1. Assume that $X_1, X_2, ..X_{10}$ is a random sample from a distribution having a p.d.f. of the form $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$$f(x) = \lambda x^{\lambda-1} = \lambda e^{(\lambda-1)\ln x} \implies \ln x \sim Exp()$$

Let $Y_i = -\ln x_i$, $0 < x < 1$, then $X = e^{-y}$, $\frac{dx}{dy} = -e^{-y}$

$g(y) = \lambda(e^{-y})^{\lambda-1}|-e^{-y}| = \lambda e^{-\lambda y} \sim Exp(\lambda), y > 0$

So $\sum_{i=1}^{10} Y_i = -\sum_{i=1}^{10} \ln x_i \sim Gamma(\alpha = 10, \beta = \frac{1}{\lambda}) = \chi^2_{20}$, then,

$$\Lambda = \frac{\sup L(\hat{\lambda}_0|x)}{\sup L(\hat{\lambda}_1|x)} = \frac{(1/2)^{10}(\prod_{i=1}^{10} x_i)^{1/2-1}}{(1)^{10}(\prod_{i=1}^{10} x_i)^{1-1}} = (1/2)^{10}(\prod_{i=1}^{10} x_i)^{-1/2} \leq C$$

Take derivative of both sides, $\Lambda \leq C$ is equivalent $-\sum_{i=1}^{10} \ln x_i \leq C_1'$ or $-\sum_{i=1}^{10} \ln x_i \geq C_2'$.

$\chi^2_{(0.05/2),20} = 34.16961$, $\chi^2_{(1-0.05/2),20} = 9.590777$

So the critical region are $-\sum_{i=1}^{10} \ln x_i \in (0, 9.590777]$ and $-\sum_{i=1}^{10} \ln x_i \in [34.16961, \infty)$

$$L(\lambda) = \lambda^{10}(\prod_{i=1}^{10} x_i)^{\lambda-1} = \lambda^{10} e^{(\lambda-1)\sum_{i=1}^{10} \ln x_i}$$

$$l(\lambda) = 10 \ln \lambda + (\lambda - 1)\sum_{i=1}^{10} \ln x_i$$

$$l'(\lambda) = \frac{10}{\lambda} - \sum_{i=1}^{10} \ln x_i \overset{\text{set}}{=} 0$$

$$\hat{\lambda}_{MLE} = \frac{10}{\sum_{i=1}^{10} \ln x_i}$$

$$E[\hat{\lambda}_{MLE}] = 10E[Y^{-1}] = 10\frac{\beta^{-1}\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{10\lambda\Gamma(10-1)}{\Gamma(10)} = \frac{10\lambda}{9}$$

---

5. Find the MUVE of λ.

- **Step1: Proof sufficient**

From *Fisher–Neyman factorization theorem* (`2019-2-14p5`)

$$f(x|\lambda) = L(\lambda) = \lambda^n(\prod_{i=1}^{n} x_i)^{\lambda-1} = \lambda^n e^{(\lambda-1)\sum_{i=1}^{n} \ln x_i} \cdot 1 = k(t|\lambda)h(\vec{x})$$

$h(\vec{x}) = 1$ is free of λ. So $T = \sum_{i=1}^{n} \ln x_i$ is a sufficient statistic for λ.

- **Step2: Proof complete**

$f(x|\lambda)$ is a member of the exponential family (`2019-2-19p12`). By the Theorem of Complete Statistics in the exponential family

$$f(x|\vec{\lambda}) = \lambda^n e^{\sum_{i=1}^n (\lambda-1)\ln x_i} = h(x)c(\vec{\lambda})e^{\sum_{j=1}^k W_j(\vec{\lambda})t_j(x)}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0, \lambda > 0$. $\{W_1(\vec{\lambda}), .., W_k(\vec{\lambda})\}$ contains an open interval in $\mathbb{R}$, so $T(\vec{x}) = \sum_{i=1}^n \ln x_i)$ is a complete sufficient statistic for λ.

6. Show that the MUVE of λ is asymptotically efficient.

---

b. Find the expected value of $\hat{\lambda}_{MLE}$.

- Method 1
- Method 2

By 2.1.10 Probability integral transformation, let $U_i = F_X(\mathbf{x}|\lambda) = \int_{-\infty}^x \lambda x^{-\lambda-1}dx = 1 - \int_1^x \lambda x^{-\lambda-1}dx = 1 - (-x^{-\lambda}|_1^x) = x^{-\lambda} \sim Uni(0,1)$,

By 5.6.3 the exponential-uniform transformation, $\sum_{i=1}^n \ln X = -\frac{1}{\lambda}\sum_{i=1}^n \ln U_i \sim Gamma(n, \frac{1}{\lambda})$; $(\sum_{i=1}^n \ln x_i)^{-1} \sim Inv-Gamma(n, \frac{1}{\lambda})$.

For a Inv-Gamma(α, β), $f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\frac{\beta}{x}}, x > 0$, $E[x^n] = \frac{\beta^n}{(\alpha-1)\cdots(\alpha-n)}$.

Thus,

$$E[\hat{\lambda}] = E\left[\frac{n}{\sum_{i=1}^n \ln x_i}\right] = nE\left[(\sum_{i=1}^n \ln x_i)^{-1}\right] = \frac{n\lambda}{n-1}$$

—-

c. Find the variance of $\hat{\lambda}_{MLE}$.

From method 1,

$$E[\hat{\lambda}^2] = n^2E[Y^{-2}] = n^2\frac{\beta^{-2}\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{n^2\lambda^2\Gamma(n-2)}{\Gamma(n)} = \frac{n^2\lambda^2(n-3)!}{(n-1)!} = \frac{n^2\lambda^2}{(n-1)(n-2)}$$

From method 2,

For Inv-Gamma(α, β), $E[x^n] = \frac{\beta^n}{(\alpha-1)\cdots(\alpha-n)}$, then

$$E[\hat{\lambda}^2] = E\left[\frac{n^2}{(\sum_{i=1}^n \ln x_i)^2}\right] = n^2E\left[(\sum_{i=1}^n \ln x_i)^{-2}\right] = \frac{n^2\lambda^2}{(n-1)(n-2)}$$

Therefore,

$$Var[\hat{\lambda}^2] = \frac{n^2\lambda^2}{(n-1)(n-2)} - \frac{n^2\lambda^2}{(n-1)^2} = \frac{n^2\lambda^2}{(n-1)}[\frac{1}{n-2} - \frac{1}{n-1}] = \frac{n^2\lambda^2}{(n-1)^2(n-2)}$$

---

d. Using $\hat{\lambda}_{MLE}$, create an unbiased estimator $\hat{\lambda}_U$.

$E[\hat{\lambda}] = \frac{n\lambda}{n-1}$ is a biased estimator.

We can set $\frac{n-1}{n}E[\hat{\lambda}] = E[\frac{n-1}{n} \cdot \frac{n\lambda}{n-1}] = \lambda$

Therefore, $E[\hat{\lambda}_U] = E[\frac{n-1}{n}E[\hat{\lambda}]] = \lambda$

$$\hat{\lambda}_U = \frac{n-1}{n}\hat{\lambda}_{MLE} \quad \text{is an unbiased estimator.}$$

---

e. Find the variance of $\hat{\lambda}_U$.`2019-3-5p12`

$$Var[\hat{\lambda}_U] = Var[\frac{n-1}{n}\hat{\lambda}_{MLE}] = (\frac{n-1}{n})^2 \frac{n^2\lambda^2}{(n-1)^2(n-2)} = \frac{\lambda^2}{n-2}$$