國立交通大學

應用數學系

博士論文

二分圖的譜半徑

Spectral Radius of a Bipartite Graph

博 士 生：鄭硯仁

指導教授：翁志文 教授

中華民國一百零七年七月

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摘要

一個方陣的譜半徑是指此方陣其特徵值最大的長度，而一個圖的譜半徑則是指此圖的鄰接矩陣的譜半徑。令 G 是個有 e 條邊且沒有孤立點的二分圖，已知 G 的譜半徑最多是 $\sqrt{e}$，而且達到此上界時若且唯若 G 是個完全二分圖，我們的第一個結果是延伸此結果當 $(e-1, e+1)$ 不是一組雙生質數時，能去找到有 e 個邊，不是完全二分圖且擁有最大譜半徑的圖。

Bhattacharya、Friedland 和 Peled 猜想當一個給定邊數 e 和兩部份的點數 (p, q) 且不是完全二分圖的二分圖，其能達到最大的譜半徑時，這個圖一定是完全二分圖再加上一個點和相應的邊數，我們找到了這個猜想的反例，並且猜測當我們把〝不是完全二分圖〞這個前提拿掉時，這個個較弱的猜想是對的。令 $p \leq q$，在 $e \geq pq - q$ 和 $p \leq 5$ 時，我們證明了這個較弱的猜想。

為了處理以上的問題，我們討論圖 G 的鄰接矩陣的平方 C。更一般性地，對於一個非負矩陣 C，藉由調整一個列中的值但使得此列的總和維持不變，我們發展出一套方法來找到一個非負矩陣的譜半徑的上下界，這個方法能讓我們很容易地得到許多已知的或新的上下界。

關鍵字：二分圖，鄰接矩陣，非負矩陣，譜半徑

Spectral Radius of a Bipartite Graph

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Abstract

The spectral radius of a square matrix C is the largest magnitude of an eigenvalue of C and the spectral radius of a graph G is the spectral radius of the adjacency matrix of G. Let G be a bipartite graph with e edges without isolated vertices. It was known that the spectral radius of G is at most the square root of e, and the upper bound is attained if and only if G is a complete bipartite graph. Our first result extends this result to find the maximum spectral radius of a non-complete bipartite graph with e edges under the assumption that $(e-1, e+1)$ is not a pair of twin primes.

Bhattacharya, Friedland and Peled conjectured that a non-complete bipartite graph which has the maximum spectral radius with given e and bi-order (p, q) is obtained from a complete bipartite graph by deleting edges incident to a common vertex. We find counter examples of this conjecture. Under the additional assumption $e \geq pq - q$ or under the assumption $p \leq 5$, where $p \leq q$, we prove a weaker version of the above conjecture that drops the non-complete assumption of the bipartite graph.

To handle the problem above, we study the spectral radius of a nonnegative matrix C which takes the square of the adjacency matrix of G as a special case.

For a general nonnegative matrix C, we give a new approach to obtain lower bounds and upper bounds of the spectral radius of C which are the spectral radii of matrices obtained by suitably reweighting the entries in a row of C keeping the row-sum unchanged. This method helps us to find many spectral bounds of C easily.

Keywords: bipartite graph, adjacency matrix, nonnegative matrix, spectral radius.

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Chapter 1

Introduction

For a square matrix C, the *spectral radius* of C is

$$\rho(C) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ. Given an undirected graph G, the *adjacency matrix* of G is the square matrix A indexed by its vertices, and

$$A_{ij} = \begin{cases} 1, & \text{if } i \text{ is adjacent to } j; \\ 0, & \text{otherwise.} \end{cases}$$

The *spectral radius* of G is the spectral radius of its adjacency matrix A.

1.1 Spectral radius of a bipartite graph

The problem of finding the maximum spectral radius of a graph with e edges was initially posed by Brualdi and Hoffman [1, p.438] in 1976. They later gave the following conjecture [5],

Conjecture 1.1.1 (BH Conjecture)**.** *The maximum spectral radius of a graph with e edges is attained by taking a complete graph and adding a new vertex which is adjacent to a corresponding number of vertices in the complete graph, probably together with some isolated vertices.*

The above conjecture was proved by Rowlinson [22] in 1988. In 2008, Bhattacharya, Friedland and Peled [2] proved that if G is a bipartite graph with e edges, then $\rho(G) \leq \sqrt{e}$ with equality if and only if G is a complete bipartite graph, possibly together with some isolated vertices. Moreover, they gave a conjecture as a bipartite graphs analogue of the BH conjecture.

The following comes from [2].

"We assume here the normalization $1 \leq p \leq q$. Let e be a positive integer satisfying $e \leq pq$. Denote by $\mathcal{K}(p, q, e)$ the family of subgraphs of $K_{p,q}$ with e edges and with no isolated vertices and which are not complete bipartite graphs.

Problem A. Let $2 \leq p \leq q$, $1 < e < pq$ be integers. Characterize the graphs which solve the maximal problem

$$\max_{G \in \mathcal{K}(p,q,e)} \lambda_{\max}(G). \tag{*}$$

We conjecture below an analog of the Brualdi-Hoffman conjecture for non-bipartite graphs, which was ....

Conjecture B. Under the assumptions of Problem A an extremal graph that solves the maximal problem (*) is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges."

Let $\mathcal{K}_0(p, q, e)$ denote the subset of $\mathcal{K}(p, q, e)$ such that each graph in the subset is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges. Note that $\mathcal{K}_0(p, q, e)$ is also the subset of $\mathcal{K}(p, q, e)$ such that each graph in the subset is obtained from a complete bipartite graph by deleting edges incident on a common vertex. Noting that $\mathcal{K}_0(p, p, p^2 - p) = \emptyset$, We restate Conjecture B with the additional assumption that $\mathcal{K}_0(p, q, e) \neq \emptyset$,:

Conjecture 1.1.2 (BFP Conjecture for $\mathcal{K}(p, q, e)$)**.** *If $G \in \mathcal{K}(p, q, e)$ such that $\rho(G) = \max_{H \in \mathcal{K}(p,q,e)} \rho(H)$ and $\mathcal{K}_0(p, q, e) \neq \emptyset$, then $G \in \mathcal{K}_0(p, q, e)$.*

In paper [2], Bhattacharya, Friedland and Peled also proved BFP Conjecture for $\mathcal{K}(p, q, e)$ with $e = st - 1$ and $2 \leq s \leq p \leq t \leq q \leq t + (t - 1)/(s - 1)$. As stated clearly above, the complete graphs are excluded in their consideration of BFP Conjecture for $\mathcal{K}(p, q, e)$. One might observe from the special case $\mathcal{K}(5,10,35)$ of their result to know that they exclude the 35-edge subgraph $K_{5,7}$ of $K_{5,10}$ in consideration and choose the 35-edge subgraph of $K_{5,10}$ obtained from $K_{4,9}$ by deleting an edge.

In Chapter 3, we will extend the above result $\rho(G) \le \sqrt{e}$ of [2] and determine the e-edge bipartite graphs G with

$$\sqrt{\frac{e+\sqrt{e^2-4(e-1-\sqrt{e-1})}}{2}} \le \rho(G) < \sqrt{e}.$$

As a byproduct, we prove BFP Conjecture for $\mathcal{K}(p,q,e)$ when $e \in \{st-1, st'+1 \mid s \le p, t \le q, t' \le q-1\}$ in Theorem 3.6.1.

In 2010, Chen, Fu, Kim, Stehr and Watts [6] proved BFP Conjecture for $\mathcal{K}(p,q,pq-2)$. In 2015, Liu and Weng [19] proved BFP Conjecture for $\mathcal{K}(p,q,e)$ under assumption $pq-p < e < pq$. Note that if $pq-p < e < pq$, there are no e-edge complete subgraphs of $K_{p,q}$. In Proposition 3.6.2, we will provide a class of counter examples that disproves BFP Conjecture for $\mathcal{K}(p,q,e)$ with $e = p(q-1)$, $p \ge 3$ and $q > p+2$.

Since complete graphs are considered in BH Conjecture, one might expect that a bipartite graphs analogue of the BH conjecture also includes complete bipartite graphs in considering. For $1 \le e \le pq$, let $\mathcal{C}(p,q,e)$ be the class of all subgraphs of $K_{p,q}$ with e edges and no isolated vertices (we do not assume $p \le q$), and $\mathcal{C}_0(p,q,e)$ be the subset of $\mathcal{C}(p,q,e)$ such that each graph in the subset is a complete bipartite graph or a graph obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges. The following is a weaker version of BFP Conjecture.

Conjecture 1.1.3 (Weak BFP Conjecture for $\mathcal{C}(p,q,e)$.)**.** *If $G \in \mathcal{C}(p,q,e)$ such that $\rho(G) = \max_{H\in\mathcal{C}(p,q,e)} \rho(H)$, then $G \in \mathcal{C}_0(p,q,e)$.*

Note that $\mathcal{C}_0(p,q,e) \ne \emptyset$ if $e \le pq$. Indeed, since $e = sq + r$ for some nonnegative integers s, r such that $s \le p, r < q$, the graph in $\mathcal{K}(p,q,e)$ obtained from $K_{s,q}$ by adding a vertex in the part of q vertices and a corresponding number of edges is in $\mathcal{K}_0(p,q,e)$. If BFP Conjecture for $\mathcal{K}(p,q,e)$ holds, then weak BFP Conjecture for $\mathcal{C}(p,q,e)$ holds with the same p, q, e. We prove weak

BFP Conjecture for $\mathcal{C}(p, q, e)$ when $p \leq 5$ in Theorem 3.7.1 and Theorem 5.2.4, and when $e \geq pq - q$ in Theorem 5.2.3.

1.2 The spectral radius of a nonnegative matrix

Although we are mainly interested in binary symmetric matrices, our results are extended to nonnegative matrices, not necessary symmetric. In Chapter 4, we give a systematic method to get upper bounds and lower bounds of the spectral radius of a nonnegative matrix. This method will be used in Chapter 5.

For real matrices $C = (c_{ij})$, $C' = (c'_{ij})$ of the same size, if $c_{ij} \leq c'_{ij}$ for all i,j, then we say $C \leq C'$. A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *rooted* if $v'_j \geq v'_n \geq 0$ for $1 \leq j \leq n-1$. For an $n \times n$ matrix C', we use $C'[-|n)$ to denote the first $n \times (n-1)$ submatrix of C', and C' is called *rooted* if the columns in $C'[-|n)$ and the row-sum vector of C' are all rooted. The main theorem of Chapter 4 is the following.

Theorem C. Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix and $C' = (c_{ij})$ be an $n \times n$ rooted matrix. Assume that

(i) $C[-|n) \leq C'[-|n)$ and $(r_1, r_2, \ldots, r_n)^T \leq (r'_1, r'_2, \ldots, r'_n)^T$, where $(r_1, r_2, \ldots, r_n)^T$ is the row-sum vector of C and $(r'_1, r'_2, \ldots, r'_n)^T$ is the row-sum vector of C';

(ii) C' has a positive rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ for λ' for some $\lambda' \in \mathbb{R}$;

Then the spectral radius $\rho(C')$ of C' is an upper bound of $\rho(C)$. $\square$

We use Theorem C to extend a classical result of Richard Stanley in 1987 [25] for a symmetric $(0, 1)$-matrix to the following theorem for a general nonnegative matrix, which is also appeared in Theorem 4.5.2 of Chapter 4.

Theorem D. Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix. Let m be the sum of entries and d (resp. f) be any number which is larger than or equal to the largest diagonal element (resp. the largest off-diagonal element) of M. Then

$$\rho(C) \leq \frac{d-f+\sqrt{(d-f)^2+4mf}}{2}.$$

Moreover, if $mf > 0$ then the above equality holds if and only if $m = k(k-1)f + kd$ and PCP^T has the form

$$\begin{pmatrix} fJ_k + (d-f)I_k & 0 \\ 0 & O_{n-k} \end{pmatrix} = (fJ_k + (d-f)I_k) \oplus O_{n-k}$$

for some permutation matrix P and some nonnegative integer k. $\square$

We apply the dual version of Theorem C to prove the following theorem, appeared in Corollary 4.8.2.

Theorem E. Let $C = (c_{ij})$ be an $n \times n$ $(0,1)$ matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n > 0$, and choose $t \geq n - r_n + 1$ and $t \leq n$. Then

$$\rho(C) \geq \frac{r_t + \sqrt{r_t^2 - 4(n-t)(r_t - r_n)}}{2}.$$

Moreover, if C is irreducible, then the equality holds if and only if $r_1 = r_n$ or

(a) $r_1 = r_t$ and $r_{t+1} = r_n$, and

(b) $\sum_{j\in[t]} c_{ij} = r_t - (n-t)$ for all $i \in [t]$, and
$\sum_{j\in[t]} c_{ij} = r_n - (n-t)$ for all $t < i \leq n$, where $[t] := \{1, 2, \ldots, t\}$.

$\square$

For a decreasing sequence $D = (d_1, d_2, \ldots, d_p)$ of positive integers, set

$$r_i = \sum_{k=1}^{p} d_k + (i-1)d_i - \sum_{k=1}^{i-1} d_k.$$

The following is an application of Theorem C, appeared in Theorem 5.1.1.

Theorem F. Let G be a bipartite graph and $D = (d_1, d_2, \ldots, d_p)$ be the degree sequence of one part of G in decreasing order. Then for $1 \leq \ell \leq p$,

$$\rho(G) \leq \sqrt{\frac{r_1 + \sqrt{(2r_\ell - r_1)^2 + 4d_\ell \sum_{i=1}^{\ell}(r_i - r_\ell)}}{2}},$$

with equality if and only if $G = G_D$, $d_1 = d_t$ and $d_{t+1} = d_p$ for some $1 \leq t \leq \ell - 1$, where G_D is the bipartite graph with degree sequence D of one part. $\square$

Chapter 2

Preliminaries

In this chapter, we shall provide the notations and properties of matrices and graphs which will be adopted in this thesis.

2.1 Matrices

In this section, we introduce matrix notations.

2.1.1 Submatrices

For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices respectively, we use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$, $C[\alpha|\beta) := C[\alpha|\overline{\beta}]$, where $\overline{\beta}$ is the complement of β in the set of column indices, and similarly, for the definitions of $C(\alpha|\beta]$ and $C(\alpha|\beta)$. For $\ell \in \mathbb{N}$, $[\ell] := \{1, 2, \ldots, \ell\}$, symbol $-$ is the complete set of indices, and we use i to denote the singleton subset $\{i\}$ to reduce the double use of parentheses. For example of the $n \times n$ matrix C, $C[-|n) = C[[n]|[n-1]]$ is the $n \times (n-1)$ submatrix of C obtained by deleting the last column of C. The following are more examples. Let

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

Then

$$C[-|[3]] = C[-|4) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, C'[-|4) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$C[-|4] = (1,1,0,0)^T$, $C'[-|4] = (1,0,-1,-1)^T$, $C[4|-] = (1\ 1\ 0\ 0)$ and $C'[4|4) = (1\ 1\ 1)$, where M^T denotes the transpose of matrix M.

2.1.2 Perron-Frobenius theorem

The following is the famous Perron-Frobenius theorem, which plays an important role in Chapter 4.

Theorem 2.1.1 ([3, Theorem 2.2.1], [16, Corollary 8.1.29, Theorem 8.3.2]). *If C is a nonnegative square matrix, then the following (i)-(iii) hold.*

(i) The spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector.

(ii) If there exists a column vector $v > 0$ and a nonnegative number λ such that $Cv \le \lambda v$, then $\rho(C) \le \lambda$.

(iii) If there exists a column vector $v \ge 0$, $v \ne 0$ and a nonnegative number λ such that $Cv \ge \lambda v$, then $\rho(C) \ge \lambda$.

Moreover, if in addition C is irreducible, then the eigenvalue $\rho(C)$ in (i) has algebraic multiplicity 1 *and its corresponding left eigenvector and right eigenvector can be chosen to be positive, and any nonnegative left or right eigenvector of C only corresponds to the eigenvalue $\rho(C)$.* □

Unless specified otherwise, by an eigenvector we mean a right eigenvector. The nonnegative eigenvectors in (i) are called *Perron eigenvectors*. The following two lemmas are well-known consequences of Theorem 2.1.1. We shall provide their proofs since they motivate our proofs of results.

Lemma 2.1.2 ([3, Theorem 2.2.1])**.** *If $0 \le C \le C'$ are square matrices, then $\rho(C) \le \rho(C')$. Moreover, if C' is irreducible, then $\rho(C') = \rho(C)$ if and only if $C' = C$.*

Proof. Let v be a nonnegative eigenvector of C for $\rho(C)$. From the assumption, $C'v \ge Cv = \rho(C)v$. By Theorem 2.1.1(iii) with $(C, \lambda) = (C', \rho(C))$, we have $\rho(C') \ge \rho(C)$. Clearly $C' = C$ implies $\rho(C') = \rho(C)$. If $\rho(C') = \rho(C)$ and C' is irreducible, then $\rho(C)v'^T v = \rho(C')v'^T v = v'^T C'v \ge v'^T Cv = \rho(C)v'^T v$, where v'^T is a positive left eigenvector of C' for $\rho(C')$. Hence the above inequality is the equality $v'^T C'v = v'^T Cv$. Assume by way of contradiction that $C' \ne C$. Then $C' - C$ is a nonzero nonnegative matrix. Hence $v'^T(C' - C)v > 0$ since v' and v are positive, a contradiction.

□

In Lemma 2.1.2, the matrix C' is a *matrix realization* of the upper bound $\rho(C')$ of $\rho(C)$ and the matrix C in Lemma 2.1.2 is a matrix realization of the lower bound $\rho(C)$ of $\rho(C')$. Hence the pair $(\rho(C), \rho(C'))$ is a pair of *reciprocal bounds*. We shall provide other matrix realizations and reciprocal bounds in Chapter 4.

Lemma 2.1.3 ([16, Theorem 8.1.22])**.** *If an $n \times n$ matrix $C = (c_{ij})$ is nonnegative with row-sum vector $(r_1, r_2, \ldots, r_n)^T$, where $r_i = \sum_{1 \le j \le n} c_{ij}$ and $r_1 \ge r_i \ge r_n$ for $1 \le i \le n$, then*

$$r_n \le \rho(C) \le r_1.$$

Moreover, if C is irreducible, then $\rho(C) = r_1$ (resp. $\rho(C) = r_n$) if and only if C has constant row-sum.

We provide a proof of the following generalized version of Lemma 2.1.3, which is due to M. N. Ellingham and Xiaoya Zha [11].

Lemma 2.1.4 ([11]). *If an $n\times n$ matrix C with row-sum vector $(r_1, r_2, \ldots, r_n)^T$, where $r_1 \geq r_i \geq r_n$ for $1 \leq i \leq n$, has a nonnegative left eigenvector $v^T = (v_1, v_2, \ldots, v_n)$ for θ, then*

$$r_n \leq \theta \leq r_1.$$

Moreover, $\theta = r_1$ (resp. $\theta = r_n$) if and only if $r_i = r_1$ (resp. $r_i = r_n$) for the indices i with $v_i \neq 0$. In particular, if v^T is positive, $\theta = r_1$ (resp. $\theta = r_n$) if and only if C has constant row-sum.

Proof. Without loss of generality, let $\sum_{i=1}^n v_i = 1$ and u be the all-one column vector. Then

$$\theta = \theta v^T u = v^T C u = \sum_{i=1}^{n} v_i r_i.$$

So θ is a convex combination of those r_i with indices i satisfying $1 \leq i \leq n$ and $v_i > 0$, and the lemma follows. $\square$

2.1.3 Quotient matrix

Define $[n] = \{1, 2, \ldots, n\}$. For a partition $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of $[n]$, the $\ell \times \ell$ matrix $\Pi(C) := (\pi_{ab})$, where π_{ab} equals the average row-sum of the submatrix $C[\pi_a|\pi_b]$ of C, is called the *quotient matrix* of C with respect to Π. In matrix notation,

$$\Pi(C) = (S^T S)^{-1} S^T C S, \tag{2.1.1}$$

where $S = (s_{jb})$ is the $n \times \ell$ *characteristic matrix* of Π, i.e.,

$$s_{jb} = \begin{cases} 1, & \text{if } j \in \pi_b; \\ 0, & \text{otherwise} \end{cases}$$

for $1 \leq j \leq n$ and $1 \leq b \leq \ell$. If

$$\pi_{ab} = \sum_{j \in \pi_b} c_{ij} \qquad (1 \leq a, b \leq \ell)$$

for all $i \in \pi_a$, then $\Pi(C) = (\pi_{ab})$ is called the *equitable quotient matrix* of C' with respect to Π. Note that $\Pi(C)$ is an equitable quotient matrix if and only if

$$S\Pi(C) = CS. \tag{2.1.2}$$

Recall that a column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called rooted if $v'_j \geq v'_n \geq 0$ for $1 \leq j \leq n-1$.

Lemma 2.1.5 ([3, Lemma 2.3.1])**.** *If an $n \times n$ matrix C has an equitable quotient matrix $\Pi(C)$ with respect to partition $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of $[n]$ with characteristic matrix S, and λ is an eigenvalue of $\Pi(C)$ with eigenvector u, then λ is an eigenvalue of C with eigenvector Su. Moreover, if u is rooted and $n \in \pi_\ell$, then Su is rooted.*

Proof. From (2.1.2), $CSu = S\Pi(C)u = \lambda Su$. □

2.1.4 Matrices C' with $\rho(C') = \rho(\Pi(C'))$

Some special matrices whose spectral radii are preserved by equitable quotient operation are given in this subsection. One example is the class of nonnegative matrices.

Lemma 2.1.6. *If $\Pi = \{\pi_1, \ldots, \pi_\ell\}$ is a partition of $[n]$ and $C' = (c'_{ij})$ is an $n \times n$ matrix satisfying $c'_{ij} = c'_{kj}$ for all i, k in the same part π_a of Π and $j \in [n]$, then C' and its quotient matrix $\Pi(C')$ with respect to Π have the same set of nonzero eigenvalues. In particular, $\rho(C') = \rho(\Pi(C'))$.*

Proof. From the construction of C', $\Pi(C')$ is clearly an equitable quotient matrix of C'. Let λ' be a nonzero eigenvalue of C' with eigenvector $v' = (v'_1, \ldots, v'_n)^T$. Then $v'_i = (Cv')_i/\lambda' = (Cv')_k/\lambda' = v'_k$ for all i, k in the same part π_a of Π. Let $u'_a = v'_i$ with any choice of $i \in \pi_a$. Then $u' := (u'_1, \ldots, u'_\ell) \neq 0$, and $\Pi(C')u' = \lambda' u'$. From this and Lemma 2.1.5, we know that C' and $\Pi(C')$ have the same set of nonzero eigenvalues, and thus $\rho(C') = \rho(\Pi(C'))$. $\square$

Lemma 2.1.6 gives a kind of matrices C' whose spectral radius remaining unchanged under some equitable quotient operation. The following shows that the equitable quotient matrix of a nonnegative matrix with respect to any partition preserves the spectral radius.

