# A matrix realization of spectral bounds of the spectral radius of a nonnegative matrix


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## Abstract

We realize many sharp spectral bounds of the spectral radius of a nonnegative square matrix $C$ by using the largest real eigenvalues of suitable matrices of smaller sizes related to $C$ that are very easy to find. As applications, we give a sharp upper bound of the spectral radius of $C$ expressed by the sum of entries, the largest off-diagonal entry $f$ and the largest diagonal entry $d$ in $C$. We also give a new class of sharp lower bounds of the spectral radius of $C$ expressed by the above $d$ and $f$, the least row-sum $r_n$ and the $t$-th largest row-sum $r_t$ in $C$ satisfying $0 < r_n-(n-t-1)f-d \leq r_t-(n-t)f$, where $n$ is the size of $C$.



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## 1. Introduction

For real matrices $C = (c_{ij})$, $C' = (c'_{ij})$ of the same size, $C'$ *majorizes* $C$, in the notation $C \leq C'$, if $c_{ij} \leq c'_{ij}$ for all $i,j$. When $C$ is a square matrix, the spectral radius $\rho(C)$ of $C$ is defined by

$$\rho(C) := \max\{ |\lambda| \mid \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number $\lambda$. This paper is motivated by the following theorem of Xing Duan and Bo Zhou in 2013 [4, Theorem 2.1].

**Theorem 1.1.** *Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$, $f := \max_{1\leq i\neq j\leq n} c_{ij}$ and $d := \max_{1\leq i\leq n} c_{ii}$. Then*

$$\rho(C) \leq \frac{r_\ell + d - f + \sqrt{(r_\ell - d + f)^2 + 4f\sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2} \tag{1.1}$$

*for $1 \leq \ell \leq n$. Moreover, if $C$ is irreducible, then the equality holds in (1.1) if and only if $r_1 = r_n$ or for $1 \leq t \leq \ell$ with $r_{t-1} \neq r_t = r_\ell$, we have $r_t = r_n$ and*

$$c_{ij} = \begin{cases} d, & \text{if } i = j \leq t-1; \\ f, & \text{if } i \neq j \text{ and } 1 \leq i \leq n,\ 1 \leq j \leq t-1. \end{cases}$$


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Theorem 1.1 generalizes the results in [1, 3, 6, 7, 10, 12, 13] and relates to the results in [9, 11, 12], while the upper bound of $\rho(C)$ expressed in (1.1) is somewhat complicate and deserves an intuitive realization.

The values on the right hand side of (1.1) is realized as the largest real eigenvalue $\rho_r(C')$ of the $n \times n$ matrix

$$C' = \left(\begin{array}{cccc|cccc c}
d & f & \cdots & f & f & f & \cdots & f & r_1 - d - (n-2)f \\
f & d & & f & f & f & \cdots & f & r_2 - d - (n-2)f \\
\vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
f & f & \cdots & d & f & f & \cdots & f & r_{\ell-1} - d - (n-2)f \\
\hline
f & f & \cdots & f & d & f & \cdots & f & r_\ell - d - (n-2)f \\
f & f & \cdots & f & f & d & & f & r_\ell - d - (n-2)f \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots & \vdots \\
f & f & \cdots & f & f & f & & d & r_\ell - d - (n-2)f \\
f & f & \cdots & f & f & f & \cdots & f & r_\ell - (n-1)f
\end{array}\right) \tag{1.2}$$

which has the following three properties:

(i) The row-sum vector $(r_1, r_2, \ldots, r_\ell, \ldots, r_\ell)^T$ of $C'$ majorizes the row-sum vector $(r_1, r_2, \ldots, r_n)^T$ of $C$,

(ii) $C'$ majorizes $C$ except the last column, and

(iii) $C'$ has a positive eigenvector $(v'_1, v'_2, \ldots, v'_n)^T$ for $\rho_r(C')$ with $v'_i \geq v'_n$ for $1 \leq i \leq n$.

Property (iii) will be checked by Lemma 5.4. Since the above matrix $C'$ is not necessary to be nonnegative, the spectra radius $\rho(C')$ of $C'$ is replaced by the largest real eigenvalue $\rho_r(C')$ in the property (iii). Our main result in Theorem 4.3 is in a more general form that will imply for any matrices $C'$ that satisfy the properties (i)-(iii) above, we have $\rho(C) \leq \rho_r(C')$. Moreover, when the matrix $C'$ and so the value $\rho_r(C')$ are fixed, the matrices $C$ with $\rho(C) = \rho_r(C')$ are completely determined. We apply Theorem 4.3 to find a sharp upper bound of $\rho(C)$ expressed by the sum of entries in $C$, the largest off-diagonal entry $f$ and the largest diagonal entry $d$ in Theorem 7.2.

Note that $\rho_r(C') = \rho_r(C'')$ for the largest real eigenvalues of $C'$ and $C''$ respectively, where $C'$ is as in (1.2) and

$$C'' = \left(\begin{array}{cccc|c}
d & f & \cdots & f & r_1 - d - (\ell-2)f \\
f & d & & f & r_2 - d - (\ell-2)f \\
\vdots & & \ddots & \vdots & \vdots \\
f & f & \cdots & d & r_{\ell-1} - d - (\ell-2)f \\
\hline
f & f & \cdots & f & r_\ell - (\ell-1)f
\end{array}\right) \tag{1.3}$$

is the equitable quotient matrix of $C'$ with respect to the partition $\{\{1\}, \{2\}, \ldots, \{\ell-1\}, \{\ell, \ell+1, \ldots, n\}\}$ of $\{1, 2, \ldots, n\}$. Moreover $\rho_r(C'') = \rho_r(C''')$, where

$$C''' = \begin{pmatrix} (\ell-2)f + d & f \\ \sum_{i=1}^{\ell-1} r_i - (\ell-1)((\ell-2)f + d) & r_\ell - (\ell-1)f \end{pmatrix}, \tag{1.4}$$

is the equitable quotient of the transpose $C''^T$ of $C''$ with respect to the partition $\{\{1, 2, \ldots, \ell-1\}, \{\ell\}\}$ of $\{1, 2, \ldots, \ell\}$. Motivated by these observations, Theorem 9.1 will provide an upper bound

$\rho_r(C'')$ of $\rho(C)$, where $C''$ is a matrix of size smaller than that of $C$ obtained by applying equitable quotient to suitable matrix $C'$ that satisfies properties (i)-(iii) described above.

Every our theorem of upper bounds of $\rho(C)$ has a dual version that deals with lower bounds. We provide a new class of sharp lower bounds of $\rho(C)$ in Theorem 10.1. Applying Theorem 10.1 to a binary matrix $C$, we improve the well known inequality $\rho(C) \geq r_n$ as stated in Corollary 10.2. We believe that many new spectral bounds of the spectral radius of a nonnegative matrix will be easily obtained by our matrix realization in this paper.

In addition to the above results, Lemma 3.1, Lemma 3.2 and Lemma 6.2 are of independent interest in matrix theory.

## 2. Preliminaries

Our study is based on the famous Perron-Frobenius Theorem, hence we shall review the necessary parts of the theorem in this section.

