1 Linear combination of fermionic operators

a. Show that those operators are fermionic

Solution: By definition, fermion operators must satisfy the following anti-commutation rules

- $\{a_{\alpha,\sigma}, a_{\alpha',\sigma'}\} = 0.$
- $\{a^\dagger_{\alpha,\sigma}, a^\dagger_{\alpha',\sigma'}\} = 0.$
- $\{a_{\alpha,\sigma}, a^\dagger_{\alpha',\sigma'}\} = \delta_{\alpha,\alpha'}\delta_{\sigma,\sigma'}.$

Now we are going to prove that if $\hat{c}_{m,\zeta}$ is a fermion operator then the linear combinations

$$\hat{d}_\mu = \sum_m \alpha_m \hat{c}_{m,\mu}, \tag{1}$$

$$\hat{d}^\dagger_\mu = \sum_m \alpha^*_m \hat{c}^\dagger_{m,\mu}, \tag{2}$$

are fermion operators too.

So we are going to prove they obey the previously mentioned anti-commutation rules. We will use the linearity property of the anti-commutator shown in eq. 3

$$\{\alpha A + \beta B, \gamma C + \delta D\} = \{\alpha A, \gamma C\} + \{\alpha A, \delta D\} + \{\beta B, \gamma C\} + \{\beta B, \delta D\}. \tag{3}$$

Let's go then,

$$\begin{aligned}
\{\hat{d}_\mu, \hat{d}_\nu\} &= \left\{\sum_m \alpha_m \hat{c}_{m,\mu}, \sum_n \alpha_n \hat{c}_{n,\nu}\right\}, \\
&= \sum_{m,n} \alpha_m \alpha_n \cancelto{0}{\{\hat{c}_{m,\mu}, \hat{c}_{n,\nu}\}}, \\
&= 0.
\end{aligned} \tag{4}$$

$$\begin{aligned}
\{\hat{d}^\dagger_\mu, \hat{d}^\dagger_\nu\} &= \left\{\sum_m \alpha^*_m \hat{c}^\dagger_{m,\mu}, \sum_n \alpha^*_n \hat{c}^\dagger_{n,\nu}\right\}, \\
&= \sum_{m,n} \alpha^*_m \alpha^*_n \cancelto{0}{\{\hat{c}^\dagger_{m,\mu}, \hat{c}^\dagger_{n,\nu}\}}, \\
&= 0.
\end{aligned} \tag{5}$$

$$\begin{aligned}
\{\hat{d}_\mu, \hat{d}^\dagger_\nu\} &= \left\{\sum_m \alpha_m \hat{c}_{m,\mu}, \sum_n \alpha^*_n \hat{c}^\dagger_{n,\nu}\right\}, \\
&= \sum_{m,n} \alpha_m \alpha^*_n \cancelto{\delta_{m,n}\delta_{\mu,\nu}}{\{\hat{c}_{m,\mu}, \hat{c}^\dagger_{n,\nu}\}}, \\
&= \sum_{m,n} \alpha_m \alpha^*_n \delta_{m,n}\delta_{\mu,\nu}, \\
&= \cancelto{1}{\sum_m \alpha_m \alpha^*_m} \delta_{\mu,\nu}, \\
&= \delta_{\mu,\nu}.
\end{aligned} \tag{6}$$

Where we used the fact that $\hat{c}$ are fermion operators, and in the last one we used additionally $\sum_m \alpha_m^2 = 1$. Since the $\hat{d}$ operators comply with the anti-commutation rules for fermion operators we conclude that they are indeed fermion operators. ■

b. Majorana operators

Solution: Now we are going to prove basically the same but for the next operators (I changed the labels to make the life easier for me)

$$\hat{\gamma}_+ = \frac{1}{\sqrt{2}}\left(\hat{d} + \hat{d}^\dagger\right) \tag{7}$$

$$\hat{\gamma}_- = \frac{i}{\sqrt{2}}\left(\hat{d} - \hat{d}^\dagger\right) \tag{8}$$

