1. $X_1, X_2, ..X_n$ is a random sample from a distribution having a p.d.f of the form. $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find a complete sufficient statistic for λ. Justify your answer

- **Step1: Proof sufficient**

From *Fisher–Neyman factorization theorem* (`2019-2-14p5`)

$$f(x|\lambda) = L(\lambda) = \lambda^n (\prod_{i=1}^{n} x_i)^{\lambda-1} = \lambda^n e^{(\lambda-1)\sum_{i=1}^{n} \ln x_i} \cdot 1 = k(t|\lambda)h(\vec{x})$$

$h(\vec{x}) = 1$ is free of λ. So $T = \sum_{i=1}^{n} \ln x_i$ is a sufficient statistic for λ.

- **Step2: Proof complete**

$f(x|\lambda)$ is a member of the exponential family (`2019-2-19p12`). By the Theorem of Complete Statistics in the exponential family

$$f(x|\vec{\lambda}) = \lambda^n e^{\sum_{i=1}^{n}(\lambda-1)\ln x_i} = h(x)c(\vec{\lambda})e^{\sum_{j=1}^{k} W_j(\vec{\lambda})t_j(x)}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0$, $\lambda > 0$. $\{W_1(\vec{\lambda}), .., W_k(\vec{\lambda})\}$ contains an open interval in $\mathbb{R}$, so $T(\vec{x}) = \sum_{i=1}^{n} \ln x_i)$ is a complete sufficient statistic for λ.

---

2. Let Y_n be the $n^t h$ order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

- **Step1: θ is a location parameter**

Let $x = y - \theta$. For $N(\theta, \sigma^2)$ is a location family of densities (`2018.11.20p7`),

$g(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = f(x) = f(y - \theta)$

Thus, θ is a location parameter.

- **Step2: $Y_n - \bar{Y}$ is location invariant**(`2019-2-21p4-6`)

Let $g = Y_n - \bar{Y} = \frac{1}{n}\sum_{i=1}^{n}(y_{(n)} - y_i)$. Consider the group of transformations defined by $\mathcal{G} = \{g_a(y_1, .., y_n), -\infty < a < \infty\}$, where $g_a = (y_1 + a, .., y_n + a)$. From Definition 6.4.2

$g_{-a}(g_a) = g_{-a}(\frac{1}{n}\sum_{i=1}^{n}[y_{(n)} - (y_1 + a)]) = \frac{1}{n}\sum_{i=1}^{n}[y_{(n)} - (y_1 + a - a)] = g$

$g_{a_2}(g_{a_1}) = g_{a_2}(\frac{1}{n}\sum_{i=1}^{n}[y_{(n)} - (y_1 + a_1)]) = \frac{1}{n}\sum_{i=1}^{n}[y_{(n)} - (y_1 + a_1 + a_2)] = g_{a_1+a_2}$

Thus, the joint distribution of $Y_n - \bar{Y}$ is in $\mathcal{F}$ and hence $\mathcal{F}$ is location invariant under $\mathcal{G}$.

- **step3: $Y_n - \bar{Y}$ is ancillary statistic for θ**

$f(y|\theta)$ is a location parameter family with cdf $F(y - \theta), -\infty < \theta < \infty$. Let $W = Y_n - \bar{Y}$. From Theorem 3.5.6, let $Z_n, \bar{Z}$ idd from $F(y)$ (corresponding to $\theta = 0$ or $f(y|0)$ `2019-2-19p6`) with $Y_n = Z_n + \theta, \bar{Y} = \bar{Z} + \theta$. Thus the cdf of the W is

$F_W(w|\theta) = P_\theta(W \le w) = P_\theta(Y_n - \bar{Y} \le w) = P_\theta(Z_n + \theta - (\bar{Z} + \theta) \le w) = P_\theta(Z_n - \bar{Z} \le w)$

The last probability does not depend on θ because the distribution of $Z_n, \bar{Z}$ does not depend on θ. Thus, the cdf of $W = Y_n - \bar{Y}$ does not depend on θ, $W = Y_n - \bar{Y}$ is an ancillary statistic.

