1 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers respectively.

Definition 1.1. Let C be an $n \times n$ real nonnegative matrix, and $u \in \mathbb{R}^n$ be a nonzero column vector. The scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of C corresponding to the *eigenvector* u, if $Cu = \lambda u$.

Definition 1.2. [2] When C is an $n \times n$ real matrix, the *spectral radius* $\rho(C)$ of C is defined by

$$\rho(C) := \max\{\ |\lambda| \ \mid \ \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ.

The next theorem is Perron – Frobenius Theorem, which provides a feature of nonnegative eigenvector to nonnegative matrices.

Theorem 1.3. *[1, 1] [3, 3] If C is nonnegative square matrix, then the spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector.*

We are interested in spectral radius of the following matrix associated with a simple graph.

Definition 1.4. Given an undirected graph G, the *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G, and

$$a_{ij} = \begin{cases} 1, \text{if } i \text{ is adjacent to } j, \\ 0, \text{otherwise.} \end{cases}$$

Definition 1.5. Given an undirected graph G, the *spectral radius* $\rho(G)$ of G is the spectral radius of the adjacency matrix of G.

We introduce a notation of submatrix, which is taken from some columns and some rows of a matrix.

Definition 1.6. For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C respectively, We use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$,

We introduce two matrices P and Q in the following Theorem, where P is a permutation matrix which is multiplied to the left side, and Q is sum of elementary matrix and certain binary matrix. In which P generalize row permutation on cases of C matrix,and Q is the transform from C' to C', which is the first n-1 columns and sum of certain columns. We aim to find C' such that C' majors C,i.e. $C \leq C'$

The following Theorem is from [1].

Theorem 1.7. *Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that* *(i)* $PCQ \leq PC'Q$;

(ii) *there exist a nonnegative column vector* $u = (u_1, u_2, \dots, u_n)^T$ *and a scalar* $\lambda' \in \mathbb{R}$ *such that* λ' *is an eigenvalue of* C' *with associated eigenvector* Qu;

(iii) *there exist a nonnegative row vector* $v^T = (v_1, v_2, \dots, v_n)$ *and a scalar* $\lambda \in \mathbb{R}$*such that* λ *is an eigenvalue of* C *with associated left eigenvector* $v^T P$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{1}$$

Proof. Multiplying the nonnegative vector u in Theorem 1.7 assumption (i), where Qu is eigenvector of C', to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{2}$$

Multiplying the nonnegative left eigenvector v^T of C for λ in assumption (iii) to the left of all terms in (2), where $v^T P$ is left eigenvector of C for λ, thus we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{3}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part. Assume that $\lambda = \lambda'$, so the inequality in (3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (3). $\square$

2 Our Method

We use $[n-1]$ as notation of the set of elements from one to $n-1$, which is $1, 2, ..., n$. Throughout fix $k \in [n-1]$.Let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn}. We will apply the previous Theorem 1.7 with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & 1 \\ & & & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}. \tag{4}$$

Definition 2.1. A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *K-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

Lemma 2.2. *If $u = (u_1, u_2, \ldots, u_n)^T$ and $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$, then*

(i) v' is K-rooted if and only if u is nonnegative;

(ii) $u_k > 0$ if and only if $v'_k > v'_n$.

Below is our first result, in which the first condition implies the first n-1 columns of C major to the columns of C', and the (k,n)-sum column of C is also major to C'. The second and the third condition suggest that C and C' have nonnegative eigenvectors which are k-rooted. And the forth condition is simplier but with the same meaning with Theorem (1.7)

Theorem 2.3. *Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ matrices. Assume that*

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ and $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ for all $1 \leq i \leq n$;

(ii) there exists a K-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';

(iii) there exists a nonnegative vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ for $1 \leq i \leq n$ with $v_i \neq 0$ and $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ for $1 \leq i \leq n,\ 1 \leq j \leq n-1, j \neq k$with $v_i \neq 0$;

(c) $c'_{ik} = c_{ik}$ for $1 \leq i \leq n$ and $v'_k > v'_n$

Proof. The proof is based on Theorem 1.7 with $P = I$ and $Q = I + E_{kn}$ in (4). The assumption (i) $PCQ \leq PC'Q$ of Theorem 1.7 holds by the condition (i) of this Theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda' Qu$ by the condition (ii) and Lemma 2.2(i). Hence the assumption (ii) of Theorem 1.7 holds. The assumptions (iii) and (iv) of Theorem 1.7 clearly hold by conditions (iii),(iv) of this Theorem since $P = I$ and $v' = Qu$ Hence $\lambda \leq \lambda'$ by the necessary condition of Theorem 1.7. Moreover $\lambda = \lambda'$ if and only if 1 holds, and this is equivalent to conditions (a),(b),(c) of this Theorem. □

We are interested in the matrices C' that have K-rooted eigenvectors. Motivated by the condition (i) of Theorem 2.3, we provide the following two definitions. This is the definition of (k,n)-sum.