$$\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\pm\} &= \left\{\frac{\alpha_\pm}{\sqrt{2}}\left(\hat{d} \pm \hat{d}^\dagger\right), \frac{\alpha_\pm}{\sqrt{2}}\left(\hat{d} \pm \hat{d}^\dagger\right)\right\}, \\
&= \frac{\cancelto{\pm 1}{\alpha_\pm^2}}{2}\left\{\hat{d} \pm \hat{d}^\dagger, \hat{d} \pm \hat{d}^\dagger\right\}, \\
&= \frac{\pm 1}{2}\left(\cancelto{0}{\left\{\hat{d}, \hat{d}\right\}} \pm \cancelto{1}{\left\{\hat{d}, \hat{d}^\dagger\right\}} \pm \cancelto{1}{\left\{\hat{d}^\dagger, \hat{d}\right\}} + \cancelto{0}{\left\{\hat{d}^\dagger, \hat{d}^\dagger\right\}}\right), \\
&= 1.
\end{aligned} \tag{9}$$

Where $\alpha_+ = 1$, and $\alpha_- = i$.

$$\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\mp\} &= \left\{\frac{\alpha_\pm}{\sqrt{2}}\left(\hat{d} \pm \hat{d}^\dagger\right), \frac{\alpha_\mp}{\sqrt{2}}\left(\hat{d} \mp \hat{d}^\dagger\right)\right\}, \\
&= \frac{\cancelto{i}{\alpha_\pm^2}}{2}\left\{\hat{d} \pm \hat{d}^\dagger, \hat{d} \mp \hat{d}^\dagger\right\}, \\
&= \frac{i}{2}\left(\cancelto{0}{\left\{\hat{d}, \hat{d}\right\}} \pm \cancelto{1}{\left\{\hat{d}, \hat{d}^\dagger\right\}} \mp \cancelto{1}{\left\{\hat{d}^\dagger, \hat{d}\right\}} + \cancelto{0}{\left\{\hat{d}^\dagger, \hat{d}^\dagger\right\}}\right), \\
&= 0.
\end{aligned} \tag{10}$$

As a conclusion we see that

$$\boxed{\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = \delta_{\mu\nu}.} \tag{11}$$

Finally we need to express the number operator in terms of $\hat{\gamma}_+$ and $\hat{\gamma}_-$. The trick we are going to use here is an old known if we have solved the harmonic oscillator with creation and annihilation operators. We can easily show that

$$\hat{d}^\dagger = \frac{\hat{\gamma}_+ + i\hat{\gamma}_-}{\sqrt{2}}, \tag{12}$$

$$\hat{d} = \frac{\hat{\gamma}_+ - i\hat{\gamma}_-}{\sqrt{2}}, \tag{13}$$

therefore

$$
\begin{aligned}
\hat{n} &= \hat{d}^\dagger \hat{d}, \\
&= \left(\frac{\hat{\gamma}_+ + i\hat{\gamma}_-}{\sqrt{2}}\right)\left(\frac{\hat{\gamma}_+ - i\hat{\gamma}_-}{\sqrt{2}}\right), \\
&= \frac{(\hat{\gamma}_+ + i\hat{\gamma}_-)(\hat{\gamma}_+ - i\hat{\gamma}_-)}{2}, \\
&= \frac{\hat{\gamma}_+\hat{\gamma}_+ - i\hat{\gamma}_+\hat{\gamma}_- + i\hat{\gamma}_-\hat{\gamma}_+ + \hat{\gamma}_-\hat{\gamma}_-}{2}
\end{aligned} \tag{14}
$$

If we use what we know from eq. 11 we will notice that

$$
\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\mp\} = \hat{\gamma}_\pm\hat{\gamma}_\mp + \hat{\gamma}_\mp\hat{\gamma}_\pm = 0, \\
\Rightarrow \hat{\gamma}_\pm\hat{\gamma}_\mp = -\hat{\gamma}_\mp\hat{\gamma}_\pm,
\end{aligned} \tag{15}
$$

and also

$$
\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\pm\} = \hat{\gamma}_\pm\hat{\gamma}_\pm + \hat{\gamma}_\pm\hat{\gamma}_\pm = 2\hat{\gamma}_\pm\hat{\gamma}_\pm = 1, \\
\Rightarrow \hat{\gamma}_\pm\hat{\gamma}_\pm = \frac{1}{2}.
\end{aligned} \tag{16}
$$

