1 Classifier

1.1 proof

$g(x) = \langle C_+ - C_-, X - C\rangle = \langle C_+, X\rangle - \langle C_-, X\rangle - \langle C_+, C\rangle + \langle C_-, C\rangle$

$\langle C_+, X\rangle = \frac{1}{n_+}\sum_{i\in I_+}^{n}\langle x_i, x\rangle;$

$\langle C_-, X\rangle = \frac{1}{n_-}\sum_{i\in I_-}^{n}\langle x_i, x\rangle;$

$\langle C_+, C\rangle = \langle C_+, \frac{1}{2}C_+\rangle + \langle C_+, \frac{1}{2}C_-\rangle = \frac{1}{2n_+^2}\sum_{(i,j)\in I_+}\langle x_i, x_j\rangle + \frac{1}{2}\langle C_+, C_-\rangle$

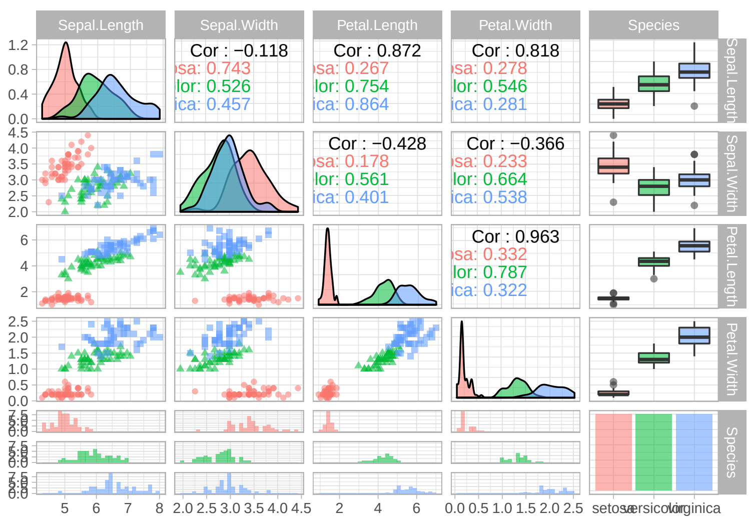
$\langle C_-, C\rangle = \langle C_-, \frac{1}{2}C_+\rangle + \langle C_-, \frac{1}{2}C_-\rangle = \frac{1}{2}\langle C_+, C_-\rangle + \frac{1}{2n_-^2}\sum_{(i,j)\in I_-}\langle x_i, x_j\rangle$

$$\begin{aligned} g(x) &= \frac{1}{n_+}\sum_{i\in I_+}^{n}\langle x_i, x\rangle - \frac{1}{n_-}\sum_{i\in I_-}^{n}\langle x_i, x\rangle - \frac{1}{2n_+^2}\sum_{(i,j)\in I_+}\langle x_i, x_j\rangle - \frac{1}{2}\langle C_+, C_-\rangle + \frac{1}{2}\langle C_+, C_-\rangle + \frac{1}{2n_-^2}\sum_{(i,j)\in I_-}\langle x_i, x_j\rangle \\ &= \sum_{i=1}^{n}\alpha_i\langle x_i, x\rangle + b \end{aligned}$$

where $\alpha_i = \begin{cases} \frac{1}{n_+} & y_i = +1 \\ -\frac{1}{n_-} & y_i = -1 \end{cases}$; $b = \frac{1}{2n_-^2}\sum_{(i,j)\in I_-}\langle x_i, x_j\rangle - \frac{1}{2n_+^2}\sum_{(i,j)\in I_+}\langle x_i, x_j\rangle$