Proposition 2.1.7. *If an $n \times n$ nonnegative matrix C' has an equitable quotient matrix $\Pi(C')$ with respect to a partition Π of [n], then $\rho(C') = \rho(\Pi(C'))$.*

Proof. For $\epsilon > 0$, we consider the matrix $C'' = C' + \epsilon J$, where J is the all-one matrix. Then C'' is irreducible with the same equitable partition of C'. Moreover the equitable quotient matrix $\Pi(C'')$ of C'' is also irreducible since $\Pi(C'')$ is positive. By Theorem 2.1.1, $\Pi(C'')$ has the eigenvalue $\rho(\Pi(C''))$ with a corresponding positive eigenvector v. Then C'' has eigenvalue $\rho(\Pi(C''))$ with a positive eigenvector Sv by Lemma 2.1.5. Therefore $\rho(C'') = \rho(\Pi(C''))$ by the irreducible case of Theorem 2.1.1. This concludes that

$$\rho(C') = \lim_{\epsilon \to 0^+} \rho(C' + \epsilon J) = \lim_{\epsilon \to 0^+} \rho(\Pi(C' + \epsilon J)) = \rho(\Pi(C'))$$

by the continuous property of complex eigenvalues [21]. $\square$

Godsil [12, Corollary 5.2.3] gave another proof of Proposition 2.1.7. We follow his proof for the general case.

Lemma 2.1.8 ([12])**.** *Let C' be an $n \times n$ matrix which has an equitable quotient matrix $\Pi(C')$ with respect to a partition Π of [n] and has a left eigenvector v^T for eigenvalue λ. If $v^T S \neq 0$, then $v^T S$ is a left eigenvector of $\Pi(C')$ for λ, where S is the characteristic matrix of Π.*

Proof. From (2.1.2), we have $S\Pi(C') = C'S$. Then

$$v^T S\Pi(C') = v^T C'S = \lambda v^T S.$$

So $v^T S$ is an left eigenvector of $\Pi(C')$ for λ. $\square$

For a square matrix with a real eigenvalue, let $\rho_r(C')$ denote the maximum real eigenvalue of C'.

Corollary 2.1.9. *Let C' be an $n \times n$ matrix with a nonnegative left eigenvector for $\rho_r(C')$. If C' has an equitable quotient matrix $\Pi(C')$ with respect to a partition Π of $[n]$, then $\rho_r(C') = \rho_r(\Pi(C'))$.*

Proof. Considering their right eigenvectors and applying Lemma 2.1.5, we have $\rho_r(C') \geq \rho_r(\Pi(C'))$. If v^T is a nonnegative left eigenvector of C' for $\rho_r(C')$, then $v^T S \neq 0$, where S is the characteristic matrix of Π. Hence $v^T S$ is a left eigenvector of $\Pi(C')$ for $\rho_r(C')$ by Lemma 2.1.8. Since $\rho_r(C')$ is also an eigenvalue of $\Pi(C')$, it follows that $\rho_r(C') = \rho_r(\Pi(C'))$. $\square$

Proposition 2.1.10. *For a partition Π of [n], if $n \times n$ matrices C' and C'' have equitable quotient matrices $\Pi(C')$ and $\Pi(C'')$ with respect to Π respectively, then $C'C''$ has equitable quotient matrix $\Pi(C'C'')$ with respect to Π and*

$$\Pi(C'C'') = \Pi(C')\Pi(C'').$$

In particular, if C'^{-1} exists and C' has equitable quotient matrix $\Pi(C')$, then $\Pi(C')^{-1} = \Pi(C'^{-1})$.

Proof. From (2.1.2), we have $S\Pi(C') = C'S$ and $S\Pi(C'') = C''S$, where S is the characteristic matrix of Π. By (2.1.1),

$$\Pi(C')\Pi(C'') = (S^T S)^{-1} S^T C' S \Pi(C'') = (S^T S)^{-1} S^T C' C'' S = \Pi(CC').$$

Hence

$$C'C''S = C'S\Pi(C'') = S\Pi(C')\Pi(C'') = S\Pi(C'C'').$$

By (2.1.2) again, $C'C''$ has the equitable quotient matrix $\Pi(C'C'')$ with respect to Π.

The second part follows from $\Pi(C)\Pi(C'^{-1}) = \Pi(CC'^{-1}) = \Pi(I_n) = I_\ell$. $\square$

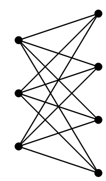


Figure 2.1: $K_{3,4}$

2.2 Bipartite graphs

A graph G is *bipartite* if its vertex set can be partitioned into two disjoint sets X and Y such that every edge of G has one endpoint in X and the other in Y, and the pair $(|X|, |Y|)$ is called the *bi-order* of G. A bipartite graph G with bipartition X, Y is called a *complete bipartite graph* if $E(G) = X \times Y$. We use the notation $K_{p,q}$ to denote a complete bipartite graph of bi-order (p, q). See Figure 2.1 for the graph $K_{3,4}$.

A bipartite graph G is *biregular* if the each of the vertices in the same part has the same degree. Let H, H' be two bipartite graphs with given ordered bipartitions $V(H) = X \bigcup Y$ and $V(H') = X' \bigcup Y'$, where $V(H) \bigcap V(H') = \phi$. The *bipartite sum* $H + H'$ of H and H' (with respect to the given ordered bipartitions) is the graph obtained from H and H' by adding an edge between x and y for each pair $(x, y) \in (X \times Y') \bigcup (X' \times Y)$. Let $N_{p,q}$ be the bipartite graph with bi-order (p, q) without any edges, The graph $K_{3,2} + N_{2,2}$ is given in Figure 2.2.

2.2.1 The bipartite graph G_D

Let $D = (d_1, d_2, \ldots, d_p)$ be a sequence of nonincreasing positive integers of length p. Let G_D denote the bipartite graph with bipartition $X \cup Y$, where $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, with $q = d_1$, and $x_i y_j$ is an edge if and only if $j \leq d_i$. The graph $G_{(4,2,2,1,1)}$ is illustrated in Figure 2.3.

Note that D is the degree sequence of the part X in the bipartition $X \cup Y$

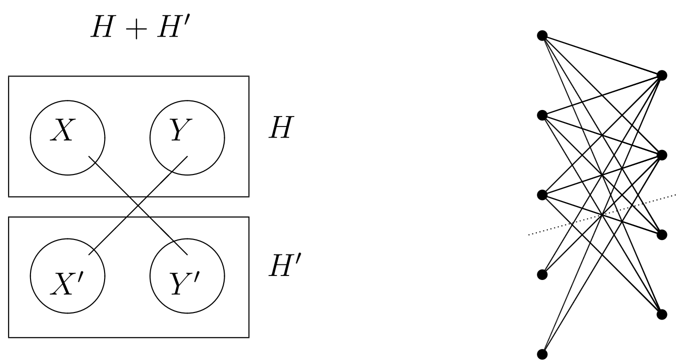
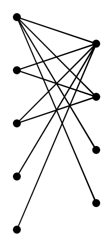


Figure 2.2: The bipartite sum and an example $K_{3,2} + N_{2,2}$



$$F(D) = \begin{array}{llll} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & & \\ \mathbf{1} & \mathbf{1} & & \\ \mathbf{1} & & & \\ \mathbf{1} & & & \end{array}$$

Figure 2.3: The graph $G_{(4,2,2,1,1)}$ and the Ferrers diagram $F(4,2,2,1,1)$

of G_D. As $e = d_1 + d_2 + \cdots + d_p$, D is a *partition* of the number e of edges in G_D. The degree sequence $D^* = (d_1^*, d_2^*, \ldots, d_q^*)$ of the other part Y forms the *conjugate partition* of e, where $e = d_1^* + d_2^* + \cdots + d_q^*$ and $d_j^* = |\{i \mid d_i \geq j\}|$. See [4, Section 8.3] for details. The sequence D will define a *Ferrers diagram* of 1's that has p rows with d_i 1's in row i for $1 \leq i \leq p$. For example, the Ferrers diagram $F(D)$ of the sequence $D = (4, 2, 2, 1, 1)$ is illustrated in Figure 2.4. One can check that $D^* = (5, 3, 1, 1)$ in the above D.

According to the order $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q$, the adjacency matrix of

G_D is of the form

$$A = \begin{pmatrix} O_p & B(D) \\ B(D)^T & O_q \end{pmatrix}, \tag{2.2.1}$$

where $B(D)$ is the $p \times q$ (0,1)-matrix obtained from the Ferrers diagram $F(D)$ by filling 0's into the empty cells. We have that

$$A^2 = \begin{pmatrix} B(D)B(D)^T & O \\ O & B(D)^T B(D) \end{pmatrix}.$$

Let $H = H(D) := B(D)B(D)^T$, which is the $p \times p$ matrix as follows:

$$H = (h_{ij}) = (\min(d_i, d_j)) = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_p \\ d_2 & d_2 & d_3 & \cdots & d_p \\ d_3 & d_3 & d_3 & \cdots & d_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p & d_p & d_p & \cdots & d_p \end{pmatrix}. \tag{2.2.2}$$

For example, if $D = (4, 2, 2, 1, 1)$, then

$$B(D) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 4 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is well-known that $B(D)B(D)^T$ and $B(D)^T B(D)$ have the same nonzero eigenvalues. Hence

$$\rho^2(G_D) = \rho(A^2) = \rho(H). \tag{2.2.3}$$

2.2.2 Spectral upper bounds by D

The graph G_D is important in the study of the spectral radius of bipartite graphs with prescribed degree sequence D of one part of the bipartition. Bhattacharya, Friedland and Peled [2] proved the following lemma.

Lemma 2.2.1. *([2, Theorem 3.1]) Let G be a bipartite graph without isolated vertices such that one part in the bipartition of G has degree sequence $D = (d_1, \ldots, d_p)$. Then $\rho(G) \leq \rho(G_D)$ with equality if and only if $G = G_D$ (up to isomorphism).*

The adjacency matrix $A(G_D)$ of G_D is a matrix realization of the upper bound of $\rho(G)$ in Lemma 2.2.1. The following lemma is used in the proof of Lemma 2.2.1 which may be traced back to [23].

Lemma 2.2.2. *Let G be a bipartite graph of bi-order (p, q) and $(u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ be a positive Perron eigenvector of the adjacency matrix of G according to the bipartition $X \bigcup Y$, where vertices in the part Y of G are ordered to ensure $v_1 \geq v_2 \geq \cdots \geq v_q$. For $1 \leq i < j \leq q$, if $x_k y_j$ is an edge and $x_k y_i$ is not an edge in G for some $x_k \in X$, then the new bipartite graph G' with the same vertex set as G obtained by deleting the edge $x_k y_j$ and adding a new edge $x_k y_i$ has spectral radius $\rho(G') \geq \rho(G)$.*

2.2.3 Spectral upper bounds by D and D'

Chia-an Liu and Chih-wen Weng [19] found the upper bounds of $\rho(G)$ expressed by degree sequences of two parts of the bipartition of G.

Lemma 2.2.3. *([19]) Let G be a bipartite graph with bipartition $X \cup Y$ of orders p and q respectively such that the part X has degree sequence $D = (d_1, \ldots, d_p)$, and the other part Y has degree sequence $D' = (d'_1, d'_2 \ldots, d'_q)$, both in nonincreasing order. For $1 \leq s \leq p$ and $1 \leq t \leq q$, let $X_{s,t} = d_s d'_t + \sum_{i=1}^{s-1} (d_i - d_s) + \sum_{j=1}^{t-1} (d'_j - d'_t)$, $Y_{s,t} = \sum_{i=1}^{s-1} (d_i - d_s) \cdot \sum_{j=1}^{t-1} (d'_j - d'_t)$. Then*

$$\rho(G) \leq \phi_{s,t} := \sqrt{\frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}}.$$

Furthermore, if G is connected then the above equality holds if and only if there exist nonnegative integers $s' < s$ and $t' < t$, and a biregular graph H of bipartition orders $p - s'$ and $q - t'$ respectively such that $G = K_{s',t'} + H$.

It worths mentioning that the graph $G = K_{s',t'} + H$ attaining the equality in Lemma 2.2.3 is not necessary to be G_D. The idea of the proof in Lemma 2.2.3 is to apply Lemma 2.1.3 for the spectral radius to matrices that are similar to the adjacency matrix of G by diagonal matrices with variables on diagonals. Results using this powerful method can also be found in [7, 9, 10, 15, 17, 18, 24].

Chapter 3

The largest spectral radius of a non-complete bipartite graph

Let G be a bipartite graph. There are several extending results of $\rho(G) \leq \sqrt{e}$. These extending results are scattered in [2, 6, 19]. We give another extending result here. To illustrate this, we need some notations. For $2 \leq s \leq t$, let $K^-_{s,t}$ denote the graph obtained from the complete bipartite graph $K_{s,t}$ of bi-order (s,t) by deleting an edge, and $K^+_{s,t}$ denote the graph obtained from $K_{s,t}$ by adding a new edge xy, where x is a new vertex and y is a vertex in the part of order s. Note that $K^-_{2,t+1} = K^+_{2,t}$, and $K^-_{s,t}$ and $K^+_{s,t}$ are not complete bipartite graphs. In Proposition 3.3.2 we shall show that $K^-_{s,t}$ and $K^+_{s,t}$ are the only two types of e-edge graphs G of order at least 4 satisfying

$$\sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}} \leq \rho(G) < \sqrt{e}.$$

For $e \geq 2$, let

$$\rho(e) := \max_{p,q} \max_{G \in \mathcal{K}(p,q,e)} \rho(G)$$

denote the maximal value $\rho(G)$ of a bipartite graph G with e edges which is not a union of a complete bipartite graph and some isolated vertices. For the case that $(e-1, e+1)$ is not a pair of *twin primes*, i.e., a pair of primes with difference two, we will describe the bipartite graph G with e edges such that $\rho(G) = \rho(e)$. Indeed, we will show in Theorem 3.4.1 that if $e \geq 3$ and $\rho(G) = \rho(e)$ then $G \in \{K^-_{s',t'}, K^+_{s'',t''}\}$, where s' and t' (resp. s'' and t'') are chosen to minimize s subject to $2 \leq s \leq t$ and $e = st - 1$ (resp. $e = st + 1$). The twin prime case is not completely solved because G could be any one of $K^-_{s',t'}$ and $K^+_{s'',t''}$. Nevertheless, we find that the values of $\rho(e)$ tend to be smaller than others when $(e-1, e+1)$ is a pair of twin primes. Indeed, this property characterizes a pair of twin primes. See Theorem 3.4.2 for the detailed description. In Section 3.6.1, we prove BFP Conjecture for $\mathcal{K}(p, q, e)$ when $e \in \{st - 1, st' + 1 \mid s \leq p, t \leq q, t' \leq q - 1\}$. Our results are the main tools in [20] for determining if $K^-_{s,t}$ and $K^+_{s,t}$ are determined by their eigenvalues.

3.1 An upper bound of $\rho(G_D)$

We have learned $\rho(G) \leq \rho(G_D)$ in Lemma 2.2.1 for a bipartite graph G with one-part degree sequence D. We shall provide an upper bound of G_D in this section.

Applying Lemma 2.2.3 to the graph $G = G_D$ for a given sequence $D = (d_1, d_2, \ldots, d_p)$ of nonincreasing positive integers of length p, one immediately finds that $d'_j = d^*_j$ and

$$\sum_{j=1}^{t-1} (d'_j - d'_t) = \sum_{i=d'_t+1}^{p} d_i,$$

where $(d^*_1, d^*_2, \ldots, d^*_{d_1})$ is the conjugate partition corresponding to D defined in Section 2.2.1. Moreover, if s is chosen such that $d_s < d_{s-1}$ and $t = d_s + 1$, then $d'_t = s - 1$ and the corresponding Ferrers diagram $F(D)$ has a blank in the (s, t) position, so

$$X_{s,t} = d_s(s-1) + \sum_{i=1}^{s-1} (d_i - d_s) + \sum_{i=s}^{p} d_i = e$$

and

$$Y_{s,t} = \sum_{i=1}^{s-1} (d_i - d_s) \cdot \sum_{i=s}^{p} d_i, \tag{3.1.1}$$

completely expressed by D. Hence we have the following simpler form of Lemma 2.2.3.

Lemma 3.1.1. *Assume that s is chosen satisfying $d_s < d_{s-1}$ in the sequence $D = (d_1, d_2, \ldots, d_p)$ of positive integers and $e = d_1 + d_2 + \cdots + d_p$. Then*

$$\rho(G_D) \leq \sqrt{\frac{e + \sqrt{e^2 - 4\sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{i=s}^{p} d_i}}{2}},$$

with equality if and only if D contains exactly two different values. $\square$

The following are a few special cases of G_D that satisfy the equality in Lemma 3.1.1.

Example 3.1.2. ([19]) Suppose that $2 \leq p \leq q$ and $K_{p,q}^e$ (resp. ${}^eK_{p,q}$) is the graph obtained from $K_{p,q}$ by deleting $k := pq - e$ edges incident on a common vertex in the part of order q (resp. p), where $k < p$ or, equivalently, $p(q-1) < e$. Then

$$\rho(K_{p,q}^e) = \sqrt{\frac{e + \sqrt{e^2 - 4k(q-1)(p-k)}}{2}} \qquad (k = pq - e < p),$$

$$\rho({}^eK_{p,q}) = \sqrt{\frac{e + \sqrt{e^2 - 4k(p-1)(q-k)}}{2}} \qquad (k = pq - e < q).$$

$\square$

3.2 $\rho(K_{p,q}^{\pm})$

Applying Example 3.1.2 to the graph $K_{p,q}^- = K_{p,q}^{pq-1} = {}^{pq-1}K_{p,q}$, one immediate finds that

$$\rho(K_{p,q}^-) = \sqrt{\frac{e + \sqrt{e^2 - 4(e - (p+q) + 2)}}{2}},$$

which obtains maximum (resp. minimum) when p is minimum (resp. p is maximum) subject to the fixed number $e = pq - 1$ of edges and $2 \leq p \leq q$. Note that

$$e - (p+q) + 2 \leq e - 2\sqrt{pq} + 2 = e - 2\sqrt{e+1} + 2 < e - 1 - \sqrt{e-1}$$

for $e \geq 6$. Hence

$$\rho(K^-_{p,q}) > \sqrt{\frac{e+\sqrt{e^2-4(e-1-\sqrt{e-1})}}{2}} \qquad (q \geq p \geq 3).$$

As $K^-_{2,2}$ has 3 edges, one can check that

$$\rho(K^-_{2,2}) = \sqrt{\frac{3+\sqrt{5}}{2}} < \sqrt{\frac{e+\sqrt{e^2-4(e-1-\sqrt{e-1})}}{2}}. \tag{3.2.1}$$

Similarly $K^+_{p,q} = K^{pq+1}_{p,q+1}$ has spectral radius

$$\rho(K^+_{p,q}) = \sqrt{\frac{e+\sqrt{e^2-4(e-1-q)}}{2}}, \tag{3.2.2}$$

which obtains the maximum (resp. minimum) when p is minimum (resp. maximum) subject to the fixed number $e = pq+1$ and $2 \leq p \leq q$. Note that $e-1-q \leq e-1-\sqrt{e-1}$ in this case. Hence

$$\rho(K^+_{p,q}) \geq \sqrt{\frac{e+\sqrt{e^2-4(e-1-\sqrt{e-1})}}{2}}$$

with equality if and only if $p = q = \sqrt{e-1}$. This proves the following lemma.

Lemma 3.2.1. *The following (i)-(iii) hold.*

(i) For all positive integers $2 \leq p' \leq q', (p',q') \neq (2,2)$, $2 \leq p'' \leq q''$ satisfying $e = p'q'-1 = p''q''+1$, we have

$$\rho(K^-_{p',q'}), \rho(K^+_{p'',q''}) \geq \sqrt{\frac{e+\sqrt{e^2-4(e-1-\sqrt{e-1})}}{2}}.$$

Moreover the above equality does not hold for $\rho(K^-_{p',q'})$, and holds for $\rho(K^+_{p'',q''})$ if and only if $p'' = q''$.

(ii) If $e+1$ is not a prime and $p' \geq 2$ is the least integer such that p' divides $e+1$ and $q' := (e+1)/p'$ so that $e = p'q'-1$, then for any positive integers $2 \leq p \leq q$ with $e = pq-1$, we have $\rho(K^-_{p,q}) \leq \rho(K^-_{p',q'})$, with equality if and only if $(p,q) = (p',q')$.

(iii) *If $e-1$ is not a prime, and $p'' \geq 2$ is the least integer such that p'' divides $e-1$ and $q'' := (e-1)/p''$ so that $e = p''q''+1$, then for positive integers $2 \leq p \leq q$ with $e = pq+1$, we have $\rho(K^+_{p,q}) \leq \rho(K^+_{p'',q''})$, with equality if and only if $(p,q) = (p'', q'')$.*

$\square$

Note that the condition $2 \leq p' \leq q', (p', q') \neq (2, 2)$ in (i) is from the previous condition $3 \leq p' \leq q'$ and $K^-_{2,q} = K^+_{2,q-1}$ for $q \geq 3$.

3.3 $\rho(G)$ with $G \neq K_{s,t}, K^{\pm}_{s,t}$

In this section, we consider bipartite graphs which are not complete bipartite and not considered in Lemma 3.2.1(i). The following lemma is for the special case that the graph has the form $G = G_D$.

Lemma 3.3.1. *Let $D = (d_1, d_2, \ldots, d_p)$ be a partition of e. Suppose that G_D is not a complete bipartite graph and is not one of the graphs $K^-_{p',q'}$ or $K^+_{p'',q''}$ for any $1 \leq p' \leq q', (p', q') \neq (2, 2)$, $1 \leq p'' \leq q''$ such that $e = p'q' - 1 = p''q'' + 1$. Then*

$$\rho(G_D) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Proof. When $e \leq 3$, $G_D = K^-_{2,2}$ is the only graph satisfies the assumption above and the inequality holds by (3.2.1). We assume that $e \geq 4$. Let $q = d_1$. The assumption implies that $q \geq 2$ and $4 \leq e \leq pq - 2$. Using D^* to replace D if necessary, we might assume that $2 \leq p \leq q$ and $q \geq 3$. Since G_D is not complete, we choose s such that $1 \leq s \leq p$ and $d_{s-1} > d_s$. Set $t = d_s + 1$. According to the partition $(s-1, 1, p-s)$ of rows and the partition $(t-1, 1, q-t)$ of columns, the Ferrers diagram $F(D)$ is divided into 9 blocks and the number b_{ij} of 1's in the block (i, j) for $1 \leq i, j \leq 3$ is shown as

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} (s-1)d_s & s-1 & \sum_{i=1}^{s-1}(d_i - d_s - 1) \\ d_s & 0 & 0 \\ \sum_{i=s+1}^{p} d_i & 0 & 0 \end{bmatrix}.$$

Note that $b_{11} = b_{12}b_{21}$ and $b_{11}+b_{12}+b_{13}+b_{21}+b_{31} = e$. Referring to Lemma 3.1.1 and (3.1.1), it suffices to show that $Y_{s,t} > e - 1 - \sqrt{e-1}$. Note that

$$\begin{aligned} Y_{s,t} =& \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{i=s}^{p} d_i = (s - 1 + \sum_{i=1}^{s-1}(d_i - d_s - 1))(d_s + \sum_{i=s+1}^{p} d_i) \\ =&(b_{12} + b_{13})(b_{21} + b_{31}) = b_{11} + b_{12}b_{31} + b_{21}b_{13} + b_{13}b_{31}, \end{aligned}$$

$b_{12}b_{21} \neq 0$, and $G \neq K^-_{p',q'}$ implies that $b_{13} \neq 0$ or $b_{31} \neq 0$. If both of b_{13} and b_{31} are not zero, then $b_{12}b_{31} \geq b_{12}+b_{31}-1$, $b_{21}b_{13} \geq b_{21}+b_{13}-1$, and $b_{13}b_{31} \geq 1$, so $Y_{s,t} \geq b_{11}+(b_{12}+b_{31}-1)+(b_{21}+b_{13}-1)+1 = e-1 > e-1-\sqrt{e-1}$. Therefore, the proof is completed. The above proof holds for any s with $d_{s-1} < d_s$. We choose the least one with such property, and might assume one of the following two cases (i)-(ii).

