

Assume that X_1, X_2, \dots, X_{10} is a random sample from a distribution having a p.d.f. of the form $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

1. Find the best critical region of level 0.05 for testing $H_0 : \lambda = 1/2$ against $H_1 : \lambda = 1$

The rejection region is $R = \{\vec{x} : \Lambda \leq C\}$

$$\Lambda = \frac{L(\hat{\lambda}_0|x)}{L(\hat{\lambda}_1|x)} = \frac{(1/2)^{10} (\prod_{i=1}^{10} x_i)^{1/2-1}}{(1)^{10} (\prod_{i=1}^{10} x_i)^{1-1}} = (1/2)^{10} e^{1/2(-\sum_{i=1}^{10} \ln x_i)} \leq C \implies -\sum_{i=1}^{10} \ln x_i \leq C'$$

Let $T_i = -\ln x_i$, $0 < x < 1$, then $X = e^{-t}$, $\frac{dx}{dt} = -e^{-t}$

$$g(t) = \lambda(e^{-t})^{\lambda-1} | -e^{-t} | = \lambda e^{-\lambda t} \sim \text{Exp}(\lambda), t > 0$$

$$\text{So } \sum_{i=1}^{10} T_i = \sum_{i=1}^{10} (-\ln x_i) \sim \text{Gamma}(\alpha = 10, \beta = \frac{1}{\lambda})$$

Under $H_0 : \lambda = 1/2$, $-\sum_{i=1}^{10} \ln x_i \sim \text{Exp}(1/2) = \text{Gamma}(\alpha = 10, \beta = 2) = \chi_{20}^2$. Then,

For Reject H_0 , $\Lambda \leq C$ is equivalent $-\sum_{i=1}^{10} \ln x_i \leq C'$, where C' cuts off the upper α area in the χ_{20}^2 distribution.

$$1 - P(-\sum_{i=1}^{10} \ln x_i \leq C' | \lambda = 1/2) = \alpha = 0.05$$

$$C' = \chi_{(1-0.05), 20}^2 = 10.85081. \text{ So the critical region are } -\sum_{i=1}^{10} \ln x_i \in (0, 10.85081].$$

2. Find the power of the test in (1)

Let $W_i = 2T_i = -2\ln x_i$, $0 < x < 1$, then $X = e^{-\frac{1}{2}w}$, $\frac{dx}{dw} = -\frac{1}{2}e^{-\frac{1}{2}w}$

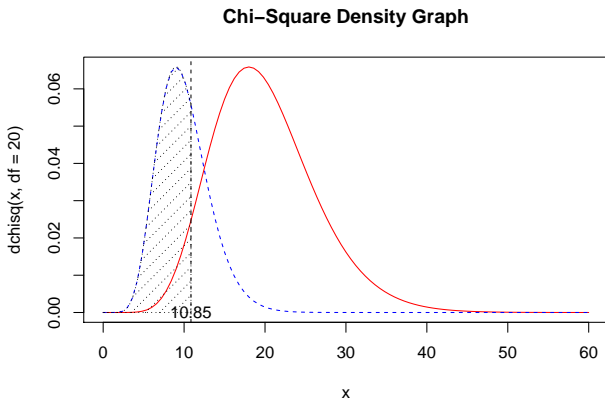
$$g(w) = \lambda(e^{-\frac{1}{2}w})^{\lambda-1} | -\frac{1}{2}e^{-\frac{1}{2}w} | = \frac{\lambda}{2} e^{-\frac{\lambda}{2}w} \sim \text{Exp}(\frac{\lambda}{2}), w > 0$$

Under $H_1 : \lambda = 1$, $W = -2\sum_{i=1}^{10} \ln x_i \sim \text{Exp}(1/2) = \text{Gamma}(\alpha = 10, \beta = 2) = \chi_{20}^2$.

The rejection rule is $-\sum_{i=1}^{10} \ln x_i \leq 10.85081$, then

Power = $P(\text{reject } H_0 | H_1 \text{ is true})$

$$= P(-\sum_{i=1}^{10} \ln x_i \leq 10.85081 | \lambda = 1) = P(-2\sum_{i=1}^{10} \ln x_i \leq 2 * 10.85081 | \lambda = 1) = 0.6430814$$



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3. Is your answer from (1) uniformly most powerful for testing $H_0 : \lambda = 1/2$ against $H_1 : \lambda > 1/2$? Explain

For $H_0 : \lambda = 1/2$, $H_1 : \lambda = \lambda_1$ where $\lambda_1 > 1/2$, Neyman-Pearson Theorem says that the test is most powerful when

$$\Lambda = \frac{\sup L(1/2|x)}{\sup L(\lambda_1|x)} = \frac{(1/2)^{10} (\prod_{i=1}^{10} x_i)^{1/2-1}}{(\lambda_1)^{10} (\prod_{i=1}^{10} x_i)^{\lambda_1-1}} = (2\lambda_1)^{-10} e^{(1/2-\lambda_1) \sum_{i=1}^{10} \ln x_i} \stackrel{set}{\leq} C$$

Take derivative of both sides, $-10 \ln(2\lambda_1)(\lambda_1 - 1/2)(-\sum_{i=1}^{10} \ln x_i) \leq \ln C$

For $\lambda_1 - 1/2 \geq 0$, $\Lambda \leq C$ is equivalent $-\sum_{i=1}^{10} \ln x_i \leq C'$, the most powerfule test is $T = -\sum_{i=1}^{10} \ln x_i$

The sturcture of the test T does not involve the actual value of λ_1 , then T is UMP.

