565 mid-term

CRD

model

$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, for $\varepsilon_{ij} \sim iidN(0, \sigma^2), i = 1, 2, , a, j = 1, 2, , n$, $\sum_{i=1}^{a} \tau_i = 0$
Random effects: $\tau_i \sim iidN(0, \sigma_\tau^2)$,$\tau_i$ and ε_{ij} indep

least square and normal equation

$SSE = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \mu - \tau_i)^2$
$\frac{\partial SSE}{\partial \mu}\Big|_{\hat\mu,\hat\tau_i} = 2 \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \hat\mu - \hat\tau_i)(-1) = 0$ $y_{..} = an\hat\mu + n \sum_{i=1}^{a} \hat\tau_i$
$\frac{\partial SSE}{\partial \tau_i}\Big|_{\hat\mu,\hat\tau_i} = 2 \sum_{j=1}^{n} (y_{ij} - \hat\mu - \hat\tau_i)(-1) = 0$ $y_{i.} = n\hat\mu + n\hat\tau_i$

ANOVA

	SS	df	MS	F
SS_{Trt}	$n \sum_{i=1}^{a} (\bar y_{i.} - \bar y_{..})^2$	$a-1$	$\frac{SS_{Trt}}{a-1}$	$\frac{MS_{Trt}}{MS_E}$
SS_E	$\sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar y_{i.})^2$	$a(n-1)$	$\frac{SS_E}{N-a}$	
SS_T	$\sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar y_{..})^2$	$an-1$		

[f.]$SS_{Trt} = \frac{1}{n} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}$

RCBD

$y_{ij} = (\mu + \tau_i + \beta_j) + \varepsilon_{ij}$, for $\varepsilon_{ij} \sim iidN(0, \sigma^2), i = 1, 2, , a, j = 1, 2, , b$
Fixed effects: $\sum_{i=1}^{a} \tau_i = 0, \sum_{j=1}^{b} \beta_j = 0$ Random effects: $\beta_j \sim iidN(0, \sigma_\beta^2)$

least square and normal equation

$SSE = \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \mu - \tau_i - \beta_j)^2$
$\frac{\partial SSE}{\partial \mu}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j} = 2 \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \hat\mu - \hat\tau_i - \hat\beta_j)(-1) = 0$
$\frac{\partial SSE}{\partial \tau_i}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j} = 2 \sum_{j=1}^{b} (y_{ij} - \hat\mu - \hat\tau_i - \hat\beta_j)(-1) = 0$
$\frac{\partial SSE}{\partial \beta_j}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j} = 2 \sum_{i=1}^{a} (y_{ij} - \hat\mu - \hat\tau_i - \hat\beta_j)(-1) = 0$
$y_{..} = ab\hat\mu + b \sum_{i=1}^{a} \hat\tau_i + a \sum_{j=1}^{b} \hat\beta_j$
$y_{i.} = b\hat\mu + b\hat\tau_i + \sum_{j=1}^{b} \hat\beta_j$
$y_{.j} = a\hat\mu + \sum_{i=1}^{a} \hat\tau_i + a\hat\beta_j$

SS_{Trt}	$b \sum_{i=1}^{a} (\bar y_{i.} - \bar y_{..})^2$	$a-1$	$\frac{SS_{Trt}}{a-1}$	$\frac{MS_{Trt}}{MS_E}$
SS_{Blk}	$a \sum_{j=1}^{b} (\bar y_{.j} - \bar y_{..})^2$	$b-1$	$\frac{SS_{Blk}}{b-1}$	$\frac{MS_{Blk}}{MS_E}$
SS_E	$SS_T - SS_{Trt} - SS_{Blk}$	$(a-1)(b-1)$	$\frac{SS_E}{df_E}$	
SS_T	$\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar y_{..})^2$	$ab-1$		

[r.]$SS_{Trt} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}$ $SS_{Blk} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N}$
$SSE = \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar y_{i.} - \bar y_{.j} + \bar y_{..})^2$
$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N}$, $S_i^2 = \frac{\sum_{j=1}^{n} (y_{ij} - \bar y_{i.})^2}{n-1}$

Latin squre

Graeco-Latin square: $y_{ijkl} = \mu + \theta_i + \tau_j + \omega_k + \psi_l + \varepsilon_{ijkl}$;$i, j, k, l = 1, 2, .., p$

