Question 1

A process engineer is testing the yield of a product manufactured on randomly selected three machines. Each machine can be operated at a large number of power settings and only two power settings are randomly selected for this study. Furthermore, a machine has only three stations on which the product is formed. An experiment is conducted in which each machine is tested at both power settings, and three observations on yield are taken from each station. The runs are made in random order. Part of the ANOVA table computed for the above experiment incorrectly treating all main effects as fixed and all combinations of factors as crossed is given below.

Source	SS
Machine	21.436
Power	845.698
Station	16.980
Machine × Power	0.383
Machine × Station	16.603
Power × Station	16.303
Machine × Power × Station	12.905
Error	61.760
Total	992.068

The model of the nested & crossed Design is

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_{k(i)} + (\tau\beta)_{ij} + (\beta\gamma)_{jk(i)} + \varepsilon_{l(ijk)}$$

for $i = 1, 2, 3$ is the number of Machine being compared; $j = 1, 2$ is the number of Power being compared; $k = 1, 2, 3$ is the number of stations; $l = 1, 2, 3$ is the number of replication;

μ is the overall true mean response;

τ_i is the random effect of i^{th} level of Machine;

β_j is the random effect of j^{th} level of Power;

$\gamma_{k(i)}$ is the main fixed effect of k^{th} stations nested in i^{th} level of Machine;

$(\tau\beta)_{ij}$ is the interaction random effect of i^{th} level of Machine and j^{th} level of Power;

$(\beta\gamma)_{jk(i)}$ is the interaction random effect of j^{th} level of Power and k^{th} stations nested in i^{th} level of Machine.

y_{ijkl} is response value for the l^{th} replication for k^{th} stations nested in i^{th} level of Machine when j^{th} level of Power is applied;

$\varepsilon_{l(ijk)}$ is random error for the l^{th} replication for k^{th} stations nested in i^{th} level of Machine when j^{th} level of Power is applied.

Assumptions:

$\varepsilon_{l(ijk)} \sim iidN(0, \sigma^2)$; $\tau_i \sim iidN(0, \sigma^2_\tau)$ $\beta_j \sim iidN(0, \sigma^2_\beta)$ $\sum_{k=1}^{3} \gamma_{k(i)} = 0$; $(\tau\beta)_{ij} \sim iidN(0, \sigma^2_{\tau\beta})$ $(\beta\gamma)_{jk(i)} \sim iidN(0, \sigma^2_{\beta\gamma})$

$\varepsilon_{l(ijk)}$, τ_i, β_j, $(\tau\beta)_{ij}$, and $(\beta\gamma)_{jk(i)}$ are independent.

Produce the corrected ANOVA table with Source, df, SS, MS, EMS, F value along with numerator and denominator df, and p value for each test.

source	i(r)	j(r)	k(f)	l(r)	df	SS	EMS	F
A	1	b	c	n	a-1	SS_A	$\sigma^2 + cn\sigma^2_{\tau\beta} + bcn\sigma^2_{\tau_i}$	$\frac{MS_A}{MS_{AB}}$
τ_i (r)		2	3	3	3-1	21.436	$\sigma^2 + 9\sigma^2_{\tau\beta} + 18\sigma^2_{\tau_i}$	$df_{2,2}$
B	a	1	c	n	b-1	SS_B	$\sigma^2 + cn\sigma^2_{\beta\gamma} + acn\sigma^2_{\tau\beta}$	$\frac{MS_B}{MS_{AB}}$
β_j (r)	3		3	3	2-1	845.698	$\sigma^2 + 9\sigma^2_{\beta\gamma} + 27\sigma^2_{\tau\beta}$	$df_{1,2}$
C(A)	1	b	0	n	a(c-1)	$SS_C + SS_{AC}$	$\sigma^2 + n\sigma^2_{\beta\gamma} + \frac{bn\sum_{k=1}^{c}\gamma^2_{k(i)}}{c-1}$	$\frac{MS_{C(A)}}{MS_{BC(A)}}$
$\gamma_{k(i)}$ (f)		2		3	3(3-1)	16.980+16.603	$\sigma^2 + 3\sigma^2_{\beta\gamma} + 3\sum_{k=1}^{3}\gamma^2_{k(i)}$	$df_{6,6}$
AB	1	1	c	n	(a-1)(b-1)	SS_{AB}	$\sigma^2 + cn\sigma^2_{\tau\beta}$	$\frac{MS_{AB}}{MS_E}$
$(\tau\beta)_{ij}$ (r)			3	3	(3-1)(2-1)	0.383	$\sigma^2 + 9\sigma^2_{\tau\beta}$	$df_{2,36}$
BC(A)	1	1	0	n	a(b-1)(c-1)	$SS_{BC} + SS_{ABC}$	$\sigma^2 + n\sigma^2_{\beta\gamma}$	$\frac{MS_{BC(A)}}{MS_E}$
$(\beta\gamma)_{jk(i)}$ (r)				3	3(2-1)(3-1)	16.303+12.905	$\sigma^2 + 3\sigma^2_{\beta\gamma}$	$df_{6,36}$
Error (r)	1	1	1	1	abc(n-1) 3×2×3(3-1)	SS_E 61.760	σ^2	
Total					abcn-1 3×2×3×3-1	SS_T 992.068		

Compute the p value assuming you don't have access to a computer. If an exact F test is not available, construct a Satterthwaite (approximate) F statistic and its df values. Provide the numerical values for df, SS, MS, F in the table. The numerical values:

source	df	SS	MS	F	p
A	2	21.436	21.436/2=10.718	$\frac{10.718}{0.1915} = 55.96867$	0.01755351
B	1	845.698	845.698	$\frac{845.698}{0.1915} = 4416.178$	0.0002263633
C(A)	6	33.583	33.583/6=5.597167	$\frac{5.597167}{4.868} = 1.149788$	0.4348901
AB	2	0.383	0.383/2=0.1915	$\frac{0.1915}{1.715556} = 0.1116256$	0.8946874
BC(A)	6	29.208	29.208/6=4.868	$\frac{4.868}{1.715556} = 2.837564$	0.02292305
Error	36	61.760	61.760/36=1.715556		
Total	53	992.068			

Question 2

Consider the linear model for a two-stage nested design with B nested in A as given below. Using only the given information here,

$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{(ij)k}$; for $i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$;

Assumptions: $\varepsilon_{(ij)k} \sim iidN(0, \sigma^2)$; $\tau_i \sim iidN(0, \sigma_\tau^2)$; $\sum_{j=1}^{b} \beta_j(i) = 0$; $\varepsilon_{(ij)k}$, τ_i are independent.