4. Find the for the Cramer-Rao lower bound for variance of an unbiased estimator of λ .

$$\ln f(x) = \ln \lambda + (\lambda - 1) \ln x_i$$

$$\frac{\partial}{\partial \lambda} \ln f(x) = \frac{1}{\lambda} + \ln x_i$$

$$\frac{\partial^2}{\partial \lambda^2} \ln f(x) = -\frac{1}{\lambda^2}$$

$$I_\lambda = -E[\frac{\partial^2}{\partial \lambda^2} \ln f(x)] = \frac{1}{\lambda^2}$$

$$CRLB = \frac{1}{nI_\lambda} = \frac{\lambda^2}{10}$$

5. Find the MUVE of λ .

$f(x) = \lambda x^{\lambda-1} = \lambda e^{-(\lambda-1)(-\ln x)}$ is a member of the exponential family.

$$L(\lambda) = \lambda^{10} e^{-(\lambda-1)(-\sum_{i=1}^{10} \ln x_i)} = h(x)c(\lambda)e^{W_i(\lambda)t_i(\vec{x})}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0$, $\lambda > 0$. $W_i(\lambda) = \lambda - 1$ contains an open interval in \mathbb{R} , so $T = -\sum_{i=1}^{10} \ln x_i$ is a complete sufficient statistic for λ .

$$L(\lambda) = \lambda^{10} (\prod_{i=1}^{10} x_i)^{\lambda-1} = \lambda^{10} e^{(\lambda-1) \sum_{i=1}^{10} \ln x_i}$$

$$l(\lambda) = 10 \ln \lambda + (\lambda - 1) \sum_{i=1}^{10} \ln x_i$$

$$l'(\lambda) = \frac{10}{\lambda} + \sum_{i=1}^{10} \ln x_i \stackrel{set}{=} 0$$

$$\hat{\lambda}_{MLE} = \frac{10}{-\sum_{i=1}^{10} \ln x_i}$$

For $-\sum_{i=1}^{10} \ln x_i \sim \text{Gamma}(\alpha = 10, \beta = \frac{1}{\lambda})$

$$E[\hat{\lambda}_{MLE}] = 10E[Y^{-1}] = 10 \frac{\beta^{-1} \Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{10\lambda \Gamma(10-1)}{\Gamma(10)} = \frac{10\lambda}{9}$$

Create an unbiased estimator $\hat{\lambda}_U = \frac{9}{10} \hat{\lambda}_{MLE}$.

$$E[\hat{\lambda}_U] = E\left[\frac{9}{10}\hat{\lambda}_{MLE}\right] = \frac{9}{10} \cdot \frac{10\lambda}{9} = \lambda$$

$$\hat{\lambda}_U = \frac{9}{10}\hat{\lambda}_{MLE} = \frac{9}{-\sum_{i=1}^{10} \ln x_i} = \frac{9}{T}$$

From Lehmann-Scheffe Theorem, $\hat{\lambda}_U = \frac{9}{T}$ is the unique MVUE of λ because it is an unbiased estimator of λ and a function of T , which is a complete sufficient statistic for λ .

6. Show that the MUVE of λ is asymptotically efficient.

When sample size $n \rightarrow \infty$, from previous results, $CRLB = \frac{1}{nI_\lambda} = \frac{\lambda^2}{n}$, $-\sum_{i=1}^n \ln x_i \sim \text{Gamma}(n, \frac{1}{\lambda})$

$$\hat{\lambda}_U = \frac{n-1}{-\sum_{i=1}^n \ln x_i} = \frac{n-1}{T}, \quad E[\hat{\lambda}_U] = \lambda$$

$$E[\hat{\lambda}_U^2] = (n-1)^2 E[T^{-2}] = (n-1)^2 \frac{\beta^{-2}\Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{(n-1)^2 \lambda^2 \Gamma(n-2)}{\Gamma(n)} = \frac{(n-1)\lambda^2}{n-2}$$

$$\text{Var}[\hat{\lambda}_U] = E[\hat{\lambda}_U^2] - E[\hat{\lambda}_U]^2 = \frac{(n-1)\lambda^2}{n-2} - \lambda^2 = \frac{\lambda^2}{n-2}$$

$$\lim_{n \rightarrow \infty} \frac{CRLB}{\text{Var}[\hat{\lambda}_U]} = \lim_{n \rightarrow \infty} \frac{\frac{\lambda^2}{n}}{\frac{\lambda^2}{n-2}} = \lim_{n \rightarrow \infty} \frac{n-2}{n} = 1$$

Therefore, the MUVE $\frac{n-1}{-\sum_{i=1}^{10} \ln x_i}$ is asymptotically efficient.