Assume that $X_1, X_2, ..X_{10}$ is a random sample from a distribution having a p.d.f. of the form $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

1. Find the best critical region of level 0.05 for testing $H_0 : \lambda = 1/2$ against $H_1 : \lambda = 1$

The regiction region is $R = \{\vec{x} : \Lambda \leq C\}$

$$\Lambda = \frac{L(\hat{\lambda}_0|x)}{L(\hat{\lambda}_1|x)} = \frac{(1/2)^{10}(\prod_{i=1}^{10} x_i)^{1/2-1}}{(1)^{10}(\prod_{i=1}^{10} x_i)^{1-1}} = (1/2)^{10}e^{1/2(-\sum_{i=1}^{10}\ln x_i)} \leq C \implies -\sum_{i=1}^{10}\ln x_i \leq C'$$

Let $T_i = -\ln x_i, 0 < x < 1$, then $X = e^{-t}$, $\frac{dx}{dt} = -e^{-t}$

$g(t) = \lambda(e^{-t})^{\lambda-1}| - e^{-t}| = \lambda e^{-\lambda t} \sim Exp(\lambda), t > 0$

So $\sum_{i=1}^{10} T_i = \sum_{i=1}^{10}(-\ln x_i) \sim Gamma(\alpha = 10, \beta = \frac{1}{\lambda})$

Under $H_0 : \lambda = 1/2, -\sum_{i=1}^{10}\ln x_i \sim Exp(1/2) = Gamma(\alpha = 10, \beta = 2) = \chi^2_{20}$. Then,

For Reject H_0, $\Lambda \leq C$ is equivalent $-\sum_{i=1}^{10}\ln x_i \leq C'$, where C' cuts off the upper α area in the χ^2_{20} distribution.

$1 - P(-\sum_{i=1}^{10}\ln x_i \leq C'|\lambda = 1/2) = \alpha = 0.05$

$C' = \chi^2_{(1-0.05),20} = 10.85081$. So the critical region are $-\sum_{i=1}^{10}\ln x_i \in (0, 10.85081]$.

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2. Find the power of the test in (1)

Let $W_i = 2T_i = -2\ln x_i, 0 < x < 1$, then $X = e^{-\frac{1}{2}w}$, $\frac{dx}{dw} = -\frac{1}{2}e^{-\frac{1}{2}w}$

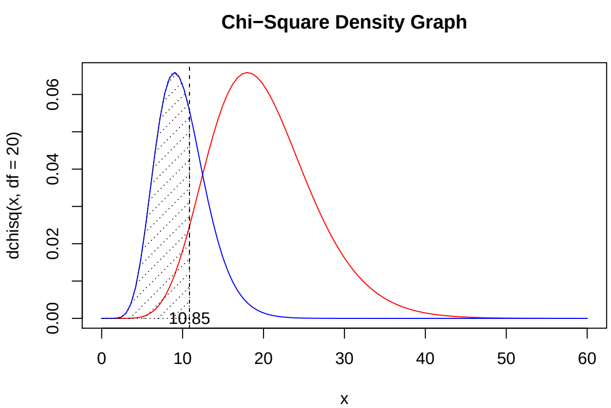
$g(w) = \lambda(e^{-\frac{1}{2}w})^{\lambda-1}| - \frac{1}{2}e^{-\frac{1}{2}w}| = \frac{\lambda}{2}e^{-\frac{\lambda}{2}w} \sim Exp(\frac{\lambda}{2}), w > 0$

Under $H_1 : \lambda = 1, W = -2\sum_{i=1}^{10}\ln x_i \sim Exp(1/2) = Gamma(\alpha = 10, \beta = 2) = \chi^2_{20}$.

The rejection rule is $-\sum_{i=1}^{10}\ln x_i \leq 10.85081$, then

Power $= P(\text{reject } H_0|H_1 \text{ is ture})$

$$= P(-\sum_{i=1}^{10}\ln x_i \leq 10.85081|\lambda = 1) = P(-2\sum_{i=1}^{10}\ln x_i \leq 2 * 10.85081|\lambda = 1) = 0.6430814$$



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3. Is your answer from (1) uniformly most powerful for testing $H_0 : \lambda = 1/2$ against $H_1 : \lambda > 1/2$? Explain

For $H_0 : \lambda = 1/2$, $H_1 : \lambda = \lambda_1$ where $\lambda_1 > 1/2$, Neyman-Pearson Theorem says that the test is most powerful when

$$\Lambda = \frac{\sup L(1/2|x)}{\sup L(\lambda_1|x)} = \frac{(1/2)^{10}(\prod_{i=1}^{10} x_i)^{1/2-1}}{(\lambda_1)^{10}(\prod_{i=1}^{10} x_i)^{\lambda_1-1}} = (2\lambda_1)^{-10} e^{(1/2-\lambda_1)\sum_{i=1}^{10} x_i} \overset{set}{\leq} C$$

Take derivative of both sides, $-10\ln(2\lambda_1)(\lambda_1 - 1/2)(-\sum_{i=1}^{10}) \ln x_i \leq \ln C$

For $\lambda_1 - 1/2 \geq 0$,$\Lambda \leq C$ is equivalent $-\sum_{i=1}^{10} \ln x_i \leq C'$, the most powerfule test is $T = -\sum_{i=1}^{10} \ln x_i$

The sturcture of the test T does not involve the actual value of λ_1, then T is UMP.

