UNIT-I

Introduction:

Learning,

Types of Machine Learning.

Concept learning:

Introduction,

Version Spaces and the Candidate Elimination Algorithm.

Learning with Trees:

Constructing Decision Trees,

CART,

Classification Example

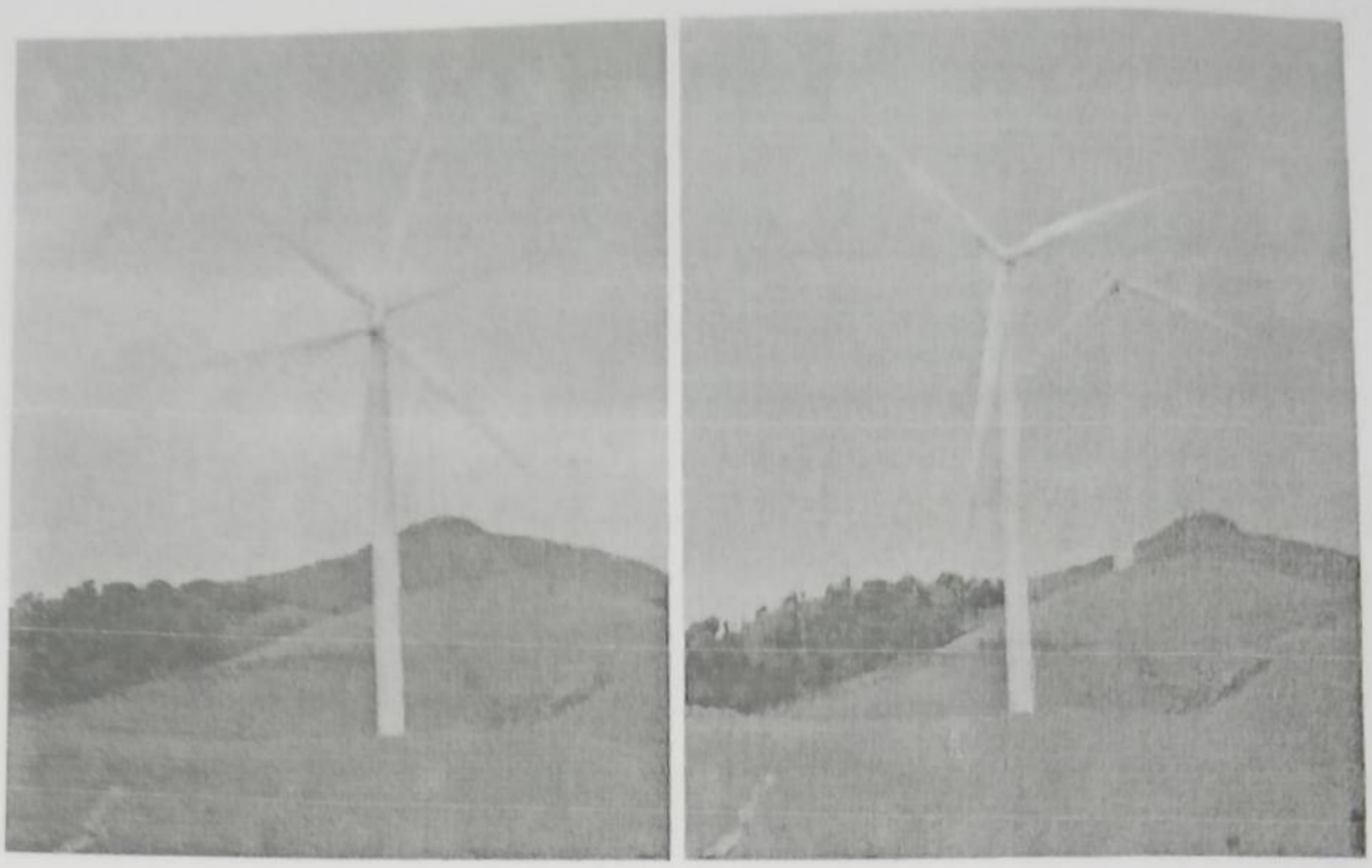

FIGURE 1.2: Two views of the same two wind turbines (Te Apiti wind farm, Ashhurst, New Zealand) taken at an angle of about 30° to each other. The two-dimensional projections of three-dimensional objects hides information.

1.2 Learning

Before we delve too much further into the topic, let's step back and think about what learning actually is. The key concept that we will need to think about for our machines is learning from data, since data is what we have: terabytes of it, in some cases. However, it isn't too large a step to put that into human behavioural terms, and talk about learning from experience. Hopefully, we all agree that humans and other animals can display behaviours that we label as intelligent by learning from experience. Learning is what gives us flexibility in our life: the fact that we can adjust and adapt to new circumstances, and learn new tricks, no matter how old a dog we are! The important parts of animal learning for this book are remembering, adapting, and generalising: recognising that last time we were in this situation (saw this data) we tried out some particular action (gave this output) and it worked (was correct), so we'll try it again, or it didn't work, so we'll try something different. The last word, generalising, is about recognising similarity between different situations, so that things that applied in one place can be used in another. This is what makes learning useful, because we can use our knowledge in lots of different places.

Of course, there are plenty of other bits to intelligence, such as reasoning, and logical deduction, but we won't worry too much about those. We

difference b/w AI and ML
AI is symbolic processing
ML is subsymbolic (NO symbols) I-5

are interested in the most fundamental parts of intelligence—learning and adapting—and how we can model them in a computer. There has also been a lot of interest in making computers reason and deduce facts. This was the basis of most early Artificial Intelligence, and is sometimes known as symbolic processing because the computer manipulates symbols that reflect the environment. In contrast, machine learning methods are sometimes called subsymbolic because no symbols or symbolic manipulation are involved.

1.2.1 Machine Learning

Machine learning, then, is about making computers modify or adapt their actions (whether these actions are making predictions, or controlling a robot) so that these actions get more accurate, where accuracy is measured by how well the chosen actions reflect the correct ones. Imagine that you are playing Scrabble (or some other game) against a computer. You might beat it every time in the beginning, but after lots of games it starts beating you, until finally you never win. Either you are getting worse, or the computer is learning how to win at Scrabble. Having learnt to beat you, it can go on and use the same strategies against other players, so that it doesn't start from scratch with each new player: this is a form of generalisation.

It is only over the past decade or so that the inherent multi-disciplinarity of machine learning has been recognised. ML merges ideas from neuroscience and biology, statistics, mathematics, and physics, to make computers learn. There is a fantastic existence proof that learning is possible, which is the bag of water and electricity (together with a few trace chemicals) sitting between your ears. In Section 1.5 we will have a brief peek inside and see if there is anything we can borrow/steal in order to make machine learning algorithms. It turns out that there is, and neural networks have grown from exactly this, although even their own father wouldn't recognise them now, after the developments that have seen them reinterpreted as statistical learners. Another thing that has driven the change in direction of machine learning research is data mining, which looks at the extraction of useful information from massive datasets (by men with computers and pocket protectors rather than pickaxes and hard hats), and which requires efficient algorithms, putting more of the emphasis back onto computer science.

The computational complexity of the machine learning methods will also be of interest to us since what we are producing is algorithms. It is particularly important because we might want to use some of the methods on very large datasets, so algorithms that have high-degree polynomial complexity in the size of the dataset (or worse) will be a problem. The complexity is often broken into two parts: the complexity of training, and the complexity of applying the trained algorithm. Training does not happen very often, and is not usually time critical, so it can take longer. However, we often want a decision about a test point quickly, and there are potentially lots of test points when an algorithm is in use, so this needs to have low computational cost.

1.3 Types of Machine Learning

In the example that started the chapter, your webpage, the aim was to predict what software a visitor to the website might buy based on information that you can collect. There are a couple of interesting things in there. The first is the data. It might be useful to know what software visitors have bought before, and how old they are. However, it is not possible to get that information from their web browser (even cookies can't tell you how old somebody is), so you can't use that information. Picking the variables that you want to use (which are called features in the jargon) is a very important part of finding good solutions to problems, and something that we will talk about in several places in the book. Equally, choosing how to process the data can be important. This can be seen in the example in the time of access. Your computer can store this down to the nearest millisecond, but that isn't very useful, since you would like to spot similar patterns between users. For this reason, in the example above I chose to quantise it down to one of the set `morning, afternoon, evening, night`; obviously I need to ensure that these times are correct for their time zones, too.

We are going to loosely define learning as meaning getting better at some task through practice. This leads to a couple of vital questions: how does the computer know whether it is getting better or not, and how does it know how to improve? There are several different possible answers to these questions, and they produce different types of machine learning. For now we will consider the quesion of knowing whether or not the machine is learning. We can tell the algorithm the correct answer for a problem so that it gets it right next time (which is what would happen in the webpage example, since we know what software the person bought). We hope that we only have to tell it a few right answers and then it can 'work out' how to get the correct answers for other problems (generalise). Alternatively, we can tell it whether or not the answer was correct, but not how to find the correct answer, so that it has to search for the right answer. A variant of this is that we give a score for the answer, according to how correct it is, rather than just a 'right or wrong' response. Finally, we might not have any correct answers, we just want the algorithm to find inputs that have something in common.

These different answers to the question provide a useful way to classify the different algorithms that we will be talking about:

Supervised learning A training set of examples with the correct responses (targets) are provided and, based on this training set, the algorithm generalises to respond correctly to all possible inputs. This is also called learning from exemplars.

Unsupervised learning Correct responses are not provided, instead the algorithm tries to identify similarities between the inputs so that inputs

that have something in common are categorised together. The statistical approach to unsupervised learning is known as density estimation.

Reinforcement learning This is somewhere between supervised and unsupervised learning. The algorithm gets told when the answer is wrong, but does not get told how to correct it. It has to explore and try out different possibilities until it works out how to get the answer right. Reinforcement learning is sometime called learning with a critic because of this monitor that scores the answer, but does not suggest improvements.

Evolutionary learning Biological evolution can be seen as a learning process: biological organisms adapt to improve their survival rates and chance of having offspring in their environment. We'll look at how we can model this in a computer, using an idea of fitness, which corresponds to a score for how good the current solution is.

The most common type of learning is supervised learning, and it is going to be the focus of the next few chapters. So, before we get started, we'll have a look at what it is, and the kinds of problems that can be solved using it.

1.4 Supervised Learning

As has already been suggested, the webpage example is a typical problem for supervised learning. There is a set of data (the training data) that consists of a set of input data that has target data, which is the answer that the algorithm should produce, attached. This is usually written as a set of data $(\mathbf{x}_i, \mathbf{t}_i)$, where the inputs are $\mathbf{x}_i$, the targets are $\mathbf{t}_i$ and the i index suggests that we have lots of pieces of data, indexed by i running from 1 to some upper limit N. Note that the inputs and targets are written in boldface font to signify vectors, since each piece of data has values for several different features; the notation used in the book is described in more detail in Section 2.1. If we had examples of every possible piece of input data, then we could put them together into a big look-up table, and there would be no need for machine learning at all. The thing that makes machine learning better than that is generalisation: the algorithm should produce sensible outputs for inputs that weren't encountered during learning. This also has the result that the algorithm can deal with noise, which is small inaccuracies in the data that are inherent in measuring any real world process. It is hard to specify rigorously what generalisation means, but let's see if an example helps.

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CHAPTER 2

CONCEPT LEARNING AND THE GENERAL-TO-SPECIFIC ORDERING

The problem of inducing general functions from specific training examples is central to learning. This chapter considers concept learning: acquiring the definition of a general category given a sample of positive and negative training examples of the category. Concept learning can be formulated as a problem of searching through a predefined space of potential hypotheses for the hypothesis that best fits the training examples. In many cases this search can be efficiently organized by taking advantage of a naturally occurring structure over the hypothesis space—a general-to-specific ordering of hypotheses. This chapter presents several learning algorithms and considers situations under which they converge to the correct hypothesis. We also examine the nature of inductive learning and the justification by which any program may successfully generalize beyond the observed training data.

2.1 INTRODUCTION

Much of learning involves acquiring general concepts from specific training examples. People, for example, continually learn general concepts or categories such as "bird," "car," "situations in which I should study more in order to pass the exam," etc. Each such concept can be viewed as describing some subset of objects or events defined over a larger set (e.g., the subset of animals that constitute

birds). Alternatively, each concept can be thought of as a boolean-valued function defined over this larger set (e.g., a function defined over all animals, whose value is true for birds and false for other animals).

In this chapter we consider the problem of automatically inferring the general definition of some concept, given examples labeled as members or nonmembers of the concept. This task is commonly referred to as *concept learning*, or approximating a boolean-valued function from examples.

Concept learning. Inferring a boolean-valued function from training examples of its input and output.

2.2 A CONCEPT LEARNING TASK

To ground our discussion of concept learning, consider the example task of learning the target concept "days on which my friend Aldo enjoys his favorite water sport." Table 2.1 describes a set of example days, each represented by a set of *attributes*. The attribute *EnjoySport* indicates whether or not Aldo enjoys his favorite water sport on this day. The task is to learn to predict the value of *EnjoySport* for an arbitrary day, based on the values of its other attributes.

What hypothesis representation shall we provide to the learner in this case? Let us begin by considering a simple representation in which each hypothesis consists of a conjunction of constraints on the instance attributes. In particular, let each hypothesis be a vector of six constraints, specifying the values of the six attributes *Sky, AirTemp, Humidity, Wind, Water,* and *Forecast*. For each attribute, the hypothesis will either

- indicate by a "?" that any value is acceptable for this attribute,
- specify a single required value (e.g., *Warm*) for the attribute, or
- indicate by a "Ø" that no value is acceptable.

If some instance x satisfies all the constraints of hypothesis h, then h classifies x as a positive example ($h(x) = 1$). To illustrate, the hypothesis that Aldo enjoys his favorite sport only on cold days with high humidity (independent of the values of the other attributes) is represented by the expression

$$\langle ?, Cold, High, ?, ?, ? \rangle$$

Example	*Sky*	*AirTemp*	*Humidity*	*Wind*	*Water*	*Forecast*	*EnjoySport*
1	Sunny	Warm	Normal	Strong	Warm	Same	Yes
2	Sunny	Warm	High	Strong	Warm	Same	Yes
3	Rainy	Cold	High	Strong	Warm	Change	No
4	Sunny	Warm	High	Strong	Cool	Change	Yes

TABLE 2.1
Positive and negative training examples for the target concept *EnjoySport*.

The most general hypothesis—that every day is a positive example—is represented by

$$\langle ?, ?, ?, ?, ?, ? \rangle$$

and the most specific possible hypothesis—that *no* day is a positive example—is represented by

$$\langle \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle$$

To summarize, the *EnjoySport* concept learning task requires learning the set of days for which $EnjoySport = yes$, describing this set by a conjunction of constraints over the instance attributes. In general, any concept learning task can be described by the set of instances over which the target function is defined, the target function, the set of candidate hypotheses considered by the learner, and the set of available training examples. The definition of the *EnjoySport* concept learning task in this general form is given in Table 2.2.

