Erdős-Gallai Theorem

A sequence of nonnegative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ can be represented as the degree sequence of a simple graph on n vertices if and only if

$$\sum_{i=1}^{n} d_i$$

is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} \qquad (1 \leq k \leq n).$$

□

A proof of necessity

Let G be a simple graph with vertex set $[n] := \{1, 2, \ldots, n\}$ with vertex i of degree d_i satisfying $d_1 \geq d_2 \geq \cdots \geq d_n$. Then

$$\sum_{i=1}^{n} d_i = 2|E(G)|$$

is even. Let $[k] := \{1, 2, \ldots, k\}$, $E([k])$ denote the set of edges on $[k]$ and $E([k], [n] - [k])$ denote the set of edges with one end in $[k]$ and the other end in $[n] - [k]$. Then

$$\begin{aligned} \sum_{i=1}^{k} d_i &= 2|E([k])| + |E([k], [n] - [k])| \\ &\leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}. \end{aligned}$$

$\square$

A proof of sufficiency

We prove by induction on s, where $2s = \sum_{i=1}^{n} d_i$. For $s = 0$ we choose the null graph of order n. Suppose $s > 0$. Note that $d_2 > 0$. Choose

$$\begin{aligned} t &= \begin{cases} \min\limits_{d_i > d_{i+1} > 0} i, & \text{if } d_j \text{ has at least two different nonzero values;} \\ \max\limits_{d_i = d_{i+1} > 0} i, & \text{if } d_j \text{ has only one nonzero value,} \end{cases} \\ p &= \max_{d_i > 0} i \end{aligned}$$

Hence $1 \leq t < p \leq n$. Set

$$d_i' = \begin{cases} d_i, & \text{if } i \notin \{t, p\}; \\ d_i - 1, & \text{if } i \in \{t, p\}. \end{cases}$$

Then

$$d_1' = \cdots = d_{t-1}' > d_t' \geq \cdots \geq d_p' \geq 0 = d_{p+1}' = \cdots = d_n'.$$

The case $k \geq t$

Clearly $\sum_{i=1}^{n} d'_i = 2(s-1)$ is even. We will show that

$$\sum_{i=1}^{k} d'_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d'_i, k\} \qquad (1 \leq k \leq n).$$

No matter $p \geq k \geq t$ or $k > p > t$, we have

$$\begin{aligned}
\sum_{i=1}^{k} d'_i &\leq d_t - 1 + \sum_{\substack{i=1 \\ i \neq t}}^{k} d_i \\
&\leq -1 + k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} \\
&\leq k(k-1) + \sum_{i=k+1}^{n} \min\{d'_i, k\}.
\end{aligned}$$

Proof of the case $k < t$ by contradiction

$$
\begin{aligned}
a \quad &= \quad k(k-1) + \sum_{i=k+1}^{n} \min\{d'_i, k\} < \sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i \\
&\leq \quad k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} = b \leq a + 2 \\
\Rightarrow \quad & \quad 0 < \sum_{i=k+1}^{n} (\min\{d_i, k\} - \min\{d'_i, k\}) \\
\Rightarrow \quad & \quad d_1 = d_t \leq k \ (b = a+2), \ \text{or} \ d_1 = d_t > k \geq d_p \ (b = a+1) \\
\Rightarrow \quad & \quad \sum_{i=1}^{k} d'_i = kd_1 \leq (k-1)k + d_1 = (k-1)k + d'_t + 1 \\
??? \leq \quad & \quad k(k-1) + \sum_{i=k+1}^{n} \min\{d'_i, k\} \quad \text{or} \quad d_1 = d_t > k \geq d_p.
\end{aligned}
$$

The case $d_1 = d_t > k \geq d_p$ and $k < t$

$$\sum_{i=1}^{k} d_i' = kd_1 = k(k-1) + k(d_1 - k + 1)$$

$$??? \leq \quad k(k-1) + \sum_{i=k+1}^{n} \min\{d_i', k\} \qquad (n - k \geq d_t - k + 1)$$

Induction hypothesis

There exists a graph G' with degree sequence $\{d'_i\}_{i=1}^n$. If tp is not an edge in G', we construct the graph G by adding the edge tp to G'.

Suppose tp is an edge in G'. As $d'_t < d'_{t-1} \leq n$, there is a vertex m such that tm is not an edge in G'. As $d'_m = d_m \geq d_p > d'_p$, there exists a vertex q such that mq is an edge and pq is not an edge in G'.