$\bar{y}_{i..} = \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}^2$ $\bar{y}_{.j.} = \frac{1}{an} \sum_{i=1}^{a} \sum_{k=1}^{n} y_{ijk}$ $\bar{y}_{ij.} = \frac{1}{n} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ijk}$ $\bar{y}_{...} = \frac{1}{abn} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}$

(a). Derive the least squares estimator of $\beta_{j(i)}$. Provide the appropriate constraints about estimators used when solving normal equations. (Use only sum to zero constraints)

This is a two-stage nested design with fixed factor B nested in random factor A. $\varepsilon_{(ij)k} = y_{ijk} - \mu - \tau_i - \beta_{j(i)}$

$$SSE = \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (y_{ijk} - \mu - \tau_i - \beta_{j(i)})^2$$

Derive For $i = 1, \ldots, a$, $j = 1, \ldots, b$,

$\frac{\partial SSE}{\partial \tau_i} = 2 \sum_j^b \sum_k^n (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_{j(i)})(-1) = 0$

For $\sum_j^b \hat{\beta}_{j(i)} = 0$,

$$\sum_{j}^{b} \sum_{k}^{n} y_{ijk} = bn\hat{\mu} + bn\hat{\tau}_i + n \sum_{j}^{b} \hat{\beta}_{j(i)} = bn\bar{y}_{i..} \implies \hat{\mu} + \hat{\tau}_i = \bar{y}_{i..}$$

$\frac{\partial SSE}{\partial \beta_{j(i)}} = 2 \sum_k^n (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_{j(i)})(-1) = 0$

$$\sum_{k}^{n} y_{ijk} = n\hat{\mu} + n\hat{\tau}_i + n\hat{\beta}_{j(i)} = n\bar{y}_{ij.} \implies \hat{\mu} + \hat{\tau}_i + \hat{\beta}_{j(i)} = \bar{y}_{ij.}$$

$$\implies \hat{\beta}_{j(i)} = \bar{y}_{ij.} - \bar{y}_{i..}$$

(b). Derive $E(MS_{B(A)})$ for this model.

For $\sum_j^b \hat{\beta}_{j(i)} = 0$,

$$\bar{y}_{i..} = \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} = \mu + \tau_i + \frac{1}{b} \sum_{j=1}^{b} \beta_{j(i)} + \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} \varepsilon_{ijk} = \mu + \tau_i + \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} \varepsilon_{ijk}$$

$$\bar{y}_{ij.} = \frac{1}{n} \sum_{k=1}^{n} y_{ijk} = \mu + \tau_i + \beta_j + \frac{1}{n} \sum_{k=1}^{n} \varepsilon_{ijk}$$

$$\bar{y}_{ij.} - \bar{y}_{i..} = \beta_{j(i)} + \frac{1}{n} \sum_{k=1}^{n} \varepsilon_{ijk} - \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} \varepsilon_{ijk}$$

For B has fixed effect, $E[\beta_{j(i)}] = \beta_{j(i)}$, $V[\beta_{j(i)}] = 0$, $Cov[\beta_{j(i)}, \frac{1}{n} \sum_{k=1}^{n} \varepsilon_{(ij)k} - \frac{1}{bn} \sum_{j=1}^{b} \sum_{k=1}^{n} \varepsilon_{ijk}] = 0$

For $\varepsilon_{(ij)k} \sim iidN(0, \sigma^2)$, $E[\varepsilon_{(ij)k}] = 0$, $V[\varepsilon_{(ij)k}] = \sigma^2$

$$E[\bar{y}_{ij.} - \bar{y}_{i..}] = E[\beta_{j(i)} + \frac{1}{n}\sum_{k=1}^{n}\varepsilon_{(ij)k} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{(ij)k}] = \beta_{j(i)}$$

$$Var[\bar{y}_{ij.} - \bar{y}_{i..}] = Var[\beta_{j(i)} + \frac{1}{n}\sum_{k=1}^{n}\varepsilon_{(ij)k} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{(ij)k}]$$

$$= Var[\beta_{j(i)}] + Var[\frac{1}{n}\sum_{k=1}^{n}\varepsilon_{ijk} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk}] + 2Cov[\beta_{j(i)}, \frac{1}{n}\sum_{k=1}^{n}\varepsilon_{ijk} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk}]$$

$$= 0 + \frac{1}{n^2}\sum_{k=1}^{n}Var[\varepsilon_{ijk}] + \frac{1}{b^2n^2}\sum_{j=1}^{b}\sum_{k=1}^{n}Var[\varepsilon_{ijk}] - \frac{2}{bn^2}Cov[\sum_{k=1}^{n}\varepsilon_{ijk}, \sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk}] + 0$$

$$= \frac{1}{n}\sigma^2 + \frac{\sigma^2}{bn} - \frac{2\sigma^2}{bn} = \frac{(b-1)\sigma^2}{bn}$$

For $MS_{B(A)} = SS_{B(A)}/df_{B(A)} = \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..})^2$

$$E(MS_{B(A)}) = \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}E[(\bar{y}_{ij.} - \bar{y}_{i..})^2] = \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\left\{Var[\bar{y}_{ij.} - \bar{y}_{i..}] + E[\bar{y}_{ij.} - \bar{y}_{i..}]^2\right\}$$

