Computer Vision - Homework 5

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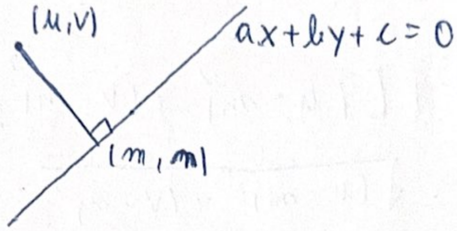
1. $d((u,v),(a,b,c)) = |au + bv + c|$, if $a^2 + b^2 = 1$



Slope of the line $ax+by+c$, $m_1 = -\frac{a}{b}$

Slope of the line $(u,v),(m,n)$, $m_2 = \frac{v-n}{u-m}$

These lines are perpendicular $\Rightarrow m_1 \cdot m_2 = -1$

$$m_1 = -\frac{1}{m_2}$$

$$\Rightarrow \frac{v-n}{u-m} = \frac{b}{a} \Leftrightarrow a(v-n) = b(u-m)$$

$$a(v-n) - b(u-m) = 0 \quad |(\,)^2$$

$$[a(v-n) - b(u-m)]^2 = 0$$

$$a^2(v-n)^2 - 2ab(v-n)(u-m) + b^2(u-m)^2 = 0$$

$$a^2(v-n)^2 + b^2(u-m)^2 = 2ab(v-n)(u-m)$$

$$[a(u-m) + b(v-n)]^2 = a^2(u-m)^2 + 2ab(u-m)(v-n) + b^2(v-n)^2$$

$$= a^2(u-m)^2 + a^2(v-n)^2 + b^2(u-m)^2 + b^2(v-n)^2 =$$

$$= a^2[(u-m)^2 + (v-n)^2] + b^2[(u-m)^2 + (v-n)^2] =$$

$$= (a^2+b^2)[(u-m)^2 + (v-n)^2]$$

$[a(u-m)+b(v-n)]^2 = (au+bv-am-bn)^2$

point $(m,n) \in$ line $(a,b,c) \Rightarrow am+bn+c=0$

$\Leftrightarrow (au+bv-\underbrace{(am+bn+c)}_{=0}+c)^2 = (au+bv+c)^2$

$\Rightarrow (au+bv+c)^2 = (a^2+b^2)[(u-m)^2+(v-n)^2]$

$\Rightarrow |au+bv+c| = \sqrt{a^2+b^2} \cdot \sqrt{(u-m)^2+(v-n)^2}$

distance from (u,v) to (m,n), $d = \sqrt{(u-m)^2+(v-n)^2}$

$\Rightarrow |au+bv+c| = d\sqrt{a^2+b^2}$

$\Rightarrow d = \dfrac{|au+bv+c|}{\sqrt{a^2+b^2}}$

knowing $\sqrt{a^2+b^2}=1$

$\Rightarrow d = |au+bv+c|$

2. $x(t) = \dfrac{1-t^2}{1+t^2}$

$y(t) = \dfrac{2t}{1+t^2}$

$x^2+y^2 = \dfrac{(1-t^2)^2+4t^2}{(1+t^2)^2} = \dfrac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \dfrac{1+2t^2+t^4}{(1+t^2)^2}$

$= \dfrac{\cancel{(1+t^2)^2}}{\cancel{(1+t^2)^2}} = 1 \Rightarrow$ equation of a circle with $r=1$

3. point (a, b)

$$x(t) = \frac{1-t^2}{1+t^2}$$

$$y(t) = \frac{2t}{1+t^2}$$

$$d = \sqrt{(a-x(t))^2 + (b-y(t))^2}$$

minimize $(a-x(t))^2 + (b-y(t))^2$

$$f(t) = \left(a - \frac{1-t^2}{1+t^2}\right)^2 + \left(b - \frac{2t}{1+t^2}\right)^2 =$$

$$= a^2 - 2a\frac{1-t^2}{1+t^2} + \frac{(1-t^2)^2}{(1+t^2)^2} + b^2 - 2b\frac{2t}{1+t^2} + \frac{4t^2}{(1+t^2)^2}$$

$$= a^2 - \frac{2a}{1+t^2} + \frac{2at^2}{1+t^2} + \frac{1-2t^2+t^4}{(1+t^2)^2} + b^2 - 2b\frac{2t}{1+t^2} + \frac{4t^2}{(1+t^2)^2}$$

$$= a^2 - \frac{2a}{1+t^2} + \frac{2at^2}{1+t^2} + \frac{1}{(1+t^2)^2} - \frac{2t^2}{(1+t^2)^2} + \frac{t^4}{(1+t^2)^2}$$