1.2 iris data clasification

- Import the iris data

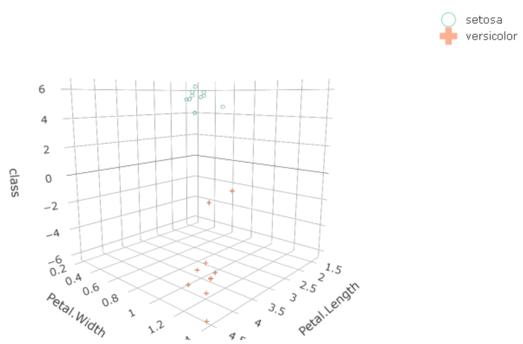


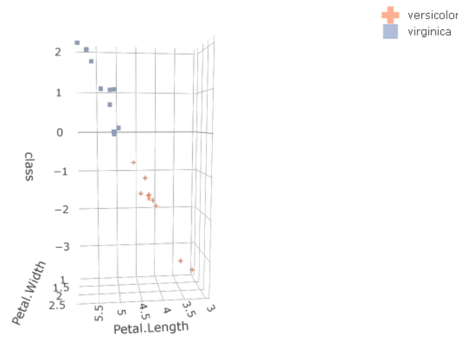

```r
# Define the train and test sets
iris$class <- NA
iris_setosa <- iris[iris$Species=="setosa",]
iris_versicolor <- iris[iris$Species=="versicolor",]
iris_virginica <- iris[iris$Species=="virginica",]
iris_train_se<- iris_setosa[1:40,]
iris_train_ve<- iris_versicolor[1:40,]
iris_train_vi<- iris_virginica[1:40,]
iris_test_se<- iris_setosa[41:50,]
iris_test_ve<- iris_versicolor[41:50,]
iris_test_vi<- iris_virginica[41:50,]
iris_train_se.ve<- rbind(iris_train_se,iris_train_ve)
iris_train_ve.vi<- rbind(iris_train_ve,iris_train_vi)
iris_test_se.ve<- rbind(iris_test_se,iris_test_ve)
iris_test_ve.vi<- rbind(iris_test_ve,iris_test_vi)
```

```r
# Define the kernel function and Computing the classifier
k = function(x,y)  return(sum(x*y))
# setosa v.s.versicolor
k.pp=outer(1:40,1:40,Vectorize(function(i,j) k(iris_train_se[i,1:4],iris_train_se[j,1:4])))
k.mm=outer(1:40,1:40,Vectorize(function(i,j) k(iris_train_ve[i,1:4],iris_train_ve[j,1:4])))
b=(sum(k.mm)/(40^2)-sum(k.pp)/(40^2))/2
alpha=ifelse(iris_train_se.ve$Species=="setosa",1/40,-1/40)
k.x=outer(1:80,1:20,Vectorize(function(i,j) k(iris_train_se.ve[i,1:4],iris_test_se.ve[j,1:4])))
iris_test_se.ve[,6]=(t(k.x)%*%alpha+b)
```

```
# virginica v.s.versicolor
k.pp=outer(1:40,1:40,Vectorize(function(i,j) k(iris_train_vi[i,1:4],iris_train_vi[j,1:4])))
k.mm=outer(1:40,1:40,Vectorize(function(i,j) k(iris_train_ve[i,1:4],iris_train_ve[j,1:4])))
b=(sum(k.mm)/(40^2)-sum(k.pp)/(40^2))/2
alpha=ifelse(iris_train_ve.vi$Species=="virginica",1/40,-1/40)
k.x=outer(1:80,1:20,Vectorize(function(i,j) k(iris_train_ve.vi[i,1:4],iris_test_ve.vi[j,1:4])))
iris_test_ve.vi[,6]=(t(k.x)%*%alpha+b)
# Evaluate the classifier
iris_test_se.ve$evaluate=ifelse(iris_test_se.ve$class>0,"setosa","versicolor")
iris_test_se.ve$evaluate=ifelse(iris_test_se.ve$Species==iris_test_se.ve$evaluate,"Rigt","Wrong")
error_rate1 <- length(which(iris_test_se.ve$evaluate=="Wrong"))/20
iris_test_ve.vi$evaluate=ifelse(iris_test_ve.vi$class>0,"virginica","versicolor")
iris_test_ve.vi$evaluate=ifelse(iris_test_ve.vi$Species==iris_test_ve.vi$evaluate,"Rigt","Wrong")
error_rate2 <- length(which(iris_test_ve.vi$evaluate=="Wrong"))/20
```

Error rate = 0% and 5% in two tests respectively.