LSD	SS	df
SS_{Trt1}	$p \sum_{i=1}^{p} (\bar y_{i..} - \bar y_{...})^2$	$p-1$
SS_{Row}	$p \sum_{j=1}^{p} (\bar y_{.j.} - \bar y_{...})^2$	$p-1$
SS_{Col}	$p \sum_{k=1}^{p} (\bar y_{..k} - \bar y_{...})^2$	$p-1$
SS_E	$SS_T - SS.$	$(p-1)(p-2)$
SS_T	$\sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} (y_{ijk} - \bar y_{..})^2$	$p^2 - 1$

df	Case1	Case2	Case3
SS_{Trt}	$p-1$	$p-1$	$p-1$
SS_{Row}	$p-1$	$n(p-1)$	$n(p-1)$
SS_{Col}	$p-1$	$p-1$	$n(p-1)$
SS_{Rep}	$n-1$	$n-1$	$n-1$
SS_E	$(p-1)(np+n-3)$	$(p-1)(np-2)$	$(p-1)(np-n-1)$
SS_T	np^2-1	np^2-1	np^2-1

Graeco-LSD	SS	df
SS_{Trt1}	$b \sum_{i=1}^{a} (\bar y_{i.} - \bar y_{..})^2$	$p-1$
SS_{Trt2}	$b \sum_{i=1}^{b} (\bar y_{i.} - \bar y_{..})^2$	$p-1$
SS_{Row}	$a \sum_{j=1}^{b} (\bar y_{.j} - \bar y_{..})^2$	$p-1$
SS_{Col}	$a \sum_{j=1}^{b} (\bar y_{.j} - \bar y_{..})^2$	$p-1$
SS_E	$SS_T - SS.$	$(p-1)(p-3)$
SS_T	$\sum_{i=1}^{p} \sum_{j=1}^{p} (y_{ijkl} - \bar y_{..})^2$	$p^2 - 1$

BIBD

BIBD: $y_{ij} = (\mu + \tau_i + \beta_j) + \varepsilon_{ij}$;$i = 1, 2, , a, j = 1, 2, , b$;$N = kb = ra, \lambda = \frac{r(k-1)}{a-1}$

BIBD	SS	df
SS_{Trt}	$\frac{k}{\lambda a} \sum_{i=1}^{a} Q_i^2 = \frac{k}{\lambda a} \sum_{i=1}^{a} (y_{i.} - \sum_{j=1}^{b} n_{ij} \bar y_{.j})^2$	$a-1$
SS_{Blk}	$a \sum_{j=1}^{b} (\bar y_{.j} - \bar y_{..})^2$	$b-1$
SS_E	$SS_T - SS_{Trt(adj)} - SS_{Blk}$	$N-a-b+1$
SS_T	$\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar y_{..})^2$	$N-1$

hypothesis test

model	H0	H1
FT-CRD	$\mu_1 = \mu_2 = .. = \mu_a$	at least two of μ_a diff
RT-CRD	$\sigma_\tau^2 = 0$	$\sigma_\tau^2 > 0$
FB-RCBD	$\mu_1 = \mu_2 = .. = \mu_a$	at least two of μ_a diff
-Block	$\beta_1 = \beta_2 = .. = \beta_j = 0$	at least one of $\beta_j \neq 0$
RB-RCBD	$\sigma_\beta^2 = 0$	$\sigma_\beta^2 > 0$
Contrast	$\Gamma = \sum_{i=1}^{a} c_i\mu_i = 0$	$\Gamma = \sum_{i=1}^{a} c_i\mu_i \neq 0$
LSD	$\mu_1 = \mu_2 = .. = \mu_p$	at least two of μ_p diff
BIBD	$\mu_1 = \mu_2 = .. = \mu_a$	at least two of μ_a diff

distribution

	y_{ij}	$\bar y_{i.}$	$\frac{y_{i.} - \mu_i}{\frac{\sigma}{\sqrt{n}}}$	$\bar y_{i.} - \bar y_{j.}$	$\frac{\bar y_{i.} - \bar y_{j.} - (\mu_i - \mu_j)}{\frac{2\sigma}{\sqrt{n}}}$
N	$(\mu + \tau_i, \sigma^2)$	$(\mu + \tau_i, \frac{\sigma^2}{n})$	$(0, 1)$	$(\mu_i - \mu_j, \frac{2\sigma^2}{n})$	$(0, 1)$

t_{df_E}	$\frac{\frac{y_{i.} - \mu_i}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{SS_E}{\sigma^2(N-a)}}} = \frac{y_{i.} - \mu_i}{\sqrt{\frac{SS_E}{n(df_E)}}} = \frac{y_{i.} - \mu_i}{\sqrt{\frac{MS_E}{n}}}$	$\frac{\bar y_{i.} - \bar y_{j.} - (\mu_i - \mu_j)}{\sqrt{\frac{2MS_E}{n}}}$	$\frac{C}{\sqrt{\widehat{Var(C)}}}$

