STAT562 Final Exam

Winter 2019

1. $X_1, X_2, ..X_n$ is a random sample from a distribution having a p.d.f of the form. $f(x) = \begin{cases} \lambda x^{\lambda-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find a complete sufficient statistic for λ. Justify your answer

- Step1: Proof sufficient

From *Fisher–Neyman factorization theorem* (`2019-2-14p5`)

$$f(x|\lambda) = L(\lambda) = \lambda^n (\prod x_i)^{\lambda-1} = \lambda^n e^{(\lambda-1)\sum_{i=1}^n \ln x_i} \cdot 1 = k(t|\lambda)h(\vec{x})$$

$h(\vec{x}) = 1$ is free of λ. So $T = \sum_{i=1}^n \ln x_i$ is a sufficient statistic for λ.

- Step2: Proof complete

$f(x|\lambda)$ is a member of the exponential family (`2019-2-19p12`),

$$f(x|\vec{\lambda}) = \lambda^n e^{\sum_{i=1}^n (\lambda-1)\ln x_i} = h(x)c(\vec{\lambda})e^{\sum_{j=1}^k W_j(\vec{\lambda})t_j(x)}$$

For pdf $f(x) > 0$ and $x^{\lambda-1} > 0$, $\lambda > 0$. $\{W_1(\vec{\lambda}), .., W_k(\vec{\lambda})\}$ contains an open interval in $\mathbb{R}$, so $T(\vec{x}) = \sum_{i=1}^n \ln x_i)$ is a complete sufficient statistic for λ.

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2. Let Y_n be the n^th order statistic of a random sample of size n from the normal distribution $N(\theta, \sigma^2)$. Prove that $Y_n - \bar{Y}$ and $\bar{Y}$ are independent.

- Step1: θ is a location parameter

Let $X = Y - \theta$. For $N(\theta, \sigma^2)$ is a location family of densities (`2018.11.20p7`),

$$g(y|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\theta)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = f(x) = f(y-\theta)$$

Thus, θ is a location parameter.

- Step2: $Y_n - \bar{Y}$ is location invariant

`2019-2-21p4-6`

Let $X = Y_n - \bar{Y}$.

Consider the group of transformations defined by $\mathcal{G} = \{Y_n - \bar{Y}, -\infty < \bar{Y} < \infty\}$, where $Y_n - \bar{Y} = (y_1 - \bar{y}, .., y_n - \bar{y})$. For $\bar{Y}$ of a random sample is a value,

$$Y_n - \bar{Y} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\bar{y}-\theta)^2}{2\sigma^2}} \sim N(\theta + \bar{Y}, \sigma^2)$$

Thus, the joint distribution of $Y_n - \bar{Y}$ is in $\mathcal{F}$ and hence $\mathcal{F}$ is invariant under $\mathcal{G}$.

$$Y_n - \bar{Y} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_n-\theta)^2}{2\sigma^2}} - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i-\theta)^2}{2\sigma^2}} \sim N(\theta + \bar{Y}, \sigma^2)$$

$$Y_n - \bar{Y} = Y_n - \frac{1}{n}(Y_1 + Y2 + .. + Y_n)$$

- step3: $Y_n - \bar{Y}$ is ancillary statistic for θ

`2019-2-19p6`

$f(y|\theta)$ is a location exponential family. Let $X_n = Y_n - \theta$ is a random sample from $f(y|0)$

$$Y_n - \bar{Y} = Y_n - \frac{1}{n}\sum_{i=1}^{n} Y_i = (Y_n - \theta) - \frac{1}{n}\sum_{i=1}^{n}(Y_i - \theta) = X_n - \frac{1}{n}\sum_{i=1}^{n} X_i$$

$Y_n - \bar{Y}$ is a function of only $X_1, .., X_n$ and be free of θ. It is an ancillary statistic.

- step4: $\bar{Y}$ is sufficient statistic for θ

Method 1:

$f(\mathbf{y}|\theta)$ is the joint pdf or pmf of $\mathbf{Y}$, $g(\bar{y}|\theta)$ is the pdf or pmf of $\bar{\mathbf{Y}}$.

$$f(\vec{y}|\theta) = \prod_{i=1}^{n}(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i-\theta)^2}{2\sigma^2}}) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2} = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\bar{y}+\bar{y}-\theta)^2}$$

For $\sum_{i=1}^{n}(y_i - \bar{y}) = 0, \quad \sum_{i=1}^{n}(\bar{y} - \theta)^2 = n(\bar{y} - \theta)^2$, the part of exponent is

$$= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}[(y_i - \bar{y})^2 + (y_i - \bar{y})(\bar{y} - \theta) + (\bar{y} - \theta)^2] = -\frac{1}{2\sigma^2}[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \theta)^2]$$

