# 565 mid-term

## model

$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, for $\varepsilon_{ij} \sim iidN(0, \sigma^2), i = 1, 2, , a, j = 1, 2, , n$, $\sum_{i=1}^{a} \tau_i = 0$

Random effects: $\tau_i \sim iidN(0, \sigma_\tau^2)$,$\tau_i$ and $\varepsilon_{ij}$ indep

$y_{ij} = (\mu + \tau_i + \beta_j) + \varepsilon_{ij}$, for $\varepsilon_{ij} \sim iidN(0, \sigma^2), i = 1, 2, , a, j = 1, 2, , b$

Fixed effects: $\sum_{i=1}^{a} \tau_i = 0, \sum_{j=1}^{b} \beta_j = 0$ Random effects: $\beta_j \sim iidN(0, \sigma_\beta^2)$

## least square and normal equation

CRD:$SSE = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \mu - \tau_i)^2$

$\frac{\partial SSE}{\partial \mu}\Big|_{\hat\mu,\hat\tau_i} = 2 \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \hat\mu - \hat\tau_i)(-1) = 0$ $y_{..} = an\hat\mu + n \sum_{i=1}^{a} \hat\tau_i$

$\frac{\partial SSE}{\partial \tau_i}\Big|_{\hat\mu,\hat\tau_i} = 2 \sum_{j=1}^{n} (y_{ij} - \hat\mu - \hat\tau_i)(-1) = 0$ $y_{i.} = n\hat\mu + n\hat\tau_i$

RCBD: $SSE = \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \mu - \tau_i - \beta_j)^2$

$\frac{\partial SSE}{\partial \mu}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j} = 2 \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \hat\mu - \hat\tau_i - \hat\beta_j)(-1) = 0$

$\frac{\partial SSE}{\partial \tau_i}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j} = 2 \sum_{j=1}^{b} (y_{ij} - \hat\mu - \hat\tau_i - \hat\beta_j)(-1) = 0$

$\frac{\partial SSE}{\partial \beta_j}\Big|_{\hat\mu,\hat\tau_i,\hat\beta_j} = 2 \sum_{i=1}^{a} (y_{ij} - \hat\mu - \hat\tau_i - \hat\beta_j)(-1) = 0$

$y_{..} = ab\hat\mu + b \sum_{i=1}^{a} \hat\tau_i + a \sum_{j=1}^{b} \hat\beta_j$

$y_{i.} = b\hat\mu + b\hat\tau_i + \sum_{j=1}^{b} \hat\beta_j$

$y_{.j} = a\hat\mu + \sum_{i=1}^{a} \hat\tau_i + a\hat\beta_j$

## hypothesis test

| model | H0 | H1 |
|---|---|---|
| FT-CRD | $\mu_1 = \mu_2 = .. = \mu_i$ | at least two of $\mu_i$ are different |
| RT-CRD | $\sigma_\tau^2 = 0$ | $\sigma_\tau^2 > 0$ |
| FB-RCBD | $\mu_1 = \mu_2 = .. = \mu_i$ | at least two of $\mu_i$ are different |
| -Block | $\beta_1 = \beta_2 = .. = \beta_j = 0$ | at least one of $\beta_j \neq 0$ |
| RB-RCBD | $\sigma_\beta^2 = 0$ | $\sigma_\beta^2 > 0$ |
| Contrast | $\Gamma = \sum_{i=1}^{a} c_i\mu_i = 0$ | $\Gamma = \sum_{i=1}^{a} c_i\mu_i \neq 0$ |

