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碩 士 論 文

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A simple method on comparison between spectral radii of two directed graphs

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摘 要

矩陣的譜半徑為其特徵值絕對值的最大值，而一有向圖的譜半徑則定義為其鄰接矩陣之譜半徑。本論文給出一個比較方陣譜半徑的方法，將此方法應用於有向圖的鄰接矩陣，我們可以簡易比較有向圖的譜半徑。

關鍵詞：譜半徑、鄰接矩陣

A simple method on comparison between spectral radii of two directed graphs

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Abstract

The spectral radius of a square matrix is the largest magnitude of its eigenvalues. And the spectral radius of a directed graph is defined as the spectral radius of the corresponding adjacency matrix. In this paper, we give an approach to compare the spectral radii of two nonnegative matrices. By applying this method on the adjacency matrix of a directed graph, we can compare the spectral radii of two directed graphs simply.

Keywords: spectral radius, adjacency matrix

Contents

1 Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers, respectively. Let C be an $n \times n$ real square matrix. If there is a nonzero column vector $u \in \mathbb{C}^n$ such that $Cu = \lambda u$ for some scalar $\lambda \in \mathbb{C}$, then the scalar λ is called the eigenvalue of C with corresponding eigenvector u. And the spectral radius of a matrix C is the largest magnitude (or complex modulus) of its eigenvalues, denoted by $\rho(C)$. We are interested in the spectral radius of the following matrix associated with a simple directed graph.

Definition 1.1. Given a directed graph G, the *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G, and

$$a_{ij} = \begin{cases} 1, \text{if } ji \text{ is an arc in } G, \\ 0, \text{otherwise.} \end{cases}$$

Given a directed graph G, the spectral radius of G is the spectral radius of the adjacency matrix of G, denoted by $\rho(G)$. Note that the spectral radius $\rho(G)$ is independent of the ordering of the vertex set of G.

Conversely, we can also define a directed graph G from a given nonnegative $n \times n$ matrix $C = (c_{ij})$ by setting the vertex set $\{1, 2, \ldots, n\}$ and edge set $\{ij : c_{ij} > 0\}$. The matrix C is *irreducible* if the defined graph G from C is strongly connected.

The spectral radius is an important indicator to specify the relation of connected vertices in a graph, so it is meaningful to find a simple method to estimate the spectral radius. A simple and excellent executable method to estimate the spectral radius has some features, first, the bios is minimized, and second, there must be a way to prove it sensible. Enumerate these factors and prove it correctly would make this method reliable.

In [1], Cheng and Weng give many bounds of the spectral radius of a nonnegative square matrix. And based on their theory and Perron-Frobenius theorem, we give another approach to obtain an upper bound of the spectral radius and apply it on the adjacency matrix of a directed graph. All theorems come from continuous discussions between C.W. Weng and K.H. Chen. These were all documented.[5]

2 Preliminaries

The following is Perron – Frobenius theorem, which provides a feature of nonnegative eigenvectors to nonnegative matrices.

Theorem 2.1. *[3] If C is a nonnegative square matrix, then the spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector. Moreover if C is irreducible the above eigenvectors can be chosen to be positive.*

A well-known application of Theorem 2.1 show that if matrix C' *majors* C (in notation $C \leq C'$), i.e. $c_{ij} \leq c'_{ij}$ for all ij, j, then $\rho(C) \leq \rho(C')$. Our main result shows that the assumption $C \leq C'$ can be a little loosen. Our theory is based on the following theorem, which is from [1].

Theorem 2.2. *Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ real matrices with real eigenvalues λ, λ' respectively such that there exist $n \times n$ matrices P and Q satisfying the following (i)-(iv).*

(i) $PCQ \leq PC'Q$;

(ii) an eigenvector Qu of C' associated with eigenvalue λ' exists for some nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$.

