

2015S  
Fountain\*, Crain

2015S4

2018S4 2019S1  
Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i, i = 1, \dots, n$ , with the restriction that  $\beta_1 - \beta_0 = 0$ . Find the least-squares estimators of the regression coefficients.

Let  $SSE = \sum_{i=1}^n (y_i - \beta_0 - \beta_0 x_i - \beta_2 x_i^2)^2$

$$\frac{\partial SSE}{\partial \beta_0} = 2 \sum_{i=1}^n (y_i - \beta_0 - \beta_0 x_i - \beta_2 x_i^2)(-1 - x_i) \stackrel{set}{=} 0; \hat{\beta}_0 = \frac{\sum_{i=1}^n (1+x_i)y_i - \hat{\beta}_2 \sum_{i=1}^n (1+x_i)x_i^2}{\sum_{i=1}^n (1+x_i)^2}$$
$$\frac{\partial SSE}{\partial \beta_2} = 2 \sum_{i=1}^n (y_i - \beta_0 - \beta_0 x_i - \beta_2 x_i^2)(-x_i^2) \stackrel{set}{=} 0; \sum_{i=1}^n x_i^2 y_i = \hat{\beta}_0 \sum_{i=1}^n x_i^2 (1+x_i) + \hat{\beta}_2 \sum_{i=1}^n x_i^4 = \frac{\sum_{i=1}^n (1+x_i)y_i - \hat{\beta}_2 \sum_{i=1}^n (1+x_i)x_i^2}{\sum_{i=1}^n (1+x_i)^2} \sum_{i=1}^n x_i^2 (1+x_i) + \hat{\beta}_2 \sum_{i=1}^n x_i^4$$
$$\hat{\beta}_2 \left[ \sum_{i=1}^n x_i^4 \sum_{i=1}^n (1+x_i)^2 + \left[ \sum_{i=1}^n x_i^2 (1+x_i) \right]^2 \right] = \sum_{i=1}^n x_i^2 y_i \sum_{i=1}^n (1+x_i)^2 - \sum_{i=1}^n (1+x_i)y_i \sum_{i=1}^n x_i^2 (1+x_i)$$
$$\hat{\beta}_2 = \frac{\sum_{i=1}^n x_i^2 y_i \sum_{i=1}^n (1+x_i)^2 - \sum_{i=1}^n (1+x_i)y_i \sum_{i=1}^n x_i^2 (1+x_i)}{\sum_{i=1}^n x_i^4 \sum_{i=1}^n (1+x_i)^2 + \left[ \sum_{i=1}^n x_i^2 (1+x_i) \right]^2}$$

2015F  
2015F1

2016S1 [566-HW2-6]  
You must design an experiment to test six factors, each having two levels. Your budget will only allow sixteen runs. Furthermore, due to time constraints, only eight runs can be done on a given day, so you will have to conduct the experiment in 2 blocks. You may assume that 4-way and higher interactions are not important.  
Show all of the following:  
• all of your generators (make sure that your resolution is at least III)  
 $2_{IV}^{6-2}$  E=ABC, F=BCD; I=ABCE=BCDF=ADEF  
• the 16 runs to conduct

A	B	C	D	E	F	Run	Block
-	-	-	-	-	-	(1)	1
+	-	-	-	+	-	ae	2
-	+	-	-	+	+	bef	1
+	+	-	-	-	+	abf	2
-	-	+	-	+	+	cef	2
+	-	+	-	-	+	acf	2
-	+	+	-	-	-	bc	1
+	+	+	-	+	-	abce	1
-	-	-	+	-	+	df	2
+	-	-	+	+	+	adef	1
-	+	-	+	+	-	bde	2
+	+	-	+	-	-	abd	1
-	-	+	+	+	-	cde	1
+	-	+	+	-	-	acd	2
-	+	+	+	-	+	bcd	1
+	+	+	+	+	+	abcdef	2

E=ABC; F=BCD; Block=ACD  
• the alias structure  
A=BCE=DEF; B=CDF=ACE; C=ABE=BDF; D=BCF=AEF; E=ABC=ADF; F=BCD=ADE;  
AB=CE; AC=BE; AD=EF; AE=BC=DF; AF=DE; BD=CF; BF=CD;  
ABD=CDE=ACF=BEF; ACD=BDE=ABF=CEF  
• the eight runs to be done on each day  
Day1: (1), bef, bc, abce, adef, abd, cde, bcd  
Day2: ae, abf, cef, acf, df, bde, acd, abcdef  
• the effects to be confounded with blocks  
ACD=BDE=ABF=CEF  
• the Source and DF columns of the ANOVA table

A	B	C	D	E	F	AB	AC	AD	AE	AF	BD	BF	ABC	Block
---	---	---	---	---	---	----	----	----	----	----	----	----	-----	-------

AB=CE, AC=BE, AD=EF, AE=BC=DF, AF=DE, BD=CF, BF=CD, ABC=CDE=ACF

2015F2

2017F1  
A company is comparing two different methods of processing their raw material. They begin with 8 batches of material, which are randomly assigned to one of the two processes. The quality of the final product is measured on a 50-point scale. There is some concern that the outside temperature on each day might have an effect on the product (and the effect might be different for the two processes), and

so it is recorded so that it can be taken into account. The following table shows an ordered pair for each batch, consisting of the quality measurement and the temperature.

Process 1	(45,81)(40,68)(41,77)(41,61)
Process 2	(42,59)(37,62)(41,83)(35,70)

- a) Write an appropriate model for the situation described above. Hint: there are 8 observations, and you will need to use at least one indicator variable.

Process 1:  $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ ; Process 2:  $y_i = \gamma_0 + \gamma_1 x_i + \varepsilon_i$ ; Let  $w_i = \begin{cases} 0 & 1 \leq i \leq 4 \\ 1 & 5 \leq i \leq 8 \end{cases}$ , overall  $y_i = \beta_0 + \beta_1 x_i + w_i \gamma_0 + w_i \gamma_1 x_i + \varepsilon_i$

- b) Write the matrix form of the appropriate model. Show the contents and dimensions of all matrices.

$$\begin{bmatrix} 45 \\ 40 \\ 41 \\ 41 \\ 42 \\ 42 \\ 37 \\ 41 \\ 35 \end{bmatrix}_{8 \times 1} = \begin{bmatrix} 1 & 81 & 0 & 0 \\ 1 & 68 & 0 & 0 \\ 1 & 77 & 0 & 0 \\ 1 & 61 & 0 & 0 \\ 1 & 59 & 0 & 59 \\ 1 & 62 & 0 & 62 \\ 1 & 63 & 0 & 63 \\ 1 & 70 & 1 & 70 \end{bmatrix}_{8 \times 4} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \gamma_0 \\ \gamma_1 \end{bmatrix}_{4 \times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix}_{8 \times 1}$$

