Title : Combinatorial Nullstellensatz and Applications
Paper by Noga Alon

$0 \neq f(x) \in F(x)$, degree t

f can't have more than t zeroes

Given $S \subset F$ with $|S| > t$, $\exists\, s \in S$ s.t. $f(s) \neq 0$

Lemma : $0 \neq f \in F[x_1, x_2, \dots, x_n]$ (Lemma 2.1 in Alon's paper)

deg of $x_i \leq t_i$

$S_1 \times S_2 \times \dots \times S_n \subset F \times F \times \dots \times F$, $|S_i| > t_i$

$\Downarrow$

$\exists\, (s_1, \dots, s_n)$ s.t. $f(s_1, s_2, \dots, s_n) \neq 0$

Theorem (Comb. Null) : $f \in F[x_1, x_2, \dots, x_n]$ (Thm 1.2 in Alon's paper)

$\deg f = \sum_{i=1}^{n} t_i$

Coeff of $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$ is nonzero

$S_1, \dots, S_n \subset F$, $|S_i| > t_i$

Then $\exists\, (s_1, \dots, s_n) \in S_1 \times \dots \times S_n$

s.t. $f(s_1, s_2, \dots, s_n) \neq 0$

" A polynomial of a certain degree cannot vanish over a large set of nos. "

Applns to Graph Theory, Number Theory, Enumerative Combinatorics

Thm asserting something is true :
Pf by Contradiction $\longrightarrow$ construct a polynomial of smaller degree that vanishes over a large enough set

Thm guaranteeing existence of something :
Construct a poly of small degree, our search space ~~corresponds~~ to looking at a large enough set of values, we are guaranteed a pt where poly is non-vanishing

① Cauchy - Davenport Theorem

(Thm 3.2 in Alon's paper)

$A, B \subset \mathbb{Z}_p$. $\quad A+B := \{a+b \mid a \in A, b \in B\}$

Then $|A+B| \geq \min\{p, |A|+|B|-1\}$

($|A|+|B|-1$ can be attained by for eg $A = \{0\}$, $B = \{1, 2, .., p-1\}$)

Proof: Thm asserts

$|A|+|B| > p \Rightarrow |A+B| = p \Rightarrow A+B = \mathbb{Z}_p$

$g \in \mathbb{Z}_p$. $\quad |g-A| + |B| = |A|+|B| > p$

$b \in (g-A) \cap B \neq \phi \qquad \underline{\underline{g-a = b \in B}}$

So assume $|A|+|B| \leq p$. Want $|A+B| \geq |A|+|B|-1$

Suppose $|A+B| \leq |A|+|B|-2 = (|A|-1)+(|B|-1)$

$C \neq A+B \qquad f(x,y) = \prod_{c \in C}(x+y-c)$ vanishes $\forall (x,y) \in A \times B$

$C \supseteq A+B$

$|C| = |A|+|B|-2$

$\deg = |C| = (|A|-1)+(|B|-1)$

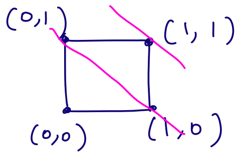
Coeff of $x^{|A|-1} y^{|B|-1}$: $\binom{|A|+|B|-2}{|A|-1}$, $\neq 0$ in $\mathbb{Z}_p$ since $|A|+|B|-2 < p$

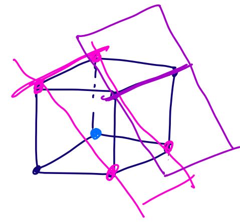
So $\exists$ non-zero pt in $A \times B$. Contradiction!

Theorem: Let $H_1, H_2, \dots, H_m$ be hyperplanes in $\mathbb{R}^n$ that cover all vertices of $[0,1]^n$ but one. Then $m \geq n$.

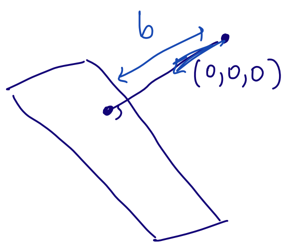
(Thm 6.3 in Alon's paper)

Proof:





(all but one vertex condition is imp — o/w $\{0,1\}^3$ for eg can be covered with 2 planes)



unit vector in this direction: a

$$a \cdot x = b \qquad (x = (x_1, x_2, \dots x_n))$$

$H_1, H_2, \dots, H_m$ covers all but $(0, 0, \dots, 0)$

$x = (x_1, \dots, x_n)$. $H_i : (a_i \cdot x) = b_i$

Suppose $m < n$.

