565 mid-term

model

$y_{ij}=\mu+\tau_i+\varepsilon_{ij}$, for $\varepsilon_{ij}\sim iidN(0,\sigma^2), i=1,2,,a, j=1,2,,n, \sum_{i=1}^a \tau_i=0$

Random effects: $\tau_i\sim iidN(0,\sigma_\tau^2)$,$\tau_i$ and ε_{ij} indep

$y_{ij}=(\mu+\tau_i+\beta_j)+\varepsilon_{ij}$, for $\varepsilon_{ij}\sim iidN(0,\sigma^2), i=1,2,,a, j=1,2,,b$

Fixed effects: $\sum_{i=1}^a\tau_i=0,\sum_{j=1}^b\beta_j=0$ Random effects: $\beta_j\sim iidN(0,\sigma_\beta^2)$

least square and normal equation

CRD:$SSE=\sum_{i=1}^a\sum_{j=1}^n(y_{ij}-\mu-\tau_i)^2$

$\frac{\partial SSE}{\partial\mu}\Big|_{\hat\mu,\hat\tau_i}=2\sum_{i=1}^a\sum_{j=1}^n(y_{ij}-\hat\mu-\hat\tau_i)(-1)=0$ $y_{..}=an\hat\mu+n\sum_{i=1}^a\hat\tau_i$

$\frac{\partial SSE}{\partial\tau_i}\Big|_{\hat\mu,\hat\tau_i}=2\sum_{j=1}^n(y_{ij}-\hat\mu-\hat\tau_i)(-1)=0$ $y_{i.}=n\hat\mu+n\hat\tau_i$

RCBD: $SSE=\sum_{i=1}^a\sum_{j=1}^b(y_{ij}-\mu-\tau_i-\beta_j)^2$

$\frac{\partial SSE}{\partial\mu}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j}=2\sum_{i=1}^a\sum_{j=1}^b(y_{ij}-\hat\mu-\hat\tau_i-\hat\beta_j)(-1)=0$

$\frac{\partial SSE}{\partial\tau_i}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j}=2\sum_{j=1}^b(y_{ij}-\hat\mu-\hat\tau_i-\hat\beta_j)(-1)=0$

$\frac{\partial SSE}{\partial\beta_j}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j}=2\sum_{i=1}^a(y_{ij}-\hat\mu-\hat\tau_i-\hat\beta_j)(-1)=0$

$y_{..}=ab\hat\mu+b\sum_{i=1}^a\hat\tau_i+a\sum_{j=1}^b\hat\beta_j$

$y_{i.}=b\hat\mu+b\hat\tau_i+\sum_{j=1}^b\hat\beta_j$

$y_{.j}=a\hat\mu+\sum_{i=1}^a\hat\tau_i+a\hat\beta_j$

hypothesis test

model	H0	H1
FT-CRD	$\mu_1=\mu_2=..=\mu_i$	at least two of μ_i are different
RT-CRD	$\sigma_\tau^2=0$	$\sigma_\tau^2>0$
FB-RCBD	$\mu_1=\mu_2=..=\mu_i$	at least two of μ_i are different
-Block	$\beta_1=\beta_2=..=\beta_j=0$	at least one of $\beta_j\neq 0$
RB-RCBD	$\sigma_\beta^2=0$	$\sigma_\beta^2>0$
Contrast	$\Gamma=\sum_{i=1}^a c_i\mu_i=0$	$\Gamma=\sum_{i=1}^a c_i\mu_i\neq 0$

ANOVA

	SS	df	MS	F
SS_{Trt}	$n\sum_{i=1}^a(\bar y_{i.}-\bar y_{..})^2$	$a-1$	$\frac{SS_{Trt}}{a-1}$	$\frac{MS_{Trt}}{MS_E}$
SS_E	$\sum_{i=1}^a\sum_{j=1}^n(y_{ij}-\bar y_{i.})^2$	$a(n-1)$	$\frac{SS_E}{N-a}$	
SS_T	$\sum_{i=1}^a\sum_{j=1}^n(y_{ij}-\bar y_{..})^2$	$an-1$		
SS_{Trt}	$b\sum_{i=1}^a(\bar y_{i.}-\bar y_{..})^2$	$a-1$	$\frac{SS_{Trt}}{a-1}$	$\frac{MS_{Trt}}{MS_E}$
SS_{Blk}	$a\sum_{j=1}^b(\bar y_{.j}-\bar y_{..})^2$	$b-1$	$\frac{SS_{Blk}}{b-1}$	$\frac{MS_{Blk}}{MS_E}$
SS_E	$SS_T-SS_{Trt}-SS_{Blk}$	$(a-1)(b-1)$	$\frac{SS_E}{df_E}$	
SS_T	$\sum_{i=1}^a\sum_{j=1}^b(y_{ij}-\bar y_{..})^2$	$ab-1$		