Using eqs. 15-16 in eq. 17 we will have

$$
\begin{aligned}
\hat{n} &= \frac{\hat{\gamma}_+\hat{\gamma}_+ - i\hat{\gamma}_+\hat{\gamma}_- + i\hat{\gamma}_-\hat{\gamma}_+ + \hat{\gamma}_-\hat{\gamma}_-}{2} \\
&= \frac{1/2 + 2i\hat{\gamma}_-\hat{\gamma}_+ + 1/2}{2} \\
&= \frac{1}{2} + i\hat{\gamma}_-\hat{\gamma}_+.
\end{aligned} \tag{17}
$$

2 Spin and spin density operators

a. Spin

Solution: The first step we need to take is expanding the field operators on the given basis

$$\Phi_\sigma(\vec{r}) = \sum_k \psi_k(\vec{r}) c_{k,\sigma}, \tag{18}$$

$$\Phi_\sigma(\vec{r}) = \sum_{n,l,m} \psi_{n,l,m}(\vec{r}) c_{n,l,m,\sigma} \tag{19}$$

And their respective dagger operators. From Fetter and Walecka we know how to obtain the second-quantized operator from the first-quantization operator.

$$\hat{s}_i = \sum_{\sigma,\sigma'} \Phi^\dagger_\sigma(\vec{r}) s_i \Phi_{\sigma'}(\vec{r}), \tag{20}$$

inserting eq. 18 into eq. 20, and pulling up the spin index explicitly, yields

$$
\begin{aligned}
\hat{s}_i &= \sum_{k,k',\sigma,\sigma'} \langle\psi_k| \langle\sigma| s_i |\sigma'\rangle |\psi_{k'}\rangle c^\dagger_{k,\sigma} c_{k',\sigma'}, \\
&= \sum_{k,k',\sigma,\sigma'} \langle\psi_k|\psi_{k'}\rangle \langle\sigma| s_i |\sigma'\rangle c^\dagger_{k,\sigma} c_{k',\sigma'}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k,k',\sigma,\sigma'} \delta_k^{k'} \langle\sigma| s_i |\sigma'\rangle c^\dagger_{k,\sigma} c_{k',\sigma'} \\
&= \boxed{\sum_{k,\sigma,\sigma'} \langle\sigma| s_i |\sigma'\rangle c^\dagger_{k,\sigma} c_{k',\sigma'}}\,. \qquad (21)
\end{aligned}
$$

We can use the same trick, but now plugging eq. 19 into eq. 20

$$
\begin{aligned}
\hat{s_i} &= \sum \langle\psi_{n,l,m}| \langle\sigma| s_i |\sigma'\rangle |\psi_{n',l',m'}\rangle c^\dagger_{n,l,m,\sigma} c_{n',l',m',\sigma'}, \\
&= \sum \langle\psi_{n,l,m}|\psi_{n',l',m'}\rangle \langle\sigma| s_i |\sigma'\rangle c^\dagger_{n,l,m,\sigma} c_{n',l',m',\sigma'}, \\
&= \sum \delta_{nlm}^{n'l'm'} \langle\sigma| s_i |\sigma'\rangle c^\dagger_{n,l,m,\sigma} c_{n',l',m',\sigma'}, \\
&= \boxed{\sum_{nlm\sigma\sigma'} \langle\sigma| s_i |\sigma'\rangle c^\dagger_{n,l,m,\sigma} c_{n,l,m,\sigma'}.} \qquad (22)
\end{aligned}
$$

Where the first three sums were performed over $n, l, m, \sigma, n', l', m', \sigma'$. The highlighted equations are indeed the total spin operators for the two chosen basis.