2.2.1 Notation

Throughout this book, we employ the following terminology when discussing concept learning problems. The set of items over which the concept is defined is called the set of *instances*, which we denote by X. In the current example, X is the set of all possible days, each represented by the attributes *Sky*, *AirTemp*, *Humidity*, *Wind*, *Water*, and *Forecast*. The concept or function to be learned is called the *target concept*, which we denote by c. In general, c can be any boolean-valued function defined over the instances X; that is, $c : X \rightarrow \{0, 1\}$. In the current example, the target concept corresponds to the value of the attribute *EnjoySport* (i.e., $c(x) = 1$ if $EnjoySport = Yes$, and $c(x) = 0$ if $EnjoySport = No$).

- **Given:**
  - Instances X: Possible days, each described by the attributes
    - *Sky* (with possible values *Sunny*, *Cloudy*, and *Rainy*),
    - *AirTemp* (with values *Warm* and *Cold*),
    - *Humidity* (with values *Normal* and *High*),
    - *Wind* (with values *Strong* and *Weak*),
    - *Water* (with values *Warm* and *Cool*), and
    - *Forecast* (with values *Same* and *Change*).
  - Hypotheses H: Each hypothesis is described by a conjunction of constraints on the attributes *Sky*, *AirTemp*, *Humidity*, *Wind*, *Water*, and *Forecast*. The constraints may be "?" (any value is acceptable), "$\emptyset$" (no value is acceptable), or a specific value.
  - Target concept c: $EnjoySport : X \rightarrow \{0, 1\}$
  - Training examples D: Positive and negative examples of the target function (see Table 2.1).
- **Determine:**
  - A hypothesis h in H such that $h(x) = c(x)$ for all x in X.

TABLE 2.2
The *EnjoySport* concept learning task.

When learning the target concept, the learner is presented a set of *training examples*, each consisting of an instance x from X, along with its target concept value $c(x)$ (e.g., the training examples in Table 2.1). Instances for which $c(x) = 1$ are called *positive examples*, or members of the target concept. Instances for which $c(x) = 0$ are called *negative examples*, or nonmembers of the target concept. We will often write the ordered pair $\langle x, c(x) \rangle$ to describe the training example consisting of the instance x and its target concept value $c(x)$. We use the symbol D to denote the set of available training examples.

Given a set of training examples of the target concept c, the problem faced by the learner is to hypothesize, or estimate, c. We use the symbol H to denote the set of *all possible hypotheses* that the learner may consider regarding the identity of the target concept. Usually H is determined by the human designer's choice of hypothesis representation. In general, each hypothesis h in H represents a boolean-valued function defined over X; that is, $h : X \rightarrow \{0, 1\}$. The goal of the learner is to find a hypothesis h such that $h(x) = c(x)$ for all x in X.

2.2.2 The Inductive Learning Hypothesis

Notice that although the learning task is to determine a hypothesis h identical to the target concept c over the entire set of instances X, the only information available about c is its value over the training examples. Therefore, inductive learning algorithms can at best guarantee that the output hypothesis fits the target concept over the training data. Lacking any further information, our assumption is that the best hypothesis regarding unseen instances is the hypothesis that best fits the observed training data. This is the fundamental assumption of inductive learning, and we will have much more to say about it throughout this book. We state it here informally and will revisit and analyze this assumption more formally and more quantitatively in Chapters 5, 6, and 7.

> **The inductive learning hypothesis.** Any hypothesis found to approximate the target function well over a sufficiently large set of training examples will also approximate the target function well over other unobserved examples.

2.3 CONCEPT LEARNING AS SEARCH

Concept learning can be viewed as the task of searching through a large space of hypotheses implicitly defined by the hypothesis representation. The goal of this search is to find the hypothesis that best fits the training examples. It is important to note that by selecting a hypothesis representation, the designer of the learning algorithm implicitly defines the space of all hypotheses that the program can ever represent and therefore can ever learn. Consider, for example, the instances X and hypotheses H in the *EnjoySport* learning task. Given that the attribute *Sky* has three possible values, and that *AirTemp*, *Humidity*, *Wind*, *Water*, and *Forecast* each have two possible values, the instance space X contains exactly

$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 96$ distinct instances. A similar calculation shows that there are $5 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 5120$ *syntactically distinct* hypotheses within H. Notice, however, that every hypothesis containing one or more "$\emptyset$" symbols represents the empty set of instances; that is, it classifies every instance as negative. Therefore, the number of *semantically distinct* hypotheses is only $1 + (4 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 973$. Our *EnjoySport* example is a very simple learning task, with a relatively small, finite hypothesis space. Most practical learning tasks involve much larger, sometimes infinite, hypothesis spaces.

If we view learning as a search problem, then it is natural that our study of learning algorithms will examine different strategies for searching the hypothesis space. We will be particularly interested in algorithms capable of efficiently searching very large or infinite hypothesis spaces, to find the hypotheses that best fit the training data.

2.3.1 General-to-Specific Ordering of Hypotheses

Many algorithms for concept learning organize the search through the hypothesis space by relying on a very useful structure that exists for any concept learning problem: a general-to-specific ordering of hypotheses. By taking advantage of this naturally occurring structure over the hypothesis space, we can design learning algorithms that exhaustively search even infinite hypothesis spaces without explicitly enumerating every hypothesis. To illustrate the general-to-specific ordering, consider the two hypotheses

$$h_1 = \langle Sunny, ?, ?, Strong, ?, ? \rangle$$

$$h_2 = \langle Sunny, ?, ?, ?, ?, ? \rangle$$

Now consider the sets of instances that are classified positive by h_1 and by h_2. Because h_2 imposes fewer constraints on the instance, it classifies more instances as positive. In fact, any instance classified positive by h_1 will also be classified positive by h_2. Therefore, we say that h_2 is more general than h_1.

This intuitive "more general than" relationship between hypotheses can be defined more precisely as follows. First, for any instance x in X and hypothesis h in H, we say that x *satisfies* h if and only if $h(x) = 1$. We now define the *more_general_than_or_equal_to* relation in terms of the sets of instances that satisfy the two hypotheses: Given hypotheses h_j and h_k, h_j is *more_general_than_or_equal_to* h_k if and only if any instance that satisfies h_k also satisfies h_j.

Definition: Let h_j and h_k be boolean-valued functions defined over X. Then h_j is **more_general_than_or_equal_to** h_k (written $h_j \geq_g h_k$) if and only if

$$(\forall x \in X)[(h_k(x) = 1) \rightarrow (h_j(x) = 1)]$$

We will also find it useful to consider cases where one hypothesis is strictly more general than the other. Therefore, we will say that h_j is (strictly) *more_general_than*

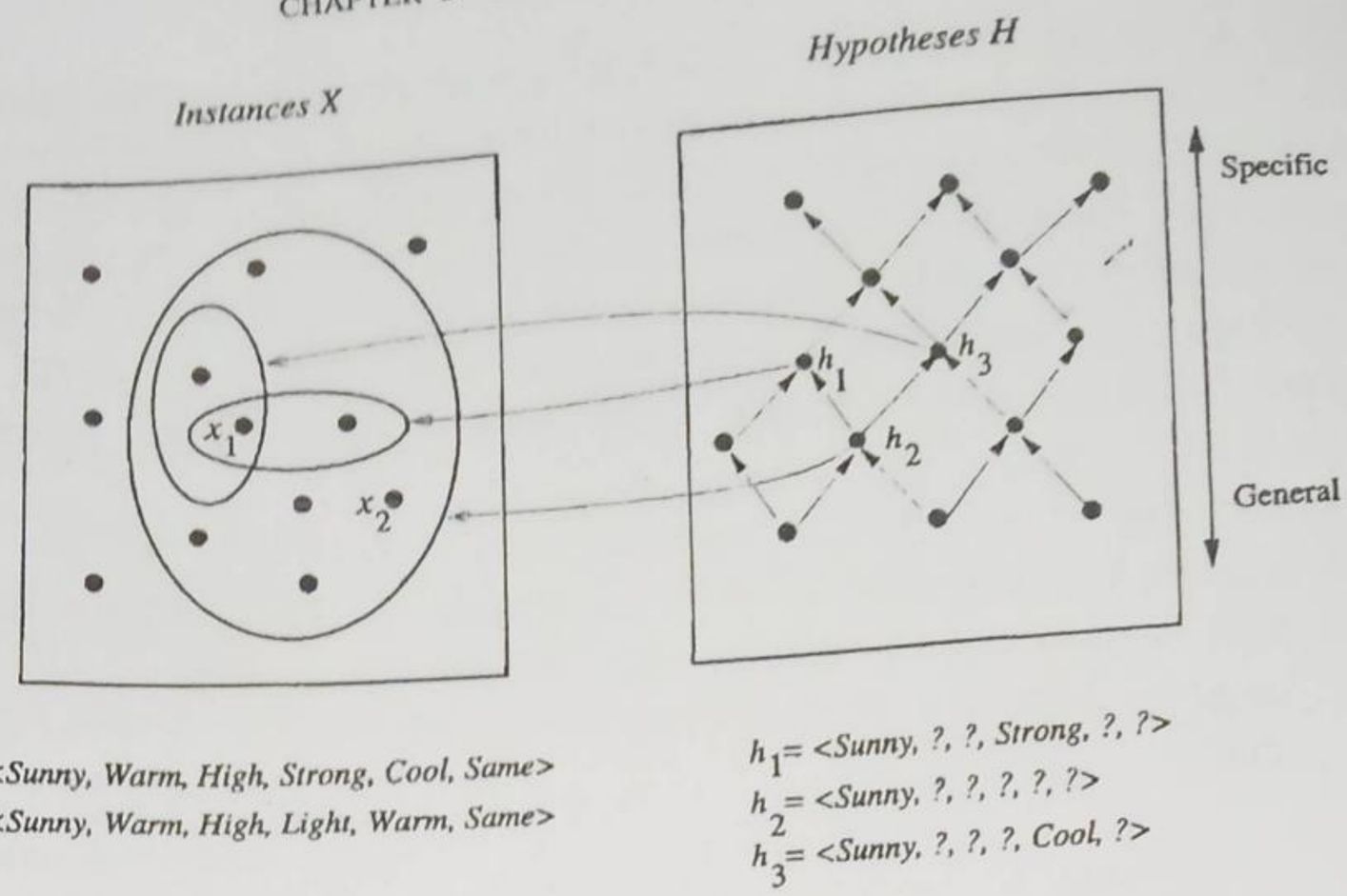


FIGURE 2.1
Instances, hypotheses, and the *more_general_than* relation. The box on the left represents the set X of all instances, the box on the right the set H of all hypotheses. Each hypothesis corresponds to some subset of X—the subset of instances that it classifies positive. The arrows connecting hypotheses represent the *more_general_than* relation, with the arrow pointing toward the less general hypothesis. Note the subset of instances characterized by h_2 subsumes the subset characterized by h_1, hence h_2 is *more_general_than* h_1.

h_k (written $h_j >_g h_k$) if and only if $(h_j \geq_g h_k) \wedge (h_k \not\geq_g h_j)$. Finally, we will sometimes find the inverse useful and will say that h_j is *more_specific_than* h_k when h_k is *more_general_than* h_j.

To illustrate these definitions, consider the three hypotheses h_1, h_2, and h_3 from our *EnjoySport* example, shown in Figure 2.1. How are these three hypotheses related by the $\geq_g$ relation? As noted earlier, hypothesis h_2 is more general than h_1 because every instance that satisfies h_1 also satisfies h_2. Similarly, h_2 is more general than h_3. Note that neither h_1 nor h_3 is more general than the other; although the instances satisfied by these two hypotheses intersect, neither set subsumes the other. Notice also that the $\geq_g$ and $>_g$ relations are defined independent of the target concept. They depend only on which instances satisfy the two hypotheses and not on the classification of those instances according to the target concept. Formally, the $\geq_g$ relation defines a partial order over the hypothesis space H (the relation is reflexive, antisymmetric, and transitive). Informally, when we say the structure is a partial (as opposed to total) order, we mean there may be pairs of hypotheses such as h_1 and h_3, such that $h_1 \not\geq_g h_3$ and $h_3 \not\geq_g h_1$.

The $\geq_g$ relation is important because it provides a useful structure over the hypothesis space H for *any* concept learning problem. The following sections present concept learning algorithms that take advantage of this partial order to efficiently organize the search for hypotheses that fit the training data.

1. Initialize h to the most specific hypothesis in H
2. For each positive training instance x
   - For each attribute constraint a_i in h
     If the constraint a_i is satisfied by x
     Then do nothing
     Else replace a_i in h by the next more general constraint that is satisfied by x
3. Output hypothesis h

TABLE 2.3
FIND-S Algorithm.

2.4 FIND-S: FINDING A MAXIMALLY SPECIFIC HYPOTHESIS

How can we use the *more_general_than* partial ordering to organize the search for a hypothesis consistent with the observed training examples? One way is to begin with the most specific possible hypothesis in H, then generalize this hypothesis each time it fails to cover an observed positive training example. (We say that a hypothesis "covers" a positive example if it correctly classifies the example as positive.) To be more precise about how the partial ordering is used, consider the FIND-S algorithm defined in Table 2.3.