[f.]$SS_{Trt}=\frac1n\sum_{i=1}^a y_{i.}^2-\frac{y_{..}^2}{N}$

[r.]$SS_{Trt}=\frac1b\sum_{i=1}^a y_{i.}^2-\frac{y_{..}^2}{N}$ $SS_{Blk}=\frac1a\sum_{j=1}^b y_{.j}^2-\frac{y_{..}^2}{N}$

$SSE=\sum_{i=1}^a\sum_{j=1}^b(y_{ij}-\bar y_{i.}-\bar y_{.j}+\bar y_{..})^2$

$SS_T=\sum_{i=1}^a\sum_{j=1}^b y_{ij}^2-\frac{y_{..}^2}{N}$, $S_i^2=\frac{\sum_{j=1}^n(y_{ij}-\bar y_{i.})^2}{n-1}$

distribution

	y_{ij}	$\bar y_{i.}$	$\frac{y_{i.}-\mu_i}{\frac{\sigma}{\sqrt n}}$	$\bar y_{i.}-\bar y_{j.}$	$\frac{\bar y_{i.}-\bar y_{j.}-(\mu_i-\mu_j)}{\frac{2\sigma}{\sqrt n}}$
N	$(\mu+\tau_i,\sigma^2)$	$(\mu+\tau_i,\frac{\sigma^2}{n})$	$(0,1)$	$(\mu_i-\mu_j,\frac{2\sigma^2}{n})$	$(0,1)$

t_{df_E}	$\frac{\frac{y_{i.}-\mu_i}{\frac{\sigma}{\sqrt n}}}{\sqrt{\frac{SS_E}{\sigma^2(N-a)}}}=\frac{y_{i.}-\mu_i}{\sqrt{\frac{SS_E}{n(df_E)}}}=\frac{y_{i.}-\mu_i}{\sqrt{\frac{MS_E}{n}}}$	$\frac{\bar y_{i.}-\bar y_{j.}-(\mu_i-\mu_j)}{\sqrt{\frac{2MS_E}{n}}}$	$\frac{C}{\sqrt{\widehat{Var(C)}}}$

χ^2	$\frac{SS_E}{\sigma^2}$	$\frac{SS_{Trt}}{\sigma^2}$	$\frac{SS_{Blk}}{\sigma^2}$	SS_C
df	E	Trt	Blk	1

F	$\frac{MS_{Trt}}{MS_E}$	$\frac{(MS_{Blk}}{MS_E}$	$\frac{(\sum_{i=1}^a c_i\bar y_{i.})^2}{\frac{MS_E}{n}\sum_{i=1}^a c_i^2}$
df	(Trt,E)	(Blk,E)	$(C=1,E)$

contrast

$\Gamma=\sum_{i=1}^a c_i\mu_i$,$C=\sum_{i=1}^a c_i\bar y_i$;$\sum_{i=1}^a c_i=0$;Orthogonal $\sum_{i=1}^a c_id_i=0$

$SS_C=\frac{(\sum_{i=1}^a c_i\bar y_i)^2}{\frac1n\sum_{i=1}^a c_i^2}$,$\sum_1^{a-1}SS_C=SS_{Trt}$

CI

CI	balanced	unbalanced
μ_i	$\bar y_{i.}\pm t_{\frac{\alpha}{2}}\sqrt{\frac{MS_E}{n}}$	$\bar y_{i.}\pm t_{\frac{\alpha}{2}}\sqrt{\frac{MS_E}{n_i}}$
$\mu_i-\mu_j$	$\bar y_{i.}-\bar y_{j.}\pm t_{\frac{\alpha}{2}}\sqrt{\frac{2MS_E}{n}}$	$\bar y_{i.}-\bar y_{j.}\pm t_{\frac{\alpha}{2}}\sqrt{MS_E(\frac{1}{n_i}+\frac{1}{n_j})}$
Γ	$\sum_{i=1}^a c_i\bar y_i\pm t_{\frac{\alpha}{2}}\sqrt{\frac{MS_E}{n}\sum_{i=1}^a c_i^2}$	

expectation and variance

$y_{..}=\sum_{i=1}^a(y_{i.}=\sum_{j=1}^n y_{ij})$,$\mu=\frac{\sum_{i=1}^a\mu_i}{a}$

