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Spectral variance estimation and the analysis of turbulence data

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The results of an asymptotic analysis for the variance of power spectra and the real and imaginary parts of complex cross spectra are presented. The formulae for the spectral variances do not require an assumption of jointly Gaussian probability density functions for the time series and, therefore, are of special importance to the analysis of turbulence data. The variances computed from the asymptotic formulae compare favorably with directly estimated variances for power spectra and a third-order cross spectrum which appear in the study of spectral energy transfer in turbulent flows. The asymptotic variance formulae can be useful tools for the design of experiments which require estimation of power and cross spectra.

I. INTRODUCTION

Spectral analysis remains an important tool in turbulence research. A Fourier representation of a turbulent field provides insight into the behavior of the continuous scales of turbulent motions and their interactions. Computer analysis of turbulence data has made accurate computation of a wide range of statistics such as higher-order correlations and spectra a simple and economical operation. Earlier computational and experimental limitations on spectral analysis which used the correlation-transform technique or banks of narrow-band electronic filters have been largely overcome by computer analysis and the fast Fourier transform computational algorithm. The spectral computations which appear in the turbulence literature rarely include estimates of their variances, but proper analysis of spectral data should include some information on the variability of the spectral estimates.

The purpose of this paper is to present some results of an asymptotic theory for the variance of power and cross spectra and to demonstrate the use of variance estimates on the interpretation of a third-order cross spectrum and related power spectra which arise naturally in the study of spectral energy transfer and higher-order isotropy. It is unusual to find measurements of spectra which are accompanied by estimates in the turbulence literature and this may be due in part to the amount of data usually available in laboratory experiments where a nearly unlimited experimental run time is often possible. Typically a spectrum is considered to be "converged" when it appears smooth or when the addition of more data produces only negligible changes in the spectrum estimate. This is an intuitive if workable approach to spectral convergence, but a reliable method for prediction and evaluation of the spectral variances is preferable to intuitive approaches and may also be useful for optimizing the data processing.

The importance of estimating spectral variance has been realized by many investigators. Goodman[1] used a jointly Gaussian assumption to derive variance estimates for power and cross spectra. Measurements in the atmosphere and the oceans are generally more difficult than in the laboratory because of limited experimental times and changing conditions. It is not surprising, then, to find the geophysical literature contains many references to variance of spectral estimators and discussion of optimum methods for performing spectral analysis. For example, Hinich and Clay[2] have used an asymptotic technique to permit making variance estimates of power spectra, coherence, and bispectra. Asymptotic estimates for the power spectra and the real and imaginary parts of cross spectra were available somewhat earlier in the probability and statistics literature, but these results do not appear to be widely available in the applied literature. The asymptotic analysis presented in Rosenblatt[3] forms the mathematical basis for the present discussion.

II. COMPUTATIONAL METHODS AND RESULTS OF ASYMPTOTIC THEORY

The asymptotic formulae for the spectral variances discussed in the present paper were originally derived in general form in Rosenblatt,[3] and the reader interested in the mathematical details of the derivation is referred to this paper and to the references contained therein. Our purpose here is to review the assumptions required for application of the asymptotic theory and to provide a consistent set of expressions for the Fourier transform, power and cross spectra, and the spectral variances. Define two time series (u_t, v_t), $t = 1, \ldots, N$. Then the discrete Fourier transform of each time series is given by

$$\begin{aligned} &F_u(\lambda) = \Delta t \Sigma u_t \exp(-i2\pi\lambda t)\,, \\ &F_v(\lambda) = \Delta t \Sigma v_t \exp(-i2\pi\lambda t)\,, \\ &\lambda = (k-1)\Delta\lambda\,, \quad k = 1, \ldots, N/2+1\,, \\ &t = (j-1)\Delta t\,, \quad j = 1, \ldots, N\,, \end{aligned} \tag{1}$$

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where $\Delta\lambda = 1/(N\Delta t)$, $\Delta t = 1/SR$, N is the number of samples in the record or "ensemble", and SR is the sampling frequency. From the transforms F_u and F_v the spectra (or periodograms) over one record are defined by:

