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碩 士 論 文

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A simple method on estimating spectral radius for some directed graphs

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摘 要

我知道譜半徑是區分圖中點連線關係的重要指標，所以找一個簡單的方法來估算譜半徑是很有意義的。一個簡單而且傑出估算譜半徑的方法有一些特徵。首先，誤差被極小化，第二必定存在一個方法正確理性地證明它。列舉這些方法並正確證明會讓這方法穩固且可信靠。在估計的領域裡，最好思考過所有的因素，再根據目標，取用估計譜半徑的方法，以利之後分析圖的連接矩陣，與其連通性有關的矩陣。

關鍵詞：譜半徑，簡易方法，有向圖，估算。

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Abstract

I know that spectral radius is an important index to specify the relation of connected vertices in the graphs, so it is meanful to find a simple method to estimate spectral readius. A simple and excellent executable method to estimate spectral radius has some features. First, the bios is minimized, and second there must be a way to prove it sensible. Enumerate these factors and prove it correctly would make this method solid and reliable. In the field of estimation, it's better to go through all the factors, and base on one's goal to apply the methods of estimating spectral radius before analyzing the adjacency matrix of the graphs, which is related to the connectness of the graph.

Keywords: spectral radius, simple method, directed graphs, estimation

Contents

1 Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers respectively.

Definition 1.1. Let C be an $n \times n$ real nonnegative matrix, and $u \in \mathbb{R}^n$ be a nonzero column vector. The scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of C corresponding to the *eigenvector* u, if $Cu = \lambda u$.

Definition 1.2. [2] When C is an $n \times n$ real matrix, the *spectral radius* $\rho(C)$ of C is defined by

$$\rho(C) := \max\{\ |\lambda|\ \ |\ \ \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ.

We are interested in spectral radius of the following matrix associated with a simple graph.

Definition 1.3. Given an directed graph G, the*adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by vertices of G, and

$$a_{ij} = \begin{cases} 1, \text{if } i \text{ is adjacent to } j, \\ 0, \text{otherwise.} \end{cases}$$

Definition 1.4. Given a directed graph G, the *spectral radius* $\rho(G)$ of G is the spectral radius of the adjacency matrix of G.

2 Preliminaries

The next theorem is Perron−Frobenius Theorem, which provides a feature of nonnegative eigenvector to nonnegative matrices.

Theorem 2.1. *[1] [3] If C is a nonnegative square matrix, then the spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnegative right eigenvector and a corresponding nonnegative left eigenvector.*

We introduce a notation of submatrix, which is taken from some columns and some rows of a matrix.

Definition 2.2. For a matrix $C = (c_{ij})$ and subsets α, β of row indices and column indices of C respectively, We use $C[\alpha|\beta]$ to denote the submatrix of C with size $|\alpha| \times |\beta|$ that has entries c_{ij} for $i \in \alpha$ and $j \in \beta$,

We introduce two matrices P and Q in the following Theorem, where P is a permutation matrix which is multiplied to the left side, and Q is sum of elementary matrix and certain binary matrix. In which P generalize row permutation on cases of C matrix,and Q is the transform from C' to C', which is the first n-1 columns and sum of certain columns. We aim to find C' such that C' majors C,i.e. $C \leq C'$

The following Theorem is from [1].

Theorem 2.3. *Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that*

(i) $PCQ \leq PC'Q$;

(ii) there exist a nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector Qu;

(iii) there exist a nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$such that λ is an eigenvalue of C with associated left eigenvector $v^T P$; and

(iv) $v^T PQu > 0$.

Then $\lambda \le \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \le i, j \le n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0. \tag{1}$$

Proof. Multiplying the nonnegative vector u in Theorem 2.3 assumption (i), where Qu is eigenvector of C', to the right of both terms of (i),

$$PCQu \le PC'Qu = \lambda' PQu. \tag{2}$$

Multiplying the nonnegative left eigenvector v^T of C for λ in assumption (iii) to the left of all terms in (2), where $v^T P$ is left eigenvector of C for λ, thus we have

$$\lambda v^T PQu = v^T PCQu \le v^T PC'Qu = \lambda' v^T PQu. \tag{3}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \le \lambda'$ and finish the proof of the first part. Assume that $\lambda = \lambda'$, so the inequality in (3) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence, $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (1) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (3). □

3 Our Method

We use $[n-1]$ as notation of the set of elements from one to $n-1$, which is $\{1, 2, ..., n-1\}$. Throughout fix $k \in [n-1]$. Let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn}. We will apply the previous Theorem 2.3 with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & 1 \\ & & & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}. \tag{4}$$

Definition 3.1. A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *K-rooted* if $v'_j \ge 0$ for $1 \le j \le n$ and $v'_k \ge v'_n$.

