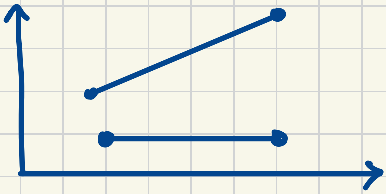
Lecture 8: December 17, 2024

Polynomial interpolation

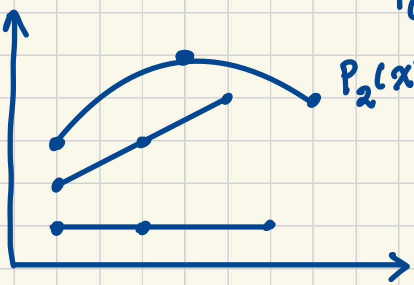


2 data

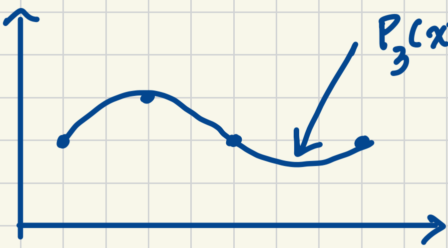
$P_1(x) = a + bx$

Polynomial of degree 1

$P_0(x) = a$



$P_2(x) = a + bx + cx^2$



$P_3(x) = a + bx + cx^2 + dx^3$

Ex: $(1,2), (2,5), (4,1), (5,4)$

$$P_3(x) = a + bx + cx^2 + dx^3$$

$$a + b.1 + c.1^2 + d.1^3 = 2$$

$$a + b.2 + c.2^2 + d.2^3 = 5$$

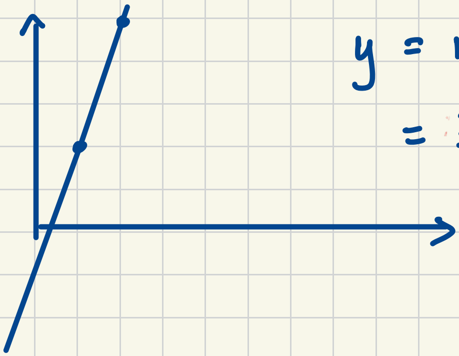
$$a + b.4 + c.4^2 + d.4^3 = 1$$

$$a + b.5 + c.5^2 + d.5^3 = 4$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 5 & 5^2 & 5^3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 4 \end{bmatrix}$$

Newton's form of polynomial

Ex: $(1, 2), (2, 5)$



$$y = mx + c$$

$$= \frac{3}{1}x + (-1)$$

$$y = 3x - 1$$

Ex: $(1,2),\ (2,5)$

$$\boxed{P_1(x) = a + bx} \quad *$$

$a = -1$

$b = 3$

$$a + 1.b = 2 \quad -①$$

$$a + 2.b = 5 \quad -②$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$a = \frac{\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{4-5}{2-1} = -1, \quad b = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}}{1} = 5-2=3$$

$$\boxed{P_1(x) = 3x - 1}$$

$(x_0, y_0), (x_1, y_1)$

$$\boxed{P_1(x) = a + (x - x_0)b} \quad **$$

$$(x_0, y_0) \Rightarrow a + (\cancel{x_0 - x_0})^{0} b = y_0 \Rightarrow \boxed{a = y_0}$$

$$(x_1, y_1) \Rightarrow \boxed{y_0} + (x_1 - x_0)b = y_1$$

$$\boxed{b = \frac{y_1 - y_0}{x_1 - x_0}}$$

Ex: $\overset{(x_0,y_0)}{(1,2)}, \overset{(x_1,y_1)}{(2,5)}$

$$P_1(x) = a + (x-1)b$$

$$a + (1-1)b = 2 \Rightarrow a = 2$$

$$\overset{2}{\cancel{a}} + (2-1)b = 5$$

$$b = \frac{5-2}{2-1} = 3$$

$$\boxed{P_1(x) = 2 + (x-1)(3)} = 2 + 3x - 3 = \underline{\underline{3x-1}}$$

Ex: $\overset{(x_0,y_0)}{(1,2)}, \overset{(x_1,y_1)}{(2,5)}, \overset{(x_2,y_2)}{(4,3)}$

$$\boxed{P_2(x) = a + (x-x_0)b + (x-x_0)(x-x_1)c}$$

$$P_2(x) = \overset{2}{a} + (x-1)\overset{3}{b} + (x-1)(x-2)c\left(-\frac{4}{3}\right)$$