Case (i). $b_{31} = 0$ and $b_{13} \neq 0$: Then $s = p = b_{12} + 1 \geq 2$, and $G = {}^{e}K_{p,q}$, where $e = pq - (q - d_p) \geq (p-1)q + 1 > (p-1)^2 + 1$. Thus

$$Y_{s,t} = b_{11} + b_{21}b_{13} \geq e - 1 - b_{12} = e - p > e - 1 - \sqrt{e-1}.$$

Case (ii). $b_{13} = 0$ and $b_{31} \neq 0$: The condition $b_{31} \neq 0$ implies that $q \geq p \geq 3$. The condition $b_{13} = 0$ implies that $t = q$ and $b_{21} = q - 1 \geq 2$. The proof is further divided into the following two cases (iia) and (iib).

Case (iia). $1 \leq b_{31} < b_{21}$: If $s < p-1$, let $s' = s+1$ and $t' = d_{s'}+1$. Then $d_{s'-1} > d_{s'}$ and $d_{s'+1} \neq 0$. Let b'_{ij} be the b_{ij} corresponding to the new choice of s' and t'. Then $b'_{13}b'_{31} \neq 0$ and the proof is completed as in the beginning. Note that $s \neq p$ since $b_{31} \neq 0$. Then we may assume $s = p - 1$. This implies that $b_{31} = d_p < q - 1$ and $e = pq - 1 - q + d_p \geq p^2 - p > (p-1)^2 + 1$. Let $s' = p$ and $t' = d_p + 1$, and then

$$Y_{s',t'} = b'_{21}(b'_{12} + b'_{13}) \geq e - 1 - b'_{12} = e - p > e - 1 - \sqrt{e-1}.$$

Case (iib). $b_{31} \geq b_{21}$: If $b_{12} = 1$ then by the assumption $G \neq K^+_{p'',q''}$, there exists another $s'' > s$ such that $d_{s''} < d_{s''-1}$. Apply the above proof on

$(s,t) = (s'', t'')$. Since $b''_{13} \geq 1$, we might assume $b''_{31} = 0$. Then $s'' = p$ and $e = (p-1)(q-1) + d_p + 1 > (p-1)^2 + 1$. Hence

$$Y_{s'',t''} = b''_{21}(b''_{12} + b''_{13}) \geq e - 1 - b''_{12} = e - p > e - 1 - \sqrt{e-1}.$$

We now assume in the last situation that $b_{12} > 1$. Then

$$Y_{s,t} = b_{11} + (b_{12} - 1)b_{31} + b_{31} \geq b_{11} + b_{12} + 2b_{31} - 2 \geq e - 2 > e - 1 - \sqrt{e-1}.$$

$\square$

We now study the general case.

Proposition 3.3.2. *Let G be a bipartite graph without isolated vertices with e edges which is neither a complete bipartite graph nor one of the graphs $K^-_{p',q'}$, $K^+_{p'',q''}$ for any $2 \leq p' \leq q', (p',q') \neq (2,2)$, $2 \leq p'' \leq q''$, such that $e = p'q' - 1 = p''q'' + 1$ is the number of edges in G. Then*

$$\rho(G) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Proof. If G is not connected, then

$$\rho(G) \leq \sqrt{e-1} < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Assume G is connected. Let G_D be the graph obtained from a degree sequence D of any part, say X, in the bipartition $X \cup Y$ of G. Then $\rho(G) \leq \rho(G_D)$ by Lemma 2.2.1. The proof is done if G_D satisfies the assumption of Lemma 3.3.1. Let D' be the degree sequence of the other part Y in the bipartition of G. Then we might assume that $G \neq G_D$, $G \neq G_{D'}$, and G_D and $G_{D'}$ are graphs of the forms $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$.

For $y_i \in Y$, let $N(y_i)$ be the set of neighbors of y_i in G. Suppose for this moment that $|N(y_i)| = |N(y_j)|$ and $N(y_i) \neq N(y_j)$ for some $y_i, y_j \in Y$. Assume that y_i is in front of y_j in the order that makes the entries in the latter

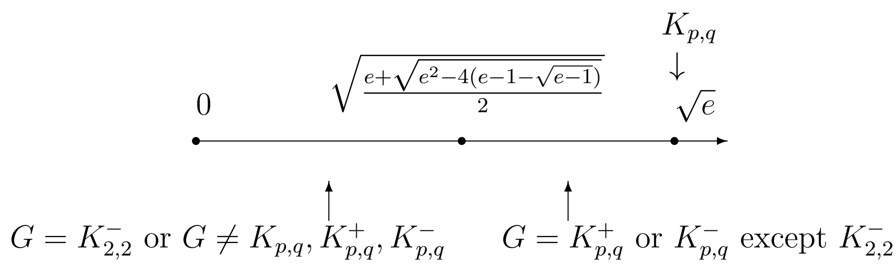


Figure 3.1: Values of $\rho(G)$

part of the positive Perron eigenvector nonincreasing. Let G'' be the bipartite graph obtained from G by moving one edge incident on y_j but not on y_i to incident on y_i, keeping the other endpoint of this edge unchanged. Let D'' be the new degree sequence on the part Y of the new bipartite graph G''. Then $\rho(G) \leq \rho(G'') \leq \rho(G_{D''})$, where the first inequality is obtained from Lemma 2.2.2. We will show that $G_{D''}$ is not of the form $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$. Thus the proof follows from Lemma 3.3.1. Suppose $G_{D''}$ is of the form $K_{p,q}$, $K^-_{p',q'}$, and $K^+_{p'',q''}$. Note that the elements in the degree sequence of any part of $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$ is of the form $k, \ldots, k, \ell$, where ℓ could be $1, k-1, k, k+1$, for some positive integer k. Noticing that D'' is obtained from D' by replacing two given equal values a by $a-1$ and $a+1$. If $a-1 > 1$, then the difference between $a+1$ and $a-1$ is two, a contradiction. If $a-1 = 1$, then $G_{D''}$ must be $K^+_{3,q-1}$ and $D' = (3, \ldots, 3, 2, 2)$. So $G_{D'}$ is not a graph of the form $K_{p,q}$, $K^-_{p',q'}$, or $K^+_{p'',q''}$, a contradiction. Hence we might assume that if $|N(y_i)| = |N(y_j)|$ then $N(y_i) = N(y_j)$ for all $y_i, y_j \in Y$. Reordering the vertices in Y such that the former has larger degree and then doing the same thing for X, we find indeed $G = G_D = G_{D'}$ since G is connected, a contradiction. $\square$

From Lemma 3.2.1(i) and Proposition 3.3.2, we can characterize the value $\rho(G)$ of a bipartite graph G as shown in Figure 3.1.

Here we provide an application of Proposition 3.3.2.

Corollary 3.3.3. *Let G be a bipartite graph without isolated vertices which is neither a complete bipartite graph nor one of the graphs $K_{p,q}$, $K^-_{p',q'}$, $K^+_{p'',q''}$ for any $1 \leq p \leq q$, $2 \leq p' \leq q'$, $2 \leq p'' \leq q''$ such that $e = pq = p'q' - 1 = p''q'' + 1$ is the number of edges in G. Assume that $e = st + 1$ (resp. $e = st - 1$) for $2 \leq s \leq t$. Then*

$$\rho(G) < \rho(K^+_{s,t}) \qquad (\textit{resp. } \rho(G) < \rho(K^-_{s,t})).$$

Proof. If $s = t = 2$ and $e = st - 1 = 3$ then either $G = 3K_2$ the disjoint union of three edges or $G = K_{1,2} \cup K_2$ the disjoint of a path of order 3 and an edge. One can easily check that $\rho(G) < \rho(K^-_{2,2})$. The remaining cases are from Proposition 3.3.2 and Lemma 3.2.1(i) and noticing that $K^-_{2,t+1} = K^+_{2,t}$ for $t \geq 2$. □

3.4 Main Theorems

For $e \geq 2$, recall that $\rho(e)$ is the maximal value $\rho(G)$ of a bipartite graph G with e edges which is not a union of a complete bipartite graph and some isolated vertices. Note that

$$\rho(2) = \rho(2K_2) = 1, \text{ and } \rho(3) = \rho(K^-_{2,2}) = \sqrt{\frac{3+\sqrt{5}}{2}}.$$

Two theorems about $\rho(e)$ are given in this section.

Theorem 3.4.1. *Let G be a bipartite graph with $e \geq 3$ edges without isolated vertices such that $\rho(G) = \rho(e)$. Then the following (i)–(iv) hold.*

(i) If e is odd then $G = K^-_{2,q}$, where $q = (e+1)/2$.

(ii) If e is even, $e-1$ is a prime and $e+1$ is not a prime, then $G = K^-_{p',q'}$, where $p' \geq 3$ is the least integer that divides $e+1$ and $q' = (e+1)/p'$.

(iii) If e is even, $e-1$ is not a prime and $e+1$ is a prime, then $G = K^+_{p'',q''}$, where $p'' \geq 3$ is the least integer that divides $e-1$ and $q'' = (e-1)/p''$.

(iv) If e is even and neither $e-1$ nor $e+1$ is a prime, then $G \in \{K^-_{p',q'}, K^+_{p'',q''}\}$, where p', q' are as in (ii) and p'', q'' are as in (iii).

Proof. By the definition of $\rho(e)$, G is not a complete graph. From Lemma 3.2.1(i) and Proposition 3.3.2, we only need to compare the spectral radii $\rho(K^-_{p,q})$ and $\rho(K^+_{p,q})$ for all possible positive integers $2 \le p \le q$ that keep the graphs having e edges. This has been done in Lemma 3.2.1(ii)-(iii). $\square$

Due to Yitang Zhang's recent result [26], the conjecture that there are infinite pairs of twin primes receives much attention. Theorem 3.4.2 provides a spectral description of the pairs of twin primes.

Theorem 3.4.2. *Let $e \ge 4$ be an integer. Then $(e-1, e+1)$ is a pair of twin primes if and only if*

$$\rho(e) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

Proof. The necessity is by Proposition 3.3.2. The sufficiency is from Theorem 3.4.1 and Lemma 3.2.1(i). $\square$

3.5 Numerical comparisons

In the case (iv) of Theorem 3.4.1, the two graphs $K^-_{p',q'}$ and $K^+_{p'',q''}$ are candidates to be extremal graph. For even $e \le 100$ and neither $e-1$ nor $e+1$ is a prime, we shall determine which graph has larger spectral radius. The symbol $-$ in the last column of the following table means that $K^-_{p',q'}$ wins, i.e. $\rho(K^-_{p',q'}) > \rho(K^+_{p'',q''})$ and $+$ otherwise.

e	$\rho(K^-_{p',q'})$	$\rho(K^+_{p'',q''})$	winner
26	$\sqrt{13+3\sqrt{17}}$	$\sqrt{13+\sqrt{149}}$	$-$
34	$\sqrt{17+\sqrt{265}}$	$\sqrt{17+\sqrt{267}}$	$+$
50	$\sqrt{25+\sqrt{593}}$	$\sqrt{25+\sqrt{583}}$	$-$
56	$\sqrt{28+\sqrt{748}}$	$\sqrt{28+\sqrt{740}}$	$-$
64	$\sqrt{32+\sqrt{976}}$	$\sqrt{32+\sqrt{982}}$	$+$
76	$\sqrt{38+\sqrt{1384}}$	$\sqrt{38+\sqrt{1394}}$	$+$
86	$\sqrt{43+\sqrt{1813}}$	$\sqrt{43+\sqrt{1781}}$	$-$
92	$\sqrt{46+\sqrt{2096}}$	$\sqrt{46+\sqrt{2078}}$	$-$
94	$\sqrt{47+\sqrt{2137}}$	$\sqrt{47+\sqrt{2147}}$	$+$

Table 3.1: Comparisons of $\rho(K^-_{p',q'})$ and $\rho(K^+_{p'',q''})$ for $e \leq 100$

3.6 BFP Conjecture for $\mathcal{K}(p, q, e)$

3.6.1 The case $e = st \pm 1$

Theorem 3.6.1. *BFP Conjecture for $\mathcal{K}(p, q, e)$ holds for $e \in \{st-1, st'+1 \mid s \leq p, t \leq q, t' \leq q-1\}$.*

Proof. When $e = st - 1$ $(s \leq p, t \leq q)$ or $e = st + 1$ $(s \leq p, t \leq q-1)$, $K^-_{s,t} \in \mathcal{K}(p, q, e)$ or $K^+_{s,t} \in \mathcal{K}_0(p, q, e)$ respectively, it implies BFP Conjecture for $\mathcal{K}_0(p, q, e)$ directly by Corollary 3.3.3. $\square$

3.6.2 Counter examples

Let $q > p \geq 3$ be two positive integers and $D_1 = (q, q-1, \ldots, q-1, q-2)$, $D_2 = (q, q, \ldots, q, q-p)$ be two sequences of p positive integers. Then $G_{D_1}, G_{D_2} \in \mathcal{K}(p, q, p(q-1))$. Note that G_{D_2} is the only graph in $\mathcal{K}_0(p, q, p(q-1))$. The following proposition shows that BFP Conjecture for $\mathcal{K}(p, q, p(q-1))$ is false when $q > p + 2$.

Proposition 3.6.2. *Suppose $q > p+2$. Then $\rho(G_{D_2}) < \rho(G_{D_1})$. In particular, BFP Conjecture for $\mathcal{K}(p, q, p(q-1))$ is false.*

Proof. Referring to H in (2.2.2), we have

$$H(D_1) = \left(\begin{array}{c|ccc|c} q & q-1 & \cdots & q-1 & q-2 \\ \hline q-1 & q-1 & \cdots & q-1 & q-2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q-1 & q-1 & \cdots & q-1 & q-2 \\ \hline q-2 & q-2 & \cdots & q-2 & q-2 \end{array}\right),$$

$$H(D_2) = \left(\begin{array}{c|ccc|c} q & q & \cdots & q & q-p \\ \hline q & q & \cdots & q & q-p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q & q & \cdots & q & q-p \\ \hline q-p & q-p & \cdots & q-p & q-p \end{array}\right).$$

Let $\Pi = \{\{1\}, \{2, \ldots, p-1\}, \{p\}\}$. Then

$$\Pi(H(D_1)) = \begin{pmatrix} q & (p-2)(q-1) & q-2 \\ q-1 & (p-2)(q-1) & q-2 \\ q-2 & (p-2)(q-2) & q-2 \end{pmatrix},$$

$$\Pi(H(D_2)) = \begin{pmatrix} q & (p-2)q & q-p \\ q & (p-2)q & q-p \\ q-p & (p-2)(q-p) & q-p \end{pmatrix}$$

are equitable quotient matrices of $H(D_1)$, $H(D_2)$ with respect to Π respectively. Note that $\rho(\Pi(H(D_i))) = \rho(H(D_i)) = \rho^2(G_{D_i})$ for $i = 1, 2$ by Proposition 2.1.7 and (2.2.3).

The characteristic polynomials of $\Pi(H(D_1))$ and $\Pi(H(D_2))$ are

$$\begin{aligned} g_1(x) &= x^3 - p(q-1)x^2 + ((2p-2)(q-1) - p)x - (p-2)(q-2), \\ g_2(x) &= x^3 - p(q-1)x^2 + p(p-1)(q-p)x \end{aligned}$$

respectively. Since $q \geq p+3 > p+2+1/(p-1) = p^2/(p-1)+1$,

$$g_2(x) = g_1(x) + (p-2)(((p-1)(q-1)-p^2)x + q - 2) > 0$$

for $x \geq \rho(H(D_1))$. Therefore the zeros of g_2 are less than $\rho(H(D_1))$ and

$$\rho(H(D_2)) < \rho(H(D_1)).$$

This implies $\rho(G_{D_2}) < \rho(G_{D_1})$. Since G_{D_2} is the unique graph in $\mathcal{K}_0(p, q, p(q-1))$, the graph with maximum spectral radius among $\mathcal{K}(p, q, p(q-1))$ in not in $\mathcal{K}_0(p, q, p(q-1))$. □

3.7 Weak BFP Conjecture for $\mathcal{C}(p, q, e)$ with $p \leq 3$

We shall prove weak BFP Conjecture for $\mathcal{C}(p, q, e)$ with $p \leq 3$ despite the existence of a counter example for BFP Conjecture for $\mathcal{K}(p, q, e)$ with $p = 3$ in the last section.

Theorem 3.7.1. *If $p \leq 3$, then weak BFP Conjecture for $\mathcal{C}(p, q, e)$ holds.*

Proof. When $p \leq 3$, there is at least one of $K_{s,t}, K^+_{s,t}, K^-_{s,t}$ in $\mathcal{C}_0(p, q, e)$ for some s, t. So weak BFP Conjecture for $\mathcal{C}(p, q, e)$ holds by Corollary 3.3.3. □

Chapter 4

Spectral bounds of a nonnegative matrix

The research in this chapter is motivated by the following theorem of Xing Duan and Bo Zhou in 2013 [10, Theorem 2.1].

Theorem 4.0.1. *Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$, $f := \max_{i,j\in[n],i\neq j} c_{ij}$ and $d := \max_{1\leq i\leq n} c_{ii}$. Then*

$$\rho(C) \leq \frac{r_\ell + d - f + \sqrt{(r_\ell - d + f)^2 + 4f\sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2} \tag{4.0.1}$$

for $1 \leq \ell \leq n$. Moreover, if C is irreducible, then the equality holds in (4.0.1) if and only if $r_1 = r_n$ or for some $2 \leq t \leq \ell$, we have $r_{t-1} > r_t = \cdots = r_\ell$ and

$$c_{ij} = \begin{cases} d, & \text{if } i = j \leq t-1; \\ f, & \text{if } i \neq j \text{ and } 1 \leq i \leq n,\ 1 \leq j \leq t-1. \end{cases}$$

Theorem 4.0.1 generalizes the results in [5, 7, 13, 14, 18, 24, 25] and relates to the results in [17, 19, 25], while the upper bound of $\rho(C)$ expressed in (4.0.1) is somewhat complicated and deserves an intuitive realization.

The values on the right hand side of (4.0.1) is realized as the largest real eigenvalue $\rho_r(C')$ of the $n \times n$ matrix

$$C' = \left(\begin{array}{cccc|cccc}
d & f & \cdots & f & f & f & \cdots & f & r_1 - d - (n-2)f \\
f & d & & f & f & f & \cdots & f & r_2 - d - (n-2)f \\
\vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
f & f & \cdots & d & f & f & \cdots & f & r_{\ell-1} - d - (n-2)f \\
\hline
f & f & \cdots & f & d & f & \cdots & f & r_\ell - d - (n-2)f \\
f & f & \cdots & f & f & d & & f & r_\ell - d - (n-2)f \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \vdots \\
f & f & \cdots & f & f & f & & d & r_\ell - d - (n-2)f \\
f & f & \cdots & f & f & f & \cdots & f & r_\ell - (n-1)f
\end{array}\right) \qquad (4.0.2)$$

which has the following three properties:

(i) $(r_1, r_2, \ldots, r_\ell, \ldots, r_\ell)^T \geq (r_1, r_2, \ldots, r_n)^T$, where $(r_1, r_2, \ldots, r_\ell, \ldots, r_\ell)^T$ and $(r_1, r_2, \ldots, r_n)^T$ are the row-sum vectors of C' and C respectively,

(ii) $C'[-|n) \geq C[-|n)$, and

(iii) C' has a positive eigenvector $(v_1', v_2', \ldots, v_n')^T$ for $\rho_r(C')$ with $v_i' \geq v_n'$ for $1 \leq i \leq n$.

Property (iii) will be checked by Lemma 4.3.4. Since the above matrix C' is not necessarily nonnegative, the spectral radius $\rho(C')$ of C' is replaced by the largest real eigenvalue $\rho_r(C')$ in the property (iii). Our main result in Theorem 4.2.3 is in a more general form that will imply for any matrix C' that satisfies the properties (i)-(iii) above, we have $\rho(C) \leq \rho_r(C')$. Moreover, when a matrix C' is fixed and C and C' satisfy (i)-(iii), the matrices C with $\rho(C) = \rho_r(C')$ are completely determined. We apply Theorem 4.2.3 to find a sharp upper bound of $\rho(C)$ expressed by the sum of entries in C, the largest off-diagonal entry f and the largest diagonal entry d in Theorem 4.5.2.

Note that $\rho_r(C') = \rho_r(C'')$ for the largest real eigenvalues of C' and C''

respectively, where C' is as in (4.0.2) and

$$C'' = \left(\begin{array}{cccc|c} d & f & \cdots & f & r_1 - d - (\ell-2)f \\ f & d & & f & r_2 - d - (\ell-2)f \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{\ell-1} - d - (\ell-2)f \\ \hline f & f & \cdots & f & r_\ell - (\ell-1)f \end{array}\right) \tag{4.0.3}$$

is the equitable quotient matrix of C' with respect to the partition $\{\{1\}, \{2\}, \ldots, \{\ell-1\}, \{\ell, \ell+1, \ldots, n\}\}$ of $\{1, 2, \ldots, n\}$. Moreover $\rho_r(C'') = \rho_r(C''')$, where

$$C''' = \begin{pmatrix} (\ell-2)f + d & f \\ \sum_{i=1}^{\ell-1} r_i - (\ell-1)((\ell-2)f + d) & r_\ell - (\ell-1)f \end{pmatrix}, \tag{4.0.4}$$

is the equitable quotient of the transpose C''^T of C'' with respect to the partition $\{\{1, 2, \ldots, \ell-1\}, \{\ell\}\}$ of $\{1, 2, \ldots, \ell\}$. Motivated by these observations, Theorem 4.7.1 will provide an upper bound $\rho_r(C'')$ of $\rho(C)$, where C'' is a matrix of size smaller than that of C obtained by applying equitable quotient to suitable matrix C' that satisfies properties (i)-(iii) described above.

Each of our theorems on upper bounds of $\rho(C)$ has a dual version that deals with lower bounds. We provide a new class of sharp lower bounds of $\rho(C)$ in Theorem 4.8.1. Applying Theorem 4.8.1 to a binary matrix C, we improve the well known inequality $\rho(C) \geq r_n$ as stated in Corollary 4.8.2. We believe that many new spectral bounds of the spectral radius of a nonnegative matrix will be easily obtained by our matrix realization in this chapter.

In addition to the above results, Lemma 4.1.1 and Lemma 4.1.2 are of independent interest in matrix theory.

4.1 The spectral bound $\rho(C')$

We generalize Lemma 2.1.2 in the sense of Lemma 2.1.4 to find spectral bounds of C, where the matrix C considered are not necessarily nonnegative, but instead, assume that C has a nonnegative eigenvector.

Lemma 4.1.1. *Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that*

(i) $PCQ \leq PC'Q$;

(ii) C' *has an eigenvector* Qu *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T$ *and* $\lambda' \in \mathbb{R}$;

(iii) C *has a left eigenvector* $v^T P$ *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n)$ *and* $\lambda \in \mathbb{R}$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{4.1.1}$$

Proof. Multiplying the nonnegative vector u in (ii) to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{4.1.2}$$

Multiplying the nonnegative left eigenvector v^T of C for λ in assumption (iii) to the left of all terms in (4.1.2), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{4.1.3}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Assume that $\lambda = \lambda'$, so the inequality in (4.1.3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$.