- c) Suppose that you wish to test for equality of the two slopes. Write the matrix form of the reduced model. What will be the numerator and denominator degrees of freedom for the additional sum of squares F test?

$$H_0 : \beta_1 = \gamma_1, y_i = \beta_0 + \beta_1 x_i + w_i \gamma_0 + \varepsilon_i$$

$$\begin{bmatrix} 45 \\ 40 \\ 41 \\ 41 \\ 42 \\ 37 \\ 41 \\ 35 \end{bmatrix}_{8 \times 1} = \begin{bmatrix} 1 & 81 & 0 & 0 \\ 1 & 68 & 0 & 0 \\ 1 & 77 & 0 & 0 \\ 1 & 61 & 0 & 0 \\ 1 & 59 & 1 & 59 \\ 1 & 62 & 1 & 62 \\ 1 & 83 & 1 & 83 \\ 1 & 70 & 1 & 70 \end{bmatrix}_{8 \times 3} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \gamma_0 \end{bmatrix}_{3 \times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix}_{8 \times 1}$$

$$\frac{8-3}{8-4}$$

**2015F3**

2016S3 2017F2

- Explain the difference between fixed and random effects in an experimental design. Give an example to illustrate your explanation.
- Explain the difference between crossed and nested effects in an experimental design. Give an example to illustrate your explanation. Make sure to discuss which terms would be absent from the model, and the resulting effect on sums of squares and degrees of freedom in the ANOVA table.

**2015F4**

2019S1

Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$ ,  $i = 1, \dots, n$  with the additional restrictions that  $\beta_1 = 0$ ,  $\beta_0 = 2\beta_2$ . Find the least-squares estimators of  $\beta_0$  and  $\beta_1$ .

$$\text{Let } SSE = \sum_{i=1}^n (y_i - \hat{y})^2 = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2 = \sum_{i=1}^n (y_i - \beta_0 - 2\beta_0 x_i^2)^2$$

$$\frac{\partial SSE}{\partial \beta_0} = \sum_{i=1}^n (y_i - \beta_0 - 2\beta_0 x_i^2)(-2x_i^2) \stackrel{set}{=} 0; \hat{\beta}_0 = \frac{\sum_{i=1}^n x_i^2 y_i}{\sum_{i=1}^n x_i^2 + 2 \sum_{i=1}^n x_i^4} \quad \hat{\beta}_2 = \frac{2 \sum_{i=1}^n x_i^2 y_i}{\sum_{i=1}^n x_i^2 + 2 \sum_{i=1}^n x_i^4}$$

# 2016S

Fountain, Tableman\*

## 2016S1

You must design an experiment to test six factors, each having two levels. Your budget will only allow sixteen runs. Furthermore, due to time constraints, only eight runs can be done on a given day, so you will have to conduct the experiment in 4 blocks. You may assume that 4-way and higher interactions are not important.

Show all of the following:

- all of your generators (make sure that your resolution is at least III)

$$2_{III}^{6-4} \text{ E=ABC, F=BCD; I=ABCE=BCDF=ADEF}$$

- the 16 runs to conduct E=ABC; F=BCD; Block=ACD

[illegible]

- the alias structure  
 $A=BCE=DEF$ ;  $B=CDF=ACE$ ;  $C=ABE=BDF$ ;  $D=BCF=AEF$ ;  $E=ABC=ADF$ ;  $F=BCD=ADE$ ;  
 $AB=CE$ ;  $AC=BE$ ;  $AD=EF$ ;  $AE=BC=DF$ ;  $AF=DE$ ;  $BD=CF$ ;  $BF=CD$ ;  
 $ABD=CDE=ACF=BEF$ ;  $ACD=BDE=ABF=CEF$
- the four runs to be done on each day  
Day1: ae, bc, df, abcdef  
Day2: abf, acd, bde, cef  
Day3: abd, acf, bef, cde  
Day4: (1), abce, adef, bcdf
- the effects to be confounded with blocks  
 $ACD=BDE=ABF=CEF$ ;  $ABD=ACF=BEF=CDE$ ;  $AD=EF$
- the Source and DF columns of the ANOVA table

A	B	C	D	E	F	AB	AC	AE	AF	BD	BF	Block
1	1	1	1	1	1	1	1	1	1	1	1	3

$AB=CE$ ,  $AC=BE$ ,  $AE=BC=DF$ ,  $AF=DE$ ,  $BD=CF$ ,  $BF=CD$

## 2016S2

2017F3 2018S3

The multiple linear regression model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \varepsilon_i$  was fit to a data set of 75 observations. The regression SS's (SSR) were partitioned sequentially into the following:

$$SSR(X_1) = 108$$

$$SSR(X_2|X_1) = 163$$

$$SSR(X_3|X_1 X_2) = 29$$

$$SSR(X_4|X_1 X_2 X_3) = 41$$

$$SSR(X_5|X_1 X_2 X_3 X_4) = 26$$

The model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_3 x_{i3} + \beta_5 x_{i5} + \varepsilon_i$  was also fit to the same data and the following ANOVA was calculated:

Source	SS
Regression	214
Residual Error	489
Total	703

Answer the following from the above information:

- Calculate the F-statistic for testing the hypothesis ( $H_0$ ) that  $X_3$ ,  $X_4$ , and  $X_5$  have no significant effect on the response  $Y$ .
- Calculate  $R^2$  for the model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_3 x_{i3} \varepsilon_i$
- Calculate the  $R^2_{adj}$  for the model in part (b).
- Calculate the F-statistic for testing  $H_0 : \beta_2 = \beta_4 = 0$ .

## 2016S3

2017F2

- Explain the difference between fixed and random effects in an experimental design. Give an example to illustrate your explanation.
- Explain the difference between crossed and nested effects in an experimental design. Give an example to illustrate your explanation. Make sure to discuss which terms would be absent from the model, and the resulting effect on sums of squares and degrees of freedom in the ANOVA table.

## 2016S4

2019S1

Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$ ,  $i = 1, \dots, n$  with the additional restrictions that  $\beta_1 = 1$ ,  $\beta_2 = \beta_0$ . Find the least-squares estimators of the coefficients.

## 2016S5

In the multiple regression model with  $p - 1$  independent variables  $X_j$ , let the  $n \times p$  matrix  $\mathbf{X}$  denote the design matrix which contains the column of 1's to fit the intercept term and has full rank. Let  $\mathbf{H}$  denote the hat matrix. Let  $h_{ii}$  denote the  $i_{th}$  diagonal element of  $\mathbf{H}$ . Prove that  $0 \leq h_{ii} \leq 1$ .