Definition 2.4. For an $n \times n$ matrix $C' = (c'_{ij})$, the (k, n)-sum vector of C' is the vector of the sum of the k-th and n-th columns of C', where $k \leq n - 1$.

Note that the last column of $C'Q$ is the (k, n)-sum vector of C' Below is the definition of k-rooted matrix.

Definition 2.5. A matrix $C' = (c'_{ij})$ is called $k - rooted$ if its columns and its (k, n)-sum vector are all K-rooted except the last column of C'.

Remark 2.6.

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

The matrix $C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} & c'_{k2} & \cdots & c'_{kn-1} & c'_{kk} + c'_{kn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

The following Theorem shows that a K-rooted matrix has a K-rooted eigenvector.

Lemma 2.7. *Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(iii) hold.*

(i) C' is a K-rooted matrix, if and only if, $Q^{-1}C'Q$ is nonnegative.

(ii) Assume that C' is K-rooted and let u be a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Then C' has a K-rooted eigenvector $v' = Qu$ for $\rho(C')$.

(iii) $\rho(C') = \rho(Q^{-1}C'Q)$

Proof. (i) and $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k}+c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k}+c'_{k-1n} \\ c'_{k1}-c'_{n1} & c'_{k2}-c'_{n2} & \cdots & c'_{kn-1}-c'_{nk-1} & c'_{kk}+c'_{kn}-c'_{nk}-x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k}+c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk}+c'_{nn} \end{pmatrix}.$$

(ii) By Lemma 2.2 $v' = Qu$ is K-rooted. Since $Q^{-1}C'Qu = \rho(C')u$ by the assumption, we have
$Q^{-1}C'Qu = Q^{-1}\rho(C')Qu = \rho(C')u$
$C'Qu = \rho(C')Qu$.

(iii) Since C' and $Q^{-1}C'Q$ have the same set of eigenvalues, clearly $\rho(C') = \rho(Q^{-1}C'Q)$.

□

Lemma 2.8. *If a square matrix C' has a rooted eigenvector for λ', then $C' + dI$ also has the same rooted eigenvector for $\lambda' + d$, where d is a constant and I is the identity matrix with the same size of C'.*

Theorem 2.9. *Let C be an $n \times n$ nonnegative matrix. For $1 \le i \le n$ and $1 \le j \le n-1$, choose c'_{ij} such that $c'_{ij} \ge c_{ij}$ and $c'_{kj} \ge c'_{nj} > 0$, and choose r'_i such that $r'_i \ge c_{ik} + c_{in}$, and $r'_k \ge r'_n$. Moreover choose $c'_{in} := r'_i - c'_{ik}$. Then $\rho(C) \le \rho(C')$, when $C' = (c'_{ij})$.*

Proof. These assumptions are necessary that $PCQ \le PC'Q$ where $P = I$,$Q = I + E_{kn}$. And C' is K-rooted,based on Lemma (2.7),
$Q^{-1}C'Q =$

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k}+c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k}+c'_{k-1n} \\ c'_{k1}-c'_{n1} & c'_{k2}-c'_{n2} & \cdots & c'_{kn-1}-c'_{nk-1} & c'_{kk}+c'_{kn}-c'_{nk}-x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k}+c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk}+c'_{nn} \end{pmatrix}.$$

$Q^{-1}C'Q \ge 0$ for that $c'_{kj} \ge c'_{nj} > 0$ when $1 \le j \le n-1$, $c'_{ij} \ge c_{ij}$ where C is nonnegative, and the last column $c'_{in} + c'_{ik} = r'_i \ge c_{in} + c_{ik}$ by assumption $r'_i \ge c_{ik} + c_{in}$

For $1 \le i \le n$ and $1 \le j \le n-1$, choose c'_{ij} such that $c'_{ij} \ge c_{ij}$,
which implies $C[[n]|[n-1]] \le C'[[n]|[n-1]]$

And under the same condition for i and j, choose r'_i such that $r'_i \geq c_{ik} + c_{in}$,

which implies $c_{ik} + c_{in} \leq c'_{ik} + c'_{in} = r'_i$;

C' is K-rooted matrix, then by Lemma (2.7) Conditions (ii) and (iii), there exists a k-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';

And since C is nonnegative, by Theorem (1.3), which is Perron−Frobenius theorem, we claim there exists $v^T = w^T P$, such that v^T is nonnegative left eigenvector of C, and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;

Due to v' and v^T are nonnegative, $v^T v' > 0.$, unless they are orthogonal, i.e, $v^T v' = 0$.

