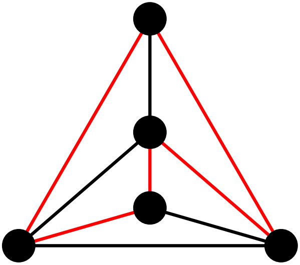
Hamiltonian graphs

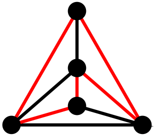
A graph is **Hamiltonian** if it contains a spanning cycle.



The red edges form a spanning cycle of the above graph

Definitions

1. A **triangulated graph** is a planar graph such that any edge is in a cycle and any face is a triangle.
2. A **chordal graph** is a graph such that every cycle of length at least 4 contains a **chord**, i.e., an edge connecting two nonadjacent vertices of the cycle.



Remark 0.1

A triangulated of order n at least 4 contains $3 + 3(n-3) = 3n - 6$ edges and $2n - 4$ faces.

Lemma

If C is a Hamiltonian cycle of a triangulated planar graph T of order n then there are $n - 2$ faces and $n - 3$ edges inside C.

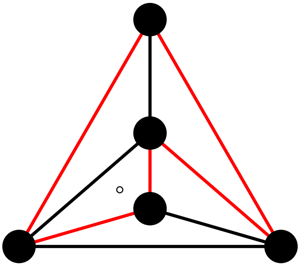
Proof.

□

Problem

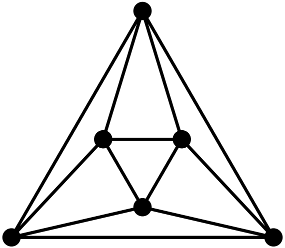
Can every edge of a triangulated graph be drawn as an edge of an outer face.

Answer: Yes

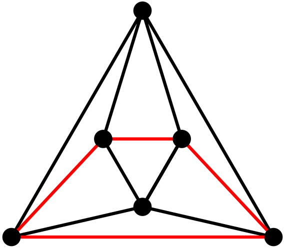


Plane is homeomorphic to the surface obtained from a sphere deleting a point.

A 4-regular triangulated graph



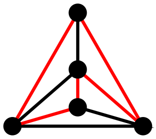
A triangulated graph which is not a chordal graph



There is no chord in the red cycle.

More definitions

1. A **weakly pancyclic graph** is a graph which contains cycles of every length between the girth and the circumference.
2. A graph is ***t*-tough** if for every integer $k > 1$, the graph cannot be split into k components by removal of fewer than tk vertices
3. The **toughness** a non-complete graph G is the maximum t such that G is t-tough. The toughness of a complete graph is ∞.



Remark 0.2

A t-tough graph with $t > 0$ is connected and a Hamiltonian graph is 1-tough.

A triangulated graph which is not weakly pancyclic

To be filled later.

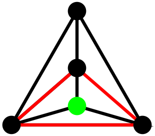
A long standing conjecture

There is a positive constant t such that every t-tough graph is Hamiltonian.

V. Chvátal, Tough graphs and hamiltonian circuits, Discrete Math. 5 (1973), 215-228.

Locally property P

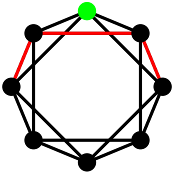
Let P be a property on graphs. A graph G has **local** P if $G_1(v)$ has property P for every vertex v in G.



Remark 0.3

A triangulated graph of order at least 4 is locally Hamiltonian.

A 1-tough and locally connected planar graph



The planar graph $DL(8; 1, 2)$ is 1-tough and locally connected.

Remark 0.4

$DL(n; 1, 2)$ is a planar graph, where $n \geq 4$ and n is even.

Remark 0.5

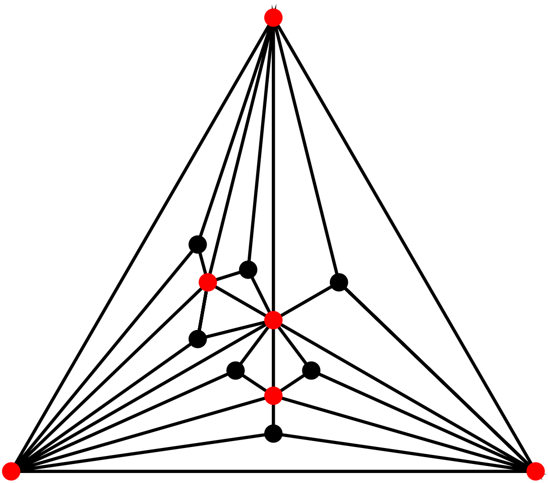
Every planar graph can be drawn by using straight lines as edges.

Question

Is a triangulated graph 1-tough?

The answer is ‘no’ as shown in the next page.

