Flow Induced Instability of the Interface between a Fluid and a Gel

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2 Abstract

Our aim is to study the stability of fluid flow adjacent to the polymer gel. The system includes a Newtonian fluid of density ρ, viscosity η and thickness R and a polymer gel of density ρ, modulus of elasticity E, viscosity η_g and thickness HR. We consider the fluid flow to be a plane Couette flow. The equations used to solve this problem are Navier- Stokes equation for the fluid and elasticity equation for the gel. A characteristic equation is formed using boundary conditions at the interface. The equation obtained is non-linear and cannot be solved directly. So, we initialise Reynold's number Re=0 and a quadratic equation is formed which is solved. Then we go on increasing Re and subsequent solutions are obtained analytically. Growth rate depends on $\Sigma = (R_e/\Lambda) = (\rho E R^2/\eta^2)$, thickness ratio H, relative viscosity $\eta_r = \eta_g/\eta$ and wave number k. At $\eta_r = 0$, when $Re > Re_t$ (transition Reynold's number), perturbations are unstable for all Σ and k. Re_c(critical Reynold's number) is the minimum of Re_t- k curve. For $\Sigma << 1$, Re_c increases proportional to Σ. For $\Sigma >> 1$, Re_c is directly proportional to Σ^β where $0.75 <= \beta <= 0.8$. For low $1 <= \Sigma <= 1000$, Re_t decreases as η_r increases (increasing gel velocity destabilizes fluid flow). For high $10,000 <= \Sigma <= 100,000$, Re_t increases as η_r increases and it starts decreasing after η_{max} (perturbations unstable when $\eta_r < \eta_{max}$).

3 Introduction

Our analysis is based on the study of fluid flow adjacent to elastic walls (example blood). Such flow is very different from fluid flow adjacent to rigid walled channel due to elasticity of walls. Linear stability analysis is used to see the effect of changing various parameters on the stability of laminar flow which has imposed perturbations. In this case we are considering the flow of a Newtonian fluid of density ρ and viscosity η in the region z = 0 to z= +R bounded by a solid wall from above (which is moving with a velocity of V) and gel from below. Gel is taken to be an incompressible viscoelastic solid with density ρ, shear modulus E and viscosity η_g. Separate equations are written for fluid (Navier- Stokes equation)
$\rho(\partial_{t^*} v_i^* + v_j^* \partial_{j^*} v_i^*) = -\partial_{i^*} p^* + \eta \partial_{j^*}^2 v_i^*$
and gel (elasticity equation)
$\rho \partial_{t^*}^2 u_i^* = -\partial_{i^*} p^* + E \partial_{j^*}^2 u_i^* + \eta_g \partial_{j^*}^2 \partial_{t^*} u_i^*$
and are solved using numerical methods like fifth order Runge − Kutta, multi variable Newton Raphson. Momentum conservation equations for these two phases include coupling force due to which relative velocity between two phases is reduced. Both dominant forces in the momentum conservation equation, elastic and surface tension forces are conservative and hence the propagation is undamped. Through this analysis, we can see that even in absence of fluid inertia, instability is present in fluid flow when strain rate exceeds the critical value.

4 Problem Formulation

We obtained a fourth order differential equation for the polymer gel
$[-(Re/\Lambda)\alpha^2 + (1+\eta'\alpha)(d_z^2 - k^2)](d_z^2 - k^2)\tilde{u_z} = 0$