## 2016F

Jong Sung Kim\*, Brad Crain

2016F1

The data for this question consist of 12 measurements on each of 2 quantitative regressor variables  $x_1$  and  $x_2$  and on a dependent variable  $y$ . The data are displayed below:

Obs	x1	x2	y
1	3	1	5
2	3	2	3
3	3	2	3
4	7	1	23
5	7	3	19
6	12	1	75
7	12	1	81
8	12	2	67
9	12	3	51
10	12	3	47
11	19	2	135
12	19	3	121

The model to be fit to the data is  $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon_i$ . What follows is partially incomplete SAS output. Although the output is incomplete, there is enough information given that you can answer the questions that follow with a minimal amount of calculation. Note that Type I SS is the same as Seq SS.

Information I: model:  $y = x_1 x_2$ ;  
Analysis of Variance

Source	SumofSquares
Model	
Error	703.87576
Total	

Root MSE	8.84355	R-Square	0.9680
Dependent Mean	52.66667	Adj R-Sq	0.9609
Coeff Var	16.79156		

Parameter Estimates

Variable	DF	ParameterEstimate	Type I SS
Intercept	1	-10.44655	33285
x1	1	8.15560	20647
x2	1	-9.56119	663.87250

Covariance of Estimates

Variable	Intercept	x1	x2
Intercept	54.81211821	-1.510773883	-16.5305567
x1	-1.510773883	0.2483463917	-0.496692783
x2	-16.5305567	-0.496692783	10.7694378

Information II: model:  $y = x_2 x_1$ ;  
Analysis of Variance

Source	SumofSquares
Model	
Error	703.87576
Total	

Root MSE	8.84355	R-Square	0.9680
Dependent Mean	52.66667	Adj R-Sq	0.9609
Coeff Var	16.79156		

Parameter Estimates

Variable	DF	ParameterEstimate	Type I SS
Intercept	1	-10.44655	33285
x2	1	-9.56119	364.50000
x1	1	8.15560	20946

- (a) Do a hypothesis test, at the .01 level of significance, of  $H_0 : \beta_1 = \beta_2 = 0$  vs.  $H_1$ : At least one of  $\beta_1$  or  $\beta_2 \neq 0$ .
- (b) What is the value of  $R^2$ ?
- (c) Do the following two hypothesis tests, each at the .05 level of significance:
  - i.  $H_0 : \beta_1 = 0$  vs.  $H_1 : \beta_1 \neq 0$

- ii.  $H_0 : \beta_2 = 0$  vs.  $H_1 : \beta_2 \neq 0$
- (d) Obtain a 99% confidence interval for  $\beta_1$ .
- (e) Give an unbiased estimate of the variance of  $\hat{\beta}_1 - \hat{\beta}_2$ .
- (f) Obtain MS(Pure Error). Hint: Pure Error can be found exactly the same way as we did for simple linear regression model. That is, group according to different combinations of levels from  $X_1$  and  $X_2$ . First compute SS(Pure Error), and then divide it by degrees of freedom.
- (g) Perform a test for lack-of-fit at the .05 level of significance. Note: If you are unable to answer part (f), use MS(Pure Error) = 7.5. This is not the correct answer to (f), but if you use it in this part of the problem, you will receive full credit on this part, provided your answer is otherwise correct.

## 2016F2

Data were collected on each of two quantitative regressor variables  $X_1$  and  $X_2$ , a dichotomous categorical variable which we shall call "group", and a dependent variable  $Y$ . The data are displayed below:

Obs	y	x1	x2	group
1	14	3.54	17	1
2	23	4.87	15	1
3	31	2.89	13	1
4	28	3.03	10	1
5	25	4.10	21	2
6	35	3.55	18	2
7	18	5.17	18	2
8	23	3.99	16	2
9	40	2.02	13	2
10	34	2.44	13	2

The model to be fit to the data is  $Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{2i}^2 + \beta_4 Z_i + \beta_5 x_{1i} Z_i + \beta_6 x_{2i} Z_i + \beta_7 x_{2i}^2 Z_i + \varepsilon_i$ , where  $Z_i = 1$ , if case  $i$  is in group 1, and  $Z_i = 0$ , otherwise.

- (a) What are the first and last rows of the X-matrix (assuming that the data are entered in the same order in which they are displayed above)?

First row:

Last row:

- (b) For each of the following objectives, give the appropriate null hypothesis.
  - i. It is desired to know whether the slope coefficient on  $x_1$  is the same for both groups.
  - ii. It is desired to know whether the entire regression models for the two groups are identical.
  - iii. It is desired to know whether a quadratic term in  $x_2$  is needed by both groups.
  - iv. It is desired to know whether the slope coefficients on  $x_1$  and  $x_2$  for the first group are equal.

## 2016F3

The following is part of the SAS output from a simple linear regression model:  $y_i = \beta_0 + \beta x_i + \varepsilon_i$ , where  $i = 1, \dots, 13$ , and  $y_i$  and  $x_i$  are the  $i$ th punter's average punting distance and right leg strength, respectively. Each punter punted 10 times and the average distance was measured. In addition, measure of right leg strength (lb lifted) was taken via a weight lifting test.

Obs	rleg	distance
1	170	162.50
2	140	144.00
3	180	147.50
4	160	163.50
5	170	192.00
6	150	171.75
7	170	162.00
8	110	104.83
9	120	105.67
10	130	117.58
11	120	140.25
12	140	150.17
13	160	165.17

Dependent Variable: distance X'X Inverse, Parameter Estimates, and SSE

Variable	Intercept	rleg	distance
Intercept	3.577777778	-0.023703704	14.906962963
rleg	-0.023703704	0.0001604938	0.9026716049
distance	14.906962963	0.9026716049	3025.6604973

Analysis of Variance

Source	DF	SumofSquares	MeanSquare	FValue	Pr>F
Model	1	5076.93063	5076.93063	18.46	0.0013
Error	11	3025.66050	275.06005		
Total	12	8102.59112			

Root MSE	16.58493	R-Square	0.6266
Dependent Mean	148.22462	Adj R-Sq	0.5926

Coeff Var	11.18906
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## Parameter Estimates

Variable	DF	ParameterEstimate	Pr> t
Intercept	1	14.90696	0.6439
rleg	1	0.90267	0.0013

