Planar Graph and plane graph

Definition 0.1

1. A graph is **planar** if it has a drawing in $\mathbb{R}^2$ without crossings.
2. A **plane graph** is a particular drawing of a planar graph in $\mathbb{R}^2$.

Definition 0.2

Let G be a plane graph. Then $\mathbb{R}^2 - G$ is a finite disjoint union of "maximal connected open sets". Each of these maximal connected open sets is called a **face** of G.

Theorem (Euler's Formula)

Let G be a connected plane graph with n vertices, e edges and f faces. Then $n - e + f = 2$.

Proof.

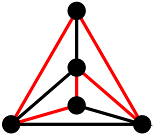
Induction on n. For $n = 1$, we have $f = e + 1$, so $n - e + f = 2$. In general suppose $n > 1$. Since G is connected, we can find an edge e which is not a loop. By **contracting** e to a point, we obtain a new graph G/e with $n' = n - 1$ vertices, $e' = e - 1$ edges and $f' = f$ faces. By induction, we have $n' - e' + f' = 2$. Hence $n - e + f = 2$. □

Remark 0.3

Without the above theorem, it is not obvious that the number f of faces in a planar graph is well-defined.

Hamiltonian graphs

A graph is **hamiltonian** if it contains a spanning cycle. A spanning path of a graph is called a **hamiltonian path**.

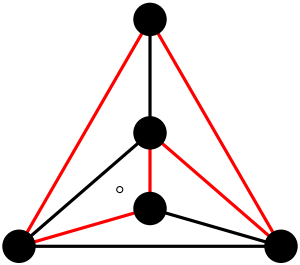


The red edges form a spanning cycle of the above graph

Problem

Can every face of a planar graph be drawn as an edge of an outer face.

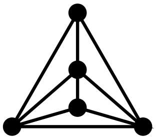
Answer: Yes



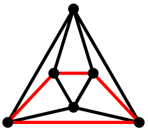
Plane is homeomorphic to the surface obtained from a sphere deleting a point.

Maximal planar graphs and chordal graphs

1. A planar graph G is **maximal** if for every pair u, v of non-adjacent vertices of G the graph $G + uv$ is non-planar.
2. A graph is **chordal** if every cycle of length at least 4 contains a **chord**, i.e., an edge connecting two non-adjacent vertices of the cycle.



A maximal planar and chordal graph.

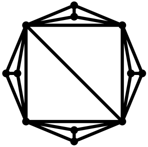


A maximal planar graph which is not a chordal graph.

(There is no chord in the red cycle of the above maximal planar graph.)

Outerplanar graphs and plane triangulations

1. An **outerplanar graph** is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing.
2. An outerplanar graph **maximal** if its interior faces in a planar drawing are all triangles.
3. A **plane triangulation** is a plane graph whose boundary of every face is a triangle except the outer face.



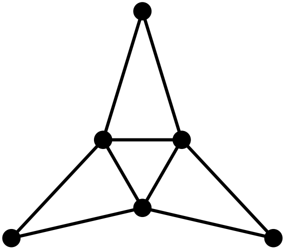
A plane triangulation.

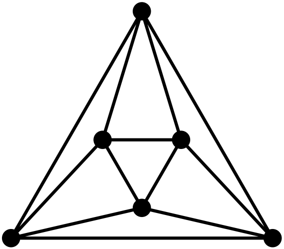
Is the above graph a maximal outerplanar graph?

Remark 0.4

A maximal outerplanar graph has a drawing to become a plane triangulation.

A maximal outerplanar graph and a maximal planar graph





Lemma

A maximal planar graph of order n has $3n-6$ edges and $2n-4$ faces.

Proof.

Let the maximal planar graph graph have e edges and f faces. Then by Euler's formula, $n-e+f=2$ and $3f=2e$. Solve the two equations for e, f in terms of n to have the results. □

Remark 0.5

In every drawing of a maximal planar graph, all faces (including the outer face) are triangles.

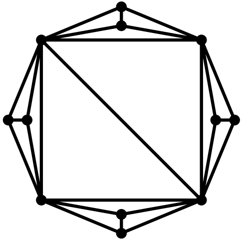
Lemma

A maximal outerplanar graph of order n contains $n-2$ interior faces and $2n-3$ edges.

Proof.

We prove by induction on n and $n=3$ is clear. In general assume $1, 2, \ldots, n$ is the vertex sequence in clockwise order along the boundary of the outer face of a drawing and assume $12i$ is a triangle face for some $i \geq 3$. Then the edge $1i$ divides the outer face in to two faces with cycles $12\cdots i1$ and $1i\cdots n1$ of lengths i and $n-i+2$ respectively as boundaries. By counting the triangle faces and edges inside these two cycles plus one more edge $1i$ and n edges in the boundary of outerface, we find
$(i-2)+(n-i+2-2)=n-2$ faces and
$(i-3)+(n-i+2-3)+1_n=2n-3$ edges. □

A plane triangulation which is not maximal outerplanar



The above planar triangulation has order 12 with 25 edges $(25 \neq 2 \cdot 12 - 3)$.

Lemma

Let G be a plane triangulation with boundary C of outer face. Then there is an interior face uvw with either the edge uv in C and vertex w not in C or uv, vw in C.