- **step4: For $Y \sim N(\theta, \sigma^2)$, $\bar{Y}$ is sufficient statistic for θ**

Let $Y = f(\mathbf{y}|\theta) \sim N(\theta, \sigma^2)$,

$$f(\vec{y}|\theta) = \prod_{i=1}^{n}(\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(y_i-\theta)^2}{2\sigma^2}}) = \frac{1}{(\sigma\sqrt{2\pi})^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2} = \frac{1}{(\sigma\sqrt{2\pi})^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\bar{y}+\bar{y}-\theta)^2}$$

For $\sum_{i=1}^{n}(y_i - \bar{y}) = 0, \quad \sum_{i=1}^{n}(\bar{y}-\theta)^2 = n(\bar{y}-\theta)^2$, the part of exponent is
$= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}[(y_i-\bar{y})^2 + 2(y_i-\bar{y})(\bar{y}-\theta) + (\bar{y}-\theta)^2] = -\frac{1}{2\sigma^2}[\sum_{i=1}^{n}(y_i-\bar{y})^2 + n(\bar{y}-\theta)^2]$

Let $\bar{Y} = g(\bar{y}|\theta) \sim N(\theta, \sigma^2/n)$, $g(\bar{y}|\theta) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}}e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}$

$\frac{f(\mathbf{y}|\theta)}{g(\bar{y}|\theta)} = \frac{\frac{1}{(\sigma\sqrt{2\pi})^n}e^{-\frac{1}{2\sigma^2}[\sum_{i=1}^{n}(y_i-\bar{y})^2+n(\bar{y}-\theta)^2]}}{\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}} = \frac{1}{\sqrt{n}(\sigma\sqrt{2\pi})^{n-1}}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\bar{y})^2}$

which is free of θ (6.2.2). For every $\vec{y}$ in the sample space, the ratio $f(\vec{y}|\theta)/g(\bar{y}|\theta)$ is constant as a function of θ, then $\bar{Y}$ is a sufficient statistic for θ.

- **step5: $\bar{Y} \sim N(\theta, \sigma^2/n)$ is a complete sufficient statistic for θ.**

Method 1: from `(2019-2-19p7-9)`

For $\bar{Y} \sim N(\theta, \sigma^2/n)$, the family of is $\{\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}} : -\infty < \theta < \infty\}$ Supporse that $E[g(\bar{Y})] = 0 \forall\theta$

$\int_0^\infty g(\bar{y})f(\bar{y})d\bar{y} = \int_0^\infty g(\bar{y})\frac{\sqrt{n}}{\sigma\sqrt{2\pi}}e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}d\bar{y} = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\int_0^\infty g(\bar{y})e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}d\bar{y} = 0$

For $\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \neq 0$, $\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f[u(x)] - v'(x)f[v(x)]$, then

$0 = \frac{d}{d\theta}E[g(\bar{Y})] = \frac{d}{d\theta}\left[\int_{-\infty}^{\infty} g(\bar{y})e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}d\bar{y}\right] = 0 - \theta'[g(\theta)e^{-\frac{n(\theta-\theta)^2}{2\sigma^2}}] = -g(\theta)$

So $g(\theta) = 0, \forall\theta$, then $P(g(T) = 0) = 1$. Thus, $\bar{Y}$ is a complete statistic.

Method 2: from `(2019-2-19p12)`

$f(y|\theta)$ is a member of the exponential family,

$f(y|\vec{\theta}) = \frac{1}{(\sigma\sqrt{2\pi})^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y-\theta)^2} = e^{-\frac{\sum_{i=1}^{n}y_i^2}{2\sigma^2}}\frac{e^{-\frac{n\theta^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}e^{\frac{\theta}{\sigma^2}n\bar{y}} = h(y)c(\vec{\theta})e^{\sum_{j=1}^{k}W_j(\vec{\theta})t_j(y)}$

For $\{W_1(\vec{\theta}), .., W_k(\vec{\theta})\}$ contains an open interval in $\mathbb{R}$, so $T(\vec{y}) = \bar{y}$ is a complete sufficient statistic.