## Output Statistics

Obs	Dep Var	Predicted Value	Std Error MeanPredict	95% CL	Mean
1	162.5000				
2	144.0000	141.2810	4.8755	130.5501	152.0119
3	147.5000	177.3879	8.1998	159.3402	195.4355
4	163.5000	159.3344	5.2769	147.7201	170.9488
5	192.0000				
6	171.7500	150.3077	4.6253	140.1274	160.4880
7	162.0000				
8	104.8300	114.2008	9.1584	94.0433	134.3583
9	105.6700	123.2276	7.4170	106.9028	139.5523
10	117.5800	132.2543	5.9141	119.2374	145.2712
11	140.2500	123.2276	7.4170	106.9028	139.5523
12	150.1700	141.2810	4.8755	130.5501	152.0119
13	165.1700	159.3344	5.2769	147.7201	170.9488

## Output Statistics

Obs	95% CL	Predict	Residual
1			
2	103.2332	179.3288	2.7190
3	136.6668	218.1089	-29.8879
4	121.0281	197.6408	4.1656
5			
6	112.4115	188.2039	21.4423
7			
8	72.5018	155.8999	-9.3708
9	83.2403	163.2148	-17.5576
10	93.4996	171.0089	-14.6743
11	83.2403	163.2148	17.0224
12	103.2332	179.3288	8.8890
13	121.0281	197.6408	5.8356

- Find the three residual values at  $x = 170$ .
- Compute a 95% confidence interval for the mean response at  $x = 170$ . Hint: Compute the variance of the estimate of the mean response at  $x = 170$ .
- Compute a 95% prediction interval on a new response observation at  $x = 170$ . Hint: You can use part of the expression in (b).

## 2016F4

Prior to 1985, Meily Lin had observed that some colors of birthday balloons seem to be harder to inflate than others. She ran this experiment to determine whether balloons of different colors are similar in terms of the time taken for inflation to a diameter of 7 inches. Four colors were selected from a single manufacturer. An assistant blew up the balloons and the experimenter recorded the times (to the nearest 1/10 second) with a stop watch. This experiment was replicated 4 times, and the data including the order are displayed in the SAS code below, where color 1 = pink, 2 = yellow, 3 = orange, and 4 = blue.

```
data balloon;
input runorder color inftime @@;
cards;
1 1 22.4 2 3 24.6 3 1 20.3 4 4 19.8 5 3 24.3 6 2 22.2 7 2 28.5 8 2 25.7
9 3 20.2 10 1 19.6 11 2 28.8 12 4 24.0 13 4 17.1 14 4 19.3 15 3 24.2 16
1 15.8 17 2 18.3 18 1 17.5 19 4 18.7 20 3 22.9 21 1 16.3 22 4 14.0 23 4
16.6 24 2 18.1 25 2 18.9 26 4 16.0 27 2 20.1 28 3 22.5 29 3 16.0 30 1
19.3 31 1 15.9 32 3 20.3
```

- Why or why not do we need to record the run order in the model?
- What kind of model would be appropriate for the above experiment?
- Read the following . If we had assumed that there is an equal slope linear relationship between the inflation time and the run order for each color, how can we test the assumption? How would you adjust the following program? Why?

```
<Program I>
proc glm data=balloon;
class color; /* color: 1 = pink, 2 = yellow, 3 = orange, 4 = blue */
model inftime = color runorder;
estimate 'pink vs. orange' color -1 0 1 0;
lsmeans color/pdiff;
run;
```

- Based on the following output from , one can apply a Bonferroni multiple comparison test with level .05. Which are significantly different and which are not?

The GLM Procedure; Least Squares Means

color	inftimeLSMEAN	LSMEANNumber
1	18.3341795	1
2	22.3883782	2

3	22.0882820	3
4	18.2141603	4

Least Squares Means for effect color  $\Pr > |t|$  for  $H_0: \text{LSMean}(i) = \text{LSMean}(j)$  Dependent Variable: inftime

i/j	1	2	3	4
1				
2	0.0043	0.0043	0.0076	0.9271
3	0.0076	0.8195	0.8195	0.0034
4	0.9271	0.0034	0.0060	0.0060

## 2016F5

Consider an experiment to compare 7 treatments in block of size 5. Taking all possible combinations of five treatments from seven gives a balanced incomplete block design with each treatment level occurring 15 times. Hint: Figure out  $p, t, k, r, \lambda$  and their relationships.

- How many blocks does the design have?
- Show that the number of times each treatment level occurs must be a multiple of five for a balanced incomplete block design with 7 treatments and blocks of size 5 to exist.
- Show that the smallest balanced incomplete block design has 15 observations per treatment.

## 2017S

Brad Crain, Jong Sung Kim\*

## 2017SR1

2018S1

A company is comparing two different methods of processing their raw material. They begin with 8 batches of material, which are randomly assigned to one of the two processes. The quality of the final product is measured on a 50-point scale. There is some concern that the outside temperature on each day might have an effect on the product (and the effect might be different for the two processes), and so it is recorded so that it can be taken into account. The following table shows an ordered pair for each batch, consisting of the quality measurement and the temperature.

Process 1	(45,81)(40,68)	(41,77)(41,61)
Process 2	(42,59)(37,62)	(41,83)(35,70)

- Write an appropriate model for the situation described above. Hint: there are 8 observations, and you will need to use at least one indicator variable.
- Write the matrix form of the appropriate model. Show the contents and dimensions of all matrices.
- Suppose that you wish to test for equality of the two slopes. Write the matrix form of the reduced model. What will be the numerator and denominator degrees of freedom for the additional sum of squares F test?

## 2017SR2

2019S1

Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i, i = 1, \dots, n$  with the additional restrictions that  $\beta_1 = 1$  and  $\beta_2 = \beta_0/2$ . Find the least-squares estimators of  $\beta_0, \beta_1, \beta_2$ .

Let

$$SSE = \sum_{i=1}^n (y_i - \beta_0 - x_i - \frac{\beta_0}{2} x_i^2)^2$$

$$\frac{\partial SSE}{\partial \beta_0} = -2 \sum_{i=1}^n (y_i - \hat{\beta}_0 - x_i - \frac{\hat{\beta}_0}{2} x_i^2) (1 + \frac{x_i^2}{2}) \stackrel{set}{=} 0$$

$$\hat{\beta}_0 = \frac{\sum_{i=1}^n (y_i - x_i)}{\sum_{i=1}^n (1 + \frac{x_i^2}{2})}, \hat{\beta}_1 = 1, \hat{\beta}_2 = \frac{\sum_{i=1}^n (y_i - x_i)}{2 \sum_{i=1}^n (1 + \frac{x_i^2}{2})}$$

## 2017SD1

Given a educational material evaluation experiment where there are three possible blocking factors [R,C,G], each with six levels  $[R_{1..6}; C_{1..6}; G_{1..6}]$ :

- Write out the model equation of the Latin Square design if the blocking factors R and C are used, and G is disregarded.
- Explain why all three blocking factors can not be used simultaneously without a modification

LT only 2 block

- What is the modification required?

add one dummy to form

- If the Relative Efficiency for the modified experiment was calculated to be 2.3, how many observations of heterogeneous experimental units in a CRD would be expected to obtain the same variance for the treatment mean as one replicate of the modified experiment.

