SLR	$y_i=\beta_0+\beta_1x_i+\varepsilon_i,i_{1:n}$
MLR	$\mathbf{Y_{n\times 1}}=\mathbf{X_{n\times(k+1)}}\boldsymbol{\beta_{(k+1)\times 1}}+\boldsymbol{\varepsilon_{n\times 1}}$
With indicator	$y_i=\beta_0+\beta_1x_i+\beta_2w_i+\beta_3w_ix_i+\varepsilon_i,\ w_i=\begin{cases}0 & 1\le i\le\frac{n}{2}\\ 1 & \frac{n}{2}+1\le i\le n\end{cases}$
$\mathbf{A_{f,r}}$	$y_{ij}=\mu+\tau_i+\varepsilon_{ij};\ i_{1:a},j_{1:n}$
$\mathbf{Trt_f}+\mathbf{Blk_f}$	$y_{ij}=\mu+\tau_i+\beta_j+\varepsilon_{ij};\ i_{1:a},j_{1:b}$
$\mathbf{A_{f,r}B_{f,r}}$	$y_{ijk}=\mu+\tau_i+\beta_j+(\tau\beta)_{ij}+\varepsilon_{ijk},\ i_{1:a},j_{1:b},k_{1:n}$
$\mathbf{A_r}(\mathbf{B_r}+\mathbf{C_r})$	$y_{ijk}=\mu+\tau_i+\beta_j+\gamma_k+(\tau\beta)_{ij}+(\tau\gamma)_{ik}+\varepsilon_{ijk}$
$\mathbf{C_r}(\mathbf{B_r}(\mathbf{A_f}))$	$y_{ijkl}=\mu+\tau_i+\beta_{j(i)}+\gamma_{k(ij)}+\varepsilon_{(ijk)l}$
Latin Square	$y_{ijk}=\mu+\tau_i+\alpha_j+\beta_k+\varepsilon_{ijk},\ i,j,k_{1:p};$
Graeco-Latin	$y_{ijkl}=\mu+\theta_i+\tau_j+\omega_k+\psi_l+\varepsilon_{ijkl};i,j,k,l_{1:p}$

Only two of the four subscripts are necessary to completely identify an observation.
rep123

BIBD$y_{ij}=\mu+\tau_i+\beta_j+\varepsilon_{ij};\ Q_i=(y_{i.}-\sum_j^b n_{ij}\bar{y}_{.j});n_{ij}=\begin{cases}1 & i^{th}\text{ trt appears in } j^{th}\text{ blk}\\ 0 & \text{o.w.}\end{cases}$

Trt$i_{1:a}$,Blk$j_{1:b}$,Size$k$;$b=\binom{a}{k}$;observations per treatment minRep$r=3\times 5$;N=ar=bk

replications of each pair$\lambda=\frac{k-1}{a-1}r=\frac{(k-1)bk}{(a-1)a}=10=\frac{2}{3}r\in\mathbf{N}^+$

μ overall mean; $y_{ijkl}(\varepsilon_{ijkl})$ is response(random error) (for the k^{th} replicate EU) when i^{th} (Latin(Greek) letter) treatment(level of Factor X) is applied (at j^{th} block) (in the k^{th} row and k^{th} column);

τ_i is fixed(random) (main) effect of i^{th} (Latin(Greek) letter) treatment (block; level of Factor X; row; column);

$(\tau\beta)_{ij}$ is interaction effect of i^{th} level of Factor A and j^{th} level of Factor B;

Assumptions: constant variance, zero mean, independent;$\varepsilon_{ijkl}\overset{iid}{\sim}N(0,\sigma^2)$

Fix	$\sum_i^a\tau_i$=0; $\sum_j^b\beta_j$=0; $\sum_i^a(\tau\beta)_{ij}$=0; $\sum_j^b(\tau\beta)_{ij}$=0;
Ran	$\tau_i\overset{iid}{\sim}N(0,\sigma_\tau^2)$; $\beta_j\overset{iid}{\sim}N(0,\sigma_\beta^2)$; $\gamma_k\overset{iid}{\sim}N(0,\sigma_\gamma^2)$; $(\tau\beta)_{ij}\overset{iid}{\sim}N(0,\sigma_{\tau\beta}^2)$ $(\tau\gamma)_{ik}\overset{iid}{\sim}N(0,\sigma_{\tau\gamma}^2)$
Mix	$\sum_i^a\tau_i$=0; $\beta_j\overset{iid}{\sim}N(0,\sigma_\beta^2)$; $\sum_{i=1}^a(\tau\beta)_{ij}$=0 (restricted model);$(\tau\beta)_{ij}\overset{iid}{\sim}N(0,\frac{a-1}{a}\sigma_{\tau\beta}^2)$

T	df	SS	
SLR	n-1	$\sum(y_i-\bar{y})^2$	EMS:$\beta_1^2S_{xx}+(n-1)\sigma^2$
MLR	n-1	$(Y-\bar{y}1)'(Y-\bar{y}1)$	
RCD	an-1	$\sum^a\sum^n(y_{ij}-\bar{y}_{..})^2$	$\sum^a\sum^n y_{ij}^2-y_{..}^2/N$
RCBD	ab-1	$\sum^a\sum^b(y_{ij}-\bar{y}_{..})^2$	
Fact	abcn-1	$\sum\sum\sum\sum(y_{ijk}-\bar{y}_{....})^2$	$\sum^a\sum^b\sum^c\sum^n y_{ijkl}^2-\frac{1}{abcn}y_{....}^2$
LSD	p^2-1	$\sum^p\sum^p\sum^p(y_{ijk}-\bar{y}_{...})^2$	
GrLa	p^2-1	$\sum\sum\sum\sum(y_{ijkl}-\bar{y}_{....})^2$	
rep123	np^2-1	BIBD	ar=bk