χ^2	$\frac{SS_E}{\sigma^2}$	$\frac{SS_{Trt}}{\sigma^2}$	$\frac{SS_{Blk}}{\sigma^2}$	SS_C
df	E	Trt	Blk	1

F	$\frac{MS_{Trt}}{MS_E}$	$\frac{(MS_{Blk}}{MS_E}$	$\frac{(\sum_{i=1}^{a} c_i\bar y_{i.})^2}{\frac{MS_E}{n} \sum_{i=1}^{a} c_i^2}$
df	(Trt, E)	(Blk, E)	$(C = 1, E)$

contrast

$\Gamma = \sum_{i=1}^{a} c_i\mu_i$,$C = \sum_{i=1}^{a} c_i\bar y_i$;$\sum_{i=1}^{a} c_i = 0$;Orthogonal $\sum_{i=1}^{a} c_id_i = 0$
$SS_C = \frac{(\sum_{i=1}^{a} c_i\bar y_i)^2}{\frac{1}{n} \sum_{i=1}^{a} c_i^2}$,$\sum_{1}^{a-1} SS_C = SS_{Trt}$

CI

CI	balanced	unbalanced
μ_i	$\bar y_{i.} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{MS_E}{n}}$	$\bar y_{i.} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{MS_E}{n_i}}$
$\mu_i - \mu_j$	$\bar y_{i.} - \bar y_{j.} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{2MS_E}{n}}$	$\bar y_{i.} - \bar y_{j.} \pm t_{\frac{\alpha}{2}} \sqrt{MS_E(\frac{1}{n_i} + \frac{1}{n_j})}$
Γ	$\sum_{i=1}^{a} c_i\bar y_i \pm t_{\frac{\alpha}{2}} \sqrt{\frac{MS_E}{n} \sum_{i=1}^{a} c_i^2}$	

Expectation and Variance

$y_{..} = \sum_{i=1}^{a} (y_{i.} = \sum_{j=1}^{n} y_{ij})$,$\mu = \frac{\sum_{i=1}^{a} \mu_i}{a}$
$\bar y_{..} = \frac{1}{N} \left[y_{..} = \sum_{i=1}^{a} (y_{i.} = \sum_{j=1}^{n} y_{ij})\right] = \frac{1}{a} \sum_{i=1}^{a} \left[\bar y_{i.} = \frac{1}{n} (y_{i.} = \sum_{j=1}^{n} y_{ij})\right]$

	y_{ij}	$\bar y_{..}$ $\frac{\sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}}{N}$	$\bar y_{i.}$ $\frac{\sum_{j=1}^{n} y_{ij}}{n}$	$\bar y_{.j}$ $\frac{\sum_{i=1}^{a} y_{ij}}{a}$	C $\sum_{i=1}^{a} c_i\bar y_{i.}$
E[f.]	$\mu + \tau_i + \beta_j$	μ	$\mu + \tau_i$	$\mu + \beta_j$	$\sum_{i=1}^{a} c_i\mu_i$
V[f.]	σ^2	$\frac{\sigma^2}{na}$	$\frac{\sigma^2}{n}$	$\frac{\sigma^2}{a}$	$\frac{\sigma^2}{n} \sum_{i=1}^{a} c_i^2$
E[r.]	μ	μ	μ	μ	
V[r.]	$\sigma^2 + \sigma_\tau^2$	$\frac{\sigma_\tau^2}{a} + \frac{\sigma^2}{na}$	$\frac{\sigma^2}{n} + \sigma_\tau^2$	$\frac{\sigma^2}{a} + \sigma_\beta^2$	