2017SD2

Given a Blocked  $2^{6-2}$  design with Factors [A,B,C,D,E,F],Generators E=ABC, F=BCD and Defining Contrasts AB, CD

1. How many blocks are included in this design?  
4
2. What is the Defining Relationship in this design?  
I=ABCE=BCDF=ADEF
3. What is the Resolution of this Design?  
IV
4. List the aliases of AE  
AE=BC=ABCDEF=DF
5. Show the effect on two-way interactions that include A, if you augment by folding on A  
I=-ABCE=BCDF=-ADEF  
AB=-CE=ACDF=-BDEF  
AC=-BE=ABDF=-CDEF  
AD=-BCDE=ABCF=-EF  
AE=-BC=ABCDEF=-DF  
AF=-BCEF=ABCD=-DE
6. List the aliases of the defining contrasts [including the generalized interaction]?  
AB=CE=ACDF=BDEF  
CD=ABDE=BF=ACEF

2017F

Robert Fountain\*, Daniel Taylor-Rodriguez

2017F1

2018S1 2019S3  
A company has developed two specialized training workshops for their employees. Each of the 12 employees is randomly assigned to one of the two workshops. The company would like to develop a model that could be used to predict each employee’s performance score (a number from 0 to 100) based on their attendance at the workshops. The previous year’s performance score is also available for use as a predictor. The following table shows an ordered pair for each employee, consisting of the current performance score and the previous year’s score.

WorkshopA	(70,58)(70,62)	(68,60)(72,65)	(72,66)(72,62)
WorkshopB	(75,60)(74,62)	(72,60)(71,60)	(73,61)(73,65)

- a) Write an appropriate model for the situation described above, allowing for different slopes and different intercepts for the two workshops. Hint: there are 12 observations, and you will need to use at least one indicator variable.
- b) Write the matrix form of the appropriate model. Show the contents and dimensions of all matrices.
- c) Suppose that you wish to test for equality of the two slopes. Write the matrix form of the reduced model. What will be the numerator and denominator degrees of freedom for the additional sum of squares F test?

2017F2

- a) Explain the difference between fixed and random effects in an experimental design. Give an example to illustrate your explanation.
- b) Explain the difference between crossed and nested effects in an experimental design. Give an example to illustrate your explanation. Make sure to discuss which terms would be absent from the model, and the resulting effect on sums of squares and degrees of freedom in the ANOVA table.

2017F3

2018S3  
The multiple linear regression model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \varepsilon_i$  was fit to a data set of 75 observations. The regression SS’s (SSR) were partitioned sequentially into the following:

- $SSR(X_1) = 108$
- $SSR(X_2|X_1) = 163$
- $SSR(X_3|X_1 X_2) = 29$
- $SSR(X_4|X_1 X_2 X_3) = 41$
- $SSR(X_5|X_1 X_2 X_3 X_4) = 26$

The model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_3 x_{i3} + \beta_5 x_{i5} + \varepsilon_i$  was also fit to the same data and the following ANOVA was calculated:

Source	SS
Regression	214
Residual Error	489
Total	703

Answer the following from the above information:

- a) Calculate the F-statistic for testing the hypothesis ( $H_0$ ) that  $X_3$ ,  $X_4$ , and  $X_5$  have no significant effect on the response  $Y$ .
- b) Calculate  $R^2$  for the model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_3 x_{i2} \varepsilon_i$
- c) Calculate the  $R^2_{adj}$  for the model in part (b).
- d) Calculate the F-statistic for testing  $H_0 : \beta_2 = \beta_4 = 0$ .



2017F4

2019S1

Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i, i = 1, \dots, n$  with the additional restrictions that  $\beta_0 = 1, \beta_1 - \beta_2 = 0$ . Find the least-squares estimators of the regression coefficients.

2018S

Robert Fountain\*, Daniel Taylor-Rodriguez

2018S1

2019S3

A company has developed three specialized training workshops for their employees. Each of the 12 employees is randomly assigned to one of the three workshops. The company would like to develop a model that could be used to predict each employee's performance score (a number from 0 to 100) based on their attendance at the workshops. The previous year's performance score is also available for use as a predictor. The following table shows an ordered pair for each employee, consisting of the current performance score and the previous year's score.

WorkshopA	(70,58)(70,62)	(68,60)(72,65)
WorkshopB	(75,60)(74,62)	(72,60)(71,60)
WorkshopC	(72,66)(72,62)	(73,61)(73,65)

- a) Write an appropriate model for the situation described above, allowing for different slopes and different intercepts for the three workshops. Hint: there are 12 observations, and you will need to use at least one indicator variable.
- b) Write the matrix form of the appropriate model. Show the contents and dimensions of all matrices.
- c) Suppose that you wish to test for equality of the three slopes. Write the matrix form of the reduced model. What will be the numerator and denominator degrees of freedom for the additional sum of squares F test?

2018S2

A company that produces textiles is trying to determine if the final quality is determined by production site (A), machine operator (B), and thread type (C). The company operates 3 production sites, and all 3 will participate in the experiment. At each of the 3 sites, 5 machine operators will be randomly selected. The company uses two different types of thread. Each of the operators will produce 3 samples of cloth using each type of thread, yielding a total of  $3 \times 5 \times 2 \times 3 = 90$  observations.

Source	SS	df	MS	F	pval<0.05
A	17	2	8.5	3.70	*
B	25	4	6.2	2.70	*
C	4	1	4.0	1.74	
AB	32	8	4.0	1.74	
AC	5	2	2.5	1.09	
BC	12	4	3.0	1.30	
ABC	12	8	1.5	0.65	
Error	138	60	2.3		
Total	245	89			

- a) State which effects are fixed at which effects are random.
- b) State which effects are nested within others and which effects are crossed.

