Sufficient; Factorization theorem

Multinomial n_i =**the number of times we get outcome** $i = 1, .., k$

$f_{\vec{\theta}}(\vec{n}) = n! \prod_{i=1}^{k} \frac{\theta_i^{n_i}}{n_i!} \mathbf{1}_{\{\sum N_i = n\}}$ $T(\vec{N}) = (N_1, .., N_{k-1}, n - \sum_{i=1}^{k-1} N_i)$ is equivalent with $(N_1, .., N_{k-1})$ is sufficient for θ.

$p(x,\theta) = \frac{1}{\sigma}\exp\{-\frac{x-\mu}{\sigma}\}\mathbf{1}_{\{x_i \geq \mu\}}$ $\theta = (\mu, \sigma)$ $-\infty < \mu < \infty, \sigma > 0$

$x_{(1)}$ is sufficient for μ when σ is fixed.

$h(x) = \sigma^{-n}\exp[-\frac{\sum_{i=1}^{n} x}{\sigma}]$, $g(T(x),\mu) = \exp[\frac{n\mu}{\sigma}]\prod_{i=1}^{n}\mathbf{1}_{\{x_i\geq\mu\}}$

$T(x) = \sum_{i=1}^{n} x_i$ is sufficient for σ when μ is fixed.

$h(x) = \prod_{i=1}^{n}\mathbf{1}_{\{x\geq\mu\}}$, $g(T(x),\sigma) = \sigma^{-n}\exp[-\frac{\sum_{i=1}^{n} x}{\sigma} + \frac{n\mu}{\sigma}]$

$T(x) = (x_{(1)}, \sum_{i=1}^{n} x_i)$ is a two-dimensional sufficient

$h(x) = 1$, $g(T(x),\mu,\sigma) = \sigma^{-n}\exp[-\frac{\sum_{i=1}^{n} x}{\sigma} + \frac{n\mu}{\sigma}]\prod_{i=1}^{n}\mathbf{1}_{\{x_{(1)}\geq\mu\}}$

Exp family:The suppose depends on the unknown θ is not a 1-para

$h(x)\exp[\eta(\theta)T(x) - B(\theta)]$, $x \in \mathcal{X} \subset \mathbb{R}$free of θ

$X = [Z, Y]^T$**, where** $Y = Z + \theta W$**,** $\theta > 0$**,**$Z \perp W \sim N(0,1)$

$f(x,\theta) = f(z,y,\theta) = f(z)f_\theta(y|z) = \phi(z)\theta^{-1}\phi(\frac{y-z}{\theta})$

$= \frac{1}{2\pi}\exp[\frac{1}{2}z^2]\exp[\frac{-1}{2\sigma^2}(y-z)^2 - \ln\theta]$

$U[\theta_1,\theta_2]$**, let** $h(x) = 1$**,** $a(\theta) = (\theta_2 - \theta_1)^{-1}$

$g(T(x),\theta_1,\theta_2) = \prod_{i=1}^{n}[\mathbf{1}_{\{x_{(n)}\leq\theta_2\}}\mathbf{1}_{\{x_{(1)}\geq\theta_1\}}][\theta_2-\theta_1]^{-n}$, $h'(x) = 1$

$T(x) = (x_{(1)}, x_{(n)})$ is a two-dimensional sufficient for θ in the $U[\theta_1,\theta_2]$ family.

$N(\mu,\sigma^2)$	σ^2 fixed	$\exp[\underbrace{\frac{\mu}{\sigma^2}}_{\eta(\mu)}\underbrace{x}_{T(x)} - \underbrace{(\frac{\mu^2}{2\sigma^2} + \ln(\sqrt{2\pi}\sigma))}_{B(\mu)}]\underbrace{\exp[-\frac{x^2}{2\sigma^2}]\mathbf{1}_{\{x\in\mathbb{R}\}}}_{h(x)}$
	μ fixed	$\exp[\underbrace{-\frac{1}{2\sigma^2}}_{\eta(\sigma^2)}\underbrace{(x-\mu)^2}_{T(x)} - \underbrace{\ln(\sqrt{2\pi}\sigma)}_{B(\sigma^2)}]\underbrace{\mathbf{1}_{\{x\in\mathbb{R}\}}}_{h(x)}$
$\Gamma(p,\lambda)$	p fixed	$\exp[\underbrace{-\lambda}_{\eta(\lambda)}\underbrace{x}_{T(x)} - \underbrace{-\ln(\frac{\lambda^p}{\Gamma p})}_{B(\lambda)}]\underbrace{x^{p-1}\mathbf{1}_{\{x\in(0,\infty)\}}}_{h(x)}$
	λ fixed	$\exp[\underbrace{(p-1)}_{\eta(p)}\underbrace{\ln(x)}_{T(x)} - \underbrace{-\ln(\frac{\lambda^p}{\Gamma p})}_{B(p)}]\underbrace{\exp[-\lambda x]\mathbf{1}_{\{x\in(0,\infty)\}}}_{h(x)}$
		$\exp[\underbrace{-\lambda x + (p-1)\ln x}_{\eta(p,\lambda)T(x)} - \underbrace{-\ln(\frac{\lambda^p}{\Gamma p})}_{B(p,\lambda)}]\underbrace{\mathbf{1}_{\{x\in(0,\infty)\}}}_{h(x)}$
$\beta(r,s)$	r fixed	$\exp[\underbrace{(s-1)}_{\eta(s)}\underbrace{\ln(1-x)}_{T(x)} - \underbrace{\ln(B(r,s))}_{B(s)}]\underbrace{x^{r-1}\mathbf{1}_{\{x\in(0,1)\}}}_{h(x)}$
	s fixed	$\exp[\underbrace{(r-1)}_{\eta(r)}\underbrace{\ln(x)}_{T(x)} - \underbrace{\ln(B(r,s))}_{B(r)}]\underbrace{(1-x)^{s-1}\mathbf{1}_{\{x\in(0,1)\}}}_{h(x)}$
		$\exp[\underbrace{(r-1)\ln(x) + (s-1)\ln(1-x)}_{\eta(r,s)T(x)} - \underbrace{\ln(B(r,s))}_{B(r,s)}]\underbrace{\mathbf{1}_{\{x\in(0,1)\}}}_{h(x)}$

Geometric $G(\theta)$ the number of failures before the first success in a sequence of Bernoulli trials with probability of success θ

$P_\theta(X=k) = \exp[\underbrace{\ln(1-\theta)}_{\eta(\theta)}\underbrace{k}_{T(k)} - \underbrace{-\ln(\theta)}_{B(\theta)}]\underbrace{\mathbf{1}_{\{k\in(0,1,2,..)\}}}_{h(k)}$

$P_\theta(X_{1:n}) = \prod_{i=1}^{n} P_\theta[X=x] = \exp[\underbrace{\ln(1-\theta)}_{\eta(\theta)}\underbrace{\sum_{i=1}^{n} x_i}_{T(x)} - \underbrace{-n\ln(\theta)}_{B(\theta)}]\underbrace{\prod_{i=1}^{n}\mathbf{1}_{\{x\in(0,1,2,..)\}}}_{h(x)}$

bivariate normal population $f(\vec{X},\vec{Y}) =$

$$\exp[\underbrace{-\frac{\sum x^2}{2(1-\rho^2)\sigma_X^2} + \frac{\sum x}{(1-\rho^2)}(\frac{\mu_X}{\sigma_X^2} - \frac{\mu_Y\rho}{\sigma_X\sigma_Y}) + \frac{\rho\sum xy}{(1-\rho^2)\sigma_X\sigma_Y} + \frac{\sum y}{(1-\rho^2)}(\frac{\mu_Y}{\sigma_Y^2} - \frac{\mu_X\rho}{\sigma_X\sigma_Y}) - \frac{\sum y^2}{2(1-\rho^2)\sigma_Y^2}}_{\eta(\rho,\mu_X,\mu_Y,\sigma_X,\sigma_Y)T(x,y)}]$$

$$\cdot\exp[\underbrace{-n\left(\frac{1}{2(1-\rho^2)}(\frac{\mu_X^2}{\sigma_X^2} - \frac{2\rho\mu_X\mu_Y}{\sigma_X\sigma_Y} + \frac{\mu_Y^2}{\sigma_Y^2}) + \ln(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2})\right)}_{nB(\rho,\mu_X,\mu_Y,\sigma_X,\sigma_Y)}]\underbrace{\mathbf{1}_{\{x,y\in\mathbb{R}^n\}}}_{h(x)}$$

$\rho \in (0,1), \mu_X \in \mathbb{R}, \mu_Y \in \mathbb{R}, \sigma_X \in \mathbb{R}^+, \sigma_Y \in \mathbb{R}^+$

$x \in \mathcal{X} \subset \mathbb{R}^n, y \in \mathcal{Y} \subset \mathbb{R}^n$

theorem 1.6.1 $\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ for a one-para exp family. the family of the distribution of $\sum_{i=1}^{n} X_i$ is a one-parameter exponential family, whose p.m.f may be written as $h^*(t)\exp[\eta(\theta)t - B(\theta)]$ for a suitable h^*.

