A simple method on comparison between spectral radii of two directed graphs

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2021.01.25

References

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Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers, respectively. Let C be an $n \times n$ real square matrix. If there is a nonzero column vector $u \in \mathbb{C}^n$ such that $Cu = \lambda u$ for some scalar $\lambda \in \mathbb{C}$, then the scalar λ is called the eigenvalue of C with corresponding eigenvector u. And the spectral radius of a matrix C is the largest magnitude (or complex modulus) of its eigenvalues, denoted by $\rho(C)$. We are interested in the spectral radius of the following matrix associated with a simple directed graph.

Definition

Given a directed graph G, the *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G, and

$$a_{ij} = \begin{cases} 1, \text{if } ji \text{ is an arc in } G, \\ 0, \text{otherwise.} \end{cases}$$

Given a directed graph G, the spectral radius of G is the spectral radius of the adjacency matrix of G, denoted by $\rho(G)$. Note that the spectral radius $\rho(G)$ is independent of the ordering of the vertex set of G.

In [1], Cheng and Weng give many bounds of the spectral radius of a nonnegative square matrix. And based on their theory and Perron-Frobenius theorem, we give another approach to obtain an upper bound of the spectral radius and apply it on the adjacency matrix of a directed graph.
All theorems come from continuous discussions between C.W. Weng and K.H. Chen. These were all documented.[5]

Preliminaries

The following is Perron–Frobenius theorem, which provides a feature of nonnegative eigenvectors to nonnegative matrices.

Theorem

[3] If C is a nonnegative square matrix, then the spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector. Moreover if C is irreducible the above eigenvectors can be chosen to be positive.

A well-known application of Theorem 6 show that if matrix C' *majors* C (in notation $C \leq C'$), i.e. $c_{ij} \leq c'_{ij}$ for all ij, j, then $\rho(C) \leq \rho(C')$. Our main result shows that the assumption $C \leq C'$ can be a little loosen. Our theory is based on the following theorem, which is from [1].

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ *be* $n \times n$ *real matrices with real eigenvalues* λ, λ' *respectively such that there exist* $n \times n$ *matrices* P *and* Q *satisfying the following (i)-(iv).*

(i) $PCQ \leq PC'Q$;

(ii) *an eigenvector* Qu *of* C' *associated with eigenvalue* λ' *exists for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T$.

(iii) *a left eigenvector* $v^T P$ *of* C' *associated with eigenvalue* λ *exists for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n)$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{1}$$

Multiplying the nonnegative vector u in assumption i to the right of both terms of i,

$$PCQu \leq PC'Qu = \lambda' PQu, \tag{2}$$

where the above equality follows by Qu being eigenvector of C' for λ'.

Multiplying the nonnegative vector v^T of C in assumption iii to the left of all terms in (2), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC' Qu = \lambda' v^T PQu, \tag{3}$$

where the above first equality follows by $v^T P$ being left eigenvector of C for λ. Now delete the positive term $v^T PQu$ by assumption iv to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Assume that $\lambda = \lambda'$, so the inequality in (3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1) implies

$$\lambda v^T PCQu = v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j$$

$$= \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu = \lambda' v^T PC'Qu,$$

so $\lambda = \lambda'$ by (3).

Our Method

We will apply Theorem 3 by using two particular square matrices P and Q to obtain our main result. We use $[n-1]$ as notation of the set of integers from 1 to $n-1$, which is $\{1, 2, \cdots, n-1\}$. Throughout this thesis we fix $k \in [n-1]$, let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn}.

Now we apply the previous Theorem 3 with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & 1 \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix}, \tag{4}$$

so the matrix $PC'Q$ in assumption i of Theorem 3 is

$$PC'Q = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1k} + c'_{1n} \\ c'_{21} & c'_{22} & \cdots & c'_{2k} + c'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{nk} + c'_{nn} \end{pmatrix}, \tag{5}$$

where c'_{ij} denotes the (i, j)-entry of C'.

Definition

A column vector $v' = (v'_1, v'_2, \dots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

Lemma

If $u = (u_1, u_2, \dots, u_n)^T$, then

(i) *Qu is k-rooted if and only if u is nonnegative;*

(ii) *$u_k > 0$ if and only if $(Qu)_k > (Qu)_n$.*

Proof.

