

1. Assume that X_1, X_2, \dots, X_{10} is a random sample from a distribution having a p.d.f. of the form $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

$$f(x) = \lambda x^{\lambda-1} = \lambda e^{(\lambda-1) \ln x} \implies \ln x \sim \text{Exp}()$$

Let $Y_i = -\ln x_i, 0 < x < 1$, then $X = e^{-y}, \frac{dx}{dy} = -e^{-y} g(Y) = \lambda(e^{-y})^{\lambda-1} | -e^{-y} | = \lambda e^{-\lambda y}, y > 0$

So $Y_i = -\sum_{i=1}^{10} \ln x_i \sim \text{Gamma}(\alpha = 10, \beta = \frac{1}{\lambda})$, then,

$$E[\hat{\lambda}_{MLE}] = 10E[Y^{-1}] = 10 \frac{\beta^{-1} \Gamma(\alpha - 1)}{\Gamma(\alpha)} = \frac{10\lambda \Gamma(10 - 1)}{\Gamma(10)} = \frac{10\lambda}{9}$$

$$L(\lambda) = \lambda^{10} \left(\prod_{i=1}^{10} x_i \right)^{\lambda-1} = \lambda^{10} e^{(\lambda-1) \sum_{i=1}^{10} \ln x_i}$$

$$l(\lambda) = 10 \ln \lambda + (\lambda - 1) \sum_{i=1}^{10} \ln x_i$$

$$l'(\lambda) = \frac{10}{\lambda} - \sum_{i=1}^{10} \ln x_i \stackrel{\text{set}}{=} 0$$

$$\hat{\lambda}_{MLE} = \frac{10}{\sum_{i=1}^{10} \ln x_i}$$

5. Find the MUVE of λ .

• **Step1: Proof sufficient**

From *Fisher–Neyman factorization theorem* (2019-2-14p5)

$$f(x|\lambda) = L(\lambda) = \lambda^n \left(\prod_{i=1}^n x_i \right)^{\lambda-1} = \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln x_i} \cdot 1 = k(t|\lambda) h(\vec{x})$$

$h(\vec{x}) = 1$ is free of λ . So $T = \sum_{i=1}^n \ln x_i$ is a sufficient statistic for λ .

• **Step2: Proof complete**

$f(x|\lambda)$ is a member of the exponential family (2019-2-19p12). By the Theorem of Complete Statistics in the exponential family

$$f(x|\vec{\lambda}) = \lambda^n e^{\sum_{i=1}^n (\lambda-1) \ln x_i} = h(x) c(\vec{\lambda}) e^{\sum_{j=1}^k W_j(\vec{\lambda}) t_j(x)}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0, \lambda > 0$. $\{W_1(\vec{\lambda}), \dots, W_k(\vec{\lambda})\}$ contains an open interval in \mathbb{R} , so $T(\vec{x}) = \sum_{i=1}^n \ln x_i$ is a complete sufficient statistic for λ .

6. Show that the MUVE of λ is asymptotically efficient.

b. Find the expected value of $\hat{\lambda}_{MLE}$.

- Method 1
- Method 2

By 2.1.10 Probability integral transformation, let $U_i = F_X(\mathbf{x}|\lambda) = \int_{-\infty}^x \lambda x^{-\lambda-1} dx = 1 - \int_1^x \lambda x^{-\lambda-1} dx = 1 - (-x^{-\lambda}|_1^x) = x^{-\lambda} \sim Uni(0, 1)$,

By 5.6.3 the exponential-uniform transformation, $\sum_{i=1}^n \ln X = -\frac{1}{\lambda} \sum_{i=1}^n \ln U_i \sim Gamma(n, \frac{1}{\lambda})$; $(\sum_{i=1}^n \ln x_i)^{-1} \sim Inv - Gamma(n, \frac{1}{\lambda})$.

For a Inv-Gamma(α, β), $f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\beta}{x}}$, $x > 0$, $E[x^n] = \frac{\beta^n}{(\alpha-1)\dots(\alpha-n)}$.

Thus,

$$E[\hat{\lambda}] = E\left[\frac{n}{\sum_{i=1}^n \ln x_i}\right] = nE\left[\left(\sum_{i=1}^n \ln x_i\right)^{-1}\right] = \frac{n\lambda}{n-1}$$

c. Find the variance of $\hat{\lambda}_{MLE}$.

From method 1,

$$E[\hat{\lambda}^2] = n^2 E[Y^{-2}] = n^2 \frac{\beta^{-2} \Gamma(-2 + \alpha)}{\Gamma(\alpha)} = \frac{n^2 \lambda^2 \Gamma(n-2)}{\Gamma(n)} = \frac{n^2 \lambda^2 (n-3)!}{(n-1)!} = \frac{n^2 \lambda^2}{(n-1)(n-2)}$$

From method 2,

For Inv-Gamma(α, β), $E[x^n] = \frac{\beta^n}{(\alpha-1)\dots(\alpha-n)}$, then

$$E[\hat{\lambda}^2] = E\left[\frac{n^2}{(\sum_{i=1}^n \ln x_i)^2}\right] = n^2 E\left[\left(\sum_{i=1}^n \ln x_i\right)^{-2}\right] = \frac{n^2 \lambda^2}{(n-1)(n-2)}$$

Therefore,

$$Var[\hat{\lambda}^2] = \frac{n^2 \lambda^2}{(n-1)(n-2)} - \frac{n^2 \lambda^2}{(n-1)^2} = \frac{n^2 \lambda^2}{(n-1)} \left[\frac{1}{n-2} - \frac{1}{n-1} \right] = \frac{n^2 \lambda^2}{(n-1)^2 (n-2)}$$

d. Using $\hat{\lambda}_{MLE}$, create an unbiased estimator $\hat{\lambda}_U$.

$E[\hat{\lambda}] = \frac{n\lambda}{n-1}$ is a biased estimator.

We can set $\frac{n-1}{n} E[\hat{\lambda}] = E[\frac{n-1}{n} \cdot \frac{n\lambda}{n-1}] = \lambda$

Therefore, $E[\hat{\lambda}_U] = E[\frac{n-1}{n} E[\hat{\lambda}]] = \lambda$

$$\hat{\lambda}_U = \frac{n-1}{n} \hat{\lambda}_{MLE} \text{ is an unbiased estimator.}$$

e. Find the variance of $\hat{\lambda}_U$. '2019-3-5p12'

$$Var[\hat{\lambda}_U] = Var[\frac{n-1}{n}\hat{\lambda}_{MLE}] = (\frac{n-1}{n})^2 \frac{n^2\lambda^2}{(n-1)^2(n-2)} = \frac{\lambda^2}{n-2}$$
