calculous

$(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$; $(uv)' = u'v - uv'$, $\int udv = uv - \int vdu$;

$(a^x)' = a^x ln(a)$, $\int a^x \ln a = a^x$; $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$;

$(e^{ax})' = ae^{ax}$, $\int e^{au} = \frac{1}{a}e^{au}$; $(x\ln x - x)' = \ln x$, $\int \ln u = u\ln u - u$;

$(x^n)' = nx^{n-1}$, $\int x^n = \frac{x^{n+1}}{n+1}$; $(\ln x)' = \frac{1}{x}$, $\int \frac{1}{ax+b} = \frac{\ln|ax+b|}{a}$;

$\lim_{n\to\infty}(1\pm\frac{a}{n})^n = e^{\pm a}$, $\sum_{n=0}^\infty \frac{(tx)^n}{n!} = e^{tx}$;

$\Gamma(n) = (n-1)! = (n-1)\Gamma(n-1)$, $\Gamma(1/2) = \sqrt{\pi}$,

$\Gamma(-1/2) = -2\Gamma(1/2)$, $\Gamma(0) = \Gamma(-1) = \infty$;

$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$, $\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$;

$\sum_{n=0}^\infty ar^n = \frac{a}{1-r}, |r|<1$; $\sum_{k=0}^{n-1} ar^n = \frac{a(1-r^n)}{1-r}$;

Finite	Binomial
$\sum_{k=1}^n k = \frac{n(n+1)}{2}$	$\sum_{k=0}^n \binom{n}{k} = 2^n$
$\sum_{k=1}^n (2k-1) = n^2$	$\sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n}$
$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$	$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$
$\sum_{k=1}^n k^3 = (\frac{n(n+1)}{2})^2$	$\sum_{k=0}^r \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$
$\sum_{k=0}^n c^k = \frac{c^{n+1}-1}{c-1}$	$\sum_{k=0}^n \binom{n}{k}a^{n-k}b^k = (a+b)^n$

1 Basic

$P(X\le y|X\ge c) = \frac{P(X\le y, X\ge c)}{P(X\ge c)} = \frac{P(c\le X\le y)}{1-P(X\le c)} = \frac{F(y)-F(c)}{1-F(c)}$

$\frac{5*0.99}{5*0.99+995*0.02} = 0.1992$;

$Pois(3), E[30-3X-X^2] = 30-3EX-EX^2 = 9$;

$U(0,1), E[Y] = E(E[Y|X]) = \alpha + \beta EX^2 = \alpha + \beta/3$

$$\begin{cases} f_T(t) = \frac{1}{1.5}e^{-\frac{1}{1.5}t}; 0 & v<5 \\ P(V=5) = P(t<3) = \int_0^3 dt = 1-e^{-2} & 5\le v<6 \\ P(V\le v) = P(2T<v) = \int_0^{v/2} dt = 1-e^{-\frac{v}{3}} & 6\le v \end{cases}$$

$\frac{2}{9}(x+1), -1\le x\le 2, Y = X^2; P(Y\le y) = P(X^2\le y) =$

$$\begin{cases} P(1\le X\le\sqrt{y}) = \int_1^{\sqrt{y}} f(x)dx = \frac{y}{9}+\frac{2\sqrt{y}}{9}-\frac{1}{3} & x\ge 1 \\ P(-\sqrt{y}\le X\le\sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx = \frac{4\sqrt{y}}{9} & |x|\le 1 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{9}+\frac{1}{9\sqrt{y}} & 1\le y\le 4 \\ \frac{2}{9\sqrt{y}} & 0\le y\le 1 \end{cases}$$

2.3 Moment

$M_X(t) = E[e^{tX}]$

$\frac{e^t}{4-3e^t} \sim Geom(\frac{1}{4})$; $EX = \frac{1}{p} = 4$; $VarX = \frac{1-p}{p^2} = 12$; $\frac{\sigma}{\mu} = \frac{\sqrt{3}}{2}$

$Expo(1)$

$E[e^{tX_{(1)}}] = \int_0^\infty e^{tx}(ne^{-nx})dx = \frac{n}{n-t}\int_0^\infty (n-t)e^{-(n-t)x}dx$

$X = e^Z \sim LN(0,1)$; $(e^{n\mu+n^2\sigma^2/2})$ $M_Z(t) = e^{\frac{1}{2}t^2}$;

$E[X^n] = E[(e^Z)^n] = M_Z(n) = e^{\frac{1}{2}n^2}$; $E[X^{1,2,3}] = e^{\frac{1}{2}}, e^2, e^{\frac{9}{2}}$

skewness $E\frac{X-\mu^3}{\sigma^3} = (e+2)\sqrt{e-1}$

Expo

$X_1 \sim \Gamma(\alpha_1,\beta)$, $X_2 \sim \Gamma(\alpha_2,\beta)$, $U = \frac{X_1}{V=X_1+X_2} \sim Beta(\alpha_1,\alpha_2)$;

$X_1, X_2 \sim Exp(1), \Gamma(1,1), U \sim \beta(1,1), U(0,1)$; $V \sim \Gamma(2,1), xe^{-x}$

$X_1, X_2 \sim Exp(\lambda)$, $Y = X_1 - X_2 \sim DExp(0,\lambda)$, $\frac{\lambda}{2}e^{-\lambda|x|}\forall x>0$

$X \sim Beta(\theta,1)$, $-\ln X \sim Expo(\theta)$, $-2\theta\sum\ln X \sim \Gamma(n,2), \chi^2_{2n}$

DExp,Laplace

$L(\theta|x) = 2^{-n}e^{-\sum|x_i-\theta|}$; $\hat\theta_{MLE} = X_{((n+1)/2)}$ sample median