$\bar y_{..}=\frac1N\left[y_{..}=\sum_{i=1}^a(y_{i.}=\sum_{j=1}^n y_{ij})\right]=\frac1a\sum_{i=1}^a\left[\bar y_{i.}=\frac1n(y_{i.}=\sum_{j=1}^n y_{ij})\right]$

	y_{ij}	$\bar y_{..}$ $\frac{\sum_{i=1}^a\sum_{j=1}^n y_{ij}}{N}$	$\bar y_{i.}$ $\frac{\sum_{j=1}^n y_{ij}}{n}$	$\bar y_{.j}$ $\frac{\sum_{i=1}^a y_{ij}}{a}$	C
E[f.]	$\mu+\tau_i+\beta_j$	μ	$\mu+\tau_i$	$\mu+\beta_j$	$\sum_{i=1}^a c_i\bar y_{i.}$ $\sum_{i=1}^a c_i\mu_i$
V[f.]	σ^2	$\frac{\sigma^2}{na}$	$\frac{\sigma^2}{n}$	$\frac{\sigma^2}{a}$	$\frac{\sigma^2}{n}\sum_{i=1}^a c_i^2$
E[r.]	μ	μ	μ	μ	
V[r.]	$\sigma^2+\sigma_\tau^2$	$\frac{\sigma_\tau^2}{a}+\frac{\sigma^2}{na}$	$\frac{\sigma^2}{n}+\sigma_\tau^2$	$\frac{\sigma^2}{a}+\sigma_\beta^2$	

	MS_E	MS_{Trt}	MS_{Trt}	MS_{Blk}
E[f.]	σ^2	$\sigma^2+\frac{n}{a-1}\sum_{i=1}^a\tau_i^2$	$\sigma^2+\frac{b}{a-1}\sum_{i=1}^a\tau_i^2$	$\sigma^2+\frac{a}{b-1}\sum_{j=1}^b\beta_j^2$
E[r.]	σ^2	$\sigma^2+n\tau_i^2$	$\sigma^2+b\tau_i^2$	$\sigma^2+a\beta_j^2$

	$\hat\sigma^2$	$\hat\sigma^2_{Trt}$	$\hat\sigma^2_{Blk}$	$\widehat{Var}(\bar y_{i.})$	$\widehat{Var}(\bar y_{.j})$
f.	MSE			$\frac{MS_E}{n}$	$\frac{MS_E}{a}$
r.	MSE	$\frac{MS_{Trt}-MS_E}{n}$	$\frac{MS_{Blk}-MS_E}{a}$	$\frac{MS_{Trt}}{n}$	$\frac{MS_{Trt}}{a}$

$\hat\mu$	$\hat\mu_i$	$\hat\tau_i$	$\hat\beta_j$	$\hat y_{ij}$	$\hat\varepsilon_{ij}$
$\bar y_{..}$	$\bar y_{i.}$	$\bar y_{i.}-\bar y_{..}$	$\bar y_{.j}-\bar y_{..}$	$\bar y_{i.}+\bar y_{.j}-\bar y_{..}$	$y_{ij}-\bar y_{..}$
$\frac{1}{an}y_{..}-\frac1a\sum_{i=1}^a\hat\tau_i$		$\frac1a y_{i.}-\hat\mu$			