Reg/Trt	df	SS	EMS	EMS$_r$
SLR	1	$\sum(\hat{y}-\bar{y})^2$,$\hat{\beta}_1^2S_{xx}$,$\hat{\beta}_1S_{xy}$	$\sigma^2+\hat{\beta}_1^2S_{xx}$	
MLR	k	$\beta'X'Y-n\bar{y}^2$		
RCD	a-1	$n\sum^a(\bar{y}_{i.}-\bar{y}_{..})^2$	$\sigma^2+\frac{n\sum\tau^2}{a-1}$	$\sigma^2+n\sigma_\tau^2$
RCBD	a-1	$b\sum^a(\bar{y}_{i.}-\bar{y}_{..})^2$	$\sigma^2+\frac{b\sum^a\tau^2}{a-1}$	$\sigma^2+b\sigma_\tau^2$
Fact-f	a-1	$bn\sum^a(\bar{y}_{i..}-\bar{y}_{...})^2;\frac{\sum^a y_{i..}^2}{bn}-\frac{y_{...}^2}{abn}$	$\sigma^2+\frac{bn\sum\tau_i^2}{a-1}$	τ_i0-b-n
Fact-f	b-1	$an\sum^b(\bar{y}_{.j.}-\bar{y}_{...})^2;\frac{\sum^b y_{.j.}^2}{an}-\frac{y_{...}^2}{abn}$	$\sigma^2+\frac{an\sum\beta_j^2}{b-1}$	β_ja-0-n
Fact-ff	(a-1)	$n\sum^a\sum^b(y_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$	$\sigma^2+\frac{n\sum\sum(\tau\beta)_{ij}}{(a-1)(b-1)}$	0-0-n
$(\tau\beta)_{ij}$	(b-1)	$n\sum\sum y_{ij.}^2-\frac{1}{abn}y_{...}^2-SS_A-SS_B$		
BIBD	a-1	$\frac{k}{\lambda a}\sum^a Q_i^2=\frac{k}{\lambda a}\sum^a(y_{i.}-\sum^b n_{ij}\bar{y}_{.j})^2$		
LaSq	p-1	$p\sum_{i=1}^p(\bar{y}_{i..}-\bar{y}_{...})^2;\frac{\sum^p y_{i..}^2}{p}-\frac{y_{...}^2}{p^2}$		
rep123	p-1	$np\sum_{i=1}^p(\bar{y}_{i..}-\bar{y}_{...})^2;\frac{\sum^p y_{i...}^2}{np}-\frac{y_{....}^2}{np^2}$		
GrLa	p-1	$p\sum_{i,j}^p(\bar{y}_{(i,j)..}-\bar{y}_{....})^2$		

Eε_{ijk}	df	SS	EMS
SLR	n-2	$\sum(y_i-\hat{y})^2$	$(n-2)\sigma^2$
MLR	n-(k+1)	$Y'(I-H)Y$	H=$X(X'X)^{-1}X'$
RCD	a(n-1)	$\sum^a\sum^n(y_{ij}-\bar{y}_{i.})^2$	σ^2
RCBD	(a-1)(b-1)	T-Trt-Blk	σ^2
BIBD	N-a-b+1	T-Trt$_{adj}$-Blk	
LaSq	(p-1)(p-2)	$SS_T-SS_.$	
Rep1	(p-1)(np+n-3)		
Rep2	(p-1)(np-1)		
Rep3	(p-1)(np-n-1)		
GrLa	(p-1)(p-3)	$SS_T-SS_.$	
Fact-f	ab(n-1)	$SST-\sum SS$;$(n-1)\sum^a\sum^b S_{ij}^2$	1-1-1,σ^2
?Fact-r	a(b-1)(c-1)	$\sum^a\sum^b\sum^c y_{ijk}^2-\frac{\sum^a\sum^b y_{ij.}^2}{c}-\frac{\sum^a\sum^c y_{i.k}^2}{b}+\frac{\sum^a y_{i..}^2}{bc}$	1-1-1, σ^2
?Fact-b	(ab-1)(n-1)		
Nested	abc(n-1)	$\sum^a\sum^b\sum^c\sum^n y_{ijkl}^2-\frac{1}{n}\sum^a\sum^b\sum^c y_{ijk.}^2$	1-1-1-1,$\varepsilon_{(ijk)l}$, σ^2
Split	0	$\varepsilon_{(ijk)h}$not estimatable	

Block	df	SS	EMS
RCBD	b-1	$a\sum^b(\bar{y}_{.j}-\bar{y}_{..})^2$	$\sigma^2+a\sigma_\beta^2$
Fact	n-1	$\frac{\sum^n y_{..k}^2}{ab}-\frac{y_{...}^2}{abn}$	$\sigma^2+ab\sigma_\delta^2$
BIBD	b-1	$k\sum_{j=1}^b(\bar{y}_{.j}-\bar{y}_{..})^2$	
LaSq-RC	p-1	$p\sum_{j,k}^p(\bar{y}_{.(j,k)}-\bar{y}_{...})^2$	$\frac{1}{p}\sum^p\bar{y}_{.(j,k)}^2-y_{...}^2/N$
nRep-RC	p-1	$np\sum_{l=1}^n(\bar{y}_{.(j,k)l}-\bar{y}_{....})^2$	$\frac{1}{np}\sum^p\bar{y}_{.(j,k)}^2-y_{...}^2/N$
Rep-RC	n(p-1)	$p\sum_l^n\sum_{j,k}^p(\bar{y}_{.(j,k)l}-\bar{y}_{....})^2$	$\frac{1}{p}\sum^n\sum^p\bar{y}_{.(j,k)}^2-\frac{1}{p^2}y_{...}^2$
Rep123	n-1	$p^2\sum_{l=1}^n(\bar{y}_{...l}-\bar{y}_{....})^2$	$\frac{1}{p^2}\sum^n\bar{y}_{...l}^2-y_{....}^2/N$
GrLa-RC	p-1	$p\sum_{k,l}^p(\bar{y}_{..(k,l)}-\bar{y}_{....})^2$	