	MS_E	MS_{Trt}	MS_{Trt}	MS_{Blk}
E[f.]	σ^2	$\sigma^2 + \frac{n}{a-1} \sum_{i=1}^{a} \tau_i^2$	$\sigma^2 + \frac{b}{a-1} \sum_{i=1}^{a} \tau_i^2$	$\sigma^2 + \frac{a}{b-1} \sum_{j=1}^{b} \beta_j^2$
E[r.]	σ^2	$\sigma^2 + n\tau_i^2$	$\sigma^2 + b\tau_i^2$	$\sigma^2 + a\beta_j^2$

	$\hat\sigma^2$	$\hat\sigma_{Trt}^2$	$\hat\sigma_{Blk}^2$	$\widehat{Var(\bar y_{i.})}$	$\widehat{Var(\bar y_{.j})}$
f.	MSE			$\frac{MS_E}{n}$	$\frac{MS_E}{a}$
r.	MSE	$\frac{MS_{Trt} - MS_E}{n}$	$\frac{MS_{Blk} - MS_E}{a}$	$\frac{MS_{Trt}}{n}$	$\frac{MS_{Trt}}{a}$

$\hat\mu$	$\hat\mu_i$	$\hat\tau_i$	$\hat\beta_j$	$\hat y_{ij}$	$\hat\varepsilon_{ij}$
$\bar y_{..}$	$\bar y_{i.}$	$\bar y_{i.} - \bar y_{..}$	$\bar y_{.j} - \bar y_{..}$	$\bar y_{i.} + \bar y_{.j} - \bar y_{..}$	$\bar y_{ij} - \bar y_{..}$
$\frac{1}{an} y_{..} - \frac{1}{a} \sum_{i=1}^{a} \hat\tau_i$		$\frac{1}{a} y_{i.} - \hat\mu$			

FT-	CRD	E	Var[]
$y_{ij}-\bar{y}_{i.}$	$\varepsilon_{ij}-\frac{\sum_{j=1}^{n}\varepsilon_{ij}}{n}$	0	$\frac{n-1}{n}\sigma^2$
$\bar{y}_{i.}-\bar{y}_{..}$	$\tau_i+\frac{\sum_{j=1}^{n}\varepsilon_{ij}}{n}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}}{an}$	τ_i	$\frac{(a-1)\sigma^2}{an}$
RT-	CRD	E	Var[]
$y_{ij}-\bar{y}_{i.}$	$\varepsilon_{ij}-\frac{\sum_{j=1}^{n}\varepsilon_{ij}}{n}$	0	$\frac{n-1}{n}\sigma^2$
$\bar{y}_{i.}-\bar{y}_{..}$	$\tau_i-\frac{\sum\limits_{i=1}^{a}\tau_i}{a}+\frac{\sum\limits_{j=1}^{n}\varepsilon_{ij}}{n}-\frac{\sum\limits_{i=1}^{a}\sum\limits_{j=1}^{n}\varepsilon_{ij}}{an}$	0	$\frac{(a-1)(n\sigma_\tau^2+\sigma^2)}{an}$
F-	RCBD	E	Var[]
$y_{ij}-\bar{y}_{i.}$	$\beta_j+\varepsilon_{ij}-\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}$	β_j	$\frac{(b-1)\sigma^2}{b}$
$\bar{y}_{i.}-\bar{y}_{..}$	$\tau_i+\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}}{ab}$	τ_i	$\frac{(a-1)\sigma^2}{ab}$
$y_{ij}-\bar{y}_{.j}$	$\tau_i+\varepsilon_{ij}-\frac{\sum_{i=1}^{a}\varepsilon_{ij}}{a}$	τ_i	$\frac{(a-1)\sigma^2}{a}$
$\bar{y}_{.j}-\bar{y}_{..}$	$\beta_j+\frac{\sum_{i=1}^{a}\varepsilon_{ij}}{a}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}}{ab}$	β_j	$\frac{(b-1)\sigma^2}{ab}$
R-	RCBD	E	Var[]
$y_{ij}-\bar{y}_{i.}$	$\beta_j-\frac{\sum_{j=1}^{b}\beta_j}{b}+\varepsilon_{ij}-\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}$	0	$\frac{(b-1)(\sigma_\beta^2+\sigma^2)}{b}$
$\bar{y}_{i.}-\bar{y}_{..}$	$\tau_i+\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}}{ab}$	τ_i	$\frac{(a-1)\sigma^2}{ab}$
$y_{ij}-\bar{y}_{.j}$	$\tau_i+\varepsilon_{ij}-\frac{\sum_{i=1}^{a}\varepsilon_{ij}}{a}$	τ_i	$\frac{(a-1)\sigma^2}{a}$
$\bar{y}_{.j}-\bar{y}_{..}$	$\beta_j-\frac{\sum\limits_{j=1}^{b}\beta_j}{b}+\frac{\sum\limits_{i=1}^{a}\varepsilon_{ij}}{a}-\frac{\sum\limits_{i=1}^{a}\sum\limits_{j=1}^{b}\varepsilon_{ij}}{ab}$	0	$\frac{(b-1)(a\sigma_\beta^2+\sigma^2)}{ab}$