$E[X_{((n+1)/2)}] = \int_{-\infty}^\theta x\frac{1}{2}e^{x-\theta}dx + \int_\theta^\infty x\frac{1}{2}e^{-x+\theta}dx = \theta$

Uniform

$F \sim Unif(0,1), Beta(1,1)$; $-\ln F, -\ln(1-F) \sim Exp(1), Gamma(1,1)$;

$Unif(0,1)$, $U = Y-X$, $0<x<y<1$, $g(u,v) = 1$, $0<v<1-u$, $f_U(u) = 1-u$, $0<u<1$; $f_{|U|}(u) = 2(1-u)$; $E[||U||] = \frac{1}{3}$;

$U(0,1), U = Y_1+Y_2, 0<u-v<1, u-1<v<u; J = 1; f(u,v) = 1$;

$$\begin{cases} f_U(u) = 0; F_U(u) = 0 & o.w. \\ \int_0^u = u; \int_0^u = \frac{u^2}{2} & u\in(0,1) \\ \int_{u-1}^1 = 2-u; F_U(1)+\int_1^u 2u-\frac{u^2}{2}-1 & u\in(1,2) \end{cases}$$

N

$V[\bar X] = \frac{\sigma^2}{n}$; $E(Z^{2k}) = \frac{(2k)!}{2^K K!}, k = 1,2,..$; $E(Z^{2k+1}) = 0$

$V[S^2] > CRLB > V[S_n^2]$; $\frac{2\sigma^4}{n-1} \ge \frac{2\sigma^4}{n} > \frac{2(n-1)\sigma^4}{n^2}$;

$E[S] = \sqrt{\frac{2\sigma^2}{n-1}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \approx \sigma - \frac{\sigma}{4(n-1)}$;

$V[S] = \sigma^2 - \frac{2\sigma^2}{n-1}\frac{\Gamma^2(\frac{n}{2})}{\Gamma^2(\frac{n-1}{2})} \approx \frac{\sigma^2}{2(n-1)}$

$E[\hat\sigma^2] = \frac{1}{n}\sum E[(x_i-\mu)^2] = \sigma^2$, μ known;

$E[\hat\sigma^2] = \frac{E[\sum(x_i-\bar x)^2]}{n} = \frac{(n-1)E[S^2]}{n} = \frac{(n-1)\sigma^2}{n}$, μ unknown

$E[\Phi^2(z)] = V[\Phi(z)] + E[\Phi(z)]^2 = \frac{1}{12}+\frac{1}{4} = \frac{1}{3}$;

$E[n\Phi^{n-1}(z)] = \int_0^1 n\Phi^{n-1}\phi(z)dz = \int_0^1 n\Phi^{n-1}d\Phi(z) = \Phi^n(z)|_0^1 = 1$;

*$Y|\theta \sim n(\theta,1), \theta \sim n(0,1)$ $Y \sim N(0,2)$ $\theta|Y \sim N(\frac{y}{2},\frac{1}{2})$

$\frac{Z_1}{Z_2}$; $f(u) = \int_{-\infty}^\infty f(1)f(2)|v|dv = 2\int_0^\infty = \frac{1}{\pi(u^2+1)} \sim Cauch(0,1)$

*$X \sim N(\mu,\sigma^2) \perp Y \sim N(\gamma,\sigma^2)$, $U = X+Y \perp V = X-Y$

Trans and Or $M_U(t) = M_X(t)\cdot M_Y(t)$, $M_V(t) = M_X(t)\cdot M_Y(-t)$;

Or $Cov(X+Y,X-Y) = Cov(X,X) - Cov(Y,Y) = V[X]-V[Y] = 0$

$U \sim N(\mu+\gamma,2\sigma^2)$; $V \sim N(\mu-\gamma,2\sigma^2)$ by factor or linear combi

$E[U] = E[X]+E[Y] = \mu+\gamma$; $E[V], \mu-\gamma$; $V[U] = V[X]+V[Y] = 2\sigma^2$

4.4 Beta-Bino

$X|P \sim Bin(n,P)$, $P \sim Beta(\alpha,\beta)$; $EX = E[E(X|P)] = E[nP] = n\frac{\alpha}{\alpha+\beta}$;

$VX = E[Var(X|P)] + Var[E(X|P)] = E[npq] + Var[np] = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$P \sim Beta(\frac{1}{2},\frac{1}{2})$, $EX = \frac{n}{2}$; $VX = \frac{n(n+1)}{8}$;

$P \sim Beta(1,1)$, $X \sim Unif(0,n+1)$, $P|X \sim Beta(x+1,n-x+1)$

Pois-Bino

$Pois(\lambda).S_{10} = \sum_{i=1}^{10} X_i \sim Pois(10\lambda)$.

$M_{S_{10}}(t) = \prod_{i=1}^{10} M_X(t) = [e^{\lambda(e^t-1)}]^{10} = e^{10\lambda(e^t-1)}$

$S_{10} = s = S_4+S_6$, $P(S_4|S_{10}) = \frac{P(S_4,S_6=s-S_{10})}{P(S_{10})} \sim Bino(s,\frac{2}{5})$

$Pois(\lambda), Pois(3\lambda)$ $P(x|x+y=5) = \frac{P(x)P(y=5-x)}{P(x+y=5)} \sim Bino(5,\frac{1}{3})$;

4.5 Cov Corr

$Cov(X,Y) = \sigma_{XY} = E((X-\mu_X)(Y-\mu_Y)) = E[XY]-\mu_X\mu_Y$;

$Cov(aX,bY) = abCov(X,Y)$; $Cov(X,Y+Z) = Cov(X,Y)+Cov(X,Z)$;