Site (A):  $\tau_i$  Fixed; Operator (B):  $\beta_{j(i)}$  Nested in A, Random; Thread Type (C):  $\gamma_k$  Crossed with B, Fixed; Replications: Random

Model:  $y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{kj(i)} + \varepsilon_{(ijk)l}, i = 1, 2, 3, j = 1, 2, 3, 4, 5, k = 1, 2, l = 1, 2, 3$

$\sum_{i=1}^a \tau_i = 0, \beta_{j(i)} \sim N(0, \sigma_{\beta}^2), (\gamma\beta)_{kj(i)} \sim N(0, \sigma_{\gamma\beta}^2), (\tau\gamma)_{ik} \sim N(0, \sigma_{\tau\gamma}^2), \varepsilon_{(ijk)l} \sim N(0, \sigma^2), \sum_{k=1}^c \gamma_k = 0$

- c) Create an abbreviated ANOVA table that has two columns: one column that lists the effects in the model (including the appropriate main effects, interactions, and nested effects), and a second column that gives the degrees of freedom for each item in the first column.

Source	SS	df	MS	F
A	17	2	8.5	8.5/5.8
B(A)	57	12	5.8	5.8/2.3
C	4	1	4.0	4.0/2.5
AC	5	2	2.5	2.5/2.0
BC(A)	24	12	2.0	2.0/2.3
Error	138	60	2.3	

term	i(f)	j(r)	k(f)	l(r)	df	EMS	F
$\tau_i f$	0	b	c	n	a-1	$\frac{bcn}{a-1} \sum_{i=1}^a \tau_i^2 + cn\sigma_{\beta}^2 + \sigma^2$	$\frac{MS_A}{MS_{B(A)}}$
$\beta_{j(i)} r$	1	1	c	n	a(b-1)	$cn\sigma_{\beta}^2 + \sigma^2$	$\frac{MS_{B(A)}}{MS_E}$
$(\gamma)_k f$	a	b	0	n	(c-1)	$\frac{abn}{c-1} \sum_{k=1}^c \gamma_k^2 + an\sigma_{\gamma}^2 + n\sigma_{\gamma\beta}^2 + \sigma^2$	$\frac{MS_C}{MS_{(AC)}}$
$(\tau\gamma)_{ik} f$	a	1	0	n	b(c-1)	$an\sigma_{\gamma}^2 + n\sigma_{\gamma\beta}^2 + \sigma^2$	$\frac{MS_{(AC)}}{MS_{BC(A)}}$

term	i(f)	j(r)	k(f)	l(r)	df	EMS	F
$(\gamma\beta)_{kj(i)}r$	1	1	0	n	ab(c-1)	$n\sigma_{\gamma\beta}^2 + \sigma^2$	$\frac{MS_{BC(A)}}{MS_E}$
$\varepsilon_{(ijk)l}$	1	1	1	1	abc(n-1)	$\sigma^2$	
Total					abcn-1		

### 2018S3

The multiple linear regression model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \varepsilon_i$  was fit to a data set of 75 observations. The regression SS's (SSR) were partitioned sequentially into the following:

$$SSR(X_1) = 108$$

$$SSR(X_2|X_1) = 163$$

$$SSR(X_3|X_1 X_2) = 29$$

$$SSR(X_4|X_1 X_2 X_3) = 41$$

$$SSR(X_5|X_1 X_2 X_3 X_4) = 26$$

The model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_3 x_{i3} + \beta_5 x_{i5} + \varepsilon_i$  was also fit to the same data and the following ANOVA was calculated:

Source	SS
Regression	214
Residual Error	489
Total	703

Answer the following from the above information:

- Calculate the F-statistic for testing the hypothesis ( $H_0$ ) that  $X_3$ ,  $X_4$ , and  $X_5$  have no significant effect on the response  $Y$ .
- Calculate  $R^2$  for the model  $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} \varepsilon_i$
- Calculate the  $R_{adj}^2$  for the model in part (b).
- Calculate the F-statistic for testing  $H_0 : \beta_2 = \beta_4 = 0$ .

### 2018S4

2019S1

Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$ ,  $i = 1, \dots, n$  with the additional restrictions that  $\beta_1 = 0$ ,  $\beta_0 = 2\beta_2$ . Find the least-squares estimators of  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$ .

### 2018F

Robert Fountain\*, Daniel Taylor-Rodriguez

### 2018F1

The weights ( $y_i$ , kilograms) and corresponding heights ( $x_i$ , centimeters) of 10 randomly sampled adolescents ( $i = 1, \dots, 10$ ) are recorded, and the following summary statistics are computed:

$$\sum_{i=1}^{10} (x_i - \bar{x})^2 = 472, \sum_{i=1}^{10} (y_i - \bar{y})^2 = 731, \sum_{i=1}^{10} (x_i - \bar{x})(y_i - \bar{y}) = 274$$

You will perform a simple linear regression of weight on height, under the usual assumption of independent, identically distributed, normal errors.

- Compute the least squares estimates for the intercept and slope parameters.
- Compute the usual unbiased estimate of the error variance.
- Compute unbiased estimates of the variances of the least squares estimates in part (a).
- Perform a two-sided test for whether or not height and weight are related (assuming the simple linear regression model holds). State the null and alternative hypotheses, and use  $\alpha = 0.05$ .
- Compute 95% simultaneous two-sided confidence intervals for the intercept and slope parameters, using the Bonferroni method.

### 2018F2

City planners are evaluating the effectiveness of a new "intelligent" traffic control system in reducing the amount of time motorists must spend on city streets. A total of 24 simulations are run: 4 simulations for each of the 6 combinations of control system (old or new) and traffic intensity (light, moderate, or heavy). All simulations use different random seeds, the combinations are run in a completely random order, and the median travel time (minutes) is recorded for each simulation. For each combination, the following table gives the average and sample standard deviation of the median travel times from the 4 simulations assigned that combination:

	Old light	System Moderate	Heavy	New light	System Moderate	Heavy
Sample Mean	13	14	15	5	8	17
Standard Deviation	1	2.5	3.5	2.5	2	3.5

- Write a (univariate) linear model equation of the usual full form for data from this experiment, with median travel time as the response. Explain each term and specify any conditions it satisfies. What crucial assumption are you making about the error variances?
- Produce an ANOVA table with all appropriate sources of variation, including the (corrected) total. Include sums of squares, degrees of freedom, and appropriate mean squares.

- Test whether your model in part (a) may be reduced to a model in which the effects of system and traffic intensity are purely additive. Remember to state the null and alternative hypotheses. Use  $\alpha = 0.05$ .
- Form a two-sided 95% confidence interval for the difference in median travel time between the new system and the old system under moderate traffic conditions.