$y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..}=\varepsilon_{ij}-\frac{\sum_{i=1}^{a}\varepsilon_{ij}}{a}-\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}+\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}}{ab}$
$E[]=0, Var=\frac{(a-1)(b-1)\sigma^2}{ab}$

covariance

R-CRD: $Cov(y_{ij},y_{kl})=\{\,0\ ,i\neq k;\sigma_\tau^2,i=k,j\neq l;\sigma^2+\sigma_\tau^2,i=k,j=l$

	ε_{ij}	$\sum_{i=1}^{a}\varepsilon_{ij}$	$\sum_{j=1}^{b}\varepsilon_{ij}$	$\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}$
ε_{ij}	σ^2	σ^2	σ^2	σ^2
$\sum_{i=1}^{a}\varepsilon_{ij}$		$a\sigma^2$	σ^2	$a\sigma^2$
$\sum_{j=1}^{b}\varepsilon_{ij}$			$b\sigma^2$	$b\sigma^2$
$\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}$				$ab\sigma^2$

$Cov[\varepsilon_{ij},\varepsilon_{ij}]=Cov[\sum_{i1}^{a}\varepsilon_{ij},\sum_{j=1}^{b}\varepsilon_{ij}]=\sigma^2$,$Cov[\tau_i,\tau_i]=\frac{2}{\tau}$,$Cov[\beta_j,\beta_j]=\frac{2}{\beta}$
$Cov[\varepsilon_{ij},\tau_i]=Cov[\varepsilon_{ij},\beta_j]=Cov[\tau_i,\beta_j]=Cov[\varepsilon_{ij},\varepsilon_{ik}]=0,k\neq j$
$Cov[\varepsilon_{ij},\sum_{j=1}^{n}\varepsilon_{ij}]=Cov[\varepsilon_{ij},(\varepsilon_{i1}+..+\varepsilon_{ij}..+\varepsilon_{in})]=Cov[\varepsilon_{ij},\varepsilon_{i1}]+..+Cov[\varepsilon_{ij},\varepsilon_{ij}]..+Cov[\varepsilon_{ij},\varepsilon_{in}]=\sigma^2$
$Cov[\sum_{j=1}^{n}\varepsilon_{ij},\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}]=Cov[(\varepsilon_{i1}+\varepsilon_{i2}..+\varepsilon_{in}),(\varepsilon_{i1}+\varepsilon_{i2}..+\varepsilon_{in})]=Cov[\varepsilon_{i1},\varepsilon_{i1}]+Cov[\varepsilon_{i2},\varepsilon_{i2}]..+Cov[\varepsilon_{in},\varepsilon_{in}]=n\sigma^2$
$Var[y_{ij}-\bar{y}_{i.}]=$
FT-CRD;RT-CRD
$Var[\varepsilon_{ij}-\frac{\sum_{j=1}^{n}\varepsilon_{ij}}{n}]=, Var[\varepsilon_{ij}]+\frac{\sum_{j=1}^{n}Var[\varepsilon_{ij}]}{n^2}-\frac{2Cov[\varepsilon_{ij},\sum_{j=1}^{n}\varepsilon_{ij}]}{n}, =\sigma^2+\frac{\sigma^2}{n}-\frac{2}{n}\sigma^2=\frac{n-1}{n}\sigma^2$
FB-RCBD
$Var[\beta_j+\varepsilon_{ij}-\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}]=, Var[\beta_j]+Var[\varepsilon_{ij}]+\frac{\sum_{j=1}^{b}Var[\varepsilon_{ij}]}{b^2}-2Cov[\beta_j,\varepsilon_{ij}]-\frac{2Cov[\beta_j,\sum_{j=1}^{b}\varepsilon_{ij}]}{b}-\frac{2Cov[\varepsilon_{ij},\sum_{j=1}^{b}\varepsilon_{ij}]}{b}, =0+\sigma^2+\frac{\sigma^2}{b}-0-0-\frac{2\sigma^2}{b}=\frac{(b-1)\sigma^2}{b}$
RB-RCBD
$Var[\beta_j-\frac{\sum_{j=1}^{b}\beta_j}{b}+\varepsilon_{ij}-\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}]=$
$Var[\beta_j]+\frac{\sum_{j=1}^{b}Var[\beta_j]}{b^2}+Var[\varepsilon_{ij}]+\frac{\sum_{j=1}^{b}Var[\varepsilon_{ij}]}{b^2}-\frac{2Cov[\beta_j,\sum_{j=1}^{b}\beta_j]}{b}+2Cov[\beta_j,\varepsilon_{ij}]-\frac{2Cov[\beta_j,\sum_{j=1}^{b}\varepsilon_{ij}]}{b}-\frac{2Cov[\sum_{j=1}^{b}\beta_j,\varepsilon_{ij}]}{b}$
$+\frac{2Cov[\sum_{j=1}^{b}\beta_j,\sum_{j=1}^{b}\varepsilon_{ij}]}{b^2}-\frac{2Cov[\varepsilon_{ij},\sum_{j=1}^{b}\varepsilon_{ij}]}{b}=\sigma_\beta^2+\frac{\sigma_\beta^2}{b}-\frac{2\sigma_\beta^2}{b}+\sigma^2+\frac{\sigma^2}{b}-\frac{2\sigma^2}{b}=\frac{(\sigma_\beta^2+\sigma^2)(b-1)}{b}$
—
$Var[\bar{y}_{i.}-\bar{y}_{..}]=$