## ANOVA

| | SS | df | MS | F |
|---|---|---|---|---|
| $SS_{Trt}$ | $n \sum_{i=1}^{a} (\bar y_{i.} - \bar y_{..})^2$ | $a - 1$ | $\frac{SS_{Trt}}{a-1}$ | $\frac{MS_{Trt}}{MS_E}$ |
| $SS_E$ | $\sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar y_{i.})^2$ | $a(n - 1)$ | $\frac{SS_E}{N-a}$ | |
| $SS_T$ | $\sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar y_{..})^2$ | $an - 1$ | | |
| $SS_{Trt}$ | $b \sum_{i=1}^{a} (\bar y_{i.} - \bar y_{..})^2$ | $a - 1$ | $\frac{SS_{Trt}}{a-1}$ | $\frac{MS_{Trt}}{MS_E}$ |
| $SS_{Blk}$ | $a \sum_{j=1}^{b} (\bar y_{.j} - \bar y_{..})^2$ | $b - 1$ | $\frac{SS_{Trt}}{b-1}$ | $\frac{MS_{Blk}}{MS_E}$ |
| $SS_E$ | $SS_T - SS_{Trt} - SS_{Blk}$ | $(a - 1)(b - 1)$ | $\frac{SS_E}{df_E}$ | |
| $SS_T$ | $\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar y_{..})^2$ | $ab - 1$ | | |

[f.]$SS_{Trt} = \frac{1}{n} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}$

[r.]$SS_{Trt} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}$ $SS_{Blk} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N}$

$SSE = \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar y_{i.} - \bar y_{.j} + \bar y_{..})^2$

$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N}$, $S_i^2 = \frac{\sum_{j=1}^{n} (y_{ij} - \bar y_{i.})^2}{n-1}$

## distribution

| | $y_{ij}$ | $\bar y_{i.}$ | $\frac{y_{i.} - \mu_i}{\frac{\sigma}{\sqrt n}}$ | $\bar y_{i.} - \bar y_{j.}$ | $\frac{\bar y_{i.} - \bar y_{j.} - (\mu_i - \mu_j)}{\frac{2\sigma}{\sqrt n}}$ |
|---|---|---|---|---|---|
| N | $(\mu + \tau_i, \sigma^2)$ | $(\mu + \tau_i, \frac{\sigma^2}{n})$ | $(0, 1)$ | $(\mu_i - \mu_j, \frac{2\sigma^2}{n})$ | $(0, 1)$ |

| | | | |
|---|---|---|---|
| $t_{df_E}$ | $\frac{\frac{y_{i.} - \mu_i}{\frac{\sigma}{\sqrt n}}}{\sqrt{\frac{SS_E}{\sigma^2(N-a)}}} = \frac{y_{i.} - \mu_i}{\sqrt{\frac{SS_E}{n(df_E)}}} = \frac{y_{i.} - \mu_i}{\sqrt{\frac{MS_E}{n}}}$ | $\frac{\bar y_{i.} - \bar y_{j.} - (\mu_i - \mu_j)}{\sqrt{\frac{2MS_E}{n}}}$ | $\frac{C}{\sqrt{\widehat{Var(C)}}}$ |

| $\chi^2$ | $\frac{SS_E}{\sigma^2}$ | $\frac{SS_{Trt}}{\sigma^2}$ | $\frac{SS_{Blk}}{\sigma^2}$ | $SS_C$ |
|---|---|---|---|---|
| df | $E$ | $Trt$ | $Blk$ | 1 |

| F | $\frac{MS_{Trt}}{MS_E}$ | $\frac{(MS_{Blk}}{MS_E}$ | $\frac{\sum_{i=1}^{a} c_i\bar y_{i.})^2}{\frac{MS_E}{n} \sum_{i=1}^{a} c_i^2}$ |
|---|---|---|---|
| df | $(Trt, E)$ | $(Blk, E)$ | $(C = 1, E)$ |

## contrast

$\Gamma = \sum_{i=1}^{a} c_i\mu_i, C = \sum_{i=1}^{a} c_i\bar y_i, E(C) = \Gamma; \sum_{i=1}^{a} c_i = 0$;orthogonal $\sum_{i=1}^{a} c_i d_i = 0$

$Var\left[\sum_{i=1}^{a} c_i y_{i.}\right] = \sigma^2 \sum_{i=1}^{a} n_i c_i^2$

$SS_C = \frac{(\sum_{i=1}^{a} c_i\bar y_i)^2}{\frac{1}{n} \sum_{i=1}^{a} c_i^2} = SS_{Trt}$