Mult$_k(n, p_1, .., p_k)$**,**$\vec{p} \in (0,1)^k$**,** $\sum p_i = 1$

$= n!\prod_{i=1}^{k}\frac{p_i^{n_i}}{x_i!}\mathbf{1}_{\{\sum x_i = n\}} = n!\prod_{i=1}^{k}(x_i!)^{-1}\exp[x_1\ln(p_1), .., x_k\ln(p_k)]$

Canonical $q(x,\eta) = h(x)\exp[\eta T(x) - A(\eta)]$**,** $x \in \mathcal{X} \subset \mathbb{R}$**free of** θ

$A(\eta) = \ln(\int_{\mathcal{X}} h(x)\exp[\eta T(x)]dx)$;$\int_{\mathcal{X}} h(x)\exp[\eta(\theta)T(x) - B(\theta)]dx = 1$

Bino:$p(x,\theta) =$**,** $\eta = \ln p1 - p$**,** $p = \frac{e^\eta}{1+e^\eta}$**,** $1 - p = \frac{1}{1+e^\eta}$

$A(\eta) = -n\ln(\frac{1}{1+e^\eta}) = n\ln(1+e^\eta)$ finite $\forall\eta \in \mathbb{R}$

Weibull:$p(x,\theta) = k\theta(\theta x)^{k-1}\exp[-(\theta x)^k], x > 0$

$p(x,\theta) = \exp[-\theta^k\sum_{i=1}^{n} xki - nk\ln\theta + X_{i=1}^{n}\ln x_i^{k-1} + n\ln k]$

$\eta = -\theta^k$, $A(\eta) = -n\ln\theta^k = -n\ln(-\eta)$,

$T = \sum_{i+1}^{n} X_i^k$ is n.s.s. $E[T] = \frac{-n}{\eta} = \frac{n}{\theta^k}$; $Var[T] = \frac{-n}{\eta^2} = \frac{n}{\theta^{2k}}$

Suppose $\eta \in \varepsilon$ **interior of** ε**, then** $M_\eta(s)$**exists for s in a nbd of 0**

$M_\eta(s) = \exp[A(s+\eta) - A(\eta)]$

Corollary $E_\eta[T(x) = \frac{\partial A(\eta)}{\partial\eta} = \dot{A}(\eta)$**;**$V_\eta[T(x)] = \frac{\partial^2 A(\eta)}{\partial\eta^2} = \ddot{A}(\eta)$

if $\eta \in \varepsilon$ has all moments

Bino $\dot{A}(\eta) = n\frac{1}{1+e^\eta}e^\eta = np$**;**$\ddot{A}(\eta) = n\frac{e^\eta}{(1+e^\eta)^2} = np(1-p)$

Pois $\dot{A}(\eta) = e^\eta = \lambda$**;**$\ddot{A}(\eta) = e^\eta = \lambda$

$\eta = \ln\lambda$;$A(\eta) = e^\eta = \lambda$

Rayleigh $\dot{A}(\eta) = -n(-\frac{1}{2\eta})(-2) = \frac{-n}{\eta} = -n(2\theta^2)$**;**$\ddot{A}(\eta) = \frac{n}{\eta^2} = 4n\theta^4$

B(μ, σ^2**)=**$-\frac{\mu^2}{2\sigma^2} + \ln(\sqrt{2\pi}\sigma) = -\frac{-\eta_1^2}{4\eta_2} - \frac{1}{2}\ln(-\eta_2) + \ln(\sqrt{\pi})$**=A**$(\eta_1,\eta_2)$

$\dot{A}(\eta) = (\frac{-\eta_1}{2\eta_2}, \frac{-1}{2\eta_2} + \frac{\eta_1^2}{4\eta_2^2})^T = (\mu, \mu^2+\sigma^2)^T$

$$\ddot{A}(\eta) = \begin{bmatrix} \frac{-1}{2\eta_2} & \frac{\eta_1}{2\eta_2^2} \\ \frac{\eta_1}{2\eta_2^2} & \frac{1}{2\eta_2} - \frac{\eta_1^2}{2\eta_2^3} \end{bmatrix} = \begin{bmatrix} \sigma^2 & 2\mu\sigma^2 \\ 2\mu\sigma^2 & 2\mu\sigma^4 + 4\mu^2\sigma^2 \end{bmatrix}$$

calculous

$(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$; $(uv)' = u'v - uv'$, $\int udv = uv - \int vdu$;
$(a^x)' = a^x ln(a)$, $\int a^x \ln a = a^x$; $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$;
$(e^{ax})' = ae^{ax}$, $\int e^{au} = \frac{1}{a}e^{au}$; $(\ln x - x)' = \ln x$, $\int \ln u = u\ln u - u$;
$(x^n)' = nx^{n-1}$, $\int x^n = \frac{x^{n+1}}{n+1}$; $(\ln x)' = \frac{1}{x}$, $\int \frac{1}{ax+b} = \frac{\ln|ax+b|}{a}$;
$\lim_{n\to\infty}(1\pm\frac{a}{n})^n = e^{\pm a}$, $\sum_{n=0}^{\infty}\frac{(tx)^n}{n!} = e^{tx}$;
$\Gamma(n) = (n-1)! = (n-1)\Gamma(n-1)$, $\Gamma(1/2) = \sqrt{\pi}$,
$\Gamma(-1/2) = -2\Gamma(1/2)$, $\Gamma(0) = \Gamma(-1) = \infty$;
$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$, $\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$;
$\Gamma(\frac{2k+1}{2}) = \frac{(2k)!\sqrt{\pi}}{4^k k!}, k>0; \frac{(-4)^{-k}(-k)!\sqrt{\pi}}{(-2k)!}, k<0$
$\sum_{n=0}^\infty ar^n = \frac{a}{1-r}, |r|<1$; $\sum_{k=0}^{n-1} ar^n = \frac{a(1-r^n)}{1-r}$;

Finite	Binomial
$\sum_{k=1}^n k = \frac{n(n+1)}{2}$	$\sum_{k=0}^n \binom{n}{k} = 2^n$
$\sum_{k=1}^n (2k-1) = n^2$	$\sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n}$
$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$	$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$
$\sum_{k=1}^n k^3 = (\frac{n(n+1)}{2})^2$	$\sum_{k=0}^r \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$
$\sum_{k=0}^n c^k = \frac{c^{n+1}-1}{c-1}$	$\sum_{k=0}^n \binom{n}{k}a^{n-k}b^k = (a+b)^n$