(i), (ii) follow immediate from the definition of k-rooted and $Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$. □

Below is our first result, in which the first condition implies the first $n-1$ columns of C' major the same columns of C, and the sum of k-th and n-th columns of C' also majors that of C. The second and the third condition suggest that C and C' have nonnegative k-rooted eigenvectors. And the forth condition is simpler but with the same meaning in Theorem 3

We need a notation of submatrix, which is taken from some columns and some rows of a matrix.

Definition

For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C, respectively, we use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$. We use the notation $C[i|j]$ for short of $C[[i]|[j]]$.

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ real matrices with real eigenvalues λ and λ' respectively. Assume that

(i) $C[n|n-1] \leq C'[n|n-1]$ *and* $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ *for all* $1 \leq i \leq n$*;*

(ii) *there exists a k-rooted eigenvector vector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *of* C' *for* λ'*;*

(iii) *there exists a nonnegative eigenvector vector* $v^T = (v_1, v_2, \ldots, v_n)$ *of* C *for* λ*;*

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$*. Moreover,* $\lambda = \lambda'$ *if and only if*

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ *for* $1 \leq i \leq n$ *with* $v_i > 0$ *and* $v'_n > 0$*;*

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1,\ j \neq k$ *with* $v_i \neq 0$ *and* $v'_j > 0$*;*

Proof.

The proof is based on Theorem 3 with $P = I$ and $Q = I + E_{kn}$ in (4). The assumption i $PCQ \le PC'Q$ of Theorem 3 holds by the condition i of this theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda' Qu$ by the condition ii and i in Lemma 5. Hence the assumption ii of Theorem 3 holds. The assumptions iii and iv of Theorem 3 clearly hold by conditions iii, iv of this theorem since $P = I$ and $v' = Qu$. Hence $\lambda \le \lambda'$ by the necessary condition of Theorem 3. Moreover, $\lambda = \lambda'$ if and only if (1) holds, and this is equivalent to conditions a, b and c of this theorem. □

We are interested in the matrices C' that have k-rooted eigenvectors. Motivated by the condition (i) of Theorem 2.3, we provide the following two definitions. The first is the definition of (k, n)-sum.

Definition

For an $n \times n$ matrix $C' = (c'_{ij})$, the (k, n)-*sum vector* of C' is the vector obtained from the sum of the k-th and n-th columns of C'.

Note that the last column of $C'Q$ is the (k, n)-sum vector of C'. Below is the definition of k-rooted matrix.

Definition

A matrix $C' = (c'_{ij})$ is called k-*rooted* if for each $i \neq k, n$ the i-th column of C' is k-rooted and the (k, n)-sum vector of C' is k-rooted.

We need a simple lemma for later use.

Lemma

$Q^{-1} = I - E_{kn}$.

Proof.

Since

$$Q(I - E_{kn} = (I + E_{kn})(I - E_{kn}) = I - E_{kn} + E_{kn} - E_{kn}E_{kn} = I,$$

we have $Q^{-1} = I - E_{kn}$. □

The matrix Q^{-1} is explicitly written as

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & -1 \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix},$$

and if $C' = (c'_{ij})$ then $Q^{-1}C'Q$ has the form

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c'_{(k-1)1} & c'_{(k-1)2} & \cdots & c'_{(k-1)(n-1)} & c'_{(k-1)k} + c'_{(k-1)n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{k(n-1)} - c'_{n(k-1)} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{(k+1)1} & c'_{(k+1)2} & \cdots & c'_{(k+1)(n-1)} & c'_{(k+1)k} + c'_{(k+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n(n-1)} & c'_{nk} + c'_{nn} \end{pmatrix}. \tag{6}$$

The following lemma shows that a k-rooted matrix has a k-rooted eigenvector.

Lemma

Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(ii) hold.

(i) *C' is a k-rooted matrix if and only if $Q^{-1}(C' + dI)Q$ is nonnegative for some $d \geq 0$, where I is the $n \times n$ identity matrix.*

(ii) *If C' is k-rooted then there exists a k-rooted eigenvector v' of C' for $\rho(C')$.*

Proof.(i).