**Theorem 2.1** ([2, Theorem 2.2.1], [8, Corollary 8.1.29, Theorem 8.3.2])**.** *If $C$ is a nonnegative square matrix, then the following (i)-(iii) hold.*

*(i) The spectral radius $\rho(C)$ is an eigenvalue of $C$ with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector.*

*(ii) If there exists a column vector $v > 0$ and a nonnegative number $\lambda$ such that $Cv \leq \lambda v$, then $\rho(C) \leq \lambda$.*

*(iii) If there exists a column vector $v \geq 0$, $v \neq 0$ and a nonnegative number $\lambda$ such that $Cv \geq \lambda v$, then $\rho(C) \geq \lambda$.*

*Moreover, if in addition $C$ is irreducible, then the eigenvalue $\rho(C)$ in (i) has multiplicity 1 and its corresponding left eigenvector and right eigenvector can be chosen to be positive, and any nonnegative left or right eigenvector of $C$ is only corresponding to the eigenvalue $\rho(C)$.* $\square$

Without further mention, an eigenvector is always a right eigenvector. The following two lemmas are well-known consequences of Theorem 2.1. We shall provide their proofs since they motivate our proofs of results.

**Lemma 2.2** ([2, Theorem 2.2.1])**.** *If $0 \leq C \leq C'$ are square matrices, then $\rho(C) \leq \rho(C')$. Moreover, if $C'$ is irreducible, then $\rho(C') = \rho(C)$ if and only if $C' = C$.*

*Proof.* Let $v$ be a nonnegative eigenvector of $C$ for $\rho(C)$. From the assumption, $C'v \geq Cv = \rho(C)v$. By Theorem 2.1(iii) with $(C, \lambda) = (C', \rho(C))$, we have $\rho(C') \geq \rho(C)$. Clearly $C' = C$ implies $\rho(C') = \rho(C)$. If $\rho(C') = \rho(C)$ and $C'$ is irreducible, then $\rho(C)v'^T v = \rho(C')v'^T v = v'^T C'v \geq v'^T Cv = \rho(C)v'^T v$, where $v'^T$ is a positive left eigenvector of $C'$ for $\rho(C')$. Hence the above inequality is the equality $v'^T C'v = v'^T Cv$. As $v'^T$ positive, $C'v = Cv = \rho(C)v = \rho(C')v$, so $v$ is a positive eigenvector of $C'$ for $\rho(C')$ and $C' = C$. $\square$

The matrix $C'$ in Lemma 2.2 is a *matrix realization* of the upper bound $\rho(C')$ of $\rho(C)$. We shall provide other matrix realizations as stated in the title.

**Lemma 2.3** ([8, Theorem 8.1.22])**.** *If an $n\times n$ matrix $C = (c_{ij})$ is nonnegative with row-sum vector $(r_1, r_2, \dots, r_n)^T$, where $r_i = \sum_{1\le j\le n} c_{ij}$ and $r_1 \ge r_i \ge r_n$ for $1 \le i \le n$, then*

$$r_n \le \rho(C) \le r_1.$$

*Moreover, if $C$ is irreducible, then $\rho(C) = r_1$ (resp. $\rho(C) = r_n$) if and only if $C$ has constant row-sum.*

We provide a proof of the following generalized version of Lemma 2.3, which is due to M. N. Ellingham and Xiaoya Zha [5].

**Lemma 2.4** ([5])**.** *If an $n \times n$ matrix $C$ with row-sum vector $(r_1, r_2, \dots, r_n)^T$, where $r_1 \ge r_i \ge r_n$ for $1 \le i \le n$, has a nonnegative left eigenvector $v^T = (v_1, v_2, \dots, v_n)$ for $\theta$, then*

$$r_n \le \theta \le r_1.$$

*Moreover, $\theta = r_1$ (resp. $\theta = r_n$) if and only if $r_i = r_1$ (resp. $r_i = r_n$) for the indices $i$ with $v_i \ne 0$. In particular, if $v^T$ is positive, $\theta = r_1$ (resp. $\theta = r_n$) if and only if $C$ has constant row-sum.*

*Proof.* Without loss of generality, let $\sum_{i=1}^{n} v_i = 1$ and $u$ be the all-one column vector. Then

$$\theta = \theta v^T u = v^T C u = \sum_{i=1}^{n} v_i r_i.$$

So $\theta$ is a convex combination of those $r_i$ with indices $i$ satisfying $1 \le i \le n$ and $v_i > 0$, and the lemma follows. □

In the sequels, we shall call two statements that resemble each other by switching $\le$ and $\ge$ and corresponding variables, like $\theta \ge r_n$ and $\theta \le r_1$, as *dual statements*, and their proofs are called *dual proofs* if one proof is obtained from the other by simply switching one of $\le$ and $\ge$ to the other.

## 3. A generalization of Lemma 2.2

We generalize Lemma 2.2 in the sense of Lemma 2.4 that the matrices considered are not necessary to be nonnegative.

**Lemma 3.1.** *Let $C = (c_{ij})$, $C' = (c'_{ij})$, $P$ and $Q$ be $n \times n$ matrices. Assume that*

- *(i) $PCQ \le PC'Q$;*
- *(ii) $C'$ has an eigenvector $Qu$ for $\lambda'$ for some nonnegative column vector $u = (u_1, u_2, \dots, u_n)^T$ and $\lambda' \in \mathbb{R}$;*
- *(iii) $C$ has a left eigenvector $v^T P$ for $\lambda$ for some nonnegative row vector $v^T = (v_1, v_2, \dots, v_n)$ and $\lambda \in \mathbb{R}$; and*
- *(iv) $v^T PQu > 0$.*

*Then $\lambda \le \lambda'$. Moreover, $\lambda = \lambda'$ if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \le i, j \le n \text{ with } v_i \ne 0 \text{ and } u_j \ne 0. \tag{3.1}$$

*Proof.* Multiplying the nonnegative vector $u$ in (ii) to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{3.2}$$

Multiplying the nonnegative left eigenvector $v^T$ of $C$ for $\lambda$ in assumption (iii) to the left of all terms in (3.2), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{3.3}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Assume that $\lambda = \lambda'$, so the inequality in (3.3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any $i$ with $v_i \neq 0$. Hence $(PCQ)_{ij} = (PC'Q)_{ij}$ for any $i$ with $v_i \neq 0$ and any $j$ with $u_j \neq 0$.

Conversely, (3.1) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (3.3). □

If $C$ is nonnegative and $P = Q = I$, where $I$ is the $n \times n$ identity matrix, then Lemma 3.1 becomes Lemma 2.2 with an additional assumption $v^T u > 0$ which immediately holds if $C$ or $C'$ is irreducible by Theorem 2.1. The following is a dual version of lemma 3.1 and its proof is by dual proof.

**Lemma 3.2.** *Let $C = (c_{ij})$, $C' = (c'_{ij})$, $P$ and $Q$ be $n \times n$ matrices. Assume that*

*(i)* $PCQ \geq PC'Q$;

*(ii)* *$C'$ has an eigenvector $Qu$ for $\lambda'$ for some nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$ and $\lambda' \in \mathbb{R}$;*

*(iii)* *$C$ has a left eigenvector $v^T P$ for $\lambda$ for some nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$ and $\lambda \in \mathbb{R}$; and*

*(iv)* $v^T PQu > 0$.