For $\bar{Y} \sim N(\theta, \sigma^2/n)$, $g(\bar{y}|\theta) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}$

$$\frac{f(\mathbf{y}|\theta)}{g(\bar{y}|\theta)} = \frac{\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}[\sum_{i=1}^{n}(y_i-\bar{y})^2 + n(\bar{y}-\theta)^2]}}{\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(\bar{y}-\theta)^2}{2\sigma^2}}} = \frac{1}{\sqrt{n}(\sigma\sqrt{2\pi})^{n-1}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\bar{y})^2}$$

which is free of θ (6.2.2). For every $\mathbf{y}$ in the sample space, the ratio $f(\mathbf{y}|\theta)/g(\bar{y}|\theta)$ is constant as a function of θ, then $\bar{Y}$ is a sufficient statistic for θ.

Method 2:

$$f(\vec{y}|\theta) = \prod_{i=1}^{n}(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i-\theta)^2}{2\sigma^2}}) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2} = \frac{e^{-\frac{n\theta^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n} e^{\frac{\theta}{\sigma^2}\sum_{i=1}^{n} y_i} e^{-\frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2}} = k(t|\theta)h(\vec{y})$$

$h(\vec{y}) = e^{-\frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2}}$ is free of θ. By the *Factorization Theorem*, $\sum_{i=1}^{n} y_i$ is a sufficient statistic for θ. $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is a function of $\sum_{i=1}^{n} y_i$ and is a minimal sufficient statistic for θ.

- step5: $\bar{Y}$ is complete statistic for θ

Method 1: `2019-2-19p7-9`

For $\bar{Y} \sim N(\theta, \sigma^2/n)$, the family of is $\{\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(y-\theta)^2}{2\sigma^2}} : -\infty < \theta < \infty\}$

Supporse that $E[g(Y)] = 0 \forall \theta$

$$\int_0^{\infty} g(y) f(y) dy = \int_0^{\infty} g(y) \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} e^{-\frac{n(y-\theta)^2}{2\sigma^2}} dy = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \int_0^{\infty} g(y) e^{-\frac{n(y-\theta)^2}{2\sigma^2}} dy = 0$$

For $\frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \neq 0$, $\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = u'(x) f[u(x)] - v'(x) f[v(x)]$, then

$$0 = \frac{d}{d\theta} E[g(Y)] = \frac{d}{d\theta} \left[\int_{-\infty}^{\infty} g(y) e^{-\frac{n(y-\theta)^2}{2\sigma^2}} dy \right] = 0 - \theta'[g(\theta) e^{-\frac{n(y-\theta)^2}{2\sigma^2}}] = -g(\theta) e^{-\frac{n(y-\theta)^2}{2\sigma^2}}$$

So $g(\theta) = 0, \forall \theta$, then $P(g(T) = 0) = 1$. Thus, $\bar{Y}$ is a complete statistic.

Method 2:

$f(y|\theta)$ is a member of the exponential family (`2019-2-19p12`),

$$f(y|\vec{\theta}) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y-\theta)^2} = e^{-\frac{\sum_{i=1}^n y_i^2}{2\sigma^2}} \frac{e^{-\frac{n\theta^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n} e^{\sum_{i=1}^n \frac{\theta}{\sigma^2} y_i} = h(y) c(\vec{\theta}) e^{\sum_{j=1}^k W_j(\vec{\theta}) t_j(y)}$$

For $\{W_1(\vec{\theta}), .., W_k(\vec{\theta})\}$ contains an open interval in $\mathbb{R}$, so $T(\vec{y}) = \bar{y})$ is a complete sufficient statistic.

- step6: By Basu's theorem, an acillary statistc $Y_n - \bar{Y}$ and a complete sufficient statistic $\bar{Y}$ are independent.`2019-2-19p10`

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3. Suppose that $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta e^{-\theta x}, x > 0$. Assume that the prior distribution of θ is $\pi(\theta) = \lambda e^{-\lambda\theta}, \theta > 0$

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

`2019-2-26p8-9,p11-p13`

For $L(\theta) = \hat{\theta} e^{-\theta \sum x_i}$, $\quad \pi(\theta) = \lambda e^{-\lambda\theta}$, and the kernel of a function is the main part of the function, the part that remains when constants are disregarded (`2019-2-28p8 Exapmle 2.3.8`). that is

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \lambda e^{-\lambda\theta} \propto \theta^{n+1-1} e^{-\theta(\lambda + \sum x_i)}$$

which is $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ distribution.

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b. Find the Bayes estimator of θ, assuming square-error loss

`2019-2-28p1`

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\lambda + \sum x_i}$$

which is the posterior mean.

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c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(\lambda + n\bar{x})} = \frac{1}{\frac{1}{n+1}(\frac{1}{1/\lambda} + \frac{n}{1/\bar{x}})}$$

which is the weighted hamonic mean of $1/\lambda$, which is the prior mean, and $1/\bar{x}$, which is the MLE of θ.