and a fourth order differential equation for fluid flowing above the polymer gel
$[-(\alpha + \Lambda\iota kz)Re/\Lambda + (d_z^2 - k^2)](d_z^2 - k^2)\tilde{v_z} = 0$
These two equations are solved using shooting method method using the following boundary conditions:
$\tilde{v_z} = \alpha\tilde{u_z}$; $\tilde{v_x} = \alpha\tilde{u_x} - \Lambda\tilde{u_z}$
$\tilde{\tau}_{zz} = \tilde{\sigma}_{zz}$; $\tilde{\tau}_{xz} = \tilde{\sigma}_{xz}$
To solve it with shooting method, we need initial values of u and v at z=-1 and z=1 respectively. For this we assume two eigenvectors for u and two for v as follows:
$\tilde{u}_z = 0$ $\quad d_z\tilde{u}_z = 0$ $\quad d_z^2\tilde{u}_z = 1$ $\quad d_z^3\tilde{u}_z = 0$ at boundary z=-1 (for gel)
$\tilde{u}_z = 0$ $\quad d_z\tilde{u}_z = 0$ $\quad d_z^2\tilde{u}_z = 0$ $\quad d_z^3\tilde{u}_z = 1$ at z=-1 (for gel)
$\tilde{v}_z = 0$ $\quad d_z\tilde{v}_z = 0$ $\quad d_z^2\tilde{v}_z = 1$ $\quad d_z^3\tilde{v}_z = 0$ at boundary z=+1 (for fluid)
$\tilde{v}_z = 0$ $\quad d_z\tilde{v}_z = 0$ $\quad d_z^2\tilde{v}_z = 0$ $\quad d_z^3\tilde{v}_z = 1$ at boundary z=+1 (for fluid)

5 Numerical Analysis

The equation for fluid is obtained through Navier-Stokes mass and momentum equations considering the perturbations in the polymer gel. The constitutive equations for the polymer gel are obtained for "infinite coupling limit" i.e. polymer and the solvent velocities in gel are equal. By scaling the length by R, the time by η/E and the pressure by E we get the following equations:
For gel:
$\partial_i u_i = 0$
$(Re/\Lambda)\partial_t^2 u_t = -\partial_i P + (1 + \eta_r\partial_t)\partial_j^2 u_i$
$\sigma_{xz} = (1 + \eta_r\partial_t)(\partial_x u_z + \partial_z u_x)$ $\qquad \sigma_{zz} = -P + 2(1 + \eta_r\partial_t)\partial_z u_z$
For fluid:
$\partial_i v_i = 0$
$(R_e/\Lambda)(\partial_t v_i + v_j\partial_j v_i) = -\partial_i P + \partial_j^2 v_i$
$\tau_{xz} = (\partial_x v_z + \partial_z v_x)$ $\qquad \tau_{zz} = -P + 2\partial_z v_z$
in linear stability analysis the expressions for velocity and displacement are:
$u_i = \bar{u}_i + u_i'$ $\qquad v_i = \bar{v}_i + v_i'$
$u_i' = \bar{u}_i(z)exp(\iota kx + \alpha t)$ $\qquad v_i' = \bar{v}_i(z)exp(\iota kx + \alpha t)$
and a fourth order differential equation for fluid flowing above the polymer gel
$[-(\alpha + \Lambda\iota kz)Re/\Lambda + (d_z^2 - k^2)](d_z^2 - k^2)\tilde{v_z} = 0$
perturbations in stress field in gel and fluid are taken into account by following expressions:
$\tilde{\sigma}_{xz} = (1 + \eta_r\alpha)(d_z\tilde{u_x} + \iota k\tilde{u_z})$ $\qquad \tilde{\sigma}_{zz} = -\tilde{P} + 2(1 + \eta_r\alpha)d_z\tilde{u_z}$
$\tilde{\tau}_{xz} = (d_z\tilde{v_x} + \iota k\tilde{v_z})$ $\qquad \tilde{\tau}_{zz} = -\tilde{P} + 2d_z\tilde{v_z}$
The fourth order differential equation for gel is solved by using the initial values at lower boundary of gel ($z = -1$) assuming a certain value of growth rate. Since we only know $u_z (= 0)$ and $du_z (= 0)$ at this point, to find the whole matrix for initial condition we consider our initial vector to be linear combination of two eigenvectors ([0 0 1 0] and [0 0 0 1]). Now, we use RK4 method and march to the interface of gel and fluid (z=0). Same steps are followed for the fluid region using the corresponding fourth order equation
$[-(Re/\Lambda)\alpha^2 + (1 + \eta'\alpha)(d_z^2 - k^2)](d_z^2 - k^2)\tilde{u_z} = 0$
and initial vector (at $z = 1$) of velocity as linear combination of eigenvectors [0 0 1 0] and [0 0 0 1]. In this case also we march from $z = 1$ to $z = 0$, i.e. interface of the two. After getting the values of $v_z$and u_z at interface, we apply the boundary conditions. On this, we get four equations which can also be denoted in matrix form with coefficients of eigenvectors for u_z and v_z as unknowns. For the assumed value of α to be true, the 4x4 coefficient matrix should have determinant equal to zero. Now, our determinant is a function of α. So to find the required value of alpha we use Newton Raphson multi-variable technique to make value of determinant equal to zero.