$$+ b^2 - 2b\frac{2t}{1+t^2} + \frac{4t^2}{(1+t^2)^2}$$

$$f'(t) = 0$$

$$f'(t) = 0 - 2a\frac{(-1)2t}{(1+t^2)^2} + \frac{4ta(1+t^2) - 2at^2 \cdot 2t}{(1+t^2)^2} + \frac{0 - 2\cancel{(1+t^2)} \cdot 2t}{(1+t^2)^{\cancel{4}3}}$$

$$-\left(\frac{4x(1+t^2)^2 - 2t^2 2(1+t^2) \cdot 2t}{(1+t^2)^4}\right) + \frac{4t^3}{(1+t^2)^2} - \frac{t^4 \cdot 4 \cdot t}{(1+t^2)^3}$$

$$+0 - \left(\frac{4b(1+t^2) - 4bt \cdot 2t}{(1+t^2)^2}\right) + \frac{8t}{(1+t^2)^2} - \frac{4t^2 \cdot 2t(1+t^2) \cdot 2}{(1+t^2)^{\cancel{4}3}}$$

$$= \frac{4at}{(1+t^2)^2} + \frac{4at}{(1+t^2)} - \frac{4at^3}{(1+t^2)^2} - \frac{4t}{(1+t^2)^3} - \frac{4t}{(1+t)^2}$$

$$+ \frac{8t^3}{(1+t^2)^3} + \frac{4t^3}{(1+t^2)^2} - \frac{4t^5}{(1+t^2)^3} - \frac{4b}{t^2+1} + \frac{8bt^2}{(t^2+1)^2}$$

$$+ \frac{8t}{(1+t^2)^2} - \frac{16t^3}{(t^2+1)^3} =$$

$$= \frac{8t^3 - 4t - 4t^5 - 16t^3}{(t^2+1)^3} \;{}^{1+t^2/}\!+ \frac{4at - 4at^3 - 4t^3 - 4t + 8bt^2 + 8t}{(1+t^2)^2}$$

$$^{(t^2+1)^2/}\!+ \frac{4at - 4b}{t^2+1}$$

$$\Rightarrow -4t^5 \cancel{+8t^3} \cancel{-4t} + 4at + \cancel{4at^3} - \cancel{4at^3} - \cancel{4at^5} - 4t^3 - 4t^4 \cancel{-4t}$$

$$-4t^3 \cancel{+8bt^2} + 8bt^4 \cancel{+8t} + \cancel{8t^3} + \cancel{4at^5} + 8at^3 + 4at$$

$$-4bt^4 - \cancel{8bt^2} - 4b = 0$$

$$-4t^5 - 4t^4 + 8bt^4 - 4bt^4 - 4t^3 - 4t^3 + 8at^3 + 4at + 4at$$

$$-4b = 0$$

$$-4t^5 + t^4(4b-4) + t^3(8a-8) + 8at - 4b = 0$$

The solution for this equation $f'(t) = 0$, is the closest point on the curve to the point (a, b)

4) Degree of this equation is 5

Depending on the discriminant of this equation, it can have

$\Delta > 0$, 3 real roots

$\Delta = 0$, 1 real root

$\Delta < 0$, 1 real root, 2 complex conjugate roots

5. In 2D, a rotation matrix is described as

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\det(R(\theta)) = \cos\theta^2 + \sin^2\theta = 1$$

$$\mathrm{Inv}(R(\theta)) = \frac{1}{\det R(\theta)} \cdot R^*(\theta)$$

$$R^*(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow \mathrm{Inv}(R(\theta)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R(\theta)^T$$

In 3D, the general rotation matrix is the multiplication of the rotation matrices around X, Y and Z axes

$$R(\alpha, \beta, \gamma) = R_Z(\gamma) \cdot R_Y(\beta) \cdot R_X(\alpha)$$

$$R_X(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_Y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_Z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(R(\alpha, \beta, \gamma)) = \det(R_Z(\gamma)) \cdot \det(R_Y(\beta)) \cdot \det(R_X(\alpha))$$

$$= 1 \cdot 1 \cdot 1 = 1$$

$$\mathrm{Inv}(R_X(\alpha)) = \frac{1}{\det(R_X(\alpha))} \cdot R_X(\alpha)^* = 1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} = R_X(\alpha)^T$$

$$\mathrm{Inv}(R_Y(\beta)) = \frac{1}{1} \cdot \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} = R_Y(\beta)^T$$

$$\mathrm{Inv}(R_Z\,\gamma) = \frac{1}{1} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_Z(\gamma)^T$$

In both cases 2D and 3D, the rotation matrices are represented by orthogonal matrices, that satisfies the property:

$$R^{-1} = R^T$$

General case:

$$R = R_1(\alpha_1) \cdot R_2(\alpha_2) \cdot \ldots \cdot R_n(\alpha_n)$$

$$R_k(\alpha_k) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & & \cos\alpha_k & 0 & \cdots & 0 & -\sin\alpha_k & \cdots & 0 \\ 0 & 0 & & 0 & 1 & \cdots & 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sin\alpha_k & 0 & \cdots & 0 & \cos\alpha_k & \cdots & 0 \\ \vdots & \vdots & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $\det R_k(\alpha_k) = 1$

and R_k is an orthogonal matrix, because

$$R_k^{-1} = R_k^T$$

=> this can be proven for any matrix in the product

$\Rightarrow R^T = R^{-1}$

$\Rightarrow \det(R) = 1$

6. 4 points in 3D: A, B, C and D

Check if the projection of D on the plane (A, B, C) falls inside the triangle (A, B, C)

Method 1.

