1 Evolution of operators

a. Time evolution of fermionic annihilator

Solution: Our hamiltonian for this problem is

$$H_0 = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma}. \tag{1}$$

We can prove trivially by using Taylor series for t that

$$[e^{\pm iH_0t}, H_0] = 0. \tag{2}$$

If we take the following relation

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC, \tag{3}$$

we can easily calculate the next commutator

$$\begin{aligned}
[e^{iH_0t}c_{k\sigma}e^{-iH_0t}, H_0] &= e^{iH_0t}c_{k\sigma}\cancelto{0}{[e^{-iH_0t}, H_0]} + e^{iH_0t}[c_{k\sigma}, H_0]e^{-iH_0t} + \cancelto{0}{[e^{iH_0t}, H_0]}c_{k\sigma}e^{-iH_0t}, \\
&= e^{iH_0t}[c_{k\sigma}, H_0]e^{-iH_0t}.
\end{aligned} \tag{4}$$

Finally we can use the relation

$$[A, BC] = B[A, C] + [A, B]C, \tag{5}$$

to calculate the following commutator

$$\begin{aligned}
[c_{k\sigma}, c^\dagger_{j\gamma}c_{j\gamma}] &= c^\dagger_{j\gamma}[c_{k\sigma}, c_{j\gamma}] + [c_{k\sigma}, c^\dagger_{j\gamma}]c_{j\gamma}, \\
&= c^\dagger_{j\gamma}(c_{k\sigma}c_{j\gamma} - c_{j\gamma}c_{k\sigma}) + (c_{k\sigma}c^\dagger_{j\gamma} - c^\dagger_{j\gamma}c_{k\sigma})c_{j\gamma}, \\
&= c^\dagger_{j\gamma}c_{k\sigma}c_{j\gamma} + c_{k\sigma}c^\dagger_{j\gamma}c_{j\gamma} - (c^\dagger_{j\gamma}c_{j\gamma}c_{k\sigma} + c^\dagger_{j\gamma}c_{k\sigma}c_{j\gamma}), \\
&= \cancelto{\delta_{jk}\delta_{\sigma\gamma}}{\{c^\dagger_{j\gamma}, c_{k\sigma}\}}c_{j\gamma} - c^\dagger_{j\gamma}\cancelto{0}{\{c_{j\gamma}c_{k\sigma}\}}, \\
&= \delta_{jk}\delta_{\sigma\gamma}c_{j\gamma}.
\end{aligned} \tag{6}$$

Where in the last step we used the anti-commutation rules of the fermionic creation and annihilation operators.

Now we can finally start. Let's consider the time evolution equation in the Heisenberg picture

$$i\frac{d}{dt}c_{k\sigma}(t) = [c_{k\sigma}(t), H_0], \tag{7}$$

taking

$$c_{k\sigma}(t) = e^{iH_0t}c_{k\sigma}e^{-iH_0t}, \tag{8}$$

the time evolution equation takes the form

$$i\frac{d}{dt}c_{k\sigma}(t) = [e^{iH_0t}c_{k\sigma}e^{-iH_0t}, H_0]. \tag{9}$$

Luckily we can use eq. 4 to move further and obtain

$$i\frac{d}{dt}c_{k\sigma}(t) = e^{iH_0t}[c_{k\sigma}, H_0]e^{-iH_0t}. \tag{10}$$

Now is time to plug the hamiltoninan into this commutator

$$i\frac{d}{dt}c_{k\sigma}(t) = e^{iH_0t}[c_{k\sigma}, \sum_{k\sigma}\varepsilon_k c^\dagger_{k\sigma}c_{k\sigma}]e^{-iH_0t}, \tag{11}$$

since the commutator is linear and $\varepsilon_k \in \mathbb{R}$ we can transform the equation to this one

$$i\frac{d}{dt}c_{k\sigma}(t) = e^{iH_0t}\sum_{k\sigma}\varepsilon_k[c_{k\sigma}, c^\dagger_{k\sigma}c_{k\sigma}]e^{-iH_0t}, \tag{12}$$

and at this point we are lucky again, because now we can use eq. 6 to continue our calculations

$$\begin{aligned} i\frac{d}{dt}c_{k\sigma}(t) &= e^{iH_0t}\sum_{k\sigma}\varepsilon_k\delta_{jk}\delta_{\sigma\gamma}c_{j\gamma}e^{-iH_0t}, \\ &= e^{iH_0t}\varepsilon_k c_{k\sigma}e^{-iH_0t}, \\ &= \varepsilon_k e^{iH_0t}c_{k\sigma}e^{-iH_0t}, \\ &= \varepsilon_k c_{k\sigma}(t), \end{aligned} \tag{13}$$

therefore

$$\frac{d}{dt}c_{k\sigma}(t) = -i\varepsilon_k c_{k\sigma}(t), \tag{14}$$

at this point the equation looks very familiar, and *if it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.* To make sure we suppose it is

$$c_{k\sigma}(t) = e^{\alpha t}c_{k\sigma}. \tag{15}$$

If we differentiate this operator with respect to t, we will have

$$\frac{d}{dt}c_{k\sigma}(t) = \frac{d}{dt}e^{\alpha t}c_{k\sigma} = \alpha e^{\alpha t}c_{k\sigma} = \alpha c_{k\sigma}(t). \tag{16}$$

Comparing eqs. 14 and 16 we finally get that

$$\alpha = -i\varepsilon_k, \tag{17}$$

so finally, our solution will be

$$c_{k\sigma}(t) = e^{-i\varepsilon_k t}c_{k\sigma}. \quad \blacksquare \tag{18}$$

There are two ways of getting the creation operator, the hard one would be to repeat similar calculations with $c^\dagger_{k\sigma}$, which we will not perform here. The easy one is just to take

$$c^\dagger_{k\sigma}(t) = (c_{k\sigma}(t))^\dagger = (e^{-i\varepsilon_k t}c_{k\sigma})^\dagger = e^{i\varepsilon_k t}c^\dagger_{k\sigma}. \tag{19}$$

2 Non-interacting Green function

a. Project exercise II

Solution:

3 Matsubara sums and chemical potential