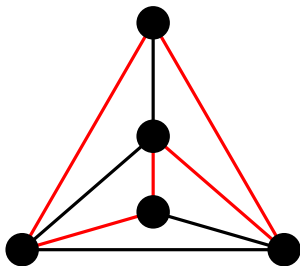


# Hamiltonian graphs

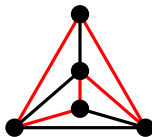
A graph is **Hamiltonian** if it contains a spanning cycle.



The red edges form a spanning cycle of the above graph

# Definitions

- 1 A **triangulated graph** is a planar graph such that any edge is in a cycle and any face is a triangle.
- 2 A **chordal graph** is a graph such that every cycle of length at least 4 contains a **chord**, i.e., an edge connecting two nonadjacent vertices of the cycle.



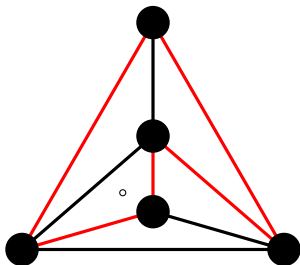
### Remark 0.1

- 1 A triangulated of order at least 4 contains  $3 + 3(n - 3) = 3n - 6$  edges and  $2n - 4$  faces.
- 2 If  $C$  is a Hamiltonian cycle of a triangulated planar graph  $T$  then there are  $n - 2$  faces and  $n - 3$  edges inside  $C$ .

## Problem

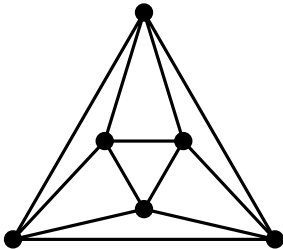
Can every edge of a triangulated graph be drawn as an edge of an outer face.

**Answer:** Yes

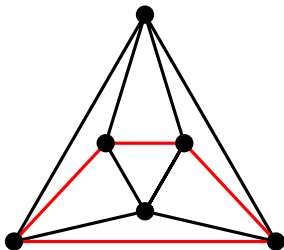


Plane is homeomorphic to the surface obtained from a sphere deleting a point.

## A 4-regular triangulated graph



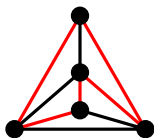
## A triangulated graph which is not a chordal graph



There is no chord in the red cycle.

## More definitions

- ① A **weakly pancyclic graph** is a graph which contains cycles of every length between the girth and the circumference.
- ② A graph is  **$t$ -tough** if for every integer  $k > 1$ , the graph cannot be split into  $k$  components by removal of fewer than  $tk$  vertices
- ③ The **toughness** of a non-complete graph  $G$  is the maximum  $t$  such that  $G$  is  $t$ -tough. The toughness of a complete graph is  $\infty$ .



### Remark 0.2

A  $t$ -tough graph with  $t > 0$  is connected and a Hamiltonian graph is 1-tough.

# A triangulated graph which is not weakly pancyclic

To be filled later.



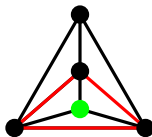
# A long standing conjecture

There is a positive constant  $t$  such that every  $t$ -tough graph is Hamiltonian.

V. Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 5 (1973), 215-228.

## Locally property $P$

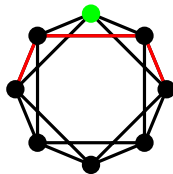
Let  $P$  be a property on graphs. A graph  $G$  has **local**  $P$  if  $G_1(v)$  has property  $P$  for every vertex  $v$  in  $G$ .



### Remark 0.3

A triangulated graph of order at least 4 is locally Hamiltonian.

# A 1-tough and locally connected planar graph



The planar graph  $DL(8; 1, 2)$  is 1-tough and locally connected.

## Remark 0.4

$DL(n; 1, 2)$  is a planar graph, where  $n \geq 4$  and  $n$  is even.

## Remark 0.5

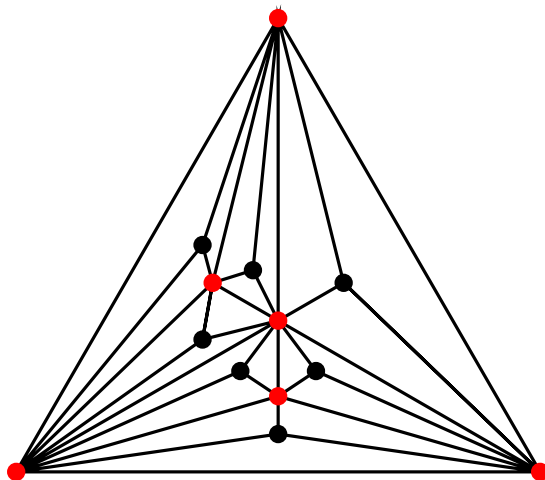
Every planar graph can be drawn by using straight lines as edges.

# Question

Is a triangulated graph 1-tough?

The answer is 'no' as shown in the next page.

## A triangulated graph which is not 1-tough



Deleting the 6 red vertices will yield 7 components.