Nested model $\mathbf{B_r}(\mathbf{A_f})\mathbf{C_f}$

F,F,R	$\sum_i^a\tau_i$=0	$\sum_k^b\gamma_k$=0	$\sum_i^a(\tau\gamma)_{ik}$=0;$\sum_k^b(\tau\gamma)_{ik}$=0
R(f),FR(f)	$\beta_{j(i)}\overset{iid}{\sim}N(0,\sigma_\beta^2)$	$\sum_i^a(\beta\gamma)_{j(i)k}$=0	$(\beta\gamma)_{j(i)k}\overset{iid}{\sim}N(0,\frac{2-1}{2}\sigma_{\beta\gamma}^2)$
r(r(f))	$\sum_i^a\tau_i$=0	$\beta_{j(i)}\overset{iid}{\sim}N(0,\sigma_\beta^2)$	$\gamma_{k(ij)}\overset{iid}{\sim}N(0,\sigma_\gamma^2)$

r(r+r)	i	j	k	l	df	SS $\mid$ EMS	F
R1	1	b	c		a-1	$\frac{1}{bc}\sum_{i=1}^a y_{i..}^2-\frac{1}{abc}y_{...}^2\mid\sigma^2+b\sigma_{\tau\gamma}^2+c\sigma_{\tau\beta}^2+bc\sigma_\tau^2$	$\frac{A}{AB+AC-E}$
R2	a	1	c		b-1	$\frac{1}{ac}\sum_{j=1}^b y_{.j.}^2-\frac{1}{abc}y_{...}^2\mid\sigma^2+c\sigma_{\tau\beta}^2+ac\sigma_\beta^2$	$\frac{B}{AB}$
R3	a	b	1		c-1	$\frac{1}{ab}\sum_{k=1}^c y_{..k}^2-\frac{1}{abc}y_{...}^2\mid\sigma^2+b\sigma_{\tau\gamma}^2+ab\sigma_\gamma^2$	$\frac{C}{AC}$
RR	1	1	c		(a-1)(b-1)	$\frac{\sum^a\sum^b y_{ij.}^2}{c}-\frac{\sum^a y_{i..}^2}{bc}-\frac{\sum^b y_{.j.}^2}{ac}+\frac{y_{...}^2}{abc}\mid\sigma^2+c\sigma_{\tau\beta}^2$	$\frac{AB}{E}$
RR	1	b	1		(a-1)(c-1)	$\frac{\sum^a\sum^c y_{i.k}^2}{b}-\frac{\sum^a y_{i..}^2}{bc}-\frac{\sum^c y_{..k}^2}{ab}+\frac{y_{...}^2}{abc}\mid\sigma^2+b\sigma_{\tau\gamma}^2$	$\frac{AC}{E}$
f(F)					a-1	$\sigma_\varepsilon^2+\frac{bn\sum^a\alpha_i^2}{a-1}$	α_i
F(f)					a(b-1)	$\sigma_\varepsilon^2+\frac{n\sum^a\sum^b\beta_{ij}^2}{a(b-1)}$	β_{ij}
r(F)	0	b	n		a-1	$\sigma_\varepsilon^2+n\sigma_\beta^2+\frac{bn\sum^a\alpha_i^2}{a-1}$	$\frac{A}{AB}$
R(f)	0	1	n		a(b-1)	$\sigma_\varepsilon^2+n\sigma_\beta^2$	$\frac{AB}{E}$
r(R)					a-1	$\sigma_\varepsilon^2+n\sigma_\beta^2+bn\sigma_\alpha^2$	
R(r)					a(b-1)	$\sigma_\varepsilon^2+n\sigma_\beta^2$	
r(r(F))	0	b	c	n	a-1	$\tau_i;\sigma^2+n\sigma_\gamma^2+cn\sigma_\beta^2+\frac{bcn\sum^a\tau_i^2}{a-1}\mid\frac{\sum^a y_{i...}^2}{bcn}-\frac{y_{....}^2}{abcn}$	$\frac{A}{B(A)}$
r(R(f))	1	1	c	n	a(b-1)	$\beta_{j(i)};\sigma^2+n\sigma_\gamma^2+cn\sigma_\beta^2\mid\frac{\sum^a\sum^b y_{ij..}^2}{cn}-\frac{\sum^a y_{i...}^2}{bcn}$	$\frac{B(A)}{C(B)}$
R(r(f))	1	1	1	n	ab(c-1)	$\gamma_{k(ji)};\sigma^2+n\sigma_\gamma^2\mid\frac{1}{n}\sum^a\sum^b\sum^c y_{ijk.}^2-\frac{1}{cn}\sum^a\sum^b y_{ij..}^2$	$\frac{C(B)}{E}$
r(F)	0	b	c	n	a-1	$\tau_i;\sigma^2+cn\sigma_\beta^2+\frac{bcn\sum_{i=1}^a\tau_i^2}{a-1}$	$\frac{A}{B(A)}$
R(f)	1	1	c	n	a(b-1)	$\beta_{j(i)};\sigma^2+cn\sigma_\beta^2$	$\frac{B(A)}{E}$
F	a	b	0	n	c-1	$(\gamma)_k;\sigma^2+n\sigma_{\gamma\beta}^2+\frac{abn\sum^c\gamma_k^2}{c-1}$	$\frac{C}{CB(A)}$
FF	0	b	0	n	(a-1)(c-1)	$(\tau\gamma)_{ik};\sigma^2+n\sigma_{\gamma\beta}^2+\frac{bn\sum^a\sum^c(\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$\frac{AC}{CB(A)}$
FR(f)	1	1	0	n	a(b-1)(c-1)	$(\gamma\beta)_{kj(i)};\sigma^2+n\sigma_{\gamma\beta}^2$	$\frac{CB(A)}{E}$
R	1	a	b		r-1	$\tau_i;\sigma^2+ab\sigma_\tau^2$	$\frac{R}{E}$
F	r	0	b		a-1	$\beta_j;\sigma^2+b\sigma_{\tau\beta}^2+\frac{rb\sum^a\beta_j^2}{a-1}$	$\frac{A}{RA}$
RF	1	0	b		(r-1)(a-1)	$(\tau\beta)_{ij};\sigma^2+b\sigma_{\tau\beta}^2$=E(Whole-plot error)	$\frac{RA}{E}$
F	r	a	0		b-1	$(\gamma)_k;\sigma^2+a\sigma_{\tau\gamma}^2+\frac{ra\sum^b\gamma_k^2}{b-1}$	$\frac{B}{RB}$
RF	1	a	0		(r-1)(b-1)	$(\tau\gamma)_{ik};\sigma^2+a\sigma_{\tau\gamma}^2$	$\frac{RB}{E}$
FF	r	0	0		(a-1)(b-1)	$(\beta\gamma)_{jk};\sigma^2+\sigma_{\tau\beta\gamma}^2+\frac{r\sum^a\sum^b(\beta\gamma)_{jk}^2}{(a-1)(b-1)}$	$\frac{AB}{RAB}$
RFF	1	0	0		(r-1)(a-1)(c-1)	$(\tau\beta\gamma)_{ijk};\sigma^2+\sigma_{\tau\beta\gamma}^2$	$\frac{RAB}{E}$