*Then $\lambda \geq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{3.4}$$

□

## 4. The special case $P = I$ and a particular $Q$

We shall applying Lemma 3.1 and Lemma 3.2 by using $P = I$ and

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{4.1}$$

Hence for $n \times n$ matrix $C' = (c'_{ij})$, the matrix $PC'Q$ in Lemma 3.1(i) is

$$C'Q = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & r'_1 \\ c'_{21} & c'_{22} & \cdots & c'_{2\ n-1} & r'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n-1\ 1} & c'_{n-1\ 2} & \cdots & c'_{n-1\ n-1} & r'_{n-1} \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & r'_n \end{pmatrix}, \tag{4.2}$$

where $(r'_1, r'_2, \ldots, r'_n)^T$ is the row-sum column vector of $C'$.

**Definition 4.1.** A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *rooted* if $v'_j \geq v'_n \geq 0$ for $1 \leq j \leq n-1$.

The following Lemma is immediate from the above definition.

**Lemma 4.2.** *If $u = (u_1, u_2, \ldots, u_n)^T$ and $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1 + u_n, u_2 + u_n, \ldots, u_{n-1} + u_n, u_n)^T$, then*

*(i) $v'$ is rooted if and only if $u$ is nonnegative;*
*(ii) $u_j > 0$ if and only if $v'_j > v'_n$ for $1 \leq j \leq n-1$.*

□

The following matrix notation will be adopted in the paper. For a matrix $C = (c_{ij})$ and subsets $\alpha$, $\beta$ of row indices and column indices respectively, we use $C[\alpha|\beta]$ to denote the submatrix of $C$ with size $|\alpha| \times |\beta|$ that has entries $c_{ij}$ for $i \in \alpha$ and $j \in \beta$, $C[\alpha|\beta) := C[\alpha|\overline{\beta}]$, where $\overline{\beta}$ is the complement of $\beta$ in the set of column indices, and similarly, for the definitions of $C(\alpha|\beta]$ and $C(\alpha|\beta)$. For $\ell \in \mathbb{N}$, $[\ell] := \{1, 2, \ldots, \ell\}$, symbol $-$ is the complete set of indices, and we use $i$ to denote the singleton subset $\{i\}$ to reduce the double use of parentheses. For example of the $n \times n$ matrix $C$, $C[-|n) = C[[n]|[n-1]]$ is the $n \times (n-1)$ submatrix of $C$ obtained by deleting the last column of $C$. The following theorem is immediate from Lemma 3.1 by applying $P = I$, the $Q$ in (4.1), $v' = Qu$ and referring to (4.2) and Lemma 4.2.

**Theorem 4.3.** *Let $C = (c_{ij})$, $C' = (c_{ij})$ be $n \times n$ matrices. Assume that*

*(i) $C[-|n) \leq C'[-|n)$ and the row-sum vector $(r'_1, r'_2, \ldots, r'_n)^T$ of $C'$ majorizes the row-sum vector $(r_1, r_2, \ldots, r_n)^T$ of $C$;*
*(ii) $C'$ has a rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ for $\lambda'$ for some $\lambda' \in \mathbb{R}$;*
*(iii) $C$ has a nonnegative left eigenvector $v^T = (v_1, v_2, \ldots, v_n)$ for $\lambda \in \mathbb{R}$;*
*(iv) $v^T v' > 0$.*

*Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if*

*(a) $r_i = r'_i$ for $1 \leq i \leq n$ with $v_i \neq 0$ when $v'_n \neq 0$;*
*(b) $c'_{ij} = c_{ij}$ for $1 \leq i \leq n,\ 1 \leq j \leq n-1$ with $v_i \neq 0$ and $v'_j > v'_n$.*

□

Note that the cases (a)-(b) in Theorem 4.3 is from the line (3.1) in Theorem 3.1. The first part of assumption (i) in Theorem 4.3 says that last column is *irrelevant* in the comparison of $C$ and $C'$. The following theorem is a dual version of Theorem 4.3.

**Theorem 4.4.** *Let $C = (c_{ij})$, $C' = (c_{ij})$ be $n \times n$ matrices. Assume that*

*(i)* $C[-|n) \geq C'[-|n)$ *and the row-sum vector* $(r_1, r_2, \ldots, r_n)^T$ *of* $C$ *majorizes the row-sum vector* $(r'_1, r'_2, \ldots, r'_n)^T$ *of* $C'$*;*

*(ii)* $C'$ *has a rooted eigenvector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *for* $\lambda'$ *for some* $\lambda' \in \mathbb{R}$*;*

*(iii)* $C$ *has a nonnegative left eigenvector* $v^T = (v_1, v_2, \ldots, v_n)$ *for* $\lambda \in \mathbb{R}$*;*

*(iv)* $v^T v' > 0$.

*Then* $\lambda \geq \lambda'$*. Moreover,* $\lambda = \lambda'$ *if and only if*

*(a)* $r_i = r'_i$ *for* $1 \leq i \leq n$ *with* $v_i \neq 0$ *when* $v'_n \neq 0$*;*

*(b)* $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1$ *with* $v_i \neq 0$ *and* $v'_j > v'_n$*.*

□

**Example 4.5.** Consider the following three matrices

$$C'_\ell = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad C'_u = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

with $C'_\ell[-|3) \leq C[-|3) \leq C'_u[-|3)$, and the same row-sum vector $(5, 3, 3)^T$. Note that $C'_\ell$ has a rooted eigenvector $v'^\ell = (1, 0, 0)^T$ for $\lambda'^\ell = 3$ and $C'_u$ has a rooted eigenvector $v'^u = (2, 1, 1)^T$ for $\lambda'^r = 4$. Since $C$ is irreducible, it has a left positive eigenvector $(v_1, v_2, v_3) > 0$. Hence assumptions (i)-(iv) in Theorem 4.3 and Theorem 4.4 hold, and we conclude that $\lambda'^\ell \leq \rho(C) \leq \lambda'^r$. Since $[3] \times [1]$ is the set of the pairs $(i, j)$ described in Theorem 4.3(b) and Theorem 4.4(b), from simple comparison of the first columns $C'_\ell[-|\,1] < C[-|\,1] = C'_u[-|\,1]$ of these three matrices, we easily conclude that $3 = \lambda'^\ell < \rho(C) = \lambda'^r = 4$ by the second part of Theorem 4.3 and that of Theorem 4.4.

## 5. Matrices with a rooted eigenvector

Before giving applications of Theorem 4.3 and Theorem 4.4, we need to construct $C'$ which possesses a rooted eigenvector for some $\lambda'$. The following lemma comes immediately.

**Lemma 5.1.** *If a square matrix* $C'$ *has a rooted eigenvector for* $\lambda'$*, then* $C' + dI$ *also has the same rooted eigenvector for* $\lambda' + d$*, where* $d$ *is a constant and* $I$ *is the identity matrix with the same size of* $C'$*.* □

A rooted column vector defined in Definition 4.1 is generalized to a rooted matrix as follows.

**Definition 5.2.** A matrix $C' = (c'_{ij})$ is called *rooted* if its columns and its row-sum vector are all rooted except the last column of $C'$.