The left figure of setosa v.s. versicolor shows that the two specises are separated by zero plane, while virginica v.s. versicolor shows that a few points are close to zero. The tables also show a misclassified point in virginica v.s. versicolor.

Table 1: Confusion matrix

	Actural Species					
	test 1	Setosa	Versicolor	test 2	Virginica	Versicolor
Test Species	Setosa	10	0	Virginica	9	0
	Versicolor	0	10	Versicolor	1	10

Table 2: setosa v.s.versicolor/ /virginica v.s.versicolor

Species	class	evaluate	Species1	class1	evaluate1
setosa	5.856	Rigt	versicolor	-1.575	Rigt
setosa	5.536	Rigt	versicolor	-0.7495	Rigt
setosa	6.35	Rigt	versicolor	-1.907	Rigt
setosa	4.665	Rigt	versicolor	-3.482	Rigt
setosa	4.135	Rigt	versicolor	-1.69	Rigt
setosa	5.429	Rigt	versicolor	-1.638	Rigt
setosa	5.215	Rigt	versicolor	-1.592	Rigt
setosa	5.869	Rigt	versicolor	-1.157	Rigt
setosa	5.239	Rigt	versicolor	-3.708	Rigt
setosa	5.548	Rigt	versicolor	-1.739	Rigt
versicolor	-5.097	Rigt	virginica	1.566	Rigt
versicolor	-6.206	Rigt	virginica	0.9795	Rigt
versicolor	-4.246	Rigt	virginica	-0.02223	Wrong
versicolor	-1.446	Rigt	virginica	1.968	Rigt
versicolor	-4.667	Rigt	virginica	1.795	Rigt
versicolor	-4.451	Rigt	virginica	0.968	Rigt
versicolor	-4.63	Rigt	virginica	0.119	Rigt
versicolor	-5.402	Rigt	virginica	0.6535	Rigt
versicolor	-0.6631	Rigt	virginica	0.9918	Rigt
versicolor	-4.411	Rigt	virginica	0.02902	Rigt

2 Perceptron

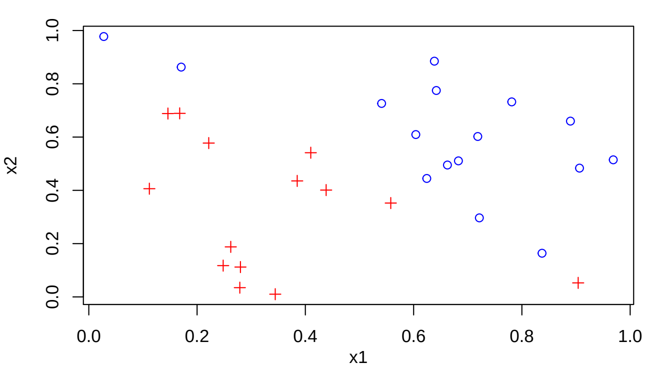
2.1 pseduo code

Init:
– define the classifier $f(x) = w^T x$
– define perceptron algorithm
– set the sample size =n
– set initial weight with $w \leftarrow y_1 x_1$
– set initial misclassified argument with TRUE
While misclassfied=TURE
– change argument with FALSE
– For $i = 2 \ldots n$ do
— If $y_i w^T x_i < 0$ then $w \leftarrow w + y_i x_i$
— change argument with TRUE
— EndIf
– EndFor
EndWhile

