Bernoulli$(y\|\theta)$	$=\theta^y(1-\theta)^{1-y}\mathbb{1}_{\{y\in\{0,1\}\}},\ \theta\in(0,1)$
Binomial$(y\|n,\theta)$	$=\binom{n}{y}\theta^y(1-\theta)^{n-y}\mathbb{1}_{\{y\in\{0,1,2,\ldots,n\}\}},\ \theta\in(0,1)$
Poisson$(y\|\theta)$	$=\frac{\theta^y e^{-\theta}}{y!}\mathbb{1}_{\{y\in\{0,1,2,\ldots\}\}},\ \theta>0$
Geometric$(y\|n,\theta)$	$=(1-\theta)^{y-1}\theta\mathbb{1}_{\{y\in\{1,2,\ldots\}\}},\ \theta\in(0,1)$
Neg.Binom$(y\|r,\theta)$	$=\binom{y+r-1}{y}(1-\theta)^r\theta^y\mathbb{1}_{\{y\in\{0,1,2,\ldots\}\}},\ r>0,\ \theta\in(0,1)$
Uniform$(y\|a,b)$	$=\frac{1}{b-a}\mathbb{1}_{\{y\in(a,b)\}},\ -\infty<a<b<\infty$
Beta$(y\|a,b)$	$=\frac{1}{B(a,b)}y^{a-1}(1-y)^{b-1}\mathbb{1}_{\{y\in(0,1)\}},\ a,b>0$
Exp$(y\|\theta)$	$=\theta e^{-y\theta}\mathbb{1}_{\{y>0\}},\ \theta>0$
Gamma$(y\|a,b)$	$=\frac{b^a}{\Gamma(a)}y^{a-1}e^{-yb}\mathbb{1}_{\{y>0\}},\ a,b>0$
InvGamma$(y\|a,b)$	$=\frac{b^a}{\Gamma(a)}y^{-a-1}e^{-\frac{b}{y}}\mathbb{1}_{\{y>0\}},\ a,b>0$
N$(y\|\mu,\sigma^2)$	$=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(y-\mu)^2},\ \mu,\sigma^2>0$
or	$\sqrt{\frac{\lambda}{2\pi}}e^{-\frac{\lambda}{2}(y-\mu)^2}$ where $\lambda=1/\sigma^2,\ \mu\in\mathbb{R},\lambda>0$
Pareto$(y\|a,b)$	$=\frac{a\,b^a}{y^{a+1}}\mathbb{1}_{\{y>b\}},\ a,b>0$

Marginal Likelihood $X_1,\ldots,X_n\sim$ Geom(θ). prior on $\theta\sim$Beta(a,b)

$p(x_{1:n})=\int_{\theta\in\Theta}p(x_{1:n}|\theta)p(\theta)d\theta=\int(\prod(1-\theta)^{x_i-1}\theta)\frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1}d\theta$

$=\frac{1}{B(a,b)}\int\theta^{n+a-1}(1-\theta)^{\sum x_i-n+b-1}d\theta=\frac{B(a+n,\sum x_i-n+b)}{B(a,b)}\int\frac{\theta^{n+a-1}(1-\theta)^{\sum x_i-n+b-1}}{B(a+n,\sum x_i-n+b)}d\theta$

Exponential family $p(y_{1:n}|\theta)=\exp\{t(y)\varphi(\theta)-n\kappa(\theta)\}h(y_{1:n})$

$N(\mu,\sigma^2)$, $\mu\in\mathbb{R},\sigma^2>0$ two-para

$$p(y_{1:n}|\mu,\lambda)=\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}}(\frac{\lambda}{2\pi})^{\frac{1}{2}}\exp[-\frac{\lambda}{2}(y_i-\mu)^2]$$

$$=\underbrace{(\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}})(2\pi)^{-\frac{n}{2}}}_{h(y)}\exp[\underbrace{-\frac{\lambda}{2}\sum_{i=1}^n y_i^2+n\lambda\mu\bar{y}}_{\phi(\mu,\lambda)t(y_{1:n})}-n\underbrace{\frac{\lambda\mu^2-\ln\lambda}{2}}_{\kappa(\mu)}]$$

$$p(y_{1:n}|\mu,\sigma^2)=\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}}(2\pi\sigma^2)^{-\frac{1}{2}}\exp[-\frac{1}{2\sigma^2}(y_i-\mu)^2]$$

$$=\underbrace{(\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}})(2\pi)^{-\frac{n}{2}}}_{h(y)}\exp[\underbrace{-\frac{\sum_{i=1}^n y_i^2}{2\sigma^2}+\frac{\mu\sum y}{\sigma^2}}_{\phi(\mu,\lambda)t(y_{1:n})}-n\underbrace{(\frac{\mu^2}{2\sigma^2}+\ln\lambda)}_{\kappa(\mu)}]$$

