1   Evolution of operators

a. Time evolution of fermionic annihilator

Solution: Our hamiltonian for this problem is

$$H_0 = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma}. \tag{1}$$

We can prove trivially by using Taylor series for t that

$$[e^{\pm iH_0 t}, H_0] = 0. \tag{2}$$

If we take the following relation

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC, \tag{3}$$

we can easily calculate the next commutator

$$[e^{iH_0 t} c_{k\sigma} e^{-iH_0 t}, H_0] = e^{iH_0 t} c_{k\sigma} \overbrace{[e^{-iH_0 t}, H_0]}^{0} + e^{iH_0 t}[c_{k\sigma}, H_0]e^{-iH_0 t} + \overbrace{[e^{iH_0 t}, H_0]}^{0} c_{k\sigma} e^{-iH_0 t},$$
$$= e^{iH_0 t}[c_{k\sigma}, H_0]e^{-iH_0 t}. \tag{4}$$

Finally we can use the relation

$$[A, BC] = B[A, C] + [A, B]C, \tag{5}$$

to calculate the following commutator

$$[c_{k\sigma}, c_{j\gamma}^\dagger c_{j\gamma}] = c_{j\gamma}^\dagger[c_{k\sigma}, c_{j\gamma}] + [c_{k\sigma}, c_{j\gamma}^\dagger]c_{j\gamma},$$
$$= c_{j\gamma}^\dagger(c_{k\sigma}c_{j\gamma} - c_{j\gamma}c_{k\sigma}) + (c_{k\sigma}c_{j\gamma}^\dagger - c_{j\gamma}^\dagger c_{k\sigma})c_{j\gamma},$$
$$= c_{j\gamma}^\dagger c_{k\sigma}c_{j\gamma} + c_{k\sigma}c_{j\gamma}^\dagger c_{j\gamma} - (c_{j\gamma}^\dagger c_{j\gamma}c_{k\sigma} + c_{j\gamma}^\dagger c_{k\sigma}c_{j\gamma}),$$
$$= \overbrace{\{c_{j\gamma}^\dagger, c_{k\sigma}\}}^{\delta_{jk}\delta_{\sigma\gamma}} c_{j\gamma} - c_{j\gamma}^\dagger \overbrace{\{c_{j\gamma}, c_{k\sigma}\}}^{0},$$
$$= \delta_{jk}\delta_{\sigma\gamma}c_{j\gamma}. \tag{6}$$

Where in the last step we used the anti-commutation rules of the fermionic creation and annihilation operators.

Now we can finally start. Let's consider the time evolution equation in the Heisenberg picture

$$i\frac{d}{dt}c_{k\sigma}(t) = [c_{k\sigma}(t), H_0], \tag{7}$$

taking

$$c_{k\sigma}(t) = e^{iH_0 t} c_{k\sigma} e^{-iH_0 t}, \tag{8}$$

the time evolution equation takes the form

$$i\frac{d}{dt}c_{k\sigma}(t) = [e^{iH_0 t} c_{k\sigma} e^{-iH_0 t}, H_0]. \tag{9}$$

Luckily we can use eq. 4 to move further and obtain

$$i\frac{d}{dt}c_{k\sigma}(t) = e^{iH_0 t}[c_{k\sigma}, H_0]e^{-iH_0 t}. \tag{10}$$

Now is time to plug the hamiltoninan into this commutator

$$i\frac{d}{dt}c_{k\sigma}(t) = e^{iH_0t}[c_{k\sigma}, \sum_{k\sigma}\varepsilon_k c_{k\sigma}^\dagger c_{k\sigma}]e^{-iH_0t}, \tag{11}$$

since the commutator is linear and $\varepsilon_k \in \mathbb{R}$ we can transform the equation to this one

$$i\frac{d}{dt}c_{k\sigma}(t) = e^{iH_0t}\sum_{k\sigma}\varepsilon_k[c_{k\sigma}, c_{k\sigma}^\dagger c_{k\sigma}]e^{-iH_0t}, \tag{12}$$

and at this point we are lucky again, because now we can use eq. 6 to continue our calculations

$$\begin{aligned}i\frac{d}{dt}c_{k\sigma}(t) &= e^{iH_0t}\sum_{k\sigma}\varepsilon_k\delta_{jk}\delta_{\sigma\gamma}c_{j\gamma}e^{-iH_0t}, \\ &= e^{iH_0t}\varepsilon_k c_{k\sigma}e^{-iH_0t}, \\ &= \varepsilon_k e^{iH_0t}c_{k\sigma}e^{-iH_0t}, \\ &= \varepsilon_k c_{k\sigma}(t),\end{aligned} \tag{13}$$

therefore

$$\frac{d}{dt}c_{k\sigma}(t) = -i\varepsilon_k c_{k\sigma}(t), \tag{14}$$

at this point the equation looks very familiar, and *if it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.* To make sure we suppose it is

$$c_{k\sigma}(t) = e^{\alpha t}c_{k\sigma}. \tag{15}$$

If we differentiate this operator with respect to t, we will have

$$\frac{d}{dt}c_{k\sigma}(t) = \frac{d}{dt}e^{\alpha t}c_{k\sigma} = \alpha e^{\alpha t}c_{k\sigma} = \alpha c_{k\sigma}(t). \tag{16}$$

Comparing eqs. 14 and 16 we finally get that

$$\alpha = -i\varepsilon_k, \tag{17}$$

so finally, our solution will be

$$c_{k\sigma}(t) = e^{-i\varepsilon_k t}c_{k\sigma}. \quad\blacksquare \tag{18}$$

There are two ways of getting the creation operator, the hard one would be to repeat similar calculations with $c_{k\sigma}^\dagger$, which we will not perform here. The easy one is just to take

$$c_{k\sigma}^\dagger(t) = (c_{k\sigma}(t))^\dagger = (e^{-i\varepsilon_k t}c_{k\sigma})^\dagger = e^{i\varepsilon_k t}c_{k\sigma}^\dagger. \tag{19}$$

2　Non-interacting Green function

Codes attached

3　Matsubara sums and chemical potential

Codes atatched