1. Compute the normal vector of the plane defined by A, B, C, by calculating the cross product of vectors $\vec{AB}$ and $\vec{AC}$.

planeNormal = crossProduct(B - A, C - A)

2. Compute the distance from point D to the plane along the normal

dist = dotProduct(D - A, planeNormal)

3. Find the projection of D onto the plane

projD = D - dist. planeNormal

4. Compute the area of PAB, PBC, PAC triangles

5. Check if the sum of areas PAB, PBC, PAC is equal to the area of the ABC triangle. If they are equal, P is inside, otherwise P is outside

Method 2

The first 3 steps are the same as in method 1.

4. Compute the area of ABC triangle

5. Compute the barycentric coordinates of point P with respect to the triangle ABC

$$a = \frac{1}{2} \frac{|PB \times AC|}{\text{area } ABC}$$

$$b = \frac{1}{2} \frac{|PC \times PA|}{\text{area } ABC}$$

$$c = \frac{1}{2} \frac{|PA \times PB|}{\text{area } ABC}$$

6. Check if a, b, c are greater than 0. If all of them are greater than 0, points P, A, B, C are on the same side $\Rightarrow$

$\Rightarrow$ P is inside ABC triangle, otherwise P is outside

7. The median filter works by sorting the pixel values in a neighbourhood and selecting the middle value as the output.

The salt-and-pepper noise introduces random high or low intensity values in the image.

0 - pepper noise

255 - salt noise

If the image has an uniform (constant) background, then all the pixels in the image will have the same value, and the noise values will be lower or ~~equal~~ greater than the background value

Sorted values in a neighbourhood:

$$\underbrace{P,P,P,P \ldots P}_{\text{pepper noise}},\ \underbrace{C,C,C,C \ldots C}_{\substack{\text{constant value}\\ \text{background}}},\ \underbrace{S\,S\,S, \ldots S}_{\text{salt noise}}$$

Ex. 3x3 kernel - 9 elements

$$\begin{bmatrix} S & C & P \\ P & S & C \\ C & C & C \end{bmatrix}$$

n salt = 2

n pepper = 2

n constant = 5

n salt + n pepper $\leq \left[\frac{n}{2}\right] = 4$

Sorted: P, P, C, C, C, C, C, P, P

middle = constant value

4 X 4 kernel - 16 elements

$$\begin{bmatrix} S & C & P & S \\ C & P & C & P \\ C & S & C & C \\ C & P & S & C \end{bmatrix}$$

n salt = 4

n pepper = 4

n constant = 8

Sorted P, P, P, P, C, C, C, C, C, C, C, C, S, S, S S

C - middle - constant value

Supposing that the number of pepper noise values and the number of salt noise values are equal, for both cases even and odd size of the kernel, the middle value will always be the constant value.

Salt and pepper values are not in equal proportions

Ex 3x3 kernel

$$\begin{bmatrix} S & S & S \\ S & C & C \\ C & C & C \end{bmatrix}$$

- S, S, S, S, C, C, C, C, C
- middle value is still C

Ex 4x4 kernel

$$\begin{matrix} C & S & C & S \\ S & C & S & C \\ C & S & C & S \\ S & C & S & C \end{matrix}$$

$S, S, S, S, S, S, S, \underbrace{S, C}_{\text{middle value}}, C, C, S, S, C, S, C$

In this case the middle value is not C

There are only 2 cases, when the middle value won't be the constant value, when

$$\text{number of salt values} = \frac{\text{numbers of pixels in the kernel}}{2}$$

or

$$\text{number of pepper values} = \frac{\text{number of pixels in the kernel}}{2}$$

But the case above represents the salt-or-pepper filter case.

For the salt-and-pepper, where values are equally distributed, the filter is always robust and will always pick the constant value.

8. $J(w) = (Aw - b)^T (Aw - b)$

$$J(w) = w^T A^T A w - w^T A^T b - b^T A w + b^T b$$

$$J'(w) = 2 A^T A w - A^T b - A^T b + 0$$

$$= 2 A^T A w - 2 A^T b$$

$$J'(w) = 0$$

$$A^T A w = A^T b$$

$$w = (A^T A)^{-1} A^T b$$

The solution that minimizes $J(w) = (Aw - b)^T (Aw - b)$ is $w = (A^T A)^{-1} A^T b$