Here we can summerize the facts we know so far as the following:

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ and $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ for all $1 \leq i \leq n$;

(ii) there exists a K-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';

(iii) there exists a nonnegative vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;

(iv) $v^T v' > 0$.

which come from Theorem (2.3).

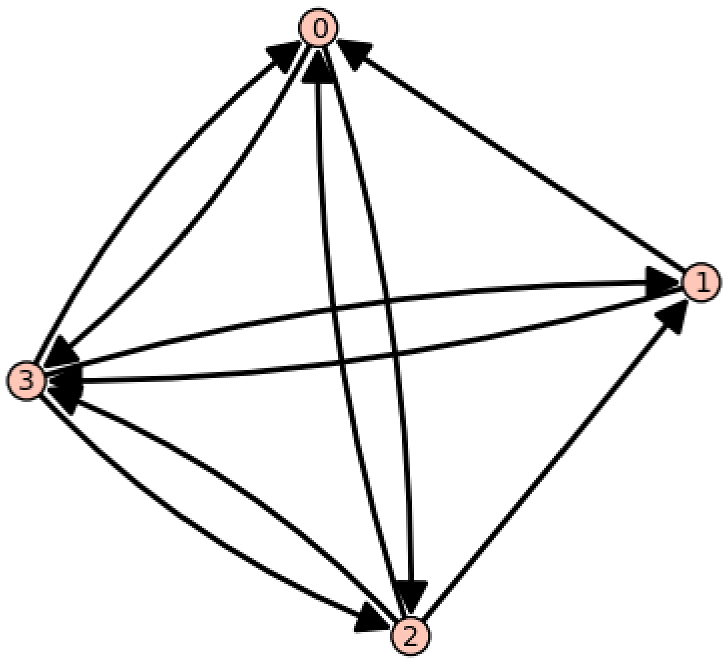
By Lemma (2.8), for certain d, if $C' + dI$ is K-rooted, then it has a K-rooted eigenvector with its spectral radius $\lambda + d$. C' would share the same eigenvector with $C' + dI$ and has eigenvalue λ. So $C' + dI$ and $C + dI$ meet the conditions of Theorem (2.3), and we can show that $\rho(C' + dI) \geq \rho(C + dI)$ and then $\rho(C') \geq \rho(C)$ $\square$

2.1 Example

For the following 4×4 matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

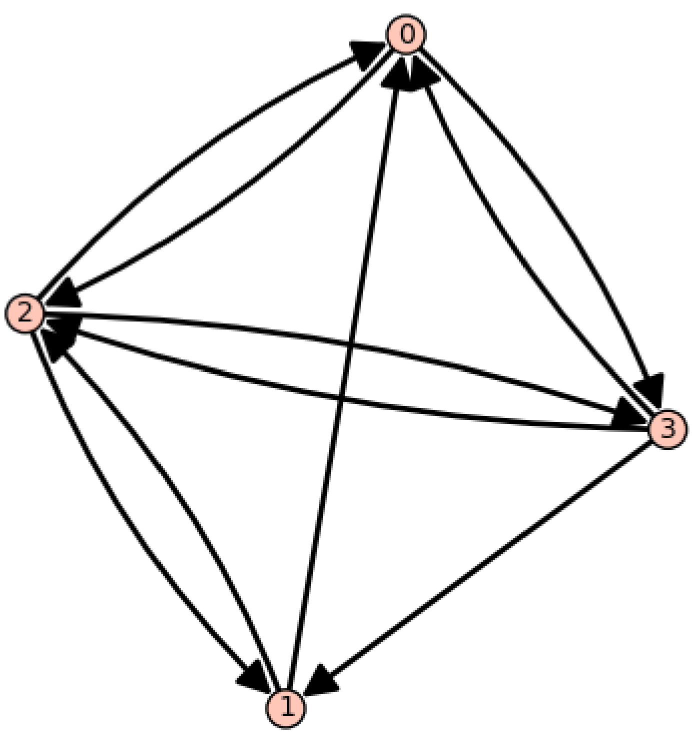
And its corresponding graph,[4, sage]



we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

also, graph,[4, sage]



Check the conditions $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ of Theorem 2.3, Then $\rho(C) \leq \rho(C')$ by previous Theorem 2.9.

2.2 Counterexample

For the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

specify n=4, k=3 in $Q = I + E_{kn} = I + E_{34}$

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

we have $CQ \leq C'Q$, but $\rho(C) = 2.234 \nleq 2.148 = \rho(C')$. This is because $c'_{33} + c'_{34} \ngeq c'_{43} + c'_{44}$.

References

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