

STAT562 Final Exam

Winter 2019

1. X_1, X_2, \dots, X_n is a random sample from a distribution having a p.d.f of the form. $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find a complete sufficient statistic for λ . Justify your answer

• **Step1: Proof sufficient**

From Fisher–Neyman factorization theorem (2019-2-14p5)

$$f(x|\lambda) = L(\lambda) = \lambda^n \left(\prod x_i \right)^{\lambda-1} = \lambda^n e^{(\lambda-1) \sum_{i=1}^n \ln x_i} \cdot 1 = k(t|\lambda)h(\vec{x})$$

$h(\vec{x}) = 1$ is free of λ . So $T = \sum_{i=1}^n \ln x_i$ is a sufficient statistic for λ .

• **Step2: Proof complete**

$f(x|\lambda)$ is a member of the exponential family (2019-2-19p12). By the Theorem of Complete Statistics in the exponential family

$$f(x|\vec{\lambda}) = \lambda^n e^{\sum_{i=1}^n (\lambda-1) \ln x_i} = h(x)c(\vec{\lambda})e^{\sum_{j=1}^k W_j(\vec{\lambda})t_j(x)}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0$, $\lambda > 0$. $\{W_1(\vec{\lambda}), \dots, W_k(\vec{\lambda})\}$ contains an open interval in \mathbb{R} , so $T(\vec{x}) = \sum_{i=1}^n \ln x_i$ is a complete sufficient statistic for λ .

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2. Let Y_n be the n^{th} order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and \bar{Y} are independent.

• **Step1: θ is a location parameter**

Let $x = y - \theta$. For $N(\theta, \sigma^2)$ is a location family of densities (2018. 11. 20p7),

$$g(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = f(x) = f(y - \theta)$$

Thus, θ is a location parameter.

• **Step2: $Y_n - \bar{Y}$ is location invariant**

2019-2-21p4-6

Let $X = Y_n - \bar{Y}$. $Y_n \sim N(\theta, \sigma^2)$, $\bar{Y} \sim N(\theta, \sigma^2/n)$

Consider the group of transformations defined by $\mathcal{G} = \{Y_n - \bar{Y}, -\infty < \bar{Y} < \infty\}$, where $Y_n - \bar{Y} = (y_1 - \bar{y}, \dots, y_n - \bar{y})$. By convolution formula,

$$f_X(x) = \int_{-\infty}^{\infty} f_{Y_n}(w) f_{\bar{Y}}(x - w) dw = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(w-\theta)^2}{2\sigma^2}} \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(x-w-\theta)^2}{2\sigma^2}} dw$$

$$Y_n - \bar{Y} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - \theta)^2}{2\sigma^2}} - \frac{1}{n} \sum_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - \theta)^2}{2\sigma^2}}$$

$$Y_n - \bar{Y} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - \theta)^2}{2\sigma^2}} - \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y} - \theta)^2}{2\sigma^2}}$$

Thus, the joint distribution of $Y_n - \bar{Y}$ is in \mathcal{F} and hence \mathcal{F} is invariant under \mathcal{G} .

- step3: $Y_n - \bar{Y}$ is ancillary statistic for θ

2019-2-19p6

$f(y|\theta)$ is a location exponential family. Let $X_n = Y_n - \theta$ is a random sample from $f(y|0)$

$$Y_n - \bar{Y} = Y_n - \frac{1}{n} \sum_{i=1}^n Y_i = (Y_n - \theta) - \frac{1}{n} \sum_{i=1}^n (Y_i - \theta) = X_n - \frac{1}{n} \sum_{i=1}^n X_i$$

$Y_n - \bar{Y}$ is a function of only X_1, \dots, X_n and be free of θ . It is an ancillary statistic.

- step4: \bar{Y} is sufficient statistic for θ

Method 1:

$f(\mathbf{y}|\theta)$ is the joint pdf or pmf of \mathbf{Y} , $g(\bar{y}|\theta)$ is the pdf or pmf of $\bar{\mathbf{Y}}$.

$$f(\bar{y}|\theta) = \prod_{i=1}^n \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - \theta)^2}{2\sigma^2}} \right) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \theta)^2} = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \bar{y} + \bar{y} - \theta)^2}$$