$\phi(\mu,\lambda)$=$(-\frac{\lambda}{2},n\lambda\mu)$, $(-\frac{1}{2\sigma^2},\frac{\mu}{\sigma^2})$; $t(y_{1:n})=(\sum_{i=1}^n y_i^2,\bar{y})^T$, $(\sum_{i=1}^n y_i^2,\sum y)^T$

Conjugate priors $p_{n_0,t_0}(\theta)\propto\exp\{t_0n_0\varphi(\theta)-n_0\kappa(\theta)\}$

Generating Family	Conjugate Family	Posterior Family
$Y\sim$ Bernoulli(θ)	$\theta\sim$ Beta(a,b)	$\theta\|Y=y\sim$ Beta$(a+y,b+1-y)$
$Y\sim$ Poisson(θ)	$\theta\sim$ Gamma(α,β)	$\theta\|Y=y\sim$ Gamma$(\alpha+y,\beta+1)$
$Y\sim$ Exp(θ)	$\theta\sim$ Gamma(α,β)	$\theta\|Y=y\sim$ Gamma$(\alpha+1,\beta+y)$
$Y\sim$ Unif$(0,\theta)$	$\theta\sim$ Pareto(a,b)	$\theta\|Y=y\sim$ Pareto$(a+n,\max\{b,x_{(n)}\})$

$Y_1,\ldots,Y_n|\theta\overset{iid}{\sim}$ U$(0,\theta)$; $p(y_i|\theta)=\frac{1}{\theta}\mathbb{1}_{\{0<y_i<\theta\}}$ $\vec{\theta}\sim$Pareto(a,b), $a>0$,$b>0$

Likelihood $p(y_{1:n}|\theta)=\theta^{-n}\prod\mathbb{1}_{\{0<y_i<\theta\}}=\theta^{-n}\mathbb{1}_{\{0<y_{(1)},y_{(n)}<\theta\}}$

Posterior $p(\theta|y_{1:n})\propto p(y_{1:n}|\theta)p(\theta)=\theta^{-n}\mathbb{1}_{\{0<y_{(1)},y_{(n)}<\theta\}}\theta^{-a-1}ab^a\mathbb{1}_{\{b<\theta\}}$

$\propto\theta^{-(n+a+1)}(a+n)b^{a+n}\mathbb{1}_{\{\max\{y_{(n)},b\}<\theta\}}$;$\theta|y_{1:n}\sim$Pareto$(a+n,\max\{y_{(n)},b\})$

Posterior predictive density $p(\tilde{y}|y_{1:n})=\int p(\tilde{y}|\theta)p(\theta|y_{1:n})d\theta$

Exp-Gamma$Y_1,\ldots,Y_n\overset{iid}{\sim}$ Exp(θ); prior $p(\theta)=$ Gamma$(\theta|a,b)$.

$p(\theta|y_{1:n})=Gamma(\theta|a_n,b_n)$, $a_n=a+n$ and $b_n=b+\sum_{i=1}^n y_i$

$p(\tilde{y}|y_{1:n})=\int_{\theta\in\Theta}$ Exp$(\tilde{y}|\theta)\Gamma(\theta|a_n,b_n)d\theta=\int\theta e^{-\theta\tilde{y}}\mathbb{1}_{\{\tilde{y}>0\}}\frac{b_n^{a_n}}{\Gamma(a_n)}\theta^{a_n-1}e^{-b_n\theta}\mathbb{1}_{\{\theta>0\}}d\theta$

$=\frac{a_n(b_n)^{a_n}}{(b_n+\tilde{y})^{a_n+1}}\mathbb{1}_{\{\tilde{y}>0\}}\int\frac{(b_n+\tilde{y})^{a_n+1}}{\Gamma(a_n+1)}\theta^{a_n}e^{-(b_n+\tilde{y})\theta}\mathbb{1}_{\{\theta>0\}}d\theta=\frac{a_n}{b_n+\tilde{y}}(\frac{b_n}{b_n+\tilde{y}})^{a_n}\mathbb{1}_{\{\tilde{y}>0\}}$

