

# Chapter 1

## Our result

We will prove the main Theorems introduced in the introduction. First, we need some notations. Throughout fix  $k \in [n-1] := \{1, 2, \dots, n-1\}$ . Let  $E_{kn}$  denote the  $n \times n$  binary matrix with a unique 1 appearing in the position  $(k, n)$ . We will apply the Theorem 1.7 with  $P = I$  and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \quad (1.1)$$

**Definition 1.1** ( $k$ -rooted vector). A column vector  $v' = (v'_1, v'_2, \dots, v'_n)^T$  is called  $k$ -rooted if  $v'_j \geq 0$  for  $1 \leq j \leq n$  and  $v'_k \geq v'_n$ .

The following Lemma is immediate from the above definition.

**Lemma 1.2.** If  $u = (u_1, u_2, \dots, u_n)^T$  and  $v' = (v'_1, v'_2, \dots, v'_n)^T := Qu = (u_1, \dots, u_{k-1}, u_k + u_n, u_{k+1}, \dots, u_n)^T$ , then

- (i)  $v'$  is  $k$ -rooted if and only if  $u$  is nonnegative;
- (ii)  $u_k > 0$  if and only if  $v'_k > v'_n$ .

□

**Theorem 1.3.** Let  $C = (c_{ij})$ ,  $C' = (c'_{ij})$  be  $n \times n$  matrices. Assume that

- (i)  $C[[n]][[n-1]] \leq C'[[n]][[n-1]]$  and  $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$  for all  $1 \leq i \leq n$ ;
- (ii) there exists a  $k$ -rooted vector  $v' = (v'_1, v'_2, \dots, v'_n)^T$  and a scalar  $\lambda' \in \mathbb{R}$  such that  $\lambda'$  is an eigenvalue of  $C'$  with associated eigenvector  $v'$ ;
- (iii) there exists a nonnegative vector  $v^T = (v_1, v_2, \dots, v_n)$  and a scalar  $\lambda \in \mathbb{R}$  such that  $\lambda$  is an eigenvalue of  $C$  with associated left eigenvector  $v^T$ ;
- (iv)  $v^T v' > 0$ .

Then  $\lambda \leq \lambda'$ . Moreover,  $\lambda = \lambda'$  if and only if

- (a)  $c_{ik} + c_{in} = c'_{ik} + c'_{in}$  for  $1 \leq i \leq n$  with  $v_i \neq 0$  and  $v'_n \neq 0$ ;
- (b)  $c'_{ij} = c_{ij}$  for  $1 \leq i \leq n$ ,  $1 \leq j \leq n-1$ ,  $j \neq k$  with  $v_i \neq 0$ ;
- (c)  $c'_{ik} = c_{ik}$  for  $1 \leq i \leq n$  and  $v'_k > v'_n$

□

*Proof.* The proof is based on Theorem 1.7 with  $P = I$  and  $Q = I + E_{kn}$  in (??). The assumption (i)  $PCQ \leq PC'Q$  of Theorem 1.7 holds by the condition (i) of this theorem. Let  $u = Q^{-1}v'$ . Then  $u$  is nonnegative and  $C'Qu = \lambda'Qu$  by the condition (ii) and Lemma 2.2(i). Hence the assumption (ii) of Theorem 1.7 holds. The assumptions (iii) and (iv) of Theorem 1.7 clearly hold by conditions (iii),(iv) of this theorem since  $P = I$  and  $v' = Qu$ . Hence  $\lambda \leq \lambda'$  by the conclusion of Theorem 1.7. Moreover  $\lambda = \lambda'$  if and only if ?? holds, and this is equivalent to conditions (a),(b),(c) of this theorem. □

We are interested in the matrices  $C'$  that have  $k$ -rooted eigenvectors. Motivated by the condition (i) of theorem 2.3, we provide the following two definitions.

**Definition 1.4** (( $k,n$ )-sum). For an  $n \times n$  matrix  $C' = (c'_{ij})$ , the  $(k, n)$ -sum vector of  $C'$  is the vector of the sum of the  $k$ -th and  $n$ -th columns of  $C'$ .

