsymbol{\beta_j}-\boldsymbol{\theta})^{\boldsymbol{T}}$$

$$\sigma^2 \sim Inverse-Gamma(\frac{1}{2}[\nu_0+\sum n_j],\frac{1}{2}[\nu_0\sigma_0^2+\mathrm{SSR}]); \quad \mathrm{SSR}=\sum_{j=1}^{m}\sum_{i=1}^{n}(y_{i,j}-\boldsymbol{\beta_j^T}\boldsymbol{x_{i,j}})^2$$

- Setup

take μ_0, the prior expectation of θ, to be equal to the average of the ordinary least squares regression estimates and the prior variance Λ_0 to be their sample covariance. Such a prior distribution represents the information of someone with unbiased but weak prior information.

```
p<-dim(X[[1]])[2]
theta<-mu0<-apply(BETA.LS,2,mean)
nu0<-1 ; s2<-s20<-mean(S2.LS)
eta0<-p+2 ; Sigma<-S0<-L0<-cov(BETA.LS) ; BETA<-BETA.LS
THETA.b<-S2.b<-NULL
iL0<-solve(L0) ; iSigma<-solve(Sigma)
Sigma.ps<-matrix(0,p,p)
SIGMA.PS<-NULL
BETA.ps<-BETA*0
BETA.pp<-NULL
set.seed(1)
mu0[2]+c(-1.96,1.96)*sqrt(L0[2,2])
## [1] -2.451  2.477
```

For example, a 95% prior confidence interval for the slope parameter θ_2 under this prior is (-2.4512, 2.4772), which is quite a large range when considering what the extremes of the interval imply in terms of average change in VMT per unit change in income. Similarly, we will take the prior sum of squares matrix S_0 to be equal to the covariance of the least squares estimate, but we'll take the prior degrees of freedom η_0 to be $p+2=4$, so that the prior distribution of Σ is reasonably diffuse but has an expectation equal to the sample covariance of the least squares estimates. Finally, we'll take σ_0^2 to be the average of the within-group sample variance but set $\nu_0=1$.

```
for(s in 1:10000) {
  ##update beta_j
  for(j in 1:m)
  {
    Vj<-solve( iSigma + t(X[[j]])%*%X[[j]]/s2 )
    Ej<-Vj%*%( iSigma%*%theta + t(X[[j]])%*%Y[[j]]/s2 )
    BETA[j,]<-rmvnorm(1,Ej,Vj)
  }
  ##

  ##update theta
  Lm<-  solve( iL0 +  m*iSigma )
  mum<- Lm%*%( iL0%*%mu0 + iSigma%*%apply(BETA,2,sum))
```

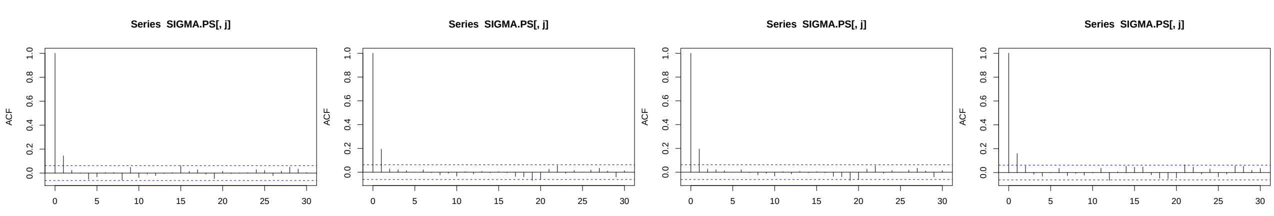
```
  theta<-t(rmvnorm(1,mum,Lm))
  ##

  ##update Sigma
  mtheta<-matrix(theta,m,p,byrow=TRUE)
  iSigma<-rwish(1, eta0+m, solve( S0+t(BETA-mtheta)%*%(BETA-mtheta) ) )
  ##

  ##update s2
  RSS<-0
  for(j in 1:m) { RSS<-RSS+sum( (Y[[j]]-X[[j]]%*%BETA[j,] )^2 ) }
  s2<-1/rgamma(1,(nu0+sum(N))/2, (nu0*s20+RSS)/2 )
  ##
  ##store results
  if(s%%10==0)
  {
   # cat(s,s2,"\n")
    S2.b<-c(S2.b,s2);THETA.b<-rbind(THETA.b,t(theta))
    Sigma.ps<-Sigma.ps+solve(iSigma) ; BETA.ps<-BETA.ps+BETA
    SIGMA.PS<-rbind(SIGMA.PS,c(solve(iSigma)))
    BETA.pp<-rbind(BETA.pp,rmvnorm(1,theta,solve(iSigma)) )
  }
  ##
}
```