$Cov(X,c) = 0$; $Cov(X,X) = E[X^2]-\mu_X^2$; $X\perp Y, Cov(X+Y,X-Y) = Cov(X,X)-Cov(Y,Y) = V[X]-V[Y]$

$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}, = \rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$

$Cov[\bar X,X_n] = \frac{Cov[\sum X_i,X_n]}{n} = \frac{\sigma^2}{n}$; $Cor[\bar X,X_n] = \frac{\sigma^2/n}{\sigma\sigma/\sqrt{n}} = \frac{1}{\sqrt{n}}$

4.7 Inequity;Chebychev;Cauchy-Schwarz;Jensen

$g(x)\ge 0\forall r>0, P(g(X)\ge r) \le \frac{Eg(X)}{r}$; $P(|x-\mu|<t\sigma) \ge 1-\frac{1}{t^2}$;

$\sigma = 3, t = 2$; $P(-4<x<8) \ge \frac{3}{4}$; $\sigma = 2, t = \frac{5}{2}, P(-2<x<8) = \frac{21}{25}$

$|EXY| \le E|XY| \le (E|X|^2)^{\frac{1}{2}}(E|Y|^2)^{\frac{1}{2}}$; $Cov(X,Y) \le \sigma_X^2\sigma_Y^2$;

Convex function $Eg(X)\ge g(EX)$

5.4 Order

$X_{(k)} = k\binom{n}{k}f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}$;

$X_{(1)} = nf(x)[1-F(x)]^{n-1}$; $X_{(n)} = nf(x)[F(x)]^{n-1}$

$U(0,1)$ $X_{(k)} \sim Beta(k,n-k+1)$; $X_{(1)} = n[1-x]^{n-1}$; $X_{(n)} = nx^{n-1}$

$EX_{(k)} = \frac{k}{n+1}$; $VarX_{(k)} = \frac{k(n-k+1)}{(n+1)^2(n+2)}$;

$P(n(1-Y_n)<y) = 1-P(Y_n<1-\frac{y}{n}) = 1-\int_0^{1-\frac{y}{n}} Y_n dy_n = $

$P(nY_1<y) = P(Y_1<\frac{y}{n}) = \int_0^{\frac{y}{n}} Y_1dy_1 = 1-(1-\frac{y}{n})^n$;

Or transf $W_1 = ny_1, f(w) = f(\frac{w}{n})|\frac{dy}{dw}| = n(1-\frac{w}{n})^{n-1}\frac{1}{n}$;

$W_n = n(1-Y_n), f(w_n) = f(1-\frac{w}{n})|\frac{dy}{dw}| = n(1-\frac{w}{n})^{n-1}\frac{1}{n}$

$\lim_{x\to\infty} nY_1 = \lim_{x\to\infty} n(1-Y_n) = 1-e^{-y} \sim Expo(1)$

$U(0,\theta)$: $X_{(n)} = \frac{n}{\theta}(\frac{x}{\theta})^{n-1} \sim Beta(n,1)$; $V[X_n] = \frac{n\theta^2}{(n+1)^2(n+2)}$;

$EX_{(n)} = \frac{n\theta}{n+1}$, $E[\frac{n+1}{n}X_{(n)}] = \theta$; $E[\frac{n+2}{n}X_{(n)}^2] = \theta^2$is UMVUE

$Expo(\lambda)$: $X_{(1)} = n\lambda e^{-n\lambda x} \sim Expo(n\lambda)$

5.5 Convergence

CLT $\frac{\sqrt{n}(\bar X_n-\mu)}{\sigma} \xrightarrow{\mathcal{D}} N(0,1)$; Stein $n(\theta,\sigma^2), E[g(X)(X-\theta)] = \sigma^2E[g'(X)]$;

Delta Method: if$\sqrt{n}(Y_n-\theta) \xrightarrow{D} N(0,\sigma^2)$,

$g'(\theta)\ne 0$, $\sqrt{n}[g(Y_n)-g(\theta)] \xrightarrow{D} N(0,\sigma^2[g'(\theta)]^2)$

$g'(\theta) = 0$, $\sqrt{n}[g(Y_n)-g(\theta)] \xrightarrow{D} \sigma^2\frac{g''(\theta)}{2}\chi_1^2)$

*$g(\mu) = \frac{1}{\mu}$, $E_\mu(\frac{1}{X}) \approx \frac{1}{\mu}$, $Var_\mu(\frac{1}{X}) \approx (\frac{1}{\mu})^4Var_\mu(X)$

$\sqrt{n}(\frac{1}{\bar X}-\frac{1}{\mu}) \xrightarrow{D} N(0,(\frac{1}{\mu})^4Var_\mu[X_1])$

$\widehat{Var}[\frac{1}{\bar X}] \approx (\frac{1}{\bar X})^4S^2$, $\frac{\sqrt{n}(\frac{1}{\bar X}-\frac{1}{\mu})}{(1/\bar X)^2S} \xrightarrow{D} N(0,1)$

*$Exp(\lambda)$, $\sqrt{n}(\bar X_n-\frac{1}{\lambda}) \xrightarrow{D} N(0,\frac{1}{\lambda^2})$, $g(u) = u^2$, $g'(u) = 2u \ne 0$

$\sqrt{n}(\bar X_n^2-(\frac{1}{\lambda})^2) \to n(0,\frac{4}{\lambda^4})$

*$Pois(\mu)$, $g(\bar X_n) = e^{-\bar X_n}$, $g(\mu) = e^{-\mu}$, $g'(\mu) = -e^{-\mu} \ne 0$

$\sqrt{n}(e^{-\bar X_n}-e^{-\mu}) \to n(0,\frac{[(e^{-\mu})']^2}{I(\mu)}) = n(0,\mu e^{-2\mu}))$

6.2.1 Suffi;6.2.2 Mini Suffi;6.2.3 Ancillary Stat

$\frac{\prod f(x|\theta)}{g(t|\theta)} = h(\vec x)$free of θ; $\frac{f(x|\theta)}{f(y|\theta)}$ constant fn of θ if $T(\mathbf{x}) = T(\mathbf{y})$.