For $\sum_{i=1}^n (y_i - \bar{y}) = 0$, $\sum_{i=1}^n (\bar{y} - \theta)^2 = n(\bar{y} - \theta)^2$, the part of exponent is

$$= -\frac{1}{2\sigma^2} \sum_{i=1}^n [(y_i - \bar{y})^2 + (y_i - \bar{y})(\bar{y} - \theta) + (\bar{y} - \theta)^2] = -\frac{1}{2\sigma^2} \left[\sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \theta)^2 \right]$$

For $\bar{Y} \sim N(\theta, \sigma^2/n)$, $g(\bar{y}|\theta) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y} - \theta)^2}{2\sigma^2}}$

$$\frac{f(\mathbf{y}|\theta)}{g(\bar{y}|\theta)} = \frac{\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} [\sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \theta)^2]}}{\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y} - \theta)^2}{2\sigma^2}}} = \frac{1}{\sqrt{n}(\sigma\sqrt{2\pi})^{n-1}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \bar{y})^2}$$

which is free of θ (6.2.2). For every \mathbf{y} in the sample space, the ratio $f(\mathbf{y}|\theta)/g(\bar{y}|\theta)$ is constant as a function of θ , then \bar{Y} is a sufficient statistic for θ .

Method 2:

$$f(\bar{y}|\theta) = \prod_{i=1}^n \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - \theta)^2}{2\sigma^2}} \right) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \theta)^2} = \frac{e^{-\frac{n\theta^2}{2\sigma^2}} e^{\frac{\theta}{\sigma^2} \sum_{i=1}^n y_i} e^{-\frac{\sum_{i=1}^n y_i^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n} = k(t|\theta)h(\bar{y})$$

$h(\vec{y}) = e^{-\frac{\sum_{i=1}^n y_i^2}{2\sigma^2}}$ is free of θ . By the *Factorization Theorem*, $\sum_{i=1}^n y_i$ is a sufficient statistic for θ . $\bar{Y} = \frac{1}{n} \sum_{i=1}^n y_i$ is a function of $\sum_{i=1}^n y_i$ and is a minimal sufficient statistic for θ .

- step5: \bar{Y} is complete statistic for θ

Method 1: 2019-2-19p7-9

For $\bar{Y} \sim N(\theta, \sigma^2/n)$, the family of is $\left\{ \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}} : -\infty < \theta < \infty \right\}$

Suppose that $E[g(\bar{Y})] = 0 \forall \theta$

$$\int_0^\infty g(\bar{y}) f(\bar{y}) d\bar{y} = \int_0^\infty g(\bar{y}) \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}} d\bar{y} = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \int_0^\infty g(\bar{y}) e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}} d\bar{y} = 0$$

For $\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \neq 0$, $\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = u'(x)f[u(x)] - v'(x)f[v(x)]$, then

$$0 = \frac{d}{d\theta} E[g(\bar{Y})] = \frac{d}{d\theta} \left[\int_{-\infty}^\infty g(\bar{y}) e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}} d\bar{y} \right] = 0 - \theta' [g(\theta) e^{-\frac{n(\theta-\theta)^2}{2\sigma^2}}] = -g(\theta)$$

So $g(\theta) = 0, \forall \theta$, then $P(g(T) = 0) = 1$. Thus, \bar{Y} is a complete statistic.

Method 2:

$f(y|\theta)$ is a member of the exponential family (2019-2-19p12),

$$f(y|\vec{\theta}) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \theta)^2} = e^{-\frac{\sum_{i=1}^n y_i^2}{2\sigma^2}} \frac{e^{-\frac{n\theta^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n} e^{\frac{\theta}{\sigma^2} \sum_{i=1}^n y_i} = h(y) c(\vec{\theta}) e^{\sum_{j=1}^k W_j(\vec{\theta}) t_j(y)}$$

For $\{W_1(\vec{\theta}), \dots, W_k(\vec{\theta})\}$ contains an open interval in \mathbb{R} , so $T(\vec{y}) = \bar{y}$ is a complete sufficient statistic.

- step6: By Basu's theorem, an ancillary statistic $Y_n - \bar{Y}$ and a complete sufficient statistic \bar{Y} are independent. 2019-2-19p10

3. Suppose that $X_1, X_2, \dots, X_n \sim \text{i.i.d.}$ $f(x|\theta) = \theta e^{-\theta x}, x > 0$. Assume that the prior distribution of θ is $\pi(\theta) = \lambda e^{-\lambda \theta}, \theta > 0$

- Find the posterior distribution $\pi(\theta|\vec{x})$.