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4. Find the for the Cramer-Rao lower bound for variance of an unbiased estimator of λ.

$\ln f(x) = \ln \lambda + (\lambda - 1) \ln x_i$

$\frac{\partial}{\partial \lambda} \ln f(x) = \frac{1}{\lambda} + \ln x_i$

$\frac{\partial^2}{\partial \lambda^2} \ln f(x) = -\frac{1}{\lambda^2}$

$I_\lambda = -E[\frac{\partial^2}{\partial \lambda^2} \ln f(x)] = \frac{1}{\lambda^2}$

$$CRLB = \frac{1}{nI_\lambda} = \frac{\lambda^2}{10}$$

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5. Find the MUVE of λ.

$f(x) = \lambda x^{\lambda-1} = \lambda e^{-(\lambda-1)(-\ln x)}$ is a member of the exponential family.

$$L(\lambda) = \lambda^{10} e^{-(\lambda-1)(-\sum_{i=1}^{10} \ln x_i)} = h(x)c(\lambda)e^{W_i(\lambda)t_i(\vec{x})}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0, \lambda > 0$. $W_i(\lambda) = \lambda - 1$ contains an open interval in $\mathbb{R}$, so $T = -\sum_{i=1}^{10} \ln x_i$ is a complete sufficient statistic for λ.

$L(\lambda) = \lambda^{10}(\prod_{i=1}^{10} x_i)^{\lambda-1} = \lambda^{10} e^{(\lambda-1)\sum_{i=1}^{10} \ln x_i}$

$l(\lambda) = 10\ln\lambda + (\lambda - 1)\sum_{i=1}^{10} \ln x_i$

$l'(\lambda) = \frac{10}{\lambda} + \sum_{i=1}^{10} \ln x_i \overset{\text{set}}{=} 0$

$\hat{\lambda}_{MLE} = \frac{10}{-\sum_{i=1}^{10} \ln x_i}$

For $-\sum_{i=1}^{10} \ln x_i \sim Gamma(\alpha = 10, \beta = \frac{1}{\lambda})$

$E[\hat{\lambda}_{MLE}] = 10E[Y^{-1}] = 10\frac{\beta^{-1}\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{10\lambda\Gamma(10-1)}{\Gamma(10)} = \frac{10\lambda}{9}$

Create an unbiased estimator $\hat{\lambda}_U = \frac{9}{10}\hat{\lambda}_{MLE}$.

$E[\hat{\lambda}_U] = E[\frac{9}{10}\hat{\lambda}_{MLE}] = \frac{9}{10} \cdot \frac{10\lambda}{9} = \lambda$

$$\hat{\lambda}_U = \frac{9}{10}\hat{\lambda}_{MLE} = \frac{9}{-\sum_{i=1}^{10} \ln x_i} = \frac{9}{T}$$

From Lehmann-Scheffe Theorem,$\hat{\lambda}_U = \frac{9}{T}$ is the unique MVUE of λ because it is an unbiased estimator of λ and a function of T, which is a complete sufficient statistic for λ.

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6. Show that the MUVE of λ is asymptotically efficient.

When sample size $n \to \infty$, from previou results, $CRLB = \frac{1}{nI_\lambda} = \frac{\lambda^2}{n}$, $-\sum_{i=1}^{n} \ln x_i \sim Gamma(n, \frac{1}{\lambda})$

$\hat{\lambda}_U = \frac{n-1}{-\sum_{i=1}^{n} \ln x_i} = \frac{n-1}{T}, \quad E[\hat{\lambda}_U] = \lambda$

$E[\hat{\lambda}_U^2] = (n-1)^2 E[T^{-2}] = (n-1)^2 \frac{\beta^{-2}\Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{(n-1)^2\lambda^2\Gamma(n-2)}{\Gamma(n)} = \frac{(n-1)\lambda^2}{n-2}$

$Var[\hat{\lambda}_U] = E[\hat{\lambda}_U^2] - E[\hat{\lambda}_U]^2 = \frac{(n-1)\lambda^2}{n-2} - \lambda^2 = \frac{\lambda^2}{n-2}$

$$\lim_{n\to\infty} \frac{CRLB}{Var[\hat{\lambda}_U]} = \lim_{n\to\infty} \frac{\frac{\lambda^2}{n}}{\frac{\lambda^2}{n-2}} = \lim_{n\to\infty} \frac{n-2}{n} = 1$$

Therefore, the MUVE $\frac{n-1}{-\sum_{i=1}^{10} \ln x_i}$ is asymptotically efficient.