Split-Plot$y_{ijkl}=\mu+\tau_i+\beta_j+(\tau\beta)_{ij}+\gamma_k+(\tau\gamma)_{ik}+(\beta\gamma)_{jk}+(\tau\beta\gamma)_{ijk}+\varepsilon_{ijk}$

$\mathbf{Run_rA_fB_f}$ $i_{1:r},j_{1:a},k_{1:b}$

R,FF	$\tau_i\overset{iid}{\sim}N(0,\sigma_\tau^2)$	$\sum_j^a(\beta\gamma)_{jk}$=0	$\sum_k^b(\beta\gamma)_{jk}$=0
F,RF	$\sum_j^a\beta_j$=0	$\sum_j^a(\tau\beta)_{ij}$=0	$(\tau\beta)_{ij}\overset{iid}{\sim}N(0,\frac{a-1}{a}\sigma_{\tau\beta}^2)$
F,RF	$\sum_k^b\gamma_k$=0	$\sum_k^b(\tau\gamma)_{ik}$=0	$(\tau\gamma)_{ik}\overset{iid}{\sim}N(0,\frac{b-1}{b}\sigma_{\tau\gamma}^2)$
RFF	$\sum_j^a(\tau\beta\gamma)_{ijk}$=0	$\sum_k^b(\tau\beta\gamma)_{ijk}$=0	$(\tau\beta\gamma)_{ijk}\overset{iid}{\sim}N(0,\frac{(a-1)(b-1)}{ab}\sigma_{\tau\beta\gamma}^2)$

$\bar{x}$; $S_{xx}=\sum(x_i-\bar{x})^2$,$S_{yy}=\sum(y_i-\bar{y})^2$, $S_{xy}=\sum(x_i-\bar{x})(y_i-\bar{y})$

Estimate

$\hat{\sigma}^2=\frac{SSE}{n-2}=\frac{1}{8}(S_{yy}-\frac{S_{xy}^2}{S_{xx}})$

$\hat{\beta}_1\sim N(\frac{S_{xy}}{S_{xx}}=\sum c_iy_i=r\frac{SD_y}{SD_x}=\frac{\sigma^2}{(n-1)S_x^2},V(\hat{\beta}_1)=\frac{\hat{\sigma}^2}{S_{xx}})$;

$\hat{\beta}_0\sim N(\bar{y}-\hat{\beta}_1\bar{x},\hat{\sigma}^2[\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}])$;

$\hat{\sigma}^2=MSE=\frac{SSE}{N-a}$; $\hat{\sigma}_\tau^2=\frac{MS_{Trt}-MSE}{n}$

$y_{ij}\sim n(\mu+\tau_i,\sigma^2)$;$\bar{y}_{i.}\sim n(\mu+\tau_i,\frac{\sigma^2}{n})$;$\bar{y}_{i.}-\bar{y}_{j.}\sim n(\mu_i-\mu_j,\frac{2\sigma^2}{n})$

$E[y_{ij}-\bar{y}_{i.}]=V[]=\frac{n-1}{n}\sigma^2$

Interval

$\hat{\beta}_1\pm t_{\frac{0.05}{k+1},n-2}se(\hat{\beta}_1)$

$\hat{y}_0=\hat{\beta}_0+\hat{\beta}_1x_0$

$\hat{y}_0\pm t_{n-2,0.025}se(y_0)$,=$\sqrt{MSE(\frac{1}{n}+\frac{(x_0-\bar{x})^2}{S_{XX}})}$(CI); $\sqrt{MSE(1+\frac{1}{n}+\frac{(x_0-\bar{x})^2}{S_{XX}})}$(PI)

CI: $\hat{\beta}_1\pm t_{\frac{\alpha}{2},n-k-1}se(\hat{\beta}_1)$, $se(\hat{\beta}_1)=\sqrt{MSE\cdot C_{22}}$; Bonferroni:$t_{\frac{\alpha}{2(k+1)}}$, Scheffe $\sqrt{2F_\alpha}$

CI of μ_i: $\bar{y}_{i.}\pm t_{\frac{\alpha}{2}}\sqrt{\frac{MS_E}{n}}$; unbalanved $\bar{y}_{i.}\pm t_{\frac{\alpha}{2}}\sqrt{\frac{MS_E}{n_i}}$

$\mu_i-\mu_j$: $\bar{y}_{i.}-\bar{y}_{j.}\pm t_{\frac{\alpha}{2}}\sqrt{\frac{2MS_E}{n}}$; unbalanved $\bar{y}_{i.}-\bar{y}_{j.}\pm t_{\frac{\alpha}{2}}\sqrt{MS_E(\frac{1}{n_i}+\frac{1}{n_j})}$