### 2018F3

Consider the linear mixed model  $y_{ijk} = \mu + \alpha_i + \beta_{ij} + \varepsilon_{ijk}$ ,  $i = 1, \dots, a$ ,  $j = 1, \dots, b$ ,  $k = 1, \dots, n$ ,  $\sum_{i=1}^a \alpha_i = 0$ ,  $\beta_{ij} \sim N(0, \sigma_\beta^2)$ ,  $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$  with all  $\beta_{ij}$ 's and  $\varepsilon_{ij}$ 's independent, where  $a \geq 2$ ,  $b \geq 2$ , and  $n \geq 2$ . The parameters  $\mu$ ,  $\alpha_i$ ,  $\sigma_\beta^2$ , and  $\sigma_\varepsilon^2$  are assumed to be unknown. Adopt the following notation:  $\bar{y}_{ij.} = \frac{1}{n} \sum_{k=1}^n y_{ijk}$ ,  $\bar{y}_{i..} = \frac{1}{bn} \sum_{j=1}^b \sum_{k=1}^n y_{ijk}$ ,  $\bar{y}_{...} = \frac{1}{abn} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}$

- In terms of the parameters, find the correlations between (i)  $y_{111}$  and  $y_{112}$ , (ii)  $y_{111}$  and  $y_{121}$ , and (iii)  $y_{111}$  and  $y_{211}$ .
- For any given value of  $i$ , specify the joint distribution of  $\bar{y}_{i1.}, \dots, \bar{y}_{ib.}$ .
- In terms of the data, write a formula for the usual unbiased estimator of  $\alpha_1 - \alpha_2$ . What is the exact distribution of this estimator?
- Show that  $E[\sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..})^2] = a(b-1)(\sigma_\beta^2 + \frac{1}{n}\sigma_\varepsilon^2)$

Justify all important steps. (Hint: Your answer to part (b) might be useful.)

- In terms of the data, write a formula for the usual unbiased (ANOVA) estimate of  $\sigma_\beta^2$ . (Define all new notation, if you use any)

### 2018F4

Consider a randomized complete block design with 12 blocks and a single treatment factor having 3 levels. Let  $X_{ij}$  denote the response measured for an experimental unit in block  $j$  that receives treatment  $i$  for  $i = 1, 2, 3$  and  $j = 1, \dots, 12$ . Suppose there is also a covariate whose value  $X_{ij}$  is measured for each experimental unit.

The following four models are fit to the data (using least squares), with the resulting residual (error) sums of squares as specified:

Model 1:  $Y_{ij} = \mu + \gamma_j + \varepsilon_{ij}$   $SS(Res) = 660$

Model 2:  $Y_{ij} = \mu + \alpha_i + \gamma_j + \varepsilon_{ij}$   $SS(Res) = 550$

Model 3:  $Y_{ij} = \mu + \alpha_i + \gamma_j + \beta x_{ij} + \varepsilon_{ij}$   $SS(Res) = 300$

Model 4:  $Y_{ij} = \mu + \gamma_j + \beta x_{ij} + \varepsilon_{ij}$   $SS(Res) = 420$

The treatment effects are  $\alpha = (\alpha_1, \alpha_2, \alpha_3)'$  and the block effects are  $\alpha = (\gamma_1, \gamma_2, \dots, \gamma_{12})'$ . The corrected total sum of squares is 820.

- Find the sequential sums of squares for  $\gamma_j$ ,  $\alpha_i$ , and  $\beta$ , in that order.
- Form an ANOVA table for the randomized complete block design without the covariate  $X_{ij}$ , that is, based on Model 2. The table should include all appropriate sources of variation (including the corrected total), with degrees of freedom, sums of squares, and mean squares where appropriate. Then test whether or not there is any treatment effect based on this model. Use  $\alpha = 0.05$ .
- Test whether there is any treatment effect, after accounting for both blocking and the covariate. Use  $\alpha = 0.05$ .
- Suppose the (possibly incorrect) model  $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$  is fit to the data. Compute the residual sum of squares for this model.

### 2019S

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### 2019S1

#### 2018S4

Assume the model  $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$ ,  $i = 1, \dots, n$ , with the restriction that  $\beta_0 = 0$ . Find the least-squares estimators of the regression coefficients.

Let  $SSE = \sum_{i=1}^n (y_i - \beta_1 x_i - \beta_2 x_i^2)^2$

$$\frac{\partial SSE}{\partial \beta_1} = 2 \sum_{i=1}^n (y_i - \beta_1 x_i - \beta_2 x_i^2)(-x_i) \stackrel{set}{=} 0; \hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i - \hat{\beta}_2 \sum_{i=1}^n x_i^3}{\sum_{i=1}^n x_i^2}$$

$$\frac{\partial SSE}{\partial \beta_2} = 2 \sum_{i=1}^n (y_i - \beta_1 x_i - \beta_2 x_i^2)(-x_i^2) \stackrel{set}{=} 0; \sum_{i=1}^n x_i^2 y_i = \hat{\beta}_1 \sum_{i=1}^n x_i^3 + \hat{\beta}_2 \sum_{i=1}^n x_i^4 = \frac{\sum_{i=1}^n x_i y_i - \hat{\beta}_2 \sum_{i=1}^n x_i^3}{\sum_{i=1}^n x_i^2} \sum_{i=1}^n x_i^3 + \hat{\beta}_2 \sum_{i=1}^n x_i^4$$

$$\hat{\beta}_2 \left[ \sum_{i=1}^n x_i^4 - \frac{(\sum_{i=1}^n x_i^3)^2}{\sum_{i=1}^n x_i^2} \right] = \sum_{i=1}^n x_i^2 y_i - \frac{\sum_{i=1}^n x_i y_i \sum_{i=1}^n x_i^3}{\sum_{i=1}^n x_i^2}$$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^n x_i^2 y_i \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i y_i \sum_{i=1}^n x_i^3}{\sum_{i=1}^n x_i^4 \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i^3)^2}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} - \frac{\sum_{i=1}^n x_i^3 [\sum_{i=1}^n x_i^2 y_i \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i y_i \sum_{i=1}^n x_i^3]}{\sum_{i=1}^n x_i^2 [\sum_{i=1}^n x_i^4 \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i^3)^2]}$$

2019S2

[566-HW2-1] [566-HW5-2]

A manufacturer wishes to know if operator (A), material (B), and heat (C) affect the outcome of the product being produced. All of the machines being used operate at the same four heat settings. Five operators are randomly chosen. There are fifteen varieties of material that can be used, and three of these are assigned to each of the five operators. For each operator, that means that there are 12 combinations of heat and material that can be used. The operator produces two replications of each of these, for a total of  $5 \times 3 \times 4 \times 2 = 120$  observations. Incorrectly treating all main effects as fixed and all combinations of factors as crossed, the statistician produces the ANOVA table shown below.