The vertex $Q$ in (4.1) is invertible with

$$Q^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Multiplying $Q^{-1}$ to $C'Q$ in (4.2), $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} - c'_{n1} & c'_{12} - c'_{n2} & \cdots & c'_{1\,n-1} - c'_{n\,n-1} & r'_1 - r'_n \\ c'_{21} - c'_{n1} & c'_{22} - c'_{n2} & \cdots & c'_{2\,n-1} - c'_{n\,n-1} & r'_2 - r'_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n-1\,1} - c'_{n1} & c'_{n-1\,2} - c'_{n2} & \cdots & c'_{n-1\,n-1} - c'_{n\,n-1} & r'_{n-1} - r'_n \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\,n-1} & r'_n \end{pmatrix}. \tag{5.1}$$

The matrices $C'$ and $Q^{-1}C'Q$ have the same set of eigenvalues. Moreover, $v'$ is an eigenvector of $C'$ for $\lambda'$ if and only if $u = Q^{-1}v'$ is an eigenvector of $Q^{-1}C'Q$ for $\lambda'$. From (5.1), $C'$ is rooted if and only if $Q^{-1}C'Q$ is nonnegative. The first part of the following lemma follows immediately from the above discussion and Theorem 2.1 by choosing $\lambda' = \rho(C')$.

**Lemma 5.3.** *If $C'$ is a rooted matrix, then $Q^{-1}C'Q$ is nonnegative, $\rho(C')$ is an eigenvalue of $C'$, and $C'$ has a rooted eigenvector $v' = Qu$ for $\rho(C')$, where $u$ is a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Moreover, with $v' = (v'_1, v'_2, \ldots, v'_n)^T$, the following (i)-(ii) hold.*

*(i) If $C'[n|n)$ is positive, then $v'$ is positive.*

*(ii) If $C'[n|n)$ is positive and $r'_i > r'_n$ for all $1 \le i \le n-1$, then $v'_j > v'_n$ for all $1 \le j \le n-1$.*

*Proof.* It remains to prove the second part.

(i) suppose that $C'[n|n)$ is positive and $v'_n = 0$. Then

$$\sum_{j=1}^{n-1} c'_{nj}v'_j = \sum_{j=1}^{n} c'_{nj}v'_j = (C'v')_n = \rho(C')v'_n = 0.$$

Hence $v'$ is a zero vector since $c'_{nj} > 0$ for $j \le n-1$, a contradiction. So $v'_n > 0$ and $v' > 0$ since $v'$ is rooted.

(ii) The assumptions imply that the matrix $Q^{-1}C'Q$ in (5.1) is irreducible. Hence $u$ is positive. By Lemma 4.2(ii), $v'_j > v'_n$ for $1 \le j < n$. □

The largest real eigenvalue of the following matrix will be used to obtain bounds of the spectral radius of a nonnegative matrix.

Fix $d, f, r_1, r_2, \ldots, r_n \ge 0$ such that $r_j \ge r_n$ for $1 \le j \le n-1$, and let

$$M_n(d, f, r_1, r_2, \ldots, r_n) = \begin{pmatrix} d & f & \cdots & f & r_1 - (d + (n-2)f) \\ f & d & & f & r_2 - (d + (n-2)f) \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{n-1} - (d + (n-2)f) \\ f & f & \cdots & f & r_n - (n-1)f \end{pmatrix} \tag{5.2}$$

be an $n \times n$ matrix with row-sum vector $(r_1, r_2, \dots, r_n)^T$.

Note that for any square matrix $C'$, it might be $\rho(C'+dI) \neq \rho(C')+d$, but $\rho_r(C'+dI) = \rho_r(C')+d$ always holds, where $\rho_r(C'+dI)$ and $\rho_r(C')$ are the largest real eigenvalues of $C'+dI$ and $C'$ respectively. Also $\rho(C') = \rho_r(C')$ if $C'$ is nonnegative.

**Lemma 5.4.** *The following (i)-(ii) hold.*

*(i) The matrix $M_n(d, f, r_1, r_2, \dots, r_n)$ has a rooted eigenvector $v' = (v'_1, v'_2, \dots, v'_n)^T$ for the largest real eigenvalue $\rho_r(M_n(d, f, r_1, r_2, \dots, r_n))$ of $M_n(d, f, r_1, r_2, \dots, r_n)$.*

*(ii) If $f > 0$, then $v' > 0$.*

*Proof.* Let $M_n := M_n(d, f, r_1, r_2, \dots, r_n)$. First assume $d \geq f$. Then $M_n$ is rooted. (i)-(ii) follows from (i)-(ii) of Lemma 5.3, in particular $\rho(M_n) = \rho_r(M_n)$. If $d < f$, then the matrix $(f-d)I+M_n$ is a rooted matrix. As in the first part, let $v'$ be a rooted eigenvector of $(f-d)I+M_n$ for $\rho((f-d)I+M_n)$. Note that $v'$ is also a rooted eigenvector of $M_n$ for $\rho_r(M_n) = \rho((f-d)I+M_n)-(f-d)$. This proves (i), and (ii) follows similarly from (ii) of Lemma 5.3. □

## 6. Equitable partition

Pattern in $C'$ will make it easier to compute its eigenvalue and to find the bound $\lambda'$ obtained in Theorem 4.3 and Theorem 4.4. For a partition $\Pi = \{\pi_1, \pi_2, \dots, \pi_\ell\}$ of $[n]$, the $\ell \times \ell$ matrix $\Pi(C') := (\pi'_{ab})$, where

$$\pi'_{ab} := \frac{1}{|\pi_a|} \sum_{i \in \pi_a, j \in \pi_b} c'_{ij},$$

is called the *quotient matrix* of $C'$ with respect to $\Pi$. In matrix notation,

$$\Pi(C') = (S^T S)^{-1} S^T C' S, \tag{6.1}$$

where $S = (s_{jb})$ is the $n \times \ell$ *characteristic matrix* of $\Pi$, i.e.,

$$s_{jb} = \begin{cases} 1, & \text{if } j \in \pi_b; \\ 0, & \text{otherwise.} \end{cases}$$

for $1 \leq j \leq n$, and $1 \leq b \leq \ell$. If

$$\pi'_{ab} = \sum_{j \in \pi_b} c'_{ij} \qquad (1 \leq a, b \leq \ell)$$

for all $i \in \pi_a$, then $\Pi(C') = (\pi'_{ab})$ is called the *equitable quotient matrix* of $C'$ with respect to $\Pi$. Note that $\Pi(C')$ is an equitable quotient matrix if and only if

$$S\Pi(C') = C'S. \tag{6.2}$$

**Lemma 6.1** ([2, Lemma 2.3.1])**.** *If an $n \times n$ matrix $C'$ has an equitable quotient matrix $\Pi(C')$ with respect to partition $\Pi = \{\pi_1, \pi_2, \dots, \pi_\ell\}$ of $[n]$ with characteristic matrix $S$, and $\lambda'$ is an eigenvalue of $\Pi(C')$ with eigenvector $u'$, then $\lambda'$ is an eigenvalue of $C'$ with eigenvector $Su'$. Moreover, if $u'$ is rooted and $n \in \pi_\ell$, then $Su'$ is rooted.*

*Proof.* From (6.2), $C'Su' = S\Pi(C')u' = \lambda' Su'$. The second statement is clear. $\square$

Under some conditions, the spectral radius is preserved by equitable quotient operation.