The following Lemma is immediate from the above definition.

Lemma 3.2. *If $u = (u_1, u_2, \ldots, u_n)^T$ and $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$, then*

(i) v' is K-rooted if and only if u is nonnegative;

(ii) $u_k > 0$ *if and only if* $v'_k > v'_n$.

Below is our first result, in which the first condition implies the first $n-1$ columns of C major to the columns of C', and the (k,n)-sum column of C is also major to C'. The second and the third condition suggest that C and C' have nonnegative eigenvectors which are k-rooted. And the forth condition is simpler but with the same meaning with Theorem (2.3)

Theorem 3.3. *Let* $C = (c_{ij})$, $C' = (c'_{ij})$ *be* $n \times n$ *matrices. Assume that*

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ *and* $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ *for all* $1 \leq i \leq n$;

(ii) *there exists a* K*-rooted vector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *and a scalar* $\lambda' \in \mathbb{R}$ *such that* λ' *is an eigenvalue of* C' *with associated eigenvector* v';

(iii) *there exists a nonnegative vector* $v^T = (v_1, v_2, \ldots, v_n)$ *and a scalar* $\lambda \in \mathbb{R}$ *such that* λ *is an eigenvalue of* C *with associated left eigenvector* v^T;

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ *for* $1 \leq i \leq n$ *with* $v_i \neq 0$ *and* $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1, j \neq k$*with* $v_i \neq 0$;

(c) $c'_{ik} = c_{ik}$ *for* $1 \leq i \leq n$ *and* $v'_k > v'_n$

Proof. The proof is based on Theorem 2.3 with $P = I$ and $Q = I + E_{kn}$ in (4). The assumption (i) $PCQ \leq PC'Q$ of Theorem 2.3 holds by the condition (i) of this Theorem. Let $u = Q^{-1}v'$. Then u is nonnegative and $C'Qu = \lambda' Qu$ by the condition (ii) and Lemma 3.2(i). Hence the assumption (ii) of Theorem 2.3 holds. The assumptions (iii) and (iv) of Theorem 2.3 clearly hold by conditions (iii),(iv) of this Theorem since $P = I$ and $v' = Qu$ Hence $\lambda \leq \lambda'$ by the necessary condition of Theorem 2.3. Moreover $\lambda = \lambda'$ if and only if 1 holds, and this is equivalent to conditions (a),(b),(c) of this Theorem. □

We are interested in the matrices C' that have K-rooted eigenvectors. Motivated by the condition (i) of Theorem 2.3, we provide the following two definitions. This is the definition of (k,n)-sum.

Definition 3.4. For an $n \times n$ matrix $C' = (c'_{ij})$, the (k,n)-sum vector of C' is the vector of the sum of the k-th and n-th columns of C', where $k \leq n-1$.

Note that the last column of $C'Q$ is the (k,n)-sum vector of C'. Below is the definition of k-rooted matrix.

Definition 3.5. A matrix $C' = (c'_{ij})$ is called $k - rooted$ if its columns and its (k,n)-sum vector are all K-rooted except the last column of C'.

Remark 3.6.

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

The matrix $C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k}+c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k}+c'_{k-1n} \\ c'_{k1} & c'_{k2} & \cdots & c'_{kn-1} & c'_{kk}+c'_{kn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k}+c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk}+c'_{nn} \end{pmatrix}.$$

The following lemma shows that a K-rooted matrix has a K-rooted eigenvector.

Lemma 3.7. *Let $C' = (c'_{ij})$ be an $n \times n$ nonnegative matrix. Then the following (i)-(iii) hold.*

(i) C' is a K-rooted matrix, if and only if, $Q^{-1}C'Q$ is nonnegative.

