Q1. Let $X_1, X_2, ..., X_n$ be i.i.d. $U(0,\theta)$ r.v.'s.

(a) Show that $X_{(n)} = \max\{X_1, X_2, ..., X_n\} \xrightarrow{\mathcal{P}} \theta$.

$X_1, X_2, ..., X_n \overset{iid}{\sim} U(0,\theta)$, $0 \leq X_{(1)}, X_{(2)}, ..., X_{(n)} \leq \theta$

$f_X(x|\theta) = \frac{1}{\theta}$; $F_X(x|\theta) = \frac{x}{\theta}$

$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!(n-n)!} f_X(y)[F_X(x)]^{n-1}[1-F_X(x)]^{n-n} = \frac{n}{\theta^n}x^{n-1}$, $0 \leq x \leq \theta$

$F_{X_{(n)}}(x) = \int_0^x \frac{n}{\theta^n} t^{n-1} dt = \frac{1}{\theta^n} x^n$

For arbitrary $\varepsilon > 0$, $P(X_{(n)} - \theta > \varepsilon) = P(X_{(n)} > \theta + \varepsilon > \theta) = 0$

If $\varepsilon > \theta > 0$, $P(X_{(n)} - \theta < -\varepsilon) = P(X_{(n)} < \theta - \varepsilon < 0) = 0$

If $\theta \geq \varepsilon > 0$,

$\lim_{n\to\infty} P(X_{(n)} - \theta < -\varepsilon) = \lim_{n\to\infty} P(X_{(n)} < \theta - \varepsilon) = \lim_{n\to\infty} F_{X_{(n)}}(\theta - \varepsilon) = \lim_{n\to\infty} (\frac{\theta-\varepsilon}{\theta})^n = 0$

- Therefore, $\lim_{n\to\infty} P(|X_{(n)} - \theta| > \varepsilon) = 0$, $X_{(n)} \xrightarrow{\mathcal{P}} \theta$

(b) Show that $X_{(1)} = \min\{X_1, X_2, ..., X_n\} \xrightarrow{\mathcal{P}} 0$.

$f_X(x|\theta) = \frac{1}{\theta}$; $F_X(x|\theta) = \frac{x}{\theta}$

$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!(n-1)!} f_X(x)[F_X(x)]^{1-1}[1-F_X(x)]^{n-1} = \frac{n}{\theta}\left(1 - \frac{x}{\theta}\right)^{n-1}$, $0 \leq x \leq \theta$

$F_{X_{(1)}}(x) = \int_0^x \frac{n}{\theta}(1 - \frac{t}{\theta})^{n-1} dt = 1 - (1 - \frac{x}{\theta})^n$

For arbitrary $\varepsilon > 0$, $P(X_{(1)} - 0 < -\varepsilon) = P(X_{(1)} < -\varepsilon < 0) = 0$

If $\varepsilon > \theta > 0$, $P(X_{(1)} > \varepsilon) = P(X_{(1)} > \varepsilon > \theta) = 0$

If $\theta > \varepsilon > 0$,

$\lim_{n\to\infty} P(X_{(1)} > \varepsilon) = \lim_{n\to\infty}[1 - P(X_{(1)} < \varepsilon)] = \lim_{n\to\infty}[1 - F_{X_{(1)}}(\varepsilon)] = \lim_{n\to\infty}[1 - 1 + (1 - \frac{\varepsilon}{\theta})^n] = 0$

- Therefore, $\lim_{n\to\infty} P(|X_{(1)} - 0| > \varepsilon) = 0$, $X_{(1)} \xrightarrow{\mathcal{P}} 0$

(c) Show that $\frac{X_{(1)}+X_{(n)}}{2} \xrightarrow{\mathcal{P}} \frac{\theta}{2}$

$X_{(1)} \xrightarrow{\mathcal{P}} 0$ and 0 is constant. $X_{(n)} \xrightarrow{\mathcal{P}} \theta$.

By the Theorem1 in Note page39, $X_{(1)} + X_{(n)} \xrightarrow{\mathcal{P}} \theta + 0 = \theta$

By the Theorem2, $y = \frac{1}{2}x$ is a continuous function, $\frac{1}{2}(X_{(1)} + X_{(n)}) \xrightarrow{\mathcal{P}} \frac{1}{2}\theta$

Or by the Corollary, for $\frac{1}{2}$ is a constant, $\frac{1}{2}(X_{(1)} + X_{(n)}) \xrightarrow{\mathcal{P}} \frac{1}{2}\theta$

Q2. Let $S_n = \sum_i^n X_i$, where $X_1, X_2, ..., X_n$ are i.i.d. Bernoulli(p) r.v.'s.