**Lemma 6.2.** *If $\Pi = \{\pi_1, \ldots, \pi_\ell\}$ is a partition of $[n]$ and $C' = (c'_{ij})$ is an $n \times n$ matrix satisfying $c'_{ij} = c'_{kj}$ for all $i, k$ in the same part $\pi_a$ of $\Pi$ and $j \in [n]$, then $C'$ and its quotient matrix $\Pi(C')$ with respect to $\Pi$ have the same set of nonzero eigenvalues. In particular, $\rho(C') = \rho(\Pi(C'))$.*

*Proof.* From the construction of $C'$, $\Pi(C')$ is clear to be an equitable quotient matrix of $C'$. Let $\lambda'$ be a nonzero eigenvalue of $C'$ with eigenvector $v' = (v'_1, \ldots, v'_n)^T$. Then $v'_i = (Cv')_i/\lambda' = (Cv')_k/\lambda' = v'_k$ for all $i, k$ in the same part $\pi_a$ of $\Pi$. Let $u'_a = v'_i$ with any choice of $i \in \pi_a$. Then $u' := (u'_1, \ldots, u'_\ell) \neq 0$, and $\Pi(C')u' = \lambda' u'$. From this and Lemma 6.1, we know that $C'$ and $\Pi(C')$ have the same set of nonzero eigenvalues, and thus $\rho(C') = \rho(\Pi(C'))$. $\square$

**Lemma 6.3.** *For $M_n = M_n(d, f, r_1, r_2, \ldots, r_n)$ defined in (5.2), where $d, f \geq 0$ and $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$, we have the following (i)-(iii).*

*(i) The largest real eigenvalue $\rho_r(M_n)$ of $M_n$ satisfies*

$$\begin{aligned}\rho_r(M_n) :=& \frac{r_n + d - f + \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2} \\ &\geq \max(d - f, r_n).\end{aligned}$$

*(ii) If $r_n = 0$, then*

$$\rho_r(M_n) = \frac{d - f + \sqrt{(d - f)^2 + 4fm}}{2},$$

*where $m := \sum_{i=1}^{n-1} r_i$ is the sum of all entries of $M_n$.*

*(iii) If $r_t = r_n$ for $t \leq n$, then $\rho_r(M_t) = \rho_r(M_n)$.*

*Proof.* (i) We consider the matrix $M_n + (f - d)I$. Note that $(M_n + (f - d)I)^T$ has equitable quotient matrix

$$\Pi((M_n + (f - d)I)^T) = \begin{pmatrix} (n-1)f & f \\ \sum_{i=1}^{n-1}(r_i - (d + (n-2)f)) & r_n - (d + (n-2)f) \end{pmatrix}$$

with respect to the partition $\Pi = \{\{1, 2, \ldots, n-1\}, \{n\}\}$ of $[n]$, which has two eigenvalues

$$\frac{r_n - d + f \pm \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2}.$$

Since $((M_n + (f - d)I)^T)_{ij} = ((M_n + (f - d)I)^T)_{kj}$ all $i, k \in [n-1]$ and $j \in [n]$ and by Lemma 6.2, $(M_n + (f - d)I)^T$ has eigenvalues

$$0^{n-2}, \frac{r_n - d + f \pm \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2},$$

and $M_n$ has eigenvalues

$$(d-f)^{n-2}, \frac{r_n + d - f \pm \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2}.$$

Note that

$$\frac{r_n + d - f + \sqrt{(r_n - d + f)^2 + 4f\sum_{i=1}^{n-1}(r_i - r_n)}}{2} \geq \max(d - f, r_n).$$

So the proof of (i) is complete.

(ii) and (iii) follow from (i) immediately. □

## 7. Spectral upper bounds with prescribed sum of entries

Let $J_k$, $I_k$ and $O_k$ be the $k \times k$ all-one matrix, the $k \times k$ identity matrix and the $k \times k$ zero matrix respectively. We recall an old result of Richard Stanley [12].

**Theorem 7.1** ([12])**.** *Let $C = (c_{ij})$ be an $n \times n$ symmetric (0,1) matrix with zero trace. Let the number of 1's of $C$ be $2e$. Then*

$$\rho(C) \leq \frac{-1 + \sqrt{1 + 8e}}{2}.$$

*Equality holds if and only if*

$$e = \binom{k}{2}$$

*and $PCP^T$ has the form*

$$\begin{pmatrix} J_k - I_k & 0 \\ 0 & O_{n-k} \end{pmatrix} = (J_k - I_k) \oplus O_{n-k}$$

*for some permutation matrix $P$ and positive integer $k$.*

The following theorem generalizes Theorem 7.1 to nonnegative matrices, not necessary symmetric.

**Theorem 7.2.** *Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix. Let $m$ be the sum of entries and $d$ (resp. $f$) be any number which is larger than or equal to the largest diagonal element (resp. the largest off-diagonal element) of $M$. Then*

$$\rho(C) \leq \frac{d - f + \sqrt{(d-f)^2 + 4mf}}{2}. \tag{7.1}$$

*Moreover, if $mf > 0$ then the equality in (7.1) holds if and only if $m = k(k-1)f + kd$ and $PCP^T$ has the form*

$$\begin{pmatrix} fJ_k + (d-f)I_k & 0 \\ 0 & O_{n-k} \end{pmatrix} = (fJ_k + (d-f)I_k) \oplus O_{n-k}$$

*for some permutation matrix $P$ and some nonnegative integer $k$.*

*Proof.* If $f = 0$ then the nonzero entries only appear in the diagonal of $C$, so $\rho(C) \leq d$ and (7.1) holds. Assume $f > 0$ for the remaining. Consider the $(n+1)\times(n+1)$ nonnegative matrix $M = C \oplus O_1$ which has row-sum vector $(r_1, r_2, \dots, r_n, r_{n+1})^T$ with $r_{n+1} = 0$ and a nonnegative left eigenvector $v^T$ for $\rho(M) = \rho(C)$. Let $C' = M_{n+1}(d, f, r_1, r_2, \dots, r_{n+1})$ as defined in (5.2) which has the same row-sum vector of $M$, and has a positive rooted eigenvector $v' = (v'_1, v'_2, \dots, v'_{n+1})^T$ for $\rho_r(C')$ by Lemma 5.4(i). Clearly $M[-|n+1) \leq C'[-|n+1)$ and $v^T v' > 0$. Hence the assumptions (i)-(iv) in Theorem 4.3 hold with $(C, \lambda, \lambda') = (M, \rho(M), \rho_r(C'))$. Now by Theorem 4.3 and Lemma 6.3(ii), we have

$$\rho(C) = \rho(M) \leq \rho_r(C') = \frac{d - f + \sqrt{(d-f)^2 + 4mf}}{2},$$

finishing the proof of the first part.

