## Q1

(a) $S \sim Bin(n, \theta)$

$E[\bar{X}] = E[\frac{S}{n}] = \theta < \infty$, $nVar[\bar{X}] = nVar[\frac{S}{n}] = \theta(1-\theta) < \infty$.
By the Central Limit Theorem, $\sqrt{n}\,(\bar{X} - \theta) \sim N(0, \theta(1-\theta))$
Let $h(\theta) = 2\sin^{-1}(\sqrt{\theta})$ is a continuous function at $E[\bar{X}] = \theta$ and differenciable $\forall \theta \in (0,1)$
$h'(\theta) = \frac{1}{\sqrt{\theta(1-\theta)}} \neq 0$ when $\theta \in (0,1)$; By Delta Method
$\sqrt{n}(h(\bar{X}) - h(\theta)) = 2\sqrt{n}(\sin^{-1}(\sqrt{\bar{X}}) - \sin^{-1}(\sqrt{\theta}))$
$nVar[\bar{X}][h'(\theta)]^2 = (\theta - \theta^2)\left[\frac{1}{\sqrt{\theta(1-\theta)}}\right]^2 = 1$
Therefore, the asymptotic distribution of $\sin^{-1}(\sqrt{\bar{X}})$ is $2\sqrt{n}\left(\sin^{-1}(\sqrt{\bar{X}}) - \sin^{-1}(\sqrt{\theta})\right) \sim N(0,1)$
$P(|2\sqrt{n}\left(\sin^{-1}(\sqrt{\bar{X}}) - \sin^{-1}(\sqrt{\theta})\right)| \leq Z_{\alpha/2}) = 1 - \alpha$
$P(\sin^{-1}(\sqrt{\bar{X}}) - \sin^{-1}(\sqrt{\theta}) \leq \frac{Z_{\alpha/2}}{2\sqrt{n}}$ or $\sin^{-1}(\sqrt{\theta}) - \sin^{-1}(\sqrt{\bar{X}}) \leq \frac{Z_{\alpha/2}}{2\sqrt{n}}) = 1 - \alpha$
$P(\sin^{-1}(\sqrt{\bar{X}}) - \frac{Z_{\alpha/2}}{2\sqrt{n}} \leq \sin^{-1}(\sqrt{\theta}) \leq \sin^{-1}(\sqrt{\bar{X}}) + \frac{Z_{\alpha/2}}{2\sqrt{n}}) = 1 - \alpha$
$I_n = \sin^{-1}(\sqrt{\bar{X}}) \pm \frac{Z_{\alpha/2}}{2\sqrt{n}}$ is an approximate $(1-\alpha)$ 100% confidence interval for $\sin^{-1}(\sqrt{\theta})$. □

(b) When $n = 100$ and $\bar{X} = .1$,

the approximate 95% confidence interval for $\theta$ is $(\sin(\sin^{-1}(\sqrt{\bar{X}}) \pm \frac{Z_{\alpha/2}}{2\sqrt{n}}))^2 = (0.0492, 0.1661)$. ■

(c) $P(|\frac{\sqrt{n}(\bar{X}-\theta)}{\sqrt{\theta(1-\theta)}}| \leq Z_{1-\alpha/2}) \approx 1 - \alpha$

$|\frac{\sqrt{n}(\bar{X}-\theta)}{\sqrt{\theta(1-\theta)}}| \leq Z_{1-\alpha/2} \implies n(\bar{X} - \theta)^2 \leq \theta(1-\theta)Z^2_{1-\alpha/2} \implies (n + Z^2_{1-\alpha/2})\theta^2 - (2n\bar{X} + Z^2_{1-\alpha/2})\theta + n\bar{X}^2 \leq 0$
For fixed $0 \leq \bar{X} \leq 1$,this a quadratic polynomial with two real roots.
$\frac{2n\bar{X}+Z^2_{\alpha/2} \pm \sqrt{(2n\bar{X}+Z^2_{\alpha/2})^2 - 4n\bar{X}^2(n+Z^2_{\alpha/2})}}{2(n+Z^2_{\alpha/2})} = \frac{2n\bar{X}+Z^2_{\alpha/2} \pm Z_{\alpha/2}\sqrt{Z^2_{\alpha/2}+4n\bar{X}-4n\bar{X}^2}}{2(n+Z^2_{\alpha/2})}$
It is confirmed by $0 \leq \lim_{n\to\infty} \frac{2\bar{X}+\frac{Z^2}{n} \pm Z\sqrt{\frac{Z^2}{n}+\frac{4\bar{X}}{n}-\frac{4\bar{X}^2}{n}}}{2+\frac{2Z^2}{n}} \leq 1$
$I_n = \frac{2n\bar{X}+Z^2_{1-\alpha/2} \pm Z_{1-\alpha/2}\sqrt{Z^2_{1-\alpha/2}+4n\bar{X}-4n\bar{X}^2}}{2(n+Z^2_{1-\alpha/2})}$ is an approximate $(1-\alpha)$ 100% confidence interval for $\theta$. □
Note: Normal distribution is symmetric. $|Z_{1-\alpha/2}| = |Z_{\alpha/2}|$ doesn't change the interval.