To illustrate this algorithm, assume the learner is given the sequence of training examples from Table 2.1 for the *EnjoySport* task. The first step of FIND-S is to initialize h to the most specific hypothesis in H

$$h \leftarrow \langle \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rangle$$

Upon observing the first training example from Table 2.1, which happens to be a positive example, it becomes clear that our hypothesis is too specific. In particular, none of the "$\emptyset$" constraints in h are satisfied by this example, so each is replaced by the next more general constraint that fits the example; namely, the attribute values for this training example.

$$h \leftarrow \langle Sunny, Warm, Normal, Strong, Warm, Same \rangle$$

This h is still very specific; it asserts that all instances are negative except for the single positive training example we have observed. Next, the second training example (also positive in this case) forces the algorithm to further generalize h, this time substituting a "?" in place of any attribute value in h that is not satisfied by the new example. The refined hypothesis in this case is

$$h \leftarrow \langle Sunny, Warm, ?, Strong, Warm, Same \rangle$$

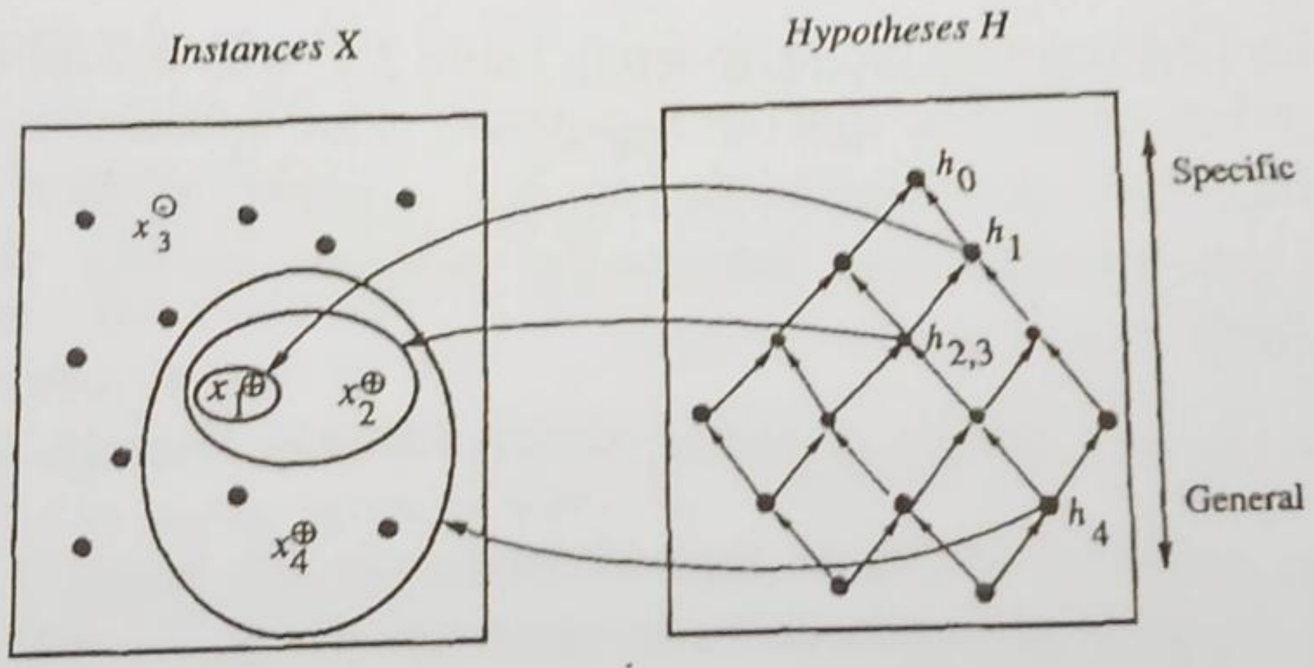
Upon encountering the third training example—in this case a negative example—the algorithm makes no change to h. In fact, the FIND-S algorithm simply *ignores every negative example*! While this may at first seem strange, notice that in the current case our hypothesis h is already consistent with the new negative example (i.e., h correctly classifies this example as negative), and hence no revision

is needed. In the general case, as long as we assume that the hypothesis space contains a hypothesis that describes the true target concept c and that the training data contains no errors, then the current hypothesis h can never require a revision in response to a negative example. To see why, recall that the current hypothesis h is the most specific hypothesis in H consistent with the observed positive examples. Because the target concept c is also assumed to be in H and to be consistent with the positive training examples, c must be *more_general_than_or_equal_to* h. But the target concept c will never cover a negative example, thus neither will h (by the definition of *more_general_than*). Therefore, no revision to h will be required in response to any negative example.

To complete our trace of FIND-S, the fourth (positive) example leads to a further generalization of h

$$h \leftarrow \langle Sunny, Warm, ?, Strong, ?, ? \rangle$$

The FIND-S algorithm illustrates one way in which the *more_general_than* partial ordering can be used to organize the search for an acceptable hypothesis. The search moves from hypothesis to hypothesis, searching from the most specific to progressively more general hypotheses along one chain of the partial ordering. Figure 2.2 illustrates this search in terms of the instance and hypothesis spaces. At each step, the hypothesis is generalized only as far as necessary to cover the new positive example. Therefore, at each stage the hypothesis is the most specific hypothesis consistent with the training examples observed up to this point (hence the name FIND-S). The literature on concept learning is



x_1 = <Sunny Warm Normal Strong Warm Same>, +
x_2 = <Sunny Warm High Strong Warm Same>, +
x_3 = <Rainy Cold High Strong Warm Change>, -
x_4 = <Sunny Warm High Strong Cool Change>, +

h_0 = <∅, ∅, ∅, ∅, ∅, ∅>
h_1 = <Sunny Warm Normal Strong Warm Same>
h_2 = <Sunny Warm ? Strong Warm Same>
h_3 = <Sunny Warm ? Strong Warm Same>
h_4 = <Sunny Warm ? Strong ? ?>

FIGURE 2.2
The hypothesis space search performed by FIND-S. The search begins (h_0) with the most specific hypothesis in H, then considers increasingly general hypotheses (h_1 through h_4) as mandated by the training examples. In the instance space diagram, positive training examples are denoted by "+," negative by "–," and instances that have not been presented as training examples are denoted by a solid circle.

populated by many different algorithms that utilize this same *more_general_than* partial ordering to organize the search in one fashion or another. A number of such algorithms are discussed in this chapter, and several others are presented in Chapter 10.

The key property of the FIND-S algorithm is that for hypothesis spaces described by conjunctions of attribute constraints (such as H for the *EnjoySport* task), FIND-S is guaranteed to output the most specific hypothesis within H that is consistent with the positive training examples. Its final hypothesis will also be consistent with the negative examples provided the correct target concept is contained in H, and provided the training examples are correct. However, there are several questions still left unanswered by this learning algorithm, such as:

- Has the learner converged to the correct target concept? Although FIND-S will find a hypothesis consistent with the training data, it has no way to determine whether it has found the *only* hypothesis in H consistent with the data (i.e., the correct target concept), or whether there are many other consistent hypotheses as well. We would prefer a learning algorithm that could determine whether it had converged and, if not, at least characterize its uncertainty regarding the true identity of the target concept.

- Why prefer the most specific hypothesis? In case there are multiple hypotheses consistent with the training examples, FIND-S will find the most specific. It is unclear whether we should prefer this hypothesis over, say, the most general, or some other hypothesis of intermediate generality.

- Are the training examples consistent? In most practical learning problems there is some chance that the training examples will contain at least some errors or noise. Such inconsistent sets of training examples can severely mislead FIND-S, given the fact that it ignores negative examples. We would prefer an algorithm that could at least detect when the training data is inconsistent and, preferably, accommodate such errors.

- What if there are several maximally specific consistent hypotheses? In the hypothesis language H for the *EnjoySport* task, there is always a unique, most specific hypothesis consistent with any set of positive examples. However, for other hypothesis spaces (discussed later) there can be several maximally specific hypotheses consistent with the data. In this case, FIND-S must be extended to allow it to backtrack on its choices of how to generalize the hypothesis, to accommodate the possibility that the target concept lies along a different branch of the partial ordering than the branch it has selected. Furthermore, we can define hypothesis spaces for which there is no maximally specific consistent hypothesis, although this is more of a theoretical issue than a practical one (see Exercise 2.7).

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2.5 VERSION SPACES AND THE CANDIDATE-ELIMINATION ALGORITHM

This section describes a second approach to concept learning, the CANDIDATE-ELIMINATION algorithm, that addresses several of the limitations of FIND-S. Notice that although FIND-S outputs a hypothesis from H, that is consistent with the training examples, this is just one of many hypotheses from H that might fit the training data equally well. The key idea in the CANDIDATE-ELIMINATION algorithm is to output a description of the set of *all hypotheses consistent with the training examples*. Surprisingly, the CANDIDATE-ELIMINATION algorithm computes the description of this set without explicitly enumerating all of its members. This is accomplished by again using the *more_general_than* partial ordering, this time to maintain a compact representation of the set of consistent hypotheses and to incrementally refine this representation as each new training example is encountered.

The CANDIDATE-ELIMINATION algorithm has been applied to problems such as learning regularities in chemical mass spectroscopy (Mitchell 1979) and learning control rules for heuristic search (Mitchell et al. 1983). Nevertheless, practical applications of the CANDIDATE-ELIMINATION and FIND-S algorithms are limited by the fact that they both perform poorly when given noisy training data. More importantly for our purposes here, the CANDIDATE-ELIMINATION algorithm provides a useful conceptual framework for introducing several fundamental issues in machine learning. In the remainder of this chapter we present the algorithm and discuss these issues. Beginning with the next chapter, we will examine learning algorithms that are used more frequently with noisy training data.

2.5.1 Representation

The CANDIDATE-ELIMINATION algorithm finds all describable hypotheses that are consistent with the observed training examples. In order to define this algorithm precisely, we begin with a few basic definitions. First, let us say that a hypothesis is *consistent* with the training examples if it correctly classifies these examples.

Definition: A hypothesis h is **consistent** with a set of training examples D if and only if $h(x) = c(x)$ for each example $\langle x, c(x)\rangle$ in D.

$$Consistent(h, D) \equiv (\forall \langle x, c(x)\rangle \in D)\ h(x) = c(x)$$

Notice the key difference between this definition of *consistent* and our earlier definition of *satisfies*. An example x is said to *satisfy* hypothesis h when $h(x) = 1$, regardless of whether x is a positive or negative example of the target concept. However, whether such an example is *consistent* with h depends on the target concept, and in particular, whether $h(x) = c(x)$.

The CANDIDATE-ELIMINATION algorithm represents the set of *all* hypotheses consistent with the observed training examples. This subset of all hypotheses is

called the *version space* with respect to the hypothesis space H and the training examples D, because it contains all plausible versions of the target concept.

Definition: The **version space**, denoted $VS_{H,D}$, with respect to hypothesis space H and training examples D, is the subset of hypotheses from H consistent with the training examples in D.

$$VS_{H,D} \equiv \{h \in H | Consistent(h, D)\}$$

2.5.2 The LIST-THEN-ELIMINATE Algorithm

One obvious way to represent the version space is simply to list all of its members. This leads to a simple learning algorithm, which we might call the LIST-THEN-ELIMINATE algorithm, defined in Table 2.4.

The LIST-THEN-ELIMINATE algorithm first initializes the version space to contain all hypotheses in H, then eliminates any hypothesis found inconsistent with any training example. The version space of candidate hypotheses thus shrinks as more examples are observed, until ideally just one hypothesis remains that is consistent with all the observed examples. This, presumably, is the desired target concept. If insufficient data is available to narrow the version space to a single hypothesis, then the algorithm can output the entire set of hypotheses consistent with the observed data.

In principle, the LIST-THEN-ELIMINATE algorithm can be applied whenever the hypothesis space H is finite. It has many advantages, including the fact that it is guaranteed to output all hypotheses consistent with the training data. Unfortunately, it requires exhaustively enumerating all hypotheses in H—an unrealistic requirement for all but the most trivial hypothesis spaces.

2.5.3 A More Compact Representation for Version Spaces

The CANDIDATE-ELIMINATION algorithm works on the same principle as the above LIST-THEN-ELIMINATE algorithm. However, it employs a much more compact representation of the version space. In particular, the version space is represented by its most general and least general members. These members form general and specific boundary sets that delimit the version space within the partially ordered hypothesis space.

The LIST-THEN-ELIMINATE Algorithm

1. $VersionSpace \leftarrow$ a list containing every hypothesis in H
2. For each training example, $\langle x, c(x)\rangle$
   remove from $VersionSpace$ any hypothesis h for which $h(x) \neq c(x)$
3. Output the list of hypotheses in $VersionSpace$

TABLE 2.4
The LIST-THEN-ELIMINATE algorithm.

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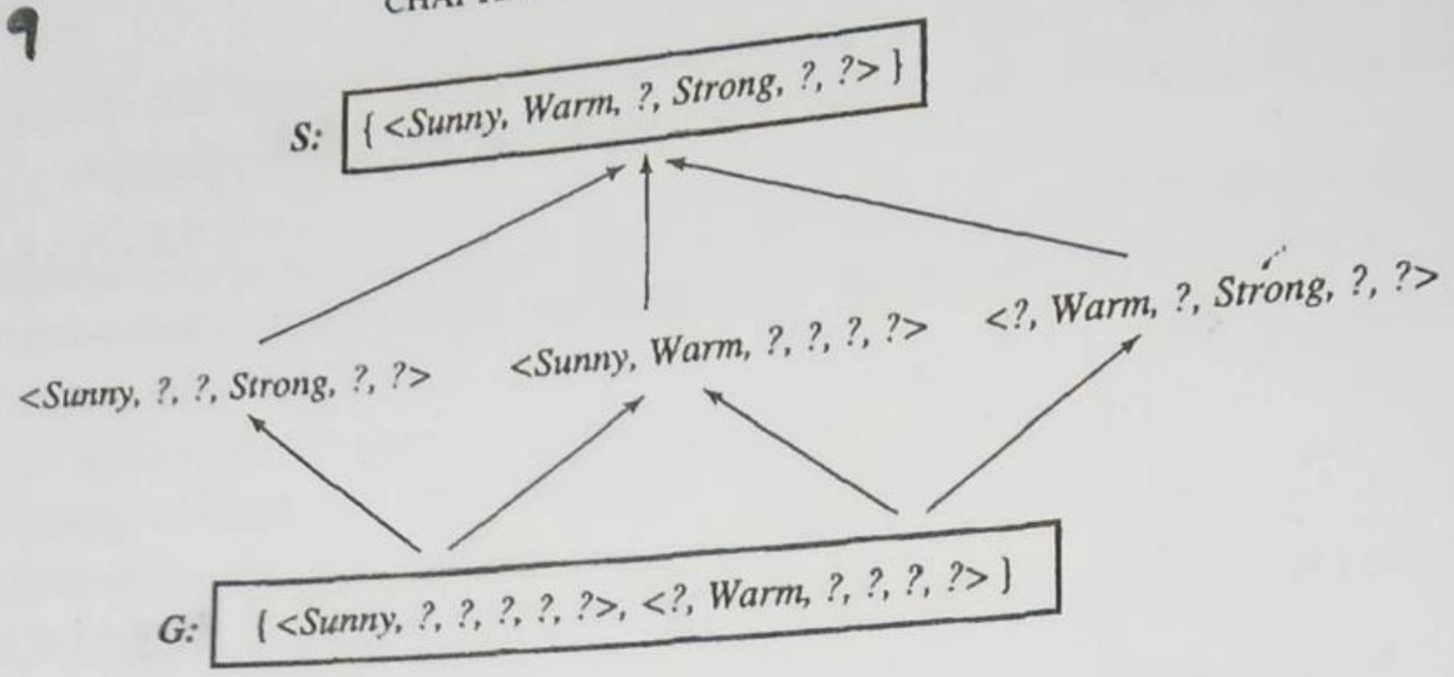

FIGURE 2.3
A version space with its general and specific boundary sets. The version space includes all six hypotheses shown here, but can be represented more simply by S and G. Arrows indicate instances of the *more_general_than* relation. This is the version space for the *EnjoySport* concept learning problem and training examples described in Table 2.1.