$g_a(\theta) = \theta+a$, $[Y_n-\bar Y](g_a(\theta)) = Y_n+a-\frac{\sum(Y_i+a)}{n} = Y_n-\bar Y$ is invariant under location parameter$\theta, g(x|\theta) = f(x-\theta)$

6.2.4 Compl Suffi; Basu

$E[g(t)] = 0$. $\forall\lambda$ iff $P(g(t) = 0) = 1$;

$f(x|\vec\theta) = h(x)c(\vec\theta)e^{\sum_{j=1}^k W_j(\vec\theta)t_j(x)}$, t free of θ W open set in $\mathbb{R}^k$;

Expo familiy $t/n = \bar Y$ is comp/suff, indep with $Y_n-\bar Y$

7 Point Estimation

7.2.1 MoM

7.2.2 MLE

*$F(x) = (\frac{x}{\theta})^2, 0\le x<\theta$

max $L(\theta) = 2^n\theta^{-2n}\prod x_iI_{(-\infty,\theta)}(x_{(n)})I_{(0,\infty)}(x_{(1)})$

min $\hat\theta_{MLE} = X_{(n)} = nf()[F()]^{n-1} = \frac{2n}{\theta^n}x^{2n-1} \sim Beta(2n,1)$

$EX_{(n)} = \frac{2n}{2n+1}\theta$;Or $= \int_0^\theta xf_{X_{(n)}}(x)dx$

$\lim_{n\to\infty}[EX_{(n)}-\theta] = 0$; $\lim_{n\to\infty}P(X_{(n)}-\theta)\ge\varepsilon) = 0$;

$\lim_{n\to\infty}V[X_{(n)}] = \lim_{n\to\infty}\frac{n}{(2n+1)^2(n+1)}\theta^2 = 0$

7.2.3 Bayes estimator

X_n iid$Expo(\theta)$, $\pi(\theta) \sim Expo(\lambda)$

$\pi(\theta|\vec x) = \frac{g(\vec x|\theta)}{m(\vec x)} = \frac{f(\vec x|\theta)h(\theta)}{\int g(\vec x|\theta)d\theta} \propto L(\theta)h(\theta) \sim \Gamma(n+1,\frac{1}{\lambda+\sum x_i})$

$\mathcal{L}_\theta(\hat\theta) = (\hat\theta-\theta)^2$; minPosterior risk $E[\mathcal{L}_\theta(\hat\theta|\vec x)]$, Posterior mean

$\hat\theta_{L_2Bayes} = E[\pi(\theta|\vec x)] = \frac{n+1}{\lambda+\sum x_i} = \frac{\lambda(\frac{1}{\lambda},E[\theta])}{\lambda+n\bar x} + \frac{n\bar x(\frac{1}{\bar x},\hat\theta_{MLE})}{\lambda+n\bar x}$

7.3.1 MSE

$MSE = V[S_k^2]+Bias^2 = [\frac{2(n-1)}{k^2}+(\frac{n-1}{k}-1)^2]\sigma^4$;

$S_k^2 = \frac{\sum(X_i-\bar X)^2}{k}$; $\frac{\partial MSE}{\partial k} \overset{set}{=} 0, S_{n+1}^2$ min(MSE)

7.3.2 CRLB

$Var_\theta W(\vec X) \ge \frac{(\frac{d}{d\theta}E_\theta W(\vec X))^2}{nE_\theta\left[(\frac{\partial}{\partial\theta}\ln f(X|\theta))^2\right]} = \frac{[\tau'(\theta)]^2}{nI_\theta}$

*$Expo(\theta)$,median$0.5 = \int_0^{\tilde x}f()dx, \tilde x = \frac{\ln 2}{\theta}$;

90^{th} $\int_0^{g(\theta)}dx = 0.9; g(\theta) = \frac{\ln 10}{\theta}$, $VX_{(90)} \ge \frac{g'(\theta)^2}{nI_\theta} = \frac{(\ln 10)^2}{n\theta^2}$

*$N(\mu,\sigma^2)$, $\frac{X_{(95)}-\mu}{\sigma} = Z_{(95)} = 1.645$; $X_{(95)} = \mu+1.645\sigma$;

$\frac{d}{d\mu}X_{(95)} = 1$; $V[\hat\mu_{(95)}] \ge \frac{\sigma^2}{n}$

$\frac{d}{d\sigma}X_{(95)} = 1.645$; $V[\hat\sigma^2_{(95)}] \ge \frac{1.645^2\cdot 2\sigma^4}{n} > \frac{2\sigma^4}{n}$

7.3.3 UMUVE;Rao-Blackwell;Lehmann-Scheffe

T Suffi for θ, $EW = \tau(\theta)$, $\phi(T) = E[W|T]$, then $E\phi(T) = \tau(\theta)$, $V_\theta\phi(T)\le V_\theta W$.

