# STAT562 Final Exam

*Winter 2019*

1. $X_1, X_2, ..X_n$ is a random sample from a distribution having a p.d.f of the form. $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find a complete sufficient statistic for $\lambda$. Justify your answer

- **Step1: Proof sufficient**

From *Fisher–Neyman factorization theorem* (`2019-2-14p5`)

$$f(x|\lambda) = L(\lambda) = \lambda^n (\prod x_i)^{\lambda-1} = \lambda^n e^{(\lambda-1)\sum_{i=1}^n \ln x_i} \cdot 1 = k(t|\lambda)h(\vec{x})$$

$h(\vec{x}) = 1$ is free of $\lambda$. So $T = \sum_{i=1}^n \ln x_i$ is a sufficient statistic for $\lambda$.

- **Step2: Proof complete**

$f(x|\lambda)$ is a member of the exponential family (`2019-2-19p12`). By the Theorem of Complete Statistics in the exponential family

$$f(x|\vec{\lambda}) = \lambda^n e^{\sum_{i=1}^n (\lambda-1)\ln x_i} = h(x)c(\vec{\lambda})e^{\sum_{j=1}^k W_j(\vec{\lambda})t_j(x)}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0$, $\lambda > 0$. $\{W_1(\vec{\lambda}), .., W_k(\vec{\lambda})\}$ contains an open interval in $\mathbb{R}$, so $T(\vec{x}) = \sum_{i=1}^n \ln x_i)$ is a complete sufficient statistic for $\lambda$.

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2. Let $Y_n$ be the $n^th$ order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

- **Step1: $\theta$ is a location parameter**

Let $x = y - \theta$. For $N(\theta, \sigma^2)$ is a location family of densities (`2018.11.20p7`),

$g(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = f(x) = f(y-\theta)$

Thus, $\theta$ is a location parameter.

- **Step2: $Y_n - \bar{Y}$ is location invariant**

For $Y_n \sim N(\theta, \sigma^2)$, $\bar{Y} \sim N(\theta, \sigma^2/n)$ `2019-2-21p4-6`

Consider the group of transformations defined by $\mathcal{G} = \{Y_n - \bar{Y}, -\infty < \bar{Y} < \infty\}$, $Y_n + a - (\bar{Y} + a) = Y_n - \bar{Y}$.

Thus, the joint distribution of $Y_n - \bar{Y}$ is in $\mathcal{F}$ and hence $\mathcal{F}$ is invariant under $\mathcal{G}$.

- **step3: $Y_n - \bar{Y}$ is ancillary statistic for $\theta$**

$f(y|\theta)$ is a location exponential family. Let $X_n = Y_n - \theta$ is a random sample from $f(y|0)$ `2019-2-19p6`

$Y_n - \bar{Y} = Y_n - \frac{1}{n}\sum_{i=1}^n Y_i = (Y_n - \theta) - \frac{1}{n}\sum_{i=1}^n (Y_i - \theta) = X_n - \frac{1}{n}\sum_{i=1}^n X_i$

$Y_n - \bar{Y}$ is a function of only $X_1, .., X_n$ and be free of $\theta$. It is an ancillary statistic for $\theta$.

- **step4: For $Y \sim N(\theta, \sigma^2)$, $\bar{Y}$ is sufficient statistic for $\theta$**
- **step5: $\bar{Y}$ is complete statistic for $\theta$**

$\bar{Y} \sim N(\theta, \sigma^2/n)$ is a member of the exponential family. It is a complete sufficient statistic.

- **step6: By Basu's theorem, an acillary statistc $Y_n - \bar{Y}$ and a complete sufficient statistic $\bar{Y}$ are independent.**`2019-2-19p10`

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3. Suppose that $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta e^{-\theta x}, x > 0$. Assume that the prior distribution of $\theta$ is $\pi(\theta) = \lambda e^{-\lambda\theta}, \theta > 0$

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

For $L(\theta) = \hat{\theta} e^{-\theta \sum x_i}, \quad \pi(\theta) = \lambda e^{-\lambda\theta}$, and the kernel of a function is the main part of the function, the part that remains when constants are disregarded (`2019-2-26p8-9,p11-p13 2019-2-28p8 Exapmle 2.3.8`). that is

$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \lambda e^{-\lambda\theta} \propto \theta^{n+1-1} e^{-\theta(\lambda + \sum x_i)}$

which is $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ distribution.