To illustrate this representation for version spaces, consider again the *EnjoySport* concept learning problem described in Table 2.2. Recall that given the four training examples from Table 2.1, FIND-S outputs the hypothesis

$$h = \langle Sunny, Warm, ?, Strong, ?, ? \rangle$$

In fact, this is just one of six different hypotheses from H that are consistent with these training examples. All six hypotheses are shown in Figure 2.3. They constitute the version space relative to this set of data and this hypothesis representation. The arrows among these six hypotheses in Figure 2.3 indicate instances of the *more_general_than* relation. The CANDIDATE-ELIMINATION algorithm represents the version space by storing only its most general members (labeled G in Figure 2.3) and its most specific (labeled S in the figure). Given only these two sets S and G, it is possible to enumerate all members of the version space as needed by generating the hypotheses that lie between these two sets in the general-to-specific partial ordering over hypotheses.

It is intuitively plausible that we can represent the version space in terms of its most specific and most general members. Below we define the boundary sets G and S precisely and prove that these sets do in fact represent the version space.

Definition: The **general boundary** G, with respect to hypothesis space H and training data D, is the set of maximally general members of H consistent with D.

$$G \equiv \{g \in H | Consistent(g, D) \wedge (\neg \exists g' \in H)[(g' >_g g) \wedge Consistent(g', D)]\}$$

Definition: The **specific boundary** S, with respect to hypothesis space H and training data D, is the set of minimally general (i.e., maximally specific) members of H consistent with D.

$$S \equiv \{s \in H | Consistent(s, D) \wedge (\neg \exists s' \in H)[(s >_g s') \wedge Consistent(s', D)]\}$$

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As long as the sets G and S are well defined (see Exercise 2.7), they completely specify the version space. In particular, we can show that the version space is precisely the set of hypotheses contained in G, plus those contained in S, plus those that lie between G and S in the partially ordered hypothesis space. This is stated precisely in Theorem 2.1.

Theorem 2.1. Version space representation theorem. Let X be an arbitrary set of instances and let H be a set of boolean-valued hypotheses defined over X. Let $c : X \rightarrow \{0, 1\}$ be an arbitrary target concept defined over X, and let D be an arbitrary set of training examples $\{\langle x, c(x)\rangle\}$. For all X, H, c, and D such that S and G are well defined,

$$VS_{H,D} = \{h \in H | (\exists s \in S)(\exists g \in G)(g \geq_g h \geq_g s)\}$$

Proof. To prove the theorem it suffices to show that (1) every h satisfying the right-hand side of the above expression is in $VS_{H,D}$ and (2) every member of $VS_{H,D}$ satisfies the right-hand side of the expression. To show (1) let g be an arbitrary member of G, s be an arbitrary member of S, and h be an arbitrary member of H, such that $g \geq_g h \geq_g s$. Then by the definition of S, s must be satisfied by all positive examples in D. Because $h \geq_g s$, h must also be satisfied by all positive examples in D. Similarly, by the definition of G, g cannot be satisfied by any negative example in D, and because $g \geq_g h$, h cannot be satisfied by any negative example in D. Because h is satisfied by all positive examples in D and by no negative examples in D, h is consistent with D, and therefore h is a member of $VS_{H,D}$. This proves step (1). The argument for (2) is a bit more complex. It can be proven by assuming some h in $VS_{H,D}$ that does not satisfy the right-hand side of the expression, then showing that this leads to an inconsistency. (See Exercise 2.6.) □

2.5.4 CANDIDATE-ELIMINATION Learning Algorithm

The CANDIDATE-ELIMINATION algorithm computes the version space containing all hypotheses from H that are consistent with an observed sequence of training examples. It begins by initializing the version space to the set of all hypotheses in H; that is, by initializing the G boundary set to contain the most general hypothesis in H

$$G_0 \leftarrow \{\langle ?, ?, ?, ?, ?, ?\rangle\}$$

and initializing the S boundary set to contain the most specific (least general) hypothesis

$$S_0 \leftarrow \{\langle \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle\}$$

These two boundary sets delimit the entire hypothesis space, because every other hypothesis in H is both more general than S_0 and more specific than G_0. As each training example is considered, the S and G boundary sets are generalized and specialized, respectively, to eliminate from the version space any hypotheses found inconsistent with the new training example. After all examples have been processed, the computed version space contains all the hypotheses consistent with these examples and only these hypotheses. This algorithm is summarized in Table 2.5.

Initialize G to the set of maximally general hypotheses in H
Initialize S to the set of maximally specific hypotheses in H
For each training example d, do

- If d is a positive example
  - Remove from G any hypothesis inconsistent with d
  - For each hypothesis s in S that is not consistent with d
    - Remove s from S
    - Add to S all minimal generalizations h of s such that
      - h is consistent with d, and some member of G is more general than h
    - Remove from S any hypothesis that is more general than another hypothesis in S
- If d is a negative example
  - Remove from S any hypothesis inconsistent with d
  - For each hypothesis g in G that is not consistent with d
    - Remove g from G
    - Add to G all minimal specializations h of g such that
      - h is consistent with d, and some member of S is more specific than h
  - Remove from G any hypothesis that is less general than another hypothesis in G

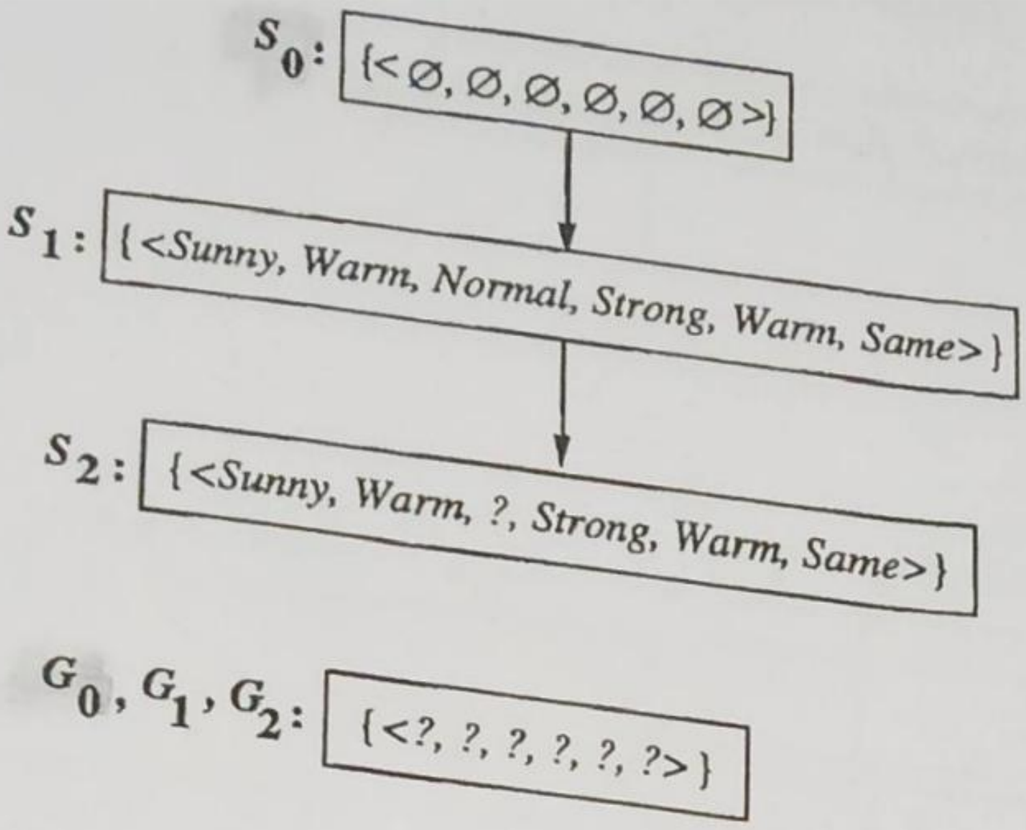
TABLE 2.5
CANDIDATE-ELIMINATION algorithm using version spaces. Notice the duality in how positive and negative examples influence S and G.

Notice that the algorithm is specified in terms of operations such as computing minimal generalizations and specializations of given hypotheses, and identifying nonminimal and nonmaximal hypotheses. The detailed implementation of these operations will depend, of course, on the specific representations for instances and hypotheses. However, the algorithm itself can be applied to any concept learning task and hypothesis space for which these operations are well-defined. In the following example trace of this algorithm, we see how such operations can be implemented for the representations used in the *EnjoySport* example problem.

2.5.5 An Illustrative Example

Figure 2.4 traces the CANDIDATE-ELIMINATION algorithm applied to the first two training examples from Table 2.1. As described above, the boundary sets are first initialized to G_0 and S_0, the most general and most specific hypotheses in H, respectively.

When the first training example is presented (a positive example in this case), the CANDIDATE-ELIMINATION algorithm checks the S boundary and finds that it is overly specific—it fails to cover the positive example. The boundary is therefore revised by moving it to the least more general hypothesis that covers this new example. This revised boundary is shown as S_1 in Figure 2.4. No update of the G boundary is needed in response to this training example because G_0 correctly covers this example. When the second training example (also positive) is observed, it has a similar effect of generalizing S further to S_2, leaving G again unchanged (i.e., $G_2 = G_1 = G_0$). Notice the processing of these first



Training examples:

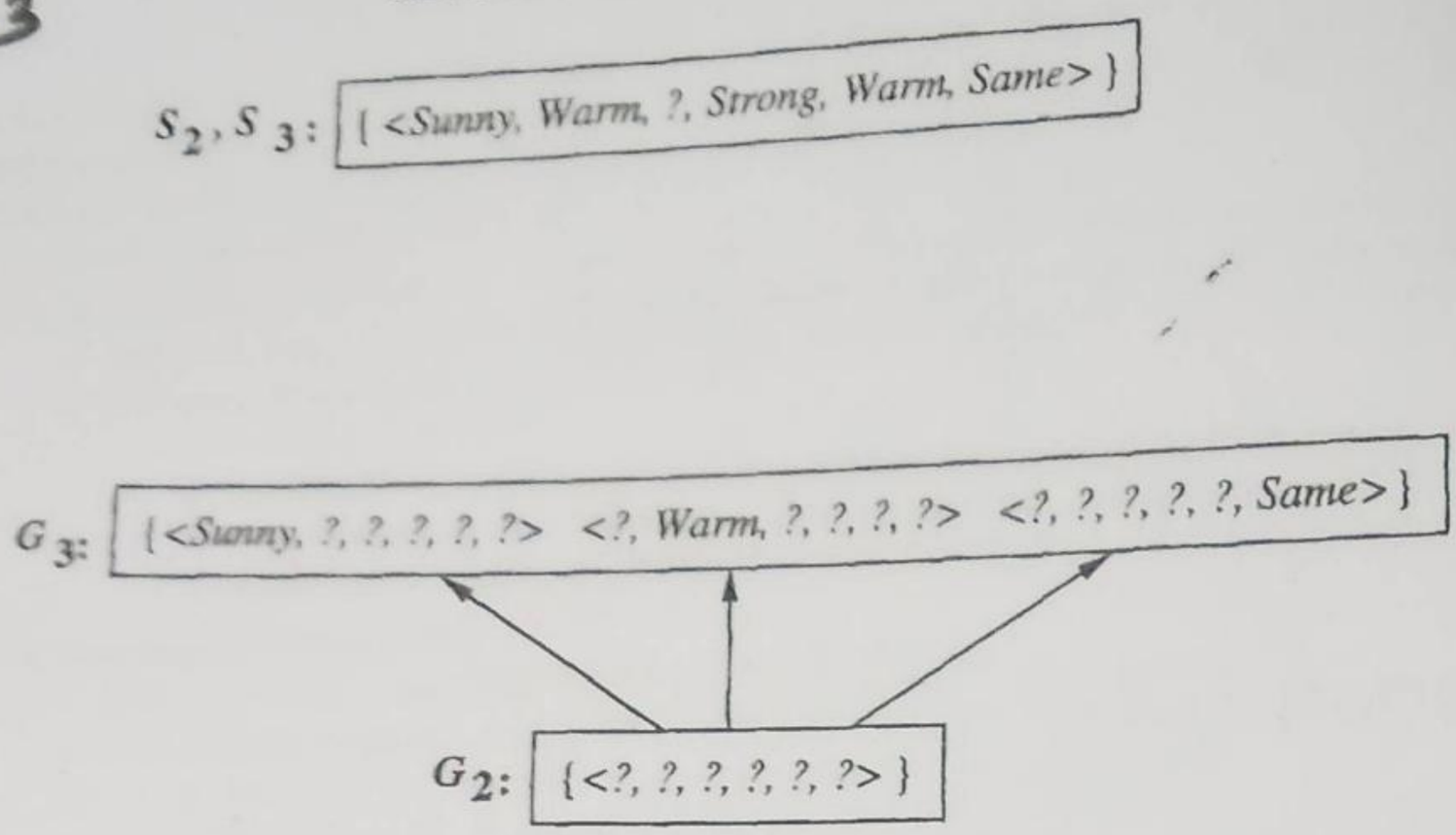
1. <*Sunny, Warm, Normal, Strong, Warm, Same*>, *Enjoy Sport = Yes*
2. <*Sunny, Warm, High, Strong, Warm, Same*>, *Enjoy Sport = Yes*

FIGURE 2.4
CANDIDATE-ELIMINATION Trace 1. S_0 and G_0 are the initial boundary sets corresponding to the most specific and most general hypotheses. Training examples 1 and 2 force the S boundary to become more general, as in the FIND-S algorithm. They have no effect on the G boundary.

two positive examples is very similar to the processing performed by the FIND-S algorithm.

As illustrated by these first two steps, positive training examples may force the S boundary of the version space to become increasingly general. Negative training examples play the complimentary role of forcing the G boundary to become increasingly specific. Consider the third training example, shown in Figure 2.5. This negative example reveals that the G boundary of the version space is overly general; that is, the hypothesis in G incorrectly predicts that this new example is a positive example. The hypothesis in the G boundary must therefore be specialized until it correctly classifies this new negative example. As shown in Figure 2.5, there are several alternative minimally more specific hypotheses. All of these become members of the new G_3 boundary set.

Given that there are six attributes that could be specified to specialize G_2, why are there only three new hypotheses in G_3? For example, the hypothesis $h = \langle ?, ?, Normal, ?, ?, ? \rangle$ is a minimal specialization of G_2 that correctly labels the new example as a negative example, but it is not included in G_3. The reason this hypothesis is excluded is that it is inconsistent with the previously encountered positive examples. The algorithm determines this simply by noting that h is not more general than the current specific boundary, S_2. In fact, the S boundary of the version space forms a summary of the previously encountered positive examples that can be used to determine whether any given hypothesis



Training Example:

3. <*Rainy, Cold, High, Strong, Warm, Change*>, *EnjoySport=No*

FIGURE 2.5
CANDIDATE-ELIMINATION Trace 2. Training example 3 is a negative example that forces the G_2 boundary to be specialized to G_3. Note several alternative maximally general hypotheses are included in G_3.

is consistent with these examples. Any hypothesis more general than S will, by definition, cover any example that S covers and thus will cover any past positive example. In a dual fashion, the G boundary summarizes the information from previously encountered negative examples. Any hypothesis more specific than G is assured to be consistent with past negative examples. This is true because any such hypothesis, by definition, cannot cover examples that G does not cover.