To prove the second part, assume $m = k(k-1)f + kd$ and $PCP^T = (fJ_k + (d-f)I_k) \oplus O_{n-k}$ for one direction. Using $\rho(C) = \rho(PCP^T) = \rho(fJ_k + (d-f)I_k)$, we have

$$\rho(C) = (k-1)f + d = \frac{d - f + \sqrt{(d-f)^2 + 4mf}}{2}.$$

For the other direction, assume $\rho(C) = \rho_r(C')$ and $mf > 0$. In particular $C \neq 0$ and $M \neq 0$. Let $(v_1, v_2, \dots, v_{n+1})$ be a nonnegative left eigenvector of $M$. Then $v_{n+1} = 0$. Write $\tilde{v}^T = (v_1, v_2, \dots, v_n)$. We first assume that $C$ has no zero row. Then $r_i > r_{n+1} = 0$ for $1 \leq i \leq n$. By Lemma 5.3(ii) with $(C', n) = (M, n+1)$, we have $v'_j > v'_{n+1}$. Then $c_{ij} = m_{ij} = c'_{ij}$ for the indices $1 \leq i \leq n$ with $v_i \neq 0$ and any $1 \leq j \leq n$ by Theorem 4.3(b). Hence

$$\rho(C)\tilde{v}^T = \tilde{v}^T C = \tilde{v}^T C'(n+1|n+1) = \tilde{v}^T (fJ + (d-f)I). \tag{7.2}$$

Since $\tilde{v}^T$ is a nonnegative left eigenvalue of the irreducible nonnegative matrix $fJ + (d-f)I$ for $\rho(C)$, we have $\tilde{v} > 0$. This together with $fJ + (d-f)I \geq C$ and (7.2) will imply $C = fJ + (d-f)I$, finishing the proof for the case under the assumption that $C$ has no zero row. Assume that $C$ has $n-k$ zero rows for some $1 \leq k \leq n-1$. Then there is a permutation matrix $P$ such that all zero rows of $PCP^T$ appear in the end, so the $(n-k)\times n$ submatrix $PCP^T([k]|-]$ of $PCP^T$ is 0 and the $k \times n$ submatrix $PCP^T[[k]|-]$ of $PCP^T$ has no zero row. Let $C_1 = PCP^T[[k]|[k]]$ and $m'$ be the sum of entries in $C_1$. Notice that $\rho(C_1) = \rho(C)$ and $m' \leq m$. Applying the first part of the theorem to $C_1$, we have

$$\rho(C_1) \leq \frac{d - f + \sqrt{(d-f)^2 + 4m'f}}{2} \leq \frac{d - f + \sqrt{(d-f)^2 + 4mf}}{2} = \rho(C) = \rho(C_1),$$

forcing $m' = m$, $C_1$ has no zero row and $C_1 = fJ_k + (d-f)I_k$. Hence $PCP^T[[k]|[k]) = 0$ and this implies $PCP^T = (fJ_k + (d-f)I_k) \oplus O_{n-k}$ and $m = k(k-1)f + kd$. □

## 8. $C'$ admitting an equitable quotient

From now on the square matrix $C$ is nonnegative, and the eigenvalue $\rho(C)$ of $C$ is corresponding to a nonnegative left eigenvector $v^T$ by Theorem 2.1(i). Hence the assumption (iii) in Theorem 4.3

and Theorem 4.4 immediately holds. In Lemma 5.1 and Lemma 5.3, we know that a rooted matrix $C'$ and its translates are possessed of a rooted eigenvector for $\rho_r(C')$. In this section, we shall apply properties of the equitable quotient to provide matrices which are not translated from a rooted matrix but still have positive rooted eigenvectors. We use this method to reduce the size of $C'$ in finding the bound $\lambda'$ of $\lambda = \rho(C)$ obtained in Theorem 4.3 and Theorem 4.4.

**Theorem 8.1.** *Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sum vector $(r_1, \ldots, r_n)^T$, and $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ a partition of $\{1, 2, \ldots, n\}$ with $n \in \pi_\ell$. Let $C' = (c'_{ij})$ be an $n \times n$ matrix that admits an $\ell \times \ell$ equitable quotient matrix $\Pi(C') = (\pi'_{ab})$ of $C'$ with respect to $\Pi$ satisfying the following (i)-(ii):*

- *(i) $C[-|n) \le C'[-|n)$ and $\Pi(C')$ has row-sum vector $\Pi(r') = (\pi(r')_1, \pi(r')_2, \ldots, \pi(r')_\ell)^T$ with $\pi(r')_a = \max_{i \in \pi_a} r_i$ for $1 \le a \le \ell$.*
- *(ii) $\Pi(C')$ has a positive rooted eigenvector $\Pi(v') = (\pi(v')_1, \pi(v')_2, \ldots, \pi(v')_\ell)^T$ for some nonnegative eigenvalue $\lambda'$.*

*Then*

$$\rho(C) \le \lambda'. \tag{8.1}$$

*Moreover, if $C$ is irreducible, then $\rho(C) = \lambda'$ if and only if*

- *(a) $r_i = \pi(r')_a$ for $1 \le a \le \ell$ and $i \in \pi_a$, and*
- *(b) $c'_{ij} = c_{ij}$ for all $1 \le i, j \le n$ such that for $1 \le b \le \ell$ with $j \in \pi_b$ we have $\pi(v')_b > \pi(v')_\ell$.*

*Proof.* Let $S$ be the $n \times \ell$ characteristic matrix of $\Pi$. From the construction of $\Pi$ and $C'$, $r' = S\Pi(r') = (r'_1, \ldots, r'_n)^T$ is the row-sum vector of $C'$, and $v' = S\Pi(v')$ is a positive rooted eigenvector of $C'$ for $\lambda'$ by Lemma 6.1. Since $C$ is nonnegative, there exists a nonnegative left eigenvector $v^T$ of $C$ for $\rho(C)$ by Theorem 2.1(i). Hence $v^T v' > 0$. Thus assumptions (i)-(iv) of Theorem 4.3 hold, concluding $\rho(C) \le \lambda'$.

Suppose that $C$ is irreducible. Then the above $v$ is positive. Hence the equivalent condition (b) of $\rho(C) = \lambda'$ in Theorem 4.3 becomes $c'_{ij} = c_{ij}$ for $1 \le i \le n, 1 \le j \le n-1$ with $v'_j > v'_n$, and this is equivalent to the (b) here from the structure of $v' = S\Pi(v')$. The equivalent condition (a) here is immediate from that in Theorem 4.3 since $r'_i = \pi(r')_a$ for $i \in \pi_a$. □

Notice that the irreducible assumption of $C$ in the second part of Theorem 7.2 is not necessary. The following example shows that this is a must in that of Theorem 8.1.

**Example 8.2.** Consider the following two $3 \times 3$ matrices

$$C = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and } \Pi(C') = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$$

is the equitable quotient matrix of $C'$ with respect to the partition $\Pi = \{\{1\}, \{2, 3\}\}$. Note that $C[-|3) = C'[-|3)$, and the row-sum vector $(3, 2, 1)^T$ of $C$ is majorized by the row-sum vector $(3, 2, 2)^T$ of $C'$. Since $\Pi(C') + I$ is positive and rooted, $\Pi(C')$ has a positive rooted eigenvector for $\lambda' = \rho(\Pi(C')) = (1 + \sqrt{13})/2$. Hence assumptions (i)-(ii) in Theorem 8.1 hold. By direct computing, $\rho(C) = (1 + \sqrt{13})/2$, so the the equality in (8.1) holds. However, $r_2 = 2 \neq 1 = r_3$, a contradiction to (a) in Theorem 8.1. This contradiction is because of the reducibility of $C$.

