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( Exercise 6.3 )

Q1 Check which of the following define linear transformations from  $\mathbb{R}^3$  to  $\mathbb{R}^2$  ?

(i)  $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Sol: Given transformation is

$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Let  $u_1 = (x_1, x_2, x_3)$

&  $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$

(i) then we prove  $T(u_1 + u_2) = T(u_1) + T(u_2)$

$$\begin{aligned} \text{Now } T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= ((x_1 + y_1) - (x_2 + y_2), (x_1 + y_1) - (x_3 + y_3)) \\ &= (x_1 + y_1 - x_2 - y_2, x_1 + y_1 - x_3 - y_3) \\ &= (x_1 - x_2 + y_1 - y_2, x_1 - x_3 + y_1 - y_3) \\ &= (x_1 - x_2, x_1 - x_3) + (y_1 - y_2, y_1 - y_3) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= T(u_1) + T(u_2) \end{aligned}$$

(ii) Let  $a \in \mathbb{R}$  &  $u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$

then we prove  $T(au_1) = aT(u_1)$

$$\begin{aligned} \text{Now } T(au_1) &= T(a(x_1, x_2, x_3)) \\ &= T(ax_1, ax_2, ax_3) \\ &= (ax_1 - ax_2, ax_1 - ax_3) \\ &= a(x_1 - x_2, x_1 - x_3) \\ &= aT(x_1, x_2, x_3) \\ &= aT(u_1) \end{aligned}$$

Hence,  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$

$$(ii) T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \text{ then we prove}$$

$$(i) T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (|x_1 + y_1|, (x_2 + y_2) - (x_3 + y_3)) \end{aligned}$$

$$\therefore T(u_1 + u_2) = (|x_1 + y_1|, x_2 + y_2 - x_3 - y_3) \quad \text{--- (1)}$$

Now

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= (|x_1|, x_2 - x_3) + (|y_1|, y_2 - y_3) \\ &= (|x_1| + |y_1|, x_2 - x_3 + y_2 - y_3) \end{aligned}$$

$$\therefore T(u_1) + T(u_2) = (|x_1| + |y_1|, x_2 + y_2 - x_3 - y_3) \quad \text{--- (2)}$$

From (1) & (2)

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

$$(iii) T(x_1, x_2, x_3) = (x_1 + 1, x_2 + x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 + 1, x_2 + x_3)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \text{ then we prove}$$

$$(i) T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$T(u_1 + u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$\begin{aligned}
 T(u_1 + u_2) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
 &= (x_1 + y_1 + 1, x_2 + y_2 + x_3 + y_3) \\
 T(u_1 + u_2) &= (x_1 + y_1 + 1, x_2 + x_3 + y_2 + y_3) \quad \text{--- (1)}
 \end{aligned}$$

Now

$$\begin{aligned}
 T(u_1) + T(u_2) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
 &= (x_1 + 1, x_2 + x_3) + (y_1 + 1, y_2 + y_3) \\
 &= (x_1 + 1 + y_1 + 1, x_2 + x_3 + y_2 + y_3) \\
 T(u_1) + T(u_2) &= (x_1 + y_1 + 2, x_2 + x_3 + y_2 + y_3) \quad \text{--- (2)}
 \end{aligned}$$

From (1) & (2)

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

$$(iv) \quad T(x_1, x_2, x_3) = (0, x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (0, x_3)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& \quad u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \quad \text{then we prove.}$$

$$(i) \quad T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned}
 T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
 &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
 &= (0, x_3 + y_3) \\
 &= (0, x_3) + (0, y_3) \\
 &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)
 \end{aligned}$$

$$\therefore T(u_1 + u_2) = T(u_1) + T(u_2)$$

$$(ii) \quad \text{Let } a \in \mathbb{R} \& \quad u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3 \quad \text{then we prove}$$

$$T(au_1) = aT(u_1)$$

$$\text{Now } T(au_1) = T(a(x_1, x_2, x_3))$$

$$\begin{aligned}
 T(au_1) &= T(ax_1, ax_2, ax_3) \\
 &= (0, ax_3) \\
 &= a(0, x_3) \\
 &= aT(x_1, x_2, x_3)
 \end{aligned}$$

$$T(au_1) = aT(u_1)$$

Hence  $T$  is a linear transformation from  $R^3$  to  $R^1$ .