T Compl/Suffi for θ, $E\phi(T) = \tau(\theta)$, $\phi(T)$ is UMVUE of $\tau(\theta)$

$Pois(\lambda>0), T = \sum x_i \sim Pois(n\lambda)$ Compl/Suffi;

$E[\frac{T}{n}] = \lambda$; $E[T^2-T] = VT+ET^2-ET = n^2\lambda^2$;

$\delta(x_i) = \begin{cases} 1 & X_1 = 0,1,k \\ 0 & X_1 \ne 0,1,k \end{cases}$, $\sum_{i=2}^n x_i \sim Pois((n-1)\lambda)$

$E(\delta|t) = \frac{P(x_i=0,1,k)P(\sum_{i=2}^n x_i=t,t-1,t-k)}{P(\sum_{i=1}^n x_i=t)} \sim Bino(t,\frac{1}{n})$

$= (\frac{n-1}{n})^t$, $\frac{t(n-1)^{t-1}}{n^t}$, $\frac{t!}{k!(n-k)!}\frac{(n-1)^{t-k}}{n^t}$

$Bino(n,p)$, Y Compl/Suffi $E[\frac{Y}{n}] = p$; $E[\frac{Y^2-Y}{n^2-n}] = p^2$;

$E[\frac{Y(n-Y)}{n(n-1)}] = pq$; $E[\frac{Y!(n-Y)!}{(Y-k)!(k-Y)!k!(n-k)!}] = \binom{n}{k}p^k(1-p)^{n-k}$

7.3.4 Loss function

absolute error loss $L(\theta,a) = |a-\theta|$

squared error loss $L(\theta,a) = (a-\theta)^2$

binary loss $L(\theta,a_0) = \begin{cases} 0 & \theta\in\Theta_0 \\ 1 & \theta\in\Theta_0^c \end{cases}$

8 Hypo test;Neyman-Pearson;MLR;Karlin-Rubin

Size:$\alpha = P_{H_0}(RR)$ Power:$\beta = 1-\alpha = P_{H_1}(RR)$

$H_0:\theta = \theta_0, H_1:\theta = \theta_1, \theta_0<\theta_1$, RR $\Lambda = \frac{L(\theta_0)}{L(\theta_1)} < C$ is MP

Expo family $g(t|\theta) = h(t)c(\theta)e^{W(\theta)t}$ T has MLR.

T free of θ is Suffi, $\Lambda = \frac{L(\theta_0)}{L(\theta_1)}$ has $\nearrow$ MLR in T,$T\le C$ $\Lambda\searrow$,$T\ge C$,

$P_{H_0}(RR) = \alpha$ is UMP

*$Beta(\theta,\theta)$,$H_0:\theta = 1$, $H_1:\theta = 2$ One Obs $X_1<\frac{2}{3}$.

$\alpha = P_{\theta_0}(x_1<\frac{2}{3}) = \int_0^{\frac{2}{3}}f(x|\theta_0) = \frac{2}{3}$

$\beta = 1-P_{\theta_1}(x_1<\frac{2}{3}) = 1-\int_0^{\frac{2}{3}}f(x|\theta_1) = \frac{7}{27}$

*$X_m \sim Expo(\frac{1}{\theta}_1)$,$Y_n \sim Expo(\frac{1}{\theta}_2)$ $H_0:\theta_1 = \theta_2$, $H_1:\theta_1\ne\theta_2$;

$\hat\theta_{MLE} = \frac{\sum x_i+\sum y_i}{m+n}$; $\hat\theta_1 = \frac{\sum x_i}{m}$; $\hat\theta_2 = \frac{\sum y_i}{n}$

$T = \frac{\sum x_i}{\sum x_i+\sum y_i} \sim \frac{\Gamma(m,\theta_1)}{\Gamma(m,\theta_1)+\Gamma(n,\theta_2)}$

$\Lambda = \frac{(m+n)^{m+n}}{m^mn^n}T^m(1-T)^n \propto Beta(m+1,n+1) \le C'$

*$N(\mu,25)$,$H_0:\mu = 50$,$H_1:\mu = 55$; $Z_{0.05} = 1.645$,$Z_{0.1} = 1.28$;

$P_{H_0}(\bar X_n\ge c) = 0.05$ $P_{H_1}(\bar X_n\ge c) = 0.90$

$n = [\frac{(Z_\alpha+Z_\beta)\sigma}{\delta=\mu_1-\mu_0}]^2 = [\frac{(1.645+1.28)5}{55-50}]^2 = 8.56\approx 9$

*$N(\theta,1)$,$N(3\theta,1)$;$H_0:\theta\le 0$, $H_1:\theta>0$;T:$\bar X+3\bar Y \sim n(10\theta,\frac{10}{m})$,$\Lambda\searrow$

$\sqrt{\frac{m}{10}}(\bar X+3\bar Y-10\theta) \xrightarrow{\mathcal{D}} n(0,1)$; $P_{\theta_0}(\sqrt{\frac{m}{10}}(\bar X+3\bar Y-10\theta_0>Z_\alpha) = \alpha$;

$\beta = P_{\theta_1}(\sqrt{\frac{m}{10}}(\bar X+3\bar Y-10\theta_1>Z_\alpha-(\theta_1-\theta_0)\sqrt{10m}$

*$Pois(\lambda)$ $H_0:\lambda\le 1$,$H_1:\lambda>1$ $\Lambda = (\frac{\lambda_0}{\lambda_1})^{\sum x_i}e^{n(\lambda_1-\lambda_0)} < C$

$T = \sum x_i \sim Pois(n\lambda)$; $\Lambda\searrow\alpha = P_{\theta_0}(T>C) = 0.05$

set $\lambda = 1$ max Λ, $P_{\theta_0}(T>4)<0.05$

*$\theta x^{\theta-1} \sim Beta(\theta,1)$,$-\ln x \sim Expo(\theta)$,$-2\theta\sum\ln x \sim \Gamma(n,2), \chi^2_{2n}$

$\beta = P_{\theta_1}(X>0.5) = 1-\int_0^{0.5}\theta x^{\theta-1}dx = 1-0.5^\theta$