- **step6: By Basu's theorem, an acillary statistc $Y_n - \bar{Y}$ and a complete sufficient statistic $\bar{Y}$ are independent.** `(2019-2-19p10)`

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3. Suppose that $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta e^{-\theta x}, x > 0$. Assume that the prior distribution of θ is $\pi(\theta) = \lambda e^{-\lambda\theta}, \theta > 0$

a. Find the posterior distribution $\pi(\theta|\vec{x})$.'2019-2-26p8-9,p11-p13' '2019-2-28p8 Exapmle 2.3.8'

For $L(\theta) = \prod_{i=1}^{n}(\theta e^{-\theta x}) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}, \quad \pi(\theta) = \lambda e^{-\lambda\theta},$

and the kernel of a function is the main part of the function, the part that remains when constants are disregarded. that is

$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \lambda e^{-\lambda\theta} \propto \theta^{n+1-1} e^{-\theta(\lambda + \sum x_i)}, \ x > 0, \theta > 0$

which is $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ distribution.

---

b. Find the Bayes estimator of θ, assuming square-error loss. '2019-2-28p1'

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. For Gamma distribution, $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\lambda + \sum_{i=1}^{n} x_i}$$

which is the posterior mean.

---

c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$f(\vec{x}|\theta) = L(\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$

$l(\theta) = n \ln \theta - \theta \sum_{i=1}^{n} \ln x_i$

$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \ln x_i \overset{\text{set}}{=} 0$

$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln x_i} = \frac{1}{\bar{x}}$

$\pi(\theta) = \lambda e^{-\lambda\theta} \sim Expo(\lambda), E[\theta] = \frac{1}{\lambda}$

Thus, we can write this estimator as

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(\lambda + n\bar{x})} = \frac{1}{\frac{1}{n+1}(\frac{1}{1/\lambda} + \frac{n}{1/\bar{x}})}$$

which is the weighted hamonic mean of $1/\lambda$ and $1/\bar{x}$. $1/\lambda$ is the prior mean and $1/\bar{x}$ is the MLE of θ.

---

d. Find the Bayes estimator of θ, assuming absolute loss. '2019-2-28p4,9'

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = median[\theta|\vec{x}]$$

The posterior median is $\hat{\theta} = F^{-1}(\frac{1}{2})$. $F(x)$ is the $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum_{i=1}^{n} x_i})$ cdf. $\hat{\theta}$ doesn't have a closed form.

---

e. Find the Bayes estimator of θ, assuming binary loss. '2019-2-28p5-6'

Suppose $L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}$.

$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$

To minimized this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = mode[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\lambda + \sum_{i=1}^{n} x_i}$$

which is the posterior mode.

---

4. Redo all of problem 3, using the non-informative prior $\pi(\theta) = 1, \theta > 0$. Note that this is not a valid density function since its intergral is infinite, but proceed with it anyway

a. Find the posterior distribution $\pi(\theta|\vec{x})$.'2019-2-26p8-9'

For $\pi(\theta) = 1, \theta > 0$, $L(\theta) = \theta^n e^{-\theta \sum x_i}$, from the kernel of function,

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^{n+1-1} e^{-\theta \sum x_i} \sim Gamma(\alpha = n+1, \beta = \frac{1}{\sum_{i=1}^{n} x_i})$$

---

b. Find the Bayes estimator of θ, assuming square-error loss

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\sum_{i=1}^{n} x_i}$$

which is the posterior mean.

---

c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

For $1/\bar{x}$ is the MLE of θ, we can write this estimator as

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(1 \times 0 + n\bar{x})} = \lim_{c \to \infty} \frac{1}{\frac{1}{n+1}(\frac{1}{c} + \frac{n}{1/\bar{x}})}$$

which is the weighted hamonic mean of c and $1/\bar{x}$. c is the prior constant as $c \to \infty$, $1/\bar{x}$ is the MLE of θ.

