STAT562 Final Exam

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1. $X_1, X_2, ..X_n$ is a random sample from a distribution having a p.d.f of the form. $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find a complete sufficient statistic for λ. Justify your answer

$$L(\lambda) = \lambda^n (\prod x_i)^{\lambda-1} = \lambda^n e^{(\lambda-1\sum_{i=1}^n \ln x_i)}$$

For $\lambda > 0$, certainly contains an open interval, so $\sum_{i=1}^n \ln x_i)$ is a complete sufficient statistic.

2. Let Y_n be the n^th order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

θ is a location parameter.

$Y_n - \bar{Y}$ is location invariant, and so it is ancillary for θ

$\bar{Y}$ is sufficient for θ, and is complete.

By Basu's theorem, $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

3. Suppose that $X_1, X_2, ..X_n \sim$ idd.

$$L(\theta) = \hat{\theta} e^{-\theta \sum x_i}, \quad \pi(\theta) = \lambda e^{-\lambda\theta}$$

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

$$\pi(\theta|\vec{x}) \propto \theta^n e^{-\theta \sum x_i} \lambda e^{-\lambda\theta} \sim Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$$

b. Find the Bayes estimator of θ, assuming square-error loss

Posterior mean $= \alpha\beta = \frac{n+1}{\lambda + \sum x_i}$

c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\frac{1}{\hat{\theta}} = \frac{\lambda + \sum x_i}{n+1} = \frac{\lambda + n\bar{x}}{n+1} = \frac{\frac{1}{\lambda} + \frac{n}{\frac{1}{\bar{x}}}}{n+1}$$

$$\hat{\theta} = \frac{1}{\frac{\frac{1}{\lambda} + \frac{n}{\frac{1}{\bar{x}}}}{n+1}}$$

which is the weighted hamonic mean of $\frac{1}{\lambda}$, which is the prior mean, and $\frac{1}{\bar{x}}$, which is the MLE of θ.

d. Find the Bayes estimator of θ, assuming absolute loss

Postmedian

$$\hat{\theta} = F^{-1}(.s)$$

where $F(x)$ is the $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ pdf, for which there is no closed form.

e. Find the Bayes estimator of θ, assuming binary loss

Posterior mode

$$\hat{\theta} = (\alpha - 1)\beta = \frac{n}{\lambda + \sum x_i}$$

4. Redo all of problem 3, using the non-informative prior

$\pi(\theta) = 1, \theta > 0$, so

$$\pi(\theta|\vec{x}) \propto L(\theta) = \theta^n e^{-\theta \sum x_i}$$

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

It must be

$$\pi(\theta|\vec{x}) \propto \theta^n e^{-\theta \sum x_i} \sim Gamma(\alpha = n+1, \beta = \frac{1}{\sum x_i} = \frac{n}{\bar{x}})$$

b. Find the Bayes estimator of θ, assuming square-error loss

$$\hat{\theta} = \alpha\beta = \frac{n+1}{\sum x_i}$$

c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\frac{1}{\hat{\theta}} = \frac{1 \times 0 + n\bar{x}}{n+1} = \lim_{c \to \infty} \frac{1 \times \frac{1}{c} + \frac{n}{\frac{1}{\bar{x}}}}{n+1}$$

So $\hat{\theta}$ is the limit as $c \to \infty$ of the hamonic mean of $\frac{1}{\bar{x}}$ (the MLE) and c.(the prior mean is infinite)

d. Find the Bayes estimator of θ, assuming absolute loss

Posterior median $\hat{\theta}$, which has no closed form.

e. Find the Bayes estimator of θ, assuming binary loss

Posterior mode

$$\hat{\theta} = (\alpha - 1)\beta = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

5. Let $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) =$

$$L(\theta) = \theta^n (\prod x_i)^{\theta - 1} = \theta^n e^{(-\theta - 1 \sum_{i=1}^n \ln x_i)}$$

$$l(\theta) = n \ln \theta - (\theta + 1) \sum_{i=1}^n \ln x_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n \ln x_i \overset{\text{set}}{=} 0$$

a. Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ.

$$\hat{\theta} = \frac{n}{\sum \ln x_i}$$

b. Find the expected value of $\hat{\theta}_{MLE}$.

