TBA

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Preliminaries

Theorem

If C is nonnegative square matrix, then the following (i)-(iii) hold.

(i) *The spectral radius $\rho(C)$ is an eigenvalue of C with a corresponding nonnogative right eigenvector and a corresponding nonnegative left eigenvector.*

(ii) *If there exists a column vector $v > 0$, and a nongative numver λ such that $Cv \leq \lambda v$, then $\rho(C) \leq \lambda$.*

(iii) *If there exists a column vector $v \geq 0, v \neq 0$ and a nonnegative number λ such that $Cv \geq \lambda v$, then $\rho(C) \leq \lambda$.*

Preliminaries

The following theorem is from [].

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$, P *and* Q *be* $n \times n$ *matrices. Assume that*

(i) $PCQ \leq PC'Q$;

(ii) *there exist a nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T$ *and a scalar* $\lambda' \in \mathbb{R}$ *such that* λ' *is an eigenvalue of* C' *with associated eigenvector* Qu;

(iii) *there exist a nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n)$ *and a scalar* $\lambda \in \mathbb{R}$ *such that* λ *is an eigenvalue of* C *with associated left eigenvector* $v^T P$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

Proof

Multiplying the nonnegative vector u in (ii) to the right of both terms of (i),

$$PCQu \leq PC'Qu = \lambda' PQu. \tag{1}$$

Multiplying the nonnegative left eigenvector v^T of C for λ in assumption (iii) to the left of all terms in (**??**), we have

$$\lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu. \tag{2}$$

Now delete the positive term $v^T PQu$ by assumption (iv) to obtain $\lambda \leq \lambda'$ and finish the proof of the first part.

Proof (Continue)

Assume that $\lambda = \lambda'$, so the inequality in (**??**) is an equality. Especially $(PCQu)_i = (PC'Qu)_i$ for any i with $v_i \neq 0$. Hence $(PCQ)_{ij} = (PC'Q)_{ij}$ for any i with $v_i \neq 0$ and any j with $u_j \neq 0$. Conversely, (**??**) implies

$$v^T PCQu = \sum_{i,j} v_i (PCQ)_{ij} u_j = \sum_{i,j} v_i (PC'Q)_{ij} u_j = v^T PC'Qu,$$

so $\lambda = \lambda'$ by (**??**).

Our Method

Throughout fix $k \in [n-1]$. Let E_{kn} denote the $n \times n$ binary matrix with a unique 1 appearing in the position k, n of E_{kn}. We will apply the previous theorem with $P = I$ and

$$Q = I + E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & 1 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

k-rooted vector

Definition

A column vector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ is called *k-rooted* if $v'_j \geq 0$ for $1 \leq j \leq n$ and $v'_k \geq v'_n$.

The following Lemma is immediate from the above definition.

Lemma

vector rooted lemma If $u = (u_1, u_2, \ldots, u_n)^T$ *and* $v' = (v'_1, v'_2, \ldots, v'_n) := Qu = (u_1, \ldots, u_{k-1}, u_k + u_n, u_{k+1}, \ldots, u_n)^T$, *then*

(i) v' *is k-rooted if and only if* u *is nonnegative;*

(ii) $u_k > 0$ *if and only if* $v'_k > v'_n$.

□

Our first result

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ *be* $n \times n$ *matrices. Assume that*

(i) $C[-|n) \leq C'[-|n)$ *and* $c_{ik} + c_{in} \leq c'_{ik} + c'_{in}$ *for all* $1 \leq i \leq n$;

(ii) *there exist a k-rooted vector* $v' = (v'_1, v'_2, \ldots, v'_n)^T$ *and a scalar* $\lambda' \in \mathbb{R}$ *such that* λ' *is an eigenvalue of* C' *with associated eigenvector* v';

(iii) *there exists a nonnegative vector* $v^T = (v_1, v_2, \ldots, v_n)$ *and a scalar* $\lambda \in \mathbb{R}$ *such that* λ *is an eigenvalue of* C *with associated left eigenvector* v^T;

(iv) $v^T v' > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

(a) $c_{ik} + c_{in} = c'_{ik} + c'_{in}$ *for* $1 \leq i \leq n$ *with* $v_i \neq 0$ *and* $v'_n \neq 0$;

(b) $c'_{ij} = c_{ij}$ *for* $1 \leq i \leq n,\ 1 \leq j \leq n-1$ *with* $v_i \neq 0$ *and* $v'_j > v'_n$.

□

k-rooted matrix

Definition ((k,n)-sum)

For a matrix $C' = (c'_{ij})$ of n columns, the (k, n)-*sum* vector of C' is the vector of the sum of the k-th and n-th columns of C', where $k \leq n - 1$.

Definition (k-rooted matrix)

A matrix $C' = (c'_{ij})$ is called *k-rooted* if its columns and its (k, n)-sum vector are all k-rooted except the last column of C'.