---

d. Find the Bayes estimator of θ, assuming absolute loss

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = median[\theta|\vec{x}]$$

The posterior median is $\hat{\theta} = F^{-1}(\frac{1}{2})$. $F(x)$ is the $Gamma(\alpha = n+1, \beta = \frac{1}{\sum_{i=1}^{n} x_i})$ cdf. $\hat{\theta}$ doesn't have a closed form.

---

e. Find the Bayes estimator of θ, assuming binary loss

Suppose $L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}$. $E[L_0(\hat{\theta})|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$ To minimized this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = mode[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

which is the posterior mode.

---

5. Let $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta x^{-\theta-1}, x_i > 1, \theta > 2$.

a. Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ.'2019-2-2p12'

$f(\vec{x}|\theta) = L(\theta) = \theta^n (\prod_{i=1}^{n} x_i)^{-\theta-1} = \theta^n e^{(-\theta-1)\sum_{i=1}^{n} \ln x_i}$ $l(\theta) = n \ln \theta - (\theta + 1) \sum_{i=1}^{n} \ln x_i$ $l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \ln x_i \overset{\text{set}}{=} 0$

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

---

b. Find the expected value of $\hat{\theta}_{MLE}$.

- Method 1

Let $Y_i = \ln x_i$, then $X = e^y$, $\frac{dx}{dy} = e^y$ $g(Y) = \theta(e^y)^{-\theta-1}e^y = \theta e^{-y\theta}, y > 0$

So $Y_i = \ln x_i \sim Gamma(\alpha = n, \beta = \frac{1}{\theta})$

We know if $Y \sim Gamma(\alpha, \beta)$, them $E[Y^k] = \frac{\beta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}$, then

$$E[\hat{\theta}] = nE[Y^{-1}] = n\frac{\beta^{-1}\Gamma(\alpha - 1)}{\Gamma(\alpha)} = \frac{n\theta\Gamma(n-1)}{\Gamma(n)} = \frac{n\theta(n-2)!}{(n-1)!} = \frac{n\theta}{n-1}$$

- Method 2

By 2.1.10 Probability integral transformation, let $U_i = F_X(\mathbf{x}|\theta) = \int_{-\infty}^{x} \theta x^{-\theta-1}dx = 1 - \int_1^x \theta x^{-\theta-1}dx = 1 - (-x^{-\theta}|_1^x) = x^{-\theta} \sim Uni(0,1)$,

By 5.6.3 the exponential-uniform transformation, $\sum_{i=1}^{n} \ln X = -\frac{1}{\theta}\sum_{i=1}^{n} \ln U_i \sim Gamma(n, \frac{1}{\theta})$; $(\sum_{i=1}^{n} \ln x_i)^{-1} \sim Inv - Gamma(n, \frac{1}{\theta})$.

For a Inv-Gamma(α, β), $f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\beta}{x}}, x > 0$, $E[x^n] = \frac{\beta^n}{(\alpha-1)\cdots(\alpha-n)}$.

Thus,

$$E[\hat{\theta}] = E\left[\frac{n}{\sum_{i=1}^{n} \ln x_i}\right] = nE\left[(\sum_{i=1}^{n} \ln x_i)^{-1}\right] = \frac{n\theta}{n-1}$$

—-

c. Find the variance of $\hat{\theta}_{MLE}$.

From method 1,

$$E[\hat{\theta}^2] = n^2 E[Y^{-2}] = n^2 \frac{\beta^{-2}\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{n^2\theta^2\Gamma(n-2)}{\Gamma(n)} = \frac{n^2\theta^2(n-3)!}{(n-1)!} = \frac{n^2\theta^2}{(n-1)(n-2)}$$

From method 2,

For Inv-Gamma(α, β), $E[x^n] = \frac{\beta^n}{(\alpha-1)\cdots(\alpha-n)}$, then

$$E[\hat{\theta}^2] = E\left[\frac{n^2}{(\sum_{i=1}^n \ln x_i)^2}\right] = n^2 E\left[(\sum_{i=1}^n \ln x_i)^{-2}\right] = \frac{n^2\theta^2}{(n-1)(n-2)}$$

Therefore,

$$Var[\hat{\theta}^2] = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)}[\frac{1}{n-2} - \frac{1}{n-1}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

---

d. Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

$E[\hat{\theta}] = \frac{n\theta}{n-1}$ is a biased estimator.