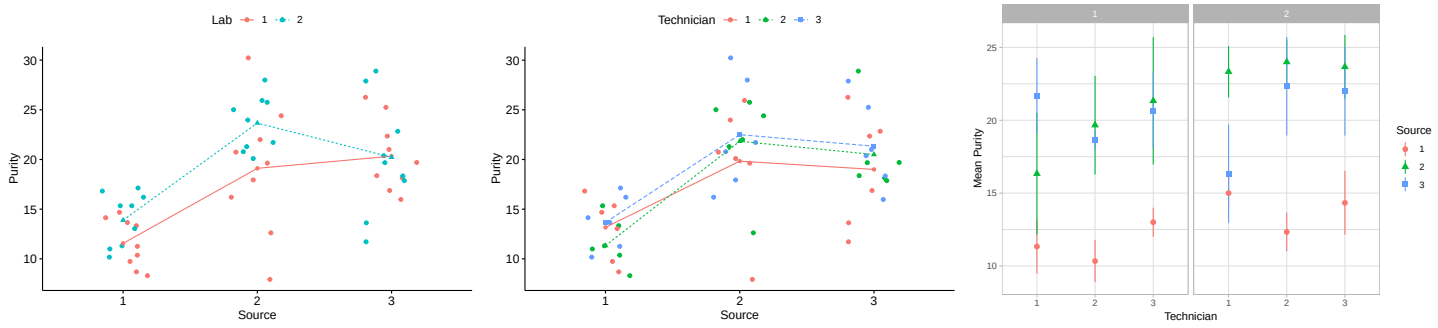
$$= \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\left\{\beta_{j(i)}^2 + \frac{(b-1)\sigma^2}{bn}\right\} = \sigma^2 + \frac{bn}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\beta_{j(i)}^2$$

Question 3

Two laboratories were used to determine the purity of a chemical compound synthesized from 3 sources. Within each of these laboratories, 3 technicians were used to carry out the analysis.

Analyze the data. If you have to decide between unrestricted and restricted models, then make a decision and provide reasons.

```
## Classes 'tbl_df', 'tbl' and 'data.frame':    54 obs. of  4 variables:
##  $ Lab       : Factor w/ 2 levels "1","2": 1 1 1 1 1 1 1 1 1 1 ...
##  $ Technician: Factor w/ 3 levels "1","2","3": 1 1 1 1 1 1 1 1 1 2 ...
##  $ Source    : Factor w/ 3 levels "1","2","3": 1 1 1 2 2 2 3 3 3 1 ...
##  $ Purity    : num  10 9 15 8 20 21 26 22 17 8 ...
```



The above plots show that:

The average purity of a chemical compound synthesized from source 2 and 3 are higher than that from source 1.

There is not much difference in the average purity between sources 2 and 3.

Not all the lines are parallel in the interaction plot. Therefore, in the model, there is the interaction effect of source level and technicians nested in the lab. The Tables show the same thing with the numerical summaries for each factor level and their combinations.

Source	min	Q1	median	Q3	max	mean	sd	n	missing
1	8	10.25	13.0	15.00	17	12.72222	2.803476	18	0
2	8	20.00	21.5	24.75	30	21.38889	5.326282	18	0
3	12	18.00	20.0	22.75	29	20.27778	4.599304	18	0

The average purity analysed in Lab 1 is lower than that from Lab 2.

There is not much difference in the average purity of sources 3 between lab 1 and 2.

Lab	min	Q1	median	Q3	max	mean	sd	n	missing
1	8	13	17	21.0	30	17.00000	5.824352	27	0
2	10	15	20	23.5	29	19.25926	5.633639	27	0

Lab.Source	min	Q1	median	Q3	max	mean	sd	n	missing
1.1	8	10	11	14	15	11.55556	2.505549	9	0
2.1	10	11	15	16	17	13.88889	2.713137	9	0
1.2	8	16	20	22	30	19.11111	6.392270	9	0
2.2	20	21	24	26	28	23.66667	2.783882	9	0
1.3	16	18	20	22	26	20.33333	3.500000	9	0
2.3	12	18	20	23	29	20.22222	5.717906	9	0

There is not much difference in the average purity among technicians.

Technician	min	Q1	median	Q3	max	mean	sd	n	missing
1	8	13.25	17	21.75	26	17.33333	5.718906	18	0
2	8	13.00	18	21.75	29	17.88889	6.057459	18	0
3	10	16.00	18	21.75	30	19.16667	5.762454	18	0

Technician.Source	min	Q1	median	Q3	max	mean	sd	n	missing
1.1	9	10.75	14.0	15.00	17	13.16667	3.125167	6	0
2.1	8	10.25	11.0	12.50	15	11.33333	2.422120	6	0
3.1	10	11.75	14.0	15.50	17	13.66667	2.732520	6	0
1.2	8	20.00	20.5	23.25	26	19.83333	6.274286	6	0
2.2	13	21.25	23.0	24.75	26	21.83333	4.708149	6	0
3.2	16	18.75	21.5	26.50	30	22.50000	5.504544	6	0
1.3	12	14.75	19.5	22.75	26	19.00000	5.513619	6	0
2.3	18	18.00	19.0	20.00	29	20.50000	4.277850	6	0
3.3	16	18.50	20.5	24.00	28	21.33333	4.457204	6	0

This is a neseted and crossed design. Three fixed sources apply on all random selected technicians are nested in fixed labs.

$$y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{j(i)k} + \varepsilon_{(ijk)l}$$

for $i = 1, 2$; $j = 1, 2, 3$; $k = 1, 2, 3$; $l = 1, 2, 3$;

μ is the overall true mean response;

τ_i is the fixed main effect of i^{th} level of labs;

$\beta_{j(i)}$ is the random effect of j^{th} level of technicians nested in i^{th} level of labs;

γ_k is the main fixed effect of k^{th} level of sources;

$(\tau\gamma)_{ik}$ is the interaction effect of i^{th} level of labs and k^{th} level of sources;

$(\beta\gamma)_{j(i)k}$ is the interaction random effect of k^{th} level of sources and j^{th} level of technicians nested in i^{th} level of labs.

y_{ijkl} is response value for the l^{th} replication for j^{th} level of technicians nested in i^{th} level of labs when k^{th} level of sources is applied;

$\varepsilon_{l(ijk)}$ is random error for the l^{th} replication for j^{th} level of technicians nested in i^{th} level of labs when k^{th} level of sources is applied.