## CI

| CI | balanced | unbalanced |
|---|---|---|
| $\mu_i$ | $\bar y_{i.} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{MS_E}{n}}$ | $\bar y_{i.} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{MS_E}{n_i}}$ |
| $\mu_i - \mu_j$ | $\bar y_{i.} - \bar y_{j.} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{2MS_E}{n}}$ | $\bar y_{i.} - \bar y_{j.} \pm t_{\frac{\alpha}{2}} \sqrt{MS_E(\frac{1}{n_i} + \frac{1}{n_j}}$ |
| $\Gamma$ | $\sum_{i=1}^{a} c_i\bar y_i \pm t_{\frac{\alpha}{2}} sqrt \frac{MS_E}{n} \sum_{i=1}^{a} c_i^2$ | |

## expectation and variance

$y_{..} = \sum_{i=1}^{a} (y_{i.} = \sum_{j=1}^{n} y_{ij})$,$\mu = \frac{\sum_{i=1}^{a} \mu_i}{a}$

$\bar y_{..} = \frac{1}{N} \left[y_{..} = \sum_{i=1}^{a} (y_{i.} = \sum_{j=1}^{n} y_{ij})\right] = \frac{1}{a} \sum_{i=1}^{a} \left[\bar y_{i.} = \frac{1}{n}(y_{i.} = \sum_{j=1}^{n} y_{ij})\right]$

| | $y_{ij}$ | $\bar y_{..}$ | $\bar y_{i.}$ | $\bar y_{.j}$ |
|---|---|---|---|---|
| | | $\frac{\sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}}{N}$ | $\frac{\sum_{j=1}^{n} y_{ij}}{n}$ | $(\sum_{i=1}^{a} y_{ij})a$ |
| E[f.] | $\mu$ | $\mu$ | $\mu + \tau_i$ | $\mu + \beta_j$ |
| V[f.] | $\sigma^2$ | $\frac{\sigma^2}{na}$ | $\frac{\sigma^2}{n}$ | $\frac{\sigma^2}{a}$ |
| E[r.] | $\mu$ | $\mu$ | $\mu$ | $\mu$ |
| V[r.] | $\sigma^2 + \sigma_\tau^2$ | $\frac{\sigma_\tau^2}{a} + \frac{\sigma^2}{na}$ | $\frac{\sigma^2}{n} + \sigma_\tau^2$ | $\frac{\sigma^2}{a} + \sigma_\beta^2$ |

| $\hat\mu$ | $\hat\mu_i$ | $\hat\tau_i$ | $\hat\beta_j$ | $\hat y_{ij}$ | $\varepsilon_{ij}$ |
|---|---|---|---|---|---|
| $\bar y_{..}$ | $\bar y_{i.}$ | $\bar y_{i.} - \bar y_{..}$ | $\bar y_{.j} - \bar y_{..}$ | $\bar y_{i.} + \bar y_{.j} - \bar y_{..}$ | $y_{ij} - \bar y_{..}$ |
| $\frac{1}{an} y_{..} - \frac{1}{a} \sum_{i=1}^{a} \hat\tau_i$ | | $\frac{1}{a} y_{i.} - \hat\mu$ | | | |

| | E[f.] | E[r.] |
|---|---|---|
| $MS_{Trt}$ | $\sigma^2 + \frac{n}{a-1} \sum_{i=1}^{a} \tau_i^2$ | $\sigma^2 + n\tau_i^2$ |
| $MS_{Trt}$ | $\sigma^2 + \frac{b}{a-1} \sum_{i=1}^{a} \tau_i^2$ | |
| $MS_{Blk}$ | $\sigma^2 + \frac{a}{b-1} \sum_{j=1}^{b} \beta_j^2$ | $\sigma^2 + a\beta_j^2$ |
| $MS_E$ | $\sigma^2$ | $\sigma^2$ |

| | $\hat\sigma^2$ | $\hat\sigma_{Trt}^2$ | $\hat\sigma_{Blk}^2$ | $\widehat{Var}(\bar y_{i.})$ | $\hat X$ |
|---|---|---|---|---|---|
| f. | $MSE$ | | | $\frac{MS_E}{n}$ | |
| r. | $MSE$ | $\frac{MS_{Trt} - MS_E}{n}$ | $\frac{MS_{Blk} - MS_E}{a}$ | $\frac{MS_{Trt}}{n}$ | $\frac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)}$ |