Let $Y_i = \ln x_i$, then $X = e^y$, $\frac{dx}{dy} e^y$

$$g(Y) = \theta (e^y)^{-\theta - 1} e^y = \theta e^{-y\theta}, y > 0$$

So $Y_i = \ln x_i \sim Gamma(\alpha = n, \beta = \frac{1}{\theta})$

$$E[\hat{\theta}] = nE[Y^{-1}] = n \frac{\beta^{-1} \Gamma(-1 + \alpha)}{\Gamma(\alpha)} = \frac{n\theta \Gamma(n-1)}{\Gamma(n)} = \frac{n\theta (n-2)!}{(n-1)!} = \frac{n\theta}{n-1}$$

c. Find the variance of $\hat{\theta}_{MLE}$.

$$E[\hat{\theta}^2] = n^2 E[Y^{-2}] = n^2 \frac{\beta^{-2} \Gamma(-2 + \alpha)}{\Gamma(\alpha)} = \frac{n^2 \theta^2 \Gamma(n-2)}{\Gamma(n)} = \frac{n^2 \theta^2 (n-3)!}{(n-1)!} = \frac{n^2 \theta^2}{(n-1)(n-2)}$$

$$Var[\hat{\theta}^2] = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)} [\frac{1}{n-2} - \frac{1}{n-1}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

d. Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

$$\hat{\theta}_U = \frac{n-1}{n} \hat{\theta}_{MLE}$$

e. Find the variance of $\hat{\theta}_U$.

$$Var[\hat{\theta}_U] = (\frac{n-1}{n})^2 \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2}$$

6. Refer to problem 5.

$$\mu = \int_1^\infty x\theta x^{-\theta-1}dx = \theta \frac{x^{-\theta+1}}{-\theta+1}\bigg|_1^\infty = \frac{\theta}{\theta-1}$$

So $\bar{X} = \frac{\theta}{\theta-1}$, $\theta\bar{x} - \bar{x} = \theta$, $\theta(\bar{x}-1) = \bar{x}$,

a. Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ.

$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1}$

b. Using the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.

$g(x) = \frac{x}{x-1}$, $g'(x) = \frac{(x-1)\times 1 - x\times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}$, $g''(x) = \frac{2}{(x-1)^3}$

$$g(x) \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(x-\mu) + \frac{2}{(\mu-1)^3}\frac{(x-\mu)^2}{2}$$

$$\frac{\bar{x}}{\bar{x}-1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(\bar{x}-\mu) + \frac{1}{(\mu-1)^3}(\bar{x}-\mu)^2$$

$$E[\frac{\bar{x}}{\bar{x}-1}] \approx \frac{\mu}{\mu-1} + \frac{1}{(\mu-1)^3}\frac{\sigma^2}{n}$$

where $\mu = \frac{\theta}{\theta-1}$

$$E[x^2] = \int_1^\infty x^2\theta x^{-\theta-1}dx = \theta\frac{x^{-\theta+2}}{-\theta+2}\bigg|_1^\infty = \frac{\theta}{\theta-2}$$

$$\sigma^2 = \frac{\theta}{\theta-2} - (\frac{\theta}{\theta-1})^2 = \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-1)^2(\theta-2)} = \frac{\theta^3 - 2\theta^2 + \theta - \theta^3 + 2\theta^2}{(\theta-1)^2(\theta-2)} = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

$$E[\hat{\theta}_{MOM}] \approx \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1}-1} + 0 + \frac{1}{(\frac{\theta}{\theta-1}-1)^3}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \theta + \frac{1}{n}\frac{\theta(\theta-1)}{\theta-2}$$

c. Using the delta method to approximate the variance of $\hat{\theta}_{MOM}$.

$$Var[\hat{\theta}_{MOM}] \approx \frac{1}{(\mu-1)^4}\frac{\sigma^2}{n} = \frac{1}{(\frac{\theta}{\theta-1}-1)^4}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \frac{1}{n}\frac{\theta(\theta-1)^2}{\theta-2}$$

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End
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