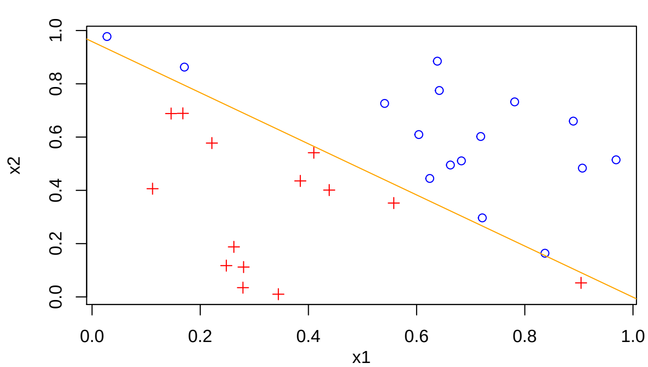
2.2 Using 3 Kernels

```
n <- 30; dim <- 2; threshold <-1            ## generate 2d data
x <- runif(n * dim)
x <- matrix(x, ncol = dim)
y <- ifelse(apply(x, 1, sum) < threshold, -1, 1)
data <- cbind(x, y, alpha = rep(1, n))
k<- function(x,w){return(sum(x*w))}         ## define the kernel function
k_perceptron <- function(data,dim) {        ## difine perceptron algorithm
n <- nrow(data)                             ## get the sample size
y <- data[ ,dim+1]                          ## select y
w <- c(data[1,1:dim]*data[1,dim+2],0)       ## initialize weight with y_1*x_1
misclassfied <- TRUE                        ## initialize the argument
  while (misclassfied) {                    ## start to loop
    misclassfied <- FALSE
    for (i in 2:n) {                        ## loop times = sample size-1
      if (y[i]*k(data[i,-dim-1],w) < 0) { ## if expectation does not match y
          w <- w + y[i] * data[i,-dim-1]    ## update the weights
          misclassfied <- TRUE              ## reset argument and check next one
  }}}
  return(w)
}
k <- function(x,y) {return(sum(x*y))} ## Use k1
(w1 <- k_perceptron(data,2))
##
##  4.004353  4.173336 -4.000000
k <- function(x,y) {return(sum(x*y)+1)} ## Use k2
(w2 <- k_perceptron(data,2))
##
##  3.957389  4.230957 -5.000000
k <- function(x,y) {return(sign(x%*%y))} ## Use k3
(w3 <- k_perceptron(data,2))
##
##  4.004353  4.173336 -4.000000
```

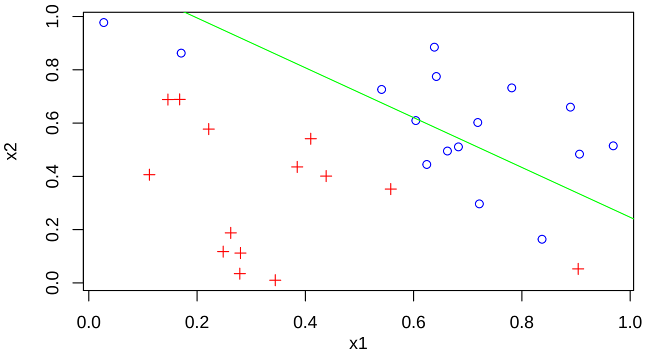
The output shows that the perceptrons with $k = x^T y$ and $k = \text{sign}(x^T y)$ can classify the data. The $k = x^T y + 1$ is biased.