Pois-Gamma $p(\tilde{y}|y_{1:n})=\int_{-\infty}^{\infty}Pois(\tilde{y}|\theta)\Gamma(\theta|a_n,b_n)d\theta$ $a_n=\sum y_i+a$,$b_n=n+b$

$=\int_{-\infty}^{\infty}\mathbb{1}_{\{\tilde{y}\in\mathbb{N}_0\}}(1/\tilde{y}!)\theta^{\tilde{y}}e^{-\theta}\frac{b_n^{a_n}}{\Gamma(a_n)}\theta^{a_n-1}e^{-b_n\theta}\mathbb{1}_{\{\theta>0\}}d\theta$

$=\mathbb{1}_{\{\tilde{y}\in\mathbb{N}_0\}}\frac{b_n^{a_n}}{\tilde{y}!\Gamma(a_n)}\int_0^{\infty}\theta^{(\tilde{y}+a_n-1)}e^{-\theta(b_n+1)}d\theta$

$$=\mathbb{1}_{\{\tilde{y}\in\mathbb{N}_0\}}\frac{\Gamma(\tilde{y}+a_n)}{\tilde{y}!\Gamma(a_n)}\frac{b_n^{a_n}}{(b_n+1)^{(\tilde{y}+a_n)}}\int_0^{\infty}\frac{(b_n+1)^{(\tilde{y}+a_n)}}{\Gamma(\tilde{y}+a_n)}\theta^{(\tilde{y}+a_n-1)}e^{-\theta(b_n+1)}d\theta$$

$=\mathbb{1}_{\{\tilde{y}\in\mathbb{N}_0\}}\binom{\tilde{y}+a_n-1}{\tilde{y}}(1-\frac{1}{b_n+1})^{a_n}(\frac{1}{b_n+1})^{\tilde{y}}$;$\tilde{y}|y_{1:n}\sim$ NegBinom$(\tilde{y}|a_n,\frac{1}{b_n+1})$.

Normal-Normal$Y_1,\ldots,Y_n|\mu,\lambda\overset{iid}{\sim}N(\mu,\lambda^{-1})$,$\lambda>0$ known

$p(y_{1:n}|\mu)=\prod_{i=1}^n\left(\mathbb{1}_{\{y_i\in\mathbb{R}\}}(\frac{\lambda}{2\pi})^{\frac{1}{2}}\exp[-\frac{\lambda}{2}(y_i-\mu)^2]\right)$

$=(\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}})(\frac{\lambda}{2\pi})^{\frac{n}{2}}\exp[-\frac{\lambda}{2}\sum_{i=1}^n(y_i-\mu)^2]$

$=(\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}})(\frac{\lambda}{2\pi})^{\frac{n}{2}}\exp[-\frac{\lambda}{2}(\sum_{i=1}^n y_i^2-2\mu\sum_{i=1}^n y_i+n\mu^2)]$

$$=\underbrace{(\prod_{i=1}^n\mathbb{1}_{\{y_i\in\mathbb{R}\}})(\frac{\lambda}{2\pi})^{\frac{n}{2}}\exp[-\frac{\lambda}{2}\sum_{i=1}^n y_i^2]}_{h(y)}\exp[\underbrace{n\lambda\mu}_{\phi(\mu)}\underbrace{\bar{y}}_{t(y)}-n\underbrace{\frac{\lambda\mu^2}{2}}_{\kappa(\mu)}]$$

$p(\mu)\propto\mathbb{1}_{\{\mu\in\mathbb{R}\}}\exp[-n_0\frac{\lambda}{2}\mu^2+n_0t_0\lambda\mu]=\mathbb{1}_{\{\mu\in\mathbb{R}\}}\exp[-\frac{n_0\lambda}{2}((\mu-t_0)^2+t_0^2)]$

$=\mathbb{1}_{\{\mu\in\mathbb{R}\}}\exp[-\frac{n_0\lambda}{2}(\mu-t_0)^2]\exp[\frac{n_0\lambda}{2}t_0^2]\propto\mathbb{1}_{\{\mu\in\mathbb{R}\}}(\frac{n_0\lambda}{2\pi})^{\frac{1}{2}}\exp[-\frac{n_0\lambda}{2}(\mu-t_0)^2]$