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d. Find the Bayes estimator of θ, assuming absolute loss

`2019-2-28p4,9`

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = median[\theta|\vec{x}]$$

For the median of Gamma distribution doesn't have a closed form, the posterior median would not have a closed form.

Postmedian $\hat{\theta} = F^{-1}(\frac{1}{2})$ where $F(x)$ is the $Gamma(\alpha = n+1, \beta = \frac{1}{\lambda + \sum x_i})$ cdf, for which there is no closed form.

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e. Find the Bayes estimator of θ, assuming binary loss

`2019-2-28p5-6`

Suppose $L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}$.

$$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$$

To minimized this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = mode[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\lambda + \sum x_i}$$

which is the posterior mode.

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4. Redo all of problem 3, using the non-informative prior $\pi(\theta) = 1, \theta > 0$. Note that this is not a valid density function since its intergral is infinite, but proceed with it anyway

a. Find the posterior distribution $\pi(\theta|\vec{x})$.

`2019-2-26p8-9`

For $\pi(\theta) = 1, \theta > 0$, $L(\theta) = \theta^n e^{-\theta \sum x_i}$, from the kernel of function,

$$\pi(\theta|\vec{x}) \propto L(\theta)\pi(\theta) = \theta^n e^{-\theta \sum x_i} \sim Gamma(\alpha = n+1, \beta = \frac{1}{\sum x_i})$$

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b. Find the Bayes estimator of θ, assuming square-error loss

Suppose $L_0(\hat{\theta}) = (\hat{\theta} - \theta)^2$. $E[L_0(\hat{\theta})|\vec{x}]$ is minimized when

$$\hat{\theta}_{Bayes} = E[\theta|\vec{x}] = \alpha\beta = \frac{n+1}{\sum x_i}$$

which is the posterior mean.

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c. writing this estimator as a weighted (arithmetic, geometric, or harmonic) average of the MLE and some prior constant

$$\hat{\theta}_{Bayes} = \frac{1}{\frac{1}{n+1}(1 \times 0 + n\bar{x})} = \lim_{c \to \infty} \frac{1}{\frac{1}{n+1}(\frac{1}{c} + \frac{n}{1/\bar{x}})}$$

which is the weighted hamonic mean of c, which is the prior mean when $c \to \infty$, and $1/\bar{x}$, which is the MLE of θ.

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d. Find the Bayes estimator of θ, assuming absolute loss

Suppose $L_1(\hat{\theta}) = |\hat{\theta} - \theta|$. $E[|\hat{\theta} - \theta|]$ is minimized when

$$\hat{\theta}_{Bayes} = median[\theta|\vec{x}]$$

For the median of Gamma distribution doesn't have a closed form, the posterior median would not have a closed form.

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e. Find the Bayes estimator of θ, assuming binary loss

Suppose $L_0(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} = \theta \\ 1 & \text{elsewhere} \end{cases}$.

$$E[L_0(\hat{\theta})|\vec{x}] = 0 \cdot P[\theta = \hat{\theta}|\vec{x}] + 1 \cdot P[\theta \neq \hat{\theta}|\vec{x}] = P[\theta \neq \hat{\theta}|\vec{x}] = 1 - P[\theta = \hat{\theta}|\vec{x}]$$

To minimized this, maximize $P[\theta = \hat{\theta}|\vec{x}]$

When $\hat{\theta}$ is the posterior mode, it is a Maximum A Posteriori estimator. For the mode of Gamma distribution is $(\alpha - 1)\beta$

$$\hat{\theta}_{Bayes} = mode[\theta|\vec{x}] = (\alpha - 1)\beta = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

which is the posterior mode.

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5. Let $X_1, X_2, ..X_n \sim$ idd. $f(x|\theta) = \theta x^{-\theta-1}, x_i > 1, \theta > 2$.

a. Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ.

$$f(\vec{x}|\theta) = L(\theta) = \theta^n (\prod x_i)^{\theta-1} = \theta^n e^{(-\theta-1)\sum_{i=1}^n \ln x_i}$$

$$l(\theta) = n \ln\theta - (\theta+1)\sum_{i=1}^n \ln x_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n \ln x_i \overset{\text{set}}{=} 0$$

$$\hat{\theta}_{MLE} = \frac{n}{\sum \ln x_i}$$

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b. Find the expected value of $\hat{\theta}_{MLE}$.

