CH # 04.

VECTOR SPACES.

FIELD :-

A field F is a set on which two operations "+" and "." (called addition and multiplication respectively) are defined so that, for each pair of elements x, y in F, there are unique elements $x+y$ and $x.y$ in F, for which following conditions hold for all elements a, b, c in F.

1) Commutative property w.r.t addition and multiplication:-

$a+b=b+a$ and $a.b=b.a$

2) Associative property w.r.t addition and multiplication:-

$(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$

3) Existence of identity element for addition and multiplication:-

There exists distinct element 0 and 1 in F, such that $a+0=a$ and $1.a=a$

4) Existence of inverse for addition and multiplication:-

For each element a in F and each non-zero element b in F, there exists element c and d in F, such that $a+c=0$ and $b.d=1$

5) Distributive property w.r.t addition and multiplication:-

$a.(b+c)=ab+ac$

EXAMPLE :-

The set of real numbers of the form $a+b\sqrt{2}$; a and b are rational numbers with addition and multiplication as in R is defined;
$A=\{a+b\sqrt{2} / a,b \in Q\}$.

Solution :-

Let, $x=a_1+b_1\sqrt{2}$ and $y=a_2+b_2\sqrt{2}$

1)
$x+y = a_1+b_1\sqrt{2}+a_2+b_2\sqrt{2}$
$=(a_1+a_2)+(b_1+b_2)\sqrt{2}$
$=(a_2+a_1)+(b_2+b_1)\sqrt{2}$
$=a_2+a_1+b_2\sqrt{2}+b_1\sqrt{2}$
$=(a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2})$

$x+y = y+x$

$x.y=(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2})$
$=a_1a_2+a_1b_2\sqrt{2}+b_1a_2\sqrt{2}+b_1b_2(\sqrt{2})^2$
$=a_2a_1+b_2a_1\sqrt{2}+a_2b_1\sqrt{2}+b_2b_1(\sqrt{2})^2$
$=(a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$

$x.y = y.x$

2) Let, $x=a_1+b_1\sqrt{2}$, $y=a_2+b_2\sqrt{2}$, $z=a_3+b_3\sqrt{2}$

$(x+y)+z=[(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})]+a_3+b_3\sqrt{2}$
$=(a_1+a_2)+(b_1+b_2)\sqrt{2}+a_3+b_3\sqrt{2}$
$=(a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{2}$
$=a_1+(a_2+a_3)+b_1\sqrt{2}+(b_2+b_3)\sqrt{2}$
$=a_1+b_1\sqrt{2}+[(a_2+a_3)+(b_2+b_3)\sqrt{2}]$
$=a_1+b_1\sqrt{2}+[(a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2})]$

$(x+y)+z = x+(y+z)$

$(x.y).z=[(a_1+b_1\sqrt{2}).(a_2+b_2\sqrt{2})].(a_3+b_3\sqrt{2})$
$=[a_1a_2+a_1b_2\sqrt{2}+b_1\sqrt{2}a_2+b_1b_2(\sqrt{2})^2]$
$.(a_3+b_3\sqrt{2})$

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$$= a_1a_2a_3 + a_1b_2a_3\sqrt{2} + b_1a_2a_3\sqrt{2} + b_1b_2a_3(\sqrt{2})^2$$
$$+ a_1a_2b_3\sqrt{2} + a_1b_2b_3(\sqrt{2})^2 + b_1a_2b_3(\sqrt{2})^2$$
$$+ b_1b_2b_3(\sqrt{2})^2$$
$$= (a_1b_1\sqrt{2})[(a_2+b_2\sqrt{2}).(a_3+b_3\sqrt{2})]$$

$$(x.y).z = x.(y.z)$$

3) Let, $x = a_1+b_1\sqrt{2}$, $0 = 0+0\sqrt{2}$

$$x+0 = a_1+b_1\sqrt{2}+0+0\sqrt{2}$$
$$= (a_1+0)+(b_1+0)\sqrt{2}$$
$$= a_1+b_1\sqrt{2}$$
$$x+0 = x$$

Let, $x = a_1+b_1\sqrt{2}$, $1 = 1+0\sqrt{2}$

$$1.x = (a_1+b_1\sqrt{2})(1+0\sqrt{2})$$
$$= a_1+b_1\sqrt{2}+0$$
$$1.x = x$$

4) Let, $x = a_1+b_1\sqrt{2}$, $x' = -a_1-b_1\sqrt{2}$

$$x+x' = a_1+b_1\sqrt{2}-a_1-b_1\sqrt{2}$$
$$= (a_1-a_1)+(b_1\sqrt{2}-b_1\sqrt{2})$$
$$= 0+0$$
$$x+x' = 0$$

Let, $x = a_1+b_1\sqrt{2}$

$$\frac{1}{x} = \frac{1}{a_1+b_1\sqrt{2}} \times \frac{a_1-b_1\sqrt{2}}{a_1-b_1\sqrt{2}}$$
$$= \frac{a_1-b_1\sqrt{2}}{(a_1)^2-(b_1\sqrt{2})^2}$$
$$= \frac{a_1-b_1\sqrt{2}}{a_1^2-2b_1^2}$$
$$x.\frac{1}{x} = a_1+b_1\sqrt{2} . \frac{a_1-b_1\sqrt{2}}{a_1^2-2b_1^2}$$
$$= \frac{\cancel{a_1^2-2b_1^2}}{\cancel{a_1^2-2b_1^2}} \Rightarrow x.\frac{1}{x} = 1$$

5) Let, $x = a_1 + b_1\sqrt{2}$, $y = a_2 + b_2\sqrt{2}$, $z = a_3 + b_3\sqrt{2}$

$$x.(y+z) = (a_1 + b_1\sqrt{2}).[(a_2 + b_2\sqrt{2}) + (a_3 + b_3\sqrt{2})]$$

$$= (a_1 + b_1\sqrt{2}).[a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2}]$$

$$= a_1a_2 + a_1b_2\sqrt{2} + a_1a_3 + a_1b_3\sqrt{2}$$
$$+ a_2b_1\sqrt{2} + b_1b_2(\sqrt{2})^2 + a_3b_1\sqrt{2} + b_1b_3(\sqrt{2})^2$$

$$= a_2(a_1 + b_1\sqrt{2}) + b_2\sqrt{2}(a_1 + b_1\sqrt{2})$$
$$+ a_3(a_1 + b_1\sqrt{2}) + b_3\sqrt{2}(a_1 + b_1\sqrt{2})$$

$$= (a_1 + b_1\sqrt{2}).(a_2 + b_2\sqrt{2}) + (a_1 + b_1\sqrt{2}).(a_3 + b_3\sqrt{2})$$

$$x.(y+z) = x.y + x.z$$