$p(\mu)=\mathbb{1}_{\{\mu\in\mathbb{R}\}}(\frac{\nu\lambda}{2\pi})^{\frac{1}{2}}\exp[-\frac{\nu\lambda}{2}(\mu-\mu_0)^2]\sim N(\mu_0,(\nu\lambda)^{-1})$,

$n_0=\nu$, $t_0=\mu_0$, $\mu_0\in\mathbb{R},\nu>0$

$p(\mu|y_{1:n})\propto p(y_{1:n}|\mu,\lambda)p(\mu)$ priors $\boldsymbol{\mu}$ is conjugate with the normal likelihood.

$\propto\exp[-\frac{n\lambda}{2}(\mu^2-2\mu\bar{y})]\exp[-\frac{\nu\lambda}{2}(\mu^2-2\mu\mu_0)]$

$\propto\exp[-\frac{\lambda}{2}(\mu^2(n+\nu)-2\mu(n\bar{y}+\nu\mu_0))]$

$\propto\exp[-\frac{\lambda(n+\nu)}{2}\left(\mu^2-2\mu\frac{n\bar{y}+\nu\mu_0}{n+\nu}+[\frac{n\bar{y}+\nu\mu_0}{n+\nu}]^2\right)]$

$\exp[\frac{\lambda(n+\nu)}{2}(\frac{n\bar{y}+\nu\mu_0}{n+\nu})^2]\propto\exp[-\frac{\lambda(n+\nu)}{2}[\mu-(\frac{n}{n+\nu}\bar{y}+\frac{\nu}{n+\nu}\mu_0)]^2]$

$\sim N(\mu^\star,\lambda^{\star-1})$, precision $\lambda^\star=\lambda(n+\nu)$,$\mu^\star=\frac{n}{n+\nu}\bar{y}+\frac{\nu}{n+\nu}\mu_0$

$p(\tilde{y}|y_{1:n})=\int_{-\infty}^{\infty}p(\tilde{y}|\theta)p(\theta|y_{1:n})d\theta=\int_{-\infty}^{\infty}N(\tilde{y}|\theta,\lambda^{-1})N(\theta|\theta^\star,\lambda^{\star-1})d\theta$

$=(2\pi)^{-\frac{1}{2}-\frac{1}{2}}(\lambda\lambda^\star)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\{-\frac{\lambda}{2}(\tilde{y}-\theta)^2-\frac{\lambda^\star}{2}(\theta-\theta^\star)^2\}d\theta$

$=(2\pi)^{-\frac{1}{2}-\frac{1}{2}}(\lambda\lambda^\star)^{1/2}\int_{-\infty}^{\infty}\exp\{-\frac{\lambda}{2}(\theta^2-2\theta\tilde{y}+\tilde{y}^2)-\frac{\lambda^\star}{2}(\theta^2-2\theta\theta^\star+\theta^{\star2})\}d\theta$

$=(2\pi)^{-1}(\lambda\lambda^\star)^{\frac{1}{2}}\int\exp\{-\frac{1}{2}\left[\theta^2(\lambda+\lambda^\star)-2\theta(\lambda\tilde{y}+\lambda^\star\theta^\star)\right]-\frac{\lambda}{2}\tilde{y}^2-\frac{\lambda^\star}{2}\theta^{\star2}\}d\theta$

$=(2\pi)^{-\frac{1}{2}}\left(\frac{\lambda\lambda^\star}{\lambda+\lambda^\star}\right)^{\frac{1}{2}}\exp\{-\frac{\lambda}{2}\tilde{y}^2-\frac{\lambda^\star}{2}\theta^{\star2}+\frac{1}{2}\frac{1}{\lambda+\lambda^\star}(\lambda\tilde{y}+\lambda^\star\theta^\star)^2\}\times$

$\int_{-\infty}^{\infty}(2\pi)^{-\frac{1}{2}}(\lambda+\lambda^\star)^{\frac{1}{2}}\exp\{-\frac{\lambda+\lambda^\star}{2}\left(\theta-\frac{1}{\lambda+\lambda^\star}(\lambda\tilde{y}+\lambda^\star\theta^\star)\right)^2\}d\theta$

