D.E

ODE ↙ ↘ PDE

Lecture No: 06 — Date: 20th Sep 2014

Example: If $w = f(x,y)$; $x = r\cos\theta$, $y = r\sin\theta$. Show that $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$ ← partial differential equation

Solution: Here w is a composite function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} \rightarrow (*)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} \rightarrow (**)$$

$$x = r\cos\theta \quad ; \quad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \quad ; \quad \frac{\partial y}{\partial r} = \sin\theta \rightarrow (1)$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \quad , \quad \frac{\partial y}{\partial \theta} = r\cos\theta \rightarrow (2)$$

Using (1) into (*) we have,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\sin\theta \rightarrow (***)$$

Using (2) into (**) we have,

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot -r\sin\theta + \frac{\partial w}{\partial y} r\cos\theta \rightarrow (****)$$

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-\sin\theta) + \frac{\partial w}{\partial y}\cos\theta$$

Squaring & adding (***) & (****) we have,

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta$$

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$$+\left(\frac{\partial w}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial w}{\partial y}\right)^2 \cos^2\theta$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

$w = f(x,y)$

$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}$

$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y}$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Hence proved.

Example.

$$f(x,y,z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$ show that

$f_x + f_y + f_z = 0$

Solution.

Expanding along R_1

$$f(x,y,z) = x^2(y-z) - y^2(x-z) + z^2(x-y)$$

$$f(x,y,z) = x^2y - x^2z - xy^2 + y^2z + xz^2 - yz^2$$

$$f_x = 2xy - 2xz - y^2 + z^2 \rightarrow (1)$$

$$f_y = x^2 - 2xy + 2yz - z^2 \rightarrow (2)$$

$$f_z = -x^2 + y^2 + 2xz - 2yz \rightarrow (3)$$

Adding (1), (2) & (3)

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$$f_x + f_y + f_z = \cancel{2xy} - \cancel{2xz} - \cancel{y^2} + \cancel{z^2} + \cancel{x^2} - \cancel{2xy} + \cancel{2yz} - \cancel{z^2} - \cancel{x^2} + \cancel{y^2} + \cancel{2xz} - \cancel{2yz}$$

Proved.

... $\cos 28^\circ \tan 44^\circ$ by using differentials. ... maximum error if the ... measurements. Also find the

Change of Variables – The Chain Rule

(9.7) Let $u = f(x, y)$ be a differentiable function. Suppose each of x and y is a differentiable function of a single variable t. Suppose t changes to $t + \Delta t$ ($\Delta t \neq 0$). Then using (1) of Theorem 9.4 we can write

$$\frac{\Delta u}{\Delta t} = \frac{\Delta u}{\Delta x}\frac{\Delta x}{\Delta t} + \frac{\Delta u}{\Delta y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

Considering u as a function of the single variable t, we let $\Delta t \longrightarrow 0$ and take limits of both sides of the above equation. Since x and y are differentiable functions of t, and ε_1 and ε_2 are both small (as $\Delta t \longrightarrow 0$ and $\Delta y \longrightarrow 0$ and so both ε_1 and ε_2 tend to zero), we conclude that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

If each of x and y is a differentiable function of variables r and s, we would similarly get,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$