σ^2: $\frac{SSE}{\chi^2_{\alpha/2,df_E}}$, $\frac{SSE}{\chi^2_{1-\alpha/2,df_E}}$ Γ: $\sum_{i=1}^a c_i\bar{y}_i \pm t_{\frac{\alpha}{2}}\sqrt{\frac{MS_E}{n}\sum_{i=1}^a c_i^2}$ $\frac{MS_{Trt}}{MS_E}\cdot\frac{\sigma^2}{n\sigma_\tau^2+\sigma^2}\sim F_{a-1,N-a}$

CI of prop of var $\frac{\sigma^2}{\sigma_\tau^2+\sigma^2}$: $\frac{L}{1+L},\frac{U}{1+U}$; $L=\frac{1}{n}\left(\frac{MS_{Trt}}{MS_E F_{\alpha/2}}-1\right)$; $U=\frac{1}{n}\left(\frac{MS_{Trt}}{MS_E F_{1-\alpha/2}}-1\right)$

$\bar{y}_{12.}-\bar{y}_{22.}\pm t_{\frac{\alpha}{2},18}\sqrt{\frac{2MSE}{n}}$

Hypothesis

$H_0:\hat{\beta}_1=0$; $t_0=\frac{\hat{\beta}_1-0}{\sqrt{Var(\hat{\beta}_1)}}<t_{\frac{0.05}{2},n-2}=2.31$Fail to reject

equality of slopes $H_0:\beta_3=0,k=3,r=1$

Partitioned regression

(df)	SS_F	$-3,4,5$	$SS_{1,2}$	$-2,4$	$SS_{1,3,5}$
R	367(5)	-96(3)	271(2)	-153(2)	214(3)
E	336(69)	+96(3)	432(72)	+153(2)	489(71)
T	703(74)				

	df	SS_F	$-\alpha$	$SS_{\beta\gamma}$	$-\beta$	$SS_{\alpha\gamma}$	$-\alpha$	SS_γ
α	2	110	-110	0		110	-110	0
γ	11	160		160		160		160
β		250	-10	240	-250	0		0
E	22	300	+120	420	+250	550	+110	660
T	35	820						

$dfE_{Ful}=n-(k+1)$,$dfE_{Red}-dfE_{Ful}=r$

$H_0:\beta_3=\beta_4=\beta_5=0;r=3;H_0':\beta_2=\beta_4=0;r=2;$

$F_{\alpha,r,dfE_{Ful}}=\frac{(SSE_{Red}-SSE_{Ful})/r}{SSE_{Ful}/dfE_{Ful}}$

$R^2=\frac{SSR}{SST};R^2_{adj}=1-\frac{SSE/dfE}{SST/dfT};R^2_{pre}=1-\frac{PRESS}{SST}$

coefficient of determination is the proportion of variation explained by regressor x

$|\mathrm{r}|=\sqrt{R^2}=\frac{S_xy}{\sqrt{S_xxS_yy}}=\frac{Cov(x,y)}{\sqrt{VXVY}}=\frac{Cov(\hat{\beta}_1,\hat{\beta}_0)}{se\hat{\beta}_1 se\hat{\beta}_0}$

ff: $H_0:(\tau\beta)_{ij}=0\forall i,j$; $F_{p,2,18}\frac{MS_{AB}}{MSE}=\frac{56}{7}=8$; $F_{0.05,2,8}=3.55$. There is enough evidence to reject H_0. The model may not be reduced, as the interaction effects is significant at 5% significance level.

r(r+r): $F_{df_n,df_d}=\frac{MS'}{MS''}$; $df_n=df_c$;$df_d=\frac{(AC+BC+ABC)^2}{\frac{AC^2}{df_{AC}}+\frac{BC^2}{df_{BC}}+\frac{ABC^2}{df_{ABC}}}$

Fixed v.s. random; crossed v.s. nested

effects	Fixed	Rrandom
across individuals	constant	vary
interest in	themselves	underlying population
a sample	exhausts the population	a small part of the population
a random variable		a realized value of

crossed(nested): Every(each) category of one factor co-occurs with every(only one) category of the other factor. There at least one observation in every combinations of categories specific (are not represented).

Example: A lab want to test the quality of products. There are only two specific test machines. Each machine assigned two operators who randomly selected from a large amont of operators. Each operator test the products in three specific temperature. In this test, the factor machine and temperature have fixed effects. The factor operator has random effects.

Three temperature are applied on all machines. Thus, temperature and machine have crossed effect. The operators were assigned to each machine are different. The operators are nested in the levels of factor machine. Thus, the effects of the factor operator are nested effect.

least-squares estimators

$SSE=\sum_{i=1}^n(y_i-\beta_0-\beta_1x_i-\beta_2x_i^2)^2$

$\frac{\partial SSE}{\partial\beta_{0,1,2}}=2\sum_{i=1}^n(y_i-\hat{\beta}_0-\hat{\beta}_1x_i-\hat{\beta}_2x_i^2)(-1,-x_i,-x_i^2)\overset{set}{=}0$

$V(\hat{\beta}_1-\hat{\beta}_2)=V(\hat{\beta}_1)+V(\hat{\beta}_2)-2Cov(\hat{\beta}_1,\hat{\beta}_2)=MSE(C_{22}+C_{33}-2C_{23})$ unbia est of var

fr: $E(SS_{AB})=E[\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^n(\widehat{\tau\beta})^2_{ij}]=n\sum_{i=1}^a\sum_{j=1}^b E[(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2]=$ $V[\sum^a\sum^b\sum^n(\widehat{\tau\beta})^2_{ij}]+0=(a-1)(b-1)(n\sigma^2_{\tau\beta}+\sigma^2)=n\sum^a\sum^b(\bar{y}_{ij.}-\bar{y}_{...})^2-SS_A-SS_B$

r(r+r):$E[SS_A]=bn\sum_{i=1}^a E[(\bar{y}_{i..}-\bar{y}_{...})^2]=bn\sum_{i=1}^a(E[]^2+V[])=bn\sum_{i=1}^a\tau_i^2+(a-1)\sigma^2$