(d) The approximate 95% confidence interval for $\theta$ is $(0.0552, 0.1744)$. ■

## Q2

(a) $X \sim Pois(\theta)$

$p(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}\ x \in \mathbb{N}_0, \theta \in \mathbb{R}^+,$

$\frac{\partial}{\partial\theta} \log p(\theta) = \frac{x-\theta}{\theta}$ exists and is finite, Regularity Assumptions I holds.

Suppose $\psi(\theta) = e^{-\theta}$, $p_\theta(x=0) = \frac{\theta^0 e^{-\theta}}{0!} = e^{-\theta}$

$T(x) = \mathbf{1}_{\{x=0\}} \sim Bern(e^{-\theta})$

$E[T] = e^{-\theta} < \infty$. $T(x)$ is an unbiased estimator for $e^{-\theta}$.

The operations of integration and differentiation by $\theta$ can be interchanged in $\frac{\partial}{\partial\theta} E[T(x)]$, Regularity Assumptions II holds.

By Theorem 3.4.1. Information Inequality $E[T(x)] = e^{-\theta}$ is differentiable $\forall\theta$,

$$I(\theta) = E[(\frac{\partial}{\partial\theta} \log p(\theta))^2] = E[\frac{(x-\theta)^2}{\theta^2}] = \frac{Var[x]}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

$$CRLB = \frac{(\psi'(\theta))^2}{I(\theta)} = \frac{(\psi'(\theta))^2}{I(\theta)} = \frac{(-e^{-\theta})^2}{\frac{1}{\theta}} = \theta e^{-2\theta}$$

For $\theta > 0$, $e^\theta = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} = 1 + \theta + \frac{\theta^2}{2} + ...$ then, $e^\theta - 1 > \theta$

$Var_\theta[T] = e^{-\theta}(1 - e^{-\theta}) = e^{-2\theta}(e^\theta - 1) > e^{-2\theta} \cdot \theta$.

This shows that UMVUE $T(x)$ doesn't attain the C-R lower bound. ∎

(b)

As a canonical 1-parameter exponential family, $p(x;\theta) = \frac{\theta^x e^{-\theta}}{x!} = \exp[x \log\theta - \theta] \frac{\mathbf{1}_{x\in\mathbb{N}_0}}{x!}$

This shows that $T(x) = \mathbf{1}_{\{x=0\}}$ is a function (not linear) of complete sufficient statistic $X$.

By Lehman-Scheffe Theorem, the unbiased estimator $T$ is UMVUE for its expectation, $e^{-\theta}$. ∎

## Q3 $X \sim Unif(0,1)$

Let $Y = -\log X \sim Expo(1)$, $E[Y] = 1 < \infty$, $Var[Y] = 1 < \infty$.