- MCMC diagnostics

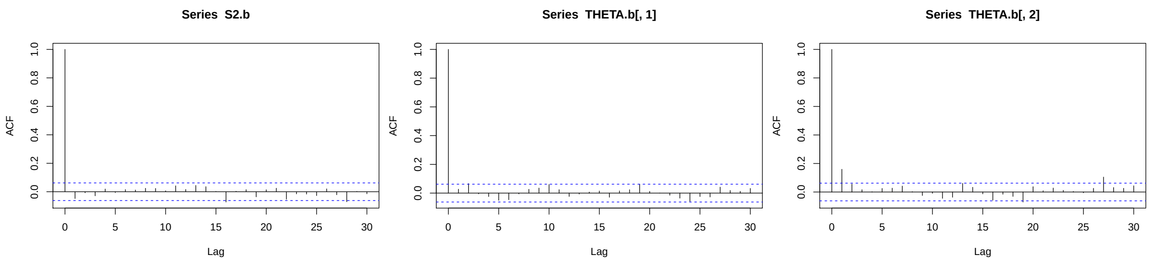
```
library(coda)
effectiveSize(S2.b)
## var1
## 1099
effectiveSize(THETA.b[,1])
##  var1
## 829.1
effectiveSize(THETA.b[,2])
##  var1
## 724.4
apply(SIGMA.PS,2,effectiveSize)
## [1] 747.2 674.0 674.0 724.0
```

```
tmp<-NULL; for(j in 1:dim(SIGMA.PS)[2]) { tmp<-c(tmp,acf(SIGMA.PS[,j])$acf[2]) }
tmp
```

```
## [1] 0.1442 0.1943 0.1943 0.1596
```

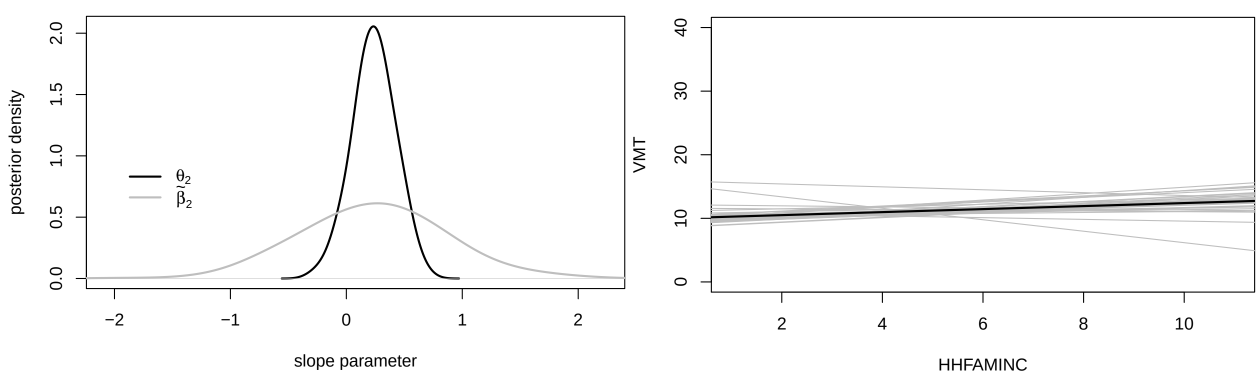


```r
acf(S2.b)
acf(THETA.b[,1])
acf(THETA.b[,2])
```