Conversely, (4.1.1) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (4.1.3) and (iv). □

In Lemma 4.1.1, the pair (λ, λ') is a pair of reciprocal bounds, the matrix C' is a realization of the upper bound λ' of λ and the matrix C is a realization of the lower bound λ of λ'. If C is nonnegative and $P = Q = I$, where I is the $n \times n$ identity matrix, then Lemma 4.1.1 becomes Lemma 2.1.2 with an additional assumption $v^T u > 0$ which immediately holds if C or C' is irreducible by Theorem 2.1.1.

In the sequels, we shall call two statements that resemble each other by switching $\leq$ and $\geq$ and corresponding variables, like $\theta \geq r_n$ and $\theta \leq r_1$, as *dual statements*, and their proofs are called *dual proofs* if one proof is obtained from the other by simply switching one of $\leq$ and $\geq$ to the other. The following is a dual version of lemma 4.1.1 and its proof is by dual proof.

Lemma 4.1.2. *Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that*

(i) $PCQ \geq PC'Q$;

(ii) C' *has an eigenvector* Qu *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T$ *and* $\lambda' \in \mathbb{R}$;

(iii) C *has a left eigenvector* $v^T P$ *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n)$ *and* $\lambda \in \mathbb{R}$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \geq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \ \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{4.1.4}$$

$\square$

4.2 The special case $P = I$ and a particular Q

We shall apply Lemma 4.1.1 and Lemma 4.1.2 by taking $P = I$ and

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{4.2.1}$$

Hence for $n \times n$ matrix $C' = (c'_{ij})$, the matrix $PC'Q$ in Lemma 4.1.1(i) is

$$C'Q = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & r'_1 \\ c'_{21} & c'_{22} & \cdots & c'_{2\ n-1} & r'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n-1\ 1} & c'_{n-1\ 2} & \cdots & c'_{n-1\ n-1} & r'_{n-1} \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & r'_n \end{pmatrix}, \tag{4.2.2}$$

where $(r'_1, r'_2, \ldots, r'_n)^T$ is the row-sum column vector of C'.

Definition 4.2.1. A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *rooted* if $v'_j \geq v'_n \geq 0$ for $1 \leq j \leq n-1$.

The following Lemma is immediate from the above definition.

Lemma 4.2.2. *If* $u = (u_1, u_2, \ldots, u_n)^T$ *and* $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1 + u_n, u_2 + u_n, \ldots, u_{n-1} + u_n, u_n)^T$*, then*

(i) v' *is rooted if and only if* u *is nonnegative;*

(ii) For $1 \leq j \leq n-1$*,* $u_j > 0$ *if and only if* $v'_j > v'_n$*.*

□

The following theorem is immediate from Lemma 4.1.1 by applying $P = I$, the Q in (4.2.1), $v' = Qu$ and referring to (4.2.2) and Lemma 4.2.2.

Theorem 4.2.3. *Let $C = (c_{ij})$, $C' = (c_{ij})$ be $n \times n$ matrices. Assume that*

(i) $C[-|n) \leq C'[-|n)$ *and* $(r_1, r_2, \ldots, r_n)^T \leq (r'_1, r'_2, \ldots, r'_n)^T$*, where* $(r_1, r_2, \ldots, r_n)^T$ *and* $(r'_1, r'_2, \ldots, r'_n)^T$ *are the row-sum vectors of C and C' respectively;*

(ii) C' *has a rooted eigenvector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *for* λ' *for some* $\lambda' \in \mathbb{R}$*;*

(iii) C *has a nonnegative left eigenvector* $v^T = (v_1, v_2, \ldots, v_n)$ *for* $\lambda \in \mathbb{R}$*;*

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

(a) $r_i = r'_i$ *for* $1 \leq i \leq n$ *with* $v_i \neq 0$ *when* $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1$ *with* $v_i \neq 0$ *and* $v'_j > v'_n$.

$\square$

Note that the cases (a)-(b) in Theorem 4.2.3 are from the line (4.1.1) in Theorem 4.1.1. The first part of assumption (i) in Theorem 4.2.3 says that the last column is *irrelevant* in the comparison of C and C'. The following theorem is a dual version of Theorem 4.2.3.

Theorem 4.2.4. *Let $C = (c_{ij})$, $C' = (c_{ij})$ be $n \times n$ matrices. Assume that*

(i) $C[-|n) \geq C'[-|n)$ *and* $(r_1, r_2, \ldots, r_n)^T \geq (r'_1, r'_2, \ldots, r'_n)^T$*, where* $(r_1, r_2, \ldots, r_n)^T$ *and* $(r'_1, r'_2, \ldots, r'_n)^T$ *are the row-sum vectors of C and C' respectively;*

(ii) C' *has a rooted eigenvector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *for* λ' *for some* $\lambda' \in \mathbb{R}$*;*

(iii) C *has a nonnegative left eigenvector* $v^T = (v_1, v_2, \ldots, v_n)$ *for* $\lambda \in \mathbb{R}$*;*

(iv) $v^T v' > 0$.

Then $\lambda \geq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if *(a)* $r_i = r'_i$ *for* $1 \le i \le n$ *with* $v_i \ne 0$ *when* $v'_n \ne 0$;

(b) $c'_{ij} = c_{ij}$ *for* $1 \le i \le n,\ 1 \le j \le n-1$ *with* $v_i \ne 0$ *and* $v'_j > v'_n$.

$\square$

Example 4.2.5. Consider the following three matrices

$$C'_\ell = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad C'_u = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

with $C'_\ell[-|3) \le C[-|3) \le C'_u[-|3)$, and the same row-sum vector $(5,3,3)^T$. Note that C'_ℓ has a rooted eigenvector $v'^\ell = (1,0,0)^T$ for $\lambda'^\ell = 3$ and C'_u has a rooted eigenvector $v'^u = (2,1,1)^T$ for $\lambda'^r = 4$. Since C is irreducible, it has a left positive eigenvector $(v_1, v_2, v_3) > 0$. Hence assumptions (i)-(iv) in Theorem 4.2.3 and Theorem 4.2.4 hold, and we conclude that $\lambda'^\ell \le \rho(C) \le \lambda'^r$. Since $[3] \times [1]$ is the set of the pairs (i,j) described in Theorem 4.2.3(b) and Theorem 4.2.4(b), from simple comparison of the first columns $C'_\ell[-|\ 1] \ne C[-|\ 1] = C'_u[-|\ 1]$ of these three matrices, we easily conclude that $3 = \lambda'^\ell < \rho(C) = \lambda'^r = 4$ by the second part of Theorem 4.2.3 and that of Theorem 4.2.4.

4.3 Matrices with a rooted eigenvector

Before giving applications of Theorem 4.2.3 and Theorem 4.2.4, we need to construct C' which possesses a rooted eigenvector for some λ'. The following lemma comes immediately.

Lemma 4.3.1. *If a square matrix C' has a rooted eigenvector for λ', then $C' + dI$ also has the same rooted eigenvector for $\lambda' + d$, where d is a constant and I is the identity matrix with the same size of C'.* $\square$

A rooted column vector defined in Definition 4.2.1 is generalized to a rooted matrix as follows.

Definition 4.3.2. A matrix $C' = (c'_{ij})$ is called *rooted* if its columns and its row-sum vector are all rooted except the last column of C'.

The matrix Q in (4.2.1) is invertible with

$$Q^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Multiplying Q^{-1} to $C'Q$ in (4.2.2), $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} - c'_{n1} & c'_{12} - c'_{n2} & \cdots & c'_{1\ n-1} - c'_{n\ n-1} & r'_1 - r'_n \\ c'_{21} - c'_{n1} & c'_{22} - c'_{n2} & \cdots & c'_{2\ n-1} - c'_{n\ n-1} & r'_2 - r'_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n-1\ 1} - c'_{n1} & c'_{n-1\ 2} - c'_{n2} & \cdots & c'_{n-1\ n-1} - c'_{n\ n-1} & r'_{n-1} - r'_n \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & r'_n \end{pmatrix}. \quad (4.3.1)$$

The matrices C' and $Q^{-1}C'Q$ have the same set of eigenvalues. Moreover, v' is an eigenvector of C' for λ' if and only if $u = Q^{-1}v'$ is an eigenvector of $Q^{-1}C'Q$ for λ'. From (4.3.1), C' is rooted if and only if $Q^{-1}C'Q$ is nonnegative. The first part of the following lemma follows immediately from the above discussion and Theorem 2.1.1 by choosing $\lambda' = \rho(C')$.

Lemma 4.3.3. *If C' is a rooted matrix, then $Q^{-1}C'Q$ is nonnegative, $\rho(C')$ is an eigenvalue of C', and C' has a rooted eigenvector $v' = Qu$ for $\rho(C')$, where u is a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Moreover, with $v' = (v'_1, v'_2, \ldots, v'_n)^T$, the following (i)-(ii) hold.*

(i) If $C'[n|n)$ is positive, then v' is positive.

(ii) If $C'[n|n)$ is positive and $r'_i > r'_n$ for all $1 \le i \le n-1$, then $v'_j > v'_n$ for all $1 \le j \le n-1$.

Proof. It remains to prove (i) and (ii).

(i) Suppose that $C'[n|n)$ is positive and $v'_n = 0$. Then

$$\sum_{j=1}^{n-1} c'_{nj} v'_j = \sum_{j=1}^{n} c'_{nj} v'_j = (C'v')_n = \rho(C') v'_n = 0.$$

Hence v' is a zero vector since $c'_{nj} > 0$ for $j \leq n-1$, a contradiction. So $v'_n > 0$ and $v' > 0$ since v' is rooted.

(ii) The assumptions imply that the matrix $Q^{-1}C'Q$ in (4.3.1) is irreducible. Hence u is positive. By Lemma 4.2.2(ii), $v'_j > v'_n$ for $1 \leq j < n$. $\square$

The largest real eigenvalue of the following matrix will be used to obtain bounds of the spectral radius of a nonnegative matrix.

Fix $d, f, r_1, r_2, \ldots, r_n \geq 0$ such that $r_j \geq r_n$ for $1 \leq j \leq n-1$, and let

$$M_n(d, f, r_1, r_2, \ldots, r_n) = \begin{pmatrix} d & f & \cdots & f & r_1 - (d + (n-2)f) \\ f & d & & f & r_2 - (d + (n-2)f) \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{n-1} - (d + (n-2)f) \\ f & f & \cdots & f & r_n - (n-1)f \end{pmatrix} \tag{4.3.2}$$

be an $n \times n$ matrix with row-sum vector $(r_1, r_2, \ldots, r_n)^T$.

Note that for any square matrix C', it might be $\rho(C' + dI) \neq \rho(C') + d$, but $\rho_r(C' + dI) = \rho_r(C') + d$ always holds, where $\rho_r(C' + dI)$ and $\rho_r(C')$ are the largest real eigenvalues of $C' + dI$ and C' respectively. Also $\rho(C') = \rho_r(C')$ if C' is nonnegative.

Lemma 4.3.4. *The following (i)-(ii) hold.*

(i) The matrix $M_n(d, f, r_1, r_2, \ldots, r_n)$ has a rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ for the largest real eigenvalue $\rho_r(M_n(d, f, r_1, r_2, \ldots, r_n))$ of $M_n(d, f, r_1, r_2, \ldots, r_n)$.

(ii) If $f > 0$, then $v' > 0$.

Proof. Let $M_n := M_n(d, f, r_1, r_2, \ldots, r_n)$. First assume $d \geq f$. Then M_n is rooted. (i)-(ii) follows from (i)-(ii) of Lemma 4.3.3, in particular $\rho(M_n) = \rho_r(M_n)$. If $d < f$, then the matrix $(f-d)I + M_n$ is a rooted matrix. As in the first part, let v' be a rooted eigenvector of $(f-d)I + M_n$ for $\rho((f-d)I + M_n)$. Note that v' is also a rooted eigenvector of M_n for $\rho_r(M_n) = \rho((f-d)I + M_n) - (f-d)$. This proves (i), and (ii) follows similarly from (ii) of Lemma 4.3.3. $\square$

Lemma 4.3.5. *For $M_n = M_n(d, f, r_1, r_2, \ldots, r_n)$ and $M_t = M_t(d, f, r_1, r_2, \ldots, r_t)$ defined in* (4.3.2)*, where $d, f \geq 0$ and $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$, we have the following (i)-(iii).*

(i) The largest real eigenvalue $\rho_r(M_n)$ of M_n satisfies

$$\begin{aligned}\rho_r(M_n) =& \frac{r_n + d - f + \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2} \\ &\geq \max(d - f, r_n).\end{aligned}$$

(ii) If $r_n = 0$, then

$$\rho_r(M_n) = \frac{d - f + \sqrt{(d-f)^2 + 4fm}}{2},$$

where $m := \sum_{i=1}^{n-1} r_i$ is the sum of all entries of M_n.

(iii) If $r_t = r_n$ for some $t \leq n$, then $\rho_r(M_t) = \rho_r(M_n)$.

Proof. (i) We consider the matrix $M_n + (f-d)I$. Note that $(M_n + (f-d)I)^T$ has equitable quotient matrix

$$\Pi((M_n + (f-d)I)^T) = \begin{pmatrix} (n-1)f & f \\ \sum_{i=1}^{n-1}(r_i - (d + (n-2)f)) & r_n - (d + (n-2)f) \end{pmatrix}$$

with respect to the partition $\Pi = \{\{1, 2, \ldots, n-1\}, \{n\}\}$ of $[n]$, which has two eigenvalues

$$\frac{r_n - d + f \pm \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2}.$$

Since $((M_n + (f-d)I)^T)_{ij} = ((M_n + (f-d)I)^T)_{kj}$ for all $i, k \in [n-1]$ and $j \in [n]$ and by Lemma 2.1.6, $(M_n + (f-d)I)^T$ has eigenvalues

$$0^{n-2}, \frac{r_n - d + f \pm \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2},$$

and M_n has eigenvalues

$$(d-f)^{n-2}, \frac{r_n + d - f \pm \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2}.$$

Note that

$$\frac{r_n + d - f + \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2} \geq \max(d-f, r_n).$$

So the proof of (i) is complete.

(ii) and (iii) follow from (i) immediately. □

4.4 Rooted matrices under equitable quotient

We have learned that the spectral radius of a nonnegative matrix is preserved under equitable quotient operation. A rooted matrix has the similar property if the partition is chosen carefully.

Lemma 4.4.1. *If an $n \times n$ rooted matrix C' has an equitable quotient matrix $\Pi(C')$ with respect to a partition $\Pi = \{\pi_1, \ldots, \pi_\ell\}$, where $\pi_\ell = \{p\}$, then $\Pi(C')$ is rooted, $\rho(\Pi(C'))$ is an eigenvalue of C' and*

$$\rho(C') = \rho(\Pi(C')).$$

Proof. Let Q be the same as in (4.2.1). It's easy to see that Q and Q^{-1} have equitable quotient matrices $\Pi(Q)$ and $\Pi(Q^{-1})$ respectively, and so does $QC'Q^{-1}$ with

$$\Pi(QC'Q^{-1}) = \Pi(Q)\Pi(C')\Pi(Q)^{-1}$$

by Proposition 2.1.10. Since $QC'Q^{-1}$ as shown in (4.3.1) is nonnegative, $\Pi(Q)\Pi(C')\Pi(Q)^{-1}$ is also nonnegative and $\Pi(C')$ is rooted. Hence $\rho(\Pi(C'))$ is an eigenvalue of $\Pi(C')$ by Lemma 4.3.3. Furthermore,

$$\rho(C') = \rho(QC'Q^{-1}) = \rho(\Pi(QC'Q^{-1})) = \rho(\Pi(Q)\Pi(C')\Pi(Q)^{-1}) = \rho(\Pi(C'))$$

by Proposition 2.1.7. □

Remark 4.4.2. In Lemma 4.4.1, we have that $\Pi(Q)\Pi(C')\Pi(Q^{-1}) = \Pi(QC'Q^{-1})$ is nonnegative and $\Pi(C')$ is rooted, but $\Pi(C'^T)^T$ may not be rooted. For example, let

$$C' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then C' has equitable quotient matrix $\Pi(C')$ with respect to the partition $\Pi = \{\{1,2\},\{3\}\}$ and $\Pi(C'^T)^T = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$ is not rooted.

Note that if π_ℓ contains p and another number, then the equality in Lemma 4.4.1 may not hold.

Example 4.4.3. Let C' be a rooted matrix with equitable quotient matrix $F(C')$ with respect to the partition $\Pi = \{\{1\},\{2,3\}\}$ as follows

$$C' = \left(\begin{array}{c|cc} 2 & 3 & 3 \\ \hline 0 & 3 & -2 \\ 0 & 0 & 1 \end{array}\right), \qquad \Pi(C') = \begin{pmatrix} 2 & 6 \\ 0 & 1 \end{pmatrix}.$$

Notice that $\rho(C') = 3 \neq 2 = \rho(\Pi(C'))$.

4.5 Spectral upper bounds with prescribed sum of entries

Let J_k, I_k and O_k be the $k \times k$ all-one matrix, the $k \times k$ identity matrix and the $k \times k$ zero matrix respectively. We recall an old result of Richard Stanley [25].

Theorem 4.5.1 ([25]). *Let $C = (c_{ij})$ be an $n \times n$ symmetric (0,1) matrix with zero trace. Let the number of 1's of C be $2e$. Then*

$$\rho(C) \leq \frac{-1 + \sqrt{1 + 8e}}{2}.$$

Equality holds if and only if

$$e = \binom{k}{2}$$

and PCP^T has the form

$$\begin{pmatrix} J_k - I_k & 0 \\ 0 & O_{n-k} \end{pmatrix} = (J_k - I_k) \oplus O_{n-k}$$

for some permutation matrix P and positive integer k.

□

The following theorem generalizes Theorem 4.5.1 to nonnegative matrices, not necessarily symmetric.

Theorem 4.5.2. *Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix. Let m be the sum of entries in C and d (resp. f) be any number which is larger than or equal to the largest diagonal element (resp. the largest off-diagonal element) of C. Then*

$$\rho(C) \leq \frac{d - f + \sqrt{(d - f)^2 + 4mf}}{2}. \tag{4.5.1}$$

Moreover, if $mf > 0$, then the equality in (4.5.1) holds if and only if $m = k(k-1)f + kd$ and PCP^T has the form

$$\begin{pmatrix} fJ_k + (d - f)I_k & 0 \\ 0 & O_{n-k} \end{pmatrix} = (fJ_k + (d - f)I_k) \oplus O_{n-k}$$

for some permutation matrix P and some nonnegative integer k.

Proof. If $f = 0$ then the nonzero entries only appear on the diagonal of C, so $\rho(C) \leq d$ and (4.5.1) holds. Assume $f > 0$ for the remaining. Consider the $(n+1) \times (n+1)$ nonnegative matrix $M = C \oplus O_1$ which has row-sum vector $(r_1, r_2, \ldots, r_n, r_{n+1})^T$ with $r_{n+1} = 0$ and a nonnegative left eigenvector v^T for $\rho(M) = \rho(C)$. Let $C' = M_{n+1}(d, f, r_1, r_2, \ldots, r_{n+1})$ as defined in (4.3.2) which has the same row-sum vector of M, and has a positive rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_{n+1})^T$ for $\rho_r(C')$ by Lemma 4.3.4(i). Clearly $M[-|n+1) \leq C'[-|n+1)$ and $v^T v' > 0$. Hence the assumptions (i)-(iv) in Theorem 4.2.3 hold with $(C, \lambda, \lambda') = (M, \rho(M), \rho_r(C'))$. Now by Theorem 4.2.3 and Lemma 4.3.5(ii), we have

$$\rho(C) = \rho(M) \leq \rho_r(C') = \frac{d - f + \sqrt{(d-f)^2 + 4mf}}{2},$$

finishing the proof of the first part.

To prove the second part, assume $m = k(k-1)f + kd$ and $PCP^T = (fJ_k + (d-f)I_k) \oplus O_{n-k}$ for one direction. Using $\rho(C) = \rho(PCP^T) = \rho(fJ_k + (d-f)I_k)$, we have

$$\rho(C) = (k-1)f + d = \frac{d - f + \sqrt{(d-f)^2 + 4mf}}{2}.$$

For the other direction, assume $\rho(C) = \rho_r(C')$ and $mf > 0$. In particular $C \neq 0$ and $M \neq 0$. Let $(v_1, v_2, \ldots, v_{n+1})$ be a nonnegative left eigenvector of M. Then $v_{n+1} = 0$. Write $\tilde{v}^T = (v_1, v_2, \ldots, v_n)$. We first assume that C has no zero row. Then $r_i > r_{n+1} = 0$ for $1 \leq i \leq n$. By Lemma 4.3.3(ii) with $(C', n) = (M, n+1)$, we have $v'_j > v'_{n+1}$. Then $c_{ij} = m_{ij} = c'_{ij}$ for the indices $1 \leq i \leq n$ with $v_i \neq 0$ and any $1 \leq j \leq n$ by Theorem 4.2.3(b). Hence

$$\rho(C)\tilde{v}^T = \tilde{v}^T C = \tilde{v}^T C'(n+1|n+1) = \tilde{v}^T(fJ + (d-f)I). \tag{4.5.2}$$

Since $\tilde{v}^T$ is a nonnegative left eigenvalue of the irreducible nonnegative matrix $fJ + (d-f)I$ for $\rho(C)$, we have $\tilde{v} > 0$. This together with $fJ + (d-f)I \geq C$ and (4.5.2) imply $C = fJ + (d-f)I$, finishing the proof for the case under the assumption that C has no zero row. Assume that C has $n-k$ zero rows for some $1 \leq k \leq n-1$. Then there is a permutation matrix P such that all zero rows of PCP^T appear in the end, so the $(n-k) \times n$ submatrix $PCP^T([k]|-]$ of PCP^T is 0 and the $k \times n$ submatrix $PCP^T[[k]|-]$ of PCP^T has no zero row. Let $C_1 = PCP^T[[k]|[k]]$ and m' be the sum of entries in C_1. Notice that $\rho(C_1) = \rho(C)$ and $m' \le m$. Applying the first part of the theorem to C_1, we have

$$\rho(C_1) \le \frac{d-f+\sqrt{(d-f)^2+4m'f}}{2} \le \frac{d-f+\sqrt{(d-f)^2+4mf}}{2} = \rho(C) = \rho(C_1).$$

Forcing $m' = m$, C_1 has no zero row and $C_1 = fJ_k + (d-f)I_k$. Hence $PCP^T[[k]|[k]]) = 0$ and this implies $PCP^T = (fJ_k + (d-f)I_k) \oplus O_{n-k}$ and $m = k(k-1)f + kd$. □

We give another proof of Theorem 4.5.2 which needs less knowledge but is tricky.

The second proof of Theorem 4.5.2. Since C in nonnegative, it has a nonnegative left eigenvector $v^T = (v_1, \ldots, v_n)$ for $\rho(C)$. Without lose of generality, we assume that for some $1 \le k \le n$, $v_i > 0$ for $1 \le i \le k$ and $v_i = 0$ for $k+1 \le i \le n$. Let $\rho = \rho(C)$. Then the matrix $C^2 - (d-f)C$ has the same nonnegative left eigenvector v^T for $\rho^2 - (d-f)\rho$. So the maximum row-sum of $C^2 - (d-f)C$ is an upper bound of $\rho^2 - (d-f)\rho$ by Lemma 2.1.4.