That is, S_n has Binomial distribution with parameters n and p. Suppose $n \to \infty$, $p \to 0$, and $np = \lambda$. Then show that $S_n \xrightarrow{\mathcal{D}} S$, where S has a Poisson distribution with λ.

For $np = \lambda$, $S_n = \sum_i^n X_i \sim Bino(n, p) = Bino(n, \frac{\lambda}{n})$

$$
\begin{aligned}
p(x; n, \lambda) &= \binom{n}{x}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x} = \binom{n}{x}(\frac{\lambda}{n})^x(\frac{n-\lambda}{n})^{-x}(1-\frac{\lambda}{n})^n \\
&= \frac{n!}{x!(n-x)!}(\frac{\lambda^x}{n^x})(\frac{n^x}{(n-\lambda)^x})(1-\frac{\lambda}{n})^n = \frac{\lambda^x}{x!}(\frac{n!}{(n-x)!(n-\lambda)^x})(1-\frac{\lambda}{n})^n
\end{aligned}
$$

$\lim_{n\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}$

$\lim_{n\to\infty} \frac{n!}{(n-x)!(n-\lambda)^x} = \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{(n-\lambda)^x} = \lim_{n\to\infty} \frac{1(1-\frac{1}{n})\cdots(1-\frac{x-1}{n})}{(1-\frac{\lambda}{n})^x} = 1$

$$
\lim_{n\to\infty} p(x; n, \lambda) = \lim_{n\to\infty} \frac{\lambda^x}{x!}(\frac{n!}{(n-x)!(n-\lambda)^x})(1-\frac{\lambda}{n})^n = \frac{\lambda^x e^{-\lambda}}{x!}
$$

By Scheffe's Theorem, if S_n has a pdf $f_n(x) \to f(x), \forall x \in$ support,

$f(x)$ is a Poisson pdf of a r.v x, then $S_n \xrightarrow{\mathcal{D}} S$

Q3 Suppose that $X_1, X_2, ..., X_n$ be i.i.d. X, where X has the following pdf:

$f(x) = \begin{cases} \frac{\beta^{\alpha_0} x^{\alpha_0 - 1}}{\Gamma(\alpha_0)}) \exp[-\beta x], & x > 0 \\ 0 & o.w. \end{cases}$, $\alpha_0 > 0$ known, and $\beta > 0$ unknown.

(a) Determine UMVUE of β^{-1}.

For Gamma Distribution (α_0, β^{-1}), $\bar{X} \sim \Gamma(n\alpha_0, (n\beta)^{-1})$, $\alpha_0 > 0$ known

$E(\bar{X}) = E(X) = \alpha_0 \cdot \beta^{-1}$, then $(\alpha_0)^{-1}\bar{X}$ is an unbiased estimator of β^{-1}.

$$f_\beta(\mathbf{x}) = \exp[\underbrace{-\beta \sum_{i=1}^{n} x_i}_{\eta(\beta)T(x)} - \underbrace{-n\alpha_0 \log(\beta)}_{A(\beta)}] \underbrace{\prod_{i=1}^{n} \frac{x_i^{\alpha_0 - 1}}{\Gamma(\alpha_0)} \mathbf{1}_{\{x_i \in (0,\infty)\}}}_{h(x)}$$

When α_0 known, Gamma is a canonical 1-parameter exponential family. $\sum_{i=1}^n x_i$ is the natural sufficient statistic where $x \in \mathcal{X} \subset \mathbb{R}_+$.

$(\alpha_0)^{-1}\bar{X}$ is a function of n.s.s and it is a complete sufficient statistic for β^{-1}.

By **Lehman-Scheffe Theorem**, $(\alpha_0)^{-1}\bar{X}$ is the UMVUE of β^{-1}.

(b) Determine the information lower bound for the estimation of β^{-1} using unbiased estimators, and determine if the UMVUE obtained in (a) attains this.