FT-	CRD	E	Var[]
$y_{ij}-\bar y_{i.}$	$\varepsilon_{ij}-\frac{\sum_{j=1}^n\varepsilon_{ij}}{n}$	0	$\frac{n-1}{n}\sigma^2$
$\bar y_{i.}-\bar y_{..}$	$\tau_i+\frac{\sum_{j=1}^n\varepsilon_{ij}}{n}-\frac{\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}}{an}$	τ_i	$\frac{(a-1)\sigma^2}{an}$
RT-	CRD	E	Var[]
$y_{ij}-\bar y_{i.}$	$\varepsilon_{ij}-\frac{\sum_{j=1}^n\varepsilon_{ij}}{n}$	0	$\frac{n-1}{n}\sigma^2$
$\bar y_{i.}-\bar y_{..}$	$\tau_i-\frac{\sum_{i=1}^a\tau_i}{a}+\frac{\sum_{j=1}^n\varepsilon_{ij}}{n}-\frac{\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}}{an}$	0	$\frac{(a-1)(n\sigma_\tau^2+\sigma^2}{an}$
F-	RCBD	E	Var[]
$y_{ij}-\bar y_{i.}$	$\beta_j+\varepsilon_{ij}-\frac{\sum_{j=1}^b\varepsilon_{ij}}{b}$	β_j	$\frac{(b-1)\sigma^2}{b}$
$\bar y_{i.}-\bar y_{..}$	$\tau_i+\frac{\sum_{j=1}^b\varepsilon_{ij}}{b}-\frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}$	τ_i	$\frac{(a-1)\sigma^2}{ab}$
$y_{ij}-\bar y_{.j}$	$\tau_i+\varepsilon_{ij}-\frac{\sum_{i=1}^a\varepsilon_{ij}}{a}$	τ_i	$\frac{(a-1)\sigma^2}{a}$
$\bar y_{.j}-\bar y_{..}$	$\beta_j+\frac{\sum_{i=1}^a\varepsilon_{ij}}{a}-\frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}$	β_j	$\frac{(b-1)\sigma^2}{ab}$
R-	RCBD	E	Var[]
$y_{ij}-\bar y_{i.}$	$\beta_j-\frac{\sum_{j=1}^b\beta_j}{b}+\varepsilon_{ij}-\frac{\sum_{j=1}^b\varepsilon_{ij}}{b}$	0	$\frac{(b-1)(\sigma_\beta^2+\sigma^2)}{b}$
$\bar y_{i.}-\bar y_{..}$	$\tau_i+\frac{\sum_{j=1}^b\varepsilon_{ij}}{b}-\frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}$	τ_i	$\frac{(a-1)\sigma^2}{ab}$
$y_{ij}-\bar y_{.j}$	$\tau_i+\varepsilon_{ij}-\frac{\sum_{i=1}^a\varepsilon_{ij}}{a}$	τ_i	$\frac{(a-1)\sigma^2}{a}$
$\bar y_{.j}-\bar y_{..}$	$\beta_j-\frac{\sum_{j=1}^b\beta_j}{b}+\frac{\sum_{i=1}^a\varepsilon_{ij}}{a}-\frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}$	0	$\frac{(b-1)(a\sigma_\beta^2+\sigma^2}{ab}$

$y_{ij}-\bar y_{i.}-\bar y_{.j}+\bar y_{..}=\varepsilon_{ij}-\frac{\sum_{i=1}^a\varepsilon_{ij}}{a}-\frac{\sum_{j=1}^b\varepsilon_{ij}}{b}+\frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}$

$E[]=0, Var=\frac{(a-1)(b-1)\sigma^2}{ab}$

covariance

R-CRD: $Cov(y_{ij},y_{kl})=\{0\ ,i\neq k;\sigma_\tau^2, i=k,j\neq l;\sigma^2+\sigma_\tau^2, i=k,j=l$

	ε_{ij}	$\sum_{i=1}^a\varepsilon_{ij}$	$\sum_{j=1}^b\varepsilon_{ij}$	$\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}$
ε_{ij}	σ^2	σ^2	σ^2	σ^2
$\sum_{i=1}^a\varepsilon_{ij}$		$a\sigma^2$	σ^2	$a\sigma^2$
$\sum_{j=1}^b\varepsilon_{ij}$			$b\sigma^2$	$b\sigma^2$
$\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}$				$ab\sigma^2$

$Cov[\varepsilon_{ij},\varepsilon_{ij}]=Cov[\sum_{i1}^a\varepsilon_{ij},\sum_{j=1}^b\varepsilon_{ij}]=\sigma^2$,$Cov[\tau_i,\tau_i]={}^2_\tau$,$Cov[\beta_j,\beta_j]={}^2_\beta$

$Cov[\varepsilon_{ij},\tau_i]=Cov[\varepsilon_{ij},\beta_j]=Cov[\tau_i,\beta_j]=Cov[\varepsilon_{ij},\varepsilon_{ik}]=0, k\neq j$

$Cov[\varepsilon_{ij},\sum_{j=1}^n\varepsilon_{ij}]=Cov[\varepsilon_{ij},(\varepsilon_{i1}+..+\varepsilon_{ij}..+\varepsilon_{in})]=Cov[\varepsilon_{ij},\varepsilon_{i1}]+..+Cov[\varepsilon_{ij},\varepsilon_{ij}]..+Cov[\varepsilon_{ij},\varepsilon_{in}]=\sigma^2$