1 Basic

$P(X\le y|X\ge c) = \frac{P(X\le y, X\ge c)}{P(X\ge c)} = \frac{P(c\le X\le y)}{1-P(X\le c)} = \frac{F(y)-F(c)}{1-F(c)}$
CDF nondecreasing, right-continuou; $\lim_{x\to\infty}F(x)=1$; $\lim_{x\to-\infty}F()=0$
$Z = X+Y, X\perp Y, g(z) = \int_{-\infty}^\infty f_X(w)f_Y(z-w)dw$
$\frac{5*0.99}{5*0.99+995*0.02} = 0.1992$; $V[X|Y] = E[X - E(X|Y)]^2|Y]$
$Pois(3), E[30-3X-X^2] = 30-3EX-EX^2 = 9$;
$U(0,1), E[Y] = E(E[Y|X]) = \alpha+\beta EX^2 = \alpha+\beta/3$

$$\begin{cases} f_T(t) = \frac{1}{1.5}e^{-\frac{t}{1.5}}; 0 & v<5 \\ P(V=5) = P(t<3) = \int_0^3 dt = 1-e^{-2} & 5\le v<6 \\ P(V\le v) = P(2T<v) = \int_0^{v/2} dt = 1-e^{-\frac{v}{3}} & 6\le v \end{cases}$$

$\frac{2}{9}(x+1), -1\le x\le 2, Y = X^2; P(Y\le y) = P(X^2\le y) =$

$$\begin{cases} P(1\le X\le\sqrt{y}) = \int_1^{\sqrt y} f(x)dx = \frac{y}{9}+\frac{2\sqrt{y}}{9}-\frac{1}{3} & x\ge1 \\ P(-\sqrt y\le X\le\sqrt y) = \int_{-\sqrt y}^{\sqrt y} f(x)dx = \frac{4\sqrt y}{9} & |x|\le1 \end{cases}$$

$$f_Y(y) = \begin{cases}\frac19+\frac{1}{9\sqrt y} & 1\le y\le4\\ \frac{2}{9\sqrt y} & 0\le y\le1\end{cases}$$

2.3 Moment

$M_X(t) = E[e^{tX}]$; $M_{\bar X}(t) = [M_X(\frac{t}{n})]^n$; $M_X^{(n)}(0) = EX^n$
$Expo(1)$
$E[e^{tX}] = \int_0^\infty e^{tx}(ne^{-nx})dx = \frac{n}{n-t}\int_0^\infty (n-t)e^{-(n-t)x}dx$
$X = e^Z \sim LN(0,1)$; $(e^{n\mu+n^2\sigma^2/2})$ $M_Z(t) = e^{\frac12 t^2}$;
$E[X^n] = E[(e^Z)^n] = M_Z(n) = e^{\frac12 n^2}$; $E[X^{1,2,3}] = e^{\frac12}, e^2, e^{\frac92}$
skewness $E\frac{X-\mu^3}{\sigma^3} = (e+2)\sqrt{e-1}$

Geom

$*\frac{e^t}{4-3e^t}\sim Geom(\frac14)$; $EX = \frac1p = 4$; $VarX = \frac{1-p}{p^2} = 12$; $\frac{\sigma}{\mu} = \frac{\sqrt3}{2}$
*2B2W,pick,replace opposite, p=0, x=1,3,5;p=$\frac14$,x=2,4,6
X=all same color,Y=$\frac{X}{2}\sim Geo(\frac14)$; EX,2$\frac1p$ = 8; VX,4$\frac{1-p}{p^2}$ = 48
$*cpq^{x-1}$,x=a,a+1, F=c[$\sum_1^\infty pq^{x-1} - \sum_1^{a-1} pq^{x-1}$] =c[1 − (1 − q^{a-1})]=1; X+1-a=Y∼ $Geo(p)$, EX=EY+a-1=$\frac1p$+a-1, VX=VY=$\frac{1-p}{p^2}$

NB

6coins,toss even head p=$\frac12$; X 2^{nd}success∼ $NB(2,\frac12)$; $EX = VX = 4$

Pois

Pois(μ);x=1 uniq-mode; $\frac{p_m}{p_{m-1}}, \frac{p_m}{p_{m+1}}\ge1, x\in[\mu-1,\mu], \mu\in(1,2)$
*Time:$\Gamma(\alpha,\beta)$,Vist:$Pois(\lambda=\frac{t}{\beta})$,$P(T>t) = P(X\le\alpha-1)$
10(α) visits longer than 8(t) min, 1.5vist/min(β)=0.2424

Expo

Exp,Geom Memoryless $P(X>a+b|x>a) = P(x>b)$
$X_1\sim\Gamma(\alpha_1,\beta)$, $X_2\sim\Gamma(\alpha_2,\beta)$, $U = \frac{X_1}{X_1+X_2}\sim B(\alpha_1,\alpha_2)$;
$X_1, X_2\sim Exp(1), \Gamma(1,1), U\sim B(1,1), U(0,1); V\sim\Gamma(2,1), xe^{-x}$
$X_1, X_2\sim Exp(\lambda), Y = X_1-X_2\sim DExp(0,\lambda), \frac{\lambda}{2}e^{-\lambda|x|}\forall x>0$
$X\sim B(\theta,1), -\ln X\sim Exo(\theta), -2\theta\sum\ln X\sim\Gamma(n,2), \chi^2_{2n}$;

DExp,Laplace

$L(\theta|x) = 2^{-n}e^{-\sum|x_i-\theta|}$; $\hat\theta_{MLE} = X_{((n+1)/2)}$ sample median
$E[X_{((n+1)/2)}] = \int_{-\infty}^\theta x\frac12 e^{x-\theta}dx + \int_\theta^\infty x\frac12 e^{-x+\theta}dx = \theta$

Gamma

$\bar X_\Gamma\sim\Gamma(n\alpha,\frac{\beta}{n})$; $\chi^2_v\sim\Gamma(\frac{v}{2},2)$

Beta

$B(\alpha,\beta)\perp B(\alpha+\beta,\gamma)$, $U = XY\sim B(\alpha,\beta+\gamma)$

4.4 Beta-Bino

$X|P\sim Bin(n,P)$, $P\sim B(\alpha,\beta)$; $EX = E[E(X|P)] = E[nP] = n\frac{\alpha}{\alpha+\beta}$;
$VX = E[Var(X|P)] + Var[E(X|P)] = E[npq] + Var[np] = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$
$P\sim B(\frac12,\frac12)$, $EX = \frac{n}{2}$; $VX = \frac{n(n+1)}{8}$;
$P\sim B(1,1)$, $X\sim U(0,n+1)$, $P|X\sim B(x+1, n-x+1)$

Pois-Bino

$Pois(\lambda).S_{10} = \sum_{i=1}^{10}X_i\sim Pois(10\lambda)$.
$M_{S_{10}}(t) = \prod_{i=1}^{10}M_X(t) = [e^{\lambda(e^t-1)}]10 = e^{10\lambda(e^t-1)}$
$S_{10} = s = S_4 + S_6$, $P(S_4|S_{10}) = \frac{P(S_4, S_6 = s-S_{10})}{P(S_{10})}\sim Bino(s,\frac25)$
$Pois(\lambda)$, $Pois(3\lambda)$ $P(x|x+y=5) = \frac{P(x)P(y=5-x)}{P(x+y=5)}\sim Bino(5,\frac13)$;

Uniform

$F\sim U(0,1), B(1,1); -\ln F, -\ln(1-F)\sim Exp(1), \Gamma(1,1)$;
$-\lambda\ln U\sim Exp(\lambda)$; $-2\sum^v\ln U\sim\chi^2_{2v}$;
$-\beta\sum^\alpha\ln U\sim\Gamma(\alpha,\beta)$; $\frac{\sum^a\ln U}{\sum^{a+b}\ln U}\sim B(a,b)$
$U(0,1), U = Y-X, 0<x<y<1, g(u,v) = 1, 0<v<1-u, f_U(u) = 1-u, 0<u<1$; $f_{|U|}(u) = 2(1-u)$; $E[|U|] = \frac13$;
$U(0,1), U = Y_1+Y_2, 0<u-v<1, u-1<v<u; J = 1; f(u,v) = 1$

$$\begin{cases} f_U(u) = 0; F_U(u) = 0 & o.w. \\ \int_0^u = u; \int_0^u = \frac{u^2}{2} & u\in(0,1) \\ \int_{u-1}^1 = 2-u; F_U(1)+\int_1^u 2u - \frac{u^2}{2}-1 & u\in(1,2) \end{cases}$$