The fourth training example, as shown in Figure 2.6, further generalizes the S boundary of the version space. It also results in removing one member of the G boundary, because this member fails to cover the new positive example. This last action results from the first step under the condition "If d is a positive example" in the algorithm shown in Table 2.5. To understand the rationale for this step, it is useful to consider why the offending hypothesis must be removed from G. Notice it cannot be specialized, because specializing it would not make it cover the new example. It also cannot be generalized, because by the definition of G, any more general hypothesis will cover at least one negative training example. Therefore, the hypothesis must be dropped from the G boundary, thereby removing an entire branch of the partial ordering from the version space of hypotheses remaining under consideration.

After processing these four examples, the boundary sets S_4 and G_4 delimit the version space of *all* hypotheses consistent with the set of incrementally observed training examples. The entire version space, including those hypotheses

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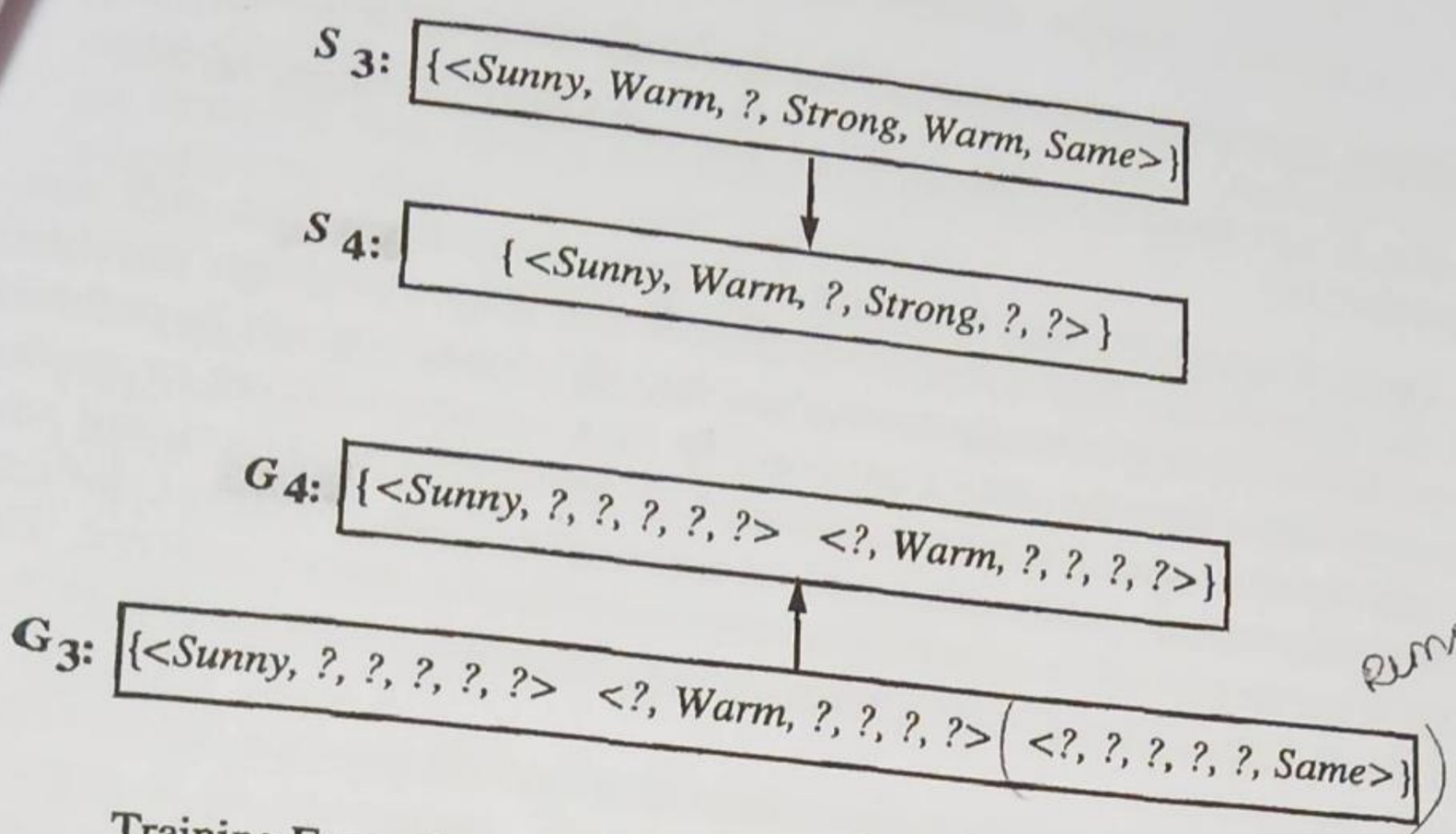

Training Example:

4. <*Sunny, Warm, High, Strong, Cool, Change*>, *EnjoySport = Yes*

FIGURE 2.6
CANDIDATE-ELIMINATION Trace 3. The positive training example generalizes the S boundary, from S_3 to S_4. One member of G_3 must also be deleted, because it is no longer more general than the S_4 boundary.

bounded by S_4 and G_4, is shown in Figure 2.7. This learned version space is independent of the sequence in which the training examples are presented (because in the end it contains all hypotheses consistent with the set of examples). As further training data is encountered, the S and G boundaries will move monotonically closer to each other, delimiting a smaller and smaller version space of candidate hypotheses.

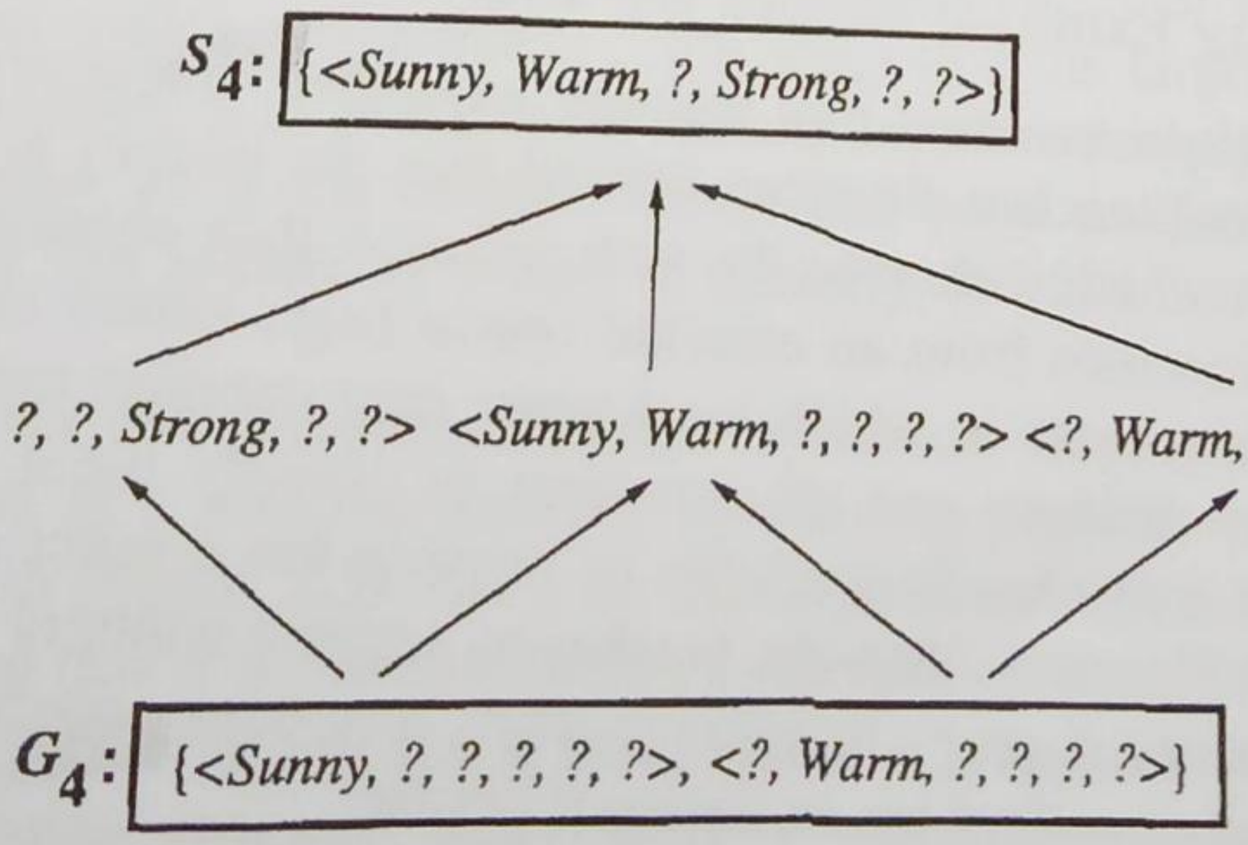

FIGURE 2.7
The final version space for the *EnjoySport* concept learning problem and training examples described earlier.

2.6 REMARKS ON VERSION SPACES AND CANDIDATE-ELIMINATION

2.6.1 Will the CANDIDATE-ELIMINATION Algorithm Converge to the Correct Hypothesis?

The version space learned by the CANDIDATE-ELIMINATION algorithm will converge toward the hypothesis that correctly describes the target concept, provided (1) there are no errors in the training examples, and (2) there is some hypothesis in H that correctly describes the target concept. In fact, as new training examples are observed, the version space can be monitored to determine the remaining ambiguity regarding the true target concept and to determine when sufficient training examples have been observed to unambiguously identify the target concept. The target concept is exactly learned when the S and G boundary sets converge to a single, identical, hypothesis.

What will happen if the training data contains errors? Suppose, for example, that the second training example above is incorrectly presented as a negative example instead of a positive example. Unfortunately, in this case the algorithm is certain to remove the correct target concept from the version space! Because it will remove every hypothesis that is inconsistent with each training example, it will eliminate the true target concept from the version space as soon as this false negative example is encountered. Of course, given sufficient additional training data the learner will eventually detect an inconsistency by noticing that the S and G boundary sets eventually converge to an empty version space. Such an empty version space indicates that there is *no* hypothesis in H consistent with all observed training examples. A similar symptom will appear when the training examples are correct, but the target concept cannot be described in the hypothesis representation (e.g., if the target concept is a disjunction of feature attributes and the hypothesis space supports only conjunctive descriptions). We will consider such eventualities in greater detail later. For now, we consider only the case in which the training examples are correct and the true target concept is present in the hypothesis space.

2.6.2 What Training Example Should the Learner Request Next?

Up to this point we have assumed that training examples are provided to the learner by some external teacher. Suppose instead that the learner is allowed to conduct experiments in which it chooses the next instance, then obtains the correct classification for this instance from an external oracle (e.g., nature or a teacher). This scenario covers situations in which the learner may conduct experiments in nature (e.g., build new bridges and allow nature to classify them as stable or unstable), or in which a teacher is available to provide the correct classification (e.g., propose a new bridge and allow the teacher to suggest whether or not it will be stable). We use the term *query* to refer to such instances constructed by the learner, which are then classified by an external oracle.

Consider again the version space learned from the four training examples of the *EnjoySport* concept and illustrated in Figure 2.3. What would be a good query for the learner to pose at this point? What is a good query strategy in

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general? Clearly, the learner should attempt to discriminate among the alternative competing hypotheses in its current version space. Therefore, it should choose an instance that would be classified positive by some of these hypotheses, but negative by others. One such instance is

$$\langle Sunny, Warm, Normal, Light, Warm, Same \rangle$$

Note that this instance satisfies three of the six hypotheses in the current version space (Figure 2.3). If the trainer classifies this instance as a positive example, the S boundary of the version space can then be generalized. Alternatively, if the trainer indicates that this is a negative example, the G boundary can then be specialized. Either way, the learner will succeed in learning more about the true identity of the target concept, shrinking the version space from six hypotheses to half this number.

In general, the optimal query strategy for a concept learner is to generate instances that satisfy exactly half the hypotheses in the current version space. When this is possible, the size of the version space is reduced by half with each new example, and the correct target concept can therefore be found with only $\lceil log_2 |VS| \rceil$ experiments. The situation is analogous to playing the game twenty questions, in which the goal is to ask yes-no questions to determine the correct hypothesis. The optimal strategy for playing twenty questions is to ask questions that evenly split the candidate hypotheses into sets that predict yes and no. While we have seen that it is possible to generate an instance that satisfies precisely half the hypotheses in the version space of Figure 2.3, in general it may not be possible to construct an instance that matches precisely half the hypotheses. In such cases, a larger number of queries may be required than $\lceil log_2 |VS| \rceil$.

2.6.3 How Can Partially Learned Concepts Be Used?

Suppose that no additional training examples are available beyond the four in our example above, but that the learner is now required to classify new instances that it has not yet observed. Even though the version space of Figure 2.3 still contains multiple hypotheses, indicating that the target concept has not yet been fully learned, it is possible to classify certain examples with the same degree of confidence as if the target concept had been uniquely identified. To illustrate, suppose the learner is asked to classify the four new instances shown in Table 2.6.

Note that although instance A was not among the training examples, it is classified as a positive instance by *every* hypothesis in the current version space (shown in Figure 2.3). Because the hypotheses in the version space unanimously agree that this is a positive instance, the learner can classify instance A as positive with the same confidence it would have if it had already converged to the single, correct target concept. Regardless of which hypothesis in the version space is eventually found to be the correct target concept, it is already clear that it will classify instance A as a positive example. Notice furthermore that we need not enumerate every hypothesis in the version space in order to test whether each

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Instance	*Sky*	*AirTemp*	*Humidity*	*Wind*	*Water*	*Forecast*	*EnjoySport*
A	Sunny	Warm	Normal	Strong	Cool	Change	?
B	Rainy	Cold	Normal	Light	Warm	Same	?
C	Sunny	Warm	Normal	Light	Warm	Same	?
D	Sunny	Cold	Normal	Strong	Warm	Same	?

TABLE 2.6
New instances to be classified.

classifies the instance as positive. This condition will be met if and only if the instance satisfies every member of S (why?). The reason is that every other hypothesis in the version space is at least as general as some member of S. By our definition of *more_general_than*, if the new instance satisfies all members of S it must also satisfy each of these more general hypotheses.

Similarly, instance B is classified as a negative instance by every hypothesis in the version space. This instance can therefore be safely classified as negative, given the partially learned concept. An efficient test for this condition is that the instance satisfies none of the members of G (why?).

Instance C presents a different situation. Half of the version space hypotheses classify it as positive and half classify it as negative. Thus, the learner cannot classify this example with confidence until further training examples are available. Notice that instance C is the same instance presented in the previous section as an optimal experimental query for the learner. This is to be expected, because those instances whose classification is most ambiguous are precisely the instances whose true classification would provide the most new information for refining the version space.