$H_0:\theta = 1$, $H_A:\theta>1$;$\Lambda\nearrow P_{H_0}(-\ln y\le C) = \alpha$

*$e^{-(x-\theta)}$, $n = 10$; $H_0:\theta = 0$, $H_1:\theta>0$; $T = X_{(1)}\ge C$

Let$Y = X-\theta$, $Y_{(1)} \sim Expo(10), \Gamma(1,\frac{1}{10})$; $20Y_{(1)} \sim \chi_2^2$;

$P_{H_0}(X_{(1)}\ge C) = P_{H_0}(20Y_{(1)}\ge 20C) \overset{set}{=} 0.05, 20C = 5.991$;

$P_{H_1}(X_{(1)}\ge 0.2996) = P_{H_1}(Y_{(1)}\ge -0.7) = 1$, min β at H_0 is unbia

*$Expo(\frac{1}{\mu})$; $H_0:\mu = 5$,$H_1:\mu\ne 5$

$\Lambda = \frac{L(\mu_0)}{L(\mu_1)} = (\frac{\mu_1}{5})^ne^{(\frac{1}{\mu_1}-\frac{1}{5})\sum x_i} \le c$

$\mu_1<5$,$\Lambda\nearrow$,$\sum x_i\le c'$; $\mu_1>5$,$\Lambda\searrow$,$\sum x_i\ge c'$;

$T = \sum x_i \sim Gamma(10,\mu)$, $\frac{2}{\mu}T \sim \chi^2_{20}$;

$P_{H_0}(\frac{2T}{5}\le\chi^2_{20,0.975}(9.59)) + P_{H_0}(\frac{2T}{5}\ge\chi^2_{20,0.025}(34.2)) = 0.05$;

$P_{H_1}(\frac{2T}{\mu_1}\le\frac{5\chi^2_{20,0.975}}{\mu_1}) + P_{H_1}(\frac{2T}{\mu_1}\ge\frac{5\chi^2_{20,0.025}}{\mu_1})$

*$Unif(\theta,\theta+1)$,$H_0:\theta = 0$,$f(z) = f(x_1+x_2) = z, z<1; 2-z, z>1$

$\alpha = P_{H_0}(x_1+x_2>c) = P_{H_0}(x_1>0.95)$;

$\int_c^2 f(z)dz = \frac{c^2}{2}-2c+2 = \int_{0.95}^1 f(x)dx = 0.05$

$c = \frac{2}{1}\pm\sqrt{2^2-4\cdot\frac{1}{2}\cdot 1.95}$, choose $2-0.3162 = 1.6838$

10 Consistency

Asym unbiased $\lim_{n\to\infty}Bias(\hat\theta) = 0$

$\lim_{n\to\infty}Var_\theta[W_n] = 0$, $\lim_{n\to\infty}Bias_\theta[W_n] = 0$,

$\lim_{n\to\infty}P(\tau(\hat\theta_{MLE})-\tau(\theta)\ge\varepsilon) = 0$

10 Efficiency

$\lim_{n\to\infty}\frac{CRLB}{Var(\hat\theta)}$; ARE: $\lim_{n\to\infty}\frac{Var(\hat\theta_2)}{Var(\hat\theta_1)}$

Asym effi of MLE: $\sqrt{n}[\tau(\hat\theta_{MLE})-\tau(\theta)] \xrightarrow{D} N(0,\frac{1}{I_\theta})$

Classical

$f(x;\theta) = (\theta+1)x^\theta, 0<x<1, Beta(\theta+1,1)$;$H_0:\theta = 0.5$,$H_1:\theta>0.5$

$\hat\theta_{MoM} = \frac{2\bar X-1}{1-\bar X}$; $\hat\theta_{MLE} = \frac{n}{-\sum\ln x_i}-1$; $\hat\theta_U = \frac{n-1}{-\sum\ln x_i}-1$;

CRLB $\frac{(\theta+1)^2}{n}$; $\sqrt{n}(\hat\theta-\theta) \sim N(0,(\theta+1)^2)$

$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = e^{(\theta_1-0.5)(-\sum x_i)} \le C\nearrow$MLR, T suff

$-\ln x^\sim Expo(\theta+1)$, $-2(\theta+1)\sum\ln x \sim Gamma(n,2), \chi^2_{2n}$,$n = 10$

$P_{H_0}(-3\sum\ln x_i\le\chi^2_{20,0.95}(10.851)) = 0.05$

$P_{H_1=1}(-4\sum\ln x_i\le\frac{4}{3}*10.851) = 1-0.806$

***$f(x;\theta) = \theta(x+1)^{-\theta-1}$, $x>0, \theta>2$; Let $y = \ln(x+1)$, $x = e^y-1$, $\frac{dx}{dy} = e^y$, $Y = \theta e^{-\theta y} \sim Expo\theta, Gamma(1,\frac{1}{\theta})$

$\hat\theta_{MOM} = \frac{1+\bar X}{\bar X}$; $\hat\theta_{MLE} = \frac{n}{\sum\ln(x+1)}$ $\hat\theta_U = \frac{n-1}{\sum\ln(x+1)}$ $CRLB = \frac{\theta^2}{n}$

$E[\hat\theta_{MLE}] = \frac{n\theta}{n-1}$, $Bias = \frac{\theta}{n-1}$; $E[\hat\theta^2_{MLE}] = \frac{n^2\theta^2}{(n-1)(n-2)}$;

$V[\hat\theta_{MLE}] = \frac{n^2\theta^2}{(n-1)^2(n-2)}$; $MSE = \frac{n+1}{(n-1)^2}\theta^2$;

$\lim_{n\to\infty}\frac{CRLB}{V\hat\theta_U} = \lim_{n\to\infty}\frac{n-2}{n} = 1$