6 Results and Discussion

Growth rate (α) is a function of Reynolds number Re, the strain rate Λ, the wave number k, the ratio of thickness of the gel and fluid H and the ratio of viscosities η_r. Variation of α has been obtained by varying Λ keeping Re fixed (In the journal the value of Re for which the curve is plotted is 100). We have tried to obtain the curves for the most unstable growth rate vs strain rate. As this growth rate is a complex number we have plotted real and imaginary part of it separately. These curves can be obtained only for some initial guess for α. As we have to obtain graphs for the most unstable growth rate i.e. the maximum value of α we have to take an initial guess above all the values of the α obtained in this range. For these curves we have taken an initial guess of

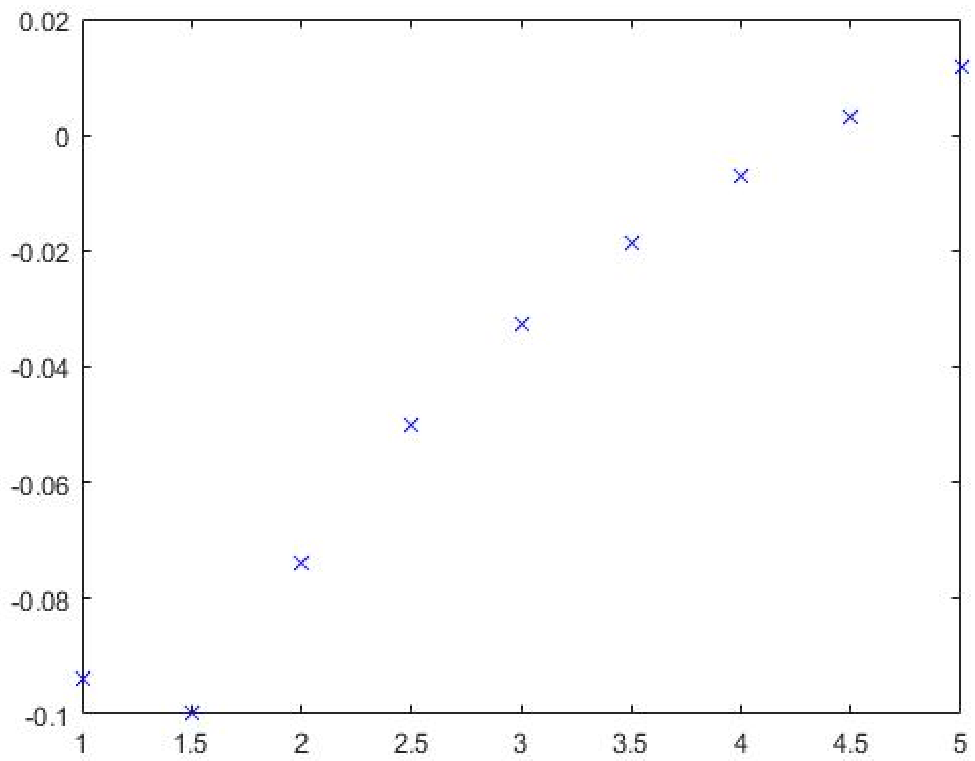

Figure 1: Variation of real part of the growth rate, α_r, as a function of dimensionless strain rate,Λ, for Re $=$ 100, H $=$ 1, $\eta_r = 0$ and $\tau = 0$. tolerance=10^{-5}