N

$E[\bar X] = \mu$, $V[\bar X] = \frac{\sigma^2}{n}$, $E[S^2] = \sigma^2$ without normality;
$\bar X\sim n(\mu,\frac{\sigma^2}{n})$; $\bar X\perp S^2$; $\frac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1}$
$E(Z^{2k}) = \frac{(2k)!}{2^K K!}, k=1,2,...$; $E(Z^{2k+1}) = 0$
$V[S^2] > CRLB > V[S_n^2]$; $\frac{2\sigma^4}{n-1}\ge\frac{2\sigma^4}{n} > \frac{2(n-1)\sigma^4}{n^2}$;
$E[S] = \sqrt{\frac{2\sigma^2}{n-1}}\frac{\Gamma(\frac n2)}{\Gamma(\frac{n-1}{2})}\approx\sigma-\frac{\sigma}{4(n-1)}$;
$V[S] = \sigma^2 - \frac{2\sigma^2}{n-1}\frac{\Gamma^2(\frac n2)}{\Gamma^2(\frac{n-1}{2})}\approx\frac{\sigma^2}{2(n-1)}$
$E[\hat\sigma^2] = \frac1n\sum E[(x_i-\mu)^2] = \sigma^2$,μ known;
$E[\hat\sigma^2] = \frac{E[\sum(x_i-\bar x)^2]}{n} = \frac{(n-1)E[S^2]}{n} = \frac{(n-1)\sigma^2}{n}$,μ unknown
$E[\Phi^2(z)] = V[\Phi(z)] + E[\Phi(z)]^2 = \frac{1}{12}+\frac14 = \frac13$;
$E[n\Phi^{n-1}(z)] = \int_0^1 n\Phi^{n-1}\phi(z)dz = \int_0^1 n\Phi^{n-1}d\Phi(z) = 1$;
$*Y|\theta\sim n(\theta,1), \theta\sim n(0,1)$ $Y\sim N(0,2)$ $\theta|Y\sim N(\frac{Y}{2},\frac12)$
$\frac{Z_1}{Z_2}$; $f(u) = \int_{-\infty}^\infty f(1)f(2)|v|dv = 2\int_0^\infty = \frac{1}{\pi(u^2+1)}\sim Cauch(0,1)$
$\bar X\sim Cauchy(0,\sigma)$, $\sum^n X\sim Cauchy(0,n\sigma)$; $Z^2\sim\chi^2_1$
$\frac{\bar X-\mu}{S/\sqrt n}, \frac{\frac{\bar x-\mu}{\sigma/\sqrt n}}{\sqrt{S^2/\sigma^2}}, t_{n-1}$; $\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}, F_{n-1,m-1}$ $\frac{S_X^2}{\sigma_X^2}, \chi^2$; $F_{(p,q)}$, $\frac{\frac pq X}{1+\frac pq X}, B(\frac p2,\frac q2)$
$t_1, Cau(0,1)$; $\chi^2_2, Exp(2)$; $\chi^2_p, \Gamma(\frac p2, 2)$; $U+V, \chi^2_{m+n}$; $T^2_{(q)}, F_{(1,q)}$;
$*X\sim N(\mu,\sigma^2)\perp Y\sim N(\gamma,\sigma^2)$, $U = X+Y\perp V = X-Y$
Trans and Or $M_U(t) = M_X(t)\cdot M_Y(t)$, $M_V(t) = M_X(t)\cdot M_Y(-t)$;
Or $Cov(X+Y, X-Y) = Cov(X,X) - Cov(Y,Y) = V[X]-V[Y] = 0$
$U\sim N(\mu+\gamma, 2\sigma^2)$; $V\sim N(\mu-\gamma,2\sigma^2)$ by factor or linear combi
$E[U] = E[X]+E[Y] = \mu+\gamma$; $E[V]_\mu-\gamma$; $V[U] = V[X]+V[Y] = 2\sigma^2$
$(X,Y) = (2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2})^{-1}\exp(-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_X})^2 - 2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y}) + (\frac{y-\mu_Y}{\sigma_Y})^2])$
$Y|X\sim n(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \sigma_Y^2(1-\rho^2))$
$aX+bY\sim n(a\mu_X+b\mu_Y, a^2\mu_X^2+b^2\mu_Y^2+2ab\rho\sigma_X\sigma_Y)$

4.5 Cov Corr

$Cov(X,Y) = \sigma_{XY} = E((X-\mu_X)(Y-\mu_Y)) = E[XY]-\mu_X\mu_Y$;
$Cov(aX,bY) = abCov(X,Y)$; $Cov(X, Y+Z) = Cov(X,Y)+Cov(X,Z)$
$Cov(X,c) = 0$; $Cov(X,X) = E[X^2]-\mu_X^2$; $X\perp Y$, $Cov(X+Y, X-Y) =$
$Cov(X,X)-Cov(Y,Y) = V[X]-V[Y]$
$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}, \rho XY$, $\frac{\sigma_{XY}}{\sigma_X\sigma_Y}$, $\frac{1/3-(7/12)^2}{11/144} = -\frac{1}{11}$
$Cov[\bar X, X_n] = \frac{Cov[\sum X_i, X_n]}{n} = \frac{\sigma^2}{n}$; $Cor[\bar X, X_n] = \frac{\sigma^2/n}{\sigma\sigma/\sqrt n} = \frac{1}{\sqrt n}$

4.7 Inequity;Chebyshev;Cauchy-Schwarz;Jensen

$g(x)\ge0\forall r>0$, $P(g(X)\ge r)\le\frac{Eg(X)}{r}$; $P(|x-\mu|<t\sigma)\ge1-\frac{1}{t^2}$;
$\sigma=3, t=2$; $P(-4<x<8)\ge\frac34$; $\sigma=2, t=\frac52$, $P(-2<x<8) = \frac{21}{25}$
$U(a,b), \sigma^2 = \frac{(b-a)^2}{12}$, $P(|x-\mu|<1.5\sigma) = P(+)-P(-) =$
$\frac{\mu+1.5\sigma}{b-a} - \frac{\mu-1.5\sigma}{b-a} = \frac{3\sigma}{b-a} = 0.866\ge0.555$ Che conservative
$|EXY|\le E|XY|\le(E|X|^2)^{\frac12}(E|Y|^2)^{\frac12}$; $Cov(X,Y)\le\sigma_X^2\sigma_Y^2$;
Convex function $Eg(X)\ge g(EX)$