$=(2\pi)^{-\frac{1}{2}}\left(\frac{\lambda\lambda^\star}{\lambda+\lambda^\star}\right)^{\frac{1}{2}}\exp\{-\frac{1}{2}\frac{\lambda\lambda^\star}{\lambda+\lambda^\star}(\tilde{y}-\theta^\star)^2\}$

$\tilde{Y}\sim N\{\tilde{y}|\theta^\star,\lambda^{\star-1}+\lambda^{-1}\}$. $\frac{\lambda+\lambda^\star}{\lambda\lambda^\star}=\lambda^{\star-1}+\lambda^{-1}$

$\tilde{Y}|\theta,\sigma^2\sim N(\theta,\lambda^{-1})\implies\tilde{Y}=\theta+\tilde{\epsilon}$, where $\tilde{\epsilon}|\theta,\sigma^2\sim N(0,\lambda^{-1})$, for a particular value of θ.

$\theta|y_{1:n}\sim N(\theta^\star,\lambda^{\star-1})$, and denoting by $\psi=\theta|y_{1:n}$, we have that $\tilde{Y}|y_{1:n}=\psi+\tilde{\epsilon}$, which by the properties of the normal (specifically the one regarding the distribution of the sum of two independent normal random variables), is distributed as follows

$\tilde{Y}|y_{1:n}\sim N\{E(\psi)+E(\tilde{\epsilon}),var(\psi)+var(\tilde{\epsilon})\}=N(\theta^\star,\lambda^{-1}+\lambda^{\star-1})$

Normal and marginal$Y_1,\ldots,Y_n\sim N(\theta,\sigma^2)$,$\sigma^2$ is known. prior$\vec{\theta}\sim N(\mu_0,\sigma_0^2)$

When $n=1$, marginal likelihood $p(y_1)$

When $n>1$, marginal likelihood factors $p(y_{1:n})\neq p(y_1)\cdots p(y_n)$

Normal-Gamma $X_1,\ldots,X_n\overset{iid}{\sim}N(\mu,\lambda^{-1})$; $\vec{\mu},\vec{\lambda}\sim$ N-G(μ_0,ν,α,β);λ unknown

prior $p(\vec{\theta},\vec{\lambda})=N(\theta|\mu_0,(\nu\lambda)^{-1})$Gamma$(\alpha,\beta)$

$=\{\sqrt{\frac{\nu\lambda}{2\pi}}\exp[-\frac{\nu\lambda}{2}(\theta-\mu_0)^2]\}\{\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\}$

$\propto\lambda^{1/2+\alpha-1}\exp[-\frac{\lambda}{2}(\nu\theta^2-2\nu\theta\mu_0+\nu\mu_0^2+2\beta)]$

likelihood

$p(y_{1:n}|\theta,\lambda)=(\frac{\lambda}{2\pi})^{\frac{n}{2}}\prod_i\exp[-\frac{\lambda}{2}(y_i-\theta)^2]\propto\lambda^{\frac{n}{2}}\exp[-\frac{\lambda}{2}(n\theta^2-2n\theta\bar{y}+\sum_i y_i^2)]$

posterior density $p(\theta,\lambda|y_{1:n})\propto p(y_{1:n}|\theta,\lambda)p(\theta,\lambda)$

$\propto\lambda^{\frac{n}{2}}\exp[-\frac{\lambda}{2}(n\theta^2-2n\theta\bar{y}+\sum_i y_i^2)]\lambda^{\frac{1}{2}+\alpha-1}\exp[-\frac{\lambda}{2}(\nu\theta^2-2\nu\theta\mu_0+\nu\mu_0^2+2\beta)]$

$\propto\lambda^{\frac{1}{2}+\alpha+\frac{n}{2}-1}\exp[-\frac{\lambda}{2}(\theta^2(n+\nu)-2\theta(n\bar{y}+\nu\mu_0)+\sum_i y_i^2+\nu\mu_0^2+2\beta)]$

$\propto\lambda^{\frac{1}{2}}\exp[-\frac{\lambda(n+\nu)}{2}(\theta-\frac{n\bar{y}+\nu\mu_0}{n+\nu})^2]\,\lambda^{\alpha+\frac{n}{2}-1}\exp[-\frac{\lambda(\sum_i y_i^2+\nu\mu_0^2+2\beta)}{2}+\frac{\lambda(n\bar{y}+\nu\mu_0)^2}{2(n+\nu)}]$