Finally, instance D is classified as positive by two of the version space hypotheses and negative by the other four hypotheses. In this case we have less confidence in the classification than in the unambiguous cases of instances A and B. Still, the vote is in favor of a negative classification, and one approach we could take would be to output the majority vote, perhaps with a confidence rating indicating how close the vote was. As we will discuss in Chapter 6, if we assume that all hypotheses in H are equally probable a priori, then such a vote provides the most probable classification of this new instance. Furthermore, the proportion of hypotheses voting positive can be interpreted as the probability that this instance is positive given the training data.

2.7 INDUCTIVE BIAS

As discussed above, the CANDIDATE-ELIMINATION algorithm will converge toward the true target concept provided it is given accurate training examples and provided its initial hypothesis space contains the target concept. What if the target concept is not contained in the hypothesis space? Can we avoid this difficulty by using a hypothesis space that includes every possible hypothesis? How does the

Chapter 6

Learning with Trees

We are now going to move away from neural networks and take a rather different approach, starting with one of the most common and powerful data structures in the whole of computer science: the binary tree. The computational cost of making the tree is fairly low, but the cost of using it is even lower: $\mathcal{O}(\log N)$, where N is the number of data points. This is important for machine learning, since querying the trained algorithm should be as fast as possible since it happens more often, and the result is often wanted immediately. This is sufficient to make trees seem attractive for machine learning. However, they do have other benefits, such as the fact that they are easy to understand (following a tree to get a classification answer is transparent, which makes people trust it more than getting an answer from a 'black box' neural network). For these reasons, classification by decision trees has grown in popularity over recent years. You are very likely to have been subjected to decision trees if you've ever phoned a helpline, for example for computer faults. The phone operators are guided through the decision tree by your answers to their questions.

The idea of a decision tree is that we break classification down into a set of choices about each feature in turn, starting at the root (base) of the tree and progressing down to the leaves, where we receive the classification decision. The trees are very easy to understand, and can even be turned into a set of if-then rules, suitable for use in a rule induction system.

6.1 Using Decision Trees

As a student it can be difficult to decide what to do in the evening. There are four things that you actually quite enjoy doing, or have to do: going to the pub, watching TV, going to a party, or even (gasp) studying. The choice is sometimes made for you—if you have an assignment due the next day, then you need to study, if you are feeling lazy then the pub isn't for you, and if there isn't a party then you can't go to it. You are looking for a nice algorithm that will let you decide what to do each evening without having to think about it every night. Figure 6.1 provides just such an algorithm.

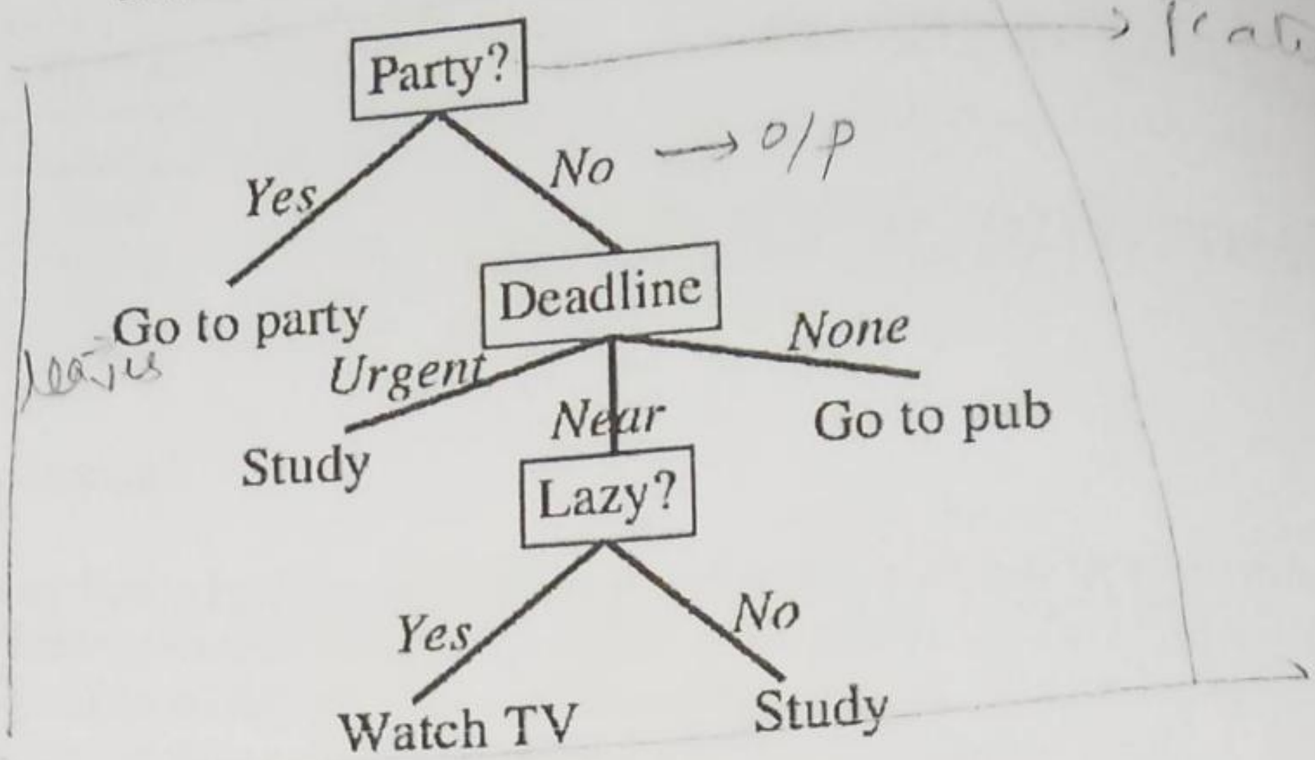

FIGURE 6.1: A simple decision tree to decide how you will spend the evening.

Each evening you start at the top (root) of the tree and check whether any of your friends know about a party that night. If there is one, then you need to go, regardless. Only if there is not a party do you worry about whether or not you have an assignment deadline coming up. If there is a crucial deadline, then you have to study, but if there is nothing that is urgent for the next few days, you think about how you feel. A sudden burst of energy might make you study, but otherwise you'll be slumped in front of the TV indulging your secret love of Shortland Street (or other soap opera of your choice) rather than studying. Of course, near the start of the semester when there are no assignments to do, and you are feeling rich, you'll be in the pub.

One of the reasons that decision trees are popular is that we can turn them into a set of logical disjunctions (`if ... then` rules) that then go into program code very simply—the first part of the tree above can be turned into:

- `if` *there is a party* `then` *go to it*
- `if` *there is not a party* `and` *you have an urgent deadline* `then` *study*
- etc.

That's all that there is to using the decision tree. The far more interesting part is how to construct the tree from data, and that is the focus of the next section.

6.2 Constructing Decision Trees

In the example above, the three features that we need for the algorithm are the state of your energy level, the date of your nearest deadline, and whether

or not there is a party tonight. The question we need to ask is how, based on those features, we can construct the tree. There are a few different decision tree algorithms, but they are almost all variants of the same principle: the algorithms build the tree in a greedy manner starting at the root, choosing the most informative feature at each step. We are going to start by focusing on the most common: Quinlan's ID3, although we'll also mention its extension, known as C4.5, and another known as CART.

There was an important word hidden in the sentence above about how the trees work, which was informative. Choosing which feature to use next in the decision tree can be thought of like playing the game '20 Questions,' where you try to elicit the item your opponent is thinking about by asking questions about it. At each stage, you choose a question that gives you the most information given what you know already. Thus, you would ask 'Is it an animal?' before you asked 'Is it a cat?'. The idea is to quantify this question of how much information is provided to you by knowing certain facts. Encoding this mathematically is the task of information theory.

6.2.1 Quick Aside: Entropy in Information Theory

Information theory was 'born' in 1948 when Claude Shannon published a paper called "A Mathematical Theory of Communication." In that paper, he proposed the measure of information entropy, which describes the amount of impurity in a set of features. The entropy H of a set of probabilities p_i is (for those who know some physics, the relation to physical entropy should be clear):

$$H(p) = -\sum_i p_i \log_2 p_i, \tag{6.1}$$

where the logarithm is base 2 because we are imagining that we encode everything using binary digits (bits), and we define $0 \log 0 = 0$. A graph of the entropy is given in Figure 6.2. Suppose that we have a set of positive and negative examples of some feature (where the feature can only take 2 values: positive and negative). If all of the examples are positive, then we don't get any extra information from knowing the value of the feature for any particular example, since whatever the value of the feature, the example will be positive. Thus, the entropy of that feature is 0. However, if the feature separates the examples into 50% positive and 50% negative, then the amount of entropy is at a maximum, and knowing about that feature is very useful to us. The basic concept is that it tells us how much *extra* information we would get from knowing the value of that feature. A function for computing the entropy is very simple, as here:

I-31

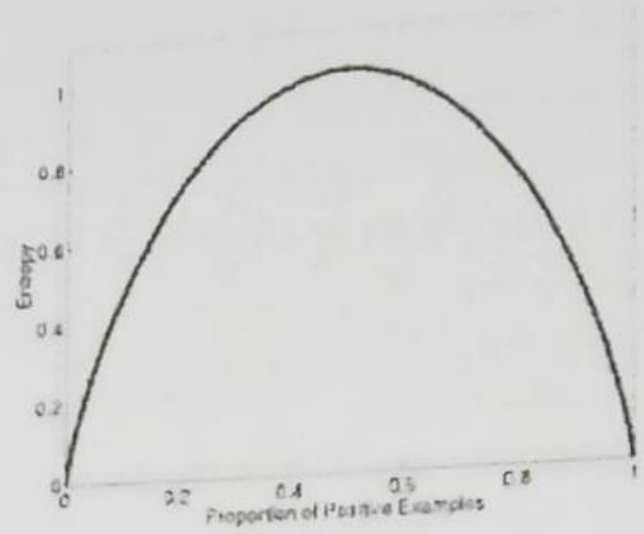

FIGURE 6.2: A graph of entropy, detailing how much information is available from finding out another piece of information given what you already know.

```
def calc_entropy(p):

    if p!=0:
        return -p * log2(p)
    else:
        return 0
```

For our decision tree, the best feature to pick as the one to classify on now is the one that gives you the most information, i.e., the one with the highest entropy. After using that feature, we re-evaluate the entropy of each feature and again pick the one with the highest entropy.

Information theory is a very interesting subject. It is possible to download Shannon's 1948 paper from the Internet, and also to find many resources showing where it has been applied. There are now whole journals devoted to information theory because it is relevant to so many areas such as computer and telecommunication networks, machine learning, and data storage. Some further readings in the area are given at the end of the chapter.

6.2.2 ID3

Now that we have a suitable measure for choosing which feature to choose next, entropy, we just have to work out how to apply it. The important idea is to work out how much the entropy of the whole training set would decrease if we choose each particular feature for the next classification step. This is known as the information gain, and it is defined as the entropy of the whole set minus the entropy when a particular feature is chosen. This is defined by (where S is the set of examples, F is a possible feature out of the set of all possible ones, and $|S_f|$ is a count of the number of members of S that have value f for feature F):

$$\text{Gain}(S, F) = \text{Entropy}(S) - \sum_{f \in values(F)} \frac{|S_f|}{|S|} \text{Entropy}(S_f). \tag{6.2}$$

As an example, suppose that we have data (with outcomes) $S = \{s_1 = \text{true}, s_2 = \text{false}, s_3 = \text{false}, s_4 = \text{false}\}$ and one feature F that can have values $\{f_1, f_2, f_3\}$. In the example, the feature value for s_1 could be f_2, for s_2 it could be f_2, for s_3, f_3 and for s_4, f_1 then we can calculate the entropy of S as (where $\oplus$ means true, of which we have one example, and $\ominus$ means false, of which we have three examples):

$$\begin{aligned}\text{Entropy}(S) &= -p_\oplus \log_2 p_\oplus - p_\ominus \log_2 p_\ominus \\ &= -\frac{1}{4}\log_2 \frac{1}{4} - \frac{3}{4}\log_2 \frac{3}{4} \\ &= 0.5 + 0.311 = 0.811.\end{aligned} \tag{6.3}$$

If you were trying to follow that calculation on a calculator, you might be wondering how to compute $\log_2 p$. The answer is to use the identity $\log_2 p = \ln p / \ln(2)$, where ln is the natural logarithm, which your calculator can produce. NumPy has the `log2()` function.

We now want to compute the information gain of F, so we now need to compute each of the values inside the summation in Equation (6.2). $\frac{|S_f|}{|S|}\text{Entropy}(S_f)$:

$$\begin{aligned}\frac{|S_{f_1}|}{|S|}\text{Entropy}(S_{f_1}) &= \frac{1}{4} \times \left(-\frac{0}{1}\log_2 \frac{0}{1} - \frac{1}{1}\log_2 \frac{1}{1}\right) \\ &= 0\end{aligned} \tag{6.4}$$

$$\begin{aligned}\frac{|S_{f_2}|}{|S|}\text{Entropy}(S_{f_2}) &= \frac{2}{4} \times \left(-\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned} \tag{6.5}$$

$$\begin{aligned}\frac{|S_{f_3}|}{|S|}\text{Entropy}(S_{f_3}) &= \frac{1}{4} \times \left(-\frac{0}{1}\log_2 \frac{0}{1} - \frac{1}{1}\log_2 \frac{1}{1}\right) \\ &= 0\end{aligned} \tag{6.6}$$

The information gain from adding this feature is the entropy of S minus the sum of the three values above:

$$\text{Gain}(S, F) = 0.811 - (0 + 0.5 + 0) = 0.311. \tag{6.7}$$

This can be computed in an algorithm using the following function (where lots of the code is to get the relevant data):

```
def calc_info_gain(data,classes,feature):
    gain = 0
    nData = len(data)
    # List the values that feature can take
    values = []
    for datapoint in data:
        if values.count(datapoint[feature])==0:
            values.append(datapoint[feature])
    featureCounts = zeros(len(values))
    entropy = zeros(len(values))
    valueIndex = 0
    # Find where those values appear in data[feature] and the
    corresponding class
    for value in values:
        dataIndex = 0
        newClasses = []
        for datapoint in data:
            if datapoint[feature]==value:
                featureCounts[valueIndex]+=1
                newClasses.append(classes[dataIndex])
            dataIndex += 1
        # Get the values in newClasses
        classValues = []
        for aclass in newClasses:
            if classValues.count(aclass)==0:
                classValues.append(aclass)

        classCounts = zeros(len(classValues))
        classIndex = 0
        for classValue in classValues:
            for aclass in newClasses:
                if aclass == classValue:
                    classCounts[classIndex]+=1
            classIndex += 1

        for classIndex in range(len(classValues)):
            entropy[valueIndex] += calc_entropy(float(classCounts[
            classIndex])/sum(classCounts))
            gain += float(featureCounts[valueIndex])/nData * entropy[
            valueIndex]
            valueIndex += 1
    return gain
```

The ID3 algorithm computes this information gain for each feature and chooses the one that produces the highest value. In essence, that is all there is to the algorithm. It searches the space of possible trees in a greedy way by choosing the feature with the highest information gain at each stage. The output of the algorithm is the tree, i.e., a list of nodes, edges, and leaves. As with any tree in computer science, it can be constructed recursively. At each stage the best feature is selected and then removed from the dataset, and the algorithm is recursively called on the rest. The recursion stops when either there is only one class remaining in the data, or there are no features left. In the first case a leaf is added with that class as its label, while in the second the most common label in the remaining data are used.