Theorem

If C' is a k-rooted matrix, then $Q^{-1}C'Q$ is nonnegative, $\rho(C')$ is an eigenvalue of C', and C' has a k-rooted eigenvector $v' = Qu$ for $\rho(C')$, where u is a nonnegative eigenvector of $Q^{-1}C'Q$ for $\rho(C')$. Moreover, with $v' = (v'_1, v'_2, \ldots, v'_n)^T$, $r'_i = C'_{ik} + C'_{in}$ the following (i)-(ii) hold.

(i) *If $C'(k|n)$ is positive, then v' is positive.*

(ii) *If $C'(k|n)$ is positive and $r'_i > r'_n$ for all $1 \le i \le n-1$, then $v'_j > v'_n$ for all $1 \le j \le n-1$.*

Proof.
(i) suppose that $C'[n|n)$ is positive and $v'_n = 0$. Then

$$\sum_{j=1}^{n-1} c'_{nj} v'_j = \sum_{j=1}^{n} c'_{nj} v'_j = (C'v')_n = \rho(C')v'_n = 0.$$

Hence v' is a zero vector since $c'_{nj} > 0$ for $j \leq n-1$, a contradiction. So $v'_n > 0$ and $v' > 0$ since v' is rooted.
(ii) The assumptions imply that the matrix $Q^{-1}C'Q$ in (**??**) is irreducible. Hence u is positive. By Lemma **??**(ii), $v'_j > v'_n$ for $1 \leq j < n$. □

background

(i) $C^{'}$*Qu $= \lambda * Qu =$ Q * λu ....times Q^{-1} to the left

(ii) $Q^{-1}C^{'}Q$* u $= \lambda * u$

Here the λ is expected to be the spectral radius of $C^{'}$ so we wish that $Q^{-1}C^{'}Q > 0$, such that this condition $Q^{-1}C^{'}Q > 0$ allows that

(i) $C^{'}$ is k-rooted $\Leftrightarrow Q^{-1}C^{'}Q > 0$

(ii) $C^{'}$ is k-rooted $\Rightarrow C^{'}$ has a rooted eigenvector

Proof.

if $C^{'}$ is k-rooted, all columns and (k,n)-sum are rooted except the last one and then $c^{'}_{kj} - c^{'}_{nj} \geq 0$ and $c^{'}_{ij} \geq 0$ for $i \neq k, i < n$, for $1 \leq j \leq n-1$. So all columns except the last of $Q^{-1}C^{'}Q$ is nonnegative. □

Remark

$$Q^{-1} = I - E_{kn} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & -1 \\ & & & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}.$$

The matrix $Q^{-1}C'Q$ is

$$\begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & c'_{1k} + c'_{1n} \\ \vdots & & & & \\ c'_{k-11} & c'_{k-12} & \cdots & c'_{k-1n-1} & c'_{k-1k} + c'_{k-1n} \\ c'_{k1} - c'_{n1} & c'_{k2} - c'_{n2} & \cdots & c'_{kn-1} - c'_{nk-1} & c'_{kk} + c'_{kn} - c'_{nk} - x'_{nn} \\ c'_{k+11} & c'_{k+12} & \cdots & c'_{k+1\ n-1} & c'_{k+1k} + c'_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c'_{n1} & c'_{n2} & \cdots & c'_{n\ n-1} & c'_{nk} + c'_{nn} \end{pmatrix}.$$

Proof.
$C^{'}$ is k-rooted also implies that (k,n)-sum vector is k-rooted, the last column of $Q^{-1}C^{'}Q$ is nonnegative. conversely, if all the columns of $Q^{-1}C^{'}Q$ are nonegative, $C^{'}$ is k-rooted. $\square$

second, $C^{'}$ is k-rooted implies that the two postulations of the lemma, $C'(k|n)$ is positive and $r'_i > r'_n$ for all $1 \leq i \leq n-1$ also note that the conclusion of the lemma !!adapt it!! is v' is positive. and $v'_j > v'_n$ for all $1 \leq j \leq n-1$. ,that is, the eigenvector of $C^{'}$ is k-rooted.

application and some summantion

Theorem

maybe the form of Q, is able to generalize the condition of some matrices with same eigenvalues
or the choice of C' and Q might affect the spectral radius. the form of eigenvector transformed but the relation in reader's mind remain, so he or she doesn't comprehense the condition difference immediately.
$Q^{-1}C'Q > 0 \Rightarrow C'$ *has a rooted eigenvector*

Theorem

Let C be an $n \times n$ nonnegative matrix. For $1 \leq i \leq n$ and $1 \leq j \leq n-1$, choose r_i', c_{ij}' such that $c_{ij}' \geq c_{ij}$, $r_i' \geq c_{ik} + c_{in}$ and $c_{kj}' \geq c_{nj}' > 0$, and let $c_{in}' := r_i' - c_{ik}'$. Then the $n \times n$ matrix $C' = (c_{ij}')$ has a positive k-rooted eigenvector for $\rho(C')$ and $\rho(C) \leq \rho(C')$.

Proof

Example

For the following 4×4 matrix

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we choose

$$C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Then $\rho(C) \leq \rho(C')$ by previous theorem.

conclusion

Theorem