1 Exercise 1 - Free electron gas in n-dimensions

a. Volume of an n-ball and area of an n-sphere

Solution: To compute the volume of an n-ball ($\mathcal{B}$) we can begin by setting the integral

$$V_n(R) = \int_{\mathcal{B}} dx_1 dx_2 \dots dx_n = \alpha_n R^n. \tag{1}$$

We know by dimensional analysis that the dimensions of the volume of the n-ball must be R, hence there must be a factor R^n embedded in the final result, times some factor dependent of n. The surface of an n-sphere will be denoted by $S_{n-1}(R)$. We can also obtain the volume of the n-ball, if we integrate very thin spherical shells of radius $r \in [0, R]$. This idea is represented by the following equation

$$V_n(R) = \int_0^R S_{n-1}(r) dr. \tag{2}$$

If we integrate this equation and rember that on $r = 0$ the sphere must have area 0, we have

$$S_{n-1}(R = \frac{dV_n(R)}{dR} = n\alpha_n R^{n-1}, \text{ using eq. 1.} \tag{3}$$

So, as we can see, our original problem is now reduced to find the functional form of α_n. To gain insight let's look at what happens if we combine eqs. 1-3.

$$\int_{\mathcal{B}} dx_1 dx_2 \dots dx_n = \int_0^R S_{n-1}(r) dr = n\alpha_n \int_0^R r^{n-1} dr. \tag{4}$$

Now let's do two things. First, we need to realize that on the right hand side of eq. 4 there are two parts, one exclusively dependent on n and another dependent on n, r (the integral). Knowing this let's take our second step and express the volume element $dx_1 dx_2 \dots dx_n$ in spherical coordinates. Therefore, it will look like this

$$dx_1 dx_2 \dots dx_n = r^{n-1} dr d\Omega_{n-1}, \tag{5}$$

where $d\Omega$ is the element of solid angle. From here we can conclude that

$$\int_\Omega d\Omega_{n-1} = n\alpha_n. \tag{6}$$

Here we need an auxiliary result. Let's calculate $\Gamma(1/2)$ from the definition

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty \exp(-t) t^{(1/2-1)} dt, \quad \text{changing } t = x^2 \text{ .} \\
&= \int_0^\infty \exp(-x^2) x^{-1} 2x dx, \\
&= 2\int_0^\infty \exp(-x^2) dx, \\
&= \int_{-\infty}^\infty \exp(-x^2) dx, \\
&= \sqrt{\pi}.
\end{aligned} \tag{7}$$

Now let's play with this idea

$$\exp(-(x_1^2 + x_2^2 + \dots + x_n^2)) = \exp(-r^2). \tag{8}$$

Which is just expressing the same function in two different coordinate systems. What would happen is we integrate such functions over all space?

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-(x_1^2 + x_2^2 + \cdots + x_n^2))dx_1 dx_2 \ldots dx_n = \int_0^{\infty} r^{n-1} dr \int_{\Omega} d\Omega_{n-1} \exp(-r^2). \tag{9}$$

With our previous knowledge from eq. 6 and integral calculus we can do this

$$\int_{-\infty}^{\infty} \exp(-x_1^2)dx_1 \cdots \int_{-\infty}^{\infty} \exp(-x_n^2)dx_n = \int_0^{\infty} r^{n-1} dr \int_{\Omega} d\Omega_{n-1} \exp(-r^2). \tag{10}$$

Using eq. 6

$$\int_{-\infty}^{\infty} \exp(-x_1^2)dx_1 \cdots \int_{-\infty}^{\infty} \exp(-x_n^2)dx_n = n\alpha_n \int_0^{\infty} r^{n-1} dr \exp(-r^2). \tag{11}$$

On the left hand side we have n times $\Gamma(1/2)$. But we already know their value, hence

$$\int_{-\infty}^{\infty} \exp(-x_1^2)dx_1 \cdots \int_{-\infty}^{\infty} \exp(-x_n^2)dx_n = (\sqrt{\pi})^n. \tag{12}$$

And from the definition of Γ we know

$$\int_0^{\infty} r^{n-1} \exp(-r^2) dr = \frac{1}{2}\Gamma\left(\frac{n}{2}\right). \tag{13}$$

And as a consequence,

$$\begin{aligned} \pi^{n/2} &= \alpha_n \frac{n}{2}\Gamma\left(\frac{n}{2}\right), \\ &= \alpha_n \Gamma\left(\frac{n}{2}+1\right). \end{aligned} \tag{14}$$

Hence

$$\alpha_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{15}$$

Now, substituting this on eqs. 1 and 3 we have

$$\boxed{V_n(R) = \frac{\pi^{n/2} R^n}{\Gamma\left(\frac{n}{2}+1\right)}.} \tag{16}$$

$$S_{n-1}(R) = \frac{n\pi^{n/2} R^{n-1}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{17}$$

Using the property $\Gamma(s+1) = s\Gamma(s)$ we can modify this last equation to get finally

$$\boxed{S_{n-1}(R) = \frac{2\pi^{n/2} R^{n-1}}{\Gamma\left(\frac{n}{2}\right)}.} \tag{18}$$