TBA

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Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers respectively.
Let C be an $n \times n$ real nonnegative matrix, and $u \in \mathbb{R}^n$ be a nonzero column vector. The scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of C corresponding to the *eigenvector* u, if $Cu = \lambda u$. Also, C has an left eigenvector v^T, when $v^T C = \lambda v^T$.
When C is an $n \times n$ real matrix, the *spectral radius* $\rho(C)$ of C is defined by

$$\rho(C) := \max\{\ |\lambda|\ \mid\ \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ.

Preliminaries

We are interested in spectral radius of the following matrix associated with a simple graph.
Given an undirected graph G, the *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G, and

$$a_{ij} = \begin{cases} 1, \text{if } i \text{ is adjacent to } j, \\ 0, \text{otherwise.} \end{cases}$$

Given an undirected graph G, the *spectral radius* $\rho(G)$ of G is the spectral radius of the adjacency matrix of G.

Preliminaries

We introduce a notation of submatrix, which is taken from some columns and some rows of a matrix. For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C respectively, We use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$,

Preliminaries

Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that

(i) $PCQ \leq PC'Q$;

(ii) there exist a nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector Qu;

(iii) there exist a nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector $v^T P$; and

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

Proof

Multiplying the nonnegative vector u in (ii) to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{1}$$

Multiplying the nonnegative left eigenvector v^T of C for λ in assumption (iii) to the left of all terms in (**??**), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{2}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Proof (Continue)

Assume that $\lambda = \lambda'$, so the inequality in (**??**) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (**??**) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (**??**).

Our Method

Throughout fix $k \in [n-1]$. Let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn}. We will apply the previous theorem with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & 1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

k-rooted vector

A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

vector rooted lemma If $u = (u_1, u_2, \ldots, u_n)^T$ and $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$, then

(i) v' is k-rooted if and only if u is nonnegative;

(ii) $u_k > 0$ if and only if $v'_k > v'_n$.

□

Our first result

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ matrices. Assume that

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ and $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ for all $1 \leq i \leq n$;

(ii) there exist a k-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';

(iii) there exists a nonnegative vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ for $1 \leq i \leq n$ with $v_i \neq 0$ and $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq n-1$ with $v_i \neq 0$ and $v'_j > v'_n$.

$\square$

Proof.

The proof is based on Theorem 1.7 with $P = I$ and $Q = I + E_{kn}$ in (**??**).The assumption (i) $PCQ \leq PC'Q$ of Theorem 1.7 holds by the condition (i) of this theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda' Qu$ by the condition (ii) and Lemma 2.2(i). Hence the assumption (ii) of Theorem 1.7 holds. The assumptions (iii) and (iv) of Theorem 1.7 clearly hold by conditions (iii),(iv) of this theorem since $P = I$ and $v' = Qu$ Hence $\lambda \leq \lambda'$ by the conclusion of Theorem 1.7. Moreover $\lambda = \lambda'$ if and only if (**??**) holds, and this is equivalent to conditions (a),(b),(c) of this theorem. $\square$

k-rooted matrix

[(k,n)-sum] For a matrix $C' = (c'_{ij})$ of n columns, the (k, n)-*sum* vector of C' is the vector of the sum of the k-th and n-th columns of C', where $k \leq n - 1$.
[k-rooted matrix] A matrix $C' = (c'_{ij})$ is called *k-rooted* if its columns and its (k, n)-sum vector are all k-rooted except the last column of C'.

Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(iii) hold.

(i) C' is a k-rooted matrix, if and only if, $Q^{-1}C'Q$ is nonnegative.

(ii) Assume that C' is k-rooted and let u be a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Then C' has a k-rooted eigenvector $v' = Qu$ for $\rho(C')$.

(iii) $\rho(C') = \rho(Q^{-1}C'Q)$

Remark

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

The matrix $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{kn-1} - c'_{nk-1} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

Remark

Proof.

(i) is immediate from Definition **??** and the observation that

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix},$$

□

Proof.

and $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k}+c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k}+c'_{k-1n} \\ c'_{k1}-c'_{n1} & c'_{k2}-c'_{n2} & \cdots & c'_{kn-1}-c'_{nk-1} & c'_{kk}+c'_{kn}-c'_{nk}-x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k}+c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk}+c'_{nn} \end{pmatrix}.$$

$\square$

Proof.

Continue (ii) By Lemma **??** $v' = Qu$ is k-rooted. Since $Q^{-1}C'Qu = \rho(C')u$ by the assumption, we have
$Q^{-1}C'Qu = Q^{-1}\rho(C')Qu = \rho(C')u$
$C'Qu = \rho(C')Qu.$
(iii) Since C' and $Q^{-1}C'Q$ have the same set of eigenvalues, clearly $\rho(C') = \rho(Q^{-1}C'Q)$.

□

If a square matrix C' has a rooted eigenvector for λ', then $C' + dI$ also has the same rooted eigenvector for $\lambda' + d$, where d is a constant and I is the identity matrix with the same size of C'.

Let C be an $n \times n$ nonnegative Tmatrix. For $1 \leq i \leq n$ and $1 \leq j \leq n-1$, choose c'_{ij} such that $c'_{ij} \geq c_{ij}$ and $c'_{kj} \geq c'_{nj} > 0$, and choose r'_i such that $r'_i \geq c_{ik} + c_{in}$, and $r'_k \geq r'_n$. Moreover choose $c'_{in} := r'_i - c'_{ik}$. Then $\rho(C) \leq \rho(C')$, when $C' = (c'_{ij})$.

Proof

The assumptions are necessary that $PCQ \leq PC'Q$, and C' is k-rooted, by *, For certain d, if C'+d*I is k-rooted, then it has a k-rooted eigenvector with its spectral radius $\lambda + d$. C' would share the same eigenvector with C'+d*I and has eigenvalue λ. So C'+d*I and C+d*I meet the conditions of (**??**), and we can show that $\rho(C' + d * I) \geq \rho(C + d * I)$ and then $\rho(C') \geq \rho(C)$ $\square$

Example

For the following 4×4 matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then $\rho(C) \leq \rho(C')$ by previous theorem.

Counterexample

For the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we have $CQ \leq C'Q$, but $\rho(C) = 2.234 \not\leq 2.148 = \rho(C')$. This is because $c'_{33} + c'_{34} \not\geq c'_{43} + c'_{44}$.