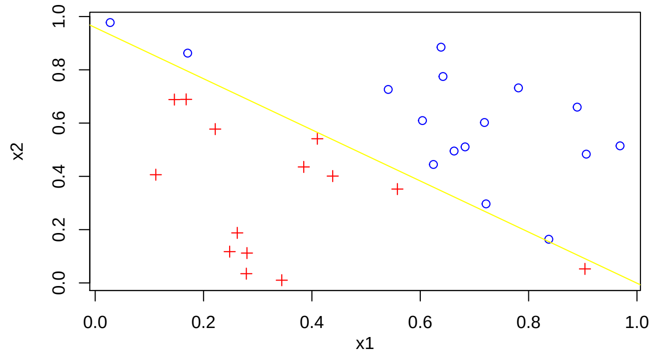
Before perceptron



Use k1 perceptron



Use k2 perceptron



Use k3 perceptron

3 Kernels over $\mathcal{X} = \mathbb{R}^2$

$x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$,

3.1 Let $\phi(x) = (x_1^2, \sqrt{2}x_1x_2, x_2^2)$,verify that $\phi(x)^T\phi(y) = (x^Ty)^2$

$$(x^Ty)^2 = (\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \begin{bmatrix} y_1 & y_2 \end{bmatrix}_{1\times 2})^2 = (x_1y_1 + x_2y_2)^2 = x_1^2y_1^2 + 2x_1x_2y_1y_2 + x_2^2y_2^2$$

$$= \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{bmatrix}_{3\times 1} \begin{bmatrix} y_1^2 & \sqrt{2}y_1y_2 & y_2^2 \end{bmatrix}_{1\times 3} = \phi(x)^T\phi(y) \quad \blacksquare$$

3.2 Find a function $\phi(x)$:$\mathbb{R}^2 \mapsto \mathbb{R}^6$ such that for any (x, y), $\phi(x)^T\phi(y) = (x^Ty + 1)^2$

$$(x^Ty + 1)^2 = (x_1y_1 + x_2y_2 + 1)^2 = x_1^2y_1^2 + 2x_1y_1 + 2x_1x_2y_1y_2 + 2x_2y_2 + x_2^2y_2^2 + 1$$

$$= \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1 \\ \sqrt{2}x_1x_2 \\ \sqrt{2}x_2 \\ x_2^2 \\ 1 \end{bmatrix}_{6\times 1} \begin{bmatrix} y_1^2 & \sqrt{2}y_1 & \sqrt{2}y_1y_2 & \sqrt{2}y_2 & y_2^2 & 1 \end{bmatrix}_{1\times 6}$$

Thus,

$$\phi(x) = (x_1^2, \sqrt{2}x_1, \sqrt{2}x_1x_2, \sqrt{2}x_2, x_2^2, 1) \quad \blacksquare$$

3.3 Find a function $\phi(x)$:$\mathbb{R}^2 \mapsto \mathbb{R}^9$ such that for any (x, y), $\phi(x)^T\phi(y) = (x^Ty + 1)^2$

$$(x^Ty + 1)^2 = x_1^2y_1^2 + 2x_1y_1 + \frac{1}{2}x_1x_2y_1y_2 + \frac{1}{2}x_2x_1y_2y_1 + \frac{1}{2}x_1^{0.5}x_2x_1^{0.5}y_1^{0.5}y_2y_1^{0.5} + \frac{1}{2}x_2^{0.5}x_1x_2^{0.5}y_2^{0.5}y_1y_2^{0.5} + 2x_2y_2 + x_2^2y_2^2 + 1$$

$$= \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1 \\ \frac{1}{\sqrt{2}}x_1x_2 \\ \frac{1}{\sqrt{2}}x_2x_1 \\ \frac{1}{\sqrt{2}}x_1^{0.5}x_2x_1^{0.5} \\ \frac{1}{\sqrt{2}}x_2^{0.5}x_1x_2^{0.5} \\ \sqrt{2}x_2 \\ x_2^2 \\ 1 \end{bmatrix}_{9\times 1} \begin{bmatrix} y_1^2 & \sqrt{2}y_1 & \frac{1}{\sqrt{2}}y_1y_2 & \frac{1}{\sqrt{2}}y_2y_1 & \frac{1}{\sqrt{2}}y_1^{0.5}y_2y_1^{0.5} & \frac{1}{\sqrt{2}}y_2^{0.5}y_1y_2^{0.5} & \sqrt{2}y_2 & y_2^2 & 1 \end{bmatrix}_{1\times 9}$$