| FT- | CRD | E | Var[] |
|---|---|---|---|
| $y_{ij} - \bar y_{i.}$ | $\varepsilon_{ij} - \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n}$ | 0 | $\frac{n-1}{n}\sigma^2$ |
| $\bar y_{i.} - \bar y_{..}$ | $\tau_i + \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}}{an}$ | $\tau_i$ | $\frac{(a-1)\sigma^2}{an}$ |
| **RT-** | **CRD** | **E** | **Var[]** |
| $y_{ij} - \bar y_{i.}$ | $\varepsilon_{ij} - \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n}$ | 0 | $\frac{n-1}{n}\sigma^2$ |
| $\bar y_{i.} - \bar y_{..}$ | $\tau_i - \frac{\sum_{i=1}^{a} \tau_i}{a} + \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}}{an}$ | 0 | $\frac{(a-1)(n\sigma_\tau^2 + \sigma^2}{an}$ |
| **F-** | **RCBD** | **E** | **Var[]** |
| $y_{ij} - \bar y_{i.}$ | $\beta_j + \varepsilon_{ij} - \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b}$ | $\beta_j$ | $\frac{(b-1)\sigma^2}{b}$ |
| $\bar y_{i.} - \bar y_{..}$ | $\tau_i + \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}$ | $\tau_i$ | $\frac{(a-1)\sigma^2}{ab}$ |
| $y_{ij} - \bar y_{.j}$ | $\tau_i + \varepsilon_{ij} - \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a}$ | $\tau_i$ | $\frac{(a-1)\sigma^2}{a}$ |
| $\bar y_{.j} - \bar y_{..}$ | $\beta_j + \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}$ | $\beta_j$ | $\frac{(b-1)\sigma^2}{ab}$ |
| **R-** | **RCBD** | **E** | **Var[]** |
| $y_{ij} - \bar y_{i.}$ | $\beta_j - \frac{\sum_{j=1}^{b} \beta_j}{b} + \varepsilon_{ij} - \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b}$ | 0 | $\frac{(b-1)(\sigma_\beta^2 + \sigma^2)}{b}$ |
| $\bar y_{i.} - \bar y_{..}$ | $\tau_i + \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}$ | $\tau_i$ | $\frac{(a-1)\sigma^2}{ab}$ |
| $y_{ij} - \bar y_{.j}$ | $\tau_i + \varepsilon_{ij} - \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a}$ | $\tau_i$ | $\frac{(a-1)\sigma^2}{a}$ |
| $\bar y_{.j} - \bar y_{..}$ | $\beta_j - \frac{\sum_{j=1}^{b} \beta_j}{b} + \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a} - \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}$ | 0 | $\frac{(b-1)(a\sigma_\beta^2 + \sigma^2}{ab}$ |

$y_{ij} - \bar y_{i.} - \bar y_{.j} + \bar y_{..} = \varepsilon_{ij} - \frac{\sum_{i=1}^{a} \varepsilon_{ij}}{a} - \frac{\sum_{j=1}^{b} \varepsilon_{ij}}{b} + \frac{\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}}{ab}$

$E[] = 0, Var = \frac{(a-1)(b-1)\sigma^2}{ab}$

## covariance

R-CRD: $Cov(y_{ij}, y_{kl}) = \{\, 0\ , i \neq k; \sigma_\tau^2, i = k, j \neq l; \sigma^2 + \sigma_\tau^2, i = k, j = l$

| | $\varepsilon_{ij}$ | $\sum_{i=1}^{a} \varepsilon_{ij}$ | $\sum_{j=1}^{b} \varepsilon_{ij}$ | $\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}$ |
|---|---|---|---|---|
| $\varepsilon_{ij}$ | $\sigma^2$ | $\sigma^2$ | $\sigma^2$ | $\sigma^2$ |
| $\sum_{i=1}^{a} \varepsilon_{ij}$ | | $a\sigma^2$ | $\sigma^2$ | $a\sigma^2$ |
| $\sum_{j=1}^{b} \varepsilon_{ij}$ | | | $b\sigma^2$ | $b\sigma^2$ |
| $\sum_{i=1}^{a} \sum_{j=1}^{b} \varepsilon_{ij}$ | | | | $ab\sigma^2$ |