2019-2-26p8-9, p11-p13

For $L(\theta) = \hat{\theta} e^{-\theta \sum x_i}$, $\pi(\theta) = \lambda e^{-\lambda \theta}$, and the kernel of a function is the main part of the function, the part that remains when constants are disregarded (2019-2-28p8 Example 2.3.8). that is

$$\pi(\theta|\vec{x}) \propto L(\theta) \pi(\theta) = \theta^n e^{-\theta \sum x_i} \lambda e^{-\lambda \theta} \propto \theta^{n+1-1} e^{-\theta(\lambda + \sum x_i)}$$

which is $\text{Gamma}(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ distribution.

- Find the Bayes estimator of θ , assuming square-error loss

2019-2-28p1

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\lambda + \sum x_i}$$

which is the posterior mean.

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- c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(\lambda + n\bar{x})} = \frac{1}{\frac{1}{n+1}(\frac{1}{1/\lambda} + \frac{n}{1/\bar{x}})}$$

which is the weighted harmonic mean of $1/\lambda$, which is the prior mean, and $1/\bar{x}$, which is the MLE of θ .

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- d. Find the Bayes estimator of θ , assuming absolute loss

2019-2-28p4, 9

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = \text{median}[\theta|\vec{x}]$$

For the median of Gamma distribution doesn't have a closed form, the posterior median would not have a closed form.

Postmedian $\hat{\theta} = F^{-1}(\frac{1}{2})$ where $F(x)$ is the $\text{Gamma}(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ cdf, for which there is no closed form.

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- e. Find the Bayes estimator of θ , assuming binary loss

2019-2-28p5-6

$$\text{Suppose } L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}.$$

$$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$$

To minimize this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = \text{mode}[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\lambda + \sum x_i}$$

which is the posterior mode.

4. Redo all of problem 3, using the non-informative prior $\pi(\theta) = 1, \theta > 0$. Note that this is not a valid density function since its integral is infinite, but proceed with it anyway

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

2019-2-26p8-9

For $\pi(\theta) = 1, \theta > 0, L(\theta) = \theta^n e^{-\theta \sum x_i}$, from the kernel of function,

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \sim \text{Gamma}(\alpha = n + 1, \beta = \frac{1}{\sum x_i})$$

b. Find the Bayes estimator of θ , assuming square-error loss

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{\text{Bayes}} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\sum x_i}$$

which is the posterior mean.

c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\hat{\theta}_{\text{Bayes}} = \frac{1}{\frac{1}{n+1}(1 \times 0 + n\bar{x})} = \lim_{c \rightarrow \infty} \frac{1}{\frac{1}{n+1}(\frac{1}{c} + \frac{n}{1/\bar{x}})}$$

which is the weighted harmonic mean of c , which is the prior mean when $c \rightarrow \infty$, and $1/\bar{x}$, which is the MLE of θ .

d. Find the Bayes estimator of θ , assuming absolute loss

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{\text{Bayes}} = \text{median}[\theta|\vec{x}]$$

For the median of Gamma distribution doesn't have a closed form, the posterior median would not have a closed form.

e. Find the Bayes estimator of θ , assuming binary loss

$$\text{Suppose } L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}.$$

$$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$$

To minimize this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = \text{mode}[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

which is the posterior mode.

5. Let $X_1, X_2, \dots, X_n \sim \text{idd.}$ $f(x|\theta) = \theta x^{-\theta-1}, x_i > 1, \theta > 2$.

a. Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ .

2019-2-2p12

$$f(\vec{x}|\theta) = L(\theta) = \theta^n (\prod x_i)^{\theta-1} = \theta^n e^{(-\theta-1) \sum_{i=1}^n \ln x_i}$$

$$l(\theta) = n \ln \theta - (\theta + 1) \sum_{i=1}^n \ln x_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n \ln x_i \stackrel{\text{set}}{=} 0$$

$$\hat{\theta}_{MLE} = \frac{n}{\sum \ln x_i}$$

b. Find the expected value of $\hat{\theta}_{MLE}$.