A triangulated graph which is not 1-tough



Deleting the 6 red vertices will yield 7 components.

Lemma

For a vertex u and a cut set S of G if $c(G-S) > |S|$ and $G-u$ is 1-tough then $u \notin S$ and $G_1(u) \subseteq S$.

Proof.

If $u \in S$ then $G-u$ has toughness at most $(|S|-1)/c(G-S) < 1$, a contradiction; if $u \notin S$ then $G-u$ being 1-tough and $|S|/c(G-S) < 1$ implies that there is a single element component $\{u\}$ in $G-S$, so $G_1(u) \subseteq S$. □

T. Nishizeki, A 1-tough nonhamiltonian maximal planar graph, Discrete Math. 30(1980), 305-307.

Proposition

Let G be a connected graph such that $G - u$ is Hamiltonian for every vertex in G. Then G is 1-tough.

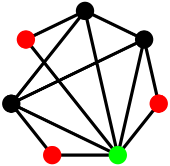
Proof.

Suppose G is not 1-tough and pick a cut set S with $c(G-S) > |S|$. As $G-u$ is 1-tough, $u \notin S$ by previous lemma. Hence $S = \emptyset$, a contradiction. $\square$

Question

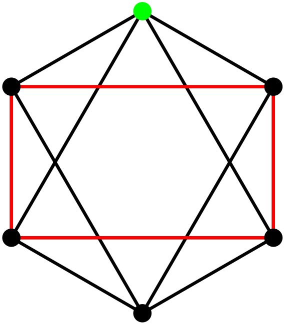
Is every 1-tough and locally connected graph Hamiltonian?

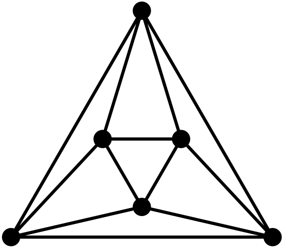
A 1-tough, locally connected, non-Hamiltonian planar graph



The graph is not Hamiltonian since the two edges of each red vertex are in every Hamiltonian cycle.

A 1-tough and locally Hamiltonian planar graph





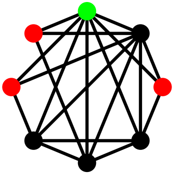
The planar graph $DL(6; 1, 2)$ is 1-tough and locally Hamiltonian, implying locally 1-tough. $DL(6; 1, 2)$ is a triangulated planar graph.

Question

Is every 1-tough and locally 1-tough graph locally Hamiltonian?

The answer is ‘no’ in the following page.

A 1-tough and locally 1-tough graph which is not locally Hamiltonian



Deleting the green vertex of the above graph yields a 1-tough non-Hamiltonian graph (The red vertices have degree 2).

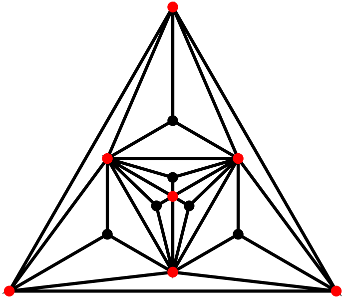
The above graph is not planar.

Question

Is a 1-tough and locally 1-tough graph Hamiltonian.

The answer is ‘no’ as shown in the next page.

A triangulated 1-tough non-Hamiltonian graph



Seven separating triangles in the above graph.

Michal Tkáč, On the shortness exponent of l-tough, maximal planar graphs, Discrete Math. 154(1996), 321-328.

Plan

We shall develop theory of 1-tough and locally 1-tough graph.

Lemma

If G is 1-tough and locally 1-tough then $G - u$ is 1-tough for every vertex u in G.

Proof.

Assume that $G - u$ is not 1-tough. Then $G - u$ is split into $k \geq 2$ components by removal of some subset S of vertices with $|S| < k$, where $u \notin S$. Assume ℓ of these k components intersecting with $G_1(u)$. If $\ell = 1$ then by connecting u to this unique component intersecting with $G_1(u)$, there are k components after removal of S from G, a contradiction to the 1-tough property of G.

Assume $\ell \geq 2$. Then $|S \cap G_1(u)| \geq \ell$ since $G_1(u)$ is 1-tough. Similarly by connecting u to these ℓ components intersecting with $G_1(u)$ to form a component, we find $k - \ell + 1$ components after the removal of $S - G_1(u)$ from G. Hence $|S - G_1(u)| \geq k - \ell + 1$ since G is 1-tough. Then $S = |S \cap G_1(u)| + |S - G_1(u)| \geq \ell + (k - \ell + 1) = k + 1$, a contradiction. $\square$

Lemma

If G is 1-tough and locally 1-tough then G has toughness greater than 1.

Proof.