Assumptions: Ususally, the nunber of technican are limited. Thus, I tend to believe that this is an unrestricted mode. We cannot expect the summation of randem effects are zero.

$\varepsilon_{l(ijk)} \sim iidN(0, \sigma^2)$	$\sum_{i=1}^{2} \tau_i = 0$	$\sum_{k=1}^{3} \gamma_k = 0$	$\beta_{j(i)} \sim iidN(0, \sigma_\beta^2)$
$\sum_{i=1}^{2} (\tau\gamma)_{ik} = 0$	$\sum_{k=1}^{3} (\tau\gamma)_{ik} = 0$	$\sum_{i=1}^{2} (\beta\gamma)_{j(i)k} = 0$	$\sum_{k=1}^{3} (\beta\gamma)_{jk(i)} = 0$

$\varepsilon_{l(ijk)}$, $\beta_{j(i)}$, and $(\beta\gamma)_{jk(i)}$ are independent.

Table 5: Analysis of Variance Table

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Lab_f	1	68.91	68.91	6.955	0.05775
Source_f	2	800.6	400.3	30.62	0.0001783
Lab_f:Technician_r	4	39.63	9.907	0.5	0.7358
Lab_f:Source_f	2	49.04	24.52	1.875	0.2148
Lab_f:Source_f:Technician_r	8	104.6	13.07	0.6598	0.7226
Residual	36	713.3	19.81	NA	NA

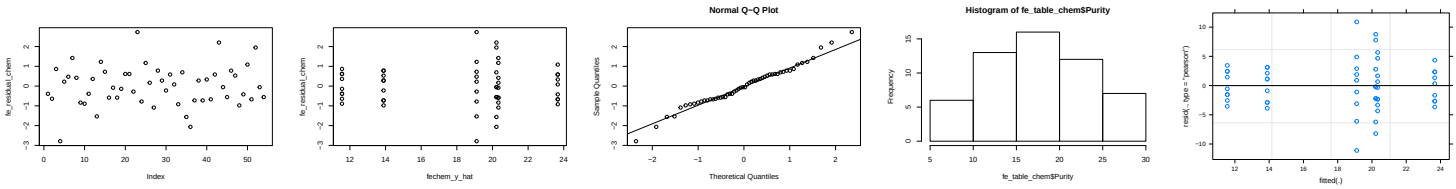
The ANOVA table shows that the average purity of a chemical compound synthesized from three sources are differnet at 0.05 significance level (p-value=0.0001783). As the main effects of sources shown in the above tables, the average purity from source 1 is lowest (12.72222). The the average purity from source 2 and 3 are 21.38889 and 20.27778 respectively.

The results of variance components and condidence intervals show that none of the effects related with technician has significant variance on average value of Hardness at 0.05 significance level. The variance of technicians nested in labs is zero. The variance of interaction effect between sources and technicians nested in labs is zero too. Their confidence intervals are both $[0, 1.589^2]$ at 0.05 significance level.

Groups	Name	Variance	Std.Dev.
Lab.Technician	(Intercept)	0.00	0.000
Lab.Technician.1	(Intercept)	0.00	0.000
Residual		17.87	4.227

Computing profile confidence intervals ...

	2.5 %	97.5 %
.sig01	0	1.589
.sig02	0	1.589
.sigma	3.337	4.873
(Intercept)	8.905	14.21
Lab2	-1.415	6.082
Source2	3.807	11.3
Source3	5.029	12.53
Lab2:Source2	-3.079	7.523
Lab2:Source3	-7.745	2.857



In the plots of residuals versus predicted value of purity, there is no significant pattern on this plot besides that more values occur when the predicted value is higher. Therefore, the fitted model is good enough to describe the relationship between the mean value of purity and the labs, technician, and sources.

The residuals in this plot are almost symmetrically distributed about zero and hence zero mean assumption is not violated. Further, the vertical deviation of the residuals from zero is about same for each predicted value and hence the constant variance assumption is not violated.

The points are along the straight line in the normal qq plot shown at bottom left and the histogram of residuals shown at the top right is about normal. These plots show no violation of normal distribution assumption of residuals.

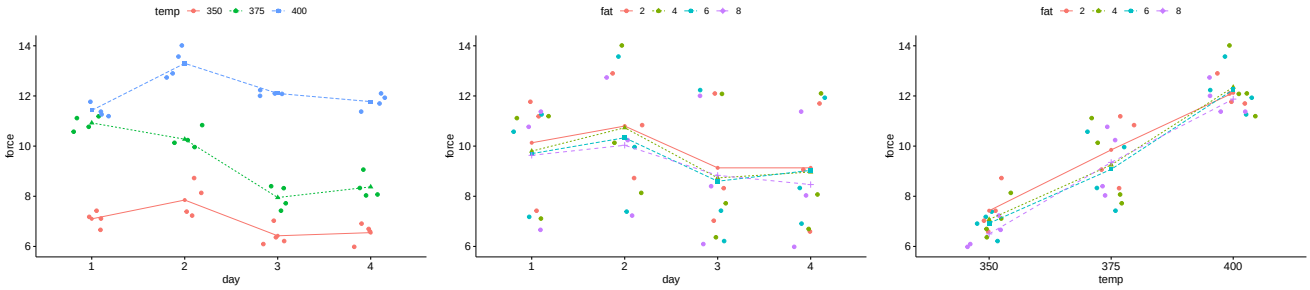
Report your code and/or output at the end of the analysis.