The following is a dual version of Theorem 8.1.

**Theorem 8.3.** *Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sum vector $(r_1, \dots, r_n)^T$, and $\Pi = \{\pi_1, \pi_2, \dots, \pi_\ell\}$ a partition of $\{1, 2, \dots, n\}$ with $n \in \pi_\ell$. Let $C'$ be an $n \times n$ matrix that admits an $\ell \times \ell$ equitable quotient matrix $\Pi(C') = (\pi'_{ab})$ of $C'$ with respect to $\Pi$ satisfying the following (i)-(ii):*

*(i) $C[-|n) \geq C'[-|n)$ and $\Pi(C')$ has row-sum vector $\Pi(r') = (\pi(r')_1, \pi(r')_2, \dots, \pi(r')_\ell)^T$ with $\pi(r')_a = \min_{i \in \pi_a} r_i$ for $1 \leq a \leq \ell$.*

*(ii) $\Pi(C')$ has a positive rooted eigenvector $\Pi(v') = (\pi(v')_1, \pi(v')_2, \dots, \pi(v')_\ell)^T$ for some nonnegative eigenvalue $\lambda'$.*

*Then*

$$\rho(C) \geq \lambda'. \tag{8.2}$$

*Moreover, if $C$ is irreducible then $\rho(C) = \lambda'$ if and only if*

*(a) $r_i = \pi(r')_a$ for $1 \leq a \leq \ell$ and $i \in \pi_a$, and*

*(b) $c'_{ij} = c_{ij}$ for all $1 \leq i, j \leq n$ such that for $1 \leq b \leq \ell$ with $j \in \pi_b$ we have $\pi(v')_b > \pi(v')_\ell$.*

□

**Remark 8.4.** (i) The positive assumption of $\Pi(v')$ in (ii) of Theorem 8.3 can be removed in concluding the first part $\rho(C) \geq \lambda'$. The following is a proof:

*Proof.* From (i) and referring to (4.2), we have $CQ \geq C'Q \geq 0$. Let $v' = S\Pi(v')$ be a rooted eigenvector of $C'$ for $\lambda'$ as shown in the above proof. Then $u = Q^{-1}v'$ is nonnegative by Lemma 4.2(i), so $Cv' = CQu \geq C'Qu = C'v' = \lambda'v'$. Since $v'$ is nonnegative, $\rho(C) \geq \lambda'$ by Theorem 2.1(iii). □

(ii) The following counterexample shows that to conclude $\rho(C) \leq \lambda'$, the positive assumption of $\Pi(v')$ in (ii) of Theorem 8.1 can not be removed:

$$C = C' = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \lambda' = 1, v' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \rho(C) = 2, v^T = (0, 1),$$

where the trivial partition $\Pi = \{\{1\}, \{2\}\}$ of $\{1, 2\}$ is adopted.

We provide an example in applying Theorem 8.1.

**Example 8.5.** Consider the following two $7 \times 7$ matrices $C$ and $C'$ expressed below under the partition $\Pi = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$ :

$$C = \left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right), \quad C' = \left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right). \tag{8.3}$$

Apparently, $C[-|7) \le C'[-|7)$, and the row-sum vector $(24, 23, 22, 20, 19, 13, 12)^T$ of $C$ is majorized by the row-sum vector $(24, 24, 24, 20, 20, 13, 13)^T$ of $C'$, so assumption (i) of Theorem 8.1 holds. Note that $C'$ is not rooted and neither of its translates. Since $C'$ has equitable quotient matrix

$$\Pi(C') = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},$$

in which $\Pi(C') + 2I$ is rooted, so assumption (ii) of Theorem 8.1 holds for $\lambda' = \rho_r(C')$ by Lemma 5.1 and Lemma 5.3(i). Hence by Theorem 8.1, $\rho(C) \le \rho_r(\Pi(C')) \approx 18.6936$.

If applying Lemma 2.2 by constructing the following nonnegative matrix $C''$ that majors $C$, and find its equitable quotient matrix $\Pi(C'')$ with respect to the above partition $\Pi$:

$$C'' = \left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 6 \\ 2 & 3 & 2 & 4 & 2 & 8 & 4 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 1 & 3 & 4 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 2 & 2 & 4 \end{array}\right), \qquad \Pi(C'') = \begin{pmatrix} 7 & 6 & 12 \\ 12 & 2 & 7 \\ 4 & 4 & 6 \end{pmatrix},$$

one will find the upper bound

$$\rho(C'') = \rho(\Pi(C'') \approx 19.4$$

of $\rho(C)$ which is larger than the previous one.

## 9. More irrelevant columns

Considering the part $\pi_\ell$ of column indices of $C$ and $C'$ in the assumption (i) of Theorem 8.1, the assumption $C[-|\pi_\ell] \le C'[-|\pi_\ell]$ for $C'$ is not really necessary. We might replace the columns indexed by $\pi_\ell$ in $C'$ by any other columns and adjust the values in the last column of keeping the row-sums of $C'$ unchanged. In this situation, the columns of $C'$ indexed by $\pi_\ell$ are irrelevant columns (in the comparison of $C$ and $C'$). For example in Example 8.5, the values in the 6-th column of $C'$ can be changed to any values (e.g., $(a, b, c, d, e, f, g)^T$), if the values in the 7-th column of $C'$ make the corresponding change (e.g., $(11-a, 11-b, 11-c, 6-d, 6-e, 5-f, 5-g)^T$ correspondingly), i.e., columns 6 and 7 of $C'$ are irrelevant. The following theorem generalizes this idea when restricting $\Pi[C']$ in Theorem 8.1 to be a rooted matrix or its translate.

**Theorem 9.1.** *Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be a partition of $[n]$ with $n \in \pi_\ell$, and $C$ be an $n \times n$ nonnegative matrix with row-sums $r_1 \ge r_2 \ge \cdots \ge r_n$. For $1 \le a \le \ell$ and $1 \le b \le \ell - 1$, choose $r''_a$, $c''_{ab}$ such that*

$$\begin{cases} r''_a & = & \max_{i \in \pi_a} r_i; & \\ c''_{ab} & \ge & \sum_{j \in \pi_b} c_{ij} & \text{for all } i \in \pi_a; \\ c''_{ab} & \ge & c''_{\ell b} > 0 & \text{for } a \ne b \end{cases}$$

*and let*

$$c''_{a\ell} = r''_a - \sum_{j=1}^{\ell-1} c''_{aj}.$$

*Then the $\ell \times \ell$ matrix $C'' = (c''_{ab})$ has a positive rooted eigenvector $v'' = (v''_1, v''_2, \ldots, v''_\ell)^T$ for $\rho_r(C'')$, and $\rho(C) \le \rho_r(C'')$. Moreover, if $C$ is irreducible, then $\rho(C) = \rho_r(C'')$ if and only if*

