Homework #2F

Solve the following system of linear equations

$$4x_1 + 4x_2 + x_3 = 5,$$
$$-2x_1 + x_2 + x_3 = -1,$$
$$-5x_1 - 4x_2 + 2x_3 = -14$$

a) Using **Doolittle** factorization method.

b) Using **Crout** factorization method.

a) Consider that $A = \begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix}$

$A = LU \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix},$

We know that $U = \begin{bmatrix} 4 & 4 & 1 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}, L_{21} = \frac{A_{21}}{U_{11}} = \frac{-2}{4} = -\frac{1}{2}, L_{31} = \frac{A_{31}}{U_{11}} = \frac{-5}{4}, U_{22} = 1 - (-\frac{1}{2} \times 4) = 3, U_{23} = 1 - (-\frac{1}{2} \times 1) = \frac{3}{2}, U_{33} = 2 - (-\frac{5}{4} \times 1) = 3.25.$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -1.25 & 0.333 & 1 \end{bmatrix}, U = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 3 & 1.5 \\ 0 & 0 & 3.25 \end{bmatrix}$

Consider $Ly = b$, $\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -1.25 & 0.333 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix} \Rightarrow \begin{cases} y_1 = 5, \\ -0.5y_1 + y_2 = -1 \\ -1.25 \times 5 + 0.333 \times (1.5) + y_3 = -14 \end{cases} \Rightarrow \begin{cases} y_1 = 5 \\ y_2 = 1.5 \\ y_3 = -5.75 \end{cases}$

Consider $Ux = y$, $\begin{bmatrix} 4 & 4 & 1 \\ 0 & 3 & 1.5 \\ 0 & 0 & 3.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1.5 \\ -5.75 \end{bmatrix} \Rightarrow \begin{cases} 3.25x_3 = -5.75 \\ 3x_2 + 1.5 \times (-1.769) = 1.5 \\ 4x_1 + 4 \times (1.69) + (-1.769) = 5 \end{cases} \Rightarrow \begin{cases} x_3 = -1.769 \\ x_2 = 1.69 \\ x_1 = -1.15 \end{cases}$

b) $A = \begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ -5 & -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix}$, Consider $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$L_{i,j} = A_{i,j} - \sum_{k=1}^{j-1} L_{i,k}U_{k,j}, \quad U_{j,k} = \frac{A_{j,k} - \sum_{m=1}^{j-1} L_{j,m}U_{m,k}}{L_{j,j}} \Rightarrow l_{11} = A_{11} = 4,$

$U_{12} = \frac{A_{12}}{l_{11}} = \frac{4}{4} = 1, U_{13} = \frac{A_{13}}{l_{11}} = \frac{1}{4} = 0.25, \quad l_{21} = A_{21} = -2, l_{22} = A_{22} - l_{21}U_{12} = 1 - (-2) \times 1 = 3$

$U_{23} = \frac{A_{23} - l_{21}U_{13}}{l_{22}} = \frac{1 - (-2 \times 0.25)}{3} = 0.5, \quad l_{31} = A_{31} = -5, l_{32} = A_{32} - l_{31}U_{12} = -4 - (-5 \times 1) = 1,$

$l_{33} = A_{33} - l_{31}U_{13} - l_{32}U_{23}, \quad l_{33} = 2 - (-5 \times 0.25) - (1 \times 0.5) = 3.25 \quad \therefore L = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ -5 & 1 & 3.25 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 & 0.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$

Consider $Ly = b \Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ -5 & 1 & 3.25 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -14 \end{bmatrix} \Rightarrow \begin{cases} 4y_1 = 5 \\ -2 \times (1.25) + 3y_2 = -1 \\ -5 \times (1.25) + 1 \times 0.5 + 3.25y_3 = -14 \end{cases} \Rightarrow \begin{cases} y_1 = 1.25 \\ y_2 = 0.5 \\ y_3 = -2.538 \end{cases}, \therefore y = \begin{bmatrix} 1.25 \\ 0.5 \\ -2.538 \end{bmatrix}$

Consider $Ux = y$, $\begin{bmatrix} 1 & 1 & 0.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.5 \\ -2.538 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2.538 \\ x_2 + 0.5 \times (-2.538) = 0.5 \\ x_1 + 1 \times (1.769) + 0.25 \times (-2.538) \end{cases} \Rightarrow \begin{cases} x_1 = -2.538 \\ x_2 = 1.769 \\ x_3 = -1.15 \end{cases} \quad \therefore x_1 = -1.15, x_2 = 1.769, x_3 = -2.538.$