Let $Y_i = \ln x_i$, then $X = e^y, \frac{dx}{dy} e^y$

$$g(Y) = \theta(e^y)^{-\theta-1} e^y = \theta e^{-y\theta}, y > 0$$

So $Y_i = \ln x_i \sim \text{Gamma}(\alpha = n, \beta = \frac{1}{\theta})$

$$E[\hat{\theta}] = nE[Y^{-1}] = n \frac{\beta^{-1} \Gamma(-1 + \alpha)}{\Gamma(\alpha)} = \frac{n\theta \Gamma(n-1)}{\Gamma(n)} = \frac{n\theta(n-2)!}{(n-1)!} = \frac{n\theta}{n-1}$$

c. Find the variance of $\hat{\theta}_{MLE}$.

$$E[\hat{\theta}^2] = n^2 E[Y^{-2}] = n^2 \frac{\beta^{-2} \Gamma(-2 + \alpha)}{\Gamma(\alpha)} = \frac{n^2 \theta^2 \Gamma(n-2)}{\Gamma(n)} = \frac{n^2 \theta^2 (n-3)!}{(n-1)!} = \frac{n^2 \theta^2}{(n-1)(n-2)}$$

$$\text{Var}[\hat{\theta}^2] = \frac{n^2 \theta^2}{(n-1)(n-2)} - \frac{n^2 \theta^2}{(n-1)^2} = \frac{n^2 \theta^2}{(n-1)} \left[\frac{1}{n-2} - \frac{1}{n-1} \right] = \frac{n^2 \theta^2}{(n-1)^2 (n-2)}$$

d. Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

$$\hat{\theta}_U = \frac{n-1}{n} \hat{\theta}_{MLE}$$

e. Find the variance of $\hat{\theta}_U$.

$$Var[\hat{\theta}_U] = \left(\frac{n-1}{n}\right)^2 \frac{n^2 \theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2}$$

6. Refer to problem 5.

a. Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ .

2019-2-21p8 7.2.1

$$EX = \mu = \int_1^\infty x \theta x^{-\theta-1} dx = \theta \frac{x^{-\theta+1}}{-\theta+1} \Big|_1^\infty = \frac{\theta}{\theta-1}$$

$$\text{Set } \bar{X} = \frac{\theta}{\theta-1} \implies \theta \bar{x} - \bar{x} = \theta \implies \theta(\bar{x} - 1) = \bar{x},$$

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x} - 1}$$

b. Using the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.

$$\text{For } EX = \mu = \frac{\theta}{\theta-1}$$

$$E[X^2] = \int_1^\infty x^2 \theta x^{-\theta-1} dx = \theta \frac{x^{-\theta+2}}{-\theta+2} \Big|_1^\infty = \frac{\theta}{\theta-2}$$

$$Var[X] = \sigma^2 = E[X^2] - E[X]^2 = \frac{\theta}{\theta-2} - \left(\frac{\theta}{\theta-1}\right)^2$$

$$= \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-1)^2(\theta-2)} = \frac{\theta^3 - 2\theta^2 + \theta - \theta^3 + 2\theta^2}{(\theta-1)^2(\theta-2)} = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

2019-3-5p1

Use a 2nd order Taylor series

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + g''(x_0) \frac{(x - x_0)^2}{2} + R$$

$$\text{Consider } g(x) = \frac{x}{x-1}, \quad g'(x) = \frac{(x-1) \times 1 - x \times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}, \quad g''(x) = \frac{2}{(x-1)^3}$$

Choose $x_0 = EX = \mu$

$$g(x) \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(x - \mu) + \frac{2}{(\mu-1)^3} \frac{(x - \mu)^2}{2}$$

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x} - 1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(\bar{x} - \mu) + \frac{1}{(\mu-1)^3}(\bar{x} - \mu)^2$$

$$E[\hat{\theta}_{MOM}] \approx \frac{\mu}{\mu-1} + 0 + \frac{1}{(\mu-1)^3} \frac{\sigma^2}{n} = \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1} - 1} + \frac{1}{\left(\frac{\theta}{\theta-1} - 1\right)^3} \frac{1}{n} \frac{\theta}{(\theta-1)^2(\theta-2)} = \theta + \frac{\theta(\theta-1)}{n(\theta-2)}$$

c. Using the delta method to approximate the variance of $\hat{\theta}_{MOM}$.

2019-3-5p3

$$\begin{aligned} \text{Var}[\hat{\theta}_{MOM}] &\approx \text{Var}[g(x_0) + g'(x_0)(x - x_0)] = \text{Var}\left[\frac{\mu}{\mu-1} + \frac{1}{(\mu-1)^2}(\bar{x} - \mu)\right] \\ &= \frac{1}{(\mu-1)^4} \frac{\sigma^2}{n} = \frac{1}{\left(\frac{\theta}{\theta-1} - 1\right)^4} \frac{1}{n} \frac{\theta}{(\theta-1)^2(\theta-2)} = \frac{\theta(\theta-1)^2}{n(\theta-2)} \end{aligned}$$

End