Question 4

A baker wanted to determine the effect that the amount of fat in a recipe of cookie dough would have on the texture of the cookie. The baker also wanted to determine whether the temperature (°F) at which the cookies were baked would have an influence on the texture of the surface. The texture of the cookie is measured by determining the amount of force (g) required to penetrate the cookie surface. On a given day, the baker made a batch of cookie dough for each of the 4 recipes and baked one cookie from each batch in the oven at one time. The baker continued this process each of 4 days so that a single cookie is baked at 3 different temperatures in one day.The data are given in Cookie Excel file. Analyze the data

and answer the baker's questions. If you have to decide between unrestricted and restricted models, then make a decision and provide reasons.

```
## Classes 'tbl_df', 'tbl' and 'data.frame':    48 obs. of  4 variables:
##  $ day  : Factor w/ 4 levels "1","2","3","4": 1 1 1 1 1 1 1 1 1 1 ...
##  $ temp : Factor w/ 3 levels "350","375","400": 1 1 1 1 2 2 2 2 3 3 ...
##  $ fat  : Factor w/ 4 levels "2","4","6","8": 1 2 3 4 1 2 3 4 1 2 ...
##  $ force: num  7.4 7.1 7.2 6.7 11.2 11.1 10.6 10.8 11.8 11.2 ...
```



The above plots show that: There is not much difference in the average force of the cookie from different days or form different fat. The tables show the same thing with the numerical summaries for each factor level and their combinations.

day	min	Q1	median	Q3	max	mean	sd	n	missing
1	6.7	7.35	10.95	11.225	11.8	9.816667	2.033284	12	0
2	7.2	8.55	10.15	12.750	14.0	10.475000	2.384467	12	0
3	6.1	6.85	8.00	12.025	12.2	8.825000	2.526271	12	0
4	6.0	6.85	8.20	11.475	12.1	8.900000	2.290693	12	0

fat	min	Q1	median	Q3	max	mean	sd	n	missing
2	6.6	8.075	9.95	11.725	12.9	9.800000	2.200413	12	0
4	6.4	7.550	9.10	11.425	14.0	9.558333	2.520988	12	0
6	6.2	7.350	9.15	11.450	13.6	9.416667	2.480408	12	0
8	6.0	7.075	9.30	11.400	12.7	9.241667	2.439992	12	0

The average force of the cookie are higher when the temperature is higher.

temp	min	Q1	median	Q3	max	mean	sd	n	missing
350	6.0	6.550	6.95	7.250	8.7	6.98125	0.7148135	16	0
375	7.4	8.250	9.55	10.650	11.2	9.38125	1.3422463	16	0
400	11.2	11.625	12.05	12.325	14.0	12.15000	0.7983316	16	0

Not all the lines are parallel in the interaction plot. Therefore, in the model, there is the interaction effect of day and temperature.

day.fat	min	Q1	median	Q3	max	mean	sd	n	missing
1.2	7.4	9.30	11.2	11.50	11.8	10.133333	2.386071	3	0
2.2	8.7	9.75	10.8	11.85	12.9	10.800000	2.100000	3	0
3.2	7.0	7.65	8.3	10.20	12.1	9.133333	2.650157	3	0
4.2	6.6	7.85	9.1	10.40	11.7	9.133333	2.550163	3	0
1.4	7.1	9.10	11.1	11.15	11.2	9.800000	2.338803	3	0
2.4	8.1	9.10	10.1	12.05	14.0	10.733333	3.000555	3	0
3.4	6.4	7.05	7.7	9.90	12.1	8.733333	2.987195	3	0
4.4	6.7	7.40	8.1	10.10	12.1	8.966667	2.802380	3	0
1.6	7.2	8.90	10.6	10.95	11.3	9.700000	2.193171	3	0
2.6	7.4	8.70	10.0	11.80	13.6	10.333333	3.113412	3	0
3.6	6.2	6.80	7.4	9.80	12.2	8.600000	3.174902	3	0
4.6	6.9	7.60	8.3	10.10	11.9	9.033333	2.579406	3	0
1.8	6.7	8.75	10.8	11.10	11.4	9.633333	2.557994	3	0
2.8	7.2	8.70	10.2	11.45	12.7	10.033333	2.753785	3	0
3.8	6.1	7.25	8.4	10.20	12.0	8.833333	2.973774	3	0
4.8	6.0	7.00	8.0	9.70	11.4	8.466667	2.730079	3	0

temp.fat	min	Q1	median	Q3	max	mean	sd	n	missing
350.2	6.6	6.900	7.20	7.725	8.7	7.425	0.9105859	4	0
375.2	8.3	8.900	9.95	10.900	11.2	9.850	1.3771952	4	0
400.2	11.7	11.775	11.95	12.300	12.9	12.125	0.5439056	4	0
350.4	6.4	6.625	6.90	7.350	8.1	7.075	0.7410578	4	0
375.4	7.7	8.000	9.10	10.350	11.1	9.250	1.6196707	4	0
400.4	11.2	11.875	12.10	12.575	14.0	12.350	1.1789826	4	0
350.6	6.2	6.725	7.05	7.250	7.4	6.925	0.5251984	4	0
375.6	7.4	8.075	9.15	10.150	10.6	9.075	1.4818344	4	0
400.6	11.3	11.750	12.05	12.550	13.6	12.250	0.9746794	4	0
350.8	6.0	6.075	6.40	6.825	7.2	6.500	0.5597619	4	0
375.8	8.0	8.300	9.30	10.350	10.8	9.350	1.3601471	4	0
400.8	11.4	11.400	11.70	12.175	12.7	11.875	0.6184658	4	0

temp.day	min	Q1	median	Q3	max	mean	sd	n	missing
350.1	6.7	7.000	7.15	7.250	7.4	7.100	0.2943920	4	0
375.1	10.6	10.750	10.95	11.125	11.2	10.925	0.2753785	4	0
400.1	11.2	11.275	11.35	11.500	11.8	11.425	0.2629956	4	0
350.2	7.2	7.350	7.75	8.250	8.7	7.850	0.6855655	4	0
375.2	10.0	10.075	10.15	10.350	10.8	10.275	0.3593976	4	0
400.2	12.7	12.850	13.25	13.700	14.0	13.300	0.6055301	4	0
350.3	6.1	6.175	6.30	6.550	7.0	6.425	0.4031129	4	0
375.3	7.4	7.625	8.00	8.325	8.4	7.950	0.4795832	4	0
400.3	12.0	12.075	12.10	12.125	12.2	12.100	0.0816497	4	0
350.4	6.0	6.450	6.65	6.750	6.9	6.550	0.3872983	4	0
375.4	8.0	8.075	8.20	8.500	9.1	8.375	0.4991660	4	0
400.4	11.4	11.625	11.80	11.950	12.1	11.775	0.2986079	4	0