$\propto\{\lambda^{1/2}\exp[-\frac{\lambda\nu^\star}{2}(\theta-\mu^\star)^2]\}\{\lambda^{\alpha^\star-1}\exp[-\lambda\beta^\star]\}$

$\propto N(\theta|\mu^\star,(\nu^\star\lambda)^{-1})$Gamma$(\lambda|\alpha^\star,\beta^\star)$

where $\mu^\star=\frac{n}{\nu+n}\bar{y}+\frac{\nu}{\nu+n}\mu_0$; $\nu^\star=\nu+n$; $\alpha^\star=\alpha+n/2$

$\beta^\star=\frac{1}{2}(\sum_i y_i^2+\nu\mu_0^2+2\beta-\nu^\star\mu^{\star2})=\beta+\frac{1}{2}\sum_i(y_i-\bar{y})^2+\frac{1}{2}\frac{\nu n}{\nu+n}(\bar{y}-\mu_0)^2$

$\vec{\lambda}|y_{1:n}\sim$ G$(\alpha^\star,\beta^\star)$; $\vec{\theta}|\lambda,y_{1:n}\sim N(\mu^\star,(\nu^\star\lambda)^{-1})$ or equivalently

$p(\theta,\lambda|y_{1:n})$ =G$(\lambda|\alpha^\star,\beta^\star)$ N$(\theta|\mu^\star,(\nu^\star\lambda)^{-1})$ =N-G$(\mu^\star,\nu^\star,\alpha^\star,\beta^\star)$

Normal-Inverse Gamma prior$Y_1,\ldots,Y_n\sim N(\mu,\sigma^2)$;$\vec{\mu},\vec{\sigma}^2\sim$ N-IG(μ_0,ν,α,β)

set $\mu_0=1$; $\nu=10$; $\alpha=3$;$\beta=2\times10^{-4}$

Prior $p(\vec{\mu},\vec{\sigma}^2)=N(\mu_0,\nu\sigma^2)IG(\alpha,\beta)$

Likelihood $P(y_{1:n}|\mu,\sigma^2)=$

Posterior $p(\vec{\mu},\vec{\sigma}^2|y_{1:n})\propto p(y_{1:n}|\mu,\sigma^2)p(\mu|\sigma^2)p(\sigma^2)$

Monte Carlo find Pr$(X>20)$,X simulating from $N(0,1)$does not work.

Express the probability as an integral and use an obvious change of variable to rewrite this integral as an expectation under a $U(0,1/20)$ distribution. Deduce a Monte Carlo approximation to Pr$(X>20)$ along with an error assessment. Compare the performance of this estimator to that of the direct Monte Carlo estimator.

$P(\tilde{X}<a|x_{1:n})=F_{\tilde{X}}(a\,|\,x_{1:n})=\int_{-\infty}^a p(\tilde{x}|x_{1:n})d\tilde{x}$,

by definition of the posterior predictive density, and by independence of the y's conditional on θ, we have that $p(\tilde{x}|x_{1:n})=\int_{\theta\in\Theta}p(\tilde{x}|\theta)p(\theta|x_{1:n})d\theta$;

$$
\begin{aligned}
P(\tilde{X} < a|x_{1:n}) &= \int_{-\infty}^{a} \int_{\theta\in\Theta} p(\tilde{x}|\theta)p(\theta|x_{1:n})d\theta d\tilde{x} \\
&= \int_{\theta\in\Theta} \int_{-\infty}^{a} p(\tilde{x}|\theta)p(\theta|x_{1:n})d\theta d\tilde{x} \\
&= \int_{\theta\in\Theta} \left(\int_{-\infty}^{a} p(\tilde{x}|\theta)d\tilde{x} \right) p(\theta|x_{1:n})d\theta \\
&= \int_{\theta\in\Theta} P(\tilde{X} < a|\theta)p(\theta|x_{1:n})d\theta \\
&= \int_{\theta\in\Theta} F_{\tilde{X}}(a|\theta)p(\theta|x_{1:n})d\theta \text{by def. of the CDF} \\
&= E\left[P(\tilde{X} < a|\,\theta)|x_{1:n}\right] \\
&\approx \frac{1}{S}\sum_{k=1}^{S} F_{\tilde{X}}(a|\theta_k), \quad \text{with } \theta_k \sim p(\theta|x_{1:n}).
\end{aligned}
$$

The expression above is easy to calculate given that it is easy to sample from $p(\theta|x_{1:n})$ and $P(\tilde{X} < a|\theta)$ is easy to compute
For $k = 1, \ldots, S$, sample θ_k from $p(\theta|x_{1:n})$.
Calculate $F_{\tilde{X}}(a|\theta_k)$ for each θ_k.
Estimate $F_{\tilde{X}}(a|x_{1:n})$ with the average of the $F_{\tilde{X}}(a|\theta_k)$ values.