We construct G by deleting the edge mq and adding two edges tm and pq to G'. Then G has degree sequence $\{d_i\}_{i=1}^n$.
This completes the proof. $\square$

ve-degree

Let G be a simple graph with vertex set V and edge set E. For a vertex $u \in V$, let $N(u) := \{v \mid uv \in E\}$ denote the open neighborhood of u. The *ve*-degree $\mathrm{vedeg}(u)$ of a vertex u in G is the number of edges incident on a vertex of $N(u)$, i.e.

$$\begin{aligned} \mathrm{vedeg}(u) &:= |\{xy \in E \mid x \in N(u)\}| \\ &= |E(G_{\leq 2}(u)) - E(G_2(u))|, \end{aligned}$$

where

$$G_{\leq 2}(u) = \{v \in V \mid \partial(u, v) \leq 2\}, \quad G_2(u) := \{v \in V \mid \partial(u, v) = 2\},$$

and $E(S)$ is the edge set in the subgraph of G induced on $S \subseteq V$.

The number of triangles in a graph

Let η_u denote the number of triangles that contain u, and $\eta(G)$ the number of triangles in G. It is known that

$$\mathrm{vedeg}(u) = \sum_{v \in N(u)} \deg(v) - \eta_u$$

and

$$\sum_{u \in V} \mathrm{vedeg}(u) = \left(\sum_{u \in V} \deg^2(u) \right) - 3\eta(G).$$

Complete graph

For the complete graph K_n of order n,

$$\sum_{u \in V} \mathrm{vedeg}(u) = \left(\sum_{u \in V} \deg^2(u) \right) - 3\eta(G) = \frac{n^2(n-1)}{2}.$$

Graph monotone of $\sum_{u \in V} \mathrm{vedeg}(u)$

If H is a subgraph of G, then
$\sum_{u \in V(H)} \mathrm{vedeg}_H(u) \leq \sum_{u \in V(G)} \mathrm{vedeg}_G(u)$.

Proof.
This is clear from

$$V(H) \subseteq V(G) \quad \text{and} \quad \mathrm{vedeg}_H(u) \leq \mathrm{vedeg}_G(u) \quad \text{if } u \in V(H).$$

□

From now on we always assume that G has no isolated vertices, so $\mathrm{vedeg}(u) > 1$ for $u \in V$.

A lower bound of $\sum_{u \in V} \mathrm{vedeg}(u)$

$$\sum_{u \in V} \mathrm{vedeg}(u) \geq \frac{4|E|^2}{|V|} - 3\eta(G)$$

with equality iff the graph is regular.

Proof.

$$\begin{aligned}
\sum_{u \in V} \mathrm{vedeg}(u) &= \left(\sum_{u \in V} \deg^2(u) \right) - 3\eta(G) \\
&\geq \frac{\left[\sum_{u \in V} \deg(u) \right]^2}{\sum_{u \in V} 1^2} - 3\eta(G) \\
&= \frac{4|E|^2}{|V|} - 3\eta(G).
\end{aligned}$$

by Cauchy inequality. Note that the above inequality is equality iff the ratio $\mathrm{vedeg}(u) : 1$ is a constant for every $u \in V$. $\square$

An upper bound of $\sum_{u \in V} \mathrm{vedeg}(u)$

$$\sum_{u \in V} \mathrm{vedeg}(u) \leq |E| \cdot |V|$$

with equality iff the graph G has diameter ay most 2 and the induced subgraph $G_2(u) := \{v \mid \partial(u, v) = 2\}$ contains no edges for every $u \in V$.

Proof.

This follows from the observation that for every $u \in V$

$$\mathrm{vedeg}(u) \leq |E|$$

with equality iff no vertex at distance more than 2 (including ∞) to u and $G_2(u) := \{v \mid \partial(u, v) = 2\}$ contains no edges for every $u \in V$.

$\square$

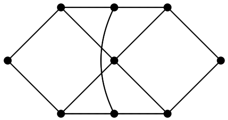
k-ve-harmonic graph

A graph G is k-ve-harmonic graph if $\mathrm{vedeg}(u) = k \cdot \deg(u)$ for any $u \in V$.

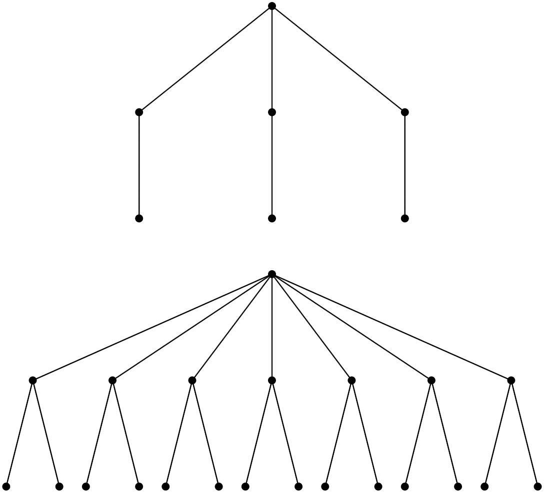
Complete graph is $\frac{n}{2}$-ve-harmonic

The complete graph K_n has $\mathrm{vedeg}(u) = \binom{n}{2}$ and $\deg(u) = n - 1$.
Hence K_n is $\frac{n}{2}$-ve-harmonic.