This is a simple Split-Plot design model (day is whole-plot factor and temperature is split-plot factor)

$$y_{ijkl} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijk}$$

for $i = 1, 2, 3, 4$; $j = 1, 2, 3$; $k = 1, 2, 3, 4$

μ is the overall true mean response;

τ_i is the effect of i^{th} replication of days;

β_j is the main effect of j^{th} level of temperature (effect of whole-plot factor);

$(\tau\beta)_{ij}$ is the interaction effect of i^{th} replication and j^{th} level of temperature (whole-plot error);

γ_k is the main effect of k^{th} level of fat;

$(\tau\gamma)_{ik}$ is the interaction effect of i^{th} replicatin and k^{th} level of fat;

$(\beta\gamma)_{jk}$ is the interaction effect of j^{th} level of temperature and k^{th} level of fat;

$(\tau\beta\gamma)_{ijk}$ is the interaction effect of i^{th} replicatin, j^{th} level of temperature and k^{th} level of fat (sub-plot error);

y_{ijk} is response value for the i^{th} replication when j^{th} level of temperature and k^{th} level of fat are applied;

$\varepsilon_{l(ijk)}$ is random error for the i^{th} replication when j^{th} level of temperature and k^{th} level of fat are applied.

Assumptions: I tend to believe that, for an experienced baker, the selection of temperature is deliberate. There are numerous days for choosing. Thus, this is a restricted model.

$\varepsilon_{l(ijk)} \sim iidN(0,\sigma^2)$	$\tau_i \sim iidN(0,\sigma^2_\tau)$	
$\sum_{j=1}^{3}\beta_j = 0$	$\sum_{j=1}^{3}(\tau\beta)_{ij} = 0$	$(\tau\beta)_{ij} \sim iidN(0, \frac{3-1}{3}\sigma^2_{\tau\beta})$
$\sum_{k=1}^{4}\gamma_k = 0$	$\sum_{k=1}^{4}(\tau\gamma)_{ik} = 0$	$(\tau\gamma)_{ik} \sim iidN(0, \frac{4-1}{4}\sigma^2_{\tau\gamma})$
$\sum_{j=1}^{3}(\beta\gamma)_{jk} = 0$	$\sum_{k=1}^{4}(\beta\gamma)_{jk} = 0$	
$\sum_{j=1}^{3}(\tau\beta\gamma)_{ijk} = 0$	$\sum_{k=1}^{4}(\tau\beta\gamma)_{ijk} = 0$	$(\tau\beta\gamma)_{ijk} \sim iidN(0, \frac{(3-1)(4-1)}{3\times4}\sigma^2_{\tau\beta\gamma})$

ε_{ijk}, τ_i, $(\tau\beta)_{ij}$, $(\tau\gamma)_{ik}$, $(\beta\gamma)_{jk}$, and $(\tau\beta\gamma)_{ijk}$ are independent.

Since this is a simple replicated factorial design, I use $(\tau\beta\gamma)_{ijk}$ to compute SSE and df.

Table 9: Analysis of Variance Table

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
day_r	3	22.4	7.466	74.5	2.415e-10
fat_f	3	2.004	0.6681	7.012	0.009915
temp_f	2	214.1	107	41.18	0.0003131
day_r:fat_f	9	0.8575	0.09528	0.9508	0.5082
day_r:temp_f	6	15.6	2.599	25.94	6.251e-08
fat_f:temp_f	6	1.59	0.2649	2.644	0.05113
Residual	18	1.804	0.1002	NA	NA

Random effects:

Groups	Name	Variance	Std.Dev.
day:fat	(Intercept)	1.697e-15	4.120e-08
day:temp	(Intercept)	6.252e-01	7.907e-01
day	(Intercept)	4.055e-01	6.368e-01
Residual		9.856e-02	3.140e-01

	2.5 %	97.5 %
.sig01	0	0.1906
.sig02	0.4371	1.242
.sig03	0	1.626
.sigma	0.2116	0.3492
(Intercept)	6.414	8.436

The results show the interaction term of days and fat is negligible. I use it to compute error and set a new model.

Table 12: Analysis of Variance Table

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
day_r	3	22.4	7.466	75.75	2.881e-13
fat_f	3	2.004	0.6681	6.778	0.00149
temp_f	2	214.1	107	41.18	0.0003131
day_r:temp_f	6	15.6	2.599	26.37	4.298e-10
fat_f:temp_f	6	1.59	0.2649	2.688	0.03544
Residual	27	2.661	0.09856	NA	NA

The ANOVA table shows that there is a significant interaction effect from the day and temperature, on average amount of force (g) (p-value=4.298×10^{-10}). This means that the effect of day and effect of temperature on the force are not independent. Similarly, there is a significant interaction effect from the fat and temperature, on average amount of force (g) (p-value=0.03544). This means that the effect of fat and effect of temperature on the force are not independent. Hence, the simple effects must be tested.

Since there are several simple effects comparison tests, the Tukey's adjustment was used to compute p value to reduce the experimentwise error rate. The Tables below show the summary of all those simple effect comparison tests.

For all levels of fat, the mean forces are significantly different between 350°F and 400°F (p value=0.0011369, 0.0005290, 0.0004966, 0.0004663 respectively).

When the fat was 4, 6 and 8, the mean forces are significantly different between 375°F and 400°F (p value=0.0145686, 0.0127251,0.0432588 respectively).

When the fat was 8, the mean forces are significantly different between 350°F and 375°F (p value=0.0231342).

When temperature was 350°F, the mean forces are significantly different between the fat of 2 and 8 (p value=0.0063518).

When temperature was 375°F, the mean forces are significantly different between the fat of 2 and 6 (p value=0.0326101).

For all the rest of temperature, the mean forces are not significantly different between any value of fat.

When the day 1, the mean forces are not significantly different between 375°F and 400°F (p value=0.3674143).

For all the rest of days, the mean forces are significantly different between any value of temperature.

For all levels of temperature, the mean forces are significantly different between the day 2 v.s. day 3, and day 2 v.s. day 4 (p value<0.0002).

