Homework #2F

Solve the following system of linear equations

$$
\begin{aligned}
4x_1 + 4x_2 + x_3 &= 5, \\
-2x_1 + x_2 + x_3 &= -1, \\
-5x_1 - 4x_2 + 2x_3 &= -14
\end{aligned}
$$

a) Using **Doolittle** factorization method.

$4x + 4y + z = 5 \rightarrow (1)$

$-2x + y + z = -1 \rightarrow (2)$

$-5x - 4y + 2z = -14 \rightarrow (3)$

Now converting given equations into matrix form

$$
\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix}
$$

Now, A = $\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix}$

Doolittle's method for LU decomposition
Let $A = LU$

$$
\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}
\times
\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}
$$

$$
\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}
=
\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}
$$

This implies

$u_{11} = 4$

$u_{12} = 4$

$u_{13} = 1$

$l_{21}u_{11} = -2 \Rightarrow l_{21} \times 4 = -2 \Rightarrow l_{21} = -\frac{1}{2}$

$l_{21}u_{12} + u_{22} = 1 \Rightarrow \left(-\frac{1}{2}\right) \times 4 + u_{22} = 1 \Rightarrow u_{22} = 3$

$l_{21}u_{13} + u_{23} = 1 \Rightarrow \left(-\frac{1}{2}\right) \times 1 + u_{23} = 1 \Rightarrow u_{23} = \frac{3}{2}$

$l_{31}u_{11} = -5 \Rightarrow l_{31} \times 4 = -5 \Rightarrow l_{31} = -\frac{5}{4}$

$l_{31}u_{12} + l_{32}u_{22} = -4 \Rightarrow \left(-\frac{5}{4}\right) \times 4 + l_{32} \times 3 = -4 \Rightarrow l_{32} = \frac{1}{3}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \Rightarrow \left(-\frac{5}{4}\right) \times 1 + \frac{1}{3} \times \frac{3}{2} + u_{33} = 2 \Rightarrow u_{33} = \frac{11}{4}$

$\therefore A = L \times U = LU$

$$\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{5}{4} & \frac{1}{3} & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 4 & 1 \\ 0 & 3 & \frac{3}{2} \\ 0 & 0 & \frac{11}{4} \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}$$

Now, $Ax = B$, and $A = LU \Rightarrow LUx = B$

let $Ux = y$, then $Ly = B \Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{5}{4} & \frac{1}{3} & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix}$$

$$\begin{aligned} y_1 \qquad\qquad\qquad &= 5 \\ -\frac{1}{2}y_1 + \quad y_2 \qquad\quad &= -1 \\ -\frac{5}{4}y_1 + \frac{1}{3}y_2 + y_3 &= -14 \end{aligned}$$

Now use forward substitution method
From (1)
$y_1 = 5$

From (2)
$-\frac{1}{2}y_1 + y_2 = -1$

$$\Rightarrow -\frac{(5)}{2} + y_2 = -1$$

$$\Rightarrow -\frac{5}{2} + y_2 = -1$$

$$\Rightarrow y_2 = -1 + \frac{5}{2}$$

$$\Rightarrow y_2 = \frac{3}{2}$$

From (3)

$$-\frac{5}{4}y_1 + \frac{1}{3}y_2 + y_3 = -14$$

$$\Rightarrow -\frac{5(5)}{4} + \frac{\left(\frac{3}{2}\right)}{3} + y_3 = -14$$

$$\Rightarrow -\frac{23}{4} + y_3 = -14$$

$$\Rightarrow y_3 = -14 + \frac{23}{4}$$

$$\Rightarrow y_3 = -\frac{33}{4}$$

Now, $Ux = y$

$$\begin{bmatrix} 4 & 4 & 1 \\ 0 & 3 & \frac{3}{2} \\ 0 & 0 & \frac{11}{4} \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{3}{2} \\ -\frac{33}{4} \end{bmatrix}$$

$$\begin{aligned} 4x + 4y + z &= 5 \\ 3y + \frac{3}{2}z &= \frac{3}{2} \\ \frac{11}{4}z &= -\frac{33}{4} \end{aligned}$$