Thus,

$$\phi(x) = (x_1^2, \sqrt{2}x_1, \frac{1}{\sqrt{2}}x_1x_2, \frac{1}{\sqrt{2}}x_2x_1, \frac{1}{\sqrt{2}}x_1^{0.5}x_2x_1^{0.5}, \frac{1}{\sqrt{2}}x_2^{0.5}x_1x_2^{0.5}, \sqrt{2}x_2, x_2^2, 1) \quad \blacksquare$$

3.4 Verify that $K(x,y) = (1 + x^T y)^d$ for $d = 1, 2\ldots$ is a positive definite kernel

We know that $x^T y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \begin{bmatrix} y_1 & y_2 \end{bmatrix}_{1\times 2}) = x_1y_1 + x_2y_2 = y^T x$, then

$$k(x,y) = (1 + x^T y)^d = (1 + y^T x)^d = k(y,x)$$

Thus, $k(x,y)$ is symmetric.

$$\exists\phi(x) = \left(\sqrt{\binom{k}{0}}x_1^k x_2^0, \sqrt{\binom{k}{1}}x_1^{k-1}x_2^1, \cdots, \sqrt{\binom{k}{k-1}}x_1^1 x_2^{k-1}, \sqrt{\binom{k}{k}}x_1^0 x_2^k\right)$$

Make

$$k(\phi(x),\phi(y)) = \phi(x)^T\phi(y) = \sum_{l=0}^{k}\binom{k}{l}(x_1y_1)^{k-l}(x_2y_2)^l = (x_1y_1 + x_2y_2)^k = (x^Ty)^k$$

$k(\phi(x),\phi(y))$ is a p.d. kernel and $\|\sum_{i=1}^2 \alpha_i\phi_i(x)\|^2 \geq 0$. Therefore,

$$k(x,y) = (1 + x^Ty)^d = \sum_{k=0}^{d}\binom{d}{k}(x^Ty)^k 1^{d-k} = \sum_{k=0}^{d}\binom{d}{k}\phi(x)^T\phi(y)$$

For $d, k \in \mathbb{N}^+$,

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i\alpha_j\langle x_i, x_j\rangle_{R^d} = \sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i\alpha_j\left[\sum_{k=0}^{d}\binom{d}{k}\phi_i(x)^T\phi_j(x)\right] = \sum_{k=0}^{d}\binom{d}{k}\left[\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i\alpha_j\langle\phi_i(x)^T\phi_j(x)\rangle_R\right]$$

$$= \sum_{k=0}^{d}\binom{d}{k}\|\sum_{i=1}^{2}\alpha_i\phi_i(x)\|^2 \geq 0$$

Therefore, for $x_1, x_2,\ x_i \in \mathbb{R};\ \alpha_1, ..\alpha_2,\ \alpha_i \in \mathbb{R}\ k(x,y) = (1 + x^Ty)^d$ is a positive definite kernel. ∎

3.5 find a function ϕ:$\mathbb{R}^2 \mapsto H$, where H is an inner product space such that for any (x,y), $\langle\phi(x),\phi(y)\rangle_H = x^Ty - 1$

$$k(x,y) = \langle\phi(x),\phi(y)\rangle_H = x^Ty - 1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}\begin{bmatrix} y_1 & y_2 \end{bmatrix}_{1\times 2} - 1 = x_1y_1 + x_2y_2 - 1$$

For all $|x| < \frac{\sqrt{2}}{2}$, when $\alpha_1 = \alpha_2 = 1$, $\langle\phi(x),\phi(x)\rangle_H = 2x^2 - 1 < 0$.
By the property of Kernel, $k(x,y)$ should be positive.
Therefore, there isn't a function ϕ:$\mathbb{R}^2 \mapsto H$ that can make $\langle\phi(x),\phi(y)\rangle_H = x^Ty - 1$. ∎