By Law of Large Number, $\lim\limits_{n\to\infty} P(|\frac{1}{n}\sum_{i=1}^n Y_i - 1| > \varepsilon) = 0$, then $\bar{Y}_n \xrightarrow{\mathcal{P}} 1$

$\bar{Y}_n = -\frac{1}{n}\sum_{i=1}^n \log x_i = -log(Z_n)$, then $Z_N = (\prod_{i=1}^n x_i)^{\frac{1}{n}} = h(\bar{Y}_n) = e^{-\bar{Y}_n}$

By Theorem 4/29, if $\bar{Y}_n \xrightarrow{\mathcal{P}} 1$ is a constant, and $h : \mathbb{R}^k \to \mathbb{R}^p$ is continuous in $1 \subseteq \mathbb{R}^k$

Then $Z_N = h(\bar{Y}_n) = e^{-\bar{Y}_n} \xrightarrow{\mathcal{P}} e^{-1}$, that means $c = e^{-1}$ ∎

**Q4** $F_n(x) = \begin{cases} 1 & x \geq n \\ \frac{x+n}{2n} = \frac{x-(-n)}{n-(-n)} & -n \leq x < n \\ 0 & x < -n \end{cases}$

$X_k \sim Unif(-k,k), k = 1,2,...,n$ do not converge in distribution to any random variable when $n \to \infty$

The characteristic function of $X_n$ is $\phi_n(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x) = \int_{-n}^{n} e^{itx} \frac{x+n}{2n} dx = \begin{cases} \frac{\sin(nt)}{nt} & t \neq 0 \\ 1 & t = 0 \end{cases}$

$\lim_{n\to\infty} \phi_n(t) = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0 \end{cases}; \lim_{n\to\infty} F_n(x) = \frac{1}{2}, \forall x \in \mathbb{R}$ □

- Try Lindeberg Central Limit Theorem

$X_1$,.. are continuous and independent random variables with density $f_k$'s $f(x;k) = \frac{1}{2k}, -k \leq x < k,$

$E[X_k] = 0 = \mu_k, \qquad Var[X_k] = \frac{(2k)^2}{12} = \frac{1}{3}k^2 = \sigma_k^2 < \infty$

$S_n^2 = \sigma_1^2 + ... + \sigma_n^2 = \frac{1}{3}\sum_{k=1}^n k^2 \to \infty$ ,as $n \to \infty$.

Assume $|k| < m, \forall k = 1,2,...,n$ is bounded. For all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^n \int_{|x_k-\mu_k|>\varepsilon S_n} (x_k - \mu_k)^2 \cdot f_k(x) = \lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^n \int_{|X_k|>\varepsilon S_n} x_k^2 \cdot \frac{1}{2k}$$
$$\leq \sum_{k=1}^n \int_{|X|>\varepsilon S_n} \frac{m^2}{2k} = m^2 \sum_{k=1}^n P(|X_k| > \varepsilon S_n) \leq \lim_{n\to\infty} \frac{m^2}{S_n^2} \sum_{k=1}^n \frac{Var[X_k]}{\varepsilon^2 S_n^2} = \lim_{n\to\infty} \frac{m^2}{\varepsilon^2 S_n^2} = 0 \quad \text{by Chebyshev's inequality}$$

The assumption holds. Therefore, Lindeberg condition is satisfied. $\frac{\sum_{k=1}^n X_k}{S_n} \xrightarrow{\mathcal{D}} N(0,1)$ □

