Stochastic processes 364-2-5431
Winter 2024
Final assignment (example)

1. Consider the Markov chain with two transient states and two absorbing states, with the matrix P equals

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   If we start at state 1 or state 2, then we will end up either in state 3 or in state 4. The questions are what is the probability for each of them, and what is the expected time until we will get there. We denote the probabilities Let α_{ij} be the probability that we will absorb in state j, given that we started at state i, and let e_i be the expected time, if we start at state i, until we are absorbed, for $i = 1, 2$ and $j = 3, 4$.

   (a) Construct a set of equations for $\alpha_{1,3}, \alpha_{2,3}$ and solve it.

   (b) The limit $\lim_{n\to\infty} P^n$ is a matrix with 0 in the columns of states $1, 2$, and 1 on elements $(3, 3)$ and $(4, 4)$ (why?). The elements in spots (i, j), for $i = 1, 2$ and $j = 3, 4$ are exactly α_{ij}. Thus, if we can find a structure of P^n we can calculate the limits. Alternatively, we can ask the computer to calculate P^n for a large n, and use the result as an approximation. Excecute the latter and compare to the result in the previous item.

   (c) An alternative way to calculate the above is simulation. We simply let the chain run, starting in state i and until it abosrbs. If we repeat the process many times, the estimator for α_{ij} is simply the fraction of times that we absobed in state j. Do so.

(d) Construct a set of questions for e_i, $i = 1, 2$ and solve it.

(e) If we let the chain run many times, starting from state i, we can record in each run how many steps it took the chain to absorb. The average is an estimator for e_i. Run it for $i = 1, 2$.

Answer

(a) By conditioning on the next step, we have in general:

$$\alpha_{ij} = \sum_k P_{ik}\alpha_{kj}.$$

In our case, $\alpha_{33} = 1$ and $\alpha_{43} = 0$. Thus, we have

$$\alpha_{13} = 0.5\alpha_{13} + 0.3\alpha_{12} + 0.1$$

$$\alpha_{23} = 0.2\alpha_{13} + 0.4\alpha_{12} + 0.4.$$

The solution is simply $\alpha_{13} = \frac{3}{4}, \alpha_{23} = \frac{11}{12}$.

(b) In the calculations file.

(c) In the calculations file. The idea behind the simulation is basic. Assume for example that we are in state 1. The next state is 1,2,3,4 with probabilities 0.5,0.3,0.1,0.1, respectively. So, generate $U \sim U(0, 1)$. Now, define the next state as follows.

$$\begin{cases} 1 & U \leq 0.5 \\ 2 & 0.5 < U \leq 0.8 \\ 3 & 0.8 < U \leq 0.9 \\ 4 & 0.9 < U \leq 1 \end{cases}.$$

(d) In general, by conditioning on the first step, we can write

$$e_i = 1 + \sum_k P_{ik}e_k.$$

That is, we have one step, and according to the state we jumped to, we still need to be absorbed from there. Of course, if k is an absorbing state, then $e_k = 0$. Thus, if we define T as the matrix of transition probabilities only between the transient states, then we can write it in matrix form:

$$\underline{e} = \underline{1} + T\underline{e} \quad \Rightarrow \quad \underline{e} = (I - T)^{-1}\underline{1}.$$

In our case, we have

$$e_1 = 1{+}0.5e1{+}0.3e2, \quad e_2 = 1{+}0.2e1{+}0.4e2 \quad \rightarrow \quad e_1 = \frac{15}{4}, \; e_2 = \frac{35}{12}.$$

In matrix form, we have

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Rightarrow$$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{pmatrix} \right) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 0.5 & -0.3 \\ -0.2 & 0.6 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{0.24} \begin{pmatrix} 0.6 & 0.3 \\ 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(e) In the calculations file.

2. In class we analyzed the $M/G/1$ queue. In this exercise we study a special case of it. We assume that the service time distribution follows the $Gamma(2, \mu)$ distribution. We can look at it as the service has two independent stages, each one of them follows the $Exp(\mu)$ distribution. We define the following continuous time Markov chain. The state 0 refer to an empty system, and for $n \geq 1$ and $i = 1, 2$, the state (n, i) refers to n customers in the system, and the one in service is in stage i.

   (a) Draw the transition diagram and write the steady-state equations.

   (b) What is the stability condition?

   (c) Define $g_i(z) = \sum_{n=1}^{\infty} \pi_{n,i} z^n$, multiply the steady state equation of state (n, i) by z_n and sum over n. You will get a set of two equations with two variables: $g_1(z), g_2(z)$. The values of $\pi_{1,1}, \pi_{1,2}$ serve as constants in these equations. Solve the equations, and use the fact that $\pi_0 = 1 - \rho$ and the steady-state equations of state 0 and $(1, 2)$ to get a complete solution.