## Lemma

For a vertex  $u$  and a cut set  $S$  of  $G$  if  $c(G - S) > |S|$  and  $G - u$  is 1-tough then  $u \notin S$  and  $G_1(u) \subseteq S$ .

Proof.

If  $u \in S$  then  $G - u$  has toughness at most  $(|S| - 1)/c(G - S) < 1$ , a contradiction; if  $u \notin S$  then  $G - u$  being 1-tough and  $|S|/c(G - S) < 1$  implies that there is a single element component  $\{u\}$  in  $G - S$ , so  $G_1(u) \subseteq S$ . □

T. Nishizeki, A 1-tough nonhamiltonian maximal planar graph, Discrete Math. 30(1980), 305-307.

# Proposition

Let  $G$  be a connected graph such that  $G - u$  is Hamiltonian for every vertex in  $G$ . Then  $G$  is 1-tough.

## Proof.

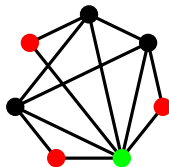
Suppose  $G$  is not 1-tough and pick a cut set  $S$  with  $c(G - S) > |S|$ . As  $G - u$  is 1-tough,  $u \notin S$  by previous lemma. Hence  $S = \emptyset$ , a contradiction. □

## Question

Is every 1-tough and locally connected graph Hamiltonian?

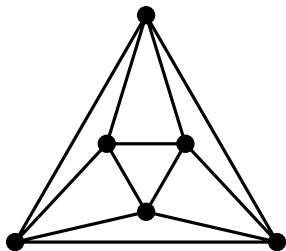
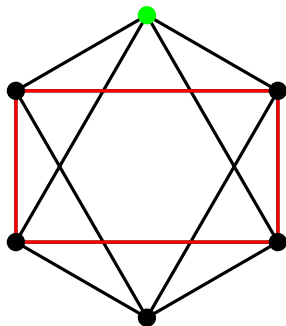


## A 1-tough, locally connected, non-Hamiltonian planar graph



The graph is not Hamiltonian since the two edges of each red vertex are in every Hamiltonian cycle.

## A 1-tough and locally Hamiltonian planar graph



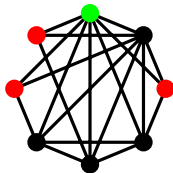
The planar graph  $DL(6; 1, 2)$  is 1-tough and locally Hamiltonian, implying locally 1-tough.  $DL(6; 1, 2)$  is a triangulated planar graph.

## Question

Is every 1-tough and locally 1-tough graph locally Hamiltonian?

The answer is 'no' in the following page.

## A 1-tough and locally 1-tough graph which is not locally Hamiltonian



Deleting the green vertex of the above graph yields a 1-tough non-Hamiltonian graph (The red vertices have degree 2).

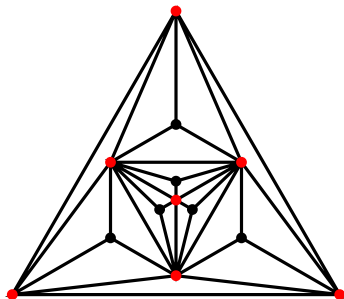
The above graph is not planar.

## Question

Is a 1-tough and locally 1-tough graph Hamiltonian.

The answer is 'no' as shown in the next page.

# A triangulated 1-tough non-Hamiltonian graph



Seven separating triangles in the above graph.

Michal Tkáč, On the shortness exponent of 1-tough, maximal planar graphs, Discrete Math. 154(1996), 321-328.

# Plan

We shall develop theory of 1-tough and locally 1-tough graph.

## Lemma

If  $G$  is 1-tough and locally 1-tough then  $G - u$  is 1-tough for every vertex  $u$  in  $G$ .

### Proof.

Assume that  $G - u$  is not 1-tough. Then  $G - u$  is split into  $k \geq 2$  components by removal of some subset  $S$  of vertices with  $|S| < k$ , where  $u \notin S$ . Assume  $\ell$  of these  $k$  components intersecting with  $G_1(u)$ . If  $\ell = 1$  then by connecting  $u$  to this unique component intersecting with  $G_1(u)$ , there are  $k$  components after removal of  $S$  from  $G$ , a contradiction to the 1-tough property of  $G$ .

Assume  $\ell \geq 2$ . Then  $|S \cap G_1(u)| \geq \ell$  since  $G_1(u)$  is 1-tough. Similarly by connecting  $u$  to these  $\ell$  components intersecting with  $G_1(u)$  to form a component, we find  $k - \ell + 1$  components after the removal of  $S - G_1(u)$  from  $G$ . Hence  $|S - G_1(u)| \geq k - \ell + 1$  since  $G$  is 1-tough. Then  $|S| = |S \cap G_1(u)| + |S - G_1(u)| \geq \ell + (k - \ell + 1) = k + 1$ , a contradiction. □



# Lemma

If  $G$  is 1-tough and locally 1-tough then  $G$  has toughness greater than 1.