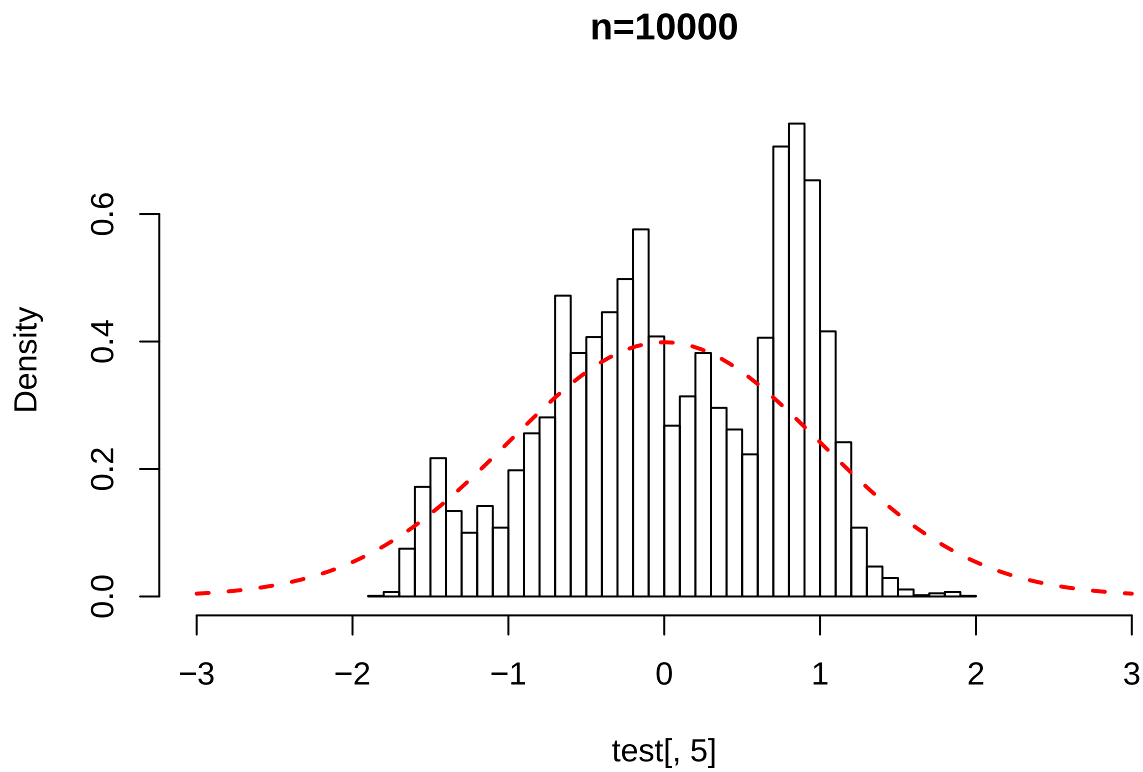
The limiting points include $x = 0$, the distribution is zero symmetric, and

$\sup \frac{\sum_{k=1}^n X_k}{S_n} = \frac{\sum_{k=1}^n k}{\sqrt{\frac{1}{3}\sum_{k=1}^n k^2}} = \frac{\frac{1}{2}n(n+1)}{\sqrt{\frac{1}{18}n(n+1)(2n+1)}} = 3\sqrt{\frac{n(n+1)}{2(2n+1)}} = \frac{3}{2}\mathcal{O}(n^{\frac{1}{2}})$

$\inf \frac{\sum_{k=1}^n X_k}{S_n} = -3\sqrt{\frac{n(n+1)}{2(2n+1)}} = -\frac{3}{2}\mathcal{O}(n^{\frac{1}{2}})$

```
Lindeberg.CLT4 <- function(n){
X <- S <- x <- sigma.sq <- lindberg<- NULL
out <- matrix(rep(0,5*n),nrow=n,ncol=5)
for (k in 1:n){
set.seed(k)
x[k] <- (runif(1,-k,k))
X[k] <- sum(x[1:k])
sigma.sq[k] <- k^2/3
S[k] <- sqrt(sum(sigma.sq[1:k]))
lindberg[k] <- X[k]/S[k]
out[k,] <- c(k,x[k],X[k],S[k],lindberg[k])
}
colnames(out) <- c("k","x_k","X","S","Lindeberg")
return(out)
}
```

```
##         k     x_k      X       S Lindeberg
##  [1,]  1 -0.4690 -0.469  0.5774   -0.8123
##  [2,]  2 -1.2605 -1.729  1.2910   -1.3396
##  [3,]  3 -1.9918 -3.721  2.1602   -1.7226
##  [4,]  4  0.6864 -3.035  3.1623   -0.9597
##  [5,]  5 -2.9979 -6.033  4.2817   -1.4089
##  [6,]  6  1.2752 -4.757  5.5076   -0.8638
##  [7,]  7  6.8447  2.087  6.8313    0.3055
##  [8,]  8 -0.5393  1.548  8.2462    0.1877
##  [9,]  9 -5.0112 -3.463  9.7468   -0.3553
## [10,] 10  0.1496 -3.314 11.3284   -0.2925
```




The histogram shows the simulated values converge weakly to a standard normal distribution.

