STAT562 Final Exam

Winter 2019

1. $X_1, X_2, ..X_n$ is a random sample from a distribution having a p.d.f of the form. $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find a complete sufficient statistic for λ. Justify your answer

- Step1: Proof sufficient

From *Fisher–Neyman factorization theorem* (`2019-2-14p5`)

$$f(x|\lambda) = L(\lambda) = \lambda^n (\prod x_i)^{\lambda-1} = \lambda^n e^{(\lambda-1)\sum_{i=1}^n \ln x_i} \cdot 1 = k(t|\lambda) h(\vec{x})$$

$h(\vec{x}) = 1$ is free of λ. So $T = \sum_{i=1}^n \ln x_i$ a sufficient statistic.

- Step2: Proof complete

$f(x|\lambda)$ is a member of the exponential family (`2019-2-19p12`),

$$f(x|\vec{\lambda}) = \lambda^n e^{(\lambda-1)\sum_{i=1}^n \ln x_i} = h(x) c(\vec{\lambda}) e^{\sum_{j=1}^k W_j(\vec{\lambda}) t_j(x)}$$

For $\lambda > 0$, $\{W_1(\vec{\lambda}), .., W_k(\vec{\lambda})\}$ contains an open interval in $\Re$, so $T(\vec{x}) = \sum_{i=1}^n \ln x_i)$ is a complete sufficient statistic.

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2. Let Y_n be the $n^t h$ order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

- Step1: θ is a location parameter

Let $X = Y_n - \bar{Y}$. For $N(\theta, \sigma^2)$ is a location family of densities (`2018.11.20p7`),

$$g(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y-\theta}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = f(x) = f(y-\theta)$$

Thus, θ is a location parameter.

- Step2: $Y_n - \bar{Y}$ is location invariant

`2019-2-21p4-6`

- step3: $Y_n - \bar{Y}$ is ancillary statistic for θ

`2019-2-19p6`

- step4: $\bar{Y}$ is sufficient statistic for θ
- step5: $\bar{Y}$ is complete statistic for θ
- step6: By Basu's theorem, $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

`2019-2-19p10`

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3. Suppose that $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta e^{-\theta x}, x > 0$. Assume that the prior distribution of θ is $\pi(\theta) = \lambda e^{-\lambda\theta}, \theta > 0$

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

`2019-2-26p8-9,p11-p13`

For $L(\theta) = \hat{\theta} e^{-\theta \sum x_i}, \quad \pi(\theta) = \lambda e^{-\lambda\theta}$, and the kernel of a function is the main part of the function, the part that remains when constants are disregarded (`2019-2-28p8 Exapmle 2.3.8`). that is

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \lambda e^{-\lambda\theta} \propto \theta^{n+1-1} e^{-\theta(\lambda + \sum x_i)}$$

which is $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$

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b. Find the Bayes estimator of θ, assuming square-error loss

`2019-2-28p1`

Posterior mean $= \alpha\beta = \frac{n+1}{\lambda + \sum x_i}$

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c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\frac{1}{\hat{\theta}} = \frac{\lambda + \sum x_i}{n+1} = \frac{\lambda + n\bar{x}}{n+1} = \frac{\frac{1}{\lambda} + \frac{n}{\frac{1}{\bar{x}}}}{n+1}$$

$$\hat{\theta} = \frac{1}{\frac{\frac{1}{\lambda} + \frac{n}{\frac{1}{\bar{x}}}}{n+1}}$$

which is the weighted hamonic mean of $\frac{1}{\lambda}$, which is the prior mean, and $\frac{1}{\bar{x}}$, which is the MLE of θ.

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d. Find the Bayes estimator of θ, assuming absolute loss

`2019-2-28p4,9`

Postmedian

$$\hat{\theta} = F^{-1}(\frac{1}{2})$$

where $F(x)$ is the $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ pdf, for which there is no closed form.

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e. Find the Bayes estimator of θ, assuming binary loss

`2019-2-28p5-6`

Posterior mode

$$\hat{\theta} = (\alpha - 1)\beta = \frac{n}{\lambda + \sum x_i}$$

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4. Redo all of problem 3, using the non-informative prior $\pi(\theta) = 1, \theta > 0$. Note that this is not a valid density function since its intergral is infinite, but proceed with it anyway

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

2019-2-26p8-9

For $\pi(\theta) = 1, \theta > 0,\ L(\theta) = \theta^n e^{-\theta \sum x_i}$

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \sim Gamma(\alpha = n+1, \beta = \frac{1}{\sum x_i} = \frac{n}{\bar{x}})$$

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b. Find the Bayes estimator of θ, assuming square-error loss

$$\hat{\theta} = \alpha\beta = \frac{n+1}{\sum x_i}$$

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c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\frac{1}{\hat{\theta}} = \frac{1 \times 0 + n\bar{x}}{n+1} = \lim_{c \to \infty} \frac{1 \times \frac{1}{c} + \frac{n}{\frac{1}{\bar{x}}}}{n+1}$$

So $\hat{\theta}$ is the limit as $c \to \infty$ of the hamonic mean of $\frac{1}{\bar{x}}$ (the MLE) and c.(the prior mean is infinite)

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d. Find the Bayes estimator of θ, assuming absolute loss

Posterior median $\hat{\theta}$, which has no closed form.