$$a + (\cancel{1-1})^{0}b + (\cancel{1-1})^{0}(1-2)c = 2 \qquad \boxed{a=2}$$

$$\overset{2}{\cancel{a}} + (2-1)b + (2-1)(\cancel{2-2})^{0}c = 5 \qquad b = \frac{5-2}{2-1}$$

$$\overset{2}{\cancel{a}} + \underbrace{(4-1)}_{3\cdot}\underset{3}{\cancel{b}} + (4-1)(4-2)c = 3 \qquad \boxed{b = 3}$$

$$c = \frac{3-2-(9)}{3\cdot 2} = \frac{-8}{6} = \boxed{-\frac{4}{3}} \quad c$$

Ex: $\overset{(x_0,y_0)}{(1,2)}$, $\overset{(x_1,y_1)}{(2,5)}$, $\overset{(x_2,y_2)}{(4,3)}$

$$P_2(x) = a + bx + cx^2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 4 & 4^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Ex: $(2,5), (6,4), (8,5)$

$$P_1(x) = a + (x-2)b = 5 + (x-2)\left(-\frac{1}{4}\right)$$

$$a + (2-2)b = 5 \Rightarrow a = 5$$

$$5 + (6-2)b = 4 \Rightarrow b = \frac{4-5}{4} = -\frac{1}{4}$$

$$\boxed{P_1(x) = 5 + \frac{1}{2} - \frac{1}{4}x = \frac{9}{2} - \frac{1}{4}x}$$

$$P_2(x) = a + (x-2)b + (x-2)(x-6)c$$

$(2,5) \Rightarrow$ $\boxed{a = 5}$

$(6,4) \Rightarrow$ $5 + (6-2)b + 0 = 4$; $\boxed{b = -\frac{1}{4}}$

$(8,5) \Rightarrow$ $5 - \frac{1}{4}(8-2) + (8-2)(8-6)c = 5$

$\boxed{c = ♡}$

Newton's divided difference.

$f(x)$ $\quad (x_0, y_0), (x_1, y_1), (x_2, y_2)$
(x_3, y_3)

$f[x] = f(x)$

$f[x_0] = f(x_0) = y_0$

$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$ $\quad 1^{st}$- difference

$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$

Newton's divided difference table

$$\begin{array}{lll} x_0 & f[x_0] & \\ & & f[x_0, x_1] \\ x_1 & f[x_1] & \end{array}$$

Ex: $\underset{(x_0, y_0)}{(1, 2)}, \underset{(x_1, y_1)}{(2, 5)}$

x_0 1 $\quad f[1] = 2$ a

x_1 2 $\quad f[2] = 5$

$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = \dfrac{5 - 2}{2 - 1} = 3$ b

Ex: $\overset{(x_0,\, y_0)}{(1,\ 2)}$, $\overset{(x_1,\, y_1)}{(2,\ 5)}$, $\overset{(x_2,\, y_2)}{(4,\ 3)}$

2

$x_0 \quad f[x_0]$

$x_1 \quad f[x_1]$

$x_2 \quad f[x_2]$

$f[x_0, x_1]$

$f[x_1, x_2]$

$f[x_0, x_1, x_2]$

x_i	$f[x_i]$		
1	2	$f[1,2] = \frac{5-2}{2-1} = 3$	$f[1,2,4]$
2	5	$f[2,4] = \frac{3-5}{4-2} = -\frac{2}{2} = -1$	$= \frac{f[2,4] - f[1,2]}{4-1}$
4	3		$= \frac{-1-3}{4-1} = -\frac{4}{3}$

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$$

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

standard form of polynomial

$$P_n(x) = a_0 + (x - x_0) a_1 + (x - x_0)(x - x_1) a_2 + \dots$$

$$+ (x - x_0)(x - x_1) \dots (x - x_{n-1}) a_n$$

$$\boxed{\begin{aligned} P_n(x) = {} & f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ & + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1}) f[x_0, x_1, x_2, \dots, x_{n-1}, x_n] \end{aligned}}$$

Ex: (1, 2), (3, 5), (6, 4), (7, 6), (8, 6)

1	2	
3	5	
6	4	
7	6	
8	6	

$f[1,3] = \frac{5-2}{3-1} = \frac{3}{2}$

$f[3,6] = \frac{4-5}{6-3} = -\frac{1}{3}$

$f[6,7] = \frac{6-4}{7-6} = 2$

$f[7,8] = \frac{6-6}{8-7} = 0$

$f[1,3,6] = \frac{f[3,6] - f[1,3]}{6-1}$

$f[3,6,7] = \frac{f[6,7] - f[3,6]}{7-3}$

$f[6,7,8] = \frac{f[7,8] - f[6,7]}{8-}$