$Cov(y_{111},y_{112})=Cov(\beta_{11}+\varepsilon_{111},\beta_{11}+\varepsilon_{112})=Var(\beta_{11})+Cov(\varepsilon_{111},\varepsilon_{112})=\sigma^2_\beta$

$Cov(y_{111},y_{121})=Cov(\beta_{11}+\varepsilon_{111},\beta_{12}+\varepsilon_{121})=Cov(\beta_{11},\beta_{12})+Cov(\varepsilon_{111},\varepsilon_{121})=0$

$Cov(y_{111},y_{211})=Cov(\beta_{11}+\varepsilon_{111},\beta_{21}+\varepsilon_{211})=Cov(\beta_{11},\beta_{21})+Cov(\varepsilon_{111},\varepsilon_{211})=0$

$Var(y_{111})=\sigma^2_\beta+\sigma^2_\varepsilon=Var(y_{112})$;

$Cor(y_{111},y_{112})=\frac{\sigma^2_\beta}{\sigma^2_\beta+\sigma^2_\varepsilon}$ $Cor(y_{111},y_{121})=Cor(y_{111},y_{211})=0$

$E[\bar{y}_{ij.}]=E[\frac{1}{n}\sum_{k=1}^n y_{ijk}]=\frac{\sum_{k=1}^n}{n}E[\mu+\alpha_i+\beta_{ij}+\varepsilon_{ijk}]=\mu+\alpha_i,\forall i=1,..,a;j=1,..,b$

$Var[\bar{y}_{ij.}]=Var[\frac{1}{n}\sum_{k=1}^n y_{ijk}]=\frac{\sum_{k=1}^n}{n^2}Var[\mu+\alpha_i+\beta_{ij}+\varepsilon_{ijk}]=\frac{1}{n}(\sigma^2_\beta+\sigma^2_\varepsilon),\forall i$

$f(\bar{y}_{i1.},..\bar{y}_{ib.})=\prod_{j=1}^b f(\bar{y}_{ij.})=(2\pi\frac{\sigma^2_\beta+\sigma^2_\varepsilon}{n})^{-\frac{b}{2}}\exp[\frac{-n}{2(\sigma^2_\beta+\sigma^2_\varepsilon)}\sum_{j=1}^b(\bar{y}_{ij.}-\mu-\alpha_i)^2]$

$\hat{\alpha}_1-\hat{\alpha}_2\sim N(\alpha_1-\alpha_2,\frac{2}{b}\sigma^2_\beta+\frac{2}{bn}\sigma^2_\varepsilon)$

$SSE=\sum^a\sum_{j=1}^b\sum_{k=1}^n(y_{ijk}-\mu-\alpha_i-\beta_{ij})^2$; $\frac{\partial SSE}{\partial\alpha_i}=2\sum_{j=1}^b\sum_{k=1}^n(y_{ijk}-\mu-\alpha_i-\beta_{ij})(-1)\overset{set}{=}0$;

$\hat{\alpha}_i=\frac{\sum_{j=1}^b\sum_{k=1}^n y_{ijk}}{bn}-\mu-\frac{\sum_{j=1}^b\beta_{ij}}{b}=\bar{y}_{i..}-\mu$

$\hat{\alpha}_1-\hat{\alpha}_2=\bar{y}_{1..}-\mu-(\bar{y}_{2..}-\mu)=\frac{1}{bn}\sum_{j=1}^b\sum_{k=1}^n(y_{1jk}-y_{2jk})$

$=\frac{1}{bn}\sum_{j=1}^b\sum_{k=1}^n(\alpha_1-\alpha_2+\beta_{1j}-\beta_{2j}+\varepsilon_{1jk}-\varepsilon_{2jk})=\alpha_1-\alpha_2+\bar{\beta}_{1.}-\bar{\beta}_{2.}+\bar{\varepsilon}_{1..}-\bar{\varepsilon}_{2..}$

$E[\hat{\alpha}_1-\hat{\alpha}_2]=\alpha_1-\alpha_2$;

$Var[\hat{\alpha}_1-\hat{\alpha}_2]=Var[\bar{\beta}_{1.}-\bar{\beta}_{2.}+\bar{\varepsilon}_{1..}-\bar{\varepsilon}_{2..}]$

$=\frac{1}{b^2}\sum_{j=1}^b(Var[\beta_{1.}]+Var[\beta_{2.}])+\frac{1}{b^2n^2}\sum_{j=1}^b\sum_{k=1}^n(Var[\varepsilon_{1..}]+Var[\varepsilon_{2..}])=\frac{2}{b}\sigma^2_\beta+\frac{2}{bn}\sigma^2_\varepsilon$

$\bar{y}_{ij.}-\bar{y}_{i..}=\mu+\alpha_i+\beta_{ij}+\bar{\varepsilon}_{ij.}-(\mu+\alpha_i+\bar{\beta}_{i.}+\bar{\varepsilon}_{i..})=\beta_{ij}-\bar{\beta}_{i.}+\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..}$

$E[\bar{y}_{ij.}-\bar{y}_{i..}]=E[\beta_{ij}-\bar{\beta}_{i.}+\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..}]=0$

$Cov(\beta_{ij},\bar{\beta}_{i.})=\frac{1}{b}Cov(\beta_{ij},\sum_{j=1}^b\beta_{ij})=\frac{1}{b}[1\cdot\sigma^2_\beta+(b-1)\cdot 0]$

$Cov(\bar{\varepsilon}_{ij.},\bar{\varepsilon}_{i..})=Cov(\frac{1}{n}\sum_{k=1}^n\varepsilon_{ijk},\frac{1}{bn}\sum_{j=1}^b\sum_{k=1}^n\varepsilon_{ijk})=\frac{\sum_{k=1}^n Cov(\varepsilon_{ijk},\sum_{j=1}^b\varepsilon_{ijk})}{bn^2}=\frac{\sigma^2_\varepsilon}{bn}$