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b. Find the Bayes estimator of $\theta$, assuming square-error loss

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. `2019-2-28p1` $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\lambda + \sum x_i}$$

which is the posterior mean.

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c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(\lambda + n\bar{x})} = \frac{1}{\frac{1}{n+1}(\frac{1}{1/\lambda} + \frac{n}{1/\bar{x}})}$$

which is the weighted hamonic mean of $1/\lambda$, which is the prior mean, and $1/\bar{x}$, which is the MLE of $\theta$.

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d. Find the Bayes estimator of $\theta$, assuming absolute loss

`2019-2-28p4,9`

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = median[\theta|\vec{x}]$$

For the median of Gamma distribution doesn't have a closed form, the posterior median would not have a closed form.

Postmedian $\hat{\theta} = F^{-1}(\frac{1}{2})$ where $F(x)$ is the $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ cdf, for which there is no closed form.

e. Find the Bayes estimator of $\theta$, assuming binary loss `2019-2-28p5-6`

Suppose $L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}$.

$$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$$

To minimized this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = mode[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\lambda + \sum x_i}$$

which is the posterior mode.

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4. Redo all of problem 3, using the non-informative prior $\pi(\theta) = 1, \theta > 0$. Note that this is not a valid density function since its intergral is infinite, but proceed with it anyway

a. Find the posterior distribution $\pi(\theta|\vec{x})$. `2019-2-26p8-9`

For $\pi(\theta) = 1, \theta > 0$, $L(\theta) = \theta^n e^{-\theta \sum x_i}$, from the kernel of function,

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \sim Gamma(\alpha = n + 1, \beta = \frac{1}{\sum x_i})$$

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b. Find the Bayes estimator of $\theta$, assuming square-error loss

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\sum x_i}$$

which is the posterior mean.

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c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(1 \times 0 + n\bar{x})} = \lim_{c \to \infty} \frac{1}{\frac{1}{n+1}(\frac{1}{c} + \frac{n}{1/\bar{x}})}$$

which is the weighted hamonic mean of $c$, which is the prior mean when $c \to \infty$, and $1/\bar{x}$, which is the MLE of $\theta$.

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d. Find the Bayes estimator of $\theta$, assuming absolute loss

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = median[\theta|\vec{x}]$$

For the median of Gamma distribution doesn't have a closed form, the posterior median would not have a closed form.

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e. Find the Bayes estimator of $\theta$, assuming binary loss

Suppose $L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}$.

$$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$$

To minimized this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = mode[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

which is the posterior mode.

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5. Let $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta x^{-\theta-1}, x_i > 1, \theta > 2$.

a. Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of $\theta$. `2019-2-2p12`

$$f(\vec{x}|\theta) = L(\theta) = \theta^n (\prod x_i)^{\theta-1} = \theta^n e^{(-\theta-1)\sum_{i=1}^n \ln x_i}$$

$$l(\theta) = n \ln \theta - (\theta + 1) \sum_{i=1}^n \ln x_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n \ln x_i \overset{\text{set}}{=} 0$$

$$\hat{\theta}_{MLE} = \frac{n}{\sum \ln x_i}$$

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b. Find the expected value of $\hat{\theta}_{MLE}$.