5.4 Order

$X_{(k)} = k\binom nk f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}$;
$X_{(1)} = nf(x)[1-F(x)]^{n-1}$; $F = 1-[1-F(x)]^n$;
$X_{(n)} = nf(x)[F(x)]^{n-1}$; $F = [F(x)]^n$; $f(1,...,n) = n!\prod f(x_i)$
$\frac{n!f(u)f(v)}{(i-1)!(j-1-i)!(n-j)!}[F(u)]^{i-1}[F(v)-F(u)]^{j-1-i}[1-F(v)]^{n-j}$
$-\infty<u<v<\infty$; $X_{(1),(n)} = n(n-1)f_{(1)}f_{(n)}[F_{(n)}-F_{(1)}]^{n-2}$;
$U(0,1)$ $X_{(k)}\sim B(k, n-k+1)$; $X_{(1)} = n[1-x]^{n-1}$; $X_{(n)} = nx^{n-1}$
$EX_{(k)} = \frac{k}{n+1}$; $VarX_{(k)} = \frac{k(n-k+1)}{(n+1)^2(n+2)}$;
$P(n(1-Y_n)<y) = 1-P(Y_n<1-\frac yn) = 1-\int_0^{1-\frac yn}Y_n dyn =$
$P(nY_1<y) = P(Y_1<\frac yn) = \int_0^{\frac yn}Y_1dy_1 = 1-(1-\frac yn)^n$;
Or transf $W_1 = ny_1, f(w) = f(\frac wn)|\frac{dy}{dw}| = n(1-\frac wn)^{n-1}\frac1n$;
$W_n = n(1-Y_n), f(w_n) = f(1-\frac wn)|\frac{dy}{dw}| = n(1-\frac wn)^{n-1}\frac1n$
$\lim_{n\to\infty}nY_1 = \lim_{n\to\infty}n(1-Y_n) = 1-e^{-y}\sim Expo(1)$
$U(0,\theta)$: $X_{(n)} = \frac n\theta(\frac x\theta)^{n-1}\sim Beta(n,1)$; $V[X_n] = \frac{n\theta^2}{(n+1)^2(n+2)}$;
$EX_{(n)} = \frac{n\theta}{n+1}$, $E[\frac{n+1}{n}X_{(n)}] = \theta$; $E[\frac{n+2}{n}X^2_{(n)}] = \theta^2$is UMVUE
$frm = \frac{n(n-1)r^{n-2}}{\theta^n}, r\in(0,\theta), m\in(\frac r2, \theta-\frac r2)$;
$f_R = \int_{\frac r2}^{\theta-\frac r2}dm = \frac{n(n-1)r^{n-2}(\theta-r)}{\theta^n}, r\in(0,\theta)$; θ=1,$Beta(n-1,2)$;
$fm = \int_0^{2m}dr = \frac{n(2m)^{n-1}}{\theta^n}, m\in(0,\frac\theta2)$; V[R]=$\frac{1}{\lambda^2}\sum_{k=1}^{n-1}\frac1k$;
$fm = \int_0^{2(\theta-m)}dr = \frac{n[2(\theta-m)]^{n-1}}{\theta^n}, m\in(\frac\theta2,\theta)$; E[R]=$\frac1\lambda\sum_{k=1}^{n-1}\frac1k$.
$Exp(\lambda)$: $X_{(1)} = n\lambda e^{-n\lambda x}\sim Exp(n\lambda)$; $F = 1-e^{-n\lambda x}$;
$X_{(n)} = n\lambda e^{-\lambda x}(1-e^{-\lambda x})^{n-1}$; $F = (1-e^{-\lambda x})^n$

5.5 Convergence;CLT;Slutsky

$X_n\xrightarrow{a.s.}X, P(\lim_{n\to\infty}|X_n-X|<\epsilon) = 1$; $\xrightarrow{P}$, $\lim_{n\to\infty}P(|X_n-X|<\epsilon) = 1$
$X_n\xrightarrow{D}X$; $\lim_{n\to\infty}F_{X_n}(x) = F_X(x)$; $\frac{\sqrt n(\bar X_n-\mu)}{\sigma}\xrightarrow{D}N(0,1)$;
$X_n\xrightarrow{D}X, Y_n\xrightarrow{P}a, X_nY_n\xrightarrow{D}aX, X_n+Y_n\xrightarrow{D}X+a$,
Delta Method: if $\sqrt n(Y_n-\theta)\xrightarrow{D}N(0,\sigma^2)$,
$g'(\theta)\ne0$, $\sqrt n[g(Y_n)-g(\theta)]\xrightarrow{D}N(0,\sigma^2[g'(\theta)]^2)$
$g'(\theta) = 0$, $\xrightarrow{D}\sigma^2\frac{g''(\theta)}{2}\chi^2_1$)
$g(x) = g(x_0)+g'(x_0)(x-x_0)+g''(x_0)\frac{(x-x_0)^2}{2}+R$
Stein $n(\theta,\sigma^2)$, $E[g(X)(X-\theta)] = \sigma^2E[g'(X)]$;
$*g(\mu) = \frac1\mu$, $E\mu(\frac1X)\approx\frac1\mu$, $Var_\mu(\frac1X)\approx(\frac1\mu)^4Var_\mu(X)$
$\sqrt n(\frac1X-\frac1\mu)\xrightarrow{D}N(0,(\frac1\mu)^4Var_\mu[X_1])$
$\hat Var[\frac1X]\approx(\frac1X)^4S^2$, $\frac{\sqrt n(\frac1X-\frac1\mu)}{(1/\bar X)^2S}\xrightarrow{D}N(0,1)$
$*Exp(\lambda)$, $\sqrt n(\bar X_n-\frac1\lambda)\xrightarrow{D}N(0,\frac1{\lambda^2})$, $g(u) = u^2$, $g'(u) = 2u\ne0$
$\sqrt n(\bar X_n^2-(\frac1\lambda)^2)\to n(0,\frac{4}{\lambda^4})$
$*Pois(\mu), g(\bar X_n) = e^{-\bar X_n}$, $g(\mu) = e^{-\mu}$, $g'(\mu) = -e^{-\mu}\ne0$
$\sqrt n(e^{-\bar X_n}-e^{-\mu})\to n(0,\frac{[(e^{-\mu})']^2}{I(\mu)}) = n(0,\mu e^{-2\mu})$

6.2.1 Suff;6.2.2 Mini Suff;6.2.3 Ancillary Stat

$\frac{\prod f(x|\theta)}{g(t|\theta)} = h(\vec x)$free of θ; $\frac{f(x|\theta)}{f(y|\theta)}$ constant fn of θ if $T(\mathbf x) = T(\mathbf y)$
$g_a(\vec x) = x_i+a$, $G = \{g_a(\vec x): a\in\mathbf R\}$ is a group,is invariant under
loc-paraθ $[Y_n-\bar Y]_{(g_a(\vec x))} = Y_n+a-\frac{\sum(Y_i+a)}{n} = (Y_n-\bar Y)_{(\vec x)}$
Loca-scale familiy $g(x|\mu,\sigma) = \frac1\sigma f(\frac{x-\mu}{\sigma})$, μlocation-par,σscale-par

6.2.4 Compl Suff; Basu

$E[g(t)] = 0$. ∀λ iff $P(g(t) = 0) = 1$;
$f(x|\vec\theta) = h(x)c(\vec\theta)e^{\sum_{j=1}^k W_j(\vec\theta)t_j(x)}$, t free of θ W open set in $\mathbb R^k$;
Expo familiy $t/n = \bar Y$ is comp/suff, indep with $Y_n-\bar Y$

7 Point Estimation

MOM $\hat\mu = EX = \bar x$; $\hat\sigma^2 = E[X^2]-EX^2 = \frac{\sum x_i^2}{n}-\bar x^2$

7.2.2 MLE - Indecator fn

$*F(x) = (\frac x\theta)^2, 0\le x<\theta$
$\max L(\theta) = 2^n\theta^{-2n}\prod x_iI_{(-\infty,\theta)}(x_{(n)})I_{(0,\infty)}(x_{(1)})$
$\min\hat\theta_{MLE} = X_{(n)} = nf()[F()]^{n-1} = \frac{2n}{\theta^n}x^{2n-1}\sim Beta(2n,1)$
$EX_{(n)} = \frac{2n}{2n+1}\theta$;Or $\int_0^\theta xf_{X_{(n)}}(x)dx$
$\lim_{n\to\infty}[EX_{(n)}-\theta] = 0$; $\lim_{n\to\infty}P(X_{(n)}-\theta)\ge\epsilon) = 0$;
$\lim_{n\to\infty}V[X_{(n)}] = \lim_{n\to\infty}\frac{n}{(2n+1)^2(n+1)}\theta^2 = 0$
$*Y_i\sim Pois(\mu_i = \gamma x_i)$, x_i known constants,
$\hat\gamma_{mle} = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i}$; $E[\hat\gamma_{mle}] = \gamma$; $V[\hat\gamma_{mle}] = \frac{\gamma}{\sum_{i=1}^n x_i}$

