






A simple method on comparison between spectral radii of two directed graphs

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References

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Introduction

Let \mathbb{R} and \mathbb{C} denote the field of real numbers and complex numbers, respectively. Let C be an $n \times n$ real square matrix. If there is a nonzero column vector $u \in \mathbb{C}^n$ such that $Cu = \lambda u$ for some scalar $\lambda \in \mathbb{C}$, then the scalar λ is called the eigenvalue of C with corresponding eigenvector u . And the spectral radius of a matrix C is the largest magnitude (or complex modulus) of its eigenvalues, denoted by $\rho(C)$. We are interested in the spectral radius of the following matrix associated with a simple directed graph.

Definition

Given a directed graph G , the *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G , and

$$a_{ij} = \begin{cases} 1, & \text{if } ji \text{ is an arc in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Given a directed graph G , the spectral radius of G is the spectral radius of the adjacency matrix of G , denoted by $\rho(G)$. Note that the spectral radius $\rho(G)$ is independent of the ordering of the vertex set of G .

In [1], Cheng and Weng give many bounds of the spectral radius of a nonnegative square matrix. And based on their theory and Perron-Frobenius theorem, we give another approach to obtain an upper bound of the spectral radius and apply it on the adjacency matrix of a directed graph.

All theorems come from continuous discussions between C.W. Weng and K.H. Chen. These were all documented.[5]

Preliminaries

The following is Perron–Frobenius theorem, which provides a feature of nonnegative eigenvectors to nonnegative matrices.

Theorem

[3] If C is a nonnegative square matrix, then the spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector. Moreover if C is irreducible the above eigenvectors can be chosen to be positive.

A well-known application of Theorem 6 show that if matrix C' *majors* C (in notation $C \leq C'$), i.e. $c_{ij} \leq c'_{ij}$ for all ij, j , then $\rho(C) \leq \rho(C')$. Our main result shows that the assumption $C \leq C'$ can be a little loosen. Our theory is based on the following theorem, which is from [1].

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ real matrices with real eigenvalues λ , λ' respectively such that there exist $n \times n$ matrices P and Q satisfying the following (i)-(iv).

- (i) $PCQ \leq PC'Q$;
- (ii) an eigenvector Qu of C' associated with eigenvalue λ' exists for some nonnegative column vector $u = (u_1, u_2, \dots, u_n)^T$.
- (iii) a left eigenvector $v^T P$ of C' associated with eigenvalue λ exists for some nonnegative row vector $v^T = (v_1, v_2, \dots, v_n)$; and
- (iv) $v^T PQ u > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \quad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \quad (1)$$

Multiplying the nonnegative vector u in assumption i to the right of both terms of i,

$$PCQu \leq PC'Qu = \lambda' PQu, \quad (2)$$

where the above equality follows by Qu being eigenvector of C' for λ' .

Multiplying the nonnegative vector v^T of C in assumption iii to the left of all terms in (2), we have

$$\lambda v^T P Q u = v^T P C Q u \leq v^T P C' Q u = \lambda' v^T P Q u, \quad (3)$$

where the above first equality follows by $v^T P$ being left eigenvector of C for λ . Now delete the positive term $v^T P Q u$ by assumption iv to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Assume that $\lambda = \lambda'$, so the inequality in (3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1) implies

$$\begin{aligned}\lambda v^T PCQu &= v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j \\ &= \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu = \lambda' v^T PC'Qu,\end{aligned}$$

so $\lambda = \lambda'$ by (3).

Our Method

We will apply Theorem 3 by using two particular square matrices P and Q to obtain our main result. We use $[n-1]$ as notation of the set of integers from 1 to $n-1$, which is $\{1, 2, \dots, n-1\}$.

Throughout this thesis we fix $k \in [n-1]$, let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn} .

Now we apply the previous Theorem 3 with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \\ & & & & \ddots \\ 0 & & & & & 1 & 0 \\ & & & & & 0 & 1 \end{pmatrix}, \quad (4)$$

so the matrix $PC'Q$ in assumption i of Theorem 3 is

$$PC'Q = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1k} + c'_{1n} \\ c'_{21} & c'_{22} & \cdots & c'_{2k} + c'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{nk} + c'_{nn} \end{pmatrix}, \quad (5)$$

where c'_{ij} denotes the (i, j) -entry of C' .