The matrix $Q^{-1}C'Q$ has ij entry

$$(Q^{-1}C'Q)_{ij} = \begin{cases} c'_{ij}, & \text{if } i \neq k \text{ and } j \neq n; \\ c'_{kj} - c'_{nj}, & \text{if } i = k \text{ and } j \neq n; \\ c'_{ik} + c'_{in}, & \text{if } i \neq k \text{ and } j = n; \\ c'_{kk} + c'_{kn} - c'_{nk} - c'_{nn}, & \text{if } i = k \text{ and } j = n, \end{cases}$$

as shown in (6). Hence $Q^{-1}(C' + dI)Q$ is nonnegative if and only if C' is k-rooted by the definition of nonnegative matrix and k-rooted matrix.

□

Proof.(ii).

Suppose C' is k-rooted. Choose $d \geq 0$ such that $Q^{-1}(C' + dI)Q$ is nonnegative. Let u be a nonnegative eigenvector of $Q^{-1}(C' + dI)Q = Q^{-1}C'Q + I$ for $\rho(C' + dI) = \rho(C') + d$. Note that $Q^{-1}C'Qu = \rho(C')u$, and Qu is k-rooted by Lemma 5. Hence $v' = Qu$ is what we want. $\square$

Note that Theorem 7 depends on eigenvectors. The following is an eigenvector-free Theorem.

Theorem

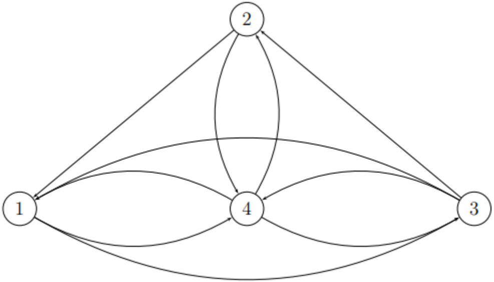
Let C be an $n \times n$ nonnegative irreducible matrix and C' be an $n \times n$ k-rooted matrix such that $C'Q$ majorizes CQ. Then $\rho(C) \leq \rho(C')$.

Proof.

Referring to (5), the assumption i in Theorem 7 holds. By Lemma 11 ii, there exists a k-rooted eigenvector v' of C' for $\rho(C')$. Since C is irreducible and nonnegative, there exists a positive eigenvector v^T of C for $\rho(C)$. Thus $v^T v' > 0$. Hence Theorem 7 i-iv hold. Hence $\rho(C) \leq \rho(C')$ by Theorem 7. $\square$

Examples

Let G be the digraph depicted below.



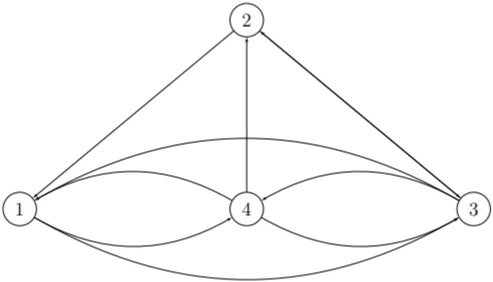
The following 4×4 matrix is the adjacency matrix of G.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We choose another matrix

$$A' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

which is the adjacency matrix of G' depicted below.


2
1
4
3

Note that both G and G' have the same number of edges, neither A majories A' nor A' majories A. Our main result still can do comparison between $\rho(A)$ and $\rho(A')$. We first observe that A is irreducible and A' is k-rooted for $k = 3$. Moreover

$$A'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = AQ.$$ Hence Hence

$\rho(A) \leq \rho(A')$ by Theorem 12 with $C = A$ and $C' = A'$. Both of the values of $\rho(C)$ and $\rho(C')$ are close to 2.511547 by calculating on computer. [4, sage]

To finish the thesis, we provide an example to show that the 'k-rooted' assumption of C' is necessary in Theorem 12. Consider the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

With $n = 4$ and $k = 3$, we have

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = C'Q.$$

Using computer, [4, sage]

$$\rho(C) \approx 2.234 \not\leq 2.148 \approx \rho(C').$$

This is because C' is not k-rooted as $c'_{33} + c'_{34} = 0 \not> 1 = c'_{43} + c'_{44}$.