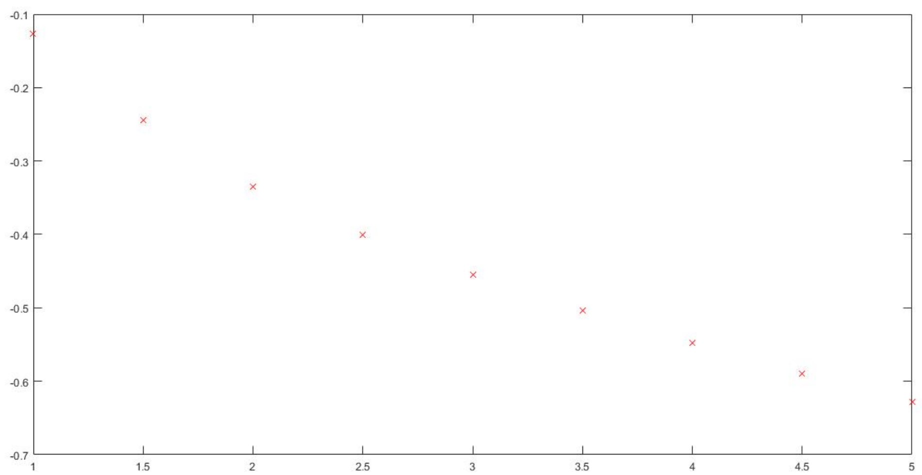

Figure 2: Variation of imaginary part of the growth rate, α_r, as a function of dimensionless strain rate,Λ, for Re $=$ 100, H $=$ 1, $\eta_r = 0$ and $\tau = 0$. tolerance=10^{-5}

0 + 0i and a tolerance value of 10^{-5} In this limit the curves obtained by us are almost similar to those obtained by the journal.
As strain rate increases the real part of the growth rate(α_r) at first decreases to the minimum value and then increases and after a particular value of strain rate α_r becomes positive. This value is called Transition Strain Rate Λ_t. If the strain rate increases above this value the the perturbations become unstable.

Imaginary part of the growth rate is negative and decreases with increasing strain rate. Hence the unstable perturbations are travelling downstream.

7 Conclusion

The low Reynolds number analysis of couette flow of a fluid adjacent to a polymer gel shows that even in absence of fluid inertia, instability is present in fluid flow when strain rate exceeds the critical value (transition strain rate). This happens due to transfer of energy from mean flow to fluctuations due to the shear work done by mean flow at the surface. In the analysis, fourth order differential equations are formed for the displacement field in the gel and were solved numerically. The characteristic matrix was obtained by applying boundary conditions for velocity and stress at the interface. The characteristic equation was non linear and hence could not be solved directly. So, it was solved by taking Re=0 for which it came out to be quadratic. Growth for subsequent Reynolds numbers were calculated by analytical continuation with Re=0 as intial guess. Critical Reynolds number depends on Σ, thickness ratio H, relative viscosity $\eta_r = \eta_g/\eta$ and wave number k. At $\eta_r = 0$, when $Re > Re_t$ (transition Reynold's number), perturbations are unstable for all Σ and k. Re_c (critical Reynold's number) is the minimum of Re_t − k curve. For $\Sigma << 1$, Re_c increases proportional to Σ. For $\Sigma >> 1$, Re_c is directly proportional to Σ^β where $0.75 <= \beta <= 0.8$. For low $1 <= \Sigma <= 1000$, Re_t decreases as η_r increases (increasing gel velocity destabilizes fluid flow). For high $10,000 <= \Sigma <= 100,000$, Re_t increases as η_r increases and it starts decreasing after $\eta_m ax$ (perturbations unstable when $\eta_r < \eta_m ax$).

8 Self-Assessment

We were able to make the code for obtaining the relation between the growth rate and Λ. The Code is running without any compilation errors and we have obtained the similar results for the most unstable growth vs strain rate curve. But this result was obtained for only some initial α which in our case was 0 + 0i. The reasons for not producing the required results for the other initial guesses could be:
• The values of v_z for the two eigenvectors calculated at interface differ largely in magnitude. This is happening because initially the eigenvectors that we have taken are orthogonal to each other. But as we proceed, with increasing iterations these vectors are increasing in different directions (other than the initial ones at z=1 and z=-1). This deviation results in two vectors not orthogonal to each other at the interface. Thus, the concept of taking our required values of u_z and v_z to be linear combination of the two vectors will become invalid and hence giving the errors.
• We have to obtain the curve for the most unstable growth rate which is the maximum of all the solution obtained for α. Hence our initial guess must be large enough for the convergence to take place only at the largest value of α.

9 References

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