Will construct a polynomial of degree n that vanishes on $\{0, 1\}^n$

$$\underbrace{(-1)^{m} \prod_{j=1}^{m} b_j \prod_{i=1}^{n} (1 - x_i)}_{\text{deg } n} - \underbrace{\prod_{i=1}^{m} \left[(a_i \cdot x) - b_i \right]}_{\text{deg } m < n}$$

(2) Want poly to vanish at $(0,0,\dots,0)$ too

(3) need this part to still vanish on non-zero pts

(1) Each pt lies on some H_i

Coeff of $x_1^1 x_2^1 \dots x_n^1 \neq 0$

So cannot vanish on $\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}$. Contradiction

III. **Theorem** : (Thm 6.1 in Alon's paper)

$G=(V,E)$ loopless graph. p prime.
Avg. degree $> 2p-2$
Max. degree $\le 2p-1$
Then G has a p-regular subgraph

Recall : $\sum_{v\in V} \deg v = 2|E| = 2n$ $\Big\}$ Hypothesis $\Rightarrow$
$2|E| > (2p-2)|V|$
$\Leftrightarrow |E| > (p-1)|V|$

In $\mathbb{F}_p$, $a^{p-1} \equiv 1 \mod p$ if $a \not\equiv 0 \mod p$
$0 \mod p$ if $a \equiv 0 \mod p$

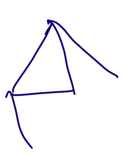
Proof : "Existence" proof. So will construct poly, will be interested in when the poly value is non-zero

$|E| = n$
$e_1, e_2, \ldots, e_n$

Subgraph $\rightsquigarrow$ collection of edges

Variables $x_{e_1}, x_{e_2}, \ldots, x_{e_n}$. Look at values in $\{0,1\}^n$
$\{0,1\}\times\{0,1\}\times\cdots\times\{0,1\}$

For an edge e, $x_e = 1$ if $e \in$ subgraph
0 if $e \notin$ subgraph

Restriction : 

If v is incident on an edge in the subgraph, then $\deg v = p$ in the subgraph
i.e. $\forall v \in V$, v is incident on exactly 0 or p of the edges in the subgraph

$\forall (v,e)\in(V,E)$, $a_{v,e} := 1$ if v incident on e
0 o/w

For fixed v, $\sum_{e \in \text{subgraph}} a_{v,e} = 0$ or p

$$\sum_{e \in E} a_{v,e} x_e = 0 \bmod p$$

$\leq \deg v < 2p$

Field : $\mathbb{F}_p$

$$\left(\sum_{e \in E} a_{v,e} x_e\right)^{p-1} = \begin{cases} 1 & \text{if sum is non-zero} \\ 0 & \text{if sum is } 0 \end{cases}$$

This is the case we want

$f(x_{e_1}, \ldots, x_{e_n})$

$$= \prod_{v \in V} \left(1 - \left(\sum_{e \in E} a_{v,e} x_e\right)^{p-1}\right) - \prod_{e \in E} (1 - x_e)$$

(1) Want non-zero value

(2) Want non-zero $\forall v$

(3) Want to guarantee that all x_e are not 0

$\deg (p-1)|V| < |E|$

$\deg |E|$, coeff of $x_{e_1}' x_{e_2}' \cdots x_{e_n}' \neq 0$

So, $\exists (s_1, s_2, \ldots, s_n) \in \{0,1\}^n$ s.t. $f(s_1, s_2, \ldots, s_n) \neq 0$

Title: Combinatorial Nullstellensatz and its Applications

Abstract: The Combinatorial Nullstellensatz is a statement about zeroes of a multi-variable polynomial over a field, and has seen a remarkable number of applications to number theory, enumerative combinatorics, and graph theory. Roughly speaking, it gives quantitative information about how a polynomial of a certain degree cannot vanish over a large enough set of values.
In this talk I will briefly explain the statement of this theorem, and then illustrate through examples a general technique for using it to prove a variety of powerful results.

~~Roughly speaking, it says the following: A single variable polynomial of degree d over a field can't have more than d zeroes. Thus the polynomial can't vanish over a set of size greater than d. When we move to multiple variables, then also it makes sense that a polynomial of certain fixed degree shouldn't be able to vanish over a large enough set. The Combinatorial Nullstellensatz gives more quantitative information about when this can happen.~~

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