FT-CRD
$Var[\tau_i+\frac{\sum_{j=1}^{n}\varepsilon_{ij}}{n}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}}{an}]=$
$Var[\tau_i]+\frac{\sum_{j=1}^{n}Var[\varepsilon_{ij}]}{n^2}+\frac{\sum_{i=1}^{a}\sum_{j=1}^{n}Var[\varepsilon_{ij}]}{a^2n^2}+\frac{2Cov[\tau_i,\sum_{j=1}^{n}\varepsilon_{ij}]}{n}-\frac{2Cov[\tau_i,\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}]}{an}-\frac{2Cov[\sum_{j=1}^{n}\varepsilon_{ij},\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}]}{an^2},$
$=0+\frac{\sigma^2}{n}+\frac{\sigma^2}{an}-\frac{2\sigma^2}{an}=\frac{(a-1)\sigma^2}{an}$
RT-CRD
$Var[\tau_i-\frac{\sum_{i=1}^{a}\tau_i}{a}+\frac{\sum_{j=1}^{n}\varepsilon_{ij}}{n}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}}{an}]=$
$Var[\tau_i]+\frac{\sum_{i=1}^{a}Var[\tau_i]}{a^2}+\frac{\sum_{j=1}^{n}Var[\varepsilon_{ij}]}{n^2}+\frac{\sum_{i=1}^{a}\sum_{j=1}^{n}Var[\varepsilon_{ij}]}{a^2n^2}-\frac{2Cov[\tau_i,\sum_{i=1}^{a}\tau_i]}{a}+\frac{2Cov[\tau_i,\sum_{j=1}^{n}\varepsilon_{ij}]}{n}-\frac{2Cov[\tau_i,\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}]}{an}-\frac{2Cov[\sum_{i=1}^{a}\tau_i,\sum_{j=1}^{n}\varepsilon_{ij}]}{an}$
$+\frac{2Cov[\sum_{i=1}^{a}\tau_i,\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}]}{a^2n}-\frac{2Cov[\sum_{j=1}^{n}\varepsilon_{ij},\sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}]}{an^2}=\sigma_\tau^2+\frac{\sigma_\tau^2}{a}-\frac{2\sigma_\tau^2}{a}+\frac{\sigma^2}{n}+\frac{\sigma^2}{an}-\frac{2\sigma^2}{an}=\frac{(n\sigma_\tau^2+\sigma^2)(a-1)}{an}$
FB-RCBD;RB-RCBD
$Var[\tau_i+\frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}}{ab}]=$
$Var[\tau_i]+\frac{\sum_{j=1}^{b}Var[\varepsilon_{ij}]}{b^2}+\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}Var[\varepsilon_{ij}]}{a^2b^2}+\frac{2Cov[\tau_i,\sum_{j=1}^{b}\varepsilon_{ij}]}{b}-\frac{2Cov[\tau_i,\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}]}{ab}-\frac{2Cov[\sum_{j=1}^{b}\varepsilon_{ij},\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}]}{ab^2}=0+\frac{\sigma^2}{b}+\frac{\sigma^2}{ab}-\frac{2\sigma^2}{ab}=\frac{(a-1)\sigma^2}{ab}$
—
$Var[y_{ij}-\bar{y}_{.j}]=$
FB-RCBD; RB-RCBD
$Var[\tau_i+\varepsilon_{ij}-\frac{\sum_{i=1}^{a}\varepsilon_{ij}}{a}]=Var[\tau_i]+Var[\varepsilon_{ij}]+\frac{\sum_{i=1}^{a}Var[\varepsilon_{ij}]}{a^2}-\frac{2Cov[\tau_i,\varepsilon_{ij}]}{a}-\frac{2Cov[\tau_i,\sum_{i=1}^{a}\varepsilon_{ij}]}{a}-\frac{2Cov[\varepsilon_{ij},\sum_{i=1}^{a}\varepsilon_{ij}]}{a}=0+\sigma^2+\frac{\sigma^2}{a}-\frac{2\sigma^2}{a}=\frac{(a-1)\sigma^2}{a}$
—
$Var[\bar{y}_{.j}-\bar{y}_{..}]=$
FB-RCBD
$Var[\beta_j+\frac{\sum_{i=1}^{a}\varepsilon_{ij}}{a}-\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}}{ab}]=$
$Var[\beta_j]+\frac{\sum_{i=1}^{a}Var[\varepsilon_{ij}]}{a^2}+\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}Var[\varepsilon_{ij}]}{a^2b^2}+\frac{2Cov[\beta_j,\sum_{i=1}^{a}\varepsilon_{ij}]}{a}-\frac{2Cov[\beta_j,\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}]}{ab}-\frac{2Cov[\sum_{i=1}^{a}\varepsilon_{ij},\sum_{i=1}^{a}\sum_{j=1}^{b}\varepsilon_{ij}]}{a^2b}$
$=0+\frac{\sigma^2}{a}+\frac{\sigma^2}{ab}-\frac{2\sigma^2}{ab}=\frac{(b-1)\sigma^2}{ab}$
RB-RCBD