(a) $r_i = r''_a$ *for $1 \le a \le \ell$ and $i \in \pi_a$, and*

(b) $\sum_{j\in\pi_b} c_{ij} = c''_{ab}$ *for all $1 \le a, b \le \ell$ with $v''_b > v''_\ell$ and $i \in \pi_a$.*

*Proof.* From the construction of $C''$, $C'' + dI$ is a positive rooted matrix for some $d$ large enough, so $C''$ has a positive rooted eigenvector for $\rho_r(C'')$ by Lemma 5.1 and Lemma 5.3(i). In view of the construction of $C'$ in Example 8.5, we construct an $n \times n$ matrix $C'$ such that $C'$ has the equitable quotient matrix $\Pi(C') = C''$ and assumptions (i)-(ii) of Theorem 8.1 hold for $\lambda' = \rho_r(\Pi(C'))$. Hence the remaining follows from the conclusion of Theorem 8.1. □

**Remark 9.2.** Theorem 1.1 is a special case of Theorem 9.1 with $\Pi = \{\{1\}, \{2\}, \ldots, \{\ell-1\}, \{\ell, \ell+1, \ldots, n\}\}$ and $C'' = M_\ell(d, f, r_1, r_2, \ldots, r_\ell)$ as shown in (1.3). The equality case needs to apply Lemma 6.3(iii) by choosing a new $\ell$ to be the least $t$ such that $r_t = r_\ell$. By using the more irrelevant columns idea in Theorem 9.1, the assumption $f := \max_{1\le i\ne j\le n} c_{ij}$ and $d := \max_{1\le i\le n} c_{ii}$ in Theorem 1.1 can be replaced by the possible smaller number $f := \max_{1\le i\le n, 1\le j\le \ell-1, i\ne j} c_{ij}$ and $d := \max_{1\le i\le \ell-1} c_{ii}$ respectively.

The following is a dual theorem of Theorem 9.1.

**Theorem 9.3.** *Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be a partition of $[n]$ with $n \in \pi_\ell$, and $C$ an $n\times n$ nonnegative matrix with row-sums $r_1 \ge r_2 \ge \cdots \ge r_n$. For $1 \le a \le \ell$ and $1 \le b \le \ell - 1$, choose $r''_a$, $c''_{ab}$ such that*

$$\begin{cases} r''_a &=& \min_{i\in\pi_a} r_i; \\ c''_{ab} &\le& \sum_{j\in\pi_b} c_{ij} \qquad \text{for all } i \in \pi_a; \\ c''_{ab} &\ge& c''_{\ell b} > 0 \qquad \text{for } a \ne b \end{cases} \tag{9.1}$$

*and let*

$$c''_{a\ell} = r''_a - \sum_{j=1}^{\ell-1} c''_{aj}. \tag{9.2}$$

*Then the $\ell\times\ell$ matrix $C'' = (c''_{ab})$ has a positive rooted eigenvector $v'' = (v''_1, v''_2, \ldots, v''_\ell)^T$ for $\rho(C'')$, and $\rho(C) \ge \rho(C'')$. Moreover, if $C$ is irreducible, then $\rho(C) = \rho(C'')$ if and only if*

(a) $r_i = r''_a$ *for $1 \le a \le \ell$ and $i \in \pi_a$, and*

(b) $\sum_{j\in\pi_b} c_{ij} = c''_{ab}$ *for all $1 \le a, b \le \ell$ with $v''_b > v''_\ell$ and $i \in \pi_a$.*

## 10. Some new lower bounds of spectral radius

We shall apply Theorem 9.3 to obtain a lower bound of $\rho(C)$ for a nonnegative matrix $C$.

**Theorem 10.1.** *Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq t < n$, let $\Pi_t = \{\{1, \ldots, t\}, \{t+1, \ldots, n\}\}$ be a partition of $[n]$. For $d \geq \max_{t<i\leq n} c_{ii}$ and $f \geq \max_{1\leq i\leq n, t<j\leq n, i\neq j} c_{ij}$, assume that $0 < r_n - (n-t-1)f - d \leq r_t - (n-t)f$. Then*

$$\rho(C) \geq \frac{r_t - f + d + \sqrt{(r_t - (2n-2t-1)f - d)^2 + 4(n-t)f(r_n - (n-t-1)f - d)}}{2}. \qquad (10.1)$$

*Moreover, if $C$ is irreducible and $f > 0$, then the equality holds in (10.1) if and only if $r_1 = r_n$ or*

*(a)* $r_1 = r_t$ *and* $r_{t+1} = r_n$, *and*

*(b)* $\sum_{j\in[t]} c_{ij} = r_t - (n-t)f$ *for all* $i \in [t]$, *and*
$\sum_{j\in[t]} c_{ij} = r_n - (n-t-1)f - d$ *for all* $t < i \leq n$.

*Proof.* The lower bound of $\rho(C)$ in (10.1) follows by applying Theorem 9.3 with the following positive rooted matrix

$$C'' = \begin{pmatrix} r_t - (n-t)f & (n-t)f \\ r_n - (n-t-1)f - d & (n-t-1)f + d \end{pmatrix},$$

which has row-sum vector $(r_t, r_n)^T$ and the assumptions in (9.1) and (9.2) of Theorem 9.3 hold from the assumptions. Note that $C''$ has a positive rooted eigenvector $(v''_1, v''_2)^T$ for $\rho(C'')$ by Lemma 5.3(i), and the value $\rho(C'')$ is as shown in the right of (10.1). To study the equality case in (10.1), we apply conditions (a)-(b) in Theorem 9.3, in which (a) is exactly the (a) of this theorem. If $v''_1 > v''_2$ then the condition (b) of this theorem is exactly the (b) of Theorem 9.3. Notice that $v''_2 = v''_1$ if and only if $\rho(C'') = r_t = r_n$ by Theorem 2.1 using the irreducible property of $C''$. This is also equivalent to $r_1 = r_n$ under the condition (a). □

The following corollary restricts Theorem 10.1 to binary matrix $C$.

**Corollary 10.2.** *Let $C = (c_{ij})$ be an $n \times n$ (0, 1) matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n > 0$, and choose $t \geq n - r_n + 1$ and $t \leq n$. Then*

$$\rho(C) \geq \frac{r_t + \sqrt{r_t^2 - 4(r_n - 1)(r_t - r_n)}}{2}. \qquad (10.2)$$

*Moreover, if $C$ is irreducible, then equality holds in (10.2) if and only if $r_1 = r_n$ or*

*(a)* $r_1 = r_t$ *and* $r_{t+1} = r_n$, *and*

*(b)* $\sum_{j\in[t]} c_{ij} = r_t - (n-t)$ *for all* $i \in [t]$, *and*
$\sum_{j\in[t]} c_{ij} = r_n - (n-t)$ *for all* $t < i \leq n$.

□

*Proof.* If $t = n$ then (10.2) becomes $\rho(C) \geq r_n$, so the corollary follows from Lemma 2.3. Assume $t < n$. Since the assumptions in Theorem 10.1 clearly holds with $d = f = 1$, the corollary also holds by (10.1) in this case. □

One can easily check that the right of (10.2) is at least $r_n$ (with equality iff $r_t = r_n$), so the above lower bound is better than the known one $r_n$ in Lemma 2.3.

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