(iii) a left eigenvector v^TP of C' associated with eigenvalue λ exists for some nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$; and

(iv) $v^TPQu > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{1}$$

Proof. Multiplying the nonnegative vector u in assumption (i)

to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu, \tag{2}$$

where the above equality follows by Qu being eigenvector of C' for λ'. Multiplying the nonnegative vector v^T of C in assumption (iii) to the left of all terms in (2), we have

$$\lambda v^TPQu = v^TPCQu \leq v^TPC'Qu = \lambda' v^TPQu, \tag{3}$$

where the above first equality follows by v^TP being left eigenvector of C for λ. Now delete the positive term v^TPQu by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part. Assume that $\lambda = \lambda'$, so the inequality in (3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1) implies

$$\lambda v^TPCQu = v^TPCQu = \sum_{i,j} v_i(PCQ)_{ij}u_j$$

$$= \sum_{i,j} v_i(PC'Q)_{ij}u_j = v^TPC'Qu = \lambda' v^TPC'Qu,$$

so $\lambda = \lambda'$ by (3). $\square$

3 Our Method

We will apply Theorem 2.2 by using two particular square matrices P and Q to obtain our main result. We use $[n-1]$ as notation of the set of integers from 1 to $n-1$, which is $\{1, 2, \cdots, n-1\}$. Throughout this thesis we fix $k \in [n-1]$, let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn}. Now we apply the previous Theorem 2.2 with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & 1 \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix}, \tag{4}$$

so the matrix $PC'Q$ in assumption (i) of Theorem 2.2 is

$$PC'Q = \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1k} + c'_{1n} \\ c'_{21} & c'_{22} & \cdots & c'_{2k} + c'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{nk} + c'_{nn} \end{pmatrix}, \tag{5}$$

where c'_{ij} denotes the (i, j)-entry of C'.

Definition 3.1. A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

Lemma 3.2. *If $u = (u_1, u_2, \ldots, u_n)^T$, then*

(i) Qu is k-rooted if and only if u is nonnegative;

(ii) $u_k > 0$ *if and only if* $(Qu)_k > (Qu)_n$.

Proof. (i), (ii) follow immediate from the definition of k-rooted and $Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$. □

Below is our first result, in which the first condition implies the first $n-1$ columns of C' major the same columns of C, and the sum of k-th and n-th columns of C' also majors that of C. The second and the third condition suggest that C and C' have nonnegative k-rooted eigenvectors. And the forth condition is simpler but with the same meaning in Theorem 2.2

We need a notation of submatrix, which is taken from some columns and some rows of a matrix.

Definition 3.3. For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C, respectively, we use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$. We use the notation $C[i|j]$ for short of $C[[i]|[j]]$.

Theorem 3.4. *Let* $C = (c_{ij})$, $C' = (c'_{ij})$ *be* $n \times n$ *real matrices with real eigenvalues* λ *and* λ' *respectively. Assume that*

(i) $C[n|n-1] \leq C'[n|n-1]$ *and* $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ *for all* $1 \leq i \leq n$;

(ii) there exists a k*-rooted eigenvector vector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *of* C' *for* λ';

(iii) there exists a nonnegative eigenvector vector $v^T = (v_1, v_2, \ldots, v_n)$ *of* C *for* λ;

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$*. Moreover,* $\lambda = \lambda'$ *if and only if*

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ *for* $1 \leq i \leq n$ *with* $v_i > 0$ *and* $v'_n > 0$*;*

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1,\ j \neq k$ *with* $v_i \neq 0$ *and* $v'_j > 0$*;*

(c) $c'_{ik} = c_{ik}$ *for* $1 \leq i \leq n$ *with* $v_i > 0$ *and* $v'_k > v'_n$

Proof. The proof is based on Theorem 2.2 with $P = I$ and $Q = I + E_{kn}$ in (4). The assumption (i) $PCQ \leq PC'Q$ of Theorem 2.2 holds by the condition (i) of this theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda' Qu$ by the condition (ii) and (i) in Lemma 3.2. Hence the assumption (ii) of Theorem 2.2 holds. The assumptions (iii) and (iv) of Theorem 2.2 clearly hold by conditions (iii), (iv) of this theorem since $P = I$ and $v' = Qu$. Hence $\lambda \leq \lambda'$ by the necessary condition of Theorem 2.2. Moreover, $\lambda = \lambda'$ if and only if (1) holds, and this is equivalent to conditions (a), (b) and (c) of this theorem. □

We are interested in the matrices C' that have k-rooted eigenvectors. Motivated by the condition (i) of Theorem 2.3, we provide the following two definitions. The first is the definition of (k, n)-sum.

Definition 3.5. For an $n \times n$ matrix $C' = (c'_{ij})$, the (k, n)-*sum vector* of C' is the vector obtained from the sum of the k-th and n-th columns of C'.

Note that the last column of $C'Q$ is the (k, n)-sum vector of C'. Below is the definition of k-rooted matrix.