Definition

A column vector $v' = (v'_1, v'_2, \dots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

Lemma

If $u = (u_1, u_2, \dots, u_n)^T$, then

- (i) Qu is *k-rooted* if and only if u is nonnegative;
- (ii) $u_k > 0$ if and only if $(Qu)_k > (Qu)_n$.

Proof.

(i), (ii) follow immediate from the definition of k -rooted and $Qu = (u_1, \dots, u_{k-1}, u_k + u_n, u_{k+1}, \dots, u_n)^T$. □

Below is our first result, in which the first condition implies the first $n - 1$ columns of C' major the same columns of C , and the sum of k -th and n -th columns of C' also majors that of C . The second and the third condition suggest that C and C' have nonnegative k -rooted eigenvectors. And the forth condition is simpler but with the same meaning in Theorem 3

We need a notation of submatrix, which is taken from some columns and some rows of a matrix.

Definition

For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C , respectively, we use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$. We use the notation $C[i|j]$ for short of $C[[i]|[j]]$.

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ real matrices with real eigenvalues λ and λ' respectively. Assume that

- (i) $C[n|n-1] \leq C'[n|n-1]$ and $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ for all $1 \leq i \leq n$;
- (ii) there exists a k -rooted eigenvector vector $\check{v} = (\check{v}_1, \check{v}_2, \dots, \check{v}_n)^T$ of C' for λ' ;
- (iii) there exists a nonnegative eigenvector vector $v^T = (v_1, v_2, \dots, v_n)$ of C for λ ;
- (iv) $v^T \check{v} > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

- (a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ for $1 \leq i \leq n$ with $v_i > 0$ and $\check{v}_n > 0$;
- (b) $c'_{ij} = c_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq n-1$, $j \neq k$ with $v_i \neq 0$ and $\check{v}_i > 0$.



Proof.

The proof is based on Theorem 3 with $P = I$ and $Q = I + E_{kn}$ in (4). The assumption i $PCQ \leq PC'Q$ of Theorem 3 holds by the condition i of this theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda'Qu$ by the condition ii and i in Lemma 5. Hence the assumption ii of Theorem 3 holds. The assumptions iii and iv of Theorem 3 clearly hold by conditions iii, iv of this theorem since $P = I$ and $v' = Qu$. Hence $\lambda \leq \lambda'$ by the necessary condition of Theorem 3. Moreover, $\lambda = \lambda'$ if and only if (1) holds, and this is equivalent to conditions a, b and c of this theorem. \square

We are interested in the matrices C' that have k -rooted eigenvectors. Motivated by the condition (i) of Theorem 2.3, we provide the following two definitions. The first is the definition of (k, n) -sum.

Definition

For an $n \times n$ matrix $C' = (c'_{ij})$, the (k, n) -sum vector of C' is the vector obtained from the sum of the k -th and n -th columns of C' .

Note that the last column of $C'Q$ is the (k, n) -sum vector of C' . Below is the definition of k -rooted matrix.

Definition

A matrix $C' = (c'_{ij})$ is called k -rooted if for each $i \neq k, n$ the i -th column of C' is k -rooted and the (k, n) -sum vector of C' is k -rooted.

We need a simple lemma for later use.

Lemma

$$Q^{-1} = I - E_{kn}.$$

Proof.

Since

$$Q(I - E_{kn}) = (I + E_{kn})(I - E_{kn}) = I - E_{kn} + E_{kn} - E_{kn}E_{kn} = I,$$

we have $Q^{-1} = I - E_{kn}$. □

The matrix Q^{-1} is explicitly written as

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & -1 \\ & & & & \ddots \\ & & & & & 1 & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix},$$

and if $C' = (c'_{ij})$ then $Q^{-1}C'Q$ has the form

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c'_{(k-1)1} & c'_{(k-1)2} & \cdots & c'_{(k-1)(n-1)} & c'_{(k-1)k} + c'_{(k-1)n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{k(n-1)} - c'_{n(k-1)} & c'_{kk} + c'_{kn} - c'_{nk} - c'_{nn} \\ c'_{(k+1)1} & c'_{(k+1)2} & \cdots & c'_{(k+1)(n-1)} & c'_{(k+1)k} + c'_{(k+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n(n-1)} & c'_{nk} + c'_{nn} \end{pmatrix}. \quad (6)$$

The following lemma shows that a k -rooted matrix has a k -rooted eigenvector.

Lemma

Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(ii) hold.