$Cov[\varepsilon_{ij}, \varepsilon_{ij}] = Cov[\sum_{i1}^{a} \varepsilon_{ij}, \sum_{j=1}^{b} \varepsilon_{ij}] = \sigma^2$,$Cov[\tau_i, \tau_i] = {}^2_\tau$,$Cov[\beta_j, \beta_j] = {}^2_\beta$

$Cov[\varepsilon_{ij}, \tau_i] = Cov[\varepsilon_{ij}, \beta_j] = Cov[\tau_i, \beta_j] = Cov[\varepsilon_{ij}, \varepsilon_{ik}] = 0, k \neq j$

$Cov[\varepsilon_{ij}, \sum_{j=1}^{n} \varepsilon_{ij}] = Cov[\varepsilon_{ij}, (\varepsilon_{i1} + .. + \varepsilon_{ij} .. + \varepsilon_{in})] = Cov[\varepsilon_{ij}, \varepsilon_{i1}] + .. + Cov[\varepsilon_{ij}, \varepsilon_{ij}] .. + Cov[\varepsilon_{ij}, \varepsilon_{in}] = \sigma^2$

$Cov[\sum_{j=1}^{n} \varepsilon_{ij}, \sum_{i=1}^{a} \sum_{j=1}^{n} \varepsilon_{ij}] = Cov[(\varepsilon_{i1} + \varepsilon_{i2} .. + \varepsilon_{in}), (\varepsilon_{i1} + \varepsilon_{i2} .. + \varepsilon_{in})] = Cov[\varepsilon_{i1}, \varepsilon_{i1}] + Cov[\varepsilon_{i2}, \varepsilon_{i2}] .. + Cov[\varepsilon_{in}, \varepsilon_{in}] = n\sigma^2$

$Var[y_{ij} - \bar{y}_{i.}] =$

FT-CRD;RT-CRD

$Var[\varepsilon_{ij} - \frac{\sum_{j=1}^{n} \varepsilon_{ij}}{n}] =, Var[\varepsilon_{ij}] + \frac{\sum_{j=1}^{n} Var[\varepsilon_{ij}]}{n^2} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^{n} \varepsilon_{ij}]}{n}, = \sigma^2 + \frac{\sigma^2}{n} - \frac{2}{n}\sigma^2 = \frac{n-1}{n}\sigma^2$

FB-RCBD

$Var[\beta_j + \varepsilon_{ij} - \frac{\sum_{j=1}^{b}\varepsilon_{ij}}{b}] =, Var[\beta_j] + Var[\varepsilon_{ij}] + \frac{\sum_{j=1}^{b} Var[\varepsilon_{ij}]}{b^2} - 2Cov[\beta_j,\varepsilon_{ij}] - \frac{2Cov[\beta_j, \sum_{j=1}^b \varepsilon_{ij}]}{b} - \frac{2Cov[\varepsilon_{ij}, \sum_{j=1}^b \varepsilon_{ij}]}{b}, = 0 + \sigma^2 + \frac{\sigma^2}{b} - 0 - 0 - \frac{2\sigma^2}{b} = \frac{(b-1)\sigma^2}{b}$

RB-RCBD

$Var[\beta_j - \frac{\sum_{j=1}^b \beta_j}{b} + \varepsilon_{ij} - \frac{\sum_{j=1}^b \varepsilon_{ij}}{b}] =$

$Var[\beta_j] + \frac{\sum_{j=1}^b Var[\beta_j]}{b^2} + Var[\varepsilon_{ij}] + \frac{\sum_{j=1}^b Var[\varepsilon_{ij}]}{b^2} - \frac{2Cov[\beta_j, \sum_{j=1}^b \beta_j]}{b} + 2Cov[\beta_j,\varepsilon_{ij}] - \frac{2Cov[\beta_j,\sum_{j=1}^b\varepsilon_{ij}]}{b} - \frac{2Cov[\sum_{j=1}^b \beta_j, \varepsilon_{ij}]}{b}$