Monte Carlo For the computation of the expectation $E[h(X)]$ when $X \sim N(0,1)$ and $h(x) = exp\{-\frac{1}{2}(x-3)^2\} + exp\{-\frac{1}{2}(x-6)^2\}$
Show that $E[h(X)]$ can be computed in closed form and derive its value.
Construct a regular Monte Carlo approximation based on a normal $N(0,1)$ sample of size $S = 10^3$ and produce an error evaluation.

Gibbs Sampling x: $p(x|a,b) = a\,b\,e^{-a\,b\,x}\mathbb{1}_{\{x>0\}}$, and suppose the prior is $p(a,b) \propto e^{-(a+b)}\mathbb{1}_{\{a,b>0\}}$ You want to sample from the posterior $p(a,b|x)$.
Derive the conditional distributions needed for implementing a Gibbs sampler, and write out the steps to implement the Gibbs Sampling algorithm.

$N(\mu,\sigma^2)$	σ^2 fixed	$\exp[\underbrace{\frac{\mu}{\sigma^2}}_{\eta(\mu)}\underbrace{x}_{T(x)}-\underbrace{(\frac{\mu^2}{2\sigma^2}+\ln(\sqrt{2\pi}\sigma))}_{B(\mu)}]\underbrace{\exp[-\frac{x^2}{2\sigma^2}]\mathbb{1}_{\{x\in\mathbb{R}\}}}_{h(x)}$
	μ fixed	$\exp[\underbrace{-\frac{1}{2\sigma^2}}_{\eta(\sigma^2)}\underbrace{(x-\mu)^2}_{T(x)}-\underbrace{\ln(\sqrt{2\pi}\sigma)}_{B(\sigma^2)}]\underbrace{\mathbb{1}_{\{x\in\mathbb{R}\}}}_{h(x)}$
$\Gamma(p,\lambda)$	p fixed	$\exp[\underbrace{-\lambda}_{\eta(\lambda)}\underbrace{x}_{T(x)}-\underbrace{-\ln(\frac{\lambda^p}{\Gamma p})}_{B(\lambda)}]\underbrace{x^{p-1}\mathbb{1}_{\{x\in(0,\infty)\}}}_{h(x)}$
	λ fixed	$\exp[\underbrace{(p-1)}_{\eta(p)}\underbrace{\ln(x)}_{T(x)}-\underbrace{-\ln(\frac{\lambda^p}{\Gamma p})}_{B(p)}]\underbrace{\exp[-\lambda x]\mathbb{1}_{\{x\in(0,\infty)\}}}_{h(x)}$
		$\exp[\underbrace{-\lambda x+(p-1)\ln x}_{\eta(p,\lambda)T(x)}-\underbrace{-\ln(\frac{\lambda^p}{\Gamma p})}_{B(p,\lambda)}]\underbrace{\mathbb{1}_{\{x\in(0,\infty)\}}}_{h(x)}$
$\beta(r,s)$	r fixed	$\exp[\underbrace{(s-1)}_{\eta(s)}\underbrace{\ln(1-x)}_{T(x)}-\underbrace{\ln(B(r,s))}_{B(s)}]\underbrace{x^{r-1}\mathbb{1}_{\{x\in(0,1)\}}}_{h(x)}$
	s fixed	$\exp[\underbrace{(r-1)}_{\eta(r)}\underbrace{\ln(x)}_{T(x)}-\underbrace{\ln(B(r,s))}_{B(r)}]\underbrace{(1-x)^{s-1}\mathbb{1}_{\{x\in(0,1)\}}}_{h(x)}$
		$\exp[\underbrace{(r-1)\ln(x)+(s-1)\ln(1-x)}_{\eta(r,s)T(x)}-\underbrace{\ln(B(r,s))}_{B(r,s)}]\underbrace{\mathbb{1}_{\{x\in(0,1)\}}}_{h(x)}$