7.2.3 Bayes estimator

$X_n$iid$Expo(\theta)$, $\pi(\theta)\sim Expo(\lambda)$
$\pi(\theta|\vec x) = \frac{g(\vec x|\theta)}{m(\vec x)} = \frac{f(\vec x|\theta)h(\theta)}{\int g(\vec x|\theta)d\theta}\propto L(\theta)h(\theta)\sim\Gamma(n+1,\frac{1}{\lambda+\sum x_i})$
$\mathcal L_\theta(\hat\theta) = (\hat\theta-\theta)^2$, to minPosterior risk $E[\mathcal L_\theta(\hat\theta)|\vec x)]$, Posterior mean
$\hat\theta_{L_2Bayes} = E[\pi(\theta|\vec x)] = \frac{n+1}{\lambda+\sum x_i} = \frac{\lambda(\frac1\lambda, E[\theta])}{\lambda+n\bar x} + \frac{n\bar x(\frac1{\bar x}, \hat\theta_{MLE})}{\lambda+n\bar x}$;
absolute error loss $|a-\theta|$, post-median$\hat\theta_{L_1Bayes} = F^{-1}(\frac12)$;
binary0, $\hat\theta = \theta$; 1, o.w. po-mode$\hat\theta_{L_0Bayes} = (\alpha-1)\beta$, $\Gamma'\overset{set}{=}0$
$X\sim Pois(\theta)$, $\theta\sim\Gamma(\alpha,\beta)$ known
$\pi(\theta|\vec x)\sim\Gamma(\alpha+\sum x_i, \frac{\beta}{n\beta+1})$; $\hat\theta_{L_2Bayes} = \frac{(\alpha+\sum x_i)\beta}{n\beta+1}$
$X\sim Bino(n,\theta)$, $\theta\sim Beta(\alpha,\beta)$;
$\pi(\theta|x)\sim B(\alpha+x, \beta+n-x)$; $\hat\theta_{L_2Bayes} = \frac{\alpha+x}{\alpha+\beta+n}$
$n(\theta,\sigma^2)$, $\theta\sim n(\mu,\tau^2)$; $f(\theta|x)\sim N(\frac{\tau^2x+\sigma^2\mu}{\tau^2+\sigma^2}, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2})$

7.3.1 MSE

$MSE = E[(T-\theta)^2] = V[S_k^2] + Bias^2 = [\frac{2(n-1)}{k^2}+(\frac{n-1}{k}-1)^2]\sigma^4$;
$S_k^2 = \frac{\sum(X_i-\bar X)^2}{k}$; $\frac{\partial MSE}{\partial k}\overset{set}{=}0, S^2_{n+1}\min(MSE)$
$kS^2_{k+1} = (n-1)S_k^2 + \frac{k}{k+1}(X_{k+1}-\bar X_k)^2$

7.3.2 CRLB

$Var_\theta W(\vec X)\ge\frac{(\frac{d}{d\theta}E_\theta W(\vec X))^2}{nE_\theta[(\frac{\partial}{\partial\theta}\ln f(X|\theta))^2]} = \frac{[\tau'(\theta)]^2}{nI_\theta}$, $h(\theta) = E[W]$
$*Expo(\theta)$,median0.5 $= \int_0^{\tilde x}f()dx$, $\tilde x = \frac{\ln2}{\theta}$;
90^{th} $\int_0^{g(\theta)}dx = 0.9$; $g(\theta) = \frac{\ln10}{\theta}$, $VX_{(90)}\ge\frac{g'(\theta)^2}{nI_\theta} = \frac{(\ln10)^2}{n\theta^2}$
$*N(\mu,\sigma^2)$, $\frac{X_{(95)}-\mu}{\sigma} = Z_{(95)} = 1.645$; $X_{(95)} = \mu+1.645\sigma$;
$E[\frac{\sum(X_{(95)}-\mu)^2}{n}] = 1.645^2\sigma^2 = g(\sigma^2)$; $g'(\sigma^2) = 1.645^2$
$V[\hat\sigma^2_{(95)}]\ge\frac{2(1.645\sigma)^4}{n} > \frac{2\sigma^4}{n}$

7.3.3 UMUVE;Rao-Blackwell;Lehmann-Scheffe

T Suffi for θ, $EW = \tau(\theta)$, $\phi(T) = E[W|T]$, then $E\phi(T) = \tau(\theta)$, $V_\theta\phi(T)\le V_\theta W$.
T Compl/Suffi for θ, $E\phi(T) = \tau(\theta)$, $\phi(T)$ is UMVUE of $\tau(\theta)$
$Pois(\lambda>0)$, $T = \sum x_i\sim Pois(n\lambda)$ Compl/Suffi;
$E[\frac Tn] = \lambda$; $E[T^2-T] = VT+ET^2-ET = n^2\lambda^2$;
$\delta(x_i) = \begin{cases}1 & X_1 = 0,1,k\\0 & X_1\ne0,1,k\end{cases}$, $\sum_{i=2}^n x_i\sim Pois((n-1)\lambda)$
$E(\delta|t) = \frac{P(x_i=0,1,k)P(\sum_{i=2}^n x_i = t, t-1, t-k)}{P(\sum_{i=1}^n x_i = t)}\sim Bino(t,\frac1n)$
$= (\frac{n-1}{n})^t$, $\frac{t(n-1)^{t-1}}{n^t}$, $\frac{t!}{k!(n-k)!}\frac{(n-1)^{t-k}}{n^t}$
$Bino(n,p)$, Y Compl/Suffi $E[\frac Yn] = p$; $E[\frac{Y^2-Y}{n^2-n}] = p^2$;
$E[\frac{Y(n-Y)}{n(n-1)}] = pq$; $E[\frac{Y!(n-Y)!}{(Y-k)!(Y-k)!k!(n-k)!}] = \binom nk p^k(1-p)^{n-k}$