$+\frac{2Cov[\sum_{j=1}^b\beta_j, \sum_{j=1}^b\varepsilon_{ij}]}{b^2} - \frac{2Cov[\varepsilon_{ij},\sum_{j=1}^b\varepsilon_{ij}]}{b} = \sigma_\beta^2 + \frac{\sigma_\beta^2}{b} - \frac{2\sigma_\beta^2}{b} + \sigma^2 + \frac{\sigma^2}{b} - \frac{2\sigma^2}{b} = \frac{(\sigma_\beta^2+\sigma^2)(b-1)}{b}$

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$Var[\bar{y}_{i.} - \bar{y}_{..}] =$

FT-CRD

$Var[\tau_i + \frac{\sum_{j=1}^n \varepsilon_{ij}}{n} - \frac{\sum_{i=1}^a\sum_{j=1}^n \varepsilon_{ij}}{an}] =$

$Var[\tau_i] + \frac{\sum_{j=1}^n Var[\varepsilon_{ij}]}{n^2} + \frac{\sum_{i=1}^a\sum_{j=1}^n Var[\varepsilon_{ij}]}{a^2n^2} + \frac{2Cov[\tau_i,\sum_{j=1}^n\varepsilon_{ij}]}{n} - \frac{2Cov[\tau_i,\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}]}{an} - \frac{2Cov[\sum_{j=1}^n\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}]}{an^2},$

$= 0 + \frac{\sigma^2}{n} + \frac{\sigma^2}{an} - \frac{2\sigma^2}{an} = \frac{(a-1)\sigma^2}{an}$

RT-CRD

$Var[\tau_i - \frac{\sum_{i=1}^a \tau_i}{a} + \frac{\sum_{j=1}^n\varepsilon_{ij}}{n} - \frac{\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}}{an}] =$

$Var[\tau_i] + \frac{\sum_{i=1}^a Var[\tau_i]}{a^2} + \frac{\sum_{j=1}^n Var[\varepsilon_{ij}]}{n^2} + \frac{\sum_{i=1}^a\sum_{j=1}^n Var[\varepsilon_{ij}]}{a^2n^2} - \frac{2Cov[\tau_i,\sum_{i=1}^a\tau_i]}{a} + \frac{2Cov[\tau_i,\sum_{j=1}^n\varepsilon_{ij}]}{n} - \frac{2Cov[\tau_i,\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}]}{an} - \frac{2Cov[\sum_{i=1}^a\tau_i,\sum_{j=1}^n\varepsilon_{ij}]}{an}$

$+\frac{2Cov[\sum_{i=1}^a\tau_i,\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}]}{a^2n} - \frac{2Cov[\sum_{j=1}^n\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^n\varepsilon_{ij}]}{an^2} = \sigma_\tau^2 + \frac{\sigma_\tau^2}{a} - \frac{2\sigma_\tau^2}{a} + \frac{\sigma^2}{n} + \frac{\sigma^2}{an} - \frac{2\sigma^2}{an} = \frac{(n\sigma_\tau^2+\sigma^2)(a-1)}{an}$

FB-RCBD;RB-RCBD

$Var[\tau_i + \frac{\sum_{j=1}^b\varepsilon_{ij}}{b} - \frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}] =$

$Var[\tau_i] + \frac{\sum_{j=1}^b Var[\varepsilon_{ij}]}{b^2} + \frac{\sum_{i=1}^a\sum_{j=1}^b Var[\varepsilon_{ij}]}{a^2b^2} + \frac{2Cov[\tau_i,\sum_{j=1}^b\varepsilon_{ij}]}{b} - \frac{2Cov[\tau_i,\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{j=1}^b\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab^2} = 0 + \frac{\sigma^2}{b} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(a-1)\sigma^2}{ab}$

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$Var[y_{ij} - \bar{y}_{.j}] =$

FB-RCBD; RB-RCBD

$Var[\tau_i + \varepsilon_{ij} - \frac{\sum_{i=1}^a\varepsilon_{ij}}{a}] = Var[\tau_i] + Var[\varepsilon_{ij}] + \frac{\sum_{i=1}^a Var[\varepsilon_{ij}]}{a^2} - \frac{2Cov[\tau_i,\varepsilon_{ij}]}{a} - \frac{2Cov[\tau_i,\sum_{i=1}^a\varepsilon_{ij}]}{a} - \frac{2Cov[\varepsilon_{ij},\sum_{i=1}^a\varepsilon_{ij}]}{a} = 0 + \sigma^2 + \frac{\sigma^2}{a} - \frac{2\sigma^2}{a} = \frac{(a-1)\sigma^2}{a}$

