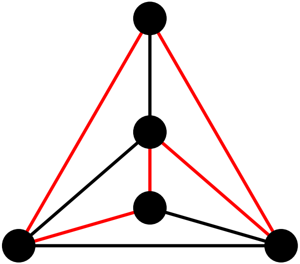
Hamiltonian graphs

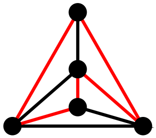
A graph is **Hamiltonian** if it contains a spanning cycle.



The graph K_5^-.

Definitions

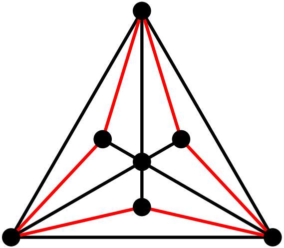
1. A **triangulated graph** is a planar graph such that any edge is in a cycle and any face is a triangle.
2. A **chordal graph** is a graph such that every cycle of length at least 4 contains a **chord**, i.e., an edge connecting two nonadjacent vertices of the cycle.



Remark 0.1

1. A triangulated of order at least 4 is not regular and contains a vertex of degree 3.
2. A triangulated of order at least 4 contains $3 + 3(n-3) = 3n - 6$ edges and $2n - 4$ faces.

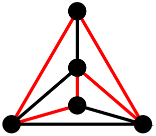
A triangulated graph which is not a chordal graph



There is no chord in the red cycle.

More definitions

1. A **weakly pancyclic graph** is a graph which contains cycles of every length between the girth and the circumference.
2. A graph is ***t*-tough** if for every integer $k > 1$, the graph cannot be split into k components by removal of fewer than tk vertices
3. The **toughness** a non-complete graph G is the maximum t such that G is t-tough. The toughness of a complete graph is ∞.



Remark 0.2

A t-tough graph with $t > 0$ is connected and a Hamiltonian graph is 1-tough.

A triangulated graph which is not weakly pancyclic

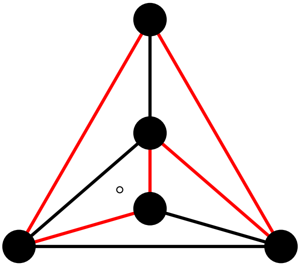
A long standing conjecture

There is a positive constant t such that every t-tough graph is Hamiltonian.

Problem

Can every edge of a triangulated graph be drawn as an edge of an outer face.

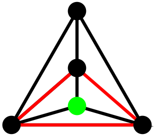
Answer: Yes



Plane is homeomorphic to the surface obtained from a sphere deleting a point.

Locally property P

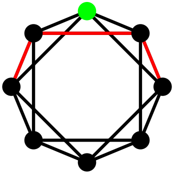
Let P be a property on graphs. A graph G has **local** P if $G_1(v)$ has property P for every vertex v in G.



Remark 0.3

A triangulated graph of order at least 4 is locally Hamiltonian.

A 1-tough and locally connected planar graph



The planar graph $DL(8;1,2)$ is 1-tough and locally connected.

Remark 0.4

$DL(n;1,2)$ is a planar graph, where $n \geq 4$ and n is even.

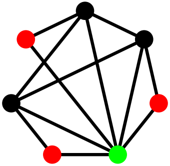
Remark 0.5

Every planar graph can be drawn by using straight lines as edges.

Question

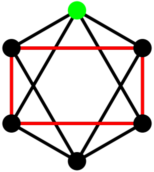
Is every 1-tough and locally connected graph Hamiltonian?

A 1-tough, locally connected, non-Hamiltonian planar graph



The red vertices have degree 2.

A 1-tough and locally Hamiltonian planar graph



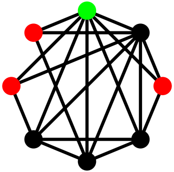
The planar graph $DL(6; 1, 2)$ is 1-tough and locally Hamiltonian, implying locally 1-tough. $DL(6; 1, 2)$ is not a triangulated graph.

Question

Is every 1-tough and locally 1-tough graph locally Hamiltonian?

The answer is ‘no’ in the following page.

A 1-tough and locally 1-tough graph which is not locally Hamiltonian



Deleting the green vertex of the above graph yields a 1-tough non-Hamiltonian graph (The red vertices have degree 2).

The above graph is not planar.

A conjecture

A 1-tough and locally 1-tough graph is Hamiltonian.

Will triangulated graphs provide many counter-examples?

Plan

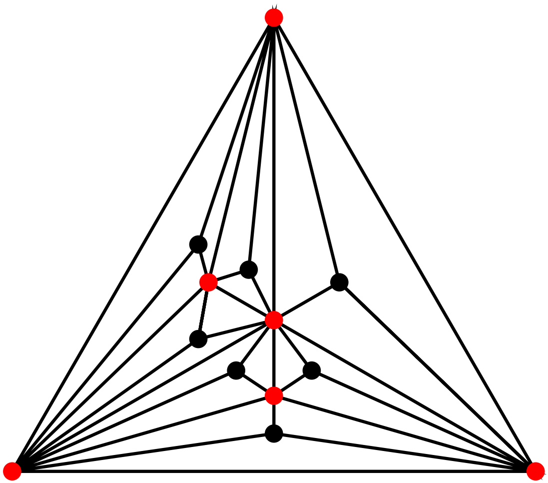
We shall develop theory of 1-tough and locally 1-tough graph.