Source	SS	df	MS	F	pval<0.05
A	34	4	8.5	3.70	*
B	12	2	6.0	2.61	
C	24	3	8.0	3.48	*
AB	32	8	4.0	1.74	
AC	30	12	2.5	1.09	
BC	18	6	3.0	1.30	
ABC	36	24	1.5	0.65	
Error	138	60	2.3		
Total	324	119			

Produce the corrected ANOVA table, including the expected mean squares and the correct F statistics. If an exact F test is not available, construct an approximate F statistic (for which you need not compute the degrees of freedom).

Operator (A): Random; Material (B): Nested in A, Random; Heat (C): Crossed with B, Fixed

Replications: Random

Model:  $y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{kj(i)} + \varepsilon_{(ijk)l}, i = 1, 2, 3, 4, j = 1, 2, 3, k = 1, 2, 3, 4, l = 1, 2$

$\tau_i \sim N(0, \sigma_\tau^2), \beta_{j(i)} \sim N(0, \sigma_\beta^2), (\gamma\beta)_{kj(i)} \sim N(0, \sigma_{\gamma\beta}^2), (\tau\gamma)_{ik} \sim N(0, \sigma_{\tau\gamma}^2), \varepsilon_{(ijk)l} \sim N(0, \sigma^2), \sum_{k=1}^c \gamma_k = 0$

Source	SS	df	MS	F
A	34	4	8.5	8.5/4.4
B+AB	44	10	4.4	4.4/2.3
C	24	3	8.0	8/2.5
AC	30	12	2.5	2.5/1.8
BC+ABC	54	30	1.8	1.8/2.3
Error	138	60	2.3	

term	i(r)	j(r)	k(f)	l(r)	df	EMS	F
$\tau_i \mathbf{r}$	1	b	c	n	a-1	$bcn\sigma_\gamma^2 + cn\sigma_\beta^2 + \sigma^2$	$\frac{MS_A}{MS_{B(A)}}$
$\beta_{j(i)} \mathbf{r}$	1	1	c	n	a(b-1)	$cn\sigma_\beta^2 + \sigma^2$	$\frac{MS_{B(A)}}{MS_E}$
$(\gamma)_k \mathbf{f}$	a	b	0	n	(c-1)	$\frac{abn}{c-1} \sum_{k=1}^c \gamma_k^2 + bn\sigma_\gamma^2 + n\sigma_{\gamma\beta}^2 + \sigma^2$	$\frac{MS_C}{MS_{(AC)}}$
$(\tau\gamma)_{ik} \mathbf{r}$	1	b	0	n	a(c-1)	$bn\sigma_\gamma^2 + n\sigma_{\gamma\beta}^2 + \sigma^2$	$\frac{MS_{(AC)}}{MS_{BC(A)}}$
$(\gamma\beta)_{kj(i)} \mathbf{r}$	1	1	0	n	ab(c-1)	$n\sigma_{\gamma\beta}^2 + \sigma^2$	$\frac{MS_{BC(A)}}{MS_E}$
$\varepsilon_{(ijk)l}$	1	1	1	1	abc(n-1)	$\sigma^2$	
Total					abcn-1		

2019S3

A company has developed two possible manufacturing processes. They will produce 5 items with each process, in a completely random order. Then they will measure the quality (Y) of each item on a 100-point scale. They suspect that the relative humidity during production might affect the outcome, so they also record it (X). They would like to develop a model that could be used to predict the outcome quality based on the process, with the relative humidity as a covariate. The following table shows an ordered pair for each item, consisting of the quality score and the humidity measurement.

Process A	(70,38)(70,55)(68,40)(72,45)(72,36)
Process B	(75,30)(74,42)(72,30)(71,30)(73,41)

- a) Write an appropriate model for the situation described above, allowing for different slopes and different intercepts for the two processes. Hint: there are 10 observations, and you will need to use at least one indicator variable.
- Process A:  $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ ; Process B:  $y_i = \gamma_0 + \gamma_1 x_i + \varepsilon_i$ ; Let  $w_i = \begin{cases} 0 & 1 \leq i \leq 6 \\ 1 & 6 \leq i \leq 10 \end{cases}$ , overall  $y_i = \beta_0 + \beta_1 x_i + w_i \gamma_0 + w_i \gamma_1 x_i + \varepsilon_i$
- b) Write the matrix form of the appropriate model. Show the contents and dimensions of all matrices.

$$\begin{bmatrix} 38 \\ 55 \\ 40 \\ 45 \\ 36 \\ 30 \\ 42 \\ 30 \\ 30 \\ 41 \end{bmatrix}_{10 \times 1} = \begin{bmatrix} 1 & 70 & 0 & 0 \\ 1 & 70 & 0 & 0 \\ 1 & 68 & 0 & 0 \\ 1 & 72 & 0 & 0 \\ 1 & 72 & 0 & 0 \\ 1 & 75 & 1 & 75 \\ 1 & 74 & 1 & 74 \\ 1 & 72 & 1 & 72 \\ 1 & 71 & 1 & 71 \\ 1 & 73 & 1 & 73 \end{bmatrix}_{10 \times 4} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \gamma_0 \\ \gamma_1 \end{bmatrix}_{4 \times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \\ \varepsilon_9 \\ \varepsilon_{10} \end{bmatrix}_{10 \times 1}$$

c) Suppose that you wish to test for equality of the two slopes. Write the matrix form of the reduced model. What will be the numerator and denominator degrees of freedom for the additional sum of squares F test?

$$H_0 : \beta_1 = \gamma_1, y_i = \beta_0 + \beta_1 x_i + w_i \gamma_0 + \varepsilon_i$$

$$\begin{bmatrix} 38 \\ 55 \\ 40 \\ 45 \\ 36 \\ 30 \\ 42 \\ 30 \\ 30 \\ 41 \end{bmatrix}_{10 \times 1} = \begin{bmatrix} 1 & 70 & 0 \\ 1 & 70 & 0 \\ 1 & 68 & 0 \\ 1 & 72 & 0 \\ 1 & 72 & 0 \\ 1 & 75 & 1 \\ 1 & 74 & 1 \\ 1 & 72 & 1 \\ 1 & 71 & 1 \\ 1 & 73 & 1 \end{bmatrix}_{10 \times 3} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \gamma_0 \end{bmatrix}_{3 \times 1} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \\ \varepsilon_9 \\ \varepsilon_{10} \end{bmatrix}_{10 \times 1}$$

$$\frac{10-3}{10-4}$$