$Cov[\sum_{j=1}^n\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}]=Cov[(\varepsilon_{i1}+\varepsilon_{i2}..+\varepsilon_{in}),(\varepsilon_{i1}+\varepsilon_{i2}..+\varepsilon_{in})]=Cov[\varepsilon_{i1},\varepsilon_{i1}]+Cov[\varepsilon_{i2},\varepsilon_{i2}]..+Cov[\varepsilon_{in},\varepsilon_{in}]=n\sigma^2$

$Var[y_{ij}-\bar y_{i.}]=$

FT-CRD;RT-CRD

$Var[\varepsilon_{ij} - \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n}] =, Var[\varepsilon_{ij}] + \frac{\sum_{j=1}^{n} Var[\varepsilon_{ij}]}{n^2} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^{n} \varepsilon_{ij}]}{n}, = \sigma^2 + \frac{\sigma^2}{n} - \frac{2}{n}\sigma^2 = \frac{n-1}{n}\sigma^2$
FB-RCBD
$Var[\beta_j + \varepsilon_{ij} - \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b}] =, Var[\beta_j] + Var[\varepsilon_{ij}] + \frac{\sum_{j=1}^{b} Var[\varepsilon_{ij}]}{b^2} - 2Cov[\beta_j, \varepsilon_{ij}] - \frac{2Cov[\beta_j, \sum_{j=1}^{b} \varepsilon_{ij}]}{b} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}]}{b}, = 0 + \sigma^2 + \frac{\sigma^2}{b} - 0 - 0 - \frac{2\sigma^2}{b} = \frac{(b-1)\sigma^2}{b}$
RB-RCBD
$Var[\beta_j - \frac{\sum_{j=1}^{b} \beta_j}{b} + \varepsilon_{ij} - \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b}] =$
$Var[\beta_j] + \frac{\sum_{j=1}^{b} Var[\beta_j]}{b^2} + Var[\varepsilon_{ij}] + \frac{\sum_{j=1}^{b} Var[\varepsilon_{ij}]}{b^2} - \frac{2Cov[\beta_j, \sum_{j=1}^{b} \beta_j]}{b} + 2Cov[\beta_j, \varepsilon_{ij}] - \frac{2Cov[\beta_j, \sum_{j=1}^{b} \varepsilon_{ij}]}{b} - \frac{2Cov[\sum_{j=1}^{b} \beta_j, \varepsilon_{ij}]}{b}$
$+ \frac{2Cov[\sum_{j=1}^{b} \beta_j, \sum_{j=1}^{b} \varepsilon_{ij}]}{b^2} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}]}{b} = \sigma_\beta^2 + \frac{\sigma_\beta^2}{b} - \frac{2\sigma_\beta^2}{b} + \sigma^2 + \frac{\sigma^2}{b} - \frac{2\sigma^2}{b} = \frac{(\sigma_\beta^2 + \sigma^2)(b-1)}{b}$
—
$Var[\bar{y}_{i.} - \bar{y}_{..}] =$
FT-CRD
$Var[\tau_i + \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}}{an}] =$
$Var[\tau_i] + \frac{\sum_{j=1}^{n} Var[\varepsilon_{ij}]}{n^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{n} Var[\varepsilon_{ij}]}{a^2 n^2} + \frac{2Cov[\tau_i, \sum_{j=1}^{n} \varepsilon_{ij}]}{n} - \frac{2Cov[\tau_i, \sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}]}{an} - \frac{2Cov[\sum_{j=1}^{n} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}]}{an^2},$
$= 0 + \frac{\sigma^2}{n} + \frac{\sigma^2}{an} - \frac{2\sigma^2}{an} = \frac{(a-1)\sigma^2}{an}$
RT-CRD
$Var[\tau_i - \frac{\sum_{i=1}^{a} \tau_i}{a} + \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}}{an}] =$
$Var[\tau_i] + \frac{\sum_{i=1}^{a} Var[\tau_i]}{a^2} + \frac{\sum_{j=1}^{n} Var[\varepsilon_{ij}]}{n^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{n} Var[\varepsilon_{ij}]}{a^2 n^2} - \frac{2Cov[\tau_i, \sum_{i=1}^{a} \tau_i]}{a} + \frac{2Cov[\tau_i, \sum_{j=1}^{n} \varepsilon_{ij}]}{n} - \frac{2Cov[\tau_i, \sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}]}{an} - \frac{2Cov[\sum_{i=1}^{a} \tau_i, \sum_{j=1}^{n} \varepsilon_{ij}]}{an}$
$+ \frac{2Cov[\sum_{i=1}^{a} \tau_i, \sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}]}{a^2 n} - \frac{2Cov[\sum_{j=1}^{n} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}]}{an^2} = \sigma_\tau^2 + \frac{\sigma_\tau^2}{a} - \frac{2\sigma_\tau^2}{a} + \frac{\sigma^2}{n} + \frac{\sigma^2}{an} - \frac{2\sigma^2}{an} = \frac{(n\sigma_\tau^2 + \sigma^2)(a-1)}{an}$
FB-RCBD;RB-RCBD
$Var[\tau_i + \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}] =$