$Var[\bar{y}_{ij.}-\bar{y}_{i..}]=Var[\beta_{ij}-\bar{\beta}_{i.}+\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..}]=Var[\beta_{ij}-\bar{\beta}_{i.}]+Var[\bar{\varepsilon}_{ij.}-\bar{\varepsilon}_{i..}]$

$=Var[\beta_{ij}]+Var[\bar{\beta}_{i.}]-2Cov(\beta_{ij},\bar{\beta}_{i.})+Var[\bar{\varepsilon}_{ij.}]+Var[\bar{\varepsilon}_{i..}]-2Cov(\bar{\varepsilon}_{ij.},\bar{\varepsilon}_{i..})$

$=\sigma^2_\beta+\frac{1}{b}\sigma^2_\beta-\frac{2}{b}\sigma^2_\beta+\frac{1}{n}\sigma^2_\varepsilon+\frac{1}{bn}\sigma^2_\varepsilon-\frac{2}{bn}\sigma^2_\varepsilon=\frac{b-1}{b}(\sigma^2_\varepsilon+\frac{1}{n}\sigma^2_\varepsilon)$

$E[\sum_{i=1}^a\sum_{j=1}^b(\bar{y}_{ij.}-\bar{y}_{i..})^2]=\sum_{i=1}^a\sum_{j=1}^b(Var[\bar{y}_{ij.}-\bar{y}_{i..}]+E[\bar{y}_{ij.}-\bar{y}_{i..}]^2)$

$=\sum_{i=1}^a\sum_{j=1}^b[\frac{b-1}{b}(\sigma^2_\varepsilon+\frac{1}{n}\sigma^2_\varepsilon)+0]=a(b-1)(\sigma^2_\beta+\frac{1}{n}\sigma^2_\varepsilon)$

$\hat{\sigma}^2_\beta=\frac{MS_{AB}-MS_E}{n}$; $E[\hat{\sigma}^2]=\frac{1}{n}(n\sigma^2_\beta+\sigma^2_\varepsilon-\sigma^2_\varepsilon)=\sigma^2_\beta$

$E(MS_{B_f(A_f)})=\frac{n}{a(b-1)}\sum_{i=1}^a\sum_{j=1}^b E[(\bar{y}_{ij.}-\bar{y}_{i..})^2]=\sigma^2+\frac{n}{a(b-1)}\sum_{i=1}^a\sum_{j=1}^b\beta^2_{j(i)}$

Factorial model

k factors,p generators; 2^p blocks/fraction;2^{k-p}Run, Blk size;

2^p-1 alias; 2^p-p-1 auto confounded;

I=ABC=BCD=AD

AD+,ABC-,BCD-;(1),bc,abd,acd;ABC+BCD+;b,c,ad,abcd

AD-,ABC-,BCD+;ab,ac,bcd,d;ABC+BCD-;a,abc,bd,cd

I=ABCDF=ABDEG=CEFG;CE=FG,CF=EG,CG=EF minimum aberration

2^{k-p}	2^k	2^p	2^{k-p}	Generator
2^{3-1}_{III}	8	2	4	C=AB
2^{4-1}_{IV}	16	2	8	D=ABC
2^{5-1}_{V}	32	2	16	E=ABCD
2^{5-2}_{III}	32	4	8	D=AB;E=AC
2^{6-1}_{VI}	64	2	32	F=ABCDE
2^{6-2}_{IV}	64	4	16	E=ABC;F=BCD
2^{6-3}_{III}	64	8	8	D=AB;E=AC;F=BC
2^{7-1}_{VII}	128	2	64	G=ABCDEF
2^{7-2}_{IV}	128	4	32	F=ABCD;G=ABDE
2^{7-3}_{IV}	128	8	16	E=ABC;F=BCD;G=ACD
2^{7-4}_{III}	128	16	8	D=AB;E=AC;F=BC;G=ABC

Run	A	B	C	D	E	F	ABD	ABF		
(1)	-	-	-	-	-	-	-	-	1	I=ABCE=BCDF=ADEF
ae	+	-	-	-	+	-	+	+	4	A = BCE = DEF = ABCDF
bef	-	+	-	-	+	+	+	-	2	B = ACE = CDF = ABDEF
abf	+	+	-	-	-	+	-	+	3	C = ABE = BDF = ACDEF
cef	-	-	+	-	+	+	-	+	3	D = BCF = AEF = ABCDE
acf	+	-	+	-	-	+	+	-	2	E = ABC = ADF = BCDEF
bc	-	+	+	-	-	-	+	+	4	F = BCD = ADE = ABCEF
abce	+	+	+	-	+	-	-	-	1	AB = CE = ACDF = BDEF
df	-	-	-	+	-	+	+	+	4	AC = BE = ABDF = CDEF
adef	+	-	-	+	+	+	-	-	1	AD = EF = BCDE = ABCF
bde	-	+	-	+	+	-	-	+	3	AF = DE = BCEF = ABCD
abd	+	+	-	+	-	-	+	-	2	BD = CF = ACDE = ABEF
cde	-	-	+	+	+	-	+	-	2	BF = CD = ACEF = ABDE
acd	+	-	+	+	-	-	-	+	3	AE = BC = DF = ABCDEF
bcdf	-	+	+	+	-	+	-	-	1	ABD = BEF = ACF = CDE
abcdef	+	+	+	+	+	+	+	+	4	ABF = BDE = ACD = CEF