   (d) Show that $\pi_0 + g_1(z) + g_2(z) = g(z)$, where $g(z)$ is the generating function of the number of customers in the system that we got in class.

answer

(a) I'll not draw here, but in my imagine there is one point at the left side, which refers to state 0. Then we have two infinite lines. The bottom is for states $(n, 1)$, and the top is for states $(n, 2)$, $n = 1, \ldots..$ We have arrows with λ from state 0 to state $(1, 1)$ and from any other state just horizontally to the right. We have arrows with rate μ from the each point at the bottom, vertically up and from each point of the top, diagonally down to the left.

From state 0 we exit only at arrival, and we enter only when a customer who is alone in the system completes service. Thus

$$\lambda\pi_0 = \mu\pi_{1,2}. \tag{1}$$

From states of the type $(n, 1)$, we exit either when there is an arrival or when the first stage of service is completed, so the total exit rate is $\lambda + \mu$. We enter such a state either at arrival (we will differ between $n = 1$ and $n > 1$) or at service completion (so the previous state was $(n + 1, 2)$. Thus, we have

$$(\lambda + \mu)\pi_{1,1} = \lambda\pi_0 + \mu\pi_{2,2}, \tag{2}$$

and

$$(\lambda + \mu)\pi_{n,1} = \lambda\pi_{n-1,1} + \mu\pi_{n+1,2}, \quad n \geq 2. \tag{3}$$

From states of the type $(n, 2)$ we exit when there is an arrival or service completion and we enter when there is an arrival (from $(n - 1, 2)$, for $n > 1$), or when is a completion of the first stage, (from $(n, 1)$). Thus, we have

$$(\lambda + \mu)\pi_{1,2} = \mu\pi_{1,1}, \tag{4}$$

and

$$(\lambda + \mu)\pi_{n,2} = \lambda\pi_{n-1,2} + \mu\pi_{n,1}, \quad n > 1. \tag{5}$$

(b) The expected service times is $\frac{2}{\mu}$ and hence the stability condition is $\rho = \frac{2\lambda}{\mu} < 1$.

(c) We multiply (2) by z, multiply (3) by z^n and sum the equations. The left hand side is simply $(\lambda + \mu)g_1(z)$. In the right hands side we have two elements. The first is

$$\lambda\pi_0 z + \lambda \sum_{n=2}^{\infty} \pi_{n-1,1} z^n = \lambda z(\left(\pi_0 + \sum_{j=1}^{\infty} \pi_j z^j\right) = \lambda z \left(\pi_0 + g_1(z)\right).$$

The second is

$$\mu \sum_{n=1}^{\infty} \pi_{n+1,2} z^n = \frac{\mu}{z} \sum_{j=2}^{\infty} \pi_{j,2} z^j = \frac{\mu}{z} \left(g_2(z) - z\pi_{1,1} \right).$$

Summarizing provides

$$(\lambda + \mu) g_1(z) = \lambda z \left(\pi_0 + g_1(z) \right) + \frac{\mu}{z} \left(g_2(z) - z\pi_{1,2} \right). \tag{6}$$

Repeating the same procedure for equations (4,5) provides

$$(\lambda + \mu) g_2(z) = \lambda z g_2(z) + \mu g_1(z). \tag{7}$$

From (7) we have

$$g_2(z) = g_1(z) \frac{\mu}{\mu + \lambda(1 - z)}. \tag{8}$$

Before Solving (6), we will obtain $\pi_{1,1}, \pi_{1,2}$. First, from (1), we have

$$\pi_{1,2} = \frac{\lambda}{\mu} \pi_0.$$

Second, from (4), we have

$$\pi_{1,1} = \frac{\mu + \lambda}{\mu} \pi_{1,2} = \frac{\mu + \lambda}{\mu} \frac{\lambda}{\mu} \pi_0.$$

We will plug the latter and (8) into (6), and will get (after using $\pi_0 = 1 - \rho$):

$$g_1(z) = \frac{\lambda z (1 - \rho)(\mu + \lambda(1 - z))}{(\mu - \lambda z)^2 - z\lambda^2} \tag{9}$$

(d) Using the fact that $\varphi_Y(\alpha) = \left(\frac{\mu}{\mu + \alpha} \right)^2$ and

$$g(z) = \frac{(1 - \rho)(z - 1)\varphi_Y(\lambda(1 - z))}{z - \varphi_Y(\lambda(1 - z))}$$

the result follows.

3. A new car costs b\$. James's strategy is as follows. If his card does not break down until some time τ, he sells it for a\$ and buys a new one. Of course, $a < b$

   (a) Assume that the life time of a car, T, follows some distribution with CDF $F(t) = P(T \leq t)$ What is James's average cost per time unit?

   (b) Help James find the value of τ that minimizes his average cost, assuming that $F(t) = t$ (in the domain (0,1), because we normalized the life time). We will get two solutions for the equation $derivative = 0$, but one of them will be ruled out.

