Linear Algebra

(Week 16-20)

Lecture 2

$$= \Sigma n \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & n-2 & -1 \\ 3 & 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & 1 & 2 & \cdots & 2 & -1 \\ n-1 & 1 & 2-n & \cdots & -2 & -1 \\ n & 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \\ \vdots \\ C_n - C_1 \end{matrix}$$

Expanding from R_1

$$= \Sigma n \begin{vmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 2 & \cdots & 2 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & 2 & -1 \\ 1 & 2-n & \cdots & -2 & -1 \\ 1-n & 2-n & \cdots & -2 & -1 \end{vmatrix}$$

$$= \Sigma n \begin{vmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -n & -1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & -n & -1 \\ 0 & -n & \cdots & -2 & -1 \\ -n & -n & \cdots & -2 & -1 \end{vmatrix} \quad \begin{matrix} C_1 + C_{n-1} \\ C_2 + 2C_{n-1} \\ \vdots \\ C_{n-2} + (n-2)C_{n-1} \end{matrix}$$

$$= \Sigma n \cdot (-1)^{n-1} \cdot (n)^{n-2} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

taking $-n$ common from C_{n-2} & -1 common from $C_1, C_2, \ldots, C_{n-1}$

Going through each of the preceding Columns, $(n-1)$th Column shifted to the place of first Column, there will be $n-2$ changes of sign. The second last Column which now is at the $(n-1)$th position shifted similarly at the position of second Column, there will be $n-3$ changes of sign etc.

So we have

$$\Delta = \Sigma n (-1)^{n-1} \cdot (n)^{n-2+r} \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\Delta = \Sigma n . (-1)^{(n-1)+r} . (n)^{n-2} . 1$$

Now $(n-1)+r = (n-1)+(n-2)+(n-3)+ \cdots +3+2+1$

$$= \frac{(n-1)(n-1+1)}{2}$$

$$= \frac{(n-1)(n)}{2}$$

$$= \frac{n(n-1)}{2}$$

So last eq. becomes

$$\Delta = \Sigma n . (-1)^{\frac{n(n-1)}{2}} . (n)^{n-2}$$

(iv)
$$\begin{vmatrix} x+1 & x & \cdots & x \\ x & x+2 & \cdots & x \\ \vdots & \vdots & & \vdots \\ x & x & \cdots & x+n \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} x+1 & x & \cdots & x & x \\ x & x+2 & \cdots & x & x \\ \vdots & \vdots & & \vdots & \vdots \\ x & x & \cdots & x+(n-1) & x \\ x & x & \cdots & \cdots & x+n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & -n \\ 0 & 2 & \cdots & 0 & -n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & -n \\ x & x & \cdots & x & x+n \end{vmatrix} \quad \begin{matrix} R_1 - R_n \\ R_2 - R_n \\ \vdots \\ R_{n-1} - R_n \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & 0 \\ x & x & \cdots & x & nx\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right) \end{vmatrix} \quad C_n + \left(nC_1 + \frac{n}{2}C_2 + \frac{n}{3}C_3 + \cdots + \frac{n}{n-1}C_{n-1}\right)$$

$$\Delta = 1.2.3 \cdots n-1.\, nx\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right) = n!.\,x.\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{x}\right)$$

Q3 If A & B are 3×3 matrices such that $\det(A^2B^3) = 108$ & $\det(A^3B^2) = 72$.
Find $\det(2A)$ & $\det(B^{-1})$.

Sol. Given

$$\left.\begin{aligned} \det(A^2B^3) &= 108 \\ \& \det(A^3B^2) &= 72 \end{aligned}\right]$$

By product theorem

$$\left.\begin{aligned} \det(A^2).\det(B^3) &= 108 \\ \det(A^3).\det(B^2) &= 72 \end{aligned}\right]$$

or

$$(\det A)^2.(\det B)^3 = 108 \quad \text{———} (1)$$

& $$(\det A)^3.(\det B)^2 = 72 \quad \text{———} (2)$$

Dividing (1) by (2)

$$\frac{\det B}{\det A} = \frac{108}{72}$$

or $$\frac{\det B}{\det A} = \frac{3}{2}$$

$$\Rightarrow \det B = \frac{3}{2}\det A$$

Put in (2)

$$(\det A)^3.\left(\frac{3}{2}\det A\right)^2 = 72$$

$$\frac{9}{4}(\det A)^5 = 72$$

$$\Rightarrow (\det A)^5 = \frac{72\times 4}{9}$$

$$(\det A)^5 = 32$$

$$\Rightarrow \boxed{\det A = 2}$$

Now $\det(2A) = 2^3.\det A$

$$\det(2A) = 8 \times 2$$

So

$$\boxed{\det(2A) = 16}$$

Now

$$\det(B^{-1}) = (\det B)^{-1}$$

$$= \left(\tfrac{3}{2}\det A\right)^{-1}$$

$$= \left(\tfrac{3}{2}\times 2\right)^{-1}$$

$$= (3)^{-1}$$

So $\det(B^{-1}) = \frac{1}{3}$

Q4 Let A be an $n\times n$ matrix. Show that

(i) $\det A^m = (\det A)^m$ for any +ve integer m

(ii) If $\det A^m = 1$ then $\det A = \pm 1$

(iii) If $\det A^m = 0$ then $\det A = 0$

Sol.-

(i) We will prove

$\det A^m = (\det A)^m$ by applying induction on m.