$H_0:\theta = 3$,$H_1:\theta = 4P_{H_0}(6\sum\ln(x+1)<\chi^2_{2n,\alpha}) = \alpha$ is UMP

$f(x,y) = 6(y-x), 0<x<y<1, Z = \frac{X+Y}{2}, W = Y; z<w<2z$;

$|J| = 2, g(z,w) = 24(w-z), g(z) = \int_z^{2z}g(z,w)dw = 12z^2, 0<z<\frac{1}{2}$;

$f_Z(z) = \int_z^1 g(z,w)dw = 12(z-1)^2, \frac{1}{2}<z<1$

*$f(y_1,y_2) = c(1-y_2), 0\le y1\le y2\le 1$; $\int_0^1\int_0^{y_2}dy_1dy_2 = 1, c = 6$;

$f(y_1) = \int_{y_1}^1 dy_2 = 3(y_1-1)^2$; $f(y_2) = \int_0^{y_2}dy_1 = 6(y_2-y_2^2)$

$P(Y_1\le\frac{1}{2}|Y_2\le\frac{3}{4}) = \frac{P(Y_1\le\frac{1}{2},Y_2\le\frac{3}{4})}{P(Y_2\le 3/4)} = \frac{P(Y_1\le\frac{1}{2},Y_2\le\frac{1}{2})+P(Y_1\le\frac{1}{2},\frac{1}{2}\le Y_2\le\frac{3}{4})}{P(Y_2\le 3/4)}$

$= \frac{\int_0^{\frac{1}{2}}\int_0^{y_2}f(,)dy_1dy_2+\int_{\frac{1}{2}}^{\frac{3}{4}}\int_0^{\frac{1}{2}}f(,)dy_1dy_2}{\int_0^{3/4}f_{Y_2}(y_2)dy_2} = \frac{25/32}{27/32} = 25/27$

*$f(y_1,y_2) = C = 2, 0\le y_1\le y_2\le 1$; $\int_0^1\int_0^{y_2}$;

$f_{Y_1} = 2-2y_1$;$f_{Y_2} = 2y_2$

$E[Y_1+Y_2] = \int_0^1 y_1(2-2y_1)dy_1+\int_0^1 y_2(2y_2)dy_2 = \frac{1}{3}+\frac{1}{2} = 1$

$P(Y_1\le\frac{3}{4}|Y_1>\frac{1}{3}) = \frac{\int_{1/3}^{3/4}}{1-\int_0^{1/3}} = \frac{55}{64}$

*$f(x;y) = Cxy, 0\le x\le 2, 0\le y\le 2, x+y\le 2$;

$0<x<2-y<2, 0<y<2-x<2$; $\int_0^2\int_0^{2-x}$, $c = \frac{3}{2}$;

$f(x) = \int_0^{2-x}f(x,y)dy = \frac{3}{4}x(x-2)^2$; $f(y)$same

$P(X<Y) = P(X<Y,Y<1)+P(X<Y,Y>1) = P(X<Y<1)+P(X<2-Y,1<Y<2) = \int_0^1\int_0^y dxdy+\int_1^2\int_0^{2-y}dxdy = \frac{1}{2}$