The ID3 Algorithm

- if all examples have the same label: //some class samples
  - return a leaf with that label
- else if there are no features left to test:
  - return a leaf with the most common label
- else:
  - choose the feature $\hat{F}$ that maximises the information gain of S to be the next node using Equation (6.2)
  - add a branch from the node for each possible value f in $\hat{F}$
  - for each branch:
    * calculate S_f by removing $\hat{F}$ from the set of features
    * recursively call the algorithm with S_f, to compute the gain relative to the current set of examples

Owing to the focus on classification for real world examples, trees are often used with text features rather than numeric values. This makes it rather difficult to use NumPy, and so the sample implementation is pretty well pure Python. It uses a feature of Python that is uncommon in other languages, which is the dictionary in order to hold the tree, which uses the braces {, }, and which is described next before we look at the decision tree implementation.

6.2.3 Implementing Trees and Graphs in Python

Trees are really just a restricted version of graphs, since they both consist of nodes and edges between the nodes. Graphs are a very useful data structure in many different areas of computer science. There are two reasonable ways to represent a graph computationally. One is as an $N \times N$ matrix, where N

is the number of nodes in the network. Each element of the matrix is a 1 if there is a link between the two nodes, and a 0 otherwise. The benefit of this approach is that it is easy to give weights to the links by changing the 1s to the values of the weights. The alternative is to store a list of nodes, following each by a list of nodes that it is linked to. Both are fairly natural in Python, with the second making use of the dictionary, a basic data structure that we have not used much, except for very simply in the decision tree (Chapter 6) that consists of a set of keys and values. For a graph, the key to each dictionary entry is the name of the node, and its value is a list of the nodes that it is connected to, as in this example:

```
graph = {'A': ['B', 'C'],'B': ['C', 'D'],'C': ['D'],'D': ['C'],
'E': ['F'],'F': ['C']}
```

That is all there is to it for creating the dictionary, and using it is not very different, since there are built-in methods to get a list of keys (`keys()`) and check if a key is in a dictionary (`in`). Code to find a path through the graph can then be written as a simple recursive function:

```
def findPath(graph, start, end, pathSoFar):
    pathSoFar = pathSoFar + [start]
    if start == end:
        return pathSoFar
    if start not in graph:
        return None
    for node in graph[start]:
        if node not in pathSoFar:
            newpath = findPath(graph, node, end, pathSoFar)
            return newpath
    return None
```

Using those methods we can now look at a Python implementation of the decision tree, which also has a recursive function call as its basis.

6.2.4 Implementation of the Decision Tree

The `make_tree()` function (which uses the `calc_entropy()` and `calc_info_gai` functions that were described previously) looks like:

```
def make_tree(data,classes,featureNames):
    # Various initialisations
    default = classes[argmax(frequency)]
    if nData==0 or nFeatures == 0:
        # Have reached an empty branch
        return default
    elif classes.count(classes[0]) == nData:
        # Only 1 class remains
        return classes[0]
    else:
    # Choose which feature is best
    gain = zeros(nFeatures)
    for feature in range(nFeatures):
        g = calc_info_gain(data,classes,feature)
        gain[feature] = totalEntropy - g

    bestFeature = argmax(gain)
    tree = {featureNames[bestFeature]:{}}
    # Find the possible feature values
    for value in values:
        # Find the datapoints with each feature value
        for datapoint in data:
            if datapoint[bestFeature]==value:
                if bestFeature==0:
                    datapoint = datapoint[1:]
                    newNames = featureNames[1:]
                elif bestFeature==nFeatures:
                    datapoint = datapoint[:-1]
                    newNames = featureNames[:-1]
                else:
                    datapoint = datapoint[:bestFeature]
                    datapoint.extend(datapoint[bestFeature+1:])
                    newNames = featureNames[:bestFeature]
                    newNames.extend(featureNames[bestFeature+1:
                    ])
                newData.append(datapoint)
                newClasses.append(classes[index])
            index += 1
        # Now recurse to the next level
        subtree = make_tree(newData,newClasses,newNames)
        # And on returning, add the subtree on to the tree
        tree[featureNames[bestFeature]][value] = subtree
    return tree
```

It is worth considering how ID3 generalises from training examples to the set of all possible inputs. It uses a method known as the **inductive bias**. The choice of the next feature to add into the tree is the one with the highest information gain, which biases the algorithm towards smaller trees, since it tries to minimise the amount of information that is left. This is consistent with a well-known principle that short solutions are usually better than longer ones (not necessarily true, but simpler explanations are usually easier to remember and understand). You might have heard of this principle as 'Occam's Razor,' although I prefer it as an acronym: KISS (Keep It Simple, Stupid). In fact, there is a sound information-theoretic way to write down this principle. It is known as the **Minimum Description Length (MDL)** and was proposed by Rissanen in 1989. In essence it says that the shortest description of something, i.e., the most compressed one, is the best description.

Note that the algorithm can deal with noise in the dataset, because the labels are assigned to the most common value of the target attribute. Another benefit of decision trees is that they can deal with missing data. Think what would happen if an example has a missing feature. In that case, we can skip that node of the tree and carry on without it, summing over all the possible values that that feature could have taken. This is virtually impossible to do with neural networks: how do you represent missing data when the computation is based on whether or not a neuron is firing? In the case of neural networks it is common to either throw away any datapoints that have missing data, or guess (more technically **impute** any missing values, either by identifying similar datapoints and using their value or by using the mean or median of the data values for that feature). This assumes that the data that is missing is randomly distributed within the dataset, not missing because of some unknown process.

Saying that ID3 is biased towards short trees is only partly true. The algorithm uses all of the features that are given to it, even if some of them are not necessary. This obviously runs the risk of overfitting, indeed it makes it very likely. There are a few things that you can do to avoid overfitting, the simplest one being to limit the size of the tree. You can also use a variant of early stopping by using a validation set and measuring the performance of the tree so far against it. However, the approach that is used in more advanced algorithms (most notably C4.5, which Quinlan invented to improve on ID3) is **pruning**.

There are a few versions of pruning, all of which are based on computing the full tree and reducing it, evaluating the error on a validation set. The most naïve version runs the decision tree algorithm until all of the features are used, so that it is probably overfitted, and then produces smaller trees by running over the tree, picking each node in turn, and replacing the subtree beneath every node with a leaf labelled with the most common classification of the sub-tree. The error of the pruned tree is evaluated on the validation set, and the pruned tree is kept if the error is the same as or less than the original tree, and rejected otherwise.

C4.5 uses a different method called rule post-pruning. This consists of taking the tree generated by ID3, converting it to a set of if-then rules, and then pruning each rule by removing preconditions if the accuracy of the rule increases without it. The rules are then sorted according to their accuracy on the training set and applied in order. The advantages of dealing with rules are that they are easier to read and their order in the tree does not matter, just their accuracy in the classification.

6.2.5 Dealing with Continuous Variables

One thing that we have not yet discussed is how to deal with continuous variables, we have only considered those with discrete sets of feature values. The simplest solution is to discretise the continuous variable. However, it is also possible to leave it continuous and modify the algorithm. For a continuous variable there is not just one place to split it: the variable can be broken between any pair of datapoints, as shown in Figure 6.3. It can, of course, be split in any of the infinite locations along the line as well, but they are no different to this smaller set of locations. Even this smaller set makes the algorithm more expensive for continuous variables than it is for discrete ones, since as well as calculating the information gain of each variable to pick the best one, the information gain of many points within each variable has to be computed. In general, only one split is made to a continuous variable, rather than allowing for threeway or higher splits, although these can be done if necessary.

The trees that these algorithms make are all univariate trees, because they pick one feature (dimension) at a time and split according to that one. There are also algorithms that make multivariate trees by picking combinations of features. This can make for considerably smaller trees if it is possible to find straight lines that separate the data well, but are not parallel to any axis. However, univariate trees are simpler and tend to get good results, so we won't consider multivariate trees any further. This fact that one feature is chosen at a time provides another useful way to visualise what the decision tree is doing. Figure 6.4 shows the idea. Given a dataset that contains three classes, the algorithm picks a feature and value for that feature to split the remaining data into two. The final tree that results from this is shown in Figure 6.5.

6.2.6 Computational Complexity

The computational cost of constructing binary trees is well known for the general case, being $\mathcal{O}(N \log N)$ for construction and $\mathcal{O}(\log N)$ for returning a particular leaf, where N is the number of nodes. However, these results are for balanced binary trees, and decision trees are often not balanced: while the information measures attempt to keep the tree balanced by finding splits that separate the data into two even parts (since that will have the largest entropy),

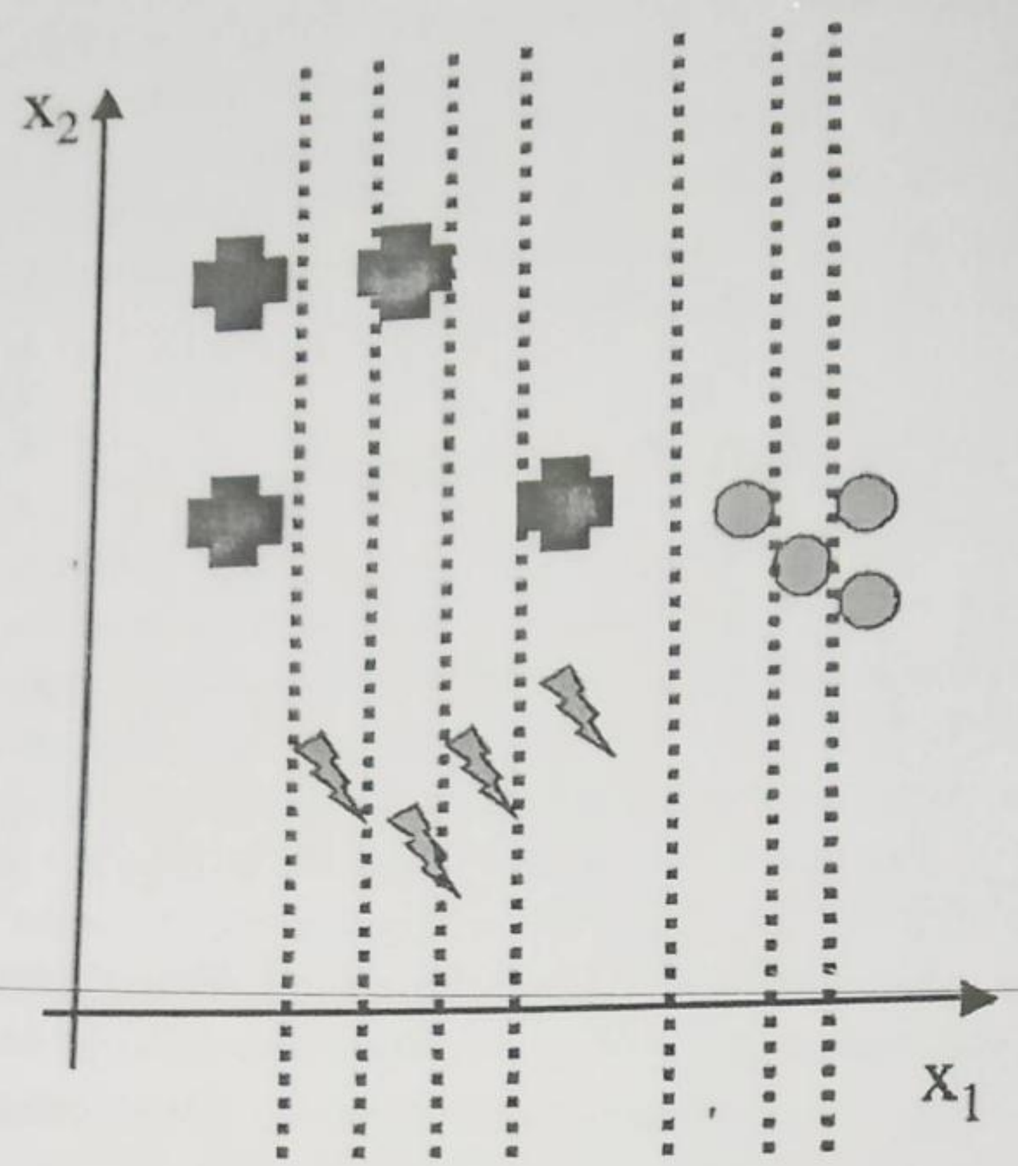

FIGURE 6.3: Possible places to split the variable x_1, between each of the datapoints as the feature value increases.

