Stochastic processes 364-2-5431
Winter 2024
Final assignment

1. Consider the nine point in the plain: (i, j), $i, j = 0, 1, 2$. You are starting at point $(1, 1)$. At each stage you take one step, either vertically or horizontally, independent of one the history. The selection which direction to go is as follows. First you choose vertically with probability p and horizontally with probability $1 - p$. Then you choose each one of the two direction (if you have this option) with equal probability. Note that if you are at point (1,0), and you need to move vertically, you have only one option. You will stop when you will reach one of the corners. We define the position as the state in a Markov chain.

   (a) Write the transition probabilities matrix.

   (b) Explain why the probability to go back to the initial point after two steps is $p^2 + (1 - p)^2$.

   (c) Let Y be the length of the game. We can write $Y = 2X$ (because the the length of the game is even). What is the distribution X? what is the generating function of Y? What is $E(Y)$?

   (d) Pick a value of p and validate your answer regarding $E(Y)$ via simulation.

2. We consider the following queueing system. There is a single server. The arrival process is Poisson, and the standard service follows the $Exp(\mu)$ distribution. At the end of service, the customer is pleased (and hence leaves) with probability $1 - p$ and displeased with probability p. If the latter is the case, the customer receives an additional service, that follows the $Exp(\theta)$ distribution.

Note that we can refer to the model as an $M/G/1$ queue, but the service time Y can be written as $Y = Y_1 + IY_2$, with I being a binary random variable.

We define the following continuous time Markov chain. The state 0 refer to an empty system, and for $n \geq 1$ and $i = 1, 2$, the state (n, i) refers to n customers in the system, and the one in service is in stage i.

(a) Draw the transition diagram and write the steady-state equations.

(b) What is the stability condition?

(c) Define $g_i(z) = \sum_{n=1}^{\infty} \pi_{n,i} z^n$, multiply the steady state equation of state (n, i) by z_n and sum over n. You will get a set of two equations with two variables: $g_1(z), g_2(z)$. The values of $\pi_{1,1}, \pi_{1,2}$ serve as constants in these equations. Solve the equations, and use the fact that $\pi_0 = 1 - \rho$ and the steady-state equations of states 0 and $(1, 2)$ to get a complete solution.

(d) Show that $\pi_0 + g_1(z) + g_2(z) = g(z)$, where $g(z)$ is the generating function of the number of customers in the system that we got in class.

3. Consider the $M/M/1/n$ queueing system. That is, the arrival process is Poisson (with rate λ) and service times follow the $Exp(\mu)$ distribution. There are n spots in system, including the one in service. We define T_n as the time from one arrival to a full system until the next arrival to a full system.

   (a) Explain why we can write $T_n = Z + I\left(T_{n-1} + \tilde{T}_n\right)$, with $Z \sim Exp(\lambda+\mu)$, I is a binary random variable and $\tilde{T}_n$ is an independent random variables with the same distribution as T_n.
   Hint: First we need an event (arrival or departure), and then, depends on what was the first event..

   (b) Use the previous item for writing a recursion for $E(T_n)$.

   (c) Compute the LST of T_1

   (d) Write a recursion for the LST of T_n.

4. A new car costs b\$. James's strategy is as follows. If his car does not break down until some time τ, he sells it for $be^{-\tau}$\$ and buys a new one.

(a) What is James's average cost per time unit?

(b) Assume that the life time of a car, $T \sim Gamma\,(2, 1)$. Help James to find the value of τ that minimizes his average cost. You will need a solver for that (Note that b does not play a role here).