Definition 3.6. A matrix $C' = (c'_{ij})$ is called *k-rooted* if for each $i \neq k, n$ the i-th column of C' is k-rooted and the (k, n)-sum vector of C' is k-rooted.

We need a simple lemma for later use.

Lemma 3.7. $Q^{-1} = I - E_{kn}$.

Proof. Since

$$Q(I - E_{kn} = (I + E_{kn})(I - E_{kn}) = I - E_{kn} + E_{kn} - E_{kn}E_{kn} = I,$$

we have $Q^{-1} = I - E_{kn}$. $\square$

The matrix Q^{-1} is explicitly written as

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & -1 \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix},$$

and if $C' = (c'_{ij})$ then $Q^{-1}C'Q$ has the form

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c'_{(k-1)1} & c'_{(k-1)2} & \cdots & c'_{(k-1)(n-1)} & c'_{(k-1)k} + c'_{(k-1)n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{k(n-1)} - c'_{n(k-1)} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{(k+1)1} & c'_{(k+1)2} & \cdots & c'_{(k+1)(n-1)} & c'_{(k+1)k} + c'_{(k+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n(n-1)} & c'_{nk} + c'_{nn} \end{pmatrix}. \quad (6)$$

The following lemma shows that a k-rooted matrix has a k-rooted eigenvector.

Lemma 3.8. *Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(ii) hold.*

(i) C' is a k-rooted matrix if and only if $Q^{-1}(C' + dI)Q$ is nonnegative for some $d \geq 0$, where I is the $n \times n$ identity matrix.

(ii) If C' is k-rooted then there exists a k-rooted eigenvector v' of C' for $\rho(C')$.

Proof. (i) The matrix $Q^{-1}C'Q$ has ij entry

$$(Q^{-1}C'Q)_{ij} = \begin{cases} c'_{ij}, & \text{if } i \neq k \text{ and } j \neq n; \\ c'_{kj} - c'_{nj}, & \text{if } i = k \text{ and } j \neq n; \\ c'_{ik} + c'_{in}, & \text{if } i \neq k \text{ and } j = n; \\ c'_{kk} + c'_{kn} - c'_{nk} - c'_{nn}, & \text{if } i = k \text{ and } j = n, \end{cases}$$

as shown in (6). Hence $Q^{-1}(C' + dI)Q$ is nonnegative if and only if C' is k-rooted by the definition of nonnegative matrix and k-rooted matrix.

(ii) Suppose C' is k-rooted. Choose $d \geq 0$ such that $Q^{-1}(C' + dI)Q$ is nonnegative. Let u be a nonnegative eigenvector of $Q^{-1}(C' + dI)Q = Q^{-1}C'Q + I$ for $\rho(C' + dI) = \rho(C') + d$. Note that $Q^{-1}C'Qu = \rho(C')u$, and Qu is k-rooted by Lemma 3.2. Hence $v' = Qu$ is what we want.

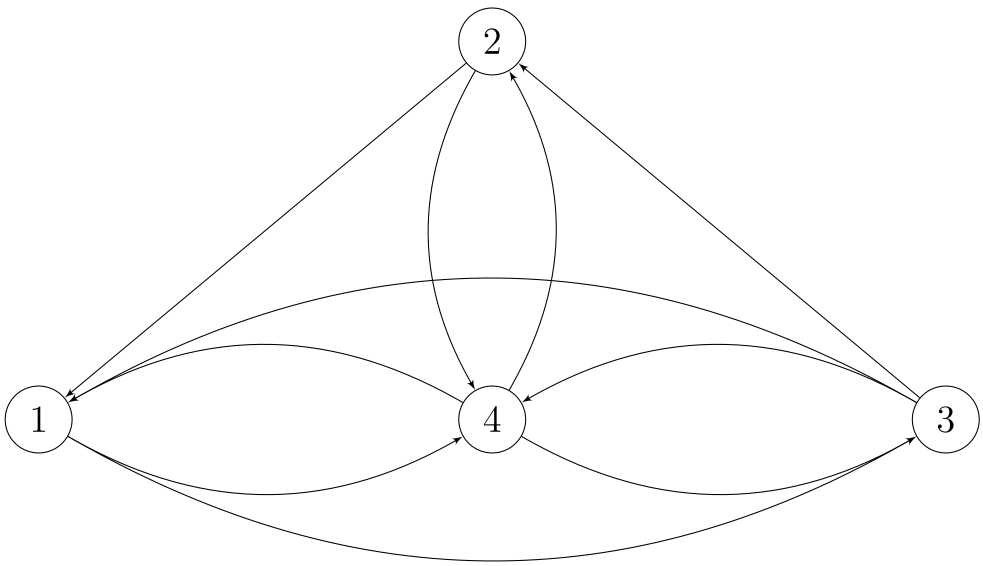
□

Note that Theorem 3.4 depends on eigenvectors. The following is an eigenvector-free Theorem.