D	$E;EX^2;V$	$f(x);F(x),P(X\leq x)$	$MLE;T;I$	$M_x(t);M'(t);M''(t);M^n(t)$
$Bern(p)$	$p;p;pq$	$p^x q^{1-x}, x=1,0$	$\bar{X};\sum x_i \sim Bino(n,p);1/pq$	pe^t+q
$Bino(n,p)$	$np;np(np+q);npq$	$\binom{n}{x}p^x q^{n-x}$,$x=0,1..n;$	$\bar{X}$or$k\geq X_{(n)};\sum x_i \sim Bino(n,p);1/pq$	$(pe^t+q)^n$
$Geom(p)$	$1/p;(p+2q)/p^2;q/p^2$	pq^{x-1},$x=1,2,..;1-q^x$	$1/\bar{X};\sum x_i;$	$\frac{pe^t}{1-qe^t}, t < -\ln q;\frac{pqe^t}{(1-qe^t)^2};\frac{2pqe^t}{(1-qe^t)^3} - M'(t)$
$NBino(r,p)$	$r/p;;rq/p^2$	$\binom{x-1}{r-1}p^r q^{x-r}$,$x=r,r+1..$	$\bar{X};\sum x_i$	$(\frac{pe^t}{1-qe^t})^r, t<\ln q$
$HGeom(N,m,k)$	$\frac{km}{N};;\mu\frac{(N-m)(N-k)}{N(N-1)}$	$\binom{m}{x}\binom{N-m}{k-x}/\binom{N}{k}$		
$Pois(\mu)$	$\mu;\mu;\mu^2+\mu$	$\frac{\mu^x}{x!}e^{-\mu}$,$x=0,1..;e^{-\mu}\sum_{i=0}^{x}\frac{\mu^i}{i!}$	$\bar{X};\sum x_i;1/\mu$	$e^{\mu(e^t-1)};\mu e^t M(t);\mu e^t(1+\mu e^t)M(t)$
$Unif(n)$	$\frac{n+1}{2};\frac{(n+1)(2n+1)}{6};\frac{n^2-1}{12}$	$\frac{1}{n}$,$x=1,2..n$,$n=b-a+1;\frac{x-a+1}{n}$	$;X_n$Comp	$\frac{1}{n}\sum_{i=1}^{n}e^{ti},\frac{e^{at}-e^{(b+1)t}}{n(1-e^t)}$
$Unif(a,b)$	$\frac{a+b}{2};;\frac{(b-a)^2}{12}$	$\frac{1}{b-a}$,$x\in(a,b);\frac{x-a}{b-a}$	$\min x_{(1)},x_{(n)}$;R ancillary	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
$Norm(\mu,\sigma^2)$	$\mu;\mu^2+\sigma^2;\sigma^2$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$		$e^{\mu t+\frac{\sigma^2 t^2}{2}};\ (\mu+\sigma^2 t)M(t);[(\mu+\sigma^2 t)^2+\sigma^2]M(t)$
$SNorm(0,1)$	$0;1;1$	$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$		$e^{\frac{t^2}{2}}$
$LNorm(\mu,\sigma^2)$	$e^{\mu+\frac{\sigma^2}{2}};e^{2\mu+2\sigma^2};E^2X(e^{\sigma^2}-1)$	$\frac{1}{x\sigma\sqrt{2\pi}}e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}}$	$\hat{\mu}=\frac{1}{n}\sum\ln x_i$,$\hat{\sigma}^2=\frac{1}{n}\sum\ln(x_i-\hat{\mu})^2$	$EX^n=e^{n\mu+n^2\sigma^2/2}$
$Cauchy(\theta,\sigma^2)$		$\frac{1}{\pi\sigma}\frac{1}{1+(\frac{x-\theta}{\sigma})^2}$		
$DExpo(\mu,\sigma^2)$	$\mu;\mu^2+2\sigma^2;2\sigma^2$	$\frac{1}{2\sigma}e^{-\lvert\frac{x-\mu}{\sigma}\rvert}$	$\hat{\mu}=median$,$\hat{\sigma}=\frac{1}{n}\sum\lvert x_i\rvert$	$\frac{e^{\mu t}}{1-\sigma^2 t^2}$
$Expo(\beta)$	$\beta;;\beta^2$	$\frac{1}{\beta}e^{-\frac{x}{\beta}}, x\in(0,\infty);1-e^{-\lambda x}$	$1/\bar{X};\sum x_i$;mean^2	$\frac{\lambda}{\lambda-t}, t<\lambda,\frac{1}{1-\beta t};\beta(1-\beta t)^{-2};2\beta^2(1-\beta t)^{-3}$
$Gamm(\alpha,\beta)$	$\alpha\beta;;\alpha\beta^2;$	$\frac{1}{\Gamma(a)\beta^\alpha}x^{a-1}e^{-x/\beta}$	$\prod x_i,\sum x_i$	$(\frac{1}{1-\beta t})^a, t<\frac{1}{\beta};EX^n=\frac{\beta^n\Gamma(\alpha+n)}{\Gamma\alpha}$
$Beta(a,b)$	$\frac{a}{a+b};\frac{a(a+1)}{(a+b)(a+b+1)};\frac{ab}{(a+b)^2(a+b+1)}$	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$		$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$,$x\in(0,1);\frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$
χ^2_p	$p;2p+p^2;2p$	$\frac{1}{\Gamma\frac{p}{2}2^{\frac{p}{2}}}x^{\frac{p}{2}-1}e^{-\frac{x}{2}}$	$;\sum\ln x_i;$	$(1-2t)^{-p/2}, t<\frac{1}{2}$
t_p	$0, p>1;;\frac{p}{p-2}, p>2$	$\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})\sqrt{p\pi}}(1+\frac{x^2}{p})^{-\frac{p+1}{2}}\ \ F$	$\frac{q}{q-2}, q>2;;2(\frac{q}{q-2})^2\frac{p+q-2}{p(q-4)}, q>4$	$\frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}(\frac{p}{q})^{\frac{p}{2}}\frac{x^{\frac{p}{2}-1}}{(1+\frac{p}{q}x)^{\frac{p+q}{2}}}$,$x>0$
$Arcsine$	$\frac{1}{2};;\frac{1}{8}$	$\frac{1}{\pi\sqrt{x(1-x)}}$,$x\in[0,1];\frac{1}{\pi\arcsin\sqrt{x}}$		$Beta(\frac{1}{2},\frac{1}{2})$
$Dirichlet$	$\frac{a_i}{\sum_k a_k}\sum_{i=1}^{k}x_i=1;;\frac{a_i(a_0-a_i)}{a_0^2(a_0+1)}$	$\frac{1}{B(a)}\Pi_{i=1}^{k}x_i^{a_i-1}$,$x\in(0,1)$,$B(a)=\frac{\Pi_{i=1}^{k}\Gamma(a_i)}{\Gamma(\sum_{i=1}^{k}a_i)}$		$Cov(X_i,X_j)=\frac{-a_i a_j}{a_0^2(a_0+1)}$,$a_0=\sum_{i=1}^{k}a_i$
$Weibull(\gamma,\beta)$	$\beta^{1/\gamma}\Gamma(1+1/\gamma);;$ $\beta^{2/\gamma}[\Gamma(1+2/\gamma)-\Gamma^2(1+1/\gamma)]$	$\frac{\gamma}{\beta}x^{\gamma-1}e^{-\frac{x^\gamma}{\beta}};1-e^{-(\frac{x}{\beta})^\gamma}$	$x^\gamma,\sum x^\gamma,\sum\ln x$	$;;\beta^{\frac{n}{\gamma}}\Gamma(1+\frac{n}{\gamma})$
$Pareto(\alpha,\beta)$	$\frac{\beta\alpha}{\beta-1}$,$\beta>1;;\frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}$,$\beta>2$	$\frac{\beta\alpha^\beta}{x^{\beta+1}};1-(\frac{\alpha}{x})^\beta$,$x\geq\alpha$	$\hat{\alpha}=\min x_i$,$\hat{\beta}=\frac{n}{\sum\ln(x_i/x_{(1)})};$ $\sum x_i$comp/suff;$1/\beta^2$	