Note that the last column of  $C'Q$  is the  $(k, n)$ -sum vector of  $C'$

**Definition 1.5.** A matrix  $C' = (c'_{ij})$  is called  $k$ -rooted if its columns and its  $(k, n)$ -sum vector are all  $k$ -rooted except the last column of  $C'$ .

The following theorem shows that a  $k$ -rooted matrix has a  $k$ -rooted eigenvector.

**Theorem 1.6.** Let  $C' = (c'_{ij})$  be an  $n \times n$  nonnegative matrix. Then the following (i)-(iii) hold.

- (i)  $C'$  is a  $k$ -rooted matrix, if and only if,  $Q^{-1}C'Q$  is nonnegative.
- (ii) Assume that  $C'$  is  $k$ -rooted and let  $u$  be a nonnegative eigenvector of  $Q^{-1}C'Q$  for  $\rho(C')$ . Then  $C'$  has a  $k$ -rooted eigenvector  $v' = Qu$  for  $\rho(C')$ .
- (iii)  $\rho(C') = \rho(Q^{-1}C'Q)$

*Proof.* (i) is immediate from Definition ?? and the observation that

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & -1 \\ & & & 1 \\ 0 & & & & 1 \end{pmatrix},$$

and  $Q^{-1}C'Q$  is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1\ n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{kn-1} - c'_{nn-1} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

(ii) By Lemma ??  $v' = Qu$  is  $k$ -rooted. Since  $Q^{-1}C'Qu = \rho(C')u$  by the assumption, we have  $Q^{-1}C'Qu = Q^{-1}\rho(C')Qu = \rho(C')u$   
 $C'Qu = \rho(C')Qu.$

(iii) Since  $C'$  and  $Q^{-1}C'Q$  have the same set of eigenvalues, clearly  $\rho(C') = \rho(Q^{-1}C'Q)$ .

□

**Lemma 1.7.** *If a square matrix  $C'$  has a rooted eigenvector for  $\lambda'$ , then  $C' + dI$  also has the same rooted eigenvector for  $\lambda' + d$ , where  $d$  is a constant and  $I$  is the identity matrix with the same size of  $C'$ .*

**Theorem 1.8.** *Let  $C$  be an  $n \times n$  nonnegative Tmatrix. For  $1 \leq i \leq n$  and  $1 \leq j \leq n - 1$ , choose  $c'_{ij}$  such that  $c'_{ij} \geq c_{ij}$  and  $c'_{kj} \geq c'_{nj} > 0$ , and choose  $r'_i$  such that  $r'_i \geq c_{ik} + c_{in}$ , and  $r'_k \geq r'_n$ . Moreover choose  $c'_{in} := r'_i - c'_{ik}$ . Then  $\rho(C) \leq \rho(C')$ , when  $C' = (c'_{ij})$ .*

*Proof.* The assumptions are necessary that  $PCQ \leq PC'Q$ , and  $C'$  is  $k$ -rooted, based on (??); by ??, For certain  $d$ , if  $C' + dI$  is  $k$ -rooted, then it has a  $k$ -rooted eigenvector with its spectral radius  $\lambda + d$ .  $C'$  would share the same eigenvector with  $C' + dI$  and has eigenvalue  $\lambda$ . So  $C' + dI$  and  $C + dI$  meet the conditions of ??, and we can show that  $\rho(C' + dI) \geq \rho(C + dI)$  and then  $\rho(C') \geq \rho(C)$  □ □

**Example** For the following  $4 \times 4$  matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Check the conditions  $C[[n][n-1]] \leq C'[[n][n-1]]$  ??, Then  $\rho(C) \leq \rho(C')$  by Theorem 4.8.

**Counterexample** For the following two  $4 \times 4$  matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

specify  $n=4$ ,  $k=3$  in  $Q = I + E_{kn} = I + E_{34}$

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

we have  $CQ \leq C'Q$ , but  $\rho(C) = 2.234 \not\leq 2.148 = \rho(C')$ . This is because  $c'_{33} + c'_{34} \not\geq c'_{43} + c'_{44}$ .