## Q5.

$P(X_n = \pm 2^{n+1}) = \frac{1}{2^{n+3}}$; $P(X_n = 0) = 1 - \frac{1}{2^{n+2}}$

$E[X_n] = -2^{n+1} \cdot \frac{1}{2^{n+3}} + 2^{n+1} \cdot \frac{1}{2^{n+3}} + 0 \cdot (1 - \frac{1}{2^{n+2}}) = 0 = \mu_n$

$Var[X_n] = E[(X_n - 0)^2] = (-2^{n+1})^2 \cdot \frac{1}{2^{n+3}} + (2^{n+1})^2 \cdot \frac{1}{2^{n+3}} + 0^2 \cdot (1 - \frac{1}{2^{n+2}}) = 2^n = \sigma_n^2$

$S_n^2 = \sigma_1^2 + ... + \sigma_n^2 = \sum_{k=1}^n 2^k = 2^{n+1} - 2 \to \infty$,as $n \to \infty$

$S_n = \sqrt{2}, \sqrt{6}, \sqrt{14}, ..., \sqrt{2^{n+1} - 2}$

$X_1$,.. are discrete with jump points $X_{kl} = -2^{k+1}, 0, 2^{k+1}$ $(l = 1, 2, 3)$ and jumps $p_{kl} = \frac{1}{2^{k+3}}, 1 - \frac{1}{2^{k+2}}, \frac{1}{2^{k+3}}$.

$|X_{k1,k3}| = 4, 8, 16, ..., 2^{k+1}$.

Let $A = \{k = 1, .., n \mid |x_{kl}| > \varepsilon S_n\}$; $\qquad B = \{k = 1, .., n\}$

When $k = \frac{n}{2}$, $\frac{|X_{k1,k3}|}{S_n} = \frac{2^{k+1}}{\sqrt{2^{n+1}-2}} = \sqrt{2}\sqrt{\frac{2^n}{2^n - 1}} > \sqrt{2}$. That means,

given $\varepsilon = \sqrt{2}$, the set $A$ contains the terms of $k \geq \frac{n}{2}$, $\emptyset \subset A \subset B$, $\sum_{k=n/2}^n \sigma_k^2 < \sum_{k=1}^n \sigma_k^2$

$0 < \frac{\sum_{k=n/2}^n \sigma_k^2}{\sum_{k=1}^n \sigma_k^2} < 1$. This inequality holds for all n.

$$\lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^{n} \sum_{|x_{kl}|>\varepsilon S_n} x_{kl}^2 \cdot p_{kl} = \lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=\frac{n}{2}}^{n} \left[\sigma_k^2\right] = \lim_{n\to\infty} \frac{\sum_{k=n/2}^n \sigma_k^2}{S_n^2} > 0$$

If $\varepsilon$ is smaller, the set $A$ will be larger. Until $\varepsilon \leq \frac{2\sqrt{2}}{\sqrt{2^n - 1}}$, $A = B$,

$\lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^n \sum_{|x_{kl}|>\varepsilon S_n} x_{kl}^2 \cdot p_{kl} = 1.$

If $\varepsilon$ is larger, the set $A$ will be smaller. Until $\varepsilon \geq \sqrt{2}\sqrt{\frac{2^{2n}}{2^n - 1}}$, $A$ is an empty set,

$\lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^n \sum_{|x_{kl}|>\varepsilon S_n} x_{kl}^2 \cdot p_{kl} = 0$

Therefore, the assumption doesn't holds, Lindeberg condition is not satisfied.

By Feller Theorem: "The triangular array satisfies Lindebergs Condition,... implies that the maximum contribution to the variance from any of the individual terms in a row becomes negligible as you go down the rows."