Power spectra of u_t and v_t:

$$F_{uu}(\lambda) = |F_u(\lambda)|^2 \Delta\lambda ,$$

$$F_{vv}(\lambda) = |F_v(\lambda)|^2 \Delta\lambda ,$$

Co-spectrum of u_t with v_t: (2)

$$\mathrm{Re}\, F_{uv}(\lambda) = \mathrm{Re}[F_u(\lambda) F_v^*(\lambda)]\Delta\lambda$$

Quad-spectrum of u_t with v_t:

$$\mathrm{Im}\, F_{uv}(\lambda) = \mathrm{Im}[F_u(\lambda) F_v^*(\lambda)]\Delta\lambda ,$$

where the asterisk denotes the complex conjugate and

$$F_{uv}(\lambda) = \mathrm{Re}\, F_{uv}(\lambda) + i\,\mathrm{Im}\, F_{uv}(\lambda) \tag{3}$$

is the complex cross spectrum between the time series u_t and v_t. It is usually desirable to smooth the spectral estimates by some form of frequency (spatial) averaging. There are many kinds of smoothing, but for our present purposes we have selected a simple boxcar averaging, but we let the width of the boxcar "window" vary logarithmically to obtain maximum smoothing at the highest frequencies and a uniform set of spectral estimates on a logarithmic scale. The one-record periodogram spectral estimates with frequency smoothing are defined by

$$\bar{G}(\bar{\lambda}) = w(\bar{\lambda}) \sum_{\lambda_1 \le \lambda \le \lambda_2} G(\lambda) , \tag{4}$$

where G is used to represent one of F_{uu}, F_{vv}, $\mathrm{Re}\, F_{uv}$, or $\mathrm{Im}\, F_{uv}$. The weighting factor $w(\bar{\lambda})$ depends on the center frequency $\bar{\lambda} = (\lambda_1 + \lambda_2)/2$ according to

$$w(\bar{\lambda}) = \Delta\lambda/(\lambda_2 - \lambda_1) = (\Delta\lambda/2\bar{\lambda})(\xi + 1)/(\xi - 1) , \tag{5}$$

where $\xi = 10^{1/n}$ and n is the selected number of smoothed spectral estimates per frequency decade. For no smoothing $\bar{G}(\bar{\lambda}) = G(\lambda)$ in Eq. (4), with $\bar{\lambda} = \lambda$, giving $w(\bar{\lambda}) = 1$. To reduce the variance of the spectral estimates further we average the spectral estimates (4) over L realizations of the time series. Define the final averaged spectral estimates by

$$f_{uu}(\bar{\lambda}) = (1/L) \sum \bar{F}_{uu}(\bar{\lambda}) ,$$

$$f_{uv}(\bar{\lambda}) = (1/L) \sum \bar{F}_{uv}(\bar{\lambda}) , \tag{6}$$

where a similar expression applies for the definition of the estimated v power spectrum f_{vv}.

Now assume that the moments up to fourth order for the time series u_t and v_t exist, and that a condition similar to strong mixing applies to the fourth order moments.[4] Then, the asymptotic variances of the smoothed and ensemble averaged spectra (6) for large N are given from Rosenblatt[3] by

$$\begin{aligned}
\mathrm{var}[f_{uu}(\bar{\lambda})] &\sim w(\bar{\lambda})[f_{uu}^2(\bar{\lambda})]/L , \\
\mathrm{var}[f_{vv}(\bar{\lambda})] &\sim w(\bar{\lambda})[f_{vv}^2(\bar{\lambda})]/L , \\
\mathrm{var}[\mathrm{Re} f_{uv}(\bar{\lambda})] &\sim w(\bar{\lambda})\{f_{uu}(\bar{\lambda}) f_{vv}(\bar{\lambda}) \\
&\quad + [\mathrm{Re} f_{uv}(\bar{\lambda})]^2 - [\mathrm{Im} f_{uv}(\bar{\lambda})]^2\}/2L , \\
\mathrm{var}[\mathrm{Im} f_{uv}(\bar{\lambda})] &\sim w(\bar{\lambda})\{f_{uu}(\bar{\lambda}) f_{vv}(\bar{\lambda}) \\
&\quad - [\mathrm{Re} f_{uv}(\bar{\lambda})]^2 + [\mathrm{Im} f_{uv}(\bar{\lambda})]^2\}/2L .
\end{aligned} \tag{7}$$

Note that these formulae for the asymptotic variances depend on the estimated spectra themselves. While initial estimates of the spectra are not known *a priori*, they can often be estimated from the data by processing a small number of records, perhaps as few as 5 or 10. The asymptotic variance formulae can then be used to estimate the total number of records and the degree of frequency smoothing required to achieve a desired accuracy in the estimated spectra. A discussion of these techniques for power spectra is given in Oppenheim and Schafer.[5] The present work provides asymptotic variance formulae which can be used for the design of experiments to measure cross spectra in a manner analogous to the methods for power spectra.