Now use back substitution method

From (3)

$$\frac{11}{4}z = -\frac{33}{4}$$

From (2)

$$3y + \frac{3}{2}z = \frac{3}{2}$$

$$\Rightarrow 3y + \frac{3(-3)}{2} = \frac{3}{2}$$

$$\Rightarrow 3y - \frac{9}{2} = \frac{3}{2}$$

$$\Rightarrow 3y = \frac{3}{2} + \frac{9}{2}$$

$$\Rightarrow 3y = 6$$

$$\Rightarrow y = \frac{6}{3} = 2$$

From (1)

$$4x + 4y + z = 5$$

$$\Rightarrow 4x + 4(2) + (-3) = 5$$

$$\Rightarrow 4x + 5 = 5$$

$$\Rightarrow 4x = 5 - 5$$

$$\Rightarrow 4x = 0$$

$$\Rightarrow x = \frac{0}{4} = 0$$

Solution by Doolittle's method is

$x = 0, y = 2$ and $z = -3$

Crout's method for LU decomposition

Let $A = LU$

$$\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \times \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

This implies

$l_{11} = 4$

$$l_{11}u_{12} = 4 \Rightarrow 4 \times u_{12} = 4 \Rightarrow u_{12} = 1$$

$$l_{11}u_{13} = 1 \Rightarrow 4 \times u_{13} = 1 \Rightarrow u_{13} = \frac{1}{4}$$

$$l_{21} = -2$$

$$l_{21}u_{12} + l_{22} = 1 \Rightarrow (-2) \times 1 + l_{22} = 1 \Rightarrow l_{22} = 3$$

$$l_{21}u_{13} + l_{22}u_{23} = 1 \Rightarrow (-2) \times \frac{1}{4} + 3 \times u_{23} = 1 \Rightarrow u_{23} = \frac{1}{2}$$

$$l_{31} = -5$$

$$l_{31}u_{12} + l_{32} = -4 \Rightarrow (-5) \times 1 + l_{32} = -4 \Rightarrow l_{32} = 1$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2 \Rightarrow (-5) \times \frac{1}{4} + 1 \times \frac{1}{2} + l_{33} = 2 \Rightarrow l_{33} = \frac{11}{4}$$

$\therefore A = L \times U = LU$

$$\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ -5 & 1 & \frac{11}{4} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}$$

Now, $Ax = B$, and $A = LU \Rightarrow LUx = B$

let $Ux = y$, then $Ly = B \Rightarrow$

$$\begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ -5 & 1 & \frac{11}{4} \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix}$$

$$\begin{aligned} 4y_1 & & & = 5 \\ -2y_1 + 3y_2 & & & = -1 \\ -5y_1 + y_2 + \frac{11}{4}y_3 & & & = -14 \end{aligned}$$

Now use forward substitution method
From (1)
$4y_1 = 5$

$\Rightarrow 4y_1 = 5$

$\Rightarrow y_1 = \frac{5}{4} = \frac{5}{4}$

From (2)

$$-2y_1 + 3y_2 = -1$$

$$\Rightarrow -2\left(\frac{5}{4}\right) + 3y_2 = -1$$

$$\Rightarrow -\frac{5}{2} + 3y_2 = -1$$

$$\Rightarrow 3y_2 = -1 + \frac{5}{2}$$

$$\Rightarrow 3y_2 = \frac{3}{2}$$

$$\Rightarrow y_2 = \frac{3}{2} \times \frac{1}{3} = \frac{1}{2}$$

From (3)

$$-5y_1 + y_2 + \frac{11}{4}y_3 = -14$$

$$\Rightarrow -5\left(\frac{5}{4}\right) + \left(\frac{1}{2}\right) + \frac{11y_3}{4} = -14$$

$$\Rightarrow -\frac{23}{4} + \frac{11y_3}{4} = -14$$

$$\Rightarrow \frac{11y_3}{4} = -14 + \frac{23}{4}$$

$$\Rightarrow \frac{11y_3}{4} = -\frac{33}{4}$$

$$\Rightarrow y_3 = -\frac{33}{4} \times \frac{4}{11} = -3$$

Now, $Ux = y$

$$\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \\ -3 \end{bmatrix}$$

$$x + y + \frac{1}{4}z = \frac{5}{4}$$
$$y + \frac{1}{2}z = \frac{1}{2}$$
$$z = -3$$

Now use back substitution method

From (2)

$$y + \frac{1}{2}z = \frac{1}{2}$$

$$\Rightarrow y + \frac{(-3)}{2} = \frac{1}{2}$$

$$\Rightarrow y = 2$$

From (1)

$$x + y + \frac{1}{4}z = \frac{5}{4}$$

$$\Rightarrow x + (2) + \frac{(-3)}{4} = \frac{5}{4}$$

$$\Rightarrow x + \frac{5}{4} = \frac{5}{4}$$

$$\Rightarrow x = \frac{5}{4} - \frac{5}{4}$$

$$\Rightarrow x = 0$$

Solution by Crout's method is

$x = 0, y = 2$ and $z = -3$