When temperature was 350°F and 400°F, the mean forces are significantly different between the day 1 and day 2 (p value=0.0421959, 0.0000001).

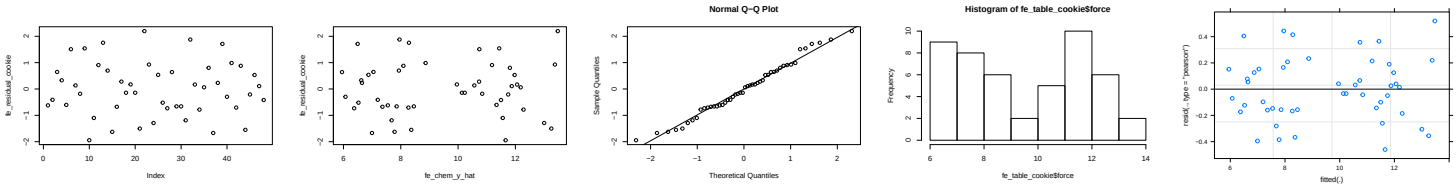
When temperature was 375°F, the mean forces are significantly different between the day 1 v.s. day 3, and day 1 v.s. day 4 (p value=0.0000000).

fat	temp	contrast	estimate	SE	df	t.ratio	p.value
2	.	350 - 375	-2.425	0.6015677	7.421373	-4.0311341	0.0527024
2	**.**	**350 - 400**	**-4.700**	**0.6015677**	**7.421373**	**-7.8129198**	**0.0011369**
2	.	375 - 400	-2.275	0.6015677	7.421373	-3.7817856	0.0711075
4	.	350 - 375	-2.175	0.6015677	7.421373	-3.6155533	0.0869670
4	**.**	**350 - 400**	**-5.275**	**0.6015677**	**7.421373**	**-8.7687557**	**0.0005290**
4	**.**	**375 - 400**	**-3.100**	**0.6015677**	**7.421373**	**-5.1532024**	**0.0145686**
6	.	350 - 375	-2.150	0.6015677	7.421373	-3.5739952	0.0914689
6	**.**	**350 - 400**	**-5.325**	**0.6015677**	**7.421373**	**-8.8518719**	**0.0004966**
6	**.**	**375 - 400**	**-3.175**	**0.6015677**	**7.421373**	**-5.2778766**	**0.0127251**
8	**.**	**350 - 375**	**-2.850**	**0.6015677**	**7.421373**	**-4.7376216**	**0.0231342**
8	**.**	**350 - 400**	**-5.375**	**0.6015677**	**7.421373**	**-8.9349880**	**0.0004663**
8	**.**	**375 - 400**	**-2.525**	**0.6015677**	**7.421373**	**-4.1973665**	**0.0432588**
.	350	2 - 4	0.350	0.2219964	27.000000	1.5766020	0.7734870
.	350	2 - 6	0.500	0.2219964	27.000000	2.2522886	0.3674143
.	**350**	**2 - 8**	**0.925**	**0.2219964**	**27.000000**	**4.1667340**	**0.0063518**
.	350	4 - 6	0.150	0.2219964	27.000000	0.6756866	0.9976187
.	350	4 - 8	0.575	0.2219964	27.000000	2.5901319	0.2117431
.	350	6 - 8	0.425	0.2219964	27.000000	1.9144453	0.5690221
.	375	2 - 4	0.600	0.2219964	27.000000	2.7027464	0.1724320
.	**375**	**2 - 6**	**0.775**	**0.2219964**	**27.000000**	**3.4910474**	**0.0326101**
.	375	2 - 8	0.500	0.2219964	27.000000	2.2522886	0.3674143
.	375	4 - 6	0.175	0.2219964	27.000000	0.7883010	0.9936958
.	375	4 - 8	-0.100	0.2219964	27.000000	-0.4504577	0.9998421
.	375	6 - 8	-0.275	0.2219964	27.000000	-1.2387587	0.9209247
.	400	2 - 4	-0.225	0.2219964	27.000000	-1.0135299	0.9723239
.	400	2 - 6	-0.125	0.2219964	27.000000	-0.5630722	0.9992813
.	400	2 - 8	0.250	0.2219964	27.000000	1.1261443	0.9511428
.	400	4 - 6	0.100	0.2219964	27.000000	0.4504577	0.9998421
.	400	4 - 8	0.475	0.2219964	27.000000	2.1396742	0.4309173
.	400	6 - 8	0.375	0.2219964	27.000000	1.6892165	0.7088627