Define $r_i(M)$ as the i-th row-sum of M. Let $r_i = r_i(C)$. We have

$$r_i(C^2) = \sum_{j=1}^{n}\sum_{s=1}^{n} c_{is}c_{sj} = \sum_{s=1}^{n}(c_{is}\sum_{j=1}^{n} c_{sj}) = c_{ii}r_i + \sum_{s\neq i} c_{is}r_s \le dr_i + f(m-r_i),$$

with equality if and only if $c_{ii}r_i = dr_i$ and $c_{is}r_s = fr_s$ for $s \neq i$. So

$$\rho^2 - (d-f)\rho \le \max_{1\le i\le n} r_i(C^2 - (d-f)C) = \max_{1\le i\le n} r_i(C^2) - (d-f)r_i \le fm.$$

This implies

$$\rho(C) \le \frac{d-f+\sqrt{(d-f)^2+4fm}}{2}.$$

By Lemma 2.1.4,

$$\rho(C) = \frac{d-f+\sqrt{(d-f)^2+4fm}}{2} \tag{4.5.3}$$

if and only if for $1 \le i \le k$

$$c_{ii}r_i = dr_i \text{ for } 1 \le j \le n \text{ and } c_{is}r_s = fr_s \text{ for } s \neq i, 1 \le j \le n. \tag{4.5.4}$$

Suppose $mf > 0$. If $C = (fJ_k + (d-f)I_k) \oplus O_{n-k}$, it is easy to see that (4.5.4) holds. Now assume (4.5.3) holds. Then (4.5.4) holds. Since $v_{k+1} = \cdots = v_n = 0$, $C[[k]|[k])$ is a zero matrix. This implies $r_s = 0$ for $k+1 \le s \le n$, otherwise $r_s \neq 0$ and $c_{is} = f \neq 0$ for $1 \le i \le k$. So $C[[k]|-]$ is a zero matrix. Since $v_1 > 0$, $c_{s1} \neq 0$ for some $1 \le s \le k$ and $r_s \neq 0$. Then $c_{is} = f \neq 0$ for $1 \le i \le k, i \neq s$. Hence $r_i \neq 0$ for $1 \le i \le k$. So $C[k]|[k] = fJ_k + (d-f)I_k$ and $C = (fJ_k + (d-f)I_k) \oplus O_{n-k}$. □

In the first proof of Theorem 4.5.2, we get an upper bound from a new matrix C' by computing its spectral radius, and in the second one, we get an upper bound by computing the maximum row-sum of $g(C)$, where $g(x) = x^2 - (d-f)x$. These two proofs are quite different. Can we find a relation between them?

Problem 4.5.3. For a bound of C given by computing the maximum row-sum of $g(C)$ from a polynomial $g(x)$, can we find C' as in the second proof such that $\rho(C')$ equals the bound? How about the converse?

4.6 The spectral bound $\rho(\Pi(C'))$

From now on we assume that the square matrix C is nonnegative, and the eigenvalue $\rho(C)$ of C corresponds to a nonnegative left eigenvector v^T by Theorem 2.1.1(i). Then the assumption (iii) in Theorem 4.2.3 and Theorem 4.2.4 immediately holds. In Lemma 4.3.1 and Lemma 4.3.3, we know that a rooted matrix C' (and its translates) has a rooted eigenvector for $\rho_r(C')$. In this section, we shall apply properties of the equitable quotient to find some matrices which are not translates of rooted matrices but still have positive rooted eigenvectors. We use this method to reduce the size of C' in finding the bound λ' of $\lambda = \rho(C)$ in Theorem 4.2.3 and Theorem 4.2.4.

Theorem 4.6.1. *Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sum vector $(r_1, \ldots, r_n)^T$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ a partition of $\{1, 2, \ldots, n\}$ with $n \in \pi_\ell$. Let $C' = (c'_{ij})$ be an $n \times n$ matrix that admits an $\ell \times \ell$ equitable quotient matrix $\Pi(C') = (\pi'_{ab})$ of C' with respect to Π satisfying the following (i)-(ii):*

(i) $C[-|n) \leq C'[-|n)$ *and* $\Pi(C')$ *has row-sum vector* $\Pi(r') = (\pi(r')_1, \pi(r')_2, \ldots, \pi(r')_\ell)^T$ *with* $\pi(r')_a \geq \max_{i\in\pi_a} r_i$ *for* $1 \leq a \leq \ell$.

(ii) $\Pi(C')$ *has a positive rooted eigenvector* $\Pi(v') = (\pi(v')_1, \pi(v')_2, \ldots, \pi(v')_\ell)^T$ *for some nonnegative eigenvalue* λ'.

Then

$$\rho(C) \leq \lambda'. \tag{4.6.1}$$

Moreover, if C *is irreducible, then* $\rho(C) = \lambda'$ *if and only if*

(a) $r_i = \pi(r')_a$ *for* $1 \leq a \leq \ell$ *and* $i \in \pi_a$, *and*

(b) $c'_{ij} = c_{ij}$ *for all* $1 \leq i, j \leq n$ *such that for* $1 \leq b \leq \ell$ *with* $j \in \pi_b$ *we have* $\pi(v')_b > \pi(v')_\ell$.

Proof. Let S be the $n \times \ell$ characteristic matrix of Π. From the construction of Π and C', $r' = S\Pi(r') = (r'_1, \ldots, r'_n)^T$ is the row-sum vector of C', and $v' = S\Pi(v')$ is a positive rooted eigenvector of C' for λ' by Lemma 2.1.5. Since C is nonnegative, there exists a nonnegative left eigenvector v^T of C for $\rho(C)$ by Theorem 2.1.1(i). Hence $v^Tv' > 0$. Thus assumptions (i)-(iv) of Theorem 4.2.3 hold, concluding $\rho(C) \leq \lambda'$.

Suppose that C is irreducible. Then the above v is positive. Hence the condition (b) of $\rho(C) = \lambda'$ in Theorem 4.2.3 becomes $c'_{ij} = c_{ij}$ for $1 \leq i \leq n$,$1 \leq j \leq n-1$ with $v'_j > v'_n$, and this is equivalent to the condition (b) here from the structure of $v' = S\Pi(v')$. The condition (a) here is immediate from that in Theorem 4.2.3 since $r'_i = \pi(r')_a$ for $i \in \pi_a$. $\square$

Notice that the irreducible assumption of C in the second part of Theorem 4.5.2 is not necessary. The following example shows that this is a must in that of Theorem 4.6.1.

Example 4.6.2. Consider the following two 3×3 matrices

$$C = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Pi(C') = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$$

is the equitable quotient matrix of C' with respect to the partition $\Pi = \{\{1\},\{2,3\}\}$. Note that $C[-|3) = C'[-|3)$ and $(3,2,1)^T \le (3,2,2)^T$, where $(3,2,1)^T$ and $(3,2,2)^T$ are the row-sum vectors of C and C' respectively. Since $\Pi(C') + I$ is positive and rooted, $\Pi(C')$ has a positive rooted eigenvector for $\lambda' = \rho(\Pi(C')) = (1+\sqrt{13})/2$ by Lemma 4.3.3(i). Hence assumptions (i)-(ii) in Theorem 4.6.1 hold. By direct computing, $\rho(C) = (1+\sqrt{13})/2$, so the the equality in (4.6.1) holds. However, $r_2 = 2 \neq 1 = r_3$, a contradiction to (a) in Theorem 4.6.1. This contradiction is because of the reducibility of C.

The following is a dual version of Theorem 4.6.1.

Theorem 4.6.3. *Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sum vector $(r_1, \ldots, r_n)^T$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ a partition of $\{1, 2, \ldots, n\}$ with $n \in \pi_\ell$. Let C' be an $n \times n$ matrix that admits an $\ell \times \ell$ equitable quotient matrix $\Pi(C') = (\pi'_{ab})$ of C' with respect to Π satisfying the following (i)-(ii):*

(i) $C[-|n) \ge C'[-|n)$ and $\Pi(C')$ has row-sum vector $\Pi(r') =(\pi(r')_1, \pi(r')_2, \ldots, \pi(r')_\ell)^T$ with $\pi(r')_a \le \min_{i\in\pi_a} r_i$ for $1 \le a \le \ell$.

(ii) $\Pi(C')$ has a positive rooted eigenvector $\Pi(v') =(\pi(v')_1, \pi(v')_2, \ldots, \pi(v')_\ell)^T$ for some nonnegative eigenvalue λ'.

Then

$$\rho(C) \ge \lambda'. \tag{4.6.2}$$

Moreover, if C is irreducible then $\rho(C) = \lambda'$ if and only if

(a) $r_i = \pi(r')_a \qquad$ for $1 \le a \le \ell$ and $i \in \pi_a$, and

(b) $c'_{ij} = c_{ij} \qquad$ for all $1 \le i, j \le n$ such that for $1 \le b \le \ell$ with $j \in \pi_b$ we have $\pi(v')_b > \pi(v')_\ell$.

□

Remark 4.6.4. (i) The positive assumption of $\Pi(v')$ in (ii) of Theorem 4.6.3 can be removed in concluding the first part $\rho(C) \ge \lambda'$. The following is a proof:

Proof. From (i) and referring to (4.2.2), we have $CQ \geq C'Q \geq 0$. Let $v' = S\Pi(v')$ be a rooted eigenvector of C' for λ' as shown in the above proof. Then $u = Q^{-1}v'$ is nonnegative by Lemma 4.2.2(i), so $Cv' = CQu \geq C'Qu = C'v' = \lambda' v'$. Since v' is nonnegative, $\rho(C) \geq \lambda'$ by Theorem 2.1.1(iii). □

(ii) The following counterexample shows that to conclude $\rho(C) \leq \lambda'$, the positive assumption of $\Pi(v')$ in (ii) of Theorem 4.6.1 can not be removed:

$$C = C' = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \lambda' = 1, v' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \rho(C) = 2, v^T = (0, 1),$$

where the trivial partition $\Pi = \{\{1\}, \{2\}\}$ of $\{1, 2\}$ is adopted.

We provide an example in applying Theorem 4.6.1.

Example 4.6.5. Consider the following two 7×7 matrices C and C' expressed below under the partition $\Pi = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$:

$$C = \left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right), \qquad C' = \left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right). \tag{4.6.3}$$

Apparently, $C[-|7) \leq C'[-|7)$, and the row-sum vector $(24, 23, 22, 20, 19, 13, 12)^T \leq (24, 24, 24, 20, 20, 13, 13)^T$, where $(24, 23, 22, 20, 19, 13, 12)^T$ and $(24, 24, 24, 20, 20, 13, 13)^T$ are the row-sum vectors of C and C' respectively. So assumption (i) of Theorem 4.6.1 holds. Note that C' is not rooted and neither of its translates. Since C' has equitable quotient matrix

$$\Pi(C') = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix}$$

with respect to Π, in which $\Pi(C') + 2I$ is rooted. So assumption (ii) of Theorem 4.6.1 holds with $\lambda' = \rho_r(C')$ by Lemma 4.3.1 and Lemma 4.3.3(i). Hence by Theorem 4.6.1, $\rho(C) \leq \rho_r(\Pi(C')) \approx 18.6936$.

If we apply Lemma 2.1.2 by constructing the following nonnegative matrix $C'' \geq C$, and find its equitable quotient matrix $\Pi(C'')$ with respect to the above partition Π:

$$C'' = \left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 6 \\ 2 & 3 & 2 & 4 & 2 & 8 & 4 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 1 & 3 & 4 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 2 & 2 & 4 \end{array}\right), \qquad \Pi(C'') = \begin{pmatrix} 7 & 6 & 12 \\ 12 & 2 & 7 \\ 4 & 4 & 6 \end{pmatrix},$$

one will find the upper bound

$$\rho(C'') = \rho(\Pi(C'') \approx 19.4$$

of $\rho(C)$ which is larger than the previous one.

Remark 4.6.6. In Theorem 4.6.1 and Theorem 4.6.3, if the condition "C is nonnegative" is replaced by "C has a nonnegative left eigenvector for λ" and $\rho(C)$ is replaced by λ, the inequality also holds. The proof is the same. For the convenience in the next section, we do not state that in Theorem 4.6.1 and Theorem 4.6.3.

4.7 More irrelevant columns

Considering the part π_ℓ of column indices of C and C' in the assumption (i) of Theorem 4.6.1, the assumption $C[-|\pi_\ell] \leq C'[-|\pi_\ell]$ for C' is not really necessary. We might replace the columns indexed by π_ℓ in C' by any other columns and adjust the values in the last column keeping the row-sums of C' unchanged.

In this situation, the columns of C' indexed by π_ℓ are irrelevant columns (in the comparison of C and C'). For example in Example 4.6.5, the values in the 6-th column of C' can be changed to any values (e.g., $(a, b, c, d, e, f, g)^T$), if the values in the 7-th column of C' make the corresponding change (e.g., $(11 - a, 11 - b, 11 - c, 6 - d, 6 - e, 5 - f, 5 - g)^T$ correspondingly), i.e., columns 6 and 7 of C' are irrelevant. The following theorem generalizes this idea when restricting $\Pi[C']$ in Theorem 4.6.1 to be a rooted matrix or its translate.

Theorem 4.7.1. *Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be a partition of $[n]$ with $n \in \pi_\ell$, and C be an $n \times n$ nonnegative matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq a \leq \ell$ and $1 \leq b \leq \ell - 1$, choose r''_a, c''_{ab} such that*

$$\begin{cases} r''_a & \geq & \max_{i \in \pi_a} r_i; \\ c''_{ab} & \geq & \sum_{j \in \pi_b} c_{ij} \qquad \text{for all } i \in \pi_a; \\ c''_{ab} & \geq & c''_{\ell b} > 0 \qquad \text{for } a \neq b; \\ r''_a & \geq & r''_\ell \end{cases}$$

and let

$$c''_{a\ell} = r''_a - \sum_{j=1}^{\ell-1} c''_{aj}.$$

Then the $\ell \times \ell$ matrix $C'' = (c''_{ab})$ has a positive rooted eigenvector $v'' = (v''_1, v''_2, \ldots, v''_\ell)^T$ for $\rho_r(C'')$ and $\rho(C) \leq \rho_r(C'')$. Moreover, if C is irreducible, then $\rho(C) = \rho_r(C'')$ if and only if

(a) $r_i = r''_a$ *for $1 \leq a \leq \ell$ and $i \in \pi_a$, and*

(b) $\sum_{j \in \pi_b} c_{ij} = c''_{ab}$ *for all $1 \leq a, b \leq \ell$ with $v''_b > v''_\ell$ and $i \in \pi_a$.*

Proof. From the construction of C'', $C'' + dI$ is a rooted matrix with $(C + dI)[n|n)$ positive for d large enough, so C'' has a positive rooted eigenvector for $\rho_r(C'')$ by Lemma 4.3.1 and Lemma 4.3.3(i). In view of the construction of C' in Example 4.6.5, we construct an $n \times n$ matrix C' such that C' has equitable quotient matrix $\Pi(C') = C''$ and assumptions (i)-(ii) of Theorem 4.6.1 hold for $\lambda' = \rho_r(\Pi(C'))$. Hence the remaining follows from the conclusion of Theorem 4.6.1. $\square$

Remark 4.7.2. Theorem 4.0.1 is a special case of Theorem 4.7.1 with $\Pi = \{\{1\}, \{2\}, \ldots, \{\ell-1\}, \{\ell, \ell+1, \ldots, n\}\}$ and $C'' = M_\ell(d, f, r_1, r_2, \ldots, r_\ell)$ as shown in (4.3.2). To characterize when the equality holds, we needs to apply Lemma 4.3.5(iii) by choosing a new ℓ to be the least t such that $r_t = r_\ell$. By using the more irrelevant columns idea in Theorem 4.7.1, the assumption $f := \max_{1\le i\ne j\le n} c_{ij}$ and $d := \max_{1\le i\le n} c_{ii}$ in Theorem 4.0.1 can be replaced by the possible smaller number $f := \max_{1\le i\le n, 1\le j\le \ell-1, i\ne j} c_{ij}$ and $d := \max_{1\le i\le \ell-1} c_{ii}$ respectively.

A new proof of the following theorem proposed by Csikvári [8] is another application of Theorem 4.7.1. This proof is systematic while the original proof is somewhat tricky. An independent set is a set of vertices in a graph, no two of which are adjacent.

Theorem 4.7.3 ([8]). *Assume that the set $K = \{v_1, v_2, \ldots, v_k\}$ forms a clique in the graph G and $V(G) \setminus K = \{v_{k+1}, \ldots, v_n\}$ forms an independent set. Let e be the number of edges between K and $V(G) \setminus K$. Then*

$$\rho(G) \le \frac{k-1+\sqrt{(k-1)^2+4e}}{2}. \tag{4.7.1}$$

Moreover, the equality holds if and only if v_i has the same neighborhood in $V(G) \setminus K$ for each $1 \le i \le k$.

Proof. Let C be the adjacency matrix of G according to the order $v_1, v_2, \ldots, v_n$ and v_{k+1} has the maximum degree among $\{v_{k+1}, \ldots, v_n\}$ without lose of generality. Let $\Pi = \{\{1\}, \{2\}, \ldots, \{k\}, \{k+1, \ldots, n\}\}$ and $C'' = (c''_{ij})$ be a $(k+1) \times (k+1)$ matrix with a rooted eigenvector for $\rho_r(C')$, where

$$c''_{ij} = \begin{cases} 1, & \text{if } i \ne j \text{ and } j \le k; \\ 0, & \text{if } i = j \text{ and } i, j \le k; \\ \deg(v_i) - k + 1, & \text{if } i \le k \text{ and } j = k+1; \\ \deg(v_{k+1}) - k, & \text{if } i = k+1 \text{ and } j = k+1. \end{cases}$$

Then $\rho(G) \le \rho_r(C'')$ by Theorem 4.7.1. We can see that

$$C''' = \begin{pmatrix} k-1 & e \\ 1 & \deg(v_{k+1}) - k \end{pmatrix}$$

is the equitable quotient matrix of C''^T with respect to the partition $\{\{1, 2, \dots k\}, \{k+1\}\}$. So

$$\rho(G) \le \rho_r(C'') = \rho_r(C''') = \frac{k-1-c+\sqrt{(k-1+c)^2+4e}}{2} \tag{4.7.2}$$

$$\le \frac{k-1+\sqrt{(k-1)^2+4e}}{2} \tag{4.7.3}$$

by Corollary 2.1.9, where $c = k - \deg(v_{k+1}) \ge 0$. Note that if $e > 0$, then the equality in (4.7.3) holds if and only if $c = 0$.

If v_i has the same neighborhood in $V(G) \setminus K$ for each $1 \le i \le k$, then

$$\Pi(C) = \begin{pmatrix} k-1 & e/k \\ k & 0 \end{pmatrix}$$

is the equitable quotient matrix of C with respect to the partition $\Pi = \{\{1, 2, \dots, k\}, \{k+1, \dots, n\}\}$ of $[n]$. By Proposition 2.1.7, $\rho(G) = \rho(\Pi(C))$ and the equality in (4.7.1) holds. For the converse, suppose the equality in (4.7.1) holds. Then the equalities in (4.7.2) and in (4.7.3) also hold. If $e = 0$, then v_i has the same neighborhood in $V(G) \setminus K$ for each $1 \le i \le k$. Now assume $e > 0$. Then the equality in (4.7.3) implies $c = 0$ and $\deg(v_{k+1}) = k$. If G is connected, then the equality in (4.7.2) and Theorem 4.7.1(a) imply that $\deg(v_j) = k$ for $k+1 \le j \le n$. So v_i has the same neighborhood in $V(G) \setminus K$ for each $1 \le i \le k$. If G is not connected, then the component G' containing K is the only component which is not an isolated vertex. Using G' instead of G, we get the conclusion we want.

□

Remark 4.7.4. If we remove the condition "$V(G)\setminus$ is an independent set" in Theorem 4.7.3, (4.7.2) still holds.

The following is the dual theorem of Theorem 4.7.1.

Theorem 4.7.5. *Let $\Pi = \{\pi_1, \pi_2, \dots, \pi_\ell\}$ be a partition of $[n]$ with $n \in \pi_\ell$, and C an $n \times n$ nonnegative matrix with row-sums $r_1 \ge r_2 \ge \cdots \ge r_n$. For*

$1 \le a \le \ell$ *and* $1 \le b \le \ell - 1$, *choose* r''_a, c''_{ab} *such that*

$$\left\{\begin{array}{lll} r''_a & \le & \min_{i\in\pi_a} r_i; \\ c''_{ab} & \le & \sum_{j\in\pi_b} c_{ij} \qquad \text{for all } i \in \pi_a; \\ c''_{ab} & \ge & c''_{\ell b} > 0 \qquad \text{for } a \ne b; \\ r''_a & \ge & r''_\ell \end{array}\right. \tag{4.7.4}$$

and let

$$c''_{a\ell} = r''_a - \sum_{j=1}^{\ell-1} c''_{aj}. \tag{4.7.5}$$

Then the $\ell \times \ell$ *matrix* $C'' = (c''_{ab})$ *has a positive rooted eigenvector* $v'' = (v''_1, v''_2, \ldots, v''_\ell)^T$ *for* $\rho(C'')$ *and* $\rho(C) \ge \rho(C'')$. *Moreover, if* C *is irreducible, then* $\rho(C) = \rho(C'')$ *if and only if*

(a) $r_i = r''_a$ *for* $1 \le a \le \ell$ *and* $i \in \pi_a$, *and*

(b) $\sum_{j\in\pi_b} c_{ij} = c''_{ab}$ *for all* $1 \le a, b \le \ell$ *with* $v''_b > v''_\ell$ *and* $i \in \pi_a$.

4.8 Some new lower bounds of spectral radius

We shall apply Theorem 4.7.5 to obtain a lower bound of $\rho(C)$ for a nonnegative matrix C.

Theorem 4.8.1. *Let* $C = (c_{ij})$ *be an* $n \times n$ *nonnegative matrix with row-sums* $r_1 \ge r_2 \ge \cdots \ge r_n$. *For* $1 \le t < n$, *let* $\Pi_t = \{\{1, \ldots, t\}, \{t+1, \ldots, n\}\}$ *be a partition of* $[n]$. *Let* $d = \max_{t<i\le n} c_{ii}$ *and* $f = \max_{1\le i\le n, t<j\le n, i\ne j} c_{ij}$. *Assume that* $0 < r_n - (n-t-1)f - d$. *Then*

$$\rho(C) \ge \frac{r_t - f + d + \sqrt{(r_t - (2n-2t-1)f - d)^2 + 4(n-t)(fr_n - (n-t-1)f - d)}}{2}. \tag{4.8.1}$$

Moreover, if C *is irreducible and* $f > 0$, *then the equality holds in (4.8.1) if and only if* $r_1 = r_n$ *or*

(a) $r_1 = r_t$ *and* $r_{t+1} = r_n$, *and*

(b) $\sum_{j\in[t]} c_{ij} = r_t - (n-t)f$ *for all* $i \in [t]$, *and*

$\sum_{j\in[t]} c_{ij} = r_n - (n-t-1)f - d$ *for all* $t < i \le n$.