Use **Theorem 1.6.2** (Bickel and Doksum)

$\eta = -\beta \implies \beta = -\eta \implies A(\eta) = -n\alpha_0 \log(\beta) = -n\alpha_0 \log(-\eta)$

η is an interior point of the natural parameter space ε.

$$E((\alpha_0)^{-1}\bar{X}) = (n\alpha_0)^{-1} E[\sum_{i=1}^{n} x_i] = (n\alpha_0)^{-1} A'(\eta) = (n\alpha_0)^{-1} \frac{n\alpha_0}{-\eta} = \beta^{-1}$$

$$Var((\alpha_0)^{-1}\bar{X}) = (n\alpha_0)^{-2} Var(\sum_{i=1}^{n} x_i) = (n\alpha_0)^{-2} A''(\eta) = (n\alpha_0)^{-2} \frac{n\alpha_0}{\eta^2} = \frac{1}{n\alpha_0 \beta^2}$$

- Therefore, The UMVUE of β^{-1}, $(\alpha_0)^{-1}\bar{X}$ has variance $\frac{1}{n\alpha_0\beta^2}$

Regularity Assumptions on the family $\{P : \beta \in \Theta\}$:

(I) The set $\mathcal{X} = \{x : p(x, \alpha_0, \beta) > 0\}$ does not depend on α_0, β, $\forall x \in \mathcal{X}, \alpha_0 > 0, \beta > 0$, score function $\frac{\partial}{\partial \beta} \log p(X, \beta) = -x + \frac{\alpha_0}{\beta}$ exists and is finite.

(II) If T is any statistic such that $E_\beta(|T|) = \beta^{-1} < \infty, \alpha_0 > 0, \beta > 0$, then the operations of integration and differentiation by β can be interchanged in $\frac{\partial}{\partial \beta} E_\beta(T(X))) = \frac{\partial}{\partial \beta} \int T(x) p(x, \beta) dx = \int T(x) \frac{\partial}{\partial \beta} p(x, \beta) dx$

Theorem 3.4.1. Information Inequality $\{P_\beta : \beta \in \Theta\}$ has density $p(x, \beta)$,$x \in \mathcal{X} \subseteq \mathbb{R}^q$, $E_\beta(T(X)) = \beta^{-1}$ is differentiable$\forall \beta$

I and II hold $0 < I(\beta) < \infty \; \forall \beta$,

$$I_1(\beta) = -E\left(\frac{\partial^2}{\partial\beta^2}\log p(X,\beta)\right) = -E\left(\frac{\partial^2}{\partial\beta^2}[-\beta x + \alpha_0\log(\beta)]\right) = -E\left(\frac{\partial}{\partial\beta}[-x + \frac{\alpha_0}{\beta}]\right) = \frac{\alpha_0}{\beta^2}$$

$$Var_\theta(T(X)) \geq \frac{(\frac{\partial}{\partial\beta}[E(\alpha_0)^{-1}\bar{X}])^2}{nI_1(\beta)} = \frac{(\frac{\partial}{\partial\beta}(\frac{1}{\beta}))^2}{n\frac{\alpha_0}{\beta^2}} = \frac{1}{n\alpha_0\beta^2}$$

- Therefore, the UMVUE $(\alpha_0)^{-1}\bar{X}$ attains the information lower bound for the estimation of β^{-1}.

Note: **Theorem 3.4.2.** also works. $\{P_\theta : \theta \in \Theta\}$ satisfies assumptions (I) and (II). There exists u.b. est. T^* of $\psi(\theta)$, which achieves the information lower bound $\forall\theta$.

Then $\{P\theta\}$ is a one-parameter exponential family of the form $p(x,\theta) = h(x)\exp\{\eta(\theta)T(x) - B(\theta)\}$

Conversely, if $\{P_\theta\}$ a one-para exp family of the form (*) with n.s.s. $T(X)$.

$\eta(\theta)$ has a continuous nonvanishing derivative on θ,

Then $T(X)$ achieves the information lower bound and is a UMVUE of $E_\theta(T(X))$

Q4. Let $X_1, X_2, ..., X_n$ be a sample from $U(\theta_1, \theta_2)$ where θ_1 and θ_2 are unknown. Show that $T(\underline{X}) = (\min(X_1, ..., X_n), \max(X_1, ..., X_n))$ is complete for (θ_1, θ_2).

$f_X(x|\theta_1,\theta_2) = \frac{1}{\theta_2-\theta_1}$; $F_X(x|\theta_1,\theta_2) = \frac{x-\theta_1}{\theta_2-\theta_1}$

Let $T(\underline{X}) = (\min(X_1, ..., X_n), \max(X_1, ..., X_n)) = (X_{(1)}, X_{(n)}) = (U, V)$. $\theta_1 \leq u < v \leq \theta_2$

$$f_{U,V}(u,v) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!}f_X(u)f_X(v)[F_X(u)]^{1-1}[F_X(v) - F_X(u)]^{n-1-1}[1-F_X(v)]^{n-n}$$

$$= n(n-1)f_X(u)f_X(v)[F_X(v) - F_X(u)]^{n-2} = \frac{n(n-1)}{(\theta_2-\theta_1)^2}\left[\frac{v-\theta_1}{\theta_2-\theta_1} - \frac{u-\theta_1}{\theta_2-\theta_1}\right]^{n-2} = \frac{n(n-1)}{(\theta_2-\theta_1)^n}[v-u]^{n-2}$$