(ii) Assume that C' is K-rooted and let u be a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Then C' has a K-rooted eigenvector $v' = Qu$ for $\rho(C')$.

(iii) $\rho(C') = \rho(Q^{-1}C'Q)$

Proof. (i) and $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k}+c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k}+c'_{k-1n} \\ c'_{k1}-c'_{n1} & c'_{k2}-c'_{n2} & \cdots & c'_{kn-1}-c'_{nk-1} & c'_{kk}+c'_{kn}-c'_{nk}-x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k}+c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk}+c'_{nn} \end{pmatrix}.$$

(ii) By Lemma 3.2 $v' = Qu$ is K-rooted. Since $Q^{-1}C'Qu = \rho(C')u$ by the assumption, we have
$Q^{-1}C'Qu = Q^{-1}\rho(C')Qu = \rho(C')u$
$C'Qu = \rho(C')Qu$.

(iii) Since C' and $Q^{-1}C'Q$ have the same set of eigenvalues, clearly $\rho(C') = \rho(Q^{-1}C'Q)$.

□

Lemma 3.8. *If a square matrix C' has a rooted eigenvector for λ', then $C' + dI$ also has the same rooted eigenvector for $\lambda' + d$, where d is a constant and I is the identity matrix with the same size of C'.*

Theorem 3.9. *Let C be an $n \times n$ nonnegative matrix. For $1 \le i \le n$ and $1 \le j \le n-1$, choose c'_{ij} such that $c'_{ij} \ge c_{ij}$ and $c'_{kj} \ge c'_{nj} > 0$, and choose r'_i such that $r'_i \ge c_{ik} + c_{in}$, and $r'_k \ge r'_n$. Moreover choose $c'_{in} := r'_i - c'_{ik}$. Then $\rho(C) \le \rho(C')$, when $C' = (c'_{ij})$.*

Proof. These assumptions are necessary that $PCQ \le PC'Q$ where $P = I$,$Q = I + E_{kn}$. And C' is K-rooted,based on Lemma (3.7),

$Q^{-1}C'Q =$

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{kn-1} - c'_{nk-1} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

$Q^{-1}C'Q \geq 0$ for that $c'_{kj} \geq c'_{nj} > 0$ when $1 \leq j \leq n-1$, $c'_{ij} \geq c_{ij}$ where C is nonnegative, and the last column $c'_{in} + c'_{ik} = r'_i \geq c_{in} + c_{ik}$ by assumption $r'_i \geq c_{ik} + c_{in}$

For $1 \leq i \leq n$ and $1 \leq j \leq n-1$, choose c'_{ij} such that $c'_{ij} \geq c_{ij}$,
which implies $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$
And under the same condition for i and j, choose r'_i such that $r'_i \geq c_{ik} + c_{in}$,
which implies $c_{ik} + c_{in} \leq c'_{ik} + c'_{in} = r'_i$;
C' is K-rooted matrix, then by Lemma (3.7) Conditions (ii) and (iii), there exists a k-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';
And since C is nonnegative, by Theorem (2.1), which is Perron–Frobenius theorem, we claim there exists $v^T = w^T P$, such that v^T is nonnegative left eigenvector of C, and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;
Due to v' and v^T are nonnegative, $v^T v' > 0.$, unless they are orthogonal, i.e, $v^T v' = 0$.

Here we can summarize the facts we know so far as the following:

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ and $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ for all $1 \leq i \leq n$;

(ii) there exists a K-rooted vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ and a scalar $\lambda' \in \mathbb{R}$ such that λ' is an eigenvalue of C' with associated eigenvector v';

(iii) there exists a nonnegative vector $v^T = (v_1, v_2, \ldots, v_n)$ and a scalar $\lambda \in \mathbb{R}$ such that λ is an eigenvalue of C with associated left eigenvector v^T;

(iv) $v^T v' > 0$.