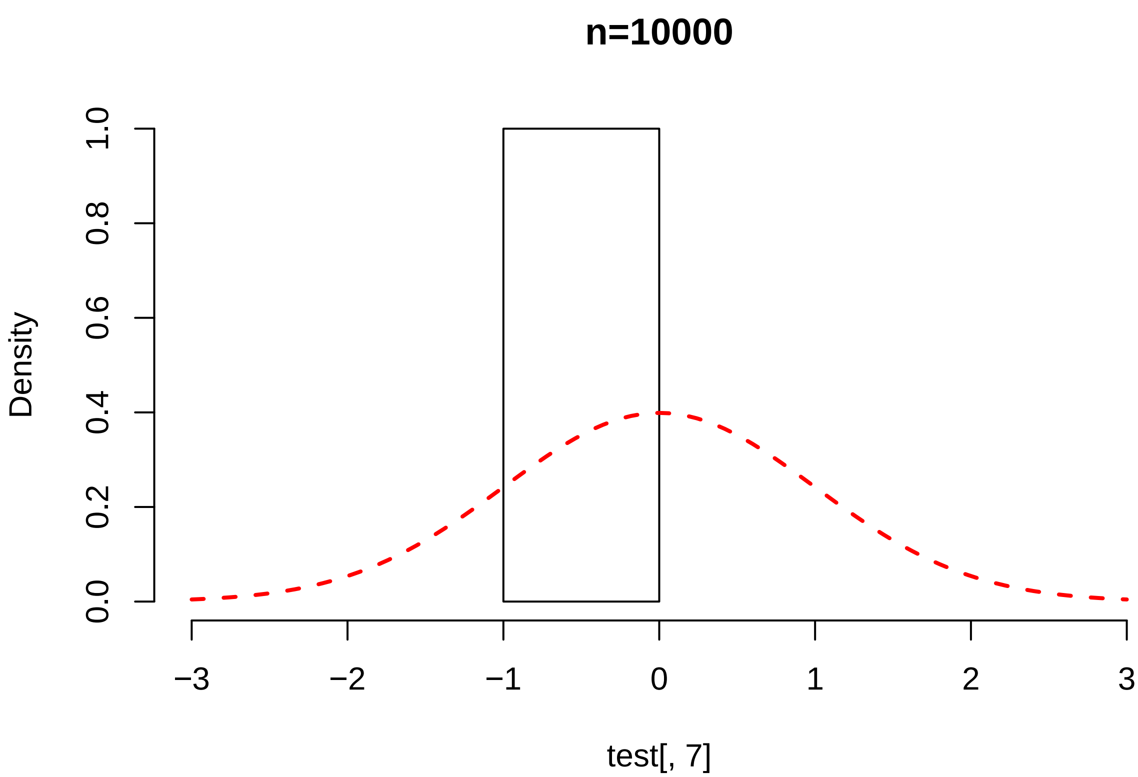
$$\lim_{n\to\infty} \frac{1}{S_n^2} \max_l E[x_{nl}^2] = \lim_{n\to\infty} \frac{1}{4} \cdot \frac{2^n}{2^n - 1} = \frac{1}{4} \neq 0$$

The condition of uniformly asymptotically negligible (UAN) doesn't hold in this case. ■

```
Lindeberg.CLT5 <- function(n){
x.upper<- p.upper <- X <- S <- x <- sigma.sq <- lindberg<- NULL
out <- matrix(rep(0,7*n),nrow=n,ncol=7)
# x <- matrix(rep(0,n*n),nrow=n,ncol=n)
for (k in 1:n){
set.seed(k)
x.upper[k] <- 2*2^k
p.upper[k] <- 1/2^k/8
x[k] <- sample(c(-2*2^k,0,2*2^k),1,prob=c(1/2^k/8,(1-1/2^k/4),1/2^k/8),replace=T)
```

```
sigma.sq[k] <- 2^k
X[k] <- sum(x[1:k])
S[k] <- sqrt(sum(sigma.sq[1:k]))
lindberg[k] <- X[k]/S[k]
out[k,] <- c(k,x[k],x.upper[k],p.upper[k],X[k],S[k],lindberg[k])
}
colnames(out) <- c("k","x_k","x.upper","p.upper","X","S","Lindberg")
return(out)
}
```

```
##        k x_k x.upper   p.upper X      S Lindberg
##  [1,]  1   0       4 0.0625000 0  1.414        0
##  [2,]  2   0       8 0.0312500 0  2.449        0
##  [3,]  3   0      16 0.0156250 0  3.742        0
##  [4,]  4   0      32 0.0078125 0  5.477        0
##  [5,]  5   0      64 0.0039062 0  7.874        0
##  [6,]  6   0     128 0.0019531 0 11.225        0
##  [7,]  7   0     256 0.0009766 0 15.937        0
##  [8,]  8   0     512 0.0004883 0 22.583        0
##  [9,]  9   0    1024 0.0002441 0 31.969        0
## [10,] 10   0    2048 0.0001221 0 45.233        0
```




The histogram shows the simulated values don't converge to a standard normal distribution. They concentrated around $x = 0$.