Question

Is a triangulated graph 1-tough?

The answer is ‘no’ as shown in the next page.

A triangulated graph which is not 1-tough



Deleting the 6 red vertices will yield 7 components.

Lemma

If G is 1-tough and locally 1-tough then $G-u$ is 1-tough for every vertex u in G.

Proof.

Assume that $G-u$ is not 1-tough. Then $G-u$ is split into $k \geq 2$ components by removal of some subset S of vertices with $|S| < k$, where $u \notin S$. Assume ℓ of these k components intersecting with $G_1(u)$. If $\ell = 1$ then by connecting u to this unique component intersecting with $G_1(u)$, there are k components after removal of S from G, a contradiction to the 1-tough property of G.

Assume $\ell \geq 2$. Then $|S \cap G_1(u)| \geq \ell$ since $G_1(u)$ is 1-tough. Similarly by connecting u to these ℓ components intersecting with $G_1(u)$ to form a component, we find $k-\ell+1$ components after the removal of $S - G_1(u)$ from G. Hence $|S - G_1(u)| \geq k-\ell+1$ since G is 1-tough. Then $S = |S \cap G_1(u)| + |S - G_1(u)| \geq \ell + (k-\ell+1) = k+1$, a contradiction. $\square$

Lemma

If G is 1-tough and locally 1-tough then G has toughness greater than 1.

Proof.

Assume that G has toughness 1 and pick a subset S of k vertices whose removal will yield k components. Pick u in S. Then the removal of $k-1$ vertices in $S-\{u\}$ from $G-u$ will yield k components, a contradiction to $G-u$ being 1-tough by previous lemma. $\square$

Toughness of a 1-tough and locally 1-tough graph

By connecting a new vertex to the complete bipartite graph $K_{t,t}$, we have a 1-tough and locally 1-tough graph with toughness $(t+1)/t$ for evert $t \geq 1$. Hence the toughness of a 1-tough and locally 1-tough graph can approximate to 1.

Lemma

If G is 1-tough and locally 1-tough of order at least 4 then G has minimum degree at least 3.

Proof.

If G has minimum degree at most 2 then G has toughness at most $1 = 2/2$ because deleting the neighbors of a vertex with minimum degree yields at least two components. $\square$

Lemma

If G is 1-tough and locally 1-tough of order at least 4 then every edge of G is in at least two triangles.

Proof.

Let $e = uv$ be an edge of G. Since u has degree at least 3, $G_1(u)$ is at least 3, so with the 1-tough property of $G_1(u)$, u has at least two neighbors in $G_1(u)$, forming two triangles containing the edge e. $\square$

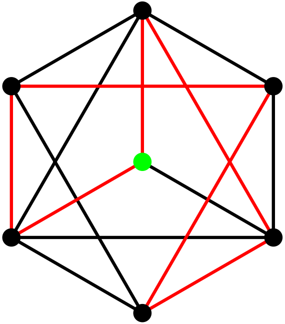
Proposition

If G is a 1-tough and locally 1-tough planar graph of order at least 4 and C is a cycle of maximum length in G, then each edge in C is in two triangles of the subgraph induced on the vertex set of C.

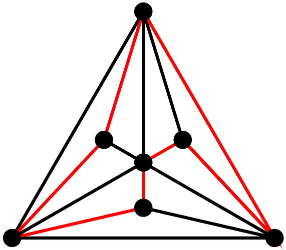
Proof.

Because every edge in a cycle of a planar graph is in exactly two faces, which are triangles by previous lemma. If the two triangles contain a vertex not in C then we have a cycle of length one more than that of C, a contradiction to the maximum length assumption of C. $\square$

1-tough and locally Hamiltonian planar graph of order 7



The graph $DL(6;1,2)^+$.



A triangulated graph of order 7 (There are many different ones including $DL(6;1,2)^+$).

Problems

1. For $n \leq 7$, find non-triangulated planar Hamiltonian graphs of order n such that each edge in its Hamiltonian cycle is in two triangles. ($DL(6; 1, 2)$ is one of the examples).
2. How many edges are possible for a planar Hamiltonian graph of order n such that each edge in its Hamiltonian cycle is in two triangles? (Note that $DL(n; 1, 2)$ has $2n$ edges and $DL(6; 1, 2)^+$ has 15 edges)
3. Determine all the non-triangulated planar Hamiltonian graphs of order n such that each edge in its Hamiltonian cycle is in two triangles.