The validity of the asymptotic formulae for the spectral variances can be tested by comparing them to variances estimated directly from the computations. If we assume that each record is uncorrelated, the directly estimated spectral variance is given by

$$\mathrm{var}[g(\bar{\lambda})] = [(1/L) \sum G^2(\bar{\lambda}) - g^2(\bar{\lambda})]/L , \tag{8}$$

where, as before, G and g are used to represent one of F_{uu}, F_{vv}, F_{uv}, and f_{uu}, f_{vv}, f_{uv}, respectively. Note that the formula (8) for the direct estimate of spectral variance can be used to measure the convergence of the mean spectra (6) whether or not the asymptotic formulae (7) are applicable. If we define

$$\epsilon(\bar{\lambda}) = |g(\bar{\lambda})| / \mathrm{var}^{1/2}[g(\bar{\lambda})] \tag{9}$$

as the resolution at frequency $\bar{\lambda}$, where g is the desired spectrum as defined previously, then we might require that $\epsilon(\bar{\lambda}) > 3$, say, before the spectrum would be said to be resolved or converged. Clearly, the magnitude of the resolvability criterion is problem dependent and can be thought of as representing the signal-to-noise ratio of a spectral estimate.

III. APPLICATION TO GRID TURBULENCE SPECTRA

As an application we present some results from measurements of grid turbulence in the University of California, San Diego low-speed windtunnel. The measurements of the longitudinal velocity component u were made at $X/M = 48$ downstream of a biplane grid with mesh size $M = 5.08$ cm with a mean speed $U = 7.7$ m/sec. These data are part of a current investigation into higher-order isotropy and energy transfer in grid turbulence. Additional details related to this investigation can be found in Helland *et al.*[6] In order to evaluate the accuracy of the formulae for the asymptotic variance estimates we present the results of calculations for

TABLE I. Power spectra of u and u^2 with corresponding asymptotic and directly estimated spectral variances.