Proof. The lower bound of $\rho(C)$ in (4.8.1) follows by applying Theorem 4.7.5 with the following positive rooted matrix

$$C'' = \begin{pmatrix} r_t - (n-t)f & (n-t)f \\ r_n - (n-t-1)f - d & (n-t-1)f + d \end{pmatrix}, \tag{4.8.2}$$

which has row-sum vector $(r_t, r_n)^T$ and the assumptions in (4.7.4) and (4.7.5) of Theorem 4.7.5 hold from the assumptions. Note that C'' has a positive rooted eigenvector $(v_1'', v_2'')^T$ for $\rho(C'')$ by Lemma 4.3.3(i), and the value $\rho(C'')$ is as shown in the right of (4.8.1). To study the equality case in (4.8.1), we apply conditions (a)-(b) in Theorem 4.7.5, in which condition (a) is exactly the condition (a) of this theorem. If $v_1'' > v_2''$ then the condition (b) of this theorem is exactly the condition (b) of Theorem 4.7.5. Notice that $v_2'' = v_1''$ if and only if $\rho(C'') = r_t = r_n$ by Theorem 2.1.1 using the irreducible property of C''. This is also equivalent to $r_1 = r_n$ under the condition (a). □

The following corollary restricts Theorem 4.8.1 to binary matrix C.

Corollary 4.8.2. *Let* $C = (c_{ij})$ *be an* $n \times n$ *(0, 1) matrix with row-sums* $r_1 \ge r_2 \ge \cdots \ge r_n > 0$, *and choose* $t \ge n - r_n + 1$ *and* $t \le n$. *Then*

$$\rho(C) \ge \frac{r_t + \sqrt{r_t^2 - 4(n-t)(r_t - r_n)}}{2}. \tag{4.8.3}$$

Moreover, if C is irreducible, then equality holds in (4.8.3) if and only if $r_1 = r_n$ *or*

(a) $r_1 = r_t$ *and* $r_{t+1} = r_n$, *and*

(b) $\sum_{j\in[t]} c_{ij} = r_t - (n-t)$ *for all* $i \in [t]$, *and*

$\sum_{j\in[t]} c_{ij} = r_n - (n-t)$ *for all* $t < i \le n$.

□

Proof. If $t = n$, then (4.8.3) becomes $\rho(C) \geq r_n$. So the corollary follows from Lemma 2.1.3. Assume $t < n$. Since the assumptions in Theorem 4.8.1 clearly hold with $d = f = 1$, the corollary also holds by (4.8.1) in this case. $\square$

One can easily check that the right hand side of (4.8.3) is at least r_n (with equality iff $r_t = r_n$) by applying Lemma 2.1.3 on (4.8.2) with $d = f = 1$, so the above lower bound is better than the known one r_n in Lemma 2.1.3.

4.9 Characterizing the eigenvector of a rooted matrix

In the second parts of Theorem 4.7.1 and Theorem 4.7.5, we need the set $K := \{b | v''_b > v''_\ell\}$ to help us to characterize when the equality holds. Sometimes we can find K from the entries in C'' by the following lemma.

Lemma 4.9.1. *Let $C'' = (c''_{ab})$ be an $\ell \times \ell$ rooted matrix with row-sums $r''_1, r''_2, \ldots, r''_\ell$ not all equal and $v'' = (v''_1, v''_2, \ldots, v''_\ell)^T$ a positive rooted eigenvector of C'' for λ', let $K = \{b | v''_b > v''_\ell\}$, $K_1 = \{b | r''_b > r''_\ell\}$ and when K_t is defined, let $K_{t+1} = \{a \notin \bigcup_{s \leq t} K_s | c''_{ab} > c''_{\ell b}$ for some $b \in \bigcup_{s \leq t} K_s\}$. Then*

(i) $\bigcup_{s \leq t} K_s \subseteq K$ *for each $k \geq 1$, and*

(ii) *if the a-th row is equal to the ℓ-th row in C'' for each $a \notin \bigcup_{s \leq t} K_s$, then $\bigcup_{s \leq t} K_s = K$.*

Proof. For $1 \leq a \leq \ell$, we have

$$\rho(C'')v''_a = (C''v'')_a = \sum_{b=1}^{\ell} c''_{ab}v''_b = \sum_{b=1}^{\ell-1} c''_{\ell b}v''_b + \sum_{b=1}^{\ell-1}(c''_{ab} - c''_{\ell b})v''_b + c''_{a\ell}v''_\ell$$

$$\geq \sum_{b=1}^{\ell-1} c''_{\ell b}v''_b + \sum_{b=1}^{\ell-1}(c''_{ab} - c''_{\ell b})v''_\ell + c''_{a\ell}v''_\ell = \sum_{b=1}^{\ell-1} c''_{\ell b}v''_b + (r''_a - (r''_\ell - c''_{\ell\ell}))v''_\ell \tag{4.9.1}$$

$$= \sum_{b=1}^{\ell} c''_{\ell b}v''_b + (r''_b - r''_\ell)v''_\ell \geq \sum_{j=1}^{\ell} c''_{\ell b}v''_b = (C''v'')_\ell = \rho(C'')v''_\ell. \tag{4.9.2}$$

If $r'_a > r'_\ell$, then the inequality in (4.9.2) is strict. Hence $\lambda' v'_a > \lambda' v'_\ell$, so $\lambda' > 0$ and $a \in K$. This proves the base case $K_1 \subset K$. Assume $K_t \subset K$. We want to prove $K_{t+1} \subset K$. Choose $a \in K_{t+1}$. Then there exists $b \in K_t$ such that $c''_{ab} > c''_{\ell b}$ and $v'_b > v'_\ell$. Hence the inequality in (4.9.1) is strict and $v''_a > v''_\ell$, proving $a \in K$. This proves $\bigcup_{s \le t} K_s \subset K$ for $t \ge 1$. If the a-th row is equal to the ℓ-th row in C'' for each $a \notin \bigcup_{s \le t} K_s$, then $v''_a = v''_\ell$ for $a \notin \bigcup_{s \le t} K_s$ and $\bigcup_{s \le t} K_s = K$. □

Corollary 4.9.2. *Under the same assumption of Theorem 4.9.1, if $K_1 = [\ell - 1]$ or the a-th row is equal to the ℓ-th row in C'' for each a satisfying $r''_a = r''_\ell$, then $K = K_1 = \{a | r''_a > r''_\ell\}$.*

4.10 Choosing C'' to get more bounds

In this section, for a $n \times n$ nonnegative matrix C, we provide a class of matrices such that $M_\ell(d, f, r_1, r_2, \ldots, r_\ell)$ is contained in it and each C'' in it satisfying the assumptions in Theorem 4.7.1 (resp. Theorem 4.7.5).

Let C be an $n \times n$ nonnegative matrix with row-sums $r_1 \ge r_2 \ge \cdots \ge r_n$ and $\Pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ a partition of $[n]$ with $n \in \pi_\ell$. For $1 \le a \le \ell$, $1 \le b \le \ell - 1$, let $s_{ab} = \sum_{i \in \pi_a, j \in \pi_b} c_{ij}$. Choose $r''_a \ge \max_{i \in \pi_a} r_i$ and d_a for $1 \le a \le \ell$ such that

(i) $d_\ell \ge \max_{1 \le b \le \ell - 1} s_{\ell b}$,

(ii) $d_a \ge \max\{\max_{1 \le b \le \ell - 1} s_{\ell b}, \max_{1 \le b \le \ell - 1} s_{ab}\}$ for $1 \le a \le \ell - 1$,

(iii) $d_a \ge d_\ell > 0$ for $1 \le a \le \ell - 1$,

(iv) $r''_a \ge r''_\ell$ for $1 \le a \le \ell - 1$,

and let $C'' = (c''_{ij})$ be the $\ell \times \ell$ matrix with row-sums $r''_1, \ldots, r''_\ell$, where $c''_{ij} = d_i$

for $1 \leq i \leq \ell, 1 \leq j \leq \ell - 1$:

$$C'' = \left(\begin{array}{ccc|c} d_1 & \cdots & d_1 & r''_1 - (\ell-1)d_1 \\ \vdots & \ddots & \vdots & \vdots \\ d_{\ell-1} & \cdots & d_{\ell-1} & r''_{\ell-1} - (\ell-1)d_{\ell-1} \\ \hline d_\ell & \cdots & d_\ell & r''_\ell - (\ell-1)d_\ell \end{array}\right). \tag{4.10.1}$$

Then C'' is rooted and (C, C'') satisfies the assumptions of Theorem 4.7.1. There is a similar construction for Theorem 4.7.5.

The following lemma shows some information about the above C'' and how to choose a better C'' for given $d_1, d_2, \ldots, d_{\ell-1}$.

Lemma 4.10.1. *For ℓ a positive integer, $d_i \geq d_\ell \geq 0$, $r''_i \geq r''_\ell \geq 0$ for $1 \leq i \leq \ell$ and (r''_i) not all equal, let $C'' = (c''_{ij})$ is of the form in (4.10.1). Then we have the following.*

(i) $\rho(C'') = \dfrac{\sum_{i=1}^{\ell-1}(d_i - d_\ell) + r''_\ell + \sqrt{(\sum_{i=1}^{\ell-1}(d_i - d_\ell) - r''_\ell)^2 + 4d_\ell \sum_{i=1}^{\ell-1}(r''_i - r''_\ell)}}{2}.$

(ii) Let $v'' = (v''_1, \ldots, v''_\ell)^T$ be a rooted eigenvector of C'' for $\rho(C'')$. Then

$$K = K_1 \bigcup K_2 = \{i | r''_i > r''_\ell\} \bigcup \{i | d_i > d_\ell\},$$

where K, K_1, K_2 are the same as in Lemma 4.9.1.

(iii) Given $d_1, \ldots, d_{\ell-1}$, $r''_1, \ldots, r''_\ell$, let $A = \sum_{i=1}^{\ell-1}(r''_\ell + (\ell-1)d_i - 2r''_i)/(\ell-1)^2$ and $B = \sum_{i=1}^{\ell-1}(r''_i - (\ell-1)d_i)$. Then for $0 \leq d_\ell \leq \min_{1 \leq i \leq \ell-1} d_i$, $\rho(C'')$

$$\begin{cases} \textit{decrease}, & \textit{if } A \geq \min_{1\leq i\leq \ell-1} d_i \textit{ or } B \leq 0; \\ \textit{increase}, & \textit{if } A \leq 0 \textit{ and } B > 0; \\ \textit{decrease before } A \textit{ and increase after } A, & \textit{if } 0 < A < \min_{1\leq i\leq \ell-1} d_i \textit{ and } B > 0. \end{cases}$$

Proof. (i) Let $\Pi = \{\{1, \ldots, \ell-1\}, \{\ell\}\}$. Then $\Pi(C''^T)$ is an equitable quotient matrix of C''^T. By Corollary 2.1.9,

$$\begin{aligned} \rho(C'') &= \rho(C''^T) = \rho(\Pi(C''^T)) = \rho(\begin{pmatrix} \sum_{i=1}^{\ell-1} d_i & d_\ell \\ \sum_{i=1}^{\ell-1}(r''_i - (\ell-1)d_i) & r''_\ell - (\ell-1)d_\ell \end{pmatrix}) \\ &= \frac{\sum_{i=1}^{\ell-1}(d_i - d_\ell) + r''_\ell + \sqrt{(\sum_{i=1}^{\ell-1}(d_i - d_\ell) - r''_\ell)^2 + 4d_\ell \sum_{i=1}^{\ell-1}(r''_i - r''_\ell)}}{2}. \end{aligned}$$

(ii) From $K_1 = \{i | r_1'' > r_\ell''\}$,

$$K_2 = \{i \notin K_1 | c_{ij}'' > c_{\ell j}'' \text{ for some } j \in K_1\} = \{i | r_i'' = r_\ell'' \text{ and } d_i > d_\ell\}.$$

So $K_1 \bigcup K_2 = \{i | r_i'' > r_\ell''\} \bigcup \{i | d_i > d_\ell\}$, $\overline{K_1 \bigcup K_2} = \{i | r_i'' = r_\ell'' \text{ and } d_i = d_\ell\}$ and the i-th row is equal to the ℓ-th row for $i \in \overline{K_1 \bigcup K_2}$. By Lemma 4.9.1, $K_1 \bigcup K_2 = K$.

(iii) Let $\ell' = \ell - 1$, $a = \sum_{i=1}^{\ell-1} d_i$, $b = \sum_{i=1}^{\ell-1} r_i''$, $b' = \sum_{i=1}^{\ell-1}(r_i'' - (\ell-1)d_i) = b - \ell' a$. Then we have $a \geq \ell' d_\ell$, $b > \ell' r_\ell''$ and

$$\begin{aligned}
\rho(C'') &= \frac{\sum_{i=1}^{\ell-1}(d_i - d_\ell) + r_\ell'' + \sqrt{(\sum_{i=1}^{\ell-1}(d_i - d_\ell) - r_\ell'')^2 + 4d_\ell \sum_{i=1}^{\ell-1}(r_i'' - r_\ell'')}}{2} \\
&= \frac{a + r_\ell'' - \ell' d_\ell + \sqrt{(a - r_\ell'' - \ell' d_\ell)^2 + 4d_\ell(b - \ell' r_\ell'')}}{2}.
\end{aligned}$$

Then

$$\frac{\partial}{\partial d_\ell}\rho(C'') = \frac{1}{2}(-\ell' + \frac{-\ell'(a - r_\ell'' - \ell' d_\ell) + 2(b - \ell' r_\ell'')}{\sqrt{(a - r_\ell'' - \ell' d_\ell)^2 + 4d_\ell(b - \ell' r_\ell'')}})$$

for $0 < d_\ell < \min_{1 \leq i \leq \ell-1} d_i$. Note that $(a - r_\ell'' - \ell' d_\ell)^2 + 4d_\ell(b - \ell' r_\ell'') > 0$ since $d_\ell > 0$. If $-\ell'(a - r_\ell'' - \ell' d_\ell) + 2(b - \ell' r_\ell'') \leq 0$, then $\frac{\partial}{\partial d_\ell}\rho(C'') < 0$. If $-\ell'(a - r_\ell'' - \ell' d_\ell) + 2(b - \ell' r_\ell'') \geq 0$, we have

$$\begin{aligned}
& (-\ell'(a - r_\ell'' - \ell' d_\ell) + 2(b - \ell' r_\ell''))^2 - \ell'^2((a - r_\ell'' - \ell' d_\ell)^2 + 4d_\ell(b - \ell' r_\ell'')) \\
= \;& 4(b - \ell' r_\ell'')(-\ell'(a - r_\ell'' - \ell' d_\ell) + (b - \ell' r_\ell'') - \ell'^2 d_\ell) \\
= \;& 4(b - \ell' r_\ell'')(b - \ell' a).
\end{aligned}$$

Since $b - \ell' r_\ell'' > 0$,

$$\begin{cases}
\dfrac{\partial}{\partial d_\ell}\rho(C'') \geq 0, & \text{if } b - \ell' a \geq 0; \\
\dfrac{\partial}{\partial d_\ell}\rho(C'') \leq 0, & \text{if } b - \ell' a \leq 0.
\end{cases}$$

Therefore, we can determine the sign of $\rho(C'')$ by the sign of

$$-\ell'(a - r_\ell'' - \ell' d_\ell) + 2(b - \ell' r_\ell'') = (\ell-1)^2 d_\ell - \sum_{i=1}^{\ell-1}(r_\ell'' + (\ell-1)d_i - 2r_i'') = (\ell-1)^2(d_\ell - A)$$

and the sign of

$$b - \ell' a = \sum_{i=1}^{\ell-1} (r_i'' - (\ell - 1)d_i) = B.$$

$\square$

Remark 4.10.2. For an $n \times n$ nonnegative matrix C and any partition Π of $[n]$, we can choose $C'' + cI$ for some c instead of C'' satisfying the condition in Theorem 4.7.1 or Theorem 4.7.5 to get an upper or lower bound of the spectral radius of C, where C'' is of the form in (4.10.1). Note that the set of such $C'' + cI$ contains $M_\ell(d, f, r_1, r_2, \ldots, r_\ell)$.

Chapter 5

BFP conjecture and weak BFP conjecture

We found counter examples of BFP Conjecture for $\mathcal{K}(p,q,e)$ when $e = p(q-1)$, $p \geq 3$ and $q > p+2$ in Proposition 3.6.2 of Chapter 3. In this chapter we devote ourselves to prove the weak BFP Conjecture for $\mathcal{C}(p,q,e)$. To do this we need better upper bounds of $\rho(G)$ for bipartite graph G, and one is given in Section 5.1. The case $e \geq pq - q$ are settled in Section 5.2, and the case $p \leq 5$ in Section 5.2.2. In Section 5.3 and Section 5.4, we provide more tools to help us get more results in Section 5.5.

5.1 Upper bounds of $\rho(G_D)$

We have learned the upper bound $\phi_{s,t}$ of $\rho(G)$ for a bipartite graph G in Lemma 2.2.3. In this section two upper bounds ϕ_ℓ and $\phi(D)$ of $\rho(G_D)$ will be introduced.

5.1.1 Upper bound ϕ_ℓ

We provide the upper bound ϕ_ℓ of $\rho(G_D)$ here.

For a decreasing sequence $D = (d_1, d_2, \ldots, d_p)$ of positive integers, let

$$H = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_p \\ d_2 & d_2 & d_3 & \cdots & d_p \\ d_3 & d_3 & d_3 & \cdots & d_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p & d_p & d_p & \cdots & d_p \end{pmatrix}$$

with row-sums $r_1 \geq r_2 \geq \cdots \geq r_p$ as also shown in (2.2.2), where

$$r_i = e + (i-1)d_i - \sum_{k=1}^{i-1} d_k = e - \sum_{k=d_i+1}^{t} d_k^*, \tag{5.1.1}$$

$e = \sum_{k=1}^{p} d_k$, $t = d_1$ and $D^* = (d_1^*, d_2^*, \ldots, d_t^*)$ is the conjugate partition of e.

Applying Theorem 4.7.1 with $n = p$, $C = H$ and the partition $\Pi = \{\{1\}, \{2\}, \ldots, \{\ell-1\}, \{\ell, \ell+1, \ldots, p\}\}$, we choose the $\ell \times \ell$ rooted matrix $C'' = (c''_{ij})$, where $c''_{ij} = d_i$ for $1 \leq i \leq \ell, 1 \leq j \leq \ell - 1$ and $c''_{i\ell} = r_i - (\ell-1)d_i$, i.e.,

$$C'' = \begin{pmatrix} d_1 & d_1 & \cdots & d_1 & r_1 - (\ell-1)d_1 \\ d_2 & d_2 & \cdots & d_2 & r_2 - (\ell-1)d_2 \\ d_3 & d_3 & \cdots & d_3 & r_3 - (\ell-1)d_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_\ell & d_\ell & \cdots & d_\ell & r_\ell - (\ell-1)d_\ell \end{pmatrix}. \tag{5.1.2}$$

Then C'^T has equitable quotient matrix

$$\Pi'(C''^T) = \begin{pmatrix} \sum_{k=1}^{\ell-1} d_k & d_\ell \\ \sum_{k=1}^{\ell-1}(r_k - (\ell-1)d_k) & \sum_{k=\ell}^{p} d_k \end{pmatrix}$$

with respect to the partition $\Pi' = \{\{1, 2, \ldots, \ell-1\}, \{\ell\}\}$ of $[p]$. Note that C'' has a rooted eigenvector for $\rho(C'')$. So C''^T has a nonnegative left eigenvector for $\rho(C'')$. Since $\Pi'(C''^T)$ has characteristic polynomial

$$\begin{aligned} & x^2 - (\textstyle\sum_{k=1}^{p} d_k)x + \sum_{k=1}^{\ell-1} d_k \cdot \sum_{k=\ell}^{p} d_k - d_\ell \sum_{k=1}^{\ell-1} r_k + d_\ell(\ell-1)\sum_{k=1}^{\ell-1} d_k \\ = \;& x^2 - (\textstyle\sum_{k=1}^{p} d_k)x + r_\ell \sum_{k=1}^{\ell-1} d_k - d_\ell \sum_{k=1}^{\ell-1} r_k \\ = \;& x^2 - (\textstyle\sum_{k=1}^{p} d_k)x + r_1 r_\ell - r_\ell^2 - d_\ell \sum_{k=1}^{\ell} r_k + \ell d_\ell r_\ell \end{aligned}$$

and by Lemma 4.4.1,

$$\phi_\ell^2 := \rho(C'') = \rho(\Pi'(C''^T)) = \frac{r_1 + \sqrt{(2r_\ell - r_1)^2 + 4d_\ell \sum_{i=1}^{\ell}(r_i - r_\ell)}}{2}. \quad (5.1.3)$$

Theorem 5.1.1. *Let G be a bipartite graph and $D = (d_1, d_2, \ldots, d_p)$ be the degree sequence of one part of G in decreasing order. Then for $1 \le \ell \le p$, we have $\rho(G) \le \phi_\ell$, with equality if and only if $G = G_D$, $d_1 = d_t$ and $d_{t+1} = d_p$ for some $1 \le t \le \ell - 1$.* □

Proof. By Lemma 2.2.1, (2.2.3), the above setting of C'', and the first conclusion of Theorem 4.7.1, we have $\rho(G) \le \rho(G_D) = \sqrt{\rho(H)} \le \sqrt{\rho(C'')} = \phi_\ell$. The first inequality is equality if and only if $G = G_D$ by Lemma 2.2.1. We apply the second conclusion of Theorem 4.7.1 to find that the second inequality is equality if and only if

(a) $r_\ell = r_p$, and

(b) $d_{\max(a,b)} = h_{ab} = c''_{ab} = d_a$ for all $1 \le a, b \le \ell - 1$ with $v''_b > v''_\ell$, and $d_i = d_{\max(i,b)} = h_{ib} = c''_{\ell b} = d_\ell$ for all $1 \le b, \le \ell - 1$ with $v''_b > v''_\ell$ and for all $i \ge \ell$, where $v'' = (v''_1, v''_2, \ldots, v''_\ell)$ is a positive rooted eigenvector of $\Pi(C'')$ for $\rho(C'')$.

Note that $r_\ell = r_p$ if and only if $d_\ell = d_p$. By Lemma 4.9.1 and the structure of C'', $\{b | v''_b > v''_\ell\} = K = K_1 = \{b | d_b > d_\ell\}$. Hence conditions (a)-(b) are equivalent to $d_1 = d_t$ and $d_{t+1} = d_p$ for t to be the largest integer with $d_t > d_\ell$, or 0 if no such t. □

Corollary 5.1.2. *Let G be a bipartite graph and $D = (d_1, d_2, \ldots, d_p)$ be the degree sequence of one part in decreasing order. Then*

$$\rho(G) \le \sqrt{\frac{r_1 + \sqrt{r_1^2 - 4d_p \sum_{i=1}^{p}(r_1 - r_i)}}{2}},$$

with equality if and only if $G = G_D$ and $d_1 = d_t$ and $d_{t+1} = \cdots = d_p$ for some $1 \le t \le p - 1$.

Proof. By Lemma 2.2.1, the case $\ell = p$ in Theorem 5.1.1 and using $r_p = pd_p$ to simplify ϕ_p in (5.1.3), we have

$$\rho(G) \le \rho(G_D) \le \phi_p = \sqrt{\frac{r_1 + \sqrt{r_1^2 - 4d_p \sum_{i=1}^{p}(r_1 - r_i)}}{2}}.$$

$\square$

5.1.2 Comparison of ϕ_ℓ and $\phi_{s,t}$

We compare the upper bound ϕ_ℓ of $\rho(G_D)$ and the bound $\phi_{s,t}$ in Lemma 2.2.3.