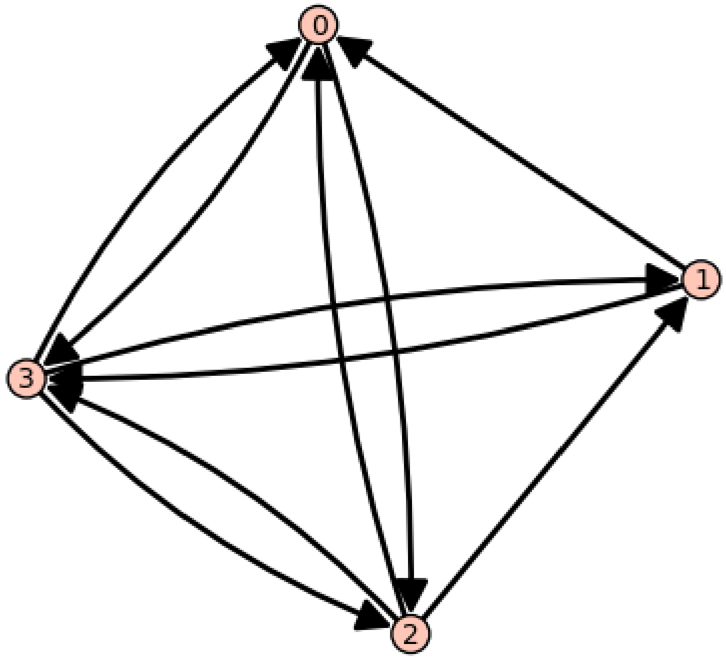
which come from Theorem (3.3).
By Lemma (3.8), for certain d, if $C' + dI$ is K-rooted, then it has a K-rooted eigenvector with its spectral radius $\lambda + d$. C' would share the same eigenvector with $C' + dI$ and has eigenvalue λ. So $C' + dI$ and $C + dI$ meet the conditions of Theorem (3.3), and we can show that $\rho(C' + dI) \geq \rho(C + dI)$ and then $\rho(C') \geq \rho(C)$ □

3.1 Example

For the following 4×4 matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

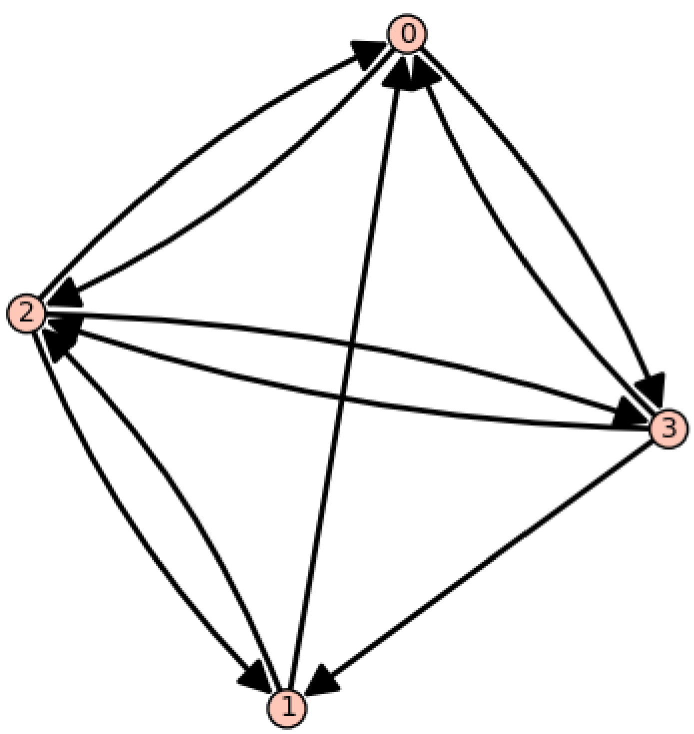
And its corresponding graph,[4, sage]



we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

also, graph,[4, sage]



Check the conditions $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$ of Theorem 3.3, Then $\rho(C) \leq \rho(C')$ by previous

Theorem 3.9.

3.2 Counterexample

For the following two 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

specify n=4, k=3 in $Q = I + E_{kn} = I + E_{34}$

$$CQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C'Q = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

we have $CQ \leq C'Q$, but $\rho(C) = 2.234 \not\leq 2.148 = \rho(C')$. This is because $c'_{33} + c'_{34} \not\geq c'_{43} + c'_{44}$.

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