$\bar{\lambda}$ (Hz)	f_{uu} (cm/sec)^2/Hz	Estimated $\mathrm{var}^{1/2}(f_{uu})$ (cm/sec)^2/Hz	Asymptotic $\mathrm{var}^{1/2}(f_{uu})$ (cm/sec)^2/Hz	$f_{u^2u^2}$ (cm/sec)^4/Hz	Estimated $\mathrm{var}^{1/2}(f_{u^2u^2})$ (cm/sec)^4/Hz	Asymptotic $\mathrm{var}^{1/2}(f_{u^2u^2})$ (cm/sec)^4/Hz
1.15	1.069×10^{0}	4.148×10^{-2}	9.407×10^{-2}	1.641×10^{2}	7.208×10^{0}	1.444×10^{1}
1.82	1.062×10^{0}	3.822×10^{-2}	7.418×10^{-2}	1.689×10^{2}	6.889×10^{0}	1.180×10^{1}
2.89	1.031×10^{0}	3.672×10^{-2}	5.722×10^{-2}	1.543×10^{2}	5.985×10^{0}	8.564×10^{0}
4.58	1.103×10^{0}	3.611×10^{-2}	4.865×10^{-2}	1.593×10^{2}	5.336×10^{0}	7.022×10^{0}
7.26	1.027×10^{0}	3.005×10^{-2}	3.596×10^{-2}	1.621×10^{2}	5.726×10^{0}	5.678×10^{0}
11.50	1.208×10^{0}	3.103×10^{-2}	3.361×10^{-2}	1.596×10^{2}	5.474×10^{0}	4.441×10^{0}
18.23	9.872×10^{-1}	1.978×10^{-2}	2.182×10^{-2}	1.580×10^{2}	3.926×10^{0}	3.492×10^{0}
28.89	6.899×10^{-1}	1.159×10^{-2}	1.211×10^{-2}	1.445×10^{2}	3.020×10^{0}	2.536×10^{0}
45.78	4.661×10^{-1}	6.324×10^{-3}	6.499×10^{-3}	1.258×10^{2}	2.301×10^{0}	1.755×10^{0}
72.56	2.794×10^{-1}	3.043×10^{-3}	3.095×10^{-3}	9.148×10^{1}	1.346×10^{0}	1.013×10^{0}
115.0	1.611×10^{-1}	1.377×10^{-3}	1.417×10^{-3}	6.397×10^{1}	7.825×10^{-1}	5.627×10^{-1}
182.3	9.111×10^{-2}	6.710×10^{-4}	6.367×10^{-4}	3.990×10^{1}	4.005×10^{-1}	2.788×10^{-1}
288.9	4.317×10^{-2}	2.471×10^{-4}	2.397×10^{-4}	2.205×10^{1}	1.852×10^{-1}	1.224×10^{-1}
457.8	1.578×10^{-2}	7.188×10^{-5}	6.958×10^{-5}	9.500×10^{0}	7.160×10^{-2}	4.189×10^{-2}
725.6	4.002×10^{-3}	1.612×10^{-5}	1.402×10^{-5}	2.745×10^{0}	1.808×10^{-2}	9.615×10^{-3}
1150.0	6.233×10^{-4}	2.545×10^{-6}	1.734×10^{-6}	4.813×10^{-1}	3.036×10^{-3}	1.339×10^{-3}
1822.6	4.488×10^{-5}	3.852×10^{-7}	9.918×10^{-8}	4.164×10^{-2}	3.126×10^{-4}	9.203×10^{-5}

the power spectra of the time series u and u^2 in Table I and the real and imaginary parts of the cross spectrum between u and u^2 in Table II. These tables include the spectral variances predicted by the asymptotic variance formulae of Eq. (7) compared with the directly estimated variances computed from Eq. (8). The logarithmic smoothing chosen for these calculations was $n = 10$, or 10 spectral estimates per decade in frequency. We should emphasize that smoothing of "spikey" data which is contaminated by 60 Hz and its harmonics must be done with great care since excessive smoothing would lead to a bias in the spectral estimate. The data used in this paper are free of any significant noise spikes of this nature and the spectra are known to behave slowly enough at large frequency that smoothing with $n = 10$ does not significantly bias the spectral estimates. This condition was verified computationally by analyzing the data with $n = 30$ and no significant change in the mean spectral levels was observed. The data had a sample rate $\mathrm{SR} = 4170.5\ \mathrm{sec}^{-1}$ with $N = 4096$ samples per record and $L = 500$ records. Each record actually contained 8192 samples but only half of each record was used in the analysis to enhance the assumption of uncorrelated records important for the comparison between asymptotic and directly estimated spectral variances. The corresponding frequency interval used in the computations was $\Delta\lambda = 1.0182$ Hz.

The asymptotic and directly estimated spectral variances presented in Tables I and II are in reasonable

TABLE II. Real and imaginary parts of the cross-spectrum between u^2 and u with the corresponding asymptotic and directly estimated spectral variances.