$Var[\tau_i] + \frac{\sum_{j=1}^{b} Var[\varepsilon_{ij}]}{b^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} Var[\varepsilon_{ij}]}{a^2 b^2} + \frac{2Cov[\tau_i, \sum_{j=1}^{b} \varepsilon_{ij}]}{b} - \frac{2Cov[\tau_i, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{j=1}^{b} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab^2} = 0 + \frac{\sigma^2}{b} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(a-1)\sigma^2}{ab}$
—
$Var[y_{ij} - \bar{y}_{.j}] =$
FB-RCBD; RB-RCBD
$Var[\tau_i + \varepsilon_{ij} - \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a}] = Var[\tau_i] + Var[\varepsilon_{ij}] + \frac{\sum_{i=1}^{a} Var[\varepsilon_{ij}]}{a^2} - \frac{2Cov[\tau_i, \varepsilon_{ij}]}{a} - \frac{2Cov[\tau_i, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} - \frac{2Cov[\varepsilon_{ij}, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} = 0 + \sigma^2 + \frac{\sigma^2}{a} - \frac{2\sigma^2}{a} = \frac{(a-1)\sigma^2}{a}$
—
$Var[\bar{y}_{.j} - \bar{y}_{..}] =$
FB-RCBD
$Var[\beta_j + \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}] =$
$Var[\beta_j] + \frac{\sum_{i=1}^{a} Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} Var[\varepsilon_{ij}]}{a^2 b^2} + \frac{2Cov[\beta_j, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} - \frac{2Cov[\beta_j, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{a^2 b}$
$= 0 + \frac{\sigma^2}{a} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(b-1)\sigma^2}{ab}$
RB-RCBD
$Var[\beta_j - \frac{\sum_{j=1}^{b} \beta_j}{b} + \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}] =,$
$Var[\beta_j] + \frac{\sum_{j=1}^{b} Var[\beta_j]}{b^2} + \frac{\sum_{i=1}^{a} Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} Var[\varepsilon_{ij}]}{a^2 b^2} - \frac{2Cov[\beta_j, \sum_{j=1}^{b} \beta_j]}{b} + \frac{2Cov[\beta_j, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} - \frac{2Cov[\beta_j, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{j=1}^{b} \beta_j, \sum_{i=1}^{a} \varepsilon_{ij}]}{ab}$
$+ \frac{2Cov[\sum_{j=1}^{b} \beta_j, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab^2} - \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{a^2 b} = \sigma_\beta^2 + \frac{\sigma_\beta^2}{b} - \frac{2\sigma_\beta^2}{b} + \frac{\sigma^2}{a} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(a\sigma_\beta^2 + \sigma^2)(b-1)}{ab}$
—
$Var[y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}] =$ FB-RCBD
$Var[\varepsilon_{ij} - \frac{1}{a} \sum_{i=1}^{a} \varepsilon_{ij} - \frac{1}{b} \sum_{j=1}^{b} \varepsilon_{ij} + \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}] =$
$Var[\varepsilon_{ij}] + \frac{\sum_{i=1}^{a} Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{j=1}^{b} Var[\varepsilon_{ij}]}{b^2} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} Var[\varepsilon_{ij}]}{a^2 b^2} - \frac{2Cov[\varepsilon_{ij}, \sum_{i=1}^{a} \varepsilon_{ij}]}{a} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}]}{b} + \frac{2Cov[\varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab} + \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}]}{ab}$
$- \frac{2Cov[\sum_{i=1}^{a} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{a^2 b} - \frac{2Cov[\sum_{j=1}^{b} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}]}{ab^2} = \sigma^2 + \frac{\sigma^2}{a} + \frac{\sigma^2}{b} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{a} - \frac{2\sigma^2}{b} + \frac{2\sigma^2}{ab} + \frac{2\sigma^2}{ab} - \frac{2\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(ab-a-b+1)\sigma^2}{ab}$