Chapter 1

Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers respectively.

Definition 1.1. Let C be an $n \times n$ real matrix, and $u \in \mathbb{R}^n$ be a nonzero column vector. The scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of C corresponding to the *eigenvector* u, if $Cu = \lambda u$.

Definition 1.2. When C is an $n \times n$ real matrix, the *spectral radius* $\rho(C)$ of C is defined by

$$\rho(C) := \max\{\ |\lambda| \ \mid\ \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ.

We are interested in spectral radius of the following matrix associated with a simple graph.

Definition 1.3. Given an undirected graph G, the *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G, and

$$a_{ij} = \begin{cases} 1, \text{if } i \text{ is adjacent to } j, \\ 0, \text{otherwise.} \end{cases}$$

Definition 1.4. Given an undirected graph G, the *spectral radius* $\rho(G)$ of G is the spectral radius of the adjacency matrix of G.

We introduce a notation of submatrix, which is taken from some columns and some rows of a matrix.

Definition 1.5. For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C respectively, We use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$,

The following theorem is a part of Perron-Frobenius theorem [**?**].

Theorem 1.6. *If C is a nonnegative square matrix, then spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector.*

The following theorem is from [**?**]. For completeness, we also provide the proof.

Theorem 1.7. *Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that*

(i) $PCQ \leq PC'Q$;

(ii) there exist a nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector Qu;

(iii) there exist a nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector $v^T P$; and

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{1.1}$$

Proof. Multiplying the nonnegative vector u in assumption (ii), where Qu is eigenvector of C', to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{1.2}$$

Multiplying the nonnegative vector v^T in assumption (iii) , where $v^T P$ is left eigenvector of C for λ,to the left of all terms in (1.2), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{1.3}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Assume that $\lambda = \lambda'$, so the inequality in (1.3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1.1) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (1.3).

$\square$