Assume that G has toughness 1 and pick a subset S of k vertices whose removal will yield k components. Pick u in S. Then the removal of $k-1$ vertices in $S-\{u\}$ from $G-u$ will yield k components, a contradiction to $G-u$ being 1-tough by previous lemma. $\square$

Toughness of a 1-tough and locally 1-tough graph

By connecting a new vertex to the complete bipartite graph $K_{t,t}$, we have a 1-tough and locally 1-tough graph with toughness $(t+1)/t$ for evert $t \geq 1$. Hence the toughness of a 1-tough and locally 1-tough graph can approximate to 1.

Lemma

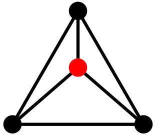
If G is 1-tough and locally 1-tough of order at least 4 then G has minimum degree at least 3.

Proof.

If G has minimum degree at most 2 then G has toughness at most $1 = 2/2$ because deleting the neighbors of a vertex with minimum degree yields at least two components. $\square$

Remark 0.6

There is only one 1-tough and locally 1-tough graph with maximum degree 3.



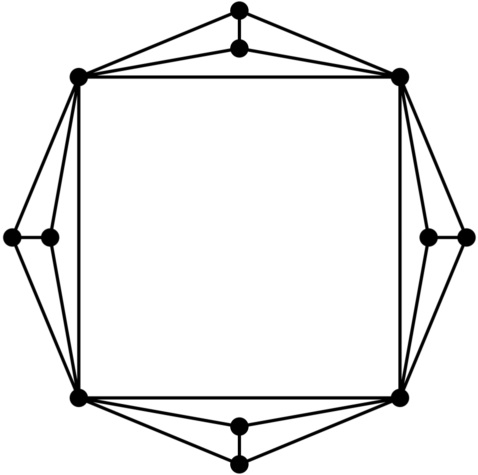
Lemma

If G is 1-tough and locally 1-tough of order at least 4 then every edge of G is in at least two triangles.

Proof.

Let $e = uv$ be an edge of G. Since u has degree at least 3, $G_1(u)$ is at least 3, so with the 1-tough property of $G_1(u)$, u has at least two neighbors in $G_1(u)$, forming two triangles containing the edge e. $\square$

Each edge is in two triangles



Each edge of the above non-triangulated graph is in two triangles. This graph is 1-tough, but not locally 1-tough.

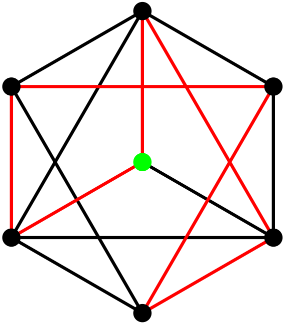
Proposition

If G is a 1-tough and locally 1-tough planar graph of order at least 4 and C is a cycle of maximum length in G, then each edge in C is in at least two triangles of the subgraph induced on the vertex set of C.

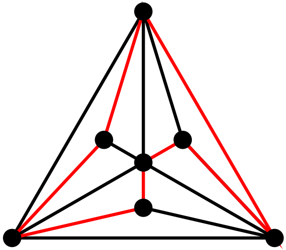
Proof.

If the two triangles of an edge in previous lemma contain a vertex not in C then we have a cycle of length one more than that of C, a contradiction to the maximum length assumption of C. $\square$

1-tough and locally Hamiltonian planar graph of order 7



The graph $DL(6;1,2)^+$.



A triangulated graph of order 7 (There are many different ones including $DL(6;1,2)^+$).

Problems

1. For $n \leq 7$, find non-triangulated planar Hamiltonian graphs of order n such that each edge in its Hamiltonian cycle is in two triangles.
2. How many edges are possible for a planar Hamiltonian graph of order n such that each edge in its Hamiltonian cycle is in two triangles?
3. Determine all the non-triangulated planar Hamiltonian graphs of order n such that each edge in its Hamiltonian cycle is in two triangles.
4. Find a 1-tough and locally 1-tough planar graph which is not triangulated.

Lemma

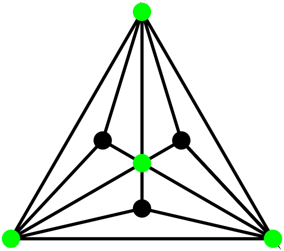
Let S be a cut set of a locally 1-tough connected graph G. If $x \in S$ is connected to at least two components in $G - S$ then x has degree at least two in the subgraph induced on S.

Proof.

The 1-tough of $G_1(x)$ implies $|G_1(x) \cap S|$ is at least the number of components in $G - S$ that are connected to x. $\square$

A cut set S of G is **effective** if each $x \in S$ is connected to at least two components in $G - S$.

Effective cut set



The green vertices form an effective cut set.