Suppose a Riemann-integrable $g(u,v)$ is a function satisfying $E[g(u,v)] = 0\ \forall(\theta_1,\theta_2)$.

$$E[g(u,v)] = \int_{\theta_1}^{\theta_2}\int_{\theta_1}^{v} g(u,v)f_{U,V}(u,v)dudv = \frac{n(n-1)}{(\theta_2-\theta_1)^n}\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{v} g(u,v)\,[v-u]^{n-2}\,dudv = 0$$

Let

$$\begin{aligned} h(\theta_1, v) &= \int_{\theta_1}^{v} g(u,v)\,[v-u]^{n-2}\,du \\ &= -g(\theta_1,v)\frac{(v-\theta_1)^{n-1}}{n-1} - \int_{\theta_1}^{v}\left[\frac{d}{du}g(u,v)\right]\frac{(v-u)^{n-1}}{n-1}du \end{aligned}$$

For $\frac{n(n-1)}{(\theta_2-\theta_1)^n} \neq 0$, then

$$\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{v} g(u,v)\,[v-u]^{n-2}\,dudv = \int_{\theta_1}^{\theta_2} h(\theta_1,v)dv = 0$$

Moreover,

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial\theta_1\partial\theta_2}\int_{\theta_1}^{\theta_2} h(\theta_1,v)dv \\
&= \frac{\partial^2}{\partial\theta_1\partial\theta_2}\left[\int_0^{\theta_2} h(\theta_1,v)dv - \int_0^{\theta_1} h(\theta_1,v)dv\right] \\
&= \frac{\partial}{\partial\theta_2}\int_0^{\theta_2}\left[\frac{\partial}{\partial\theta_1}h(\theta_1,v)\right]dv - \frac{\partial}{\partial\theta_1}\int_0^{\theta_1}\left[\frac{\partial}{\partial\theta_2}h(\theta_1,v)\right]dv \\
&= \frac{\partial}{\partial\theta_1}h(\theta_1,\theta_2) + \int_0^{\theta_2}\left[\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1}h(\theta_1,v)\right]dv \\
&= \frac{\partial}{\partial\theta_1}\left[-g(\theta_1,\theta_2)\frac{(\theta_2-\theta_1)^{n-1}}{n-1} - \int_{\theta_1}^{\theta_2}\left(\frac{\partial}{\partial u}g(u,\theta_2)\right)\frac{(\theta_2-u)^{n-1}}{n-1}du\right] \\
&= -\frac{\partial}{\partial\theta_1}\left[g(\theta_1,\theta_2)\frac{(\theta_2-\theta_1)^{n-1}}{n-1}\right] + \frac{\partial}{\partial\theta_1}\int_{\theta_2}^{\theta_1}\left(\frac{\partial}{\partial u}g(u,\theta_2)\right)\frac{(\theta_2-u)^{n-1}}{n-1}du \\
&= -\left[\frac{\partial}{\partial\theta_1}g(\theta_1,\theta_2)\right]\frac{(\theta_2-\theta_1)^{n-1}}{n-1} + g(\theta_1,\theta_2)(\theta_2-\theta_1)^{n-2} \\
&\quad + \left[\frac{\partial}{\partial\theta_1}g(\theta_1,\theta_2)\right]\frac{(\theta_2-\theta_1)^{n-1}}{n-1} + \int_{\theta_2}^{\theta_1}\frac{\partial}{\partial\theta_1}\left[\left(\frac{\partial}{\partial u}g(u,\theta_2)\right)\frac{(\theta_2-u)^{n-1}}{n-1}\right]du \\
&= g(\theta_1,\theta_2)(\theta_2-\theta_1)^{n-2}
\end{aligned}
$$

For $\theta_1 \neq \theta_2$, the solution to $E[g(T(X))] = 0, \theta \in \Theta$ is $g(u,v) = 0$ almost sure.

That is, if $E[g(u,v)] = 0, \forall(\theta_1,\theta_2)$ implies $P(g(u,v) = 0) = 1, \forall(\theta_1,\theta_2)$.

Thus, $T(\underline{X}) = (\min(X_1,...,X_n), \max(X_1,...,X_n))$ is complete for (θ_1,θ_2).