8 Hypo test;Neyman-Pearson;MLR;Karlin-Rubin

Size$\alpha_I = P_{H_0}(RR)$ Power$=1-\beta_{II} = P_{H_1}(RR)$; $Z_{1-\beta} = -Z_\beta$
$H_0:\theta = \theta_0, H_1:\theta = \theta_1, \theta_0<\theta_1$, RR $\Lambda = \frac{\sup L(\theta_0)}{\sup L(\theta_1)} < C$ is MP
Expo family $g(t|\theta) = h(t)c(\theta)e^{W(\theta)t}$ T has MLR.
T free of θ is Suffi, $\Lambda = \frac{L(\theta_0)}{L(\theta_1)}$ has ↗ MLR in T,$T\le C$ Λ ↘,$T\ge C$
$P_{H_0}(RR) = \alpha$ is UMP; $-2\ln\Lambda\xrightarrow{D}\chi^2_1$
$*Beta(\theta,\theta)$, $H_0:\theta = 1$, $H_1:\theta = 2$ One Obs $X_1<\frac23$.
$\alpha = P_{\theta_0}(x_1<\frac23) = \int_0^{\frac23}f(x|\theta_0) = \frac23$
Power$P_{\theta_1}(x_1<\frac23) = 1-\int_0^{\frac23}f(x|\theta_1) = \frac{20}{27}$
$*X_m\sim Expo(\frac{1}{\theta_1})$,$Y_n\sim Expo(\frac1{\theta_2})$ $H_0:\theta_1 = \theta_2$, $H_1:\theta_1\ne\theta_2$;
$\hat\theta_{MLE} = \frac{\sum x_i+\sum y_i}{m+n}$; $\hat\theta_1 = \frac{\sum x_i}{m}$; $\hat\theta_2 = \frac{\sum y_i}{n}$
$T = \frac{\sum x_i}{\sum x_i+\sum y_i}\sim\frac{\Gamma(m,\theta_1)}{\Gamma(m,\theta_1)+\Gamma(n,\theta_2)}$ suff
$\Lambda = \frac{(m+n)^{m+n}}{m^mn^n}T^m(1-T)^n\propto B(m+1,n+1)$, $F\le C'$
$*N(\mu,25)$, $H_0:\mu = 50$, $H_1:\mu = 55$; $Z_{0.05} = 1.645$, $Z_{0.1} = 1.28$;
$P_{H_0}(\bar X_n\ge c) = 0.05$; $\frac{\sqrt n(\bar X_n-50)}{5}\ge\frac{\sqrt n(c-50)}{5} = Z_\alpha = 1.645$;
$P_{H_1}(\bar X_n\ge c) = 0.90$; $\frac{\sqrt n(\bar X_n-55)}{5}\ge\frac{\sqrt n(c-55)}{5} = -Z_\beta = -1.28$
$n = [\frac{(Z_\alpha+Z_\beta)\sigma}{\delta=\mu_1-\mu_0}]^2 = [\frac{(1.645+1.28)5}{55-50}]^2 = 8.56\approx9, c = 52.74$ 87
$*N(120,\sigma^2)$; $\Lambda = (\frac{\sigma^2}{4})^{\frac n2}e^{(\frac{1}{2\sigma^2}-\frac18)\sum(x-120)^2}$ $T = \sum(x-120)^2$ suff;
$\sigma^2<4$ MLR Λ ↗ $T\le c$, $\sigma^2>4$ MLR Λ ↘ $T\ge c$ Under H_0, $\frac T4\sim\chi^2_n$
$*\bar X = 6.7, n = 18, \sigma = 2$, $\frac{\bar x-\mu_0}{\sigma/\sqrt n} = 1.48 < 1.645$
If $\mu_1 = 7.5$, Power=$P(Z>Z_\alpha-\frac{\delta\sqrt n}{\sigma}) = P(Z>1.53) = 0.937$
$*N(\theta,1), N(3\theta,1)$; $H_0:\theta\le0$, $H_1:\theta>0$; T:$\bar X+3\bar Y\sim n(10\theta,\frac{10}{m})$, Λ ↘
$\sqrt{\frac{m}{10}}(\bar X+3\bar Y-10\theta)\xrightarrow{D}n(0,1)$; $P_{\theta_0}(\sqrt{\frac m{10}}(\bar X+3\bar Y-10\theta_0>Z_\alpha) = \alpha$;
$P_{\theta_1}(\sqrt{\frac m{10}}(\bar X+3\bar Y-10\theta_1>Z_\alpha-(\theta_1-\theta_0)\sqrt{10m})$
$*Pois(\lambda)$ $H_0:\lambda\le1$, $H_1:\lambda>1$ $\Lambda = (\frac{\lambda_0}{\lambda_1})^{\sum x_i}e^{n(\lambda_1-\lambda_0)}<C$
$T = \sum x_i\sim Pois(n\lambda)$; Λ ↘ $\alpha = P_{\theta_0}(T>C) = 0.05$
$\lambda = 1$ max Λ, $P_{\theta_0}(T\ge4)<0.05$ ifT=3let 0.019+p0.061=0.05,$p = \frac{31}{61}$
$*\theta x^{\theta-1}\sim Beta(\theta,1)$, $-\ln x\sim Expo(\theta)$, $-2\theta\sum\ln x\sim\Gamma(n,2), \chi^2_{2n}$
$P_{\theta_1}(X>0.5) = 1-\int_0^{0.5}\theta x^{\theta-1}dx = 1-0.5^\theta$
$H_0:\theta$=1, $H_1:\theta>1$;Λ ↗ $P_{H_0}(-\ln y\le C) = \alpha$
$H_0:\lambda = \frac12$, $H_1:\lambda = 1$; $P_{H_0}(-\sum\ln x\le\chi^2_{20,0.95}(10.85)) = 0.05$;
$P_{H_1}(-2\sum\ln x\le2*10.85) = 0.64$;
$\hat\lambda_{mle}\frac nT$; $E\hat\lambda_{mle}\frac{n\lambda}{n-1}$; Bias $\frac{\lambda}{n-1}$; V $\frac{n^2\lambda^2}{(n-1)^2(n-2)}$; $MSE\frac{(n+2)\lambda^2}{(n-1)(n-2)}$;
$\hat\lambda_u\frac{n-1}{T}$; $V\hat\theta_u\frac{\lambda^2}{n-2}$; $I_\lambda = \frac{1}{\lambda^2}$; effi $\frac{n-2}{n}$
$*e^{-(x-\theta)}$, n = 10; $H_0:\theta = 0$, $H_1:\theta>0$; $T = X_{(1)}\ge C$
Let$Y = X-\theta$, $Y_{(1)}\sim Expo(10)$, $\Gamma(1,\frac1{10})$; $20Y_{(1)}\sim\chi^2_2$;
$P_{H_0}(X_{(1)}\ge C) = P_{H_0}(20Y_{(1)}\ge20C)\overset{set}{=}0.05$,$20C = 5.991$;
$P_{H_1}(X_{(1)}\ge0.2996) = P_{H_1}(Y_{(1)}\ge-0.7) = 1$, min β at H_0 is unbia
A test whose power never drops below α is unbiased
$*Expo(\frac1\mu)$; $H_0:\mu = 5$,$H_1:\mu\ne5$
$\Lambda = \frac{L(\mu_0)}{L(\mu_1)} = (\frac{\mu_1}{5})^ne^{(\frac{1}{\mu_1}-\frac15)\sum x_i}\le c$
$\mu_1<5$,Λ ↗,$\sum x_i\le c'$; $\mu_1>5$,Λ ↘,$\sum x_i\ge c'$;
$T = \sum x_i\sim Gamma(10,\mu)$, $\frac2\mu T\sim\chi^2_{20}$;
$P_{H_0}(\frac{2T}{5}\le\chi^2_{20,0.975}(9.59)) + P_{H_0}(\frac{2T}{5}\ge\chi^2_{20,0.025}(34.2)) = 0.05$;
$P_{H_1}(\frac{2T}{\mu_1}\le\frac{5\chi^2_{20,0.975}}{\mu_1}) + P_{H_1}(\frac{2T}{\mu_1}\ge\frac{5\chi^2_{20,0.025}}{\mu_1})$
$*U(\theta,\theta+1)$, $H_0:\theta = 0$, $f(z) = f(x_1+x_2) = z, z<1; 2-z, z>1$
$\alpha = P_{H_0}(x_1+x_2>c) = P_{H_0}(x_1>0.95)$;
$\int_c^2 f(z)dz = \frac{c^2}{2}-2c+2 = \int_{0.95}^1 f(x)dx = 0.05$
$c = \frac{2}{1}\pm\sqrt{2^2-4\cdot\frac12\cdot1.95}$, choose $2-0.3162 = 1.6838$

10 Consistency

Asym unbiased $\lim_{n\to\infty}Bias(\hat\theta) = 0$
$\lim_{n\to\infty}Var_\theta[W_n] = 0$, $\lim_{n\to\infty}Bias_\theta[W_n] = 0$,
$\lim_{n\to\infty}P(\tau(\hat\theta_{MLE})-\tau(\theta)\ge\varepsilon) = 0$

10 Efficiency

$\lim_{n\to\infty}\frac{CRLB}{Var(\hat\theta)}$; $U(a,b)$, $m_n\approx N(M,\frac{1}{4nf^2(M)})$,$V[\bar X] = \frac{(b-a)^2}{12n}$,
$V[m_n] = \frac{(b-a)^2}{4n}$; ARE of m_n to $\bar X = \lim_{n\to\infty}\frac{V[\bar X]}{Var[m_n]} = \frac13$
Asym effi of MLE: $\sqrt n[\tau(\hat\theta_{MLE})-\tau(\theta)]\xrightarrow{D}N(0,\frac{1}{I_\theta})$

Robust if $l'(\theta) = \sum\psi(x_i-\theta)$, $W(x_i-\theta)\frac{\psi(x_i-\theta)}{x-\theta}$ bounded;
$n0\psi(x_i-\theta) = \frac{1}{\sigma^2}x$; DE= $\frac{sgn(x-\theta)}{x-\theta}$✓; Cau= $\frac{2}{1+(x-\theta)^2}$✓
Breakdown corrupt 1 (m) value in order to corrupt Mean(Median) Absolute Deviation $\frac{\sum_{i=1}^n|X_i-\bar X|}{n}$, finite breakdown point is $\frac1n(\frac mn)$

