# TBA


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# Preliminaries

**Theorem**

*If $C$ is nonnegative square matrix, then the following holds.*
*The spectral radius $\rho(C)$ is an eigenvalue of $C$ with a corresponding nonnogative right eigenvector and a corresponding nonnegative left eigenvector.*

# Preliminaries

The following theorem is from [].

**Theorem**

*Let $C = (c_{ij})$, $C' = (c'_{ij})$, $P$ and $Q$ be $n \times n$ matrices. Assume that*

(i) $PCQ \leq PC'Q$;

(ii) *there exist a nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that $\lambda'$ is an eigenvalue of $C'$ with associated eigenvector $Qu$;*

(iii) *there exist a nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that $\lambda$ is an eigenvalue of $C$ with associated left eigenvector $v^T P$; and*

(iv) $v^T PQu > 0$.

*Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

# Proof

Multiplying the nonnegative vector $u$ in (ii) to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{1}$$

Multiplying the nonnegative left eigenvector $v^T$ of $C$ for $\lambda$ in assumption (iii) to the left of all terms in (1), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{2}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

# Proof (Continue)

Assume that $\lambda = \lambda'$, so the inequality in (2) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any $i$ with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any $i$ with $v_i \neq 0$ and any $j$ with $u_j \neq 0$. Conversely, (0.2) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (2).

# Our Method

Throughout fix $k \in [n-1]$. Let $E_{kn}$ denote the $n \times n$ binary matrix with a unique $1$ appearing in the position $k, n$ of $E_{kn}$. We will apply the previous theorem with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & 1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

# $k$-rooted vector

**Definition**

A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

**Lemma**

*vector rooted lemma If* $u = (u_1, u_2, \ldots, u_n)^T$ *and* $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$, *then*

(i) $v'$ *is k-rooted if and only if* $u$ *is nonnegative;*

(ii) $u_k > 0$ *if and only if* $v'_k > v'_n$.

$\square$

# Our first result

Theorem

*Let* $C = (c_{ij})$, $C' = (c'_{ij})$ *be* $n \times n$ *matrices. Assume that*

(i) $C[-|n) \leq C'[-|n)$ *and* $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ *for all* $1 \leq i \leq n$;

(ii) *there exist a k-rooted vector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *and a scalar* $\lambda' \in \mathbb{R}$ *such that* $\lambda'$ *is an eigenvalue of* $C'$ *with associated eigenvector* $v'$;

(iii) *there exists a nonnegative vector* $v^T = (v_1, v_2, \ldots, v_n)$ *and a scalar* $\lambda \in \mathbb{R}$ *such that* $\lambda$ *is an eigenvalue of* $C$ *with associated left eigenvector* $v^T$;

(iv) $v^T v' > 0$.

*Then* $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ *for* $1 \leq i \leq n$ *with* $v_i \neq 0$ *and* $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1$ *with* $v_i \neq 0$ *and* $v'_j > v'_n$.

$\square$

# $k$-rooted matrix

**Definition ((k,n)-sum)**

For a matrix $C' = (c'_{ij})$ of $n$ columns, the $(k, n)$-*sum* vector of $C'$ is the vector of the sum of the $k$-th and $n$-th columns of $C'$, where $k \leq n - 1$.

**Definition (k-rooted matrix)**

A matrix $C' = (c'_{ij})$ is called *$k$-rooted* if its columns and its $(k, n)$-sum vector are all $k$-rooted except the last column of $C'$.

## Theorem

(i) *$C'$ is a k-rooted matrix, if and only if, $Q^{-1}C'Q$ is nonnegative.*

(ii) *$C'$ has a k-rooted eigenvector $v' = Qu$ for $\rho(C')$ as an eigenvalue. $u$ is a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$.*

(iii) $\rho(C') = \rho(Q^{-1}C'Q)$

# Remark

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

The matrix $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{kn-1} - c'_{nk-1} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

Proof.

For (i), the form of $Q^{-1}C'Q$ in last page shows that $Q^{-1}C'Q \geq 0$ if and only if C' is k-rooted. For (ii), C' is k-rooted and it has an k-rooted eigenvector Qu. And base on (i), $Q^{-1}C'Q$ is nonnegative implies that it has an eigenvector corresponding to spectral radius of $Q^{-1}C'Q$

(i) $C^{'} * Qu = \lambda \ * \ Qu = Q * \lambda u$ ....times $Q^{-1}$ to the left

(ii) $Q^{-1}C^{'}Q$* u $= \lambda \ * \ u$

where $\lambda$ is the spectral readius of C', so it is necessary that (iii) holds.

□

### Lemma

*If a square matrix $C'$ has a rooted eigenvector for $\lambda'$, then $C' + dI$ also has the same rooted eigenvector for $\lambda' + d$, where $d$ is a constant and $I$ is the identity matrix with the same size of $C'$.*

### Theorem

*Let $C$ be an $n \times n$ nonnegative matrix. For $1 \leq i \leq n$ and $1 \leq j \leq n-1$, choose $c'_{ij}$ such that $c'_{ij} \geq c_{ij}$, and then choose $r'_i$ such that $r'_i \geq c_{ik} + c_{in}$ and $c'_{kj} \geq c'_{nj} > 0$, and let $c'_{in} := r'_i - c'_{ik}$ with $r'_k \geq r'_n$. Whereas $c'_{kk} \geq c'_{nk}$ is not necessary. Then the $n \times n$ matrix $C' = (c'_{ij})$ has a positive $k$-rooted eigenvector for $\rho(C')$ and $\rho(C) \leq \rho(C')$.*

# Proof

The assumptions are necessary that $PCQ \leq PC'Q$, and C' is k-rooted, by 0.9, For certain d, if C'+d*I is k-rooted, then it has a k-rooted eigenvector with its spectral radius $\lambda + d$. C' would share the same eigenvector with C'+d*I and has eigenvalue $\lambda$. So C'+d*I and C+d*I meet the conditions of 0.5, and we can show that $\rho(C' + d * I) \geq \rho(C + d * I)$ and then $\rho(C') \geq \rho(C)$ □

## Example

For the following $4 \times 4$ matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then $\rho(C) \leq \rho(C')$ by previous theorem.

# Counterexample

For the following two $4 \times 4$ matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we have $CQ \leq C'Q$, but $\rho(C) = 2.234 \not\leq 2.148 = \rho(C')$. This is because $c'_{33} + c'_{34} \not\geq c'_{43} + c'_{44}$.