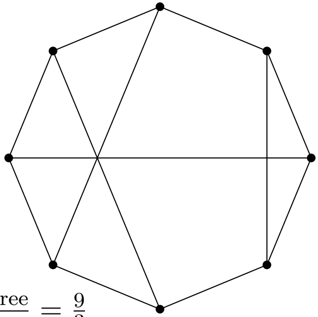
A 3-ve-harmonic graph



2- and 3-ve-harmonic trees



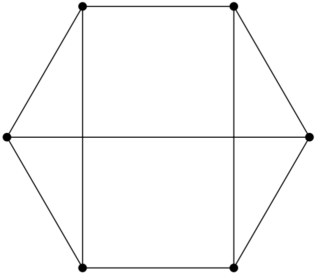
A 3-regular, but not ve-harmonic graph



$\frac{\text{vegdegree}}{3} = \frac{8}{3}$

$\frac{\text{vegdegree}}{3} = \frac{9}{3}$

A 3-regular and $\frac{8}{3}$-ve-harmonic graph



$$\frac{\text{vegdegree}}{3} = \frac{8}{3}$$

The average 2-degree

For a vertex u of G, the value

$$m(u) := \frac{1}{\deg(u)} \sum_{v:vu \in E} \deg(v)$$

is called the average 2-degree of u.

$$\begin{pmatrix} m(1) \\ \vdots \\ m(n) \end{pmatrix} = \begin{pmatrix} \deg(1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \deg(n)^{-1} \end{pmatrix} A \begin{pmatrix} \deg(1) & & 0 \\ & \ddots & \\ 0 & & \deg(n) \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Relation between $m(u)$ and $\mathrm{vedegree}(u)$

$$\eta_u = 0 \qquad \Rightarrow \qquad \mathrm{vedegree}(u) = \deg(u) \cdot m(u).$$

Proof.

$$\begin{aligned}
\mathrm{vedeg}(u) &= \sum_{v \in N(u)} \deg(v) - \eta_u \\
&= \sum_{v \in N(u)} \deg(v) \\
&= \deg(u) \cdot m(u).
\end{aligned}$$

$\square$

k-harmonic graph

A graph is called k-harmonic if $m(u) = k$ for every $u \in V$.

Integrality

If G is k-harmonic then $k \in \mathbb{N}$.

Proof.

Write $V = \{1, 2, \ldots, n\}$ and $d_i = \deg(i)$ and A the adjacency matrix of G. Then

$$k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} m(1) \\ \vdots \\ m(n) \end{pmatrix}$$

$$= \begin{pmatrix} \deg(1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \deg(n)^{-1} \end{pmatrix} A \begin{pmatrix} \deg(1) & & 0 \\ & \ddots & \\ 0 & & \deg(n) \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

implying that k is an eigenvalue of A. Since k is a rational number and also is a root of a monic integral polynomial, the characteristic polynomial of A, k is an integer. $\square$

Integrality for triangle-free ve-harmonic graphs

If G is k-ve-harmonic and $\eta(G) = 0$ then $k \in \mathbb{N}$.

Conjecture

$$\sum_{i=1}^{k} \operatorname{vedeg}(i) \leq \frac{k^2(k-1)}{2} + \sum_{i=k+1}^{n} \min\left\{\operatorname{vedeg}(i), \frac{(k+1)^2 k}{2}\right\}.$$

$\square$

The conjecture is wrong for $k = 1$ since

$$\operatorname{vedeg}(1) \not\leq 0 + (n-1) \cdot 2$$

for a complete complete graph.

Unknown property

Proof.

$$
\begin{aligned}
& \sum_{u \in S} \operatorname{vedeg}(u) \\
= \;& \sum_{u \in S} \left[\sum_{v \in N(u)} \deg(v) - \eta_u \right] \\
= \;& \sum_{u \in V} \deg(u)|N(u) \cap S| - \sum_{u \in S} \eta_u \\
= \;& \sum_{u \in S} [\deg(u)|N(u) \cap S| - \eta_u] + \sum_{u \notin S} \deg(u)|N(u) \cap S| \\
\leq \;& \begin{cases} \sum_{u \in S} \deg(u)(|S|-1) + \sum_{u \notin S} \deg(u) \min(\deg(u), |S|), \\ \ldots\ldots \end{cases} . \\
\leq \;& \begin{cases} 2(|S|-1)|E| + \sum_{u \notin S} \deg(u), & \text{(useless)} \\ \ldots\ldots \end{cases}
\end{aligned}
$$

Reference

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