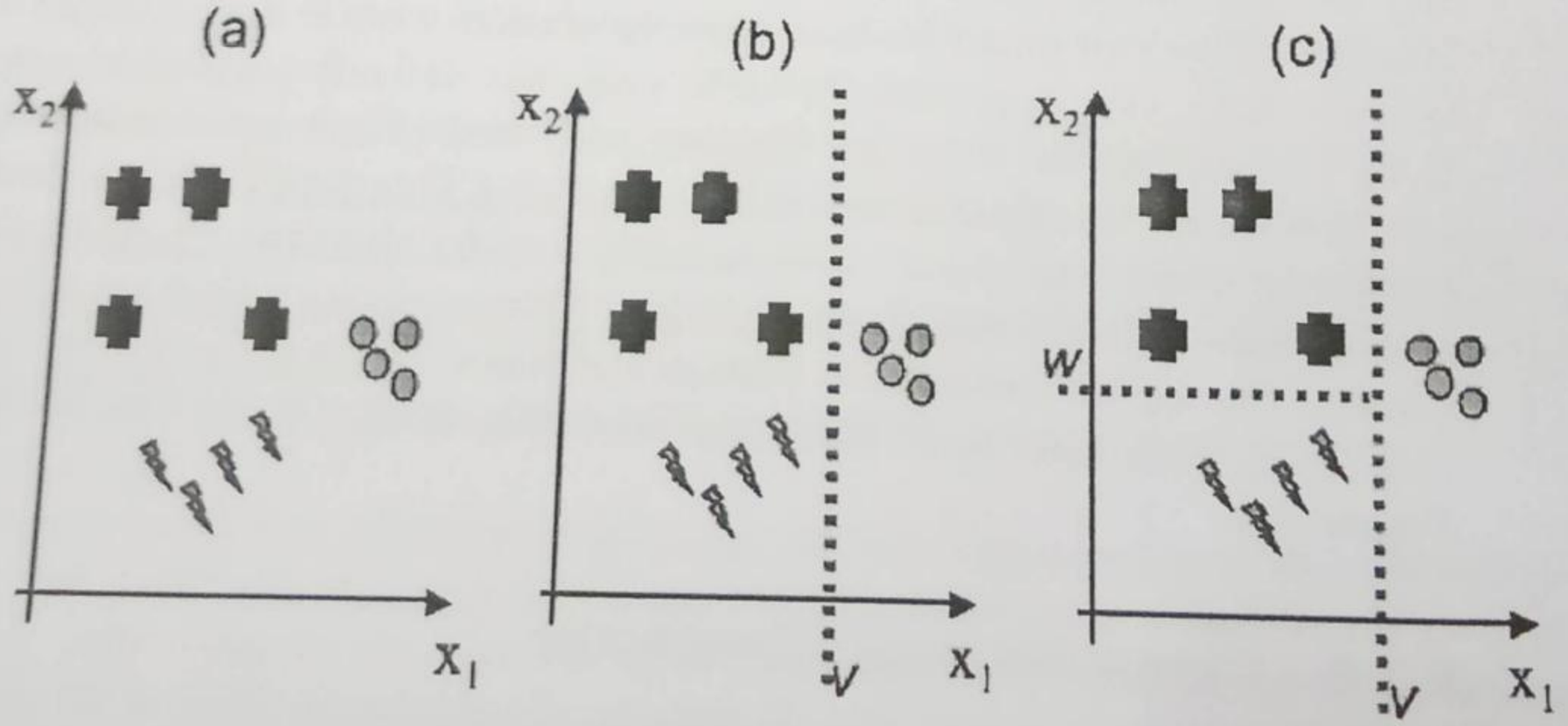

FIGURE 6.4: The effect of decision tree choices. The two-dimensional dataset shown in(a) is split first by choosing feature x_1 (b) and then x_2, (c) which separates out the three classes. The final tree is shown in Figure 6.5.

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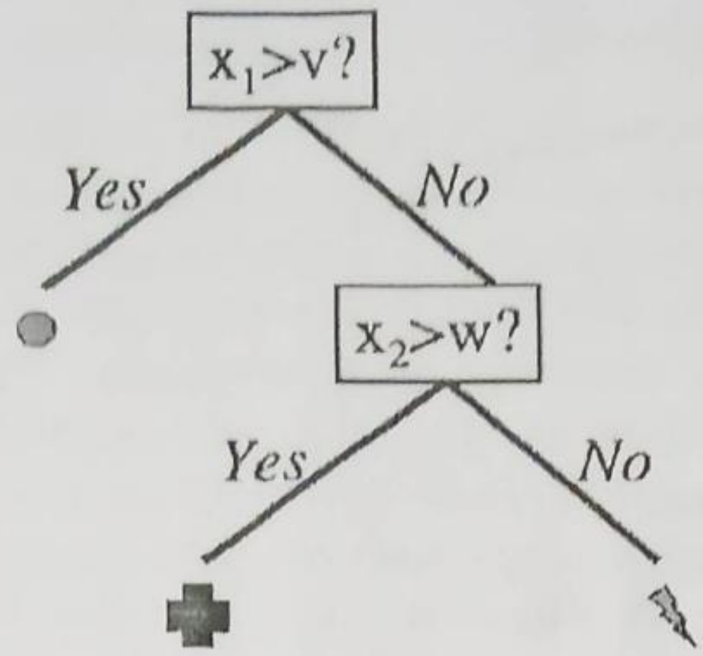

FIGURE 6.5: The final tree created by the splits in Figure 6.4.

there is no guarantee of this. Nor are they necessarily binary, especially for ID3 and C4.5, as our example shows.

If we assume that the tree is approximately balanced, then the cost at each node consists of searching through the d possible features (although this decreases by 1 at each level, that doesn't affect the complexity in the $\mathcal{O}(\cdot)$ notation) and then computing the information gain for the dataset for each split. This has cost $\mathcal{O}(dn \log n)$, where n is the size of the dataset at that node. For the root, $n = N$, and if the tree is balanced then n is divided by 2 at each stage down the tree. Summing this over the approximately $\log N$ levels in the tree gives computational cost $\mathcal{O}(dN^2 \log N)$.

6.3 Classification and Regression Trees (CART)

There is another well-known tree-based algorithm, CART, whose name indicates that it can be used for both classification and regression. Classification is not wildly different in CART, although it is usually constrained to construct binary trees. This might seem odd at first, but there are sound computer science reasons why binary trees are good, as suggested in the computational cost discussion above, and it is not a real limation. Even in the example that we started the chapter with, we can always turn questions into binary decisions by splitting the question up a little. Thus, a question that has three answers (say the question about when your nearest assignment deadline is, which is either 'urgent', 'near', or 'none') can be split into two questions: first, 'is the deadline urgent?', and then if the answer to that is 'no', second 'is the deadline near?' The only real difference with classification in CART is that a different information measure is commonly used. This is discussed next, before we look briefly at regression with trees.

6.3.1 Gini Impurity

The entropy that was used in ID3 as the information measure is not the only way to pick features. Another possibility is something known as the Gini impurity. The 'impurity' in the name suggests that the aim of the decision tree is to have each leaf node represent a set of datapoints that are in the same class, so that there are no mismatches. This is known as purity. If a leaf is pure then all of the training data within it have just one class. In which case, if we count the number of datapoints at the node (or better, the fraction of the number of datapoints) that belong to a class i (call it $N(i)$), then it should be 0 for all except one value of i. So suppose that you want to decide on which feature to choose for a split. The algorithm loops over the different features and checks how many points belong to each class. If the node is pure, then $N(i) = 0$ for all values of i except one particular one. So for any particular feature k you can compute:

$$G_k = \sum_{i=1}^{c} \sum_{j \neq i} N(i)N(j), \tag{6.8}$$

where c is the number of classes. In fact, you can reduce the algorithmic effort required by noticing that $\sum_i N(i) = 1$ (since there has to be some output class) and so $\sum_{j \neq i} N(j) = 1 - N(i)$. Then Equation (6.8) is equivalent to:

$$G_k = 1 - \sum_{i=1}^{c} N(i)^2. \tag{6.9}$$

Either way, the Gini impurity is equivalent to computing the expected error rate if the classification was picked according to the class distribution. The information gain can then be measured in the same way, subtracting each value G_i from the total Gini impurity.

The information measure can be changed in another way, which is to add a weight to the misclassifications. The idea is to consider the cost of misclassifying an instance of class i as class j (which we will call the risk in Section 8.1.1) and add a weight that says how important each datapoint is. It is typically labelled as λ_{ij} and is presented as a matrix, with element λ_{ij} representing the cost of misclassifying i as j. Using it is simple, modifying the Gini impurity (Equation (6.8)) to be:

$$G_i = \sum_{j \neq i} \lambda_{ij} N(i)N(j). \tag{6.10}$$

We will see in Section 7.1 that there is another benefit to using these weights which is to successively improve the classification ability by putting high weight on datapoints that the algorithm is getting wrong.

6.3.2 Regression in Trees

The new part about CART is its application in regression. While it might seem strange to use trees for regression, it turns out to require only a simple modification to the algorithm. Suppose that the outputs are continuous, so that a regression model is appropriate. None of the node impurity measures that we have considered so far will work. Instead, we'll go back to our old favourite—the sum-of-squares error. To evaluate the choice of which feature to use next, we also need to find the value at which to split the dataset according to that feature. Remember that the output is a value at each leaf. In general, this is just a constant value for the output, computed as the mean average of all the data points that are situated in that leaf. This is the optimal choice in order to minimise the sum-of-squares error, but it also means that we can choose the split point quickly for a given feature, by choosing it to minimise the sum-of-squares error. We can then pick the feature that has the split point that provides the best sum-of-squares error, and continue to use the algorithm as for classification.

6.4 Classification Example

We'll work through an example using ID3 in this section. The data that we'll use will be a continuation of the one we started the chapter with, about what to do in the evening. When we want to construct the decision tree to decide what to do in the evening, we start by listing everything that we've done for the past few days to get a suitable dataset (here, the last ten days):

Deadline?	Is there a party?	Lazy?	Activity
Urgent	Yes	Yes	Party
Urgent	No	Yes	Study
Near	Yes	Yes	Party
None	Yes	No	Party
None	No	Yes	Pub
None	Yes	No	Party
Near	No	No	Study
Near	No	Yes	TV
Near	Yes	Yes	Party
Urgent	No	No	Study

To produce a decision tree for this problem, the first thing that we need to do is work out which feature to use as the root node. We start by computing the entropy of S:

$$\begin{aligned}\text{Entropy}(S) &= -p_{\text{party}}\log_2 p_{\text{party}} - p_{\text{study}}\log_2 p_{\text{study}} \\ &\quad - p_{\text{pub}}\log_2 p_{\text{pub}} - p_{\text{TV}}\log_2 p_{\text{TV}} \\ &= -\frac{5}{10}\log_2\frac{5}{10} - \frac{3}{10}\log_2\frac{3}{10} - \frac{1}{10}\log_2\frac{1}{10} - \frac{1}{10}\log_2\frac{1}{10} \\ &= 0.5 + 0.5211 + 0.3322 + 0.3322 = 1.6855\end{aligned} \tag{6.11}$$

and then find which feature has the maximal information gain:

$$\begin{aligned}\text{Gain}(S, \text{Deadline}) &= 1.6855 - \frac{|S_{\text{urgent}}|}{10}\text{Entropy}(S_{\text{urgent}}) \\ &\quad - \frac{|S_{\text{near}}|}{10}\text{Entropy}(S_{\text{near}}) - \frac{|S_{\text{none}}|}{10}\text{Entropy}(S_{\text{none}}) \\ &= 1.6855 - \frac{3}{10}\left(-\frac{2}{3}\log_2\frac{2}{3} - \frac{1}{3}\log_2\frac{1}{3}\right) \\ &\quad - \frac{4}{10}\left(-\frac{2}{4}\log_2\frac{2}{4} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{4}\log_2\frac{1}{4}\right) \\ &\quad - \frac{3}{10}\left(-\frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3}\right) \\ &= 1.6855 - 0.2755 - 0.6 - 0.2755 \\ &= 0.5345\end{aligned} \tag{6.12}$$

$$\begin{aligned}\text{Gain}(S, \text{Party}) &= 1.6855 - \frac{5}{10}\left(-\frac{5}{5}\log_2\frac{5}{5}\right) \\ &\quad - \frac{5}{10}\left(-\frac{3}{5}\log_2\frac{3}{5} - \frac{1}{5}\log_2\frac{1}{5} - \frac{1}{5}\log_2\frac{1}{5}\right) \\ &= 1.6855 - 0 - 0.6855 \\ &= 1.0\end{aligned} \tag{6.13}$$

$$\begin{aligned}\text{Gain}(S, \text{Lazy}) &= 1.6855 - \frac{6}{10}\left(-\frac{3}{6}\log_2\frac{3}{6} - \frac{1}{6}\log_2\frac{1}{6} - \frac{1}{6}\log_2\frac{1}{6} - \frac{1}{6}\log_2\frac{1}{6}\right) \\ &\quad - \frac{4}{10}\left(-\frac{2}{4}\log_2\frac{2}{4} - \frac{2}{4}\log_2\frac{2}{4}\right) \\ &= 1.6855 - 1.0755 - 0.4 \\ &= 0.21\end{aligned} \tag{6.14}$$

Therefore, the root node will be the party feature, which has two feature values ('yes' and 'no'), so it will have two branches coming out of it (see Figure 6.6). When we look at the 'yes' branch, we see that in all five cases where there was a party we went to it, so we just put a leaf node there, saying

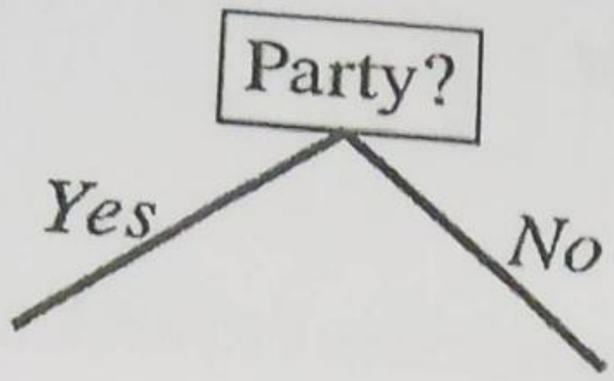

FIGURE 6.6: The decision tree after one step of the algorithm.

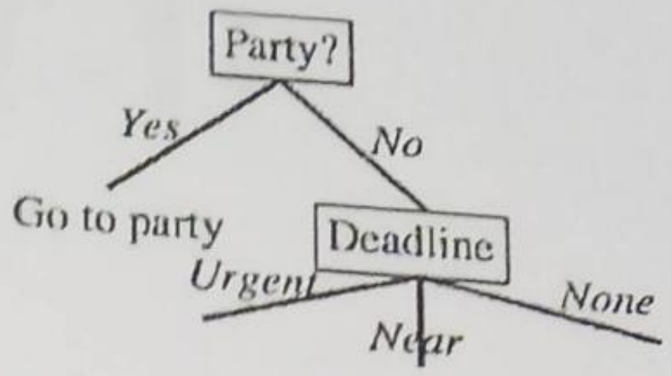

FIGURE 6.7: The tree after another step.

'party'. For the 'no' branch, out of the five cases there are three different outcomes, so now we need to choose another feature. The five cases we are looking at are:

Deadline?	Is there a party?	Lazy?	Activity
Urgent	No	Yes	Study
None	No	Yes	Pub
Near	No	No	Study
Near	No	Yes	TV
Urgent	No	Yes	Study

We've used the party feature, so we just need to calculate the information gain of the other two over these five examples:

$$\begin{aligned}\text{Gain}(S, \text{Deadline}) &= 1.371 - \frac{2}{5}\left(-\frac{2}{2}\log_2\frac{2}{2}\right) \\ &\quad - \frac{2}{5}\left(-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2}\right) - \frac{1}{5}\left(-\frac{1}{1}\log_2\frac{1}{1}\right) \\ &= 1.371 - 0 - 0.4 - 0 \\ &= 0.971 \end{aligned} \tag{6.15}$$

$$\begin{aligned}\text{Gain}(S, \text{Lazy}) &= 1.371 - \frac{4}{5}\left(-\frac{2}{4}\log_2\frac{2}{4} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{4}\log_2\frac{1}{4}\right) \\ &\quad - \frac{1}{5}\left(-\frac{1}{1}\log_2\frac{1}{1}\right) \\ &= 1.371 - 1.2 - 0 \\ &= 0.1710 \end{aligned} \tag{6.16}$$

This leads to the tree shown in Figure 6.7. From this point it is relatively simple to complete the tree, leading to the one that was shown in Figure 6.1.