Answer

(a) The expected cost per car is $b - a(1 - F(\tau)$. The expected lifetime of a car is $E \min\{T, \tau\})$. The question is how do we express the latter in terms of F. The answer is quite short. We go back to the first exercise:

$$E\left(\min\{T,\tau\}\right) = \int_{t=0}^{\infty} P\left(\min\{T,\tau\} > t\right) = \int_{t=0}^{\tau} (1 - F(t))dt.$$

The reason for the last equality:

$$P\left(\min\{T,\tau\} > t\right) = \begin{cases} P(T > t) & t < \tau \\ 0 & t \geq \tau \end{cases}.$$

We now insert $F(t) = t$, and the expression we want to minimize is

$$r(\tau) = \frac{b - a(1-\tau)}{\tau - \frac{\tau^2}{2}}.$$

We have, after simplification,

$$r'(\tau) = \frac{a+b}{(2-\tau)^2} - \frac{b-a}{\tau^2}.$$

Solving $\rho'(\tau) = 0$ provides $\tau_{1,2} = \frac{a-b\pm\sqrt{b^2-a^2}}{a}$. One of the solutions is negative, so we go for the solution with '-'. Instead of verifying that indeed it is a minimum by r'', we simply observe that $r(0) = \infty$, so only extremum point must be a minimum.

4. Condiser the $M/M/\infty$ system with arrival rate λ and service rate μ. In introductory courses, the steady-state distribution is obtained by using cuts. In this question we will obtain it in two other ways. The first is by guessing a solution and the second is by using generating functions.

   (a) Write the steady state equations.

   (b) Guess a solution of the type $\pi_n = c\frac{\alpha^n}{n!}$, insert it within the equations, and then solve a quadratic equation for α. The constant c is a normalizing constant. You will get that one of the roots depends on n so it is irrelevant. The other one is the familiar one.

   (c) We define $g(z) = \sum_{n=0}^{\infty} z^n \pi_n$. Multiply the steady-state equation os state n by z^n and sum over n. You will get a simple differential equation in the variable $g(z)$. solve it.

Answer

(a) If there are n customers in the system, then we move to $n+1$ with rate λ and to $n-1$ with rate $n\mu$. Thus, the exit rate from state n is $\lambda + n\mu$ and the we enter to state n from state $n-1$ with rate λ and from state $n+1$ with rate $(n+1)\mu$. The equation, for $n > 0$, is

$$(\lambda + n\mu)\pi_n = \lambda\pi_{n-1} + (n+1)\mu\pi_{n+1}.$$

For $n = 0$ we have $\lambda\pi_0 = \mu\pi_1$.

(b) Inserting $\pi_n = c\frac{\alpha^n}{n!}$ we get

$$(\lambda + n\mu)c\frac{\alpha^n}{n!} = \lambda c\frac{\alpha^{n-1}}{(n-1)!} + (n+1)\mu c\frac{\alpha^{n+1}}{(n+1)!}.$$

Dividing both sides by $c\frac{\alpha^{n-1}}{(n-1)!}$, we get

$$\left(\frac{\lambda}{n} + \mu\right)\alpha = \lambda + \frac{\mu}{n}\alpha^2 \Rightarrow (\lambda - \mu\alpha)\left(\frac{\alpha}{n} - 1\right) = 0.$$

One of the multipliers must be 0, so the solutions are $\alpha_{1,2} = \frac{\lambda}{\mu}, n$. The second one is ruled out.

(c) Multiplying the steady-state equation of state n by z_n and summing, gives

$$\lambda \sum_{n=0}^{\infty} \pi_n z^n + \mu \sum_{n=1}^{\infty} \pi_n n z^n = \lambda \sum_{n=1}^{\infty} \pi_{n-1} z^n + \mu \sum_{n=0}^{\infty} \pi_{n+1}(n+1) z^n.$$

We will simplify each of the four elements separately. The first one is simply $\lambda g(z)$. The second is

$$\mu z \sum_{n=1}^{\infty} \pi_n n z^{n-1} = \mu z \sum_{n=0}^{\infty} \pi_n n z^{n-1} = \mu z \sum_{n=0}^{\infty} \pi_n \left(z^n\right)' = \mu z g'(z).$$

The third is

$$\lambda \sum_{n=1}^{\infty} \pi_{n-1} z^n = \lambda \sum_{j=0}^{\infty} \pi_j z^{j+1} = \lambda z \sum_{j=0}^{\infty} \pi_j z^j = \lambda z g(z).$$

The last is simply (after we simplified the second) $\mu g'(z)$.
So we have

$$\lambda(1-z) g(z) = \mu(1-z) g'(z) \Rightarrow g'(z) = \frac{\lambda}{\mu} g(z).$$

The solution is $g(z) = a e^{\frac{\lambda}{\mu} z}$. The boundary condition $g(1) = 1$ provides that $a = e^{-\frac{\lambda}{\mu}}$ and hence $g(z) = e^{-\frac{\lambda}{\mu}(1-z)}$.