* Adjusted by Tukey's Method

day_r	temp_f	contrast	estimate	SE	df	t.ratio	p.value
1	**.**	**350 - 375**	**-3.825**	**0.2219964**	**27**	**-17.2300081**	**0.0000000**
1	**.**	**350 - 400**	**-4.325**	**0.2219964**	**27**	**-19.4822968**	**0.0000000**
1	.	375 - 400	-0.500	0.2219964	27	-2.2522886	0.3674143
2	**.**	**350 - 375**	**-2.425**	**0.2219964**	**27**	**-10.9235999**	**0.0000000**
2	**.**	**350 - 400**	**-5.450**	**0.2219964**	**27**	**-24.5499462**	**0.0000000**
2	**.**	**375 - 400**	**-3.025**	**0.2219964**	**27**	**-13.6263463**	**0.0000000**
3	**.**	**350 - 375**	**-1.525**	**0.2219964**	**27**	**-6.8694804**	**0.0000060**
3	**.**	**350 - 400**	**-5.675**	**0.2219964**	**27**	**-25.5634761**	**0.0000000**
3	**.**	**375 - 400**	**-4.150**	**0.2219964**	**27**	**-18.6939957**	**0.0000000**
4	**.**	**350 - 375**	**-1.825**	**0.2219964**	**27**	**-8.2208535**	**0.0000002**
4	**.**	**350 - 400**	**-5.225**	**0.2219964**	**27**	**-23.5364163**	**0.0000000**
4	**.**	**375 - 400**	**-3.400**	**0.2219964**	**27**	**-15.3155628**	**0.0000000**
.	**350**	**1 - 2**	**-0.750**	**0.2219964**	**27**	**-3.3784330**	**0.0421959**
.	350	1 - 3	0.675	0.2219964	27	3.0405897	0.0881989
.	350	1 - 4	0.550	0.2219964	27	2.4775175	0.2573498
.	**350**	**2 - 3**	**1.425**	**0.2219964**	**27**	**6.4190226**	**0.0000188**
.	**350**	**2 - 4**	**1.300**	**0.2219964**	**27**	**5.8559505**	**0.0000805**
.	350	3 - 4	-0.125	0.2219964	27	-0.5630722	0.9992813
.	375	1 - 2	0.650	0.2219964	27	2.9279752	0.1112061
.	**375**	**1 - 3**	**2.975**	**0.2219964**	**27**	**13.4011174**	**0.0000000**
.	**375**	**1 - 4**	**2.550**	**0.2219964**	**27**	**11.4866721**	**0.0000000**
.	**375**	**2 - 3**	**2.325**	**0.2219964**	**27**	**10.4731422**	**0.0000000**
.	**375**	**2 - 4**	**1.900**	**0.2219964**	**27**	**8.5586968**	**0.0000001**
.	375	3 - 4	-0.425	0.2219964	27	-1.9144453	0.5690221
.	**400**	**1 - 2**	**-1.875**	**0.2219964**	**27**	**-8.4460824**	**0.0000001**
.	400	1 - 3	-0.675	0.2219964	27	-3.0405897	0.0881989
.	400	1 - 4	-0.350	0.2219964	27	-1.5766021	0.7734870
.	**400**	**2 - 3**	**1.200**	**0.2219964**	**27**	**5.4054927**	**0.0002601**
.	**400**	**2 - 4**	**1.525**	**0.2219964**	**27**	**6.8694804**	**0.0000060**
.	400	3 - 4	0.325	0.2219964	27	1.4639876	0.8314608

* Adjusted by Tukey's Method



In the plots of residuals versus predicted value of purity, there is no significant pattern on this plot besides that more values occur when the predicted value is higher. Therefore, the fitted model is good enough to describe the relationship between the mean value of purity and the labs, technician, and sources.

The residuals in this plot are almost symmetrically distributed about zero and hence zero mean assumption is not violated. Further, the vertical deviation of the residuals from zero is about same for each predicted value and hence the constant variance assumption is not violated.

The points are along the straight line in the normal qq plot shown at bottom left and the histogram of residuals shown at the top right is about normal. These plots show no violation of normal distribution assumption of residuals.

For $\sum_j^b \hat{\beta}_{j(i)} = 0$,

$$\bar{y}_{i..} = \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} = \mu + \tau_i + \frac{1}{b}\sum_{j=1}^{b}\beta_{j(i)} + \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk} = \mu + \tau_i + \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$\bar{y}_{ij.} = \frac{1}{n}\sum_{k=1}^{n} y_{ijk} = \mu + \tau_i + \beta_j + \frac{1}{n}\sum_{k=1}^{n} \varepsilon_{ijk}$$

$$\bar{y}_{ij.} - \bar{y}_{i..} = \beta_{j(i)} + \frac{1}{n}\sum_{k=1}^{n} \varepsilon_{ijk} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{ijk}$$

For B nested in the random factor A, $\beta_j(i) \sim iid\ N(0, \sigma_\beta^2)$, $E[\beta_{j(i)}] = 0$, $V[\beta_{j(i)}] = \sigma_\beta^2$

For τ_i and $\varepsilon_{(ij)k}$ are independent, $\beta_{j(i)}$ and $\varepsilon_{(ij)k}$ are independent. $Cov[\beta_{j(i)}, \frac{1}{n}\sum_{k=1}^{n} \varepsilon_{(ij)k} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{ijk}] = 0$

For $\varepsilon_{(ij)k} \sim iidN(0, \sigma^2)$, $E[\varepsilon_{(ij)k}] = 0$, $V[\varepsilon_{(ij)k}] = \sigma^2$

$$E[\bar{y}_{ij.} - \bar{y}_{i..}] = E[\beta_{j(i)} + \frac{1}{n}\sum_{k=1}^{n} \varepsilon_{(ij)k} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{(ij)k}] = 0$$

$$Var[\bar{y}_{ij.} - \bar{y}_{i..}] = Var[\beta_{j(i)} + \frac{1}{n}\sum_{k=1}^{n} \varepsilon_{(ij)k} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{(ij)k}]$$

$$= Var[\beta_{j(i)}] + Var[\frac{1}{n}\sum_{k=1}^{n} \varepsilon_{ijk} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{ijk}] + 2Cov[\beta_{j(i)}, \frac{1}{n}\sum_{k=1}^{n} \varepsilon_{ijk} - \frac{1}{bn}\sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{ijk}]$$

$$= \sigma_\beta^2 + \frac{1}{n^2}\sum_{k=1}^{n} Var[\varepsilon_{ijk}] + \frac{1}{b^2n^2}\sum_{j=1}^{b}\sum_{k=1}^{n} Var[\varepsilon_{ijk}] - \frac{2}{bn^2}Cov[\sum_{k=1}^{n} \varepsilon_{ijk}, \sum_{j=1}^{b}\sum_{k=1}^{n} \varepsilon_{ijk}] + 0$$

$$= \sigma_\beta^2 + \frac{1}{n}\sigma^2 + \frac{\sigma^2}{bn} - \frac{2\sigma^2}{bn} = \sigma_\beta^2 + \frac{(b-1)\sigma^2}{bn}$$

For $MS_{B(A)} = SS_{B(A)}/df_{B(A)} = \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..})^2$

$$E(MS_{B(A)}) = \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b} E[(\bar{y}_{ij.} - \bar{y}_{i..})^2] = \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\left\{Var[\bar{y}_{ij.} - \bar{y}_{i..}] + E[\bar{y}_{ij.} - \bar{y}_{i..}]^2\right\}$$

$$= \frac{n}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\left\{\sigma_\beta^2 + \frac{(b-1)\sigma^2}{bn}\right\} = \sigma^2 + \frac{bn}{b-1}\sigma_\beta^2$$