Running a Gibbs sampler for 10,000 scans and saving every 10th scan produces a sequence of 1,000 values for each parameter, each sequence having a fairly low autocorrelation. For example, the lag-10 autocorrelations of θ_1 and θ_2 are 0.0273. We can use these simulated values to make Monte Carlo approximations to various posterior quantities of interest.

```r
plot(density(THETA.b[,2],adj=2),xlim=range(BETA.pp[,2]),
      main="",xlab="slope parameter",ylab="posterior density",lwd=2)
lines(density(BETA.pp[,2],adj=2),col="gray",lwd=2)
legend( -2 ,1.0 ,legend=c( expression(theta[2]),expression(tilde(beta)[2])),
        lwd=c(2,2),col=c("black","gray"),bty="n")

quantile(THETA.b[,2],prob=c(.025,.5,.975))
##    2.5%     50%   97.5%
## -0.1237  0.2391  0.5817
mean(BETA.pp[,2]<0)
## [1] 0.346

BETA.PM<-BETA.ps/1000
plot( range(nhts2017[,4]),c(0,40),type="n",xlab="HHFAMINC", ylab="VMT") # range(nels[,3]),range(nels[,4
for(j in 1:m) {    abline(BETA.PM[j,1],BETA.PM[j,2],col="gray")  }
abline( mean(THETA.b[,1]),mean(THETA.b[,2]),lwd=2 )
```



The first panel plots the posterior density of the expected within-school slope θ_2 of a randomly sampled income, as well as the posterior predictive distribution of a randomly sampled slope. A 95% quantile-based posterior confidence interval for θ_2 is (-0.1237, 0.5817), which, compared to our prior interval of (-2.4512, 2.4772), indicates a strong alteration in our information about θ_2.

The fact that θ_2 is unlikely to be negative only indicates that the population average of CBSA-level slopes is positive. It does not indicate that any given within-CSA slope cannot be negative. To clarify this distinction, the posterior predictive distribution of $\tilde{\beta}_2$, the slope for a to-be-sampled CBSA, is plotted in the same figure. Samples from this distribution can be generated by sampling a value $\tilde{\boldsymbol{\beta}}^{(s)}$ from a multivariate normal$(\boldsymbol{\theta}^{(s)}, \Sigma^{(s)})$ distribution for each scan s of the Gibbs sampler. Notice that this posterior predictive distribution is much more spread out than the posterior distribution of θ_2, reflecting the heterogeneity in slopes across CBSA.

Using the Monte Carlo approximation, we have $Pr(\tilde{\beta}_2 < 0|\mathbf{y_{1:m}}, \mathbf{X_{1:m}}) \approx 0.346$, which is small but not negligible.

The second panel gives posterior expectations of the 52 CBSA-specific regression lines, with the average line given by the posterior mean of θ in black. Comparing this to the first panel indicates how the hierarchical model is able to share information across groups, shrinking extreme regression lines towards the across-group average. In particular, hardly any of the slopes are negative when we share information across groups.

Generalized linear mixed effects models

- Gibbs steps for $\boldsymbol{\theta}, \Sigma)$
- Metropolis step for $\boldsymbol{\beta}_j$
- A Metropolis-Hastings approximation algorithm

1. Sample $\boldsymbol{\theta}^{(s+1)}$ from its full conditional distribution.
2. Sample $\Sigma^{(s+1)}$ from its full conditional distribution.
3. For each $j \in \{1, ..., m\}$,

    a) propose a new value $\beta_j^\star$;

    b) set $\beta_j^{(s+1)}$ equal to $\beta_j^\star$ or $\beta_j^{(s)}$ with the appropriate probability.

Compare with the results using 'brms'