Let $Y_i = \ln x_i$, then $X = e^y$, $\frac{dx}{dy} e^y$

$$g(Y) = \theta(e^y)^{-\theta-1} e^y = \theta e^{-y\theta}, y > 0$$

So $Y_i = \ln x_i \sim Gamma(\alpha = n, \beta = \frac{1}{\theta})$

$$E[\hat{\theta}] = nE[Y^{-1}] = n\frac{\beta^{-1}\Gamma(-1+\alpha)}{\Gamma(\alpha)} = \frac{n\theta\Gamma(n-1)}{\Gamma(n)} = \frac{n\theta(n-2)!}{(n-1)!} = \frac{n\theta}{n-1}$$

c. Find the variance of $\hat{\theta}_{MLE}$.

$$E[\hat{\theta}^2] = n^2 E[Y^{-2}] = n^2 \frac{\beta^{-2}\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{n^2\theta^2\Gamma(n-2)}{\Gamma(n)} = \frac{n^2\theta^2(n-3)!}{(n-1)!} = \frac{n^2\theta^2}{(n-1)(n-2)}$$
$$Var[\hat{\theta}^2] = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)}[\frac{1}{n-2} - \frac{1}{n-1}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

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d. Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

$$\hat{\theta}_U = \frac{n-1}{n}\hat{\theta}_{MLE}$$

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e. Find the variance of $\hat{\theta}_U$.

$$Var[\hat{\theta}_U] = (\frac{n-1}{n})^2 \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2}$$

6. Refer to problem 5.

a. Find $\hat{\theta}_{MOM}$, the method of moments estimator of $\theta$. `2019-2-21p8 7.2.1`

$EX = \mu = \int_1^\infty x\theta x^{-\theta-1}dx = \theta \frac{x^{-\theta+1}}{-\theta+1}\Big|_1^\infty = \frac{\theta}{\theta-1}$

Set $\bar{X} = \frac{\theta}{\theta-1} \implies \theta\bar{x} - \bar{x} = \theta \implies \theta(\bar{x}-1) = \bar{x}$,

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1}$$

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b. Using the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.

For $EX = \mu = \frac{\theta}{\theta-1}$

$E[X^2] = \int_1^\infty x^2\theta x^{-\theta-1}dx = \theta \frac{x^{-\theta+2}}{-\theta+2}\Big|_1^\infty = \frac{\theta}{\theta-2}$

$Var[X] = \sigma^2 = E[X^2] - E[X]^2 = \frac{\theta}{\theta-2} - (\frac{\theta}{\theta-1})^2$

$$= \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-1)^2(\theta-2)} = \frac{\theta^3 - 2\theta^2 + \theta - \theta^3 + 2\theta^2}{(\theta-1)^2(\theta-2)} = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

Use a $2^{nd}$ order Taylar series `2019-3-5p1`

$$g(x) = g(x_0) + g'(x_0)(x-x_0) + g''(x_0)\frac{(x-x_0)^2}{2} + R$$

Consider $g(x) = \frac{x}{x-1}$, $\quad g'(x) = \frac{(x-1)\times 1 - x\times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}$, $\quad g''(x) = \frac{2}{(x-1)^3}$

Choose $x_0 = EX = \mu$

$$g(x) \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(x-\mu) + \frac{2}{(\mu-1)^3}\frac{(x-\mu)^2}{2}$$

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(\bar{x}-\mu) + \frac{1}{(\mu-1)^3}(\bar{x}-\mu)^2$$

$$E[\hat{\theta}_{MOM}] \approx \frac{\mu}{\mu-1} + 0 + \frac{1}{(\mu-1)^3}\frac{\sigma^2}{n} = \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1}-1} + \frac{1}{(\frac{\theta}{\theta-1}-1)^3}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \theta + \frac{\theta(\theta-1)}{n(\theta-2)}$$

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c. Using the delta method to approximate the variance of $\hat{\theta}_{MOM}$. `2019-3-5p3`

$Var[\hat{\theta}_{MOM}] \approx Var[g(x_0) + g'(x_0)(x-x_0)] = Var[\frac{\mu}{\mu-1} + \frac{1}{(\mu-1)^2}(\bar{x}-\mu)]$

$$= \frac{1}{(\mu-1)^4}\frac{\sigma^2}{n} = \frac{1}{(\frac{\theta}{\theta-1}-1)^4}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \frac{\theta(\theta-1)^2}{n(\theta-2)}$$

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End
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