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e. Find the Bayes estimator of θ, assuming binary loss

Posterior mode

$$\hat{\theta} = (\alpha - 1)\beta = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

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5. Let $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta x^{-\theta-1}, x_i > 1, \theta > 2$.

a. Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ.

2019-2-2p12

$$f(\vec{x}|\theta) = L(\theta) = \theta^n (\prod x_i)^{\theta - 1} = \theta^n e^{(-\theta - 1 \sum_{i=1}^n \ln x_i)}$$

$$l(\theta) = n \ln \theta - (\theta + 1) \sum_{i=1}^n \ln x_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n \ln x_i \overset{\text{set}}{=} 0$$

$$\hat{\theta} = \frac{n}{\sum \ln x_i}$$

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b. Find the expected value of $\hat{\theta}_{MLE}$.

Let $Y_i = \ln x_i$, then $X = e^y$, $\frac{dx}{dy} e^y$

$$g(Y) = \theta(e^y)^{-\theta-1} e^y = \theta e^{-y\theta}, y > 0$$

So $Y_i = \ln x_i \sim Gamma(\alpha = n, \beta = \frac{1}{\theta})$

$$E[\hat{\theta}] = nE[Y^{-1}] = n\frac{\beta^{-1}\Gamma(-1+\alpha)}{\Gamma(\alpha)} = \frac{n\theta\Gamma(n-1)}{\Gamma(n)} = \frac{n\theta(n-2)!}{(n-1)!} = \frac{n\theta}{n-1}$$

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c. Find the variance of $\hat{\theta}_{MLE}$.

$$E[\hat{\theta}^2] = n^2 E[Y^{-2}] = n^2\frac{\beta^{-2}\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{n^2\theta^2\Gamma(n-2)}{\Gamma(n)} = \frac{n^2\theta^2(n-3)!}{(n-1)!} = \frac{n^2\theta^2}{(n-1)(n-2)}$$

$$Var[\hat{\theta}^2] = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)}[\frac{1}{n-2} - \frac{1}{n-1}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

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d. Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

$$\hat{\theta}_U = \frac{n-1}{n}\hat{\theta}_{MLE}$$

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e. Find the variance of $\hat{\theta}_U$.

$$Var[\hat{\theta}_U] = (\frac{n-1}{n})^2\frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2}$$

6. Refer to problem 5.

a. Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ.

`2019-2-21p8 7.2.1`

$$\mu = \int_1^\infty x\theta x^{-\theta-1}dx = \theta\frac{x^{-\theta+1}}{-\theta+1}\Big|_1^\infty = \frac{\theta}{\theta-1}$$

Set $\bar{X} = \frac{\theta}{\theta-1}$, $\theta\bar{x} - \bar{x} = \theta$, $\theta(\bar{x}-1) = \bar{x}$,

$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1}$

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b. Using the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.

`2019-3-5p1`

$$g(x) = \frac{x}{x-1}, \quad g'(x) = \frac{(x-1)\times 1 - x \times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}, \quad g''(x) = \frac{2}{(x-1)^3}$$

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x-x_0)^2}{2} + R$$

Choose $x_0 = EX = \mu$

$$g(x) \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(x-\mu) + \frac{2}{(\mu-1)^3}\frac{(x-\mu)^2}{2}$$

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(\bar{x}-\mu) + \frac{1}{(\mu-1)^3}(\bar{x}-\mu)^2$$

For $EX = \mu = \frac{\theta}{\theta-1}$

$$E[X^2] = \int_1^\infty x^2\theta x^{-\theta-1}dx = \theta\frac{x^{-\theta+2}}{-\theta+2}\Big|_1^\infty = \frac{\theta}{\theta-2}$$

$$Var[X] = \sigma^2 = E[X^2] - E[X]^2 = \frac{\theta}{\theta-2} - (\frac{\theta}{\theta-1})^2$$

$$= \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-1)^2(\theta-2)} = \frac{\theta^3 - 2\theta^2 + \theta - \theta^3 + 2\theta^2}{(\theta-1)^2(\theta-2)} = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

$$E[\hat{\theta}_{MOM}] \approx \frac{\mu}{\mu-1} + 0 + \frac{1}{(\mu-1)^3}\frac{\sigma^2}{n} = \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1}-1} + \frac{1}{(\frac{\theta}{\theta-1}-1)^3}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \theta + \frac{\theta(\theta-1)}{n(\theta-2)}$$

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c. Using the delta method to approximate the variance of $\hat{\theta}_{MOM}$.

`2019-3-5p3`

$$Var[\hat{\theta}_{MOM}] \approx Var[g(x_0) + g'(x_0)(x-x_0)] = Var[\frac{\mu}{\mu-1} + \frac{1}{(\mu-1)^2}(\bar{x}-\mu)]$$

$$= \frac{1}{(\mu-1)^4}\frac{\sigma^2}{n} = \frac{1}{(\frac{\theta}{\theta-1}-1)^4}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \frac{\theta(\theta-1)^2}{n(\theta-2)}$$

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End
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