$\bar{\lambda}$ (Hz)	$\mathrm{Re}f_{u^2u}$ (cm/sec)^3/Hz	Estimated $\mathrm{var}^{1/2}(\mathrm{Re}f_{u^2u})$ (cm/sec)^3/Hz	Asymptotic $\mathrm{var}^{1/2}(\mathrm{Re}f_{u^2u})$ (cm/sec)^3/Hz	$\mathrm{Im}f_{u^2u}$ (cm/sec)^3/Hz	Estimated $\mathrm{var}^{1/2}(\mathrm{Im}f_{u^2u})$ (cm/sec)^3/Hz	Asymptotic $\mathrm{var}^{1/2}(\mathrm{Im}f_{u^2u})$ (cm/sec)^3/Hz
1.15	2.121×10^{-1}	3.814×10^{-1}	8.235×10^{-1}	5.611×10^{-1}	3.840×10^{-1}	8.246×10^{-1}
1.82	2.607×10^{-1}	3.777×10^{-1}	6.611×10^{-1}	5.902×10^{-1}	3.438×10^{-1}	6.628×10^{-1}
2.89	-4.536×10^{-2}	3.396×10^{-1}	4.950×10^{-1}	4.032×10^{-2}	3.435×10^{-1}	4.950×10^{-1}
4.58	-7.427×10^{-1}	3.042×10^{-1}	4.139×10^{-1}	6.261×10^{-2}	3.185×10^{-1}	4.126×10^{-1}
7.26	-4.830×10^{-1}	2.691×10^{-1}	3.193×10^{-1}	-6.716×10^{-1}	2.605×10^{-1}	3.189×10^{-1}
11.50	-1.488×10^{-1}	2.738×10^{-1}	2.713×10^{-1}	-1.664×10^{0}	2.419×10^{-1}	2.712×10^{-1}
18.23	-2.641×10^{-1}	1.809×10^{-1}	1.924×10^{-1}	-2.116×10^{0}	1.818×10^{-1}	1.923×10^{-1}
28.89	-1.848×10^{-1}	1.241×10^{-1}	1.230×10^{-1}	-1.240×10^{0}	1.121×10^{-1}	1.229×10^{-1}
45.78	6.101×10^{-2}	7.998×10^{-2}	7.514×10^{-2}	-7.610×10^{-1}	7.041×10^{-2}	7.513×10^{-2}
72.56	9.713×10^{-2}	4.297×10^{-2}	3.953×10^{-2}	-3.081×10^{-1}	3.691×10^{-2}	3.951×10^{-2}
115.0	6.291×10^{-2}	2.242×10^{-2}	1.996×10^{-2}	-9.696×10^{-2}	1.653×10^{-2}	1.996×10^{-2}
182.3	3.981×10^{-2}	1.134×10^{-2}	9.422×10^{-3}	-2.495×10^{-2}	7.675×10^{-3}	9.418×10^{-3}
288.9	2.488×10^{-2}	4.635×10^{-3}	3.831×10^{-3}	3.044×10^{-3}	2.860×10^{-3}	3.828×10^{-3}
457.8	1.524×10^{-2}	1.682×10^{-3}	1.208×10^{-3}	7.316×10^{-3}	9.472×10^{-4}	1.206×10^{-3}
725.6	5.321×10^{-3}	3.602×10^{-4}	2.597×10^{-4}	4.680×10^{-3}	2.067×10^{-4}	2.590×10^{-4}
1150.0	1.404×10^{-3}	5.518×10^{-5}	3.408×10^{-5}	1.389×10^{-3}	2.895×10^{-5}	3.385×10^{-5}
1822.6	1.738×10^{-4}	7.756×10^{-6}	2.139×10^{-6}	1.602×10^{-4}	2.508×10^{-6}	2.104×10^{-6}