$Var[\beta_j - \frac{\sum_{j=1}^{b} \beta_j}{b} + \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}] =,$
$Var[\beta_j] + \frac{\sum_{j=1}^{b} Var[\beta_j]}{b^2} + \frac{\sum_{i=1}^{a} Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} Var[\varepsilon_{ij}]}{a^2 b^2} - \frac{2Cov[\beta_j, \sum_{j=1}^{b} \beta_j]}{b} + \frac{2Cov[\beta_j, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} - \frac{2Cov[\beta_j, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{j=1}^{b} \beta_j, \sum_{i=1}^{a} \varepsilon_{ij}]}{ab}$
$+ \frac{2Cov[\sum_{j=1}^{b} \beta_j, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab^2} - \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{a^2 b} = \sigma_\beta^2 + \frac{\sigma_\beta^2}{b} - \frac{2\sigma_\beta^2}{b} + \frac{\sigma^2}{a} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(a\sigma_\beta^2 + \sigma^2)(b-1)}{ab}$

—

$Var[y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}] =$ FB-RCBD
$Var[\varepsilon_{ij} - \frac{1}{a} \sum_{i=1}^{a} \varepsilon_{ij} - \frac{1}{b} \sum_{j=1}^{b} \varepsilon_{ij} + \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}] =$
$Var[\varepsilon_{ij}] + \frac{\sum_{i=1}^{a} Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{j=1}^{b} Var[\varepsilon_{ij}]}{b^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} Var[\varepsilon_{ij}]}{a^2 b^2} - \frac{2Cov[\varepsilon_{ij}, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}]}{b} + \frac{2Cov[\varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab} + \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}]}{ab}$
$- \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{a^2 b} - \frac{2Cov[\sum_{j=1}^{b} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab^2} = \sigma^2 + \frac{\sigma^2}{a} + \frac{\sigma^2}{b} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{a} - \frac{2\sigma^2}{b} + \frac{2\sigma^2}{ab} + \frac{2\sigma^2}{ab} - \frac{2\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(ab-a-b+1)\sigma^2}{ab}$