1 Linear combination of fermionic operators

a. Show that those operators are fermionic

Solution: By definition, fermion operators must satisfy the following anti-commutation rules

- $\{a_{\alpha,\sigma}, a_{\alpha',\sigma'}\} = 0.$
- $\{a^\dagger_{\alpha,\sigma}, a^\dagger_{\alpha',\sigma'}\} = 0.$
- $\{a_{\alpha,\sigma}, a^\dagger_{\alpha',\sigma'}\} = \delta_{\alpha,\alpha'}\delta_{\sigma,\sigma'}.$

Now we are going to prove that if $\hat{c}_{m,\zeta}$ is a fermion operator then the linear combinations

$$\hat{d}_\mu = \sum_m \alpha_m \hat{c}_{m,\mu}, \tag{1}$$

$$\hat{d}^\dagger_\mu = \sum_m \alpha^*_m \hat{c}^\dagger_{m,\mu}, \tag{2}$$

are fermion operators too.

So we are going to prove they obey the previously mentioned anti-commutation rules. We will use the linearity property of the anti-commutator shown in eq. 3

$$\{\alpha A + \beta B, \gamma C + \delta D\} = \{\alpha A, \gamma C\} + \{\alpha A, \delta D\} + \{\beta B, \gamma C\} + \{\beta B, \delta D\}. \tag{3}$$

Let's go then,

$$\begin{aligned}
\{\hat{d}_\mu, \hat{d}_\nu\} &= \left\{\sum_m \alpha_m \hat{c}_{m,\mu}, \sum_n \alpha_n \hat{c}_{n,\nu}\right\}, \\
&= \sum_{m,n} \alpha_m \alpha_n \cancelto{0}{\{\hat{c}_{m,\mu}, \hat{c}_{n,\nu}\}}, \\
&= 0.
\end{aligned} \tag{4}$$

$$\begin{aligned}
\{\hat{d}^\dagger_\mu, \hat{d}^\dagger_\nu\} &= \left\{\sum_m \alpha^*_m \hat{c}^\dagger_{m,\mu}, \sum_n \alpha^*_n \hat{c}^\dagger_{n,\nu}\right\}, \\
&= \sum_{m,n} \alpha^*_m \alpha^*_n \cancelto{0}{\{\hat{c}^\dagger_{m,\mu}, \hat{c}^\dagger_{n,\nu}\}}, \\
&= 0.
\end{aligned} \tag{5}$$

$$\begin{aligned}
\{\hat{d}_\mu, \hat{d}^\dagger_\nu\} &= \left\{\sum_m \alpha_m \hat{c}_{m,\mu}, \sum_n \alpha^*_n \hat{c}^\dagger_{n,\nu}\right\}, \\
&= \sum_{m,n} \alpha_m \alpha^*_n \cancelto{\delta_{m,n}\delta_{\mu,\nu}}{\{\hat{c}_{m,\mu}, \hat{c}^\dagger_{n,\nu}\}}, \\
&= \sum_{m,n} \alpha_m \alpha^*_n \delta_{m,n}\delta_{\mu,\nu}, \\
&= \cancelto{1}{\sum_m \alpha_m \alpha^*_m} \delta_{\mu,\nu}, \\
&= \delta_{\mu,\nu}.
\end{aligned} \tag{6}$$

Where we used the fact that $\hat{c}$ are fermion operators, and in the last one we used additionally $\sum_m \alpha_m^2 = 1$. Since the $\hat{d}$ operators comply with the anti-commutation rules for fermion operators we conclude that they are indeed fermion operators. ■

b. Majorana operators

Solution: Now we are going to prove basically the same but for the next operators (I changed the labels to make the life easier for me)

$$\hat{\gamma}_+ = \frac{1}{\sqrt{2}}\left(\hat{d} + \hat{d}^\dagger\right) \tag{7}$$

$$\hat{\gamma}_- = \frac{i}{\sqrt{2}}\left(\hat{d} - \hat{d}^\dagger\right) \tag{8}$$

$$\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\pm\} &= \left\{\frac{\alpha_\pm}{\sqrt{2}}\left(\hat{d} \pm \hat{d}^\dagger\right), \frac{\alpha_\pm}{\sqrt{2}}\left(\hat{d} \pm \hat{d}^\dagger\right)\right\}, \\
&= \frac{\cancelto{\pm 1}{\alpha_\pm^2}}{2}\left\{\hat{d} \pm \hat{d}^\dagger, \hat{d} \pm \hat{d}^\dagger\right\}, \\
&= \frac{\pm 1}{2}\left(\cancelto{0}{\left\{\hat{d}, \hat{d}\right\}} \pm \cancelto{1}{\left\{\hat{d}, \hat{d}^\dagger\right\}} \pm \cancelto{1}{\left\{\hat{d}^\dagger, \hat{d}\right\}} + \cancelto{0}{\left\{\hat{d}^\dagger, \hat{d}^\dagger\right\}}\right), \\
&= 1.
\end{aligned} \tag{9}$$