Classical

$f(x;\theta) = (\theta+1)x^\theta$, $0<x<1$, $B(\theta+1,1)$; $H_0:\theta = 0.5$, $H_1:\theta>0.5$
$\hat\theta_{mom} = \frac{2\bar X-1}{1-\bar X}$; $\hat\theta_{mle} = \frac{n}{-\sum\ln x_i}-1$; $\hat\theta_u = \frac{n-1}{-\sum\ln x_i}-1$;
CRLB $\frac{(\theta+1)^2}{n}$; $\sqrt n(\hat\theta-\theta)\sim N(0,(\theta+1)^2)$
$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = e^{(\theta_1-0.5)(-\sum\ln x_i)}\le C$ ↗MLR, T suff
$-\ln x\sim Exp(\theta+1)$, $-2(\theta+1)\sum\ln x\sim\Gamma(n,2)$, χ^2_{2n}, n = 10
$P_{H_0}(-3\sum\ln x_i\le\chi^2_{20,0.95}(10.851)) = 0.05$
P_{H_1}=1$(-4\sum\ln x_i\le\frac43*10.851) = 1-0.806$
$*f(x;\theta) = \theta(x+1)^{-\theta-1}$, $x>0$, $\theta>2$; Let $y = \ln(x+1)$, $x = e^y-1$, $\frac{dx}{dy} = e^y$, $Y = \theta e^{-\theta y}\sim Expo(\theta)$, $\Gamma(1,\frac1\theta)$;
$\hat\theta_u\frac{n-1}{\sum\ln(x+1)}$; $V\hat\theta_u\frac{\theta^2}{n-2}$; CRLB $\frac{\theta^2}{n}$ $\lim_{n\to\infty}\frac{CRLB}{V\hat\theta_U} = \lim_{n\to\infty}\frac{n-2}{n} = 1$
$\hat\theta_{mle}\frac{n}{\sum\ln(x+1)}$; $E[\hat\theta_{mle}]\frac{n\theta}{n-1}$, Bias $\frac{\theta}{n-1}$; $E[\hat\theta^2_{mle}]\frac{n^2\theta^2}{(n-1)(n-2)}$;
$V[\hat\theta_{mle}]\frac{n^2\theta^2}{(n-1)^2(n-2)}$; $MSE\frac{(n+2)\theta^2}{(n-1)(n-2)}$;
$H_0:\theta = 3$, $H_1:\theta = 4P_{H_0}(6\sum\ln(x+1)<\chi^2_{2n,\alpha}) = \alpha$ is UMP
$,\theta x^{-\theta-1}$, $x>1$
EX=$\int_1^\infty f_X = \frac{1}{\theta-1}$, $\int_1^\infty f_X = 1+\frac{1}{\theta-1} = \mu$;
$\hat\theta_{mom} = 1+\frac1{\bar X}$, $1+\frac{1}{\bar X-1} = g(\bar x)$;
VX= $\frac{\theta}{(\theta-1)^2(\theta-2)} = \sigma^2$; $\sqrt n[\bar x-\mu]\xrightarrow{D}N(0,\sigma^2)$
$g(\mu) = 1+\frac1\mu$, $1+\frac{1}{\mu-1} = \theta$; $g'(\mu) = \frac{-1}{\mu^2}$, $\frac{-1}{(\mu-1)^2} = -(\theta-1)^2\ne0$
$\sqrt n[\hat\theta_{mom}-\theta-\frac{\theta(\theta-1)}{n(\theta-2)}]\xrightarrow{D}N(0,\frac{\theta(\theta-1)^2}{(\theta-2)})$
$\hat\theta_{mom} = \frac{\bar x}{\bar x-1}\approx\frac{\mu}{\mu-1}+\frac{-1}{(\mu-1)^2}(\bar x-\mu)+\frac{1}{(\mu-1)^3}(\bar x-\mu)^2$
$E[\hat\theta_{mom}]\approx\frac{\mu}{\mu-1}+0+\frac{1}{(\mu-1)^3}\frac{\sigma^2}{n} = \theta+\frac{\theta(\theta-1)}{n(\theta-2)}$
$V[\hat\theta_{mom}]\approx V[\frac{\mu}{\mu-1}+\frac{1}{(\mu-1)^2}(\bar x-\mu)] = \frac{\theta(\theta-1)^2}{n(\theta-2)}$
$*f(x,y) = 6(y-x), 0<x<y<1, Z = \frac{X+Y}{2}, W = Y; z<w<2z$;
$|J| = 2, g(z,w) = 24(w-z), g(z) = \int_z^{2z}g(z,w)dw = 12z^2, 0<z<\frac12$;
$f_Z(z) = \int_z^1 g(z,w)dw = 12(z-1)^2, \frac12<z<1$
$*f(x,y) = c(x+2y), y\in(0,1), x\in(0,2)$;c=$\frac14$; $f_X = \frac14(x+1)$,0,o.w.

$$F = \begin{cases} 0, y\le0, x\le0 & 1, y\ge1, x\ge2 \\ \int_0^x\int_0^y dydx = \frac{x^2y}{8}+\frac{xy^2}{4} & y\in(0,1), x\in(0,2) \\ \int_0^2\int_0^y dydx = \frac{y^2}{2}+\frac{y}{2} & y\in(0,1), 2\le x \\ \int_0^x\int_0^1 dydx = \frac{x^2}{8}+\frac{x}{4} & 1\le y, x\in(0,2) \end{cases}$$

$Z = \frac{9}{(X+1)^2}$, $f_Z = \frac98z^{-2}, z\in(1,9)$,0,o.w.
$*f(y_1,y_2) = c(1-y_2), 0\le y1\le y2\le1; \int_0^1\int_0^{y_2}dy_1dy_2 = 1, c = 6$;
$f(y_1) = \int_{y_1}^1 dy_2 = 3(y_1-1)^2$; $f(y_2) = \int_0^{y_2}dy_1 = 6(y_2-y_2^2)$
$P(Y_1\le\frac12|Y_2\le\frac34) = \frac{P(Y_1\le\frac12, Y_2\le\frac34)}{P(Y_2\le3/4)} = \frac{P(Y_1\le\frac12, Y_2\le\frac12)+P(Y_1\le\frac12, \frac12\le Y_2\le\frac34)}{P(Y_2\le3/4)}$
$= \frac{\int_0^{\frac12}\int_0^{y_2}f(,)dy_1dy_2+\int_{\frac12}^{\frac34}\int_0^{\frac12}f(,)dy_1dy_2}{\int_0^{3/4}f_{Y_2}(y_2)dy_2} = \frac{25/32}{27/32} = 25/27$
$*f(y_1,y_2) = C = 2, 0\le y_1\le y_2\le1$; $\int_0^1\int_0^{y_2}$;
$f_{Y_1} = 2-2y_1$; $f_{Y_2} = 2y_2$
$E[Y_1+Y_2] = \int_0^1 y_1(2-2y_1)dy_1+\int_0^1 y_2(2y_2)dy_2 = \frac13+\frac23 = 1$
$P(Y_1\le\frac34|Y_1>\frac13) = \frac{\int_{1/3}^{3/4}}{1-\int_0^{1/3}} = \frac{55}{64}$
$*f(x;y) = Cxy, 0\le x\le2, 0\le y\le2, x+y\le2$;
$0<x<2-y<2, 0<y<2-x<2$; $\int_0^2\int_0^{2-x}$, $c = \frac32$;
$f(x) = \int_0^{2-x}f(x,y)dy = \frac34x(x-2)^2$; $f(y)$same
$P(X<Y) = P(X<Y, Y<1)+P(X<Y, Y>1) = P(X<Y<1)+P(X<2-Y, 1<Y<2) = \int_0^1\int_0^y dxdy+\int_1^2\int_0^{2-y}dxdy = \frac12$
$*f_X$=$3kxe^{-3x^2}$; k=2; m=$\sqrt{\frac{\ln2}{3}}$; f'_X=0,mode= $\frac{1}{\sqrt6}$ Y=X^2, f_Y=$3e^{-3y}$