Theorem 3.9. *Let C be an $n \times n$ nonnegative irreducible matrix and C' be an $n \times n$ k-rooted matrix such that $C'Q$ majorizes CQ. Then $\rho(C) \le \rho(C')$.*

Proof. Referring to (5), the assumption (i) in Theorem 3.4 holds. By Lemma 3.8 (ii), there exists a k-rooted eigenvector v' of C' for $\rho(C')$. Since C is irreducible and nonnegative, there exists a positive eigenvector v^T of C for $\rho(C)$. Thus $v^T v' > 0$. Hence Theorem 3.4 (i)-(iv) hold. Hence $\rho(C) \le \rho(C')$ by Theorem 3.4. □

Example 3.10. Let G be the digraph depicted below.



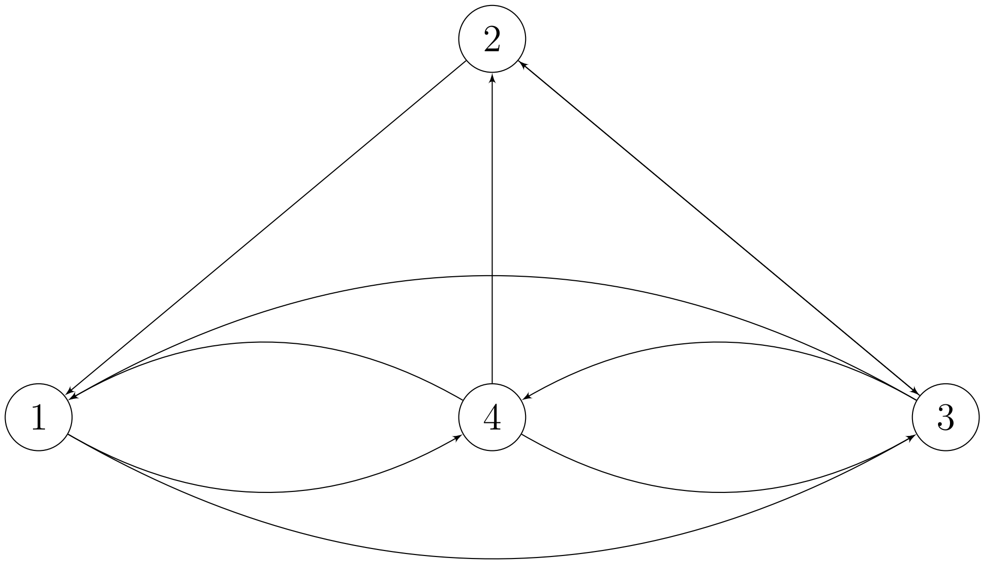
The following 4×4 matrix is the adjacency matrix of G.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We choose another matrix

$$A' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

which is the adjacency matrix of G' depicted below.



Note that both G and G' have the same number of edges, neither A majories A' nor A' majories A. Our main result still can do comparison between $\rho(A)$ and $\rho(A')$. We first observe that A is irreducible and A' is k-rooted for $k = 3$. Moreover $A'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = AQ.$ Hence Hence $\rho(A) \leq \rho(A')$ by Theorem 3.9 with $C = A$ and $C' = A'$.

Both of the values of $\rho(C)$ and $\rho(C')$ are close to 2.511547 by calculating on computer. [4, sage]

To finish the thesis, we provide an example to show that the 'k-rooted' assumption of C' is necessary in Theorem 3.9.

Example 3.11. Consider the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

With $n = 4$ and $k = 3$, we have

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = C'Q.$$

Using computer, [4, sage]

$$\rho(C) \approx 2.234 \not\leq 2.148 \approx \rho(C').$$

This is because C' is not k-rooted as $c'_{33} + c'_{34} = 0 \not\geq 1 = c'_{43} + c'_{44}$.

All theorems come from continuous discussions between C.W. Weng and K.H. Chen. These were all documented.[5]

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