Let $Y_i = \ln x_i$, then $X = e^y$, $\frac{dx}{dy} e^y$

$$g(Y) = \theta(e^y)^{-\theta-1} e^y = \theta e^{-y\theta}, y > 0$$

So $Y_i = \ln x_i \sim Gamma(\alpha = n, \beta = \frac{1}{\theta})$

$$E[\hat{\theta}] = nE[Y^{-1}] = n\frac{\beta^{-1}\Gamma(-1+\alpha)}{\Gamma(\alpha)} = \frac{n\theta\Gamma(n-1)}{\Gamma(n)} = \frac{n\theta(n-2)!}{(n-1)!} = \frac{n\theta}{n-1}$$

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c. Find the variance of $\hat{\theta}_{MLE}$.

$$E[\hat{\theta}^2] = n^2 E[Y^{-2}] = n^2\frac{\beta^{-2}\Gamma(-2+\alpha)}{\Gamma(\alpha)} = \frac{n^2\theta^2\Gamma(n-2)}{\Gamma(n)} = \frac{n^2\theta^2(n-3)!}{(n-1)!} = \frac{n^2\theta^2}{(n-1)(n-2)}$$

$$Var[\hat{\theta}^2] = \frac{n^2\theta^2}{(n-1)(n-2)} - \frac{n^2\theta^2}{(n-1)^2} = \frac{n^2\theta^2}{(n-1)}[\frac{1}{n-2} - \frac{1}{n-1}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

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d. Using $\hat{\theta}_{MLE}$, create an unbiased estimator $\hat{\theta}_U$.

$$\hat{\theta}_U = \frac{n-1}{n}\hat{\theta}_{MLE}$$

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e. Find the variance of $\hat{\theta}_U$.

$$Var[\hat{\theta}_U] = (\frac{n-1}{n})^2 \frac{n^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{n-2}$$

6. Refer to problem 5.

a. Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ.

`2019-2-21p8 7.2.1`

$EX = \mu = \int_1^\infty x\theta x^{-\theta-1}dx = \theta \frac{x^{-\theta+1}}{-\theta+1}\Big|_1^\infty = \frac{\theta}{\theta-1}$

Set $\bar{X} = \frac{\theta}{\theta-1} \implies \theta\bar{x} - \bar{x} = \theta \implies \theta(\bar{x}-1) = \bar{x}$,

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1}$$

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b. Using the delta method to approximate the expected value of $\hat{\theta}_{MOM}$.

For $EX = \mu = \frac{\theta}{\theta-1}$

$E[X^2] = \int_1^\infty x^2\theta x^{-\theta-1}dx = \theta \frac{x^{-\theta+2}}{-\theta+2}\Big|_1^\infty = \frac{\theta}{\theta-2}$

$Var[X] = \sigma^2 = E[X^2] - E[X]^2 = \frac{\theta}{\theta-2} - (\frac{\theta}{\theta-1})^2$

$$= \frac{\theta(\theta-1)^2 - \theta^2(\theta-2)}{(\theta-1)^2(\theta-2)} = \frac{\theta^3 - 2\theta^2 + \theta - \theta^3 + 2\theta^2}{(\theta-1)^2(\theta-2)} = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

`2019-3-5p1`

Use a 2^{nd} order Taylar series

$$g(x) = g(x_0) + g'(x_0)(x-x_0) + g''(x_0)\frac{(x-x_0)^2}{2} + R$$

Consider $g(x) = \frac{x}{x-1}$, $\quad g'(x) = \frac{(x-1)\times 1 - x\times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}$, $\quad g''(x) = \frac{2}{(x-1)^3}$

Choose $x_0 = EX = \mu$

$$g(x) \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(x-\mu) + \frac{2}{(\mu-1)^3}\frac{(x-\mu)^2}{2}$$

$$\hat{\theta}_{MOM} = \frac{\bar{x}}{\bar{x}-1} \approx \frac{\mu}{\mu-1} + \frac{-1}{(\mu-1)^2}(\bar{x}-\mu) + \frac{1}{(\mu-1)^3}(\bar{x}-\mu)^2$$

$$E[\hat{\theta}_{MOM}] \approx \frac{\mu}{\mu-1} + 0 + \frac{1}{(\mu-1)^3}\frac{\sigma^2}{n} = \frac{\frac{\theta}{\theta-1}}{\frac{\theta}{\theta-1}-1} + \frac{1}{(\frac{\theta}{\theta-1}-1)^3}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \theta + \frac{\theta(\theta-1)}{n(\theta-2)}$$

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c. Using the delta method to approximate the variance of $\hat{\theta}_{MOM}$.

`2019-3-5p3`

$Var[\hat{\theta}_{MOM}] \approx Var[g(x_0) + g'(x_0)(x-x_0)] = Var[\frac{\mu}{\mu-1} + \frac{1}{(\mu-1)^2}(\bar{x}-\mu)]$

$$= \frac{1}{(\mu-1)^4}\frac{\sigma^2}{n} = \frac{1}{(\frac{\theta}{\theta-1}-1)^4}\frac{1}{n}\frac{\theta}{(\theta-1)^2(\theta-2)} = \frac{\theta(\theta-1)^2}{n(\theta-2)}$$

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End
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