Where $\alpha_+ = 1$, and $\alpha_- = i$.

$$\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\mp\} &= \left\{\frac{\alpha_\pm}{\sqrt{2}}\left(\hat{d} \pm \hat{d}^\dagger\right), \frac{\alpha_\mp}{\sqrt{2}}\left(\hat{d} \mp \hat{d}^\dagger\right)\right\}, \\
&= \frac{\cancelto{i}{\alpha_\pm^2}}{2}\left\{\hat{d} \pm \hat{d}^\dagger, \hat{d} \mp \hat{d}^\dagger\right\}, \\
&= \frac{i}{2}\left(\cancelto{0}{\left\{\hat{d}, \hat{d}\right\}} \pm \cancelto{1}{\left\{\hat{d}, \hat{d}^\dagger\right\}} \mp \cancelto{1}{\left\{\hat{d}^\dagger, \hat{d}\right\}} + \cancelto{0}{\left\{\hat{d}^\dagger, \hat{d}^\dagger\right\}}\right), \\
&= 0.
\end{aligned} \tag{10}$$

As a conclusion we see that

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = \delta_{\mu\nu}. \tag{11}$$

Finally we need to express the number operator in terms of $\hat{\gamma}_+$ and $\hat{\gamma}_-$. The trick we are going to use here is an old known if we have solved the harmonic oscillator with creation and annihilation operators. We can easily show that

$$\hat{d}^\dagger = \frac{\hat{\gamma}_+ + i\hat{\gamma}_-}{\sqrt{2}}, \tag{12}$$

$$\hat{d} = \frac{\hat{\gamma}_+ - i\hat{\gamma}_-}{\sqrt{2}}, \tag{13}$$

therefore

$$
\begin{aligned}
\hat{n} &= \hat{d}^\dagger \hat{d}, \\
&= \left(\frac{\hat{\gamma}_+ + i\hat{\gamma}_-}{\sqrt{2}}\right)\left(\frac{\hat{\gamma}_+ - i\hat{\gamma}_-}{\sqrt{2}}\right), \\
&= \frac{(\hat{\gamma}_+ + i\hat{\gamma}_-)(\hat{\gamma}_+ - i\hat{\gamma}_-)}{2}, \\
&= \frac{\hat{\gamma}_+\hat{\gamma}_+ - i\hat{\gamma}_+\hat{\gamma}_- + i\hat{\gamma}_-\hat{\gamma}_+ + \hat{\gamma}_-\hat{\gamma}_-}{2}
\end{aligned} \tag{14}
$$

If we use what we know from eq. 11 we will notice that

$$
\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\mp\} = \hat{\gamma}_\pm\hat{\gamma}_\mp + \hat{\gamma}_\mp\hat{\gamma}_\pm = 0, \\
\Rightarrow \hat{\gamma}_\pm\hat{\gamma}_\mp = -\hat{\gamma}_\mp\hat{\gamma}_\pm,
\end{aligned} \tag{15}
$$

and also

$$
\begin{aligned}
\{\hat{\gamma}_\pm, \hat{\gamma}_\pm\} = \hat{\gamma}_\pm\hat{\gamma}_\pm + \hat{\gamma}_\pm\hat{\gamma}_\pm = 2\hat{\gamma}_\pm\hat{\gamma}_\pm = 1, \\
\Rightarrow \hat{\gamma}_\pm\hat{\gamma}_\pm = \frac{1}{2}.
\end{aligned} \tag{16}
$$

Using eqs. 15-16 in eq. 17 we will have

$$
\begin{aligned}
\hat{n} &= \frac{\hat{\gamma}_+\hat{\gamma}_+ - i\hat{\gamma}_+\hat{\gamma}_- + i\hat{\gamma}_-\hat{\gamma}_+ + \hat{\gamma}_-\hat{\gamma}_-}{2} \\
&= \frac{1/2 + 2i\hat{\gamma}_-\hat{\gamma}_+ + 1/2}{2} \\
&= \frac{1}{2} + i\hat{\gamma}_-\hat{\gamma}_+.
\end{aligned} \tag{17}
$$

2 Spin and spin density operators

3 Θ step function