Lemma 5.1.3. *Let $D = (d_1, d_2, \ldots, d_p)$ be a decreasing sequence of positive integers and $D' = (d'_1, d'_2, \ldots, d'_q)$ be the degree sequence of the other part of G_D. Then for $1 \le s \le p$, $1 \le t \le q$ with $s - 1 = d'_t$,*

$$\phi_s \le \phi_{s,t}.$$

Proof. Let C_1 be an $s \times s$ matrix with

$$(C_1)_{ij} = \begin{cases} d_i, & \text{if } 1 \le i \le s, 1 \le j \le s-1; \\ d_i d'_t + \sum_{k=1}^{t-1}(d'_k - d'_t) - (s-1)d_i, & \text{if } 1 \le i \le s \text{ and } j = t, \end{cases}$$

i.e.,

$$C_1 = \begin{pmatrix} d_1 & \cdots & d_1 & d_1 d'_t + \sum_{j=1}^{t-1}(d'_j - d'_t) - (s-1)d_1 \\ d_2 & \cdots & d_2 & d_2 d'_t + \sum_{j=1}^{t-1}(d'_j - d'_t) - (s-1)d_2 \\ \vdots & \ddots & \vdots & \vdots \\ d_{s-1} & \cdots & d_{s-1} & d_{s-1} d'_t + \sum_{j=1}^{t-1}(d'_j - d'_t) - (s-1)d_{s-1} \\ d_s & \cdots & d_s & d_s d'_t + \sum_{j=1}^{t-1}(d'_j - d'_t) - (s-1)d_s \end{pmatrix}.$$

We will show that C_1 realizes the upper bound $\phi_{s,t}$. Note that the transpose of the equitable quotient matrix of C_1^T with respect to the partition $\Pi = \{\{1, \ldots, s-1\}, \{s\}\}$ is

$$\Pi(C_1^T)^T = \begin{pmatrix} \sum_{i=1}^{s-1} d_i & (\sum_{i=1}^{s-1} d_i)d'_t + (s-1)\sum_{j=1}^{t-1}(d'_j - d'_t) - (s-1)\sum_{i=1}^{s-1} d_i \\ d_s & d_s d'_t + \sum_{j=1}^{t-1}(d'_j - d'_t) - (s-1)d_s \end{pmatrix}.$$

Since C_1 is rooted,

$$\rho(C_1) = \rho(\Pi(C_1^T)) = \frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2} = \phi_{s,t}^2.$$

Note that the first $s-1$ column of C_1 is the first $s-1$ column of C'' in (5.1.2) with $\ell = s$, the s-th row-sum of C_1 is equal to r_s in (5.1.1), and $(d_i - d_s)d'_t \geq r_i - r_s$. Referring to Q in (4.2.1) with $n = s$, to C'' in (5.1.2), and to the form of $Q^{-1}C'Q$ in (4.3.1) with $C' = C_1$ and $C' = C''$, we have $Q^{-1}C_1Q \leq Q^{-1}C''Q$. Thus $\phi_s^2 = \rho(C'') \leq \rho(C_1) = \phi_{s,t}^2$. □

5.1.3 Upper bound $\phi(D)$

For a decreasing sequence $D = (d_1, d_2, \ldots, d_p)$ of positive integers, define

$$f(D) := d_p \sum_{1 \leq i < j \leq p} (d_i - d_j) \tag{5.1.4}$$

and

$$\phi(D) := \sqrt{\frac{e + \sqrt{e^2 - 4d_p \sum_{1\leq i<j\leq p}(d_i - d_j)}}{2}} = \sqrt{\frac{e + \sqrt{e^2 - 4f(D)}}{2}}. \tag{5.1.5}$$

Note that $e = \sum_{i=1}^{p} d_i$ is the number of edges in G_D.

Lemma 5.1.4. *Let G be a bipartite graph and D be the degree sequence of one part of G in decreasing order. Then*

$$\rho(G) \leq \phi(D)$$

with equality if and only if $G = G_D$ and D has at most two values.

Proof. This is from Corollary 5.1.2 by using $r_1 - r_i = \sum_{k=1}^{i-1}(d_k - d_i)$ and $r_1 = e$. □

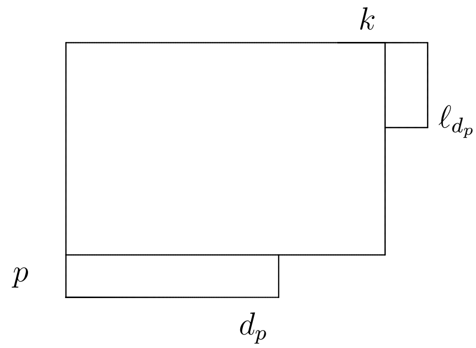


Figure 5.1: The ferrers diagram $F(D^\natural)$ of $D^\natural$

5.2 Weak BFP conjecture for $\mathcal{C}(p, q, e)$

We will show that the weak BFP conjecture for $\mathcal{C}(p, q, e)$ is true when $e \geq pq - q$ or $p \leq 5$, where $p \leq q$. We need first a general setting.

When e and d_p are fixed, to increase the value $\phi(D)$ in (5.1.5), we need to decrease the value $d_p \sum_{1 \leq i < j \leq p}(d_i - d_j)$, i.e., making the values (d_i) as closed as possible. The sequence $D^\natural$ below is for such purpose. Let

$$\ell_{d_p} = e - d_p - (p-1)k \tag{5.2.1}$$

denote the remainder of $e - d_p$ dividing by $p - 1$, where

$$k = \lfloor \frac{e - d_p}{p - 1} \rfloor$$

is the quotient. Define the sequence $D^\natural = (d_1^\natural . d_2^\natural, \ldots, d_p^\natural)$, where $d_p^\natural = d_p$ and

$$d_i^\natural = \begin{cases} \lfloor \frac{e-d_p}{p-1} \rfloor + 1, & \text{if } 1 \leq i \leq \ell_{d_p}; \\ \lfloor \frac{e-d_p}{p-1} \rfloor, & \text{if } \ell_{d_p} + 1 \leq i \leq p-1. \end{cases} \tag{5.2.2}$$

Note that $e = d_1^\natural + d_2^\natural + \cdots + d_p^\natural$, and the Ferrers diagram $F(D^\natural)$ has the form shown in Figure 5.1.

Proposition 5.2.1. *With the notation above,*

$$\phi(D) \leq \phi(D^\natural).$$

Moreover, if $d_p \neq 0$, the above equality holds if and only if $D = D^\natural$.

Proof. If $d_p = 0$ then $\phi(D) = \sqrt{e} = \phi(D^\natural)$ by (5.1.4), (5.1.5). Assume $d_p > 0$. We will show that $d_p \sum_{1\leq i<j\leq p}(d_i - d_j)$ takes minimum value if and only if $D = D^\natural$ for given e and d_p. If there is a pair $(d_i, d_j), i < j < p$ such that $d_i \geq d_j + 2$, then choose one such pair (i, j) with $j - i$ minimum and use $(d_i - 1, d_j + 1)$ to replace (d_i, d_j) in the original sequence. It strictly decreases the value $2d_p(i - j)$ of $d_p \sum_{j=1}^{p-1}(d_j - d_p)$. We can always find the above pair (i, j) unless $D = D^\natural$. This completes the proof. $\square$

Set $\phi(e, p, d_p) := \phi(D^\natural)$ and $G(e, p, d_p) := G_{D^\natural}$. Proposition 5.2.1 and the proof of Theorem 5.1.1 imply that $\rho(G_D) \leq \phi(e, p, d_p)$ with equality if and only if $D = D^\natural$ has at most two values. We shall compare the values $\phi(e, p, d_p)$ if e, p are fixed and d_p is a variable. Referring to (5.1.5), it is easier to compare the values

$$f(e, p, d_p) := d_p \sum_{1\leq i<j\leq p} (d_i^\natural - d_j^\natural), \tag{5.2.3}$$

where $d_i^\natural$ is defined in (5.2.2). Note that

$$f(e, p, d_p) = d_p \sum_{1\leq i<j\leq p} (d_i^\natural - d_j^\natural) = d_p \left[(e - pd_p) + \ell_{d_p}(p - 1 - \ell_{d_p})\right], \tag{5.2.4}$$

and

$$\rho(G_D) \leq \phi(e, p, d_p) = \sqrt{\frac{e + \sqrt{e^2 - 4f(e, p, d_p)}}{2}}. \tag{5.2.5}$$

To prove weak BFP Conjecture for $\mathcal{C}(p, q, e)$, we need to show that for any graph $G \in \mathcal{C}(p, q, e) \setminus \mathcal{C}_0(p, q, e)$, there is a graph $G^\natural \in \mathcal{C}_0(p, q, e)$ such that $\rho(G) < \rho(G^\natural)$. The following lemma can help us find this $G^\natural$.

Lemma 5.2.2. *Given e, p, d_p, choose integers k, ℓ_{d_p} with $0 \leq \ell_{d_p} \leq p - 1$ such that $e - d_p = (p - 1)k + \ell_{d_p}$. Assume $a := d_p + \ell_{d_p} - p + 1 > 0$ and*

$b := \min(k, d_p + \ell_{d_p})$. *Then* $d_p \in [a, b]$, $G(e,p,a), G(e,p,b) \in \mathcal{C}_0(p,q,e)$ *and*

$$\phi(e,p,d_p) \leq \max(\rho(G(e,p,a)), \rho(G(e,p,b))).$$

Moreover, if the equality holds, then $d_p = a$ *or* $d_p = b$.

Proof. Clearly $d_p \in [a,b]$ and $G(e,p,a), G(e,p,b) \in \mathcal{C}_0(p, k+1, e)$ from the setting. Also if $d_p + \ell_{d_p} - p + 1 > 0$, then $G(e,p,a) \in \mathcal{C}_0(p,k+1,e)$; in particular, $\rho(G(e,p,c)) = \phi(e,p,c)$ for $c \in \{a,b\}$. To study $\phi(e,p,d_p)$, we only need to compare the values $f(e,p,d_p) = d_p((e - pd_p) + \ell_{d_p}(p - 1 - \ell_{d_p}))$ in (5.2.4), where $\ell_{d_p} = e - k(p-1) - d_p$, and $\partial \ell_{d_p}/\partial d_p = -1$. Then

$$\frac{\partial f(e,p,d_p)}{\partial d_p} = e + (2\ell_{d_p} - 3p + 1)d_p + \ell_{d_p}(p - 1 - \ell_{d_p})$$

and

$$\frac{\partial^2 f(e,p,d_p)}{\partial d_p^2} = 4(\ell_{d_p} - p) - 2d_p + 2 < 0$$

since $\ell_{d_p} \leq p - 1$. So $f(e,p,d_p)$ takes the minimum value (resp. $\phi(e,p,d_p)$ takes maximum value) only if d_p is an end point of interval $[a,b]$. Then

$$\phi(e,p,d_p) \leq \max(\phi(G(e,p,a)), \phi(G(e,p,b))) = \max(\rho(G(e,p,a)), \rho(G(e,p,b)))$$

with equality only if $d_p = a$ or $d_p = b$. $\square$

Note that if $a = 0$ in the above lemma, then $\phi(e,p,a) = \sqrt{e}$.

5.2.1 The case $e \geq pq - q$

We consider the case $e \geq pq - q$ in this subsection.

Theorem 5.2.3. *If* $p \leq q$ *and* $e \geq pq - q$ *then weak BFP Conjecture for* $\mathcal{C}(p,q,e)$ *is true.*

Proof. Let $G \in \mathcal{C}(p,q,e)$ with one part degree sequence $D = (d_1, d_2, \ldots, d_p)$. If $e = pq - q$, then we choose $G_0 = K_{p-1,q}$. Assume $e > pq - q$. Referring to (5.2.1), $a := d_p + \ell_{d_p} - p + 1 = e - (p-1)(k+1) \geq e - (p-1)q > 0$, where

$k = \lfloor \frac{e-d_p}{p-1} \rfloor$. Let b as defined in Lemma 5.2.2. By Lemma 2.2.1, Lemma 5.1.4, Lemma 5.2.1 and Lemma 5.2.2, we have

$$\rho(G) \leq \rho(G_D) \leq \phi(D) \leq \phi(D^\natural) = \phi(e,p,d_p) \leq \max(\rho(G(e,p,a)), \rho(G(e,p,b))),$$

and $\rho(G) \neq \max(\rho(G(e,p,a)), \rho(G(e,p,b)))$ unless $G = G_D = G_{D^\natural} \in \mathcal{C}_0(p,q,e)$. □

5.2.2 The case $p \leq 5$

We will show that weak BFP conjecture for $\mathcal{C}(p,q,e)$ is true when $p \leq 5$ in this subsection.

Theorem 5.2.4. *Weak BFP conjecture for $\mathcal{C}(p,q,e)$ is true when $p \leq 5$.*

Proof. We prove by induction on q. The case $q = 1$ is trivial. Recall from Section 3.7, the case $p \leq 3$ is done for any q. Assume $p \in \{4,5\}$. Pick $G \in \mathcal{C}(p,q,e)$ with one part degree sequence $D = (d_1, d_2, \ldots, d_p)$. We might assume $d_p > 0$. Then $\rho(G) \leq \rho(G_D) \leq \phi(D) \leq \phi(D^\natural)$, and $\rho(G) = \phi(D^\natural)$ if and only if $G = G_D = G_{D^\natural}$ and $D^\natural$ has at most two distinct values; in particular $G \in \mathcal{C}_0(p,q,e)$. Hence we might assume $G = G_{D^\natural}$ and $D^\natural$ has three different values. We assume $d_1 = q$, otherwise $G \in \mathcal{C}(p, q-1, e)$ and the proof is finished by induction hypothesis. Then $d_1 = d_t = q$ and $d_{t+1} = d_{p-1} = q-1$, where $1 \leq t = \ell_{d_p} < p-1$ as defined in (5.2.1). By Lemma 5.2.2, we might also assume $d_p + \ell_{d_p} - p + 1 \leq 0$. There are only a few cases remaining. If $d_p + \ell_{d_p} = p - 1$ then we choose $G_0 = K_{p-1,q}$ and $\rho(G) < \rho(G_0) = \sqrt{e}$. There are only four cases $(p, \ell_{d_p}, d_p, e) \in \{(4,1,1,3q-1), (5,1,1,4q-2), (5,1,2,4q-1), (5,2,1,4q-1)\}$ remaining. Let $d'_1 = d'_{d_p+\ell_{d_p}} = q$, $d'_{d_p+\ell_{d_p}+1} = q-1 = d'_{p-1}$, and $D' = (d'_1, d'_2, \ldots, d'_{p-1})$ be a sequence of length $p-1$. Note that $G_{D'} \in \mathcal{C}_0(p-1,q,e) \subseteq \mathcal{C}_0(p,q,e)$. We compare the f-values to find $\rho(G) < \phi(D^\natural) \leq \rho(G_{D'}) = \phi(e, p-1, q-1)$ with referring to (5.2.3) and (5.2.5) in the following cases of (p, ℓ_{d_p}, d_p, e):

Case $(4,1,1,3q-1)$: $f(3q-1,4,1) = 3q-3 > 2q-2 = f(3q-1,3,q-1)$, Case $(5,1,1,4q-2)$: $f(4q-2,5,1) = 4q-4 = 4q-4 = f(4q-2,4,q-1)$,

Case $(5,1,2,4q-1)$: $f(4q-1,5,2) = 38q-16 > 3q-3 = f(4q-1,4,q-1)$,

Case $(5,2,1,4q-1)$: $f(4q-1,5,1) = 4q-2 > 3q-3 = f(4q-1,4,q-1)$.

$\square$

5.3 A condition to reduce p

To prove weak BFP conjecture for $\mathcal{C}(p,q,e)$, the following properties may be useful for doing induction on $p+q$.

Lemma 5.3.1. *Given $D = (d_1, d_2, \ldots, d_p)$ with $d_1 \geq d_2 \geq \cdots \geq d_p > 0$, $\tilde{p} < p$, $0 \leq s_1 \leq s_2 \leq \cdots \leq s_{\tilde{p}}$ with $\sum_{i=\tilde{p}+1}^{p} d_i = \sum_{k=1}^{\tilde{p}} s_k$, and $d_1 + s_1 \geq d_2 + s_2 \geq \cdots \geq d_{\tilde{p}} + s_{\tilde{p}}$, let $\tilde{D} = (d_1 + s_1, d_2 + s_2, \ldots, d_{\tilde{p}} + s_{\tilde{p}})$. Then*

$$\rho(G_D) \leq \rho(G_{\tilde{D}})$$

and $\rho(G_D) = \rho(G_{\tilde{D}})$ if and only if G_D and $G_{\tilde{D}}$ are complete bipartite graphs.

Proof. Let $\Pi = \{\{1\}, \ldots, \{\tilde{p}\}, \{\tilde{p}+1, \ldots, p\}\}$ be a partition of $[p]$ and $C'' = (c''_{ij})$ be a $(\tilde{p}+1) \times (\tilde{p}+1)$ matrix, where

$$c''_{ij} = \begin{cases} H(D)_{ij} + s_j, & \text{if } j \leq \tilde{p}; \\ 0, & \text{if } j = \tilde{p}+1. \end{cases}$$

Note that

$$H(D) = \left(\begin{array}{ccc|ccc} d_1 & \cdots & d_{\tilde{p}} & d_{\tilde{p}+1} & \cdots & d_p \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ d_{\tilde{p}} & \cdots & d_{\tilde{p}} & d_{\tilde{p}+1} & \cdots & d_p \\ \hline d_{\tilde{p}+1} & \cdots & d_{\tilde{p}+1} & d_{\tilde{p}+1} & \cdots & d_p \\ & & & & \ddots & \vdots \\ d_p & \cdots & d_p & d_p & \cdots & d_p \end{array}\right),$$

$$C'' = \begin{pmatrix} d_1+s_1 & d_2+s_2 & \cdots & d_{\tilde{p}-1}+s_{\tilde{p}-1} & d_{\tilde{p}}+s_{\tilde{p}} & 0 \\ d_2+s_1 & d_2+s_2 & \cdots & d_{\tilde{p}-1}+s_{\tilde{p}-1} & d_{\tilde{p}}+s_{\tilde{p}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{\tilde{p}-1}+s_1 & d_{\tilde{p}-1}+s_2 & \cdots & d_{\tilde{p}-1}+s_{\tilde{p}-1} & d_{\tilde{p}}+s_{\tilde{p}} & 0 \\ d_{\tilde{p}}+s_1 & d_{\tilde{p}}+s_2 & \cdots & d_{\tilde{p}}+s_{\tilde{p}-1} & d_{\tilde{p}}+s_{\tilde{p}} & 0 \\ d_{\tilde{p}+1}+s_1 & d_{\tilde{p}+1}+s_2 & \cdots & d_{\tilde{p}+1}+s_{\tilde{p}-1} & d_{\tilde{p}+1}+s_{\tilde{p}} & 0 \end{pmatrix}.$$

By Theorem 4.7.1, we have

$$\rho(H(D)) \leq \rho(C''). \tag{5.3.1}$$

Since $H(D)_{ij} + s_j \leq H(D) + s_{\max(i,j)}$, we have $C''(\tilde{p}+1|\tilde{p}+1) \leq H(\tilde{D})$ and

$$\rho(C''(\tilde{p}+1|\tilde{p}+1)) \leq \rho(H(\tilde{D})). \tag{5.3.2}$$

So

$$\rho(H(D)) \leq \rho(C'') = \rho(C''(\tilde{p}+1|\tilde{p}+1)) \leq \rho(H(\tilde{D})).$$

If G_D and $G_{\tilde{D}}$ are complete bipartite graphs, say they are of size e, then $\rho(G_D) = \rho(G_{\tilde{D}}) = \sqrt{e}$. For the converse, if $\rho(G_D) = \rho(G_{\tilde{D}})$, then the equality in (5.3.2) implies $s_1 = s_2 = \cdots = s_{\tilde{p}} > 0$ since $C''(\tilde{p}+1|\tilde{p}+1)$ is irreducible. Suppose $d_i \neq d_{\tilde{p}+1}$ for some i. Then the i-th row-sum of C'' is larger than the $(\tilde{p}+1)$-th row-sum of C''. By Lemma 4.9.1 and Theorem 4.7.1, $H(D)[[\tilde{p}]|i] = C''[[\tilde{p}]|i]$ and $s_i = 0$, a contradiction. So $G_{\tilde{D}}$ is a complete bipartite graph and so is G_D. □

5.4 Cubic bounds

The following are upper bounds of $\rho(G_D)$ which is a zero of a cubic polynomial.