Step ① Let $m = 1$

So $\det A^1 = (\det A)^1$

or $\det A = \det A$

Hence it is true for $m = 1$

Step ② Suppose it is true for $m = K$

i.e., $\det A^K = (\det A)^K$ ———— ① for $K \geq 1$

Step ③ Now we prove it for $m = K+1$

Now

$$\det(A^{k+1}) = \det(A^k \cdot A)$$

$$= (\det A^k) \cdot (\det A) \qquad \text{By product theorem}$$

$$= (\det A)^k \cdot (\det A) \qquad \text{using } ①$$

$$\therefore \det(A^{k+1}) = (\det A)^{k+1}$$

So it is true for $m = k+1$

Hence

$$\det(A^m) = (\det A)^m \quad \text{for all +ve integers } m.$$

(ii) If $\det A^m = 1$ then $\det A = \pm 1$

Sol:-

Since $\det A^m = 1$

So

$$(\det A)^m = 1$$

So $\boxed{\det A = \pm 1}$ where m is an even integer

(iii) If $\det A^m = 0$ then $\det A = 0$

Sol:-

Since $\det A^m = 0$

$$\Rightarrow (\det A)^m = 0$$

$$\Rightarrow \det A = 0$$

Q5 For any non singular matrix C, show that

(i) $\det(C^{-1}) = (\det C)^{-1}$

(ii) $\det(CAC^{-1}) = \det A$

Sol.-

(i) Since C is a non singular matrix, so C^{-1} exists such that

$$CC^{-1} = I$$

$$\Rightarrow \det(CC^{-1}) = \det(I)$$

or $\det(C).\det(C^{-1}) = \det(I)$ (By product theorem)

$$\det C \cdot \det(C^{-1}) = 1$$

$$\det(C^{-1}) = \frac{1}{\det C}$$

$$\det(C^{-1}) = (\det C)^{-1}$$

(ii) $\det(CAC^{-1}) = \det A$

Sol.- using product theorem

$$\det(CAC^{-1}) = \det C \cdot \det A \cdot \det C^{-1}$$

$= \det C \cdot \det C^{-1} \cdot \det A$ $\because$ det. of an element is an element of field & so they commute

$= \det(CC^{-1}) \cdot \det A$ By product theorem

$= \det I \cdot \det A$

$= 1 \cdot \det A$

so $\det(CAC^{-1}) = \det A$

8.

Q6 For what value of α is the matrix

$$A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix}$$ singular ?

Sol.

Given $$A = \begin{bmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{bmatrix}$$

Since A is singular

So $\det A = 0$

$$\Rightarrow \begin{vmatrix} -\alpha & \alpha-1 & \alpha+1 \\ 1 & 2 & 3 \\ 2-\alpha & \alpha+3 & \alpha+7 \end{vmatrix} = 0$$

$$\begin{vmatrix} -\alpha & 3\alpha-1 & 4\alpha+1 \\ 1 & 0 & 0 \\ 2-\alpha & 3\alpha-1 & 4\alpha+1 \end{vmatrix} = 0 \qquad \begin{matrix} C_2-2C_1 \\ C_3-3C_1 \end{matrix}$$

Expanding from R_2

$$-\begin{vmatrix} 3\alpha-1 & 4\alpha+1 \\ 3\alpha-1 & 4\alpha+1 \end{vmatrix} = 0$$

$$-(3\alpha-1)(4\alpha+1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

obviously matrix A is singular for all values of α.

$$\frac{\text{Adj}A}{\det A}\cdot A = \frac{-1}{28}\begin{bmatrix} -2 & -2 & -9 \\ 20 & -8 & 6 \\ -2 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \frac{-1}{28}\begin{bmatrix} -2-8-18 & 2-2+0 & -6-12+18 \\ 20-32+12 & -20-8 & 60-48-12 \\ -2-8+10 & 2-2+0 & -6-12-10 \end{bmatrix}$$

$$= \frac{-1}{28}\begin{bmatrix} -28 & 0 & 0 \\ 0 & -28 & 0 \\ 0 & 0 & -28 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\text{Adj}A}{\det A}\cdot A = I \quad \text{———} \quad (2)$$

from (1) & (2)

$$A\cdot\frac{\text{Adj}A}{\det A} = \frac{\text{Adj}A}{\det A}\cdot A = I$$

$\Rightarrow$

$$A^{-1} = \frac{\text{Adj}A}{\det A}$$

Q8 Evaluate

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix}$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & -6 \\ 2 & 4 & 4 & -3 \\ 3 & 1 & 8 & -9 \end{vmatrix} \qquad \begin{matrix} C_3 + C_1 \\ C_4 - 2C_1 \end{matrix}$$

Expanding from R_1

$$= \begin{vmatrix} 3 & 4 & -6 \\ 4 & 4 & -3 \\ 1 & 8 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -20 & 21 \\ 0 & -28 & 33 \\ 1 & 8 & -9 \end{vmatrix} \qquad \begin{matrix} R_1 - 3R_3 \\ R_2 - 4R_3 \end{matrix}$$

Expanding from C_1

$$= \begin{vmatrix} -20 & 21 \\ -28 & 33 \end{vmatrix}$$

$$= -660 + 588$$

$$\Delta = -72$$