We can set $\frac{n-1}{n}E[\hat{\theta}] = E[\frac{n-1}{n} \cdot \frac{n\theta}{n-1}] = \theta$

Therefore, $E[\hat{\theta}_U] = E[\frac{n-1}{n}E[\hat{\theta}]] = \theta$

$$\hat{\theta}_U = \frac{n-1}{n}\hat{\theta}_{MLE} \quad \text{is an unbiased estimator.}$$

---

e. Find the variance of $\hat{\theta}_U$.‘2019-3-5p12‘

$$Var[\hat{\theta}_U] = Var[\frac{n-1}{n}\hat{\theta}_{MLE}] = (\frac{n-1}{n})^2 \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2}$$

6. Refer to problem 5.

a. Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ.‘2019-2-21p8 7.2.1‘

$EX = \mu = \int_1^\infty x\theta x^{-\theta-1}dx = \theta\frac{1}{-\theta+1}x^{-\theta+1}\Big|_1^\infty = \frac{\theta}{\theta-1}$

Set $\bar{X} = \frac{\theta}{\theta-1} \implies \theta\bar{x} - \bar{x} = \theta \implies \theta(\bar{x}-1) = \bar{x}$,

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1}$$

---

b. Using the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.‘2019-3-5p1,2019-3-12‘

For $EX = \mu = \frac{\theta}{\theta-1}$

$E[X^2] = \int_1^\infty x^2 \theta x^{-\theta-1} dx = \theta \frac{1}{-\theta+2} x^{-\theta+2}\Big|_1^\infty = \frac{\theta}{\theta-2}$

$Var[X] = \sigma^2 = E[X^2] - E[X]^2 = \frac{\theta}{\theta-2} - (\frac{\theta}{\theta-1})^2$

$$= \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-1)^2(\theta-2)} = \frac{\theta^3 - 2\theta^2 + \theta - \theta^3 + 2\theta^2}{(\theta-1)^2(\theta-2)} = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

Use a 2^{nd} order Taylar series

$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0)\frac{(x-x_0)^2}{2} + R$

Consider $g(x) = \frac{x}{x-1}, \quad g'(x) = \frac{(x-1)\times 1 - x \times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}, \quad g''(x) = \frac{2}{(x-1)^3}$

Choose $x_0 = EX = \mu$

$g(x) = \frac{x}{x-1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(x-\mu) + \frac{2}{(\mu-1)^3}\frac{(x-\mu)^2}{2}$

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(\bar{x}-\mu) + \frac{1}{(\mu-1)^3}(\bar{x}-\mu)^2$$

$$E[\hat{\theta}_{MOM}] \approx \frac{\mu}{\mu-1} + 0 + \frac{1}{(\mu-1)^3}\frac{\sigma^2}{n} = \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1} - 1} + \frac{1}{(\frac{\theta}{\theta-1} - 1)^3} \cdot \frac{1}{n} \cdot \frac{\theta}{(\theta-1)^2(\theta-2)} = \theta + \frac{\theta(\theta-1)}{n(\theta-2)}$$

---

c. Using the delta method to approximate the variance of $\hat{\theta}_{MOM}$.'2019-3-5p3'

$Var[\hat{\theta}_{MOM}] \approx Var[g(x_0) + g'(x_0)(x - x_0)] = Var[\frac{\mu}{\mu-1} + \frac{1}{(\mu-1)^2}(\bar{x}-\mu)]$

$$= \frac{1}{(\mu-1)^4}\frac{\sigma^2}{n} = \frac{1}{(\frac{\theta}{\theta-1} - 1)^4} \cdot \frac{1}{n} \cdot \frac{\theta}{(\theta-1)^2(\theta-2)} = \frac{\theta(\theta-1)^2}{n(\theta-2)}$$

.