Chapter 1

Our result

Remark 1.0.1 (theorem in previous slide)**.** Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that

(i) $PCQ \leq PC'Q$;

(ii) there exist a nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector Qu;

(iii) there exist a nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$such that λ is an eigenvalue of C with associated left eigenvector $v^T P$; and

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{1.1}$$

Proof. 1.0.2. Multiplying the nonnegative vector u in theorem 1.0.1 assumption (i), where Qu is eigenvector of C', to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{1.2}$$

Multiplying the nonnegative left eigenvector v^T of C for λ in assumption (iii) to the left of all terms in (1.2), where $v^T P$ is left eigenvector of C for λ, thus we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{1.3}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part. Assume that $\lambda = \lambda'$, so the inequality in (1.3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1.1) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (1.3).

We will prove the main Theorems introduced in the introduction. First, we need some natations. Throughout fix $k \in [n-1] := \{1, 2, ..., n-1\}$. Let E_{kn} denote the $n \times n$ binary matrix with an unique 1 appearing in the position (k, n). We will apply the Theorem 1.7 with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & 1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \begin{array}{l} \\ \\ k \\ \\ \\ \end{array} \tag{1.4}$$

Definition 1.0.3 (k-rooted vector)**.** A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

Lemma 1.0.4. *If $u = (u_1, u_2, \ldots, u_n)^T$ and $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$, then*

(i) v' is k-rooted if and only if u is nonnegative;

(ii) $u_k > 0$ if and only if $v'_k > v'_n$.

$\square$

Theorem 1.0.5. *Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ matrices. Assume that*

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ *and* $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ *for all* $1 \leq i \leq n$*;*

(ii) there exists a k-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';

(iii) there exists a nonnegative vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ *for* $1 \leq i \leq n$ *with* $v_i \neq 0$ *and* $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1, j \neq k$*with* $v_i \neq 0$*;*

(c) $c'_{ik} = c_{ik}$ *for* $1 \leq i \leq n$ *and* $v'_k > v'_n$

$\square$

Proof. 1.0.6. The proof is based on Theorem 1.0.1 with $P = I$ and $Q = I + E_{kn}$ in (1). The assumption (i) $PCQ \leq PC'Q$ of Theorem 1.0.1 holds by the condition (i) of this theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda' Qu$ by the condition (ii) and Lemma 1.0.4(i). Hence the assumption (ii) of Theorem 1.0.1 holds. The assumptions (iii) and (iv) of Theorem 1.0.1 clearly hold by conditions (iii),(iv) of this theorem since $P = I$ and $v' = Qu$ Hence $\lambda \leq \lambda'$ by the necessary condition of Theorem 1.0.1. Moreover $\lambda = \lambda'$ if and only if 1.1 holds, and this is equivalent to conditions (a),(b),(c) of this theorem.

We are interested in the matrices C' that have k-rooted eigenvectors. Motivated by the condition (i) of theorem 2.3, we provide the following two definitions.

Definition 1.0.7 ((k,n)-sum)**.** For an $n \times n$ matrix $C' = (c'_{ij})$, the $(k,n) - sum$ vector of C' is the vector of the sum of the k-th and n-th columns of C'.

Note that the last column of $C'Q$ is the $(k,n)-sum$ vector of C'

Definition 1.0.8. A matrix $C' = (c'_{ij})$ is called $k - rooted$ if its columns and its $(k,n) - sum$ vector are all $k - rooted$ except the last column of C'.

The following theorem shows that a k-rooted matrix has a k-rooted eigenvector.

Lemma 1.0.9. *Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(iii) hold.*

(i) C' is a k-rooted matrix, if and only if, $Q^{-1}C'Q$ is nonnegative.

(ii) Assume that C' is k-rooted and let u be a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Then C' has a k-rooted eigenvector $v' = Qu$ for $\rho(C')$.

(iii) $\rho(C') = \rho(Q^{-1}C'Q)$

Proof. 1.0.10. (i) is immediate from Definition 1.0.8 and the observation that

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix},$$

and $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{kn-1} - c'_{nk-1} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

(ii) By Lemma 1.0.4 $v' = Qu$ is k-rooted. Since $Q^{-1}C'Qu = \rho(C')u$ by the assumption, we have $Q^{-1}C'Qu = Q^{-1}\rho(C')Qu = \rho(C')u$
$C'Qu = \rho(C')Qu$.

(iii) Since C' and $Q^{-1}C'Q$ have the same set of eigenvalues, clearly $\rho(C') = \rho(Q^{-1}C'Q)$.

Lemma 1.0.11. *If a square matrix C' has a rooted eigenvector for λ', then $C' + dI$ also has the same rooted eigenvector for $\lambda' + d$, where d is a constant and I is the identity matrix with the same size of C'.*

Theorem 1.0.12. *Let C be an $n \times n$ nonnegative matrix. For $1 \leq i \leq n$ and $1 \leq j \leq n-1$, choose c'_{ij} such that $c'_{ij} \geq c_{ij}$ and $c'_{kj} \geq c'_{nj} > 0$, and choose r'_i such that $r'_i \geq c_{ik} + c_{in}$, and $r'_k \geq r'_n$. Moreover choose $c'_{in} := r'_i - c'_{ik}$. Then $\rho(C) \leq \rho(C')$, when $C' = (c'_{ij})$.*

Proof. 1.0.13. These assumptions are necessary that $PCQ \leq PC'Q$, and C' is k-rooted,based on (1.0.9);
calculation:

by 1.0.11, For certain d, if $C' + dI$ is k-rooted, then it has a k-rooted eigenvector with its spectral radius $\lambda + d$. C' would share the same eigenvector with $C' + dI$ and has eigenvalue λ. So $C' + dI$ and $C + dI$ meet the conditions of 1.0.5, and we can show that $\rho(C' + dI) \geq \rho(C + dI)$ and then $\rho(C') \geq \rho(C)$ □

1.1 Example

For the following 4×4 matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Check the conditions of theorem 1.0.5, Then $\rho(C) \leq \rho(C')$ by Theorem 4.8.

1.2 Counterexample

For the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

specify n=4, k=3 in $Q = I + E_{kn} = I + E_{34}$

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

we have $CQ \leq C'Q$, but $\rho(C) = 2.234 \not\leq 2.148 = \rho(C')$. This is because $c'_{33} + c'_{34} \not\geq c'_{43} + c'_{44}$.