Theorem 5.4.1. *Given a decreasing sequence $D = (d_1, d_2, \ldots, d_p)$ of positive integers and $1 \leq s \leq p-1$, let λ be the largest zero of the following polynomial*

$$x^3 - r_1 x^2 + [\sum_{i=s}^{p-1} d_i \cdot \sum_{i=1}^{s-1}(d_i - d_s) + d_p \sum_{i=1}^{p-1}(d_i - (r_i - \tilde{r}_i))]x - d_p \sum_{i=1}^{s-1}(d_i - d_s) \sum_{i=s}^{p-1}(d_i - (r_i - \tilde{r}_i)),$$

where r_i is the i-th row sum of $H(D)$ and

$$\tilde{r}_i = \begin{cases} (s-1)d_i + (p-s)d_s & \text{if } i \le s-1; \\ (p-1)d_i & \text{if } s \le i \le p. \end{cases}$$

Then

$$\rho(G_D) \le \sqrt{\lambda}.$$

Proof. Let $\Pi = \{\{1,\dots,s-1\},\{s,\dots,p-1\},\{p\}\}$ and $C'' = (c''_{ij})$, where

$$c''_{ij} = \begin{cases} d_s, & \text{if } i \le s-1 \text{ and } s \le j \le p-1; \\ d_i, & \text{if } j \le s-1 \text{ or } s \le i \wedge j \le p-1; \\ r_i - \tilde{r}_i, & \text{if } j = p. \end{cases}$$

That is,

$$C'' = \left(\begin{array}{cccc|cccc|c} d_1 & d_1 & \cdots & d_1 & d_s & d_s & \cdots & d_s & r_1 - \tilde{r}_1 \\ d_2 & d_2 & \cdots & d_2 & d_s & d_s & \cdots & d_s & r_2 - \tilde{r}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ d_{s-1} & d_{s-1} & \cdots & d_{s-1} & d_s & d_s & \cdots & d_s & r_{s-1} - \tilde{r}_{s-1} \\ \hline d_s & d_s & \cdots & d_s & d_s & d_s & \cdots & d_s & r_s - \tilde{r}_s \\ d_{s+1} & d_{s+1} & \cdots & d_{s+1} & d_{s+1} & d_{s+1} & \cdots & d_{s+1} & r_{s+1} - \tilde{r}_{s+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{p-1} & d_{p-1} & \cdots & d_{p-1} & d_{p-1} & d_{p-1} & \vdots & d_{p-1} & r_{p-1} - \tilde{r}_{p-1} \\ \hline d_p & d_p & \cdots & d_p & d_p & d_p & \cdots & d_p & d_p \end{array}\right).$$

Then $(H(D), C'')$ satisfies the assumption in Theorem 4.7.1 for $\ell = p$ and

$$\rho(G_D) = \sqrt{H(D)} \le \sqrt{\rho(C'')}$$

by Theorem 4.7.1. Note that C''^T has the equitable quotient matrix $\Pi(C'')$ with respect to partition $\Pi = \{\{1,\dots,s-1\},\{s,\dots,p-1\},\{p\}\}$, where

$$\Pi(C''^T) = \begin{pmatrix} \sum_{i=1}^{s-1} d_i & (s-1)d_s & \sum_{i=1}^{s-1}(r_i - \tilde{r}_i) \\ \sum_{i=s}^{p-1} d_i & \sum_{i=s}^{p-1} d_i & \sum_{i=s}^{p-1}(r_i - \tilde{r}_i) \\ d_p & d_p & d_p \end{pmatrix}^T,$$

which has characteristic polynomial

$$x^3-r_1x^2+[\sum_{i=s}^{p-1} d_i\cdot\sum_{i=1}^{s-1}(d_i-d_s)+d_p\sum_{i=1}^{p-1}(d_i-(r_i-\tilde{r}_i))]x-d_p\sum_{i=1}^{s-1}(d_i-d_s)\sum_{i=s}^{p-1}(d_s-(r_i-\tilde{r}_i)).$$

By Corollary 2.1.9, $\rho(C'') = \rho(\Pi(C''^T))$ and the conclusion follows. $\square$

Lemma 5.4.2. *Recall the symbols in Theorem 5.4.1, we have the following.*

(i) $\displaystyle\sum_{i=1}^{s-1}(d_i+\tilde{r}_i-r_i) = \sum_{1\le i<j\le s-1}(d_i-d_j)+(s-1)\sum_{i=s}^{p-1}(d_s-d_i)+\sum_{i=1}^{s-1}(d_i-d_p),$

(ii) $\displaystyle\sum_{i=s}^{p-1}(d_i+\tilde{r}_i-r_i) = \sum_{s\le i<j\le p}(d_i-d_j) = \sum_{s\le i<j\le p-1}(d_i-d_j)+\sum_{i=s}^{p-1}(d_i-d_p).$

Proof. If $i \le s-1$, then

$$\begin{aligned}
& d_i+\tilde{r}_i-r_i \\
= \ & (d_i+(s-1)d_i+(p-s)d_s)-(id_i+\sum_{j=i+1}^{p} d_j) \\
= \ & \sum_{j=i+1}^{s-1}(d_i-d_j)+\sum_{j=s}^{p-1}(d_s-d_j)+(d_i-d_p)
\end{aligned}$$

and

$$\sum_{i=1}^{s-1}(d_i+\tilde{r}_i-r_i) = \sum_{1\le i<j\le s-1}(d_i-d_j)+(s-1)\sum_{i=s}^{p-1}(d_s-d_i)+\sum_{i=1}^{s-1}(d_i-d_p).$$

If $s \le i \le p-1$, then

$$\begin{aligned}
& d_i+\tilde{r}_i-r_i \\
= \ & (d_i+(p-1)d_i)-(id_i+\sum_{j=i+1}^{p} d_j) \\
= \ & \sum_{j=i+1}^{p}(d_i-d_j)
\end{aligned}$$

and

$$\sum_{i=s}^{p-1}(d_i+\tilde{r}_i-r_i) = \sum_{s\le i<j\le p}(d_i-d_j) = \sum_{s\le i<j\le p-1}(d_i-d_j)+\sum_{i=s}^{p-1}(d_i-d_p).$$

$\square$

5.5 Other partial results

To prove weak BFP conjecture for $\mathcal{C}(p,q,e)$, we have to show that for every $G_D \in \mathcal{C}(p,q,e)\setminus\mathcal{C}_0(p,q,e)$, there is a $G^\natural \in \mathcal{C}_0(p,q,e)$ such that $\rho(G_D) < \rho(G^\natural)$. Let $D = (d_1, d_2, \dots, d_p)$ (we say $d_p = 0$ if the length of D is less then p). If $d_p = 0$, then $G_D \in \mathcal{C}(p-1,q,e) \setminus \mathcal{C}_0(p-1,q,e)$. We replace p by $p-1$ and prove it by induction on p. So we assume $d_p \geq 1$. Let $e = \sum_{i=1}^{p} d_i$ and $e = (p-1)q' + m$ with $1 \leq m < p-1$. If $d_p \geq m$, then there exists $G^\natural \in \mathcal{C}_0(p,q,e)$ such that $\rho(G_D) < \rho(G^\natural)$ by Lemma 5.2.2. So we assume $d_p < m$. That is, in this section, we assume

(i) $D = (d_1, d_2, \dots, d_p)$ is a decreasing sequence of positive integers and $G_D \in \mathcal{C}(p,q,e) \setminus \mathcal{C}_0(p,q,e)$,

(ii) $e = (p-1)q' + m$, $1 \leq m < p-1$, $q' \leq q$,

(iii) $1 \leq d_p < m$.

Let

(iv) $\tilde{p} = p-1$, $e^c = \sum_{i \leq p-1 : d_i < q'} (q' - d_i)$.

Note that $G(e, \tilde{p}, q') = G(e, p-1, q') \in \mathcal{C}_0(p,q,e)$. Similarly, if $m \leq q'$, $G(e,p,m) \in \mathcal{C}_0(p,q,e)$.

Proposition 5.5.1.
$$\rho^2(G(e,\tilde{p},q')) > \tilde{p}q'$$
and $\rho^2(G(e,\tilde{p},q'))$ *is the largest zero of*
$$x^2 - ex + m(\tilde{p} - m)q'.$$
If $m \leq q'$, *then*
$$\rho^2(G(e,p,m)) > \tilde{p}q'$$
and $\rho^2(G(e,\tilde{p},m))$ *is the largest zero of*
$$x^2 - ex + m(q' - m)\tilde{p}.$$

Proof. We have

$$\begin{aligned} & \rho^2(G(e,\tilde{p},q')) \\ = \quad & \tfrac{e+\sqrt{e^2-f(e,\tilde{p},q')}}{2} \\ = \quad & \tfrac{e+\sqrt{e^2-4m(\tilde{p}-m)q'}}{2} \\ = \quad & \tfrac{e+\sqrt{(\tilde{p}q'+m)^2-4m(\tilde{p}-m)q'}}{2} \\ = \quad & \tfrac{\tilde{p}q'+m+\sqrt{(\tilde{p}q'-m)^2+4m^2q'}}{2} \\ > \quad & \tilde{p}q' \end{aligned}$$

and $\rho^2(G(e,\tilde{p},q'))$ is the largest zero of

$$x^2 - ex + m(\tilde{p}-m)q'.$$

If $m \leq q'$, the proof is similar. □

The following are some partial results.

Lemma 5.5.2. *If $e^c \leq d_p$, then $G(e,p-1,q') \in \mathcal{C}_0(p,q,e)$ and $\rho(G) < \rho(G(e,p-1,q'))$.*

Proof. It's easy to see $G(e,p-1,q') \in \mathcal{C}_0(p,q,e)$ under the assumptions. Since $e^c \leq d_p$, we have

$$\sum_{i:d_{\geq}q'} (d_i - q') \leq m - d_p + d_p = m$$

and $d_{m+1} \leq q'$. If $d_{m+1} < q'$, then

$$\begin{aligned} & \sum_{i=1}^{m}(d_i - d_{m+1}) \sum_{i=m+1} d_i \\ \geq & m \cdot (q'(\tilde{p}-m) + d_p - e^c) \qquad (5.5.1) \\ \geq & mq'(\tilde{p}-m) = f(e,p-1,q'). \end{aligned}$$

If the equality in (5.5.1) holds, then D has at least three different values. By Lemma 3.1.1, $\rho(G_D) < \rho(G(e,p-1,q'))$.

Now assume $d_{m+1} = q'$. We have

$$\sum_{i=q'+1}^{q} d'_i = m - d_p + e^c \leq m \leq d'_{q'},$$

where $D' = (d'_1, d'_2, \dots, d'_{q'})$ is the degree sequence of the other part of G_D. Let $\widetilde{D'} = (d'_1, d'_2, \dots d_{q'}, \sum_{i=q'+1}^{q} d'_i)$. By Lemma 5.3.1,

$$\rho(G_D) = \rho(G_{D'}) \le \rho(G_{\widetilde{D'}}).$$

So we assume $G_{D'} = G_{\widetilde{D'}}$. That is,

$$d_1 = d_2 = \cdots = d_{m-d_p+e^c} = q' + 1, d_{m-d_p+e^c+1} = \cdots = d_{m+1} = q'.$$

Without lose of generality, let $\tilde{p} \le q'$. If not, we use the graph $G_{D'}$ instead of G_D.

Let $s = m+1$, $g(x) = x^3 - ex^2 + Ax - B$ and λ be the largest zero of $g(x)$, where

$$A = \sum_{i=s}^{p-1} d_i \cdot \sum_{i=1}^{s-1}(d_i - d_s) + d_p \sum_{i=1}^{p-1}(d_i - (r_i - \tilde{r}_i)),$$
$$B = d_p \sum_{i=1}^{s-1}(d_i - d_s) \sum_{i=s}^{p-1}(d_i - (r_i - \tilde{r}_i)).$$

By Theorem 5.4.1, we have $\rho(G_D) \le \lambda$. By Lemma 5.4.2,

$$\begin{aligned} A &= \sum_{i=m+1}^{p-1} d_i \cdot \sum_{i=1}^{m}(d_i - d_{m+1}) + d_p \sum_{i=1}^{p-1}(d_i - (r_i - \tilde{r}_i)) \\ &= \sum_{i=m+1}^{p-1} d_i \cdot \sum_{i=1}^{m}(d_i - d_{m+1}) + d_p \sum_{1 \le i < j \le m} (d_i - d_j) \\ &\quad + md_p \sum_{i=m+1}^{p-1} (d_{m+1} - d_i) + d_p \sum_{i=1}^{p-1}(d_i - d_p) + d_p \sum_{m+1 \le i < j \le p} (d_i - d_j) \end{aligned}$$

and

$$B = d_p \sum_{i=1}^{m}(d_i - d_{m+1})(\sum_{m+1 \le i < j \le p-1} (d_i - d_j) + \sum_{i=m+1}^{p-1} (d_i - d_p)).$$

Claim that

(1) $A - f(e, p-1, q') \ge d_p \tilde{p}$ and

(2) $B < d_p m \tilde{p} q'$.

(1) We have

$$\sum_{i=m+1}^{p-1} d_i \cdot \sum_{i=1}^{m}(d_i - d_{m+1}) = (q'(\tilde{p} - m) - e^c)(m - d_p + e^c),$$

$$d_p \sum_{1\le i<j\le m}(d_i - d_j) = 0,\ \ d_p \sum_{m+1\le i<j\le p}(d_i - d_j) \ge e^c d_p,$$

$$md_p \sum_{i=m+1}^{p-1}(d_{m+1} - d_i) = e^c d_p m,$$

$$d_p \sum_{i=1}^{p-1}(d_i - d_p) = d_p((q' - d_p)\tilde{p} + m - d_p).$$

Then

$$\begin{aligned}
A \ &\ge (q'(\tilde{p} - m) - e^c)(m - d_p + e^c) + e^c d_p m + d_p((q' - d_p)\tilde{p} + m - d_p) + e^c d_p \\
&= (m + e^c - d_p)\tilde{p}q' - mq'(m - d_p + e^c) - e^c(m - d_p + e^c) + d_p\tilde{p}q' \\
&\quad - d_p^2\tilde{p} + d_p(m - d_p + e^c) + e^c d_p m \\
&= (m + e^c)\tilde{p}q' - m^2q' + mq'(d_p - e^c) + (d_p - e^c)(m - d_p + e^c) - d_p^2\tilde{p} + e^c d_p m \\
&\ge m\tilde{p}q' - m^2q' + e^c\tilde{p}q' + mq'(d_p - e^c) - d_p^2\tilde{p} \\
&\ge m(\tilde{p} - m)q' + e^c mq' + mq'(d_p - e^c) - d_p^2\tilde{p} \\
&= f(e, p-1, q') + d_p mq' - d_p^2\tilde{p} \\
&\ge f(e, p-1, q') + d_p\tilde{p}(m - d_p) \\
&\ge f(e, p-1, q') + d_p\tilde{p}.
\end{aligned}$$

(2) We have

$$\sum_{i=1}^{m}(d_i - d_{m+1}) = m - d_p + e^c,$$

$$\sum_{m+1\le i<j\le p-1}(d_i - d_j) \le (\tilde{p} - m - 2)\sum_{m+1<j\le p-1}(d_{m+1} - d_j) = e^c(\tilde{p} - m - 2),$$

$$\sum_{i=m+1}^{p-1}(d_i - d_p) = (\tilde{p} - m)(q' - d_p) - e^c.$$

Then

$$\begin{aligned}
B \ &\le d_p(m - d_p + e^c)(e^c(\tilde{p} - m - 2) + (\tilde{p} - m)(q' - d_p) - e^c) \\
&\le d_p(m - d_p + e^c)(\tilde{p} - m)(q - d_p + e^c) \\
&< d_p m\tilde{p}q'.
\end{aligned}$$

Now we have (1)(2). Therefore, for $x \geq \rho^2(G(e, p-1, q'))$,

$$\begin{aligned} g(x) &= x(x^2 - ex + m(\tilde{p} - m)q') + (A - m(\tilde{p} - m)q')x - B \\ &> 0 + d_p\tilde{p} \cdot \tilde{p}q' - d_p m\tilde{p}q' > 0 \end{aligned}$$

by Proposition 5.5.1. So

$$\rho(G_D) \leq \lambda < \rho(G(e, p-1, q')).$$

□

Lemma 5.5.3. *If $m \leq q'$ and $d_p \geq \frac{q'}{2}$, then $\rho(G_D) < \rho(G(e, p, m)) \in \mathcal{C}_0(p, q, e)$.*

Proof. Note that

$$f(e, p, d_p) - f(e, p, m) \geq d_p\tilde{p}(q' - d_p) - m\tilde{p}(q' - m) = \tilde{p}(d_p - m)(q' - m - d_p) > 0$$

since $d_p - m < 0$ and $q' - m - d_p \leq q' - (d_p + 1) - d_p \leq -1$. So $f(e, p, d_p) > f(e, p, m)$ and

$$\rho(e, p, d_p) \leq \phi(e, p, d_p) < \phi(e, p, m) = \rho(e, p, m).$$

□

Lemma 5.5.4. *Suppose $e^c \geq d_p$ and $d_{m+1} \leq q'$. Let $s \leq m + 1$ be the least number such that $d_s \leq q'$. Then we have the following.*

(i) If $q'(\tilde{p} - s + 1) + d_p - e^c \geq m$, then $\rho(G_D) < G(e, p-1, q') \in \mathcal{C}_0(p, q, e)$.

(ii) If $q'(\tilde{p} - s + 1) + d_p - e^c \leq q'$, then $\rho(G_D) < \rho(G_{\tilde{D}})$ and $G_{\tilde{D}} \in \mathcal{C}(e, p-1, q)$, where $\tilde{D} = (d_1, d_2, \ldots, d_{s-1}, \sum_{i=s}^{p} d_i)$.

Proof. (i) Note that

$$\begin{aligned} &\sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{i=s}^{p} d_i \\ \geq &(m - d_p + e^c)(q'(\tilde{p} - s + 1) + d_p - e^c) \qquad (5.5.2) \\ = &mq'(\tilde{p} - s + 1) + (e^c - d_p)(q'(\tilde{p} - s + 1) + d_p - e^c - m) \\ \geq &mq'(\tilde{p} - m) + (e^c - d_p)(q'(\tilde{p} - s + 1) + d_p - e^c - m) \\ \geq &mq'(\tilde{p} - m) = f(e, p-1, q'). \end{aligned}$$

If the equality in (5.5.2) holds, then $d_s = q'$ and D has at least three different values. By Lemma 3.1.1, $\rho(G_D) < \rho(G(e, p-1, q'))$. Clearly $G(e, p-1, q') \in \mathcal{C}_0(p, q, e)$.

(ii) Note that $q'(\tilde{p}-s+1)+d_p-e^c = \sum_{i=s}^{p} d_i$. Then $G_{\tilde{D}} \in \mathcal{C}(e, p-1, q)$ and $G_{\tilde{D}}$ is not a complete bipartite graph. By Lemma 5.3.1, $\rho(G_D) < \rho(G_{\tilde{D}})$ $\square$

Chapter 6

Conclusion

The following is a summary of this dissertation.

1. The extremal non-complete bipartite graph which has the maximum spectral radius with e edges

   - We show that bipartite graphs with e edges of the form $K_{p,q}$, $K^-_{p',q'}$ or $K^+_{p'',q''}$ have larger spectral radii than the others in Corollary 3.3.3, and characterize the value $\rho(G)$ of a bipartite graph G with e edges in Figure 3.1.
   - We also use the above properties to characterize the extremal non-complete bipartite graph which has the maximum spectral radius with e edges in Theorem 3.4.1.
   - When e is even and neither $e-1$ nor $e+1$ is a prime, the two graphs $K^-_{p',q'}$ and $K^+_{p'',q''}$ are candidates to be extremal graph. We determine which graph has larger spectral radius for $e \leq 100$ in Section 3.5.

2. The extremal bipartite graph which has the maximum spectral radius with e edges and bi-order (p, q)

   - We prove BFP conjecture for $\mathcal{K}(p, q, e)$ when $e \in \{st-1, st'+1 \mid s \leq p, t \leq q, t' \leq q-1\}$ in Theorem 3.6.1.

- We give counter examples to BFP conjecture for $\mathcal{K}(p,q,e)$ when $q > p+2$ and $p \geq 3$ in Proposition 3.6.2.
- We prove weak BFP conjecture for $\mathcal{C}(p,q,e)$ in the cases $e \geq pq-q$ with $p \leq q$ or $p \leq 5$ in Theorem 5.2.3 and Theorem 5.2.4, respectively.
- Weak BFP conjecture for $\mathcal{C}(p,q,e)$ is still open. To prove weak BFP conjecture for $\mathcal{C}(p,q,e)$ in general, we use Lemma 5.3.1 and Lemma 5.4.2 as tools and get some partial results. We show that if a bipartite graph $G \in \mathcal{C}(p,q,e)$ satisfies some condition, then there exists $G^\natural \in \mathcal{C}_0(p,q,e)$ such that $\rho(G) \leq \rho(G^\natural)$ with equality only if $G^\natural \in \mathcal{C}_0(p,q,e)$ in Lemma 5.5.2, Lemma 5.5.3 and Lemma 5.5.4.

3. The method to find spectral bounds of a nonnegative matrix

- The most important theorem are Theorem 4.2.3 and Theorem 4.2.4. They are the most general results in Chapter 4 but it is not so easy to use them.
- From Theorem 4.2.3, Theorem 4.2.4 and some properties in Section 4.3, Section 4.4 and Section 4.6, we get Theorem 4.7.1 and Theorem 4.7.5 which are not so general but more convenient than the first two.
- We can use above theorems to get many spectral bounds, for instance Theorem 4.0.1, Theorem 4.5.2, Theorem 4.7.3, Theorem 4.8.1, Theorem 5.1.1 and Theorem 5.4.1.
- In Section 4.9, we give a lemma to help us to determine the set $K := \{b | v''_b > v''_\ell\}$ without computing the eigenvector $v'' = (v''_1, \ldots, v''_\ell)^T$. This is used on characterising when the inequality in Theorem 4.7.1 or Theorem 4.7.5 is equality.
- Section 4.10 is about how to choose a C'' in Theorem 4.7.1 or Theorem 4.7.5 to get a better spectral bound.

Bibliography

[1] J. C. Bermond, J. C. Fournier, M. Las Vergnas, D. Sotteau (Eds.), *Problèmes Combinatoires et Theorie des Graphes*, Orsay, 1976, Coll. Int. C.N.R.S., vol.260, C.N.R.S. Publ., 1978.

[2] A. Bhattacharya, S. Friedland, U. N. Peled, On the first eigenvalue of bipartite graphs, *Electron. J. Combin.* 15 (2008), R144.

[3] A. E. Brouwer, W. H. Haemers, *Spectra of graphs*, Springer, 2012.

[4] R. A. Brualdi, *Introductory Combinatorics* (5th Edition), Pearson Prentice Hall, 2012.

[5] R. A. Brualdi, A. J. Hoffman, on the spectral radius of (0,1)- matrices, *Linear Algebra Appl.* 65 (1985), 133-146.

[6] Y.-F. Chen, H.-L. Fu, I.-J. Kim, E. Stehr, B. Watts, On the largest eigenvalues of bipartite graphs which are nearly complete, *Linear Algebra Appl.* 432 (2010), 606-614.

[7] Y. Chen, H. Lin, J. Shu, Sharp upper bounds on the distance spectral radius of a graph, *Linear Algebra Appl.* 439 (2013), 2659-2666.

[8] P. Csikvári, On a conjecture of Nikiforov, *Discret Math.* 309 (2009), 4522-4526.

[9] S.-Y. Cui, G.-X. Tian, J.-J. Guo, A sharp upper bound on the signless Laplacian spectral radius of graphs, *Linear Algebra Appl.* 439 (2013), 2442-2447.

[10] X. Duan, B. Zhou, Sharp bounds on the spectral radius of a nonnegative matrix, *Linear Algebra Appl.* 439 (2013), 2961-2970.

[11] M. N. Ellingham, X. Zha, The Spectral Radius of Graphs on Surfaces, *J. Combin. Theory Ser.* B 78 (2000), 45-56.

[12] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.

[13] Y. Hong, Upper bounds of the spectral radius of graphs in terms of genus, *J. Combin. Theory Ser.* B 74 (1998), 153-159.

[14] Y. Hong, J.-L. Shu, Kunfu Fang, A sharp upper bound of the spectral radius of graphs, *J. Combin. Theory Ser.* B 81 (2001), 177-183.

[15] W. Hong, L. You, Spectral radius and signless Laplacian spectral radius of strongly connected digraphs, *Linear Algebra Appl.* 457 (2014), 93-113.

[16] R. A. Horn , C. R. Johnson, *Matrix analysis*, Cambrigde University Press, 1985.

[17] Y.-P. Huang, C.-W. Weng, Spectral radius and average 2-degree sequence of a graph, *Discrete Math. Algorithms Appl.* 6 (2014), no. 2, 1450029.

[18] C.-A. Liu, C.-W. Weng, Spectral radius and degree sequence of a graph, *Linear Algebra Appl.* 438 (2013), 3511-3515.

[19] C.-A. Liu, C.-W. Weng, Spectral Radius of Bipartite Graphs, *Linear Algebra Appl.* 474 (2015), 30-43.

[20] C.-A. Liu, C.-W. Weng, Spectral characterization of two families of nearly complete bipartite graphs, *Annals of Mathematical Sciences and Applications* 2 (2017), 241-254.

[21] R. Naulin, C. Pabst, The roots of a polynomial depend continuously on its coefficients. *Rev. Colombiana Mat.* 28 (1994), 35–37.

[22] P. Rowlinson, On the maximal index of graphs with a prescribed number of edges, *Linear Algebra Appl.* 110 (1988), 43–53.

[23] B. Schwarz, Rearrangement of square matrices with nonnegative elements, *Duke Math. J.* 31 (1964), 45-62.

[24] J. Shu, Y. Wu, Sharp upper bounds on the spectral radius of graphs, *Linear Algebra Appl.* 377 (2004), 241-248.

[25] R. P. Stanley, A bound on the spectral radius of graphs with e edges, *Linear Algebra Appl.* 87 (1987), 267-269.

[26] Yitang Zhang, Bounded gaps between primes, *Annals of Mathematics* 179 (2014), 1121-1174.