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$Var[\bar{y}_{.j} - \bar{y}_{..}] =$

FB-RCBD

$Var[\beta_j + \frac{\sum_{i=1}^a\varepsilon_{ij}}{a} - \frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}] =$

$Var[\beta_j] + \frac{\sum_{i=1}^a Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{i=1}^a\sum_{j=1}^b Var[\varepsilon_{ij}]}{a^2b^2} + \frac{2Cov[\beta_j,\sum_{i=1}^a\varepsilon_{ij}]}{a} - \frac{2Cov[\beta_j,\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{i=1}^a\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{a^2b}$

$= 0 + \frac{\sigma^2}{a} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(b-1)\sigma^2}{ab}$

RB-RCBD

$Var[\beta_j - \frac{\sum_{j=1}^b\beta_j}{b} + \frac{\sum_{i=1}^a\varepsilon_{ij}}{a} - \frac{\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}}{ab}] =,$

$Var[\beta_j] + \frac{\sum_{j=1}^b Var[\beta_j]}{b^2} + \frac{\sum_{i=1}^a Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{i=1}^a\sum_{j=1}^b Var[\varepsilon_{ij}]}{a^2b^2} - \frac{2Cov[\beta_j,\sum_{j=1}^b\beta_j]}{b} + \frac{2Cov[\beta_j,\sum_{i=1}^a\varepsilon_{ij}]}{a} - \frac{2Cov[\beta_j,\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab} - \frac{2Cov[\sum_{j=1}^b\beta_j,\sum_{i=1}^a\varepsilon_{ij}]}{ab}$

$+\frac{2Cov[\sum_{j=1}^b\beta_j,\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab^2} - \frac{2Cov[\sum_{i=1}^a\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{a^2b} = \sigma_\beta^2 + \frac{\sigma_\beta^2}{b} - \frac{2\sigma_\beta^2}{b} + \frac{\sigma^2}{a} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(a\sigma_\beta^2+\sigma^2)(b-1)}{ab}$

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$Var[y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}] =$ FB-RCBD

$Var[\varepsilon_{ij} - \frac{1}{a}\sum_{i=1}^a\varepsilon_{ij} - \frac{1}{b}\sum_{j=1}^b\varepsilon_{ij} + \frac{1}{ab}\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}] =$

$Var[\varepsilon_{ij}] + \frac{\sum_{i=1}^a Var[\varepsilon_{ij}]}{a^2} + \frac{\sum_{j=1}^b Var[\varepsilon_{ij}]}{b^2} + \frac{\sum_{i=1}^a\sum_{j=1}^b Var[\varepsilon_{ij}]}{a^2b^2} - \frac{2Cov[\varepsilon_{ij},\sum_{i=1}^a\varepsilon_{ij}]}{a} - \frac{2Cov[\varepsilon_{ij},\sum_{j=1}^b\varepsilon_{ij}]}{b} + \frac{2Cov[\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab} + \frac{2Cov[\sum_{i=1}^a\varepsilon_{ij},\sum_{j=1}^b\varepsilon_{ij}]}{ab}$

$-\frac{2Cov[\sum_{i=1}^a\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{a^2b} - \frac{2Cov[\sum_{j=1}^b\varepsilon_{ij},\sum_{i=1}^a\sum_{j=1}^b\varepsilon_{ij}]}{ab^2} = \sigma^2 + \frac{\sigma^2}{a} + \frac{\sigma^2}{b} + \frac{\sigma^2}{ab} - \frac{2\sigma^2}{a} - \frac{2\sigma^2}{b} + \frac{2\sigma^2}{ab} + \frac{2\sigma^2}{ab} - \frac{2\sigma^2}{ab} - \frac{2\sigma^2}{ab} = \frac{(ab-a-b+1)\sigma^2}{ab}$