4Blk:ABD-ABF:1–;2+-;3-+;4++. $df_{noAE,ABD,ABF}$=1, df_{Blk}=3, df_T=15

2Blk:ABD:1+3,2+4; ABF:1+2,3+4. $df_{noABDorABF}$=1, df_{Blk}=1, df_T=15

ANOVA-df: T= 2^kn-1;Blk= 2^p-1;E= $2^k(n-1)$;others=1

$SS_{Block}=\frac{1}{2^3}C^2_{ABC}=SS_{ABC}$; $SS_{AB}=\frac{n}{4}C^2_{AB}=\frac{n}{4}(\bar{y}_{(1)}-\bar{y}_a-\bar{y}_b+\bar{y}_{ab})^2$

Lack of fit

H_0: There is no lack of fit, the model is appropriate

$\sum^m\sum_{j=1}^{n_i}(y_{ij}-\hat{y}_i)^2(SSE)=\sum^m\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2(SS_{PE})+\sum^m n_i(y_{ij}-\hat{y}_i)^2(SS_{LOF})$

$SS_{PE}=\sum_{i=1}^m\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_i)^2=28$; $SS_{LOF}=703.87576-28=675.8758$;

$df_{PE}=n-m=12-9=3$; $MS_{PE}=28/3$; $df_{LOF}=dfE-df_{PE}=m-(k+1)=6$

$F=\frac{SS_{LOF}/df_{LOF}}{MS_{PE}}=\frac{675.87576/7}{28/3}=10.34504>F(0.05,6,3)=8.94$. Reject

contrast

$H_0:\beta_1=2\beta_3,\beta_2=\beta_3,\beta_5=0$

$\mathbf{T}=\begin{bmatrix}0&1&0&-2&0&0&0\\0&0&1&-1&0&0&0\\0&0&0&0&0&1&0\end{bmatrix}_{3\times7}$ $\mathbf{fi}=\begin{bmatrix}\beta_0\\ \vdots\\ \beta_6\end{bmatrix}_{7\times1}$ $\mathbf{C}=\begin{bmatrix}0\\0\\0\end{bmatrix}_{3\times1}$ $rank(T)=3$

$\Gamma=\sum_{i=1}^a c_i\mu_i$,$C=\sum_{i=1}^a c_i\bar{y}_i$;$\sum_{i=1}^a c_i=0$;Orthogonal $\sum_{i=1}^a c_id_i=0$

$SS_C=\frac{(\sum_{i=1}^a c_i\bar{y}_i)^2}{\frac{1}{n}\sum_{i=1}^a c_i^2}$,$\sum_1^{a-1}SS_C=SS_{Trt}$; $V[\sum_{i=1}^a c_iy_{i.}]=\sigma^2\sum_{i=1}^a n_ic_i^2$

$\hat{\tau}_1+\hat{\tau}_2+\hat{\tau}_3$=0	$\hat{\mu}$=13.8	(a)	Contrast	(b)	$\hat{\mu}$=8.4	$\hat{\tau}_3$=0
$\hat{\mu}+\hat{\tau}_1=\bar{y}_{1.}$=10.8	$\hat{\tau}_1$=-3.0	10.8	$\mu+\tau_1$	10.8	$\hat{\tau}_1$=2.4	$\hat{\mu}+\hat{\tau}_1=\bar{y}_{1.}$=10.8
$\hat{\mu}+\hat{\tau}_2=\bar{y}_{2.}$=22.2	$\hat{\tau}_2$=8.4	-9.0	$2\tau_1-\tau_2-\tau_3$	-9.0	$\hat{\tau}_2$=13.8	$\hat{\mu}+\hat{\tau}_2=\bar{y}_{2.}$=22.2
$\hat{\mu}+\hat{\tau}_3=\bar{y}_{3.}$=8.4	$\hat{\tau}_3$=-5.4	19.2	$\mu+\tau_1+\tau_2$	24.6	$\hat{\tau}_3$=0	$\hat{\mu}+\hat{\tau}_3=\bar{y}_{3.}$=8.4

$H_{C0,D0}$; $\sum_{i=1}^3 c_id_i$=0	C,D$\sum_{i=1}^a c_i,d_i\bar{y}_{i.}$	$F_{1,12}$	$SS_{C,D}/MS_E$
$\mu_1-2\mu_2+\mu_3=0$	$\bar{y}_{1.}-2\bar{y}_{2.}+\bar{y}_{3.}$	$C^2/\frac{MS_E}{n}\sum_{i=1}^a c_i^2$	$(-25.2)^2/\frac{16.9}{5}6$
$\mu_1-\mu_3=0$	$\bar{y}_{1.}-\bar{y}_{3.}$	$D^2/\frac{MS_E}{n}\sum_{i=1}^a d_i^2$	$2.4^2/\frac{16.9}{5}2$

Missing Values

Exact(partial) method $F_0=\frac{(SSE_{red}-SSE_{ful})/r}{MSE_{ful}}=\frac{1403.7-921.5}{184.30*(a-1)}$;

Apporximation method $\hat{x}=\frac{ay'_{1.}+by'_{.1}-y'_{..}}{(a-1)(b-1)}$,$F_{adj}=\frac{MS_{Trt}}{SSE/dfE_{adj}}=\frac{162.08}{921.5/(6-1)}$

Relative Efficiency

$\frac{(df_{E(LS)}+1)(df_{E(CRD)}+3)MS_{CRD}}{(df_{E(LS)}+3)(df_{E(CRD)}+1)MS_{LS}}=2.3$

$df_{E(LS)}=(p-1)(p-2)$=20,$df_{E(GS)}=(p-1)(p-3)$=15, $df_{E(CRD)}$=a(n-1)=7.2,32

The Latin-Squre design can only use 2 blocking factors, as we distribute the levels of the treatment factor on a table with rows of one blocking factor (each row is fore one block) and columns of the other blocking factor (each column is one block)

To test three blocking factors by turning the Latin-square design into a Graeco-Latin square design, which allows to add a Greek letter to each entry in the table, where each Greek letter stands for a block of the factor G.

Aα	Bβ	Cγ	Dδ
Bδ	Aγ	Dβ	Cα
Cβ	Dα	Aδ	Bγ
Dγ	Cδ	Bα	Aβ