- (i) C' is a k -rooted matrix if and only if $Q^{-1}(C' + dI)Q$ is nonnegative for some $d \geq 0$, where I is the $n \times n$ identity matrix.
- (ii) If C' is k -rooted then there exists a k -rooted eigenvector v' of C' for $\rho(C')$.

Proof.(i).

The matrix $Q^{-1}C'Q$ has ij entry

$$(Q^{-1}C'Q)_{ij} = \begin{cases} c'_{ij}, & \text{if } i \neq k \text{ and } j \neq n; \\ c'_{kj} - c'_{nj}, & \text{if } i = k \text{ and } j \neq n; \\ c'_{ik} + c'_{in}, & \text{if } i \neq k \text{ and } j = n; \\ c'_{kk} + c'_{kn} - c'_{nk} - c'_{nn}, & \text{if } i = k \text{ and } j = n, \end{cases}$$

as shown in (6). Hence $Q^{-1}(C' + dI)Q$ is nonnegative if and only if C' is k -rooted by the definition of nonnegative matrix and k -rooted matrix.



Proof.(ii).

Suppose C' is k -rooted. Choose $d \geq 0$ such that $Q^{-1}(C' + dI)Q$ is nonnegative. Let u be a nonnegative eigenvector of $Q^{-1}(C' + dI)Q = Q^{-1}C'Q + I$ for $\rho(C' + dI) = \rho(C') + d$. Note that $Q^{-1}C'Qu = \rho(C')u$, and Qu is k -rooted by Lemma 5. Hence $v' = Qu$ is what we want. □

Note that Theorem 7 depends on eigenvectors. The following is an eigenvector-free Theorem.

Theorem

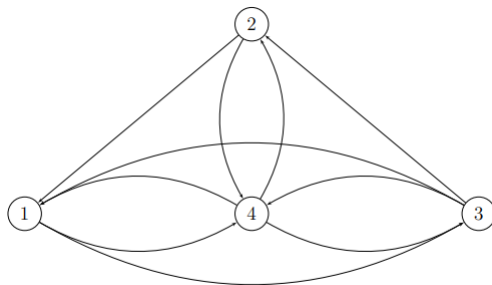
Let C be an $n \times n$ nonnegative irreducible matrix and C' be an $n \times n$ k -rooted matrix such that $C'Q$ majorizes CQ . Then $\rho(C) \leq \rho(C')$.

Proof.

Referring to (5), the assumption i in Theorem 7 holds. By Lemma 11 ii, there exists a k -rooted eigenvector v' of C' for $\rho(C')$. Since C is irreducible and nonnegative, there exists a positive eigenvector v^T of C for $\rho(C)$. Thus $v^T v' > 0$. Hence Theorem 7 i-iv hold. Hence $\rho(C) \leq \rho(C')$ by Theorem 7. \square

Examples

Let G be the digraph depicted below.



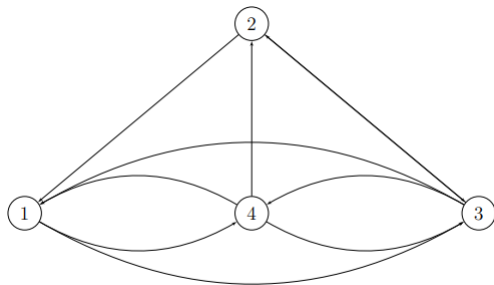
The following 4×4 matrix is the adjacency matrix of G .

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We choose another matrix

$$A' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

which is the adjacency matrix of G' depicted below.



Note that both G and G' have the same number of edges, neither A majorities A' nor A' majorities A . Our main result still can do comparison between $\rho(A)$ and $\rho(A')$. We first observe that A is irreducible and A' is k -rooted for $k = 3$. Moreover

$$A'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = AQ. \text{ Hence Hence}$$

$\rho(A) \leq \rho(A')$ by Theorem 12 with $C = A$ and $C' = A'$. Both of the values of $\rho(C)$ and $\rho(C')$ are close to 2.511547 by calculating on computer. [4, sage]

To finish the thesis, we provide an example to show that the 'k-rooted' assumption of C' is necessary in Theorem 12.

Consider the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

With $n = 4$ and $k = 3$, we have

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = C'Q.$$

Using computer, [4, sage]

$$\rho(C) \approx 2.234 \not\leq 2.148 \approx \rho(C').$$

This is because C' is not k -rooted as $c'_{22} + c'_{21} = 0 \not\geq 1 = c'_{12} + c'_{11}$. 