## Proof.

Assume that  $G$  has toughness 1 and pick a subset  $S$  of  $k$  vertices whose removal will yield  $k$  components. Pick  $u$  in  $S$ . Then the removal of  $k - 1$  vertices in  $S - \{u\}$  from  $G - u$  will yield  $k$  components, a contradiction to  $G - u$  being 1-tough by previous lemma.  $\square$

# Toughness of a 1-tough and locally 1-tough graph

By connecting a new vertex to the complete bipartite graph  $K_{t,t}$ , we have a 1-tough and locally 1-tough graph with toughness  $(t+1)/t$  for every  $t \geq 1$ . Hence the toughness of a 1-tough and locally 1-tough graph can approximate to 1.

## Lemma

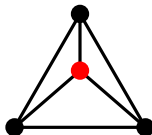
If  $G$  is 1-tough and locally 1-tough of order at least 4 then  $G$  has minimum degree at least 3.

### Proof.

If  $G$  has minimum degree at most 2 then  $G$  has toughness at most  $1 = 2/2$  because deleting the neighbors of a vertex with minimum degree yields at least two components.  $\square$

### Remark 0.6

There is only one 1-tough and locally 1-tough graph with maximum degree 3.



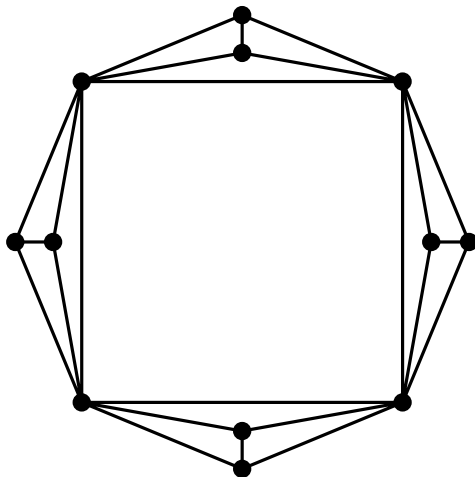
## Lemma

If  $G$  is 1-tough and locally 1-tough of order at least 4 then every edge of  $G$  is in at least two triangles.

### Proof.

Let  $e = uv$  be an edge of  $G$ . Since  $u$  has degree at least 3,  $G_1(u)$  is at least 3, so with the 1-tough property of  $G_1(u)$ ,  $u$  has at least two neighbors in  $G_1(u)$ , forming two triangles containing the edge  $e$ . □

## Each edge is in two triangles



Each edge of the above non-triangulated graph is in two triangles. This graph is 1-tough, but not locally 1-tough.

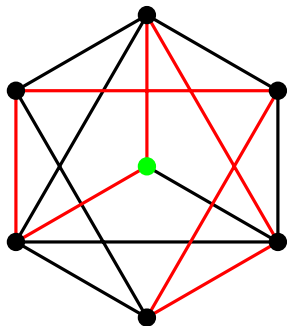
# Proposition

If  $G$  is a 1-tough and locally 1-tough planar graph of order at least 4 and  $C$  is a cycle of maximum length in  $G$ , then each edge in  $C$  is in at least two triangles of the subgraph induced on the vertex set of  $C$ .

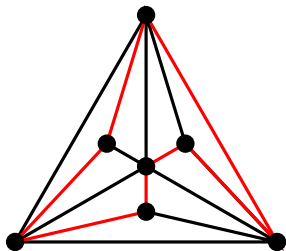
## Proof.

If the two triangles of an edge in previous lemma contain a vertex not in  $C$  then we have a cycle of length one more than that of  $C$ , a contradiction to the maximum length assumption of  $C$ . □

# 1-tough and locally Hamiltonian planar graph of order 7



The graph  $DL(6; 1, 2)^+$ .



A triangulated graph of order 7  
(There are many different ones  
including  $DL(6; 1, 2)^+$ ).

# Problems

- ① For  $n \leq 7$ , find non-triangulated planar Hamiltonian graphs of order  $n$  such that each edge in its Hamiltonian cycle is in two triangles.
- ② How many edges are possible for a planar Hamiltonian graph of order  $n$  such that each edge in its Hamiltonian cycle is in two triangles?
- ③ Determine all the non-triangulated planar Hamiltonian graphs of order  $n$  such that each edge in its Hamiltonian cycle is in two triangles.
- ④ Find a 1-tough and locally 1-tough planar graph which is not triangulated.



# Lemma

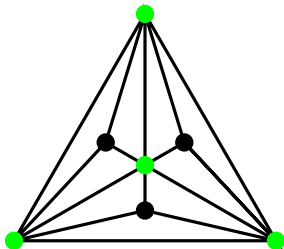
Let  $S$  be a cut set of a locally 1-tough connected graph  $G$ . If  $x \in S$  is connected to at least two components in  $G - S$  then  $x$  has degree at least two in the subgraph induced on  $S$ .

Proof.

The 1-tough of  $G_1(x)$  implies  $|G_1(x) \cap S|$  is at least the number of components in  $G - S$  that are connected to  $x$ . □

A cut set  $S$  of  $G$  is **effective** if each  $x \in S$  is connected to at least two components in  $G - S$ .

## Effective cut set



The green vertices form an effective cut set.