Defination:

The differential (or total differential) 'du' of a differentiable function $u = f(x,y)$ of two variables is defined as

$$du = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$du = U_x d_x + U_y dy$$

Theorem:

Let $y = F(x)$ be a differentiable function defined by the equation $f(x,y) = 0$ where $f(x,y)$ is differentiable and $f_x \neq 0$ then $\frac{dy}{dx} = -\frac{f_x}{f_y}$ $\forall$ $f_y \neq 0$

Example:

Let $xy^2 + yx^2 = a^3$ find $\frac{dy}{dx}$

Let

$$u(x,y) = xy^2 + yx^2 - a^3 = 0$$

$$U_x = y^2 + 2xy$$

$$U_y = 2xy + x^2$$

we know that

$$\frac{dy}{dx} = -\frac{U_x}{U_y}$$

$$\frac{dy}{dx} = -\frac{(y^2 + 2xy)}{}$$

Example:

Find $\frac{dy}{dx}$ where $f(x,y) = x^y - y^x \rightarrow$ (A)

We know that

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \longrightarrow (*)$$

Diff. P. w.r.t 'x' of (A)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^y) - \frac{\partial}{\partial x}(y^x)$$

$$f_x = yx^{y-1} - y^x \ln y \longrightarrow (**)$$

Again Diff. P. w.r.t 'y' of (A)

$$f_y = \frac{\partial}{\partial y}(x^y) - \frac{\partial}{\partial y}(y^x)$$

$$f_y = x^y \ln x - xy^{x-1} \longrightarrow (***)$$

Using (**) & (***) into (*), we get

$$\frac{dy}{dx} = -\frac{(yx^{y-1} - y^x \ln y)}{(x^y \ln x - xy^{x-1})}$$

$$= -\frac{y(x^{y-1} - y^{x-1}\ln y)}{x(x^{y-1}\ln x - y^{x-1})} \quad \text{Required.}$$

- Composite Function:

If u is given to be a function of the variables x, y and these variables themselves are given be to the function of the variable t, then u is the said to

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then the relation.

$u = f(x,y)$; $x = \phi(t)$; $y = \psi(t)$

define u as a composite function of t.

- Differentiation of composite function:

If u is a composite function of t, defined by the relations,

$u = f(x,y)$; $x = \phi(t)$; $y = \psi(t)$

when u passesses continuous 1st order P. derivative w.r.t 'x' and 'y' & x, y passes continuous derivative. w.r.t 't' then

$$\frac{du}{dt} = \frac{\partial v}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dt}$$

In each x and y is a differentiable function of variable 'r' and 's', we would similarly get

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Example.

Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y$

$x = at^2$, $y = 2at$.

Sol: we know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \rightarrow (*)$$

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$z'(t) = \frac{dz}{dt} = z_x \cdot x'(t) + z_y \cdot y'(t)$

$\frac{\partial z}{\partial x} = y^2 + 2xy$

$\frac{\partial z}{\partial y} = 2xy + x^2$ $\longrightarrow$ (*)(**)

$\frac{dx}{dt} = 2at$

$\frac{dy}{dt} = 2a$

Using * & ** we get

$\frac{dz}{dt} = (y^2+2xy) \cdot 2at + (2xy+x^2)(2a)$

$= 2aty^2 + 4xyat + 4axy + 2x^2a$

$= 2a(ty^2 + 2txy + 2xy + x^2)$

Now, put

$x = at^2$ $\qquad y = 2at$

$= 2a\left(t(2at)^2 + 2t(at^2)(2at) + 2(at^2)(2at) + (at^2)^2\right)$

$= 2a\left(t(4a^2t^2) + 2t(2a^2t^3) + 2(2a^2t^3) + a^2t^4\right)$

$= 2a(4a^2t^3 + 4a^2t^4 + 4a^2t^3 + a^2t^4)$

$= 2a(8a^2t^3 + 4a^2t^4 + a^2t^4)$

$$= 2a(8a^2t^3 + 5a^2t^4)$$

$$= a^3(16t^3 + 10t^4)$$

Verification:

$$z = (at^2)(2at)^2 + (at^2)^2(2at)$$

$$= (at^2)(4a^2t^2) + (a^2t^4)(2at)$$

$$= 4a^3t^4 + 2a^3t^5$$

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4$$

$$= a^3(16t^3 + 10t^4)$$

Example:

Let $z = f(x,y)$; $x = e^{u} + e^{-v}$; $y = e^{-u} - e^{v}$

$\frac{\partial y}{\partial u} = -e^{-u}$

Show that

$\frac{\partial x}{\partial v} = -e^{-v}$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$\frac{\partial y}{\partial v} = -e^{v}$

$x = e^{u} + e^{-v} \Rightarrow \frac{\partial x}{\partial u} = e^{u}$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \longrightarrow (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \longrightarrow (2)$$

$$(1) \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^{u} + \frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial u} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \longrightarrow (*)$$

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$$(2) \Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^{v}\right)$$

$$\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y} \longrightarrow (**)$$

$(*) - (**)$ implies,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(e^{u} - e^{-v}\right) + \frac{\partial z}{\partial y}\left(e^{v} - e^{-u}\right)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} \mp y\frac{\partial z}{\partial y}$$

$f(r, \theta, z)$ Circular

$f(r, \theta, \phi)$ spherical

$f(x, y, z)$ coordinate equation.