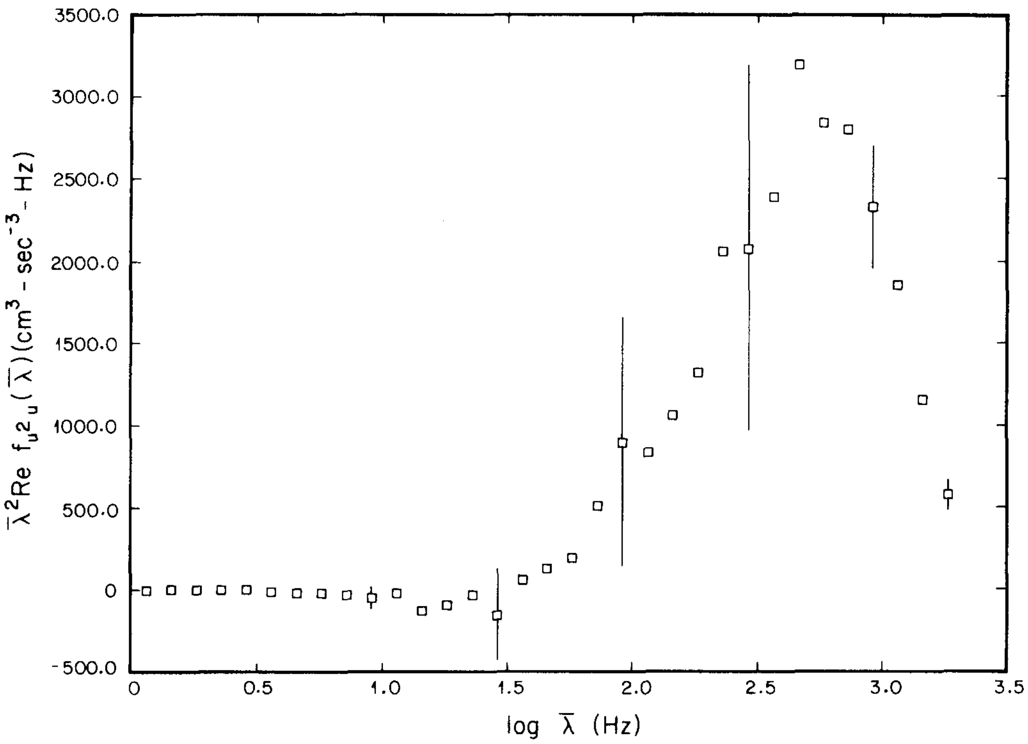

FIG. 1. $\bar{\lambda}^2 \mathrm{Re} f_{u^2u}(\bar{\lambda})$. The vertical bars denote ± 3 standard deviations of the spectrum.

agreement over most of the frequency range for both the power and cross spectra. Some differences do exist between the asymptotic and directly estimated variances for frequencies greater than about a few hundred Hz and also at the lowest frequencies. Based on the results presented here as well as a large number of other computations of a similar nature we suggest that the asymptotic formulae for spectral variance can provide useful predictions of the variances or standard deviations, but they must certainly be used with some caution. The asymptotic prediction can sometimes over or underestimate the standard deviations by about a factor or two, usually at the lowest and highest frequencies in the spectrum. The deviations between asymptotic and directly estimated variances at the highest frequencies can be reduced by increasing the resolution of the frequency analysis. For example, by increasing the number of frequencies computed per

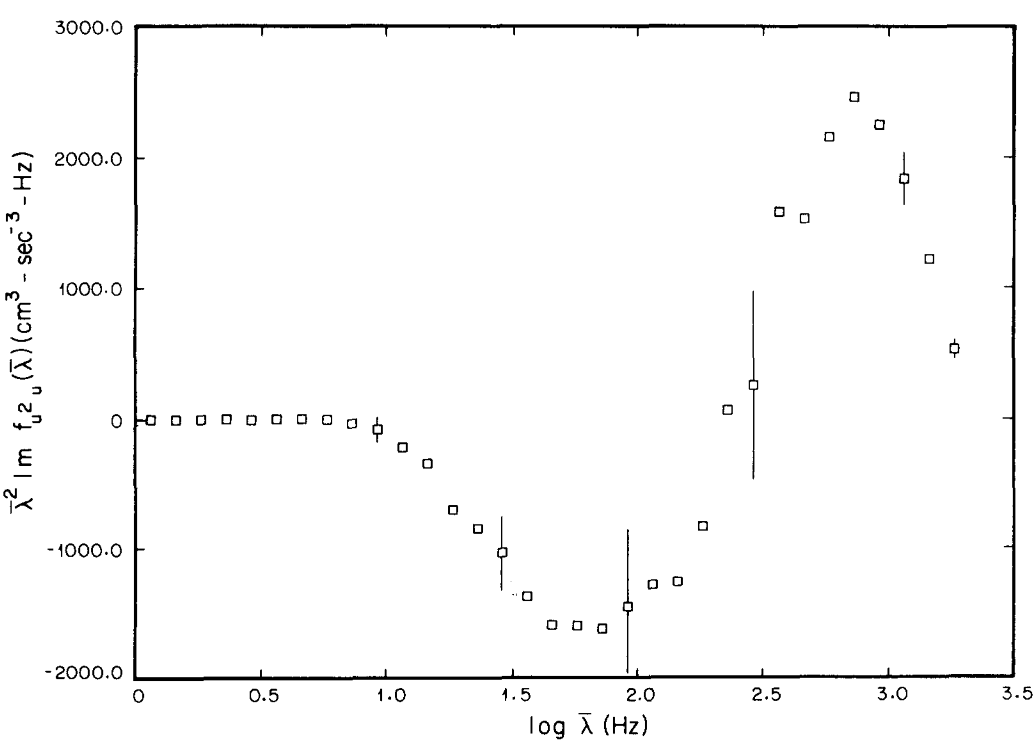

FIG. 2. $\bar{\lambda}^2 \mathrm{Im} f_{u^2u}(\bar{\lambda})$. The vertical bars denote ± 3 standard deviations of the spectrum.