Proof.

If the interior face uvw of every edge uv in C has the third vertex w always in C, the cycle C is divided into two smaller cycles with common edge wu. Applying this process repeatedly in either one of the smaller cycles, we will find a face containing three consecutive vertices of C. □

Lemma

Let G be a plane triangulation with external cycle C of length ℓ. Then G contains cycles of every length i for $3 \leq i \leq \ell$.

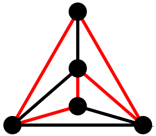
Proof.

We prove this by induction on the number of interior faces in G. If there is only one face then $\ell = 3$ and C is a cycle of length 3 . Assume $\ell > 3$. There are two cases.
Case 1. There is a triangle uvw with uv in C and w not in C. Then the boundary of outer face of $G - uv$ is the cycle $C - uv + \{uw, wv\}$ of length $\ell + 1$ and the number of interior faces is one less than that of C. By induction, $G - uv$ contains cycles of every length i for $3 \leq i \leq \ell + 1$.
Case 2. There are three consecutive vertices uvw in C that form an interior face. Then the boundary of outer face of $G - v$ is the cycle $C - \{uw, wv\} + uw$ of length $\ell - 1$ and the number of interior faces is one less than that of C. By induction, $G - v$ contains cycles of every length i for $3 \leq i \leq \ell - 1$.

□

weakly pancyclic graph

A graph of order n is **pancyclic** if it contains cycles of every length ℓ for $3 \leq \ell \leq n$. A **weakly pancyclic graph** is a graph which contains cycles of every length between the girth and the circumference.



Proposition

Every plane triangulation G of order n is weakly pancyclic.

Proof.

Let C be a cycle of maximum length ℓ in G. Since the subgraph induced on C with its interior vertices is a planar triangulation, the proposition follows from previous lemma. □

S. L. Hakimi and E. F. Schmeichel, On the number of cycles of length k in a maximal planar graph. J. Graph Theory 3 (1979), 69-86.

Lemma

Let G be a maximal planar graph with a vertex x of degree 3. Then

(i) **the subgraph induced on $G_1[x]$ is the complete graph K_4;**

(ii) **G is hamiltonian iff $G - x$ has a hamiltonian cycle that uses at least one edge in $G_1(x)$.**

Proof.

(i) is clear from the planar construction of graphs. Let $a, b \in G_1(x)$ be distinct. Then the switching of paths axb and ab makes corresponding in hamiltonian cycles of G and $G - x$ respectively. This prove (ii). □

More definitions

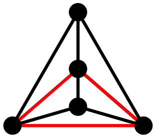
1. A graph is ***t*-tough** if for every integer $k > 1$, the graph cannot be split into k components by removal of fewer than tk vertices.
2. The **toughness** a non-complete graph G is the maximum t such that G is t-tough. The toughness of a complete graph is ∞.

Remark 0.6

A t-tough graph with $t > 0$ is connected and a Hamiltonian graph is 1-tough.

Separating triangle

A triangle c in a maximal planar graph T is a **separating triangle** if $T - C$ is disconnected.



Remark 0.7

A maximal planar graph with a separating triangle has toughness at most $3/2$.

A long standing conjecture

There is a positive constant t such that every t-tough graph is Hamiltonian.

V. Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 5 (1973), 215-228.

Known result

A chordal planar graph with toughness more than 1 is Hamiltonian.

J.C. Bermond, Hamiltonian graphs, in: L. Beinecke, R.J. Wilson (Eds.), Selected Topics in Graph Theory, Academic Press, London, New York, 1978, pp. 127–167.

Connectivity

Definition 0.8

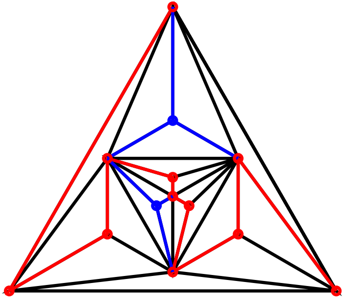
1. The **connectivity** of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex.
2. A graph is **t-connected** if it has connectivity at least t.

Remark 0.9

1. Deleting $t - 1$ vertices from a t-connected graph yields a connected graph.
2. A t-tough graph is $2t$-connected.

Tutte path in a plane graph

A **Tutte path** of a plane graph G with external cycle C is a path P of G such that each component of $G - P$ has at most three neighbors in P, and has at most two neighbors in P if it contains an edge of C.



Known result

Let G be a 2-connected plane graph with external cycle C. For any $v \in V(C)$, $e \in E(C)$ and $u \in V(G - v)$, there is a Tutte path from v to u using e.

C. Thomassen. A theorem on paths in planar graphs. *Journal of Graph Theory*, 7(2):169–176, 1983.

Known result

A 4-connected plane graph is Hamiltonian.

Proof.

Let *vu* and *e* be distinct edges in external cycle of 4-connected plane graph *G*. Let *P* be a Tutte path from *v* to *u* using *e*. Since *G* is 4-connected, *P* is a hamiltonian path. Adding the edge *vu* to *P*, we have a hamiltonian cycle. □

W.T. Tutte, A theorem on planar graphs. *Trans. Amer. Math. Soc.* 82 (1956), 99-116.