3 Θ step function

a. Heaviside step

Solution: Let's analyze the given integral

$$\Theta(x) = \int_{-\infty}^{x} \delta(y) dy. \qquad (23)$$

From the hint we know that

$$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x - x_0) dx, \qquad (24)$$

moreover, we know that if $x_0 \in (a, b)$ then

$$f(x_0) = \int_{a}^{b} f(x)\delta(x - x_0) dx, \qquad (25)$$

if we compare eqs. 25 and 23 we will see that $x_0 = 0$, and $f(x) := 1$. Finally, let's remember that $\delta(x - x_0)$ is zero for all x except x_0. Now we are ready to solve this. if $0 \notin (-\infty, x)$ then $\Theta(x) = 0$, because $\delta(y) = 0, \quad \forall y \in (-\infty, x)$. If $0 \in (-\infty, x)$ then using eq. 25 we conclude that $\Theta(x) = f(x) = 1$. Hence

$$\Theta(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases} \qquad (26)$$

We can see the plot in fig. 1

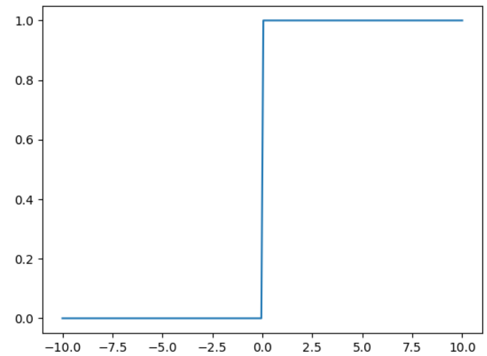

Figure 1: Plot of the function defined in eq. 23.

b. Fermi function

Solution: Consider the Fermi function

$$n(x) = \frac{1}{1 + e^{(x-\mu)/k_B T}}. \tag{27}$$

And let us analyze the behavior of $n(x)$. First consider the case $x < \mu$, then $x - \mu < 0$. If $T \to 0$ then the factor $(x-\mu)/k_B T \to -\infty$, and $e^{(x-\mu)/k_B T} \to 0$ and as a consequence $1/(1 + e^{(x-\mu)/k_B T}) \to 1$. On the other hand if $x - \mu > 0$ the factor $(x-\mu)/k_B T \to \infty$ as $T \to 0$, and therefore $e^{(x-\mu)/k_B T} \to \infty$ causing $1/(1 + e^{(x-\mu)/k_B T}) \to 0$

So we see that in the limit $T \to 0$

$$n(x) = \begin{cases} 1 & x - \mu < 0, \\ 0 & x - \mu > 0. \end{cases} \tag{28}$$

Now just multiplying the conditions by a -1 factor we will have

$$n(x) = \Theta(\mu - x) \begin{cases} 0 & \mu - x < 0, \\ 1 & \mu - x > 0. \end{cases} \quad \blacksquare \tag{29}$$

Now let's take the derivative of eq. 27.

$$\begin{aligned} \frac{d}{dx}\left(1 + e^{(x-\mu)/k_B T}\right)^{-1} &= -(1 + e^{(x-\mu)/k_B T})^2 e^{(x-\mu)/k_B T} \frac{1}{k_B T}, \\ &= -\frac{e^{(x-\mu)/k_B T}}{k_B T (1 + e^{(x-\mu)/k_B T})^2}. \end{aligned} \tag{30}$$

if $x - \mu < 0$ then

$$\lim_{T \to 0} e^{(x-\mu)/k_B T} = 0, \tag{31}$$

$$\lim_{T \to 0} \frac{1}{(1 + e^{(x-\mu)/k_B T})^2} = 1 \tag{32}$$

$$\lim_{T \to 0} \frac{e^{(x-\mu)/k_B T}}{k_B T} = 0, \tag{33}$$

and as a consequence

$$\boxed{\lim_{T\to 0} \frac{dn(x)}{dx} = 0.} \tag{34}$$

If $x - \mu > 0$ then

$$\lim_{T\to 0} e^{(x-\mu)/k_B T} = \infty, \tag{35}$$

$$\lim_{T\to 0} \frac{1}{(1+e^{(x-\mu)/k_B T})^2} \approx \lim_{T\to 0} \frac{1}{(e^{(x-\mu)/k_B T})^2} \tag{36}$$

$$\lim_{T\to 0} -\frac{e^{(x-\mu)/k_B T}}{k_B T(1+e^{(x-\mu)/k_B T})^2}. \approx \lim_{T\to 0} -\frac{1}{k_B T e^{(x-\mu)/k_B T}} = 0. \tag{37}$$

and as a consequence

$$\boxed{\lim_{T\to 0} \frac{dn(x)}{dx} = 0.} \tag{38}$$

So, **for both cases the derivative tends to zero.**