decade to $n = 30$, we obtained excellent agreement between the predicted and estimated spectral variances. It appears that rapid variations of a spectral magnitude within the smoothing interval reduces the accuracy of the asymptotic variance estimate.

Finally, we present the real and imaginary parts of the cross spectra of u^2 with u multiplied by $\bar{\lambda}^2$ in Figs. 1 and 2, respectively. The squared frequency factor has been used to emphasize the spectra at the intermediate and high frequencies. The vertical bars represent ± 3 standard deviations of the measured spectra using the direct variance estimation technique of Eq. (8). For an isotropic flow the $\mathrm{Re} f_{u^2 u}$ in Fig. 1 should be identically zero for all $\bar{\lambda}$. This is equivalent to obtaining an antisymmetric triple correlation $R_{u^2 u}(\tau) = -R_{u^2 u}(-\tau)$, where τ is the correlation time delay. It appears that the frequencies $\bar{\lambda} > 100$ Hz are resolved contrary to the assumption of full isotropy at all frequencies. The magnitude of $\mathrm{Im} f_{u^2 u}$ in Fig. 2 appears to be comparable to the real part, however, notice that the imaginary part is more highly resolved than the real part. We say a spectrum is highly resolved when the ratio of the spectrum to its local standard deviation $\epsilon(\bar{\lambda})$ is large. The nonzero real part does imply anisotropy in the grid turbulence, however, these results are not necessarily in contradiction with local isotropy at high frequencies despite the large values of $\bar{\lambda}^2 \mathrm{Re} f_{u^2 u}$ at frequencies near 1000 Hz. The spectrum $f_{u^2 u}(\bar{\lambda})$ is difficult to interpret in a locally isotropic flow field because it represents an integration over one of the two frequency axes in the appropriate bispectrum. The integrated bispectrum, or marginal bispectrum as $f_{u^2 u}$ may properly be called, mixes low frequency interactions with high frequency interactions thus smearing the meaning of the localness of its frequency argument. Readers interested in additional elaboration on bispectrum estimates in grid turbulence are referred to Helland *et al.*[7] The nonzero $\mathrm{Re} f_{u^2 u}$ for large $\bar{\lambda}$ may signify two possible conclusions for the small scale grid turbulence velocity field. Either the smaller scales are showing a deviation from local isotropy or there is a significant interaction between large and small scales, or possibly a combination of both of these effects.

IV. CONCLUDING REMARKS

We have shown that asymptotic formulae for spectral variances are in reasonable agreement with directly estimated variances of power spectra and the real and imaginary parts of a complex cross spectrum. The asymptotic formulae can provide a useful technique for the design of experiments by providing a nearly *a priori* estimate of the spectral resolution or "error" $\epsilon(\bar{\lambda})$ as a function of smoothing interval and the number of records. Alternatively, good estimates of the spectral variances can be obtained *a posteriori* from the computed spectra. For example, estimates of the standard deviation of published spectra can be calculated if a few parameters of the data analysis are known. While the asymptotic variance formulae for power spectra have been available for many years, the comparable results for the cross spectrum do not appear to be widely known even in the applied spectrum analysis literature. Caution must be used when applying the asymptotic variance results to spectral analysis for small N. We have not investigated the effect of variations in N and can offer no guidance for the lower limit of the asymptotic theory. It should be emphasized that the asymptotic formulae can be useful tools for data analysis, but even if they are not precise predictors of spectral variances, the directly estimated variance method of Eq. (8) can always be used and probably should be used for critical data analysis tasks. Finally, we emphasize that the basic time series need not be Gaussian for the asymptotic theory to apply, while some turbulence signals are nearly Gaussian, most are not.

ACKNOWLEDGMENTS

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