$$(v) \quad T(x_1, x_2, x_3) = \left( \frac{x_1 + x_2}{x_3}, x_3 \right)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = \left( \frac{x_1 + x_2}{x_3}, x_3 \right)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\text{\& } u_2 = (y_1, y_2, y_3) \in R^3 \text{ then we prove}$$

$$(i) T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned}
 T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
 &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
 &= \left( \frac{x_1 + y_1 + x_2 + y_2}{x_3 + y_3}, x_3 + y_3 \right) \quad \text{--- (1)}
 \end{aligned}$$

Now

$$\begin{aligned}
 T(u_1) + T(u_2) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
 &= \left( \frac{x_1 + x_2}{x_3}, x_3 \right) + \left( \frac{y_1 + y_2}{y_3}, y_3 \right) \\
 &= \left( \frac{x_1 + x_2}{x_3} + \frac{y_1 + y_2}{y_3}, x_3 + y_3 \right) \\
 &= \left( \frac{y_3(x_1 + x_2) + x_3(y_1 + y_2)}{x_3 y_3}, x_3 + y_3 \right) \quad \text{--- (2)}
 \end{aligned}$$

From (1) & (2)

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $R^3$  to  $R^1$ .

$$(vi) \quad T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& \quad u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \quad \text{then we prove}$$

$$(i) \quad T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (3(x_1 + y_1) - 2(x_2 + y_2) + (x_3 + y_3), (x_3 + y_3) - 3(x_2 + y_2) - 2(x_1 + y_1)) \\ &= (3x_1 - 2x_2 + x_3 + 3y_1 - 2y_2 + y_3, x_3 - 3x_2 - 2x_1 + y_3 - 3y_2 - 2y_1) \\ &= (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1) + (3y_1 - 2y_2 + y_3, y_3 - 3y_2 - 2y_1) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= T(u_1) + T(u_2) \end{aligned}$$

$$(ii) \quad \text{Let } a \in \mathbb{R} \& \quad u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3 \quad \text{then we prove}$$

$$T(au_1) = aT(u_1)$$

$$\begin{aligned} \text{Now } T(au_1) &= T(a(x_1, x_2, x_3)) \\ &= T(ax_1, ax_2, ax_3) \\ &= (3ax_1 - 2ax_2 + ax_3, ax_3 - 3ax_2 - 2ax_1) \\ &= a(3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1) \\ &= aT(x_1, x_2, x_3) \\ &= aT(u_1) \end{aligned}$$

Hence  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

Q2 Show that each of the following defines linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

$$(i) \quad T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$$

Let  $u_1 = (x_1, x_2, x_3)$

&  $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$  then we prove

(i)  $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= ((x_1 + y_1) - (x_2 + y_2), (x_2 + y_2) - (x_3 + y_3), x_1 + y_1) \\ &= (x_1 + y_1 - x_2 - y_2, x_2 + y_2 - x_3 - y_3, x_1 + y_1) \\ &= (x_1 - x_2 + y_1 - y_2, x_2 - x_3 + y_2 - y_3, x_1 + y_1) \\ &= (x_1 - x_2, x_2 - x_3, x_1) + (y_1 - y_2, y_2 - y_3, y_1) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= T(u_1) + T(u_2) \end{aligned}$$

Now

$$\begin{aligned} T(au_1) &= T(a(x_1, x_2, x_3)) \\ &= T(ax_1, ax_2, ax_3) \\ &= (ax_1 - ax_2, ax_2 - ax_3, ax_1) \\ &= a(x_1 - x_2, x_2 - x_3, x_1) \\ &= aT(x_1, x_2, x_3) \\ &= aT(u_1) \end{aligned}$$

Hence  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$

(ii)  $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$$

Let  $u_1 = (x_1, x_2, x_3)$

&  $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$  then we prove

(i)  $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned}
 T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
 &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
 &= (x_1 + y_1 + x_2 + y_2, -(x_1 + y_1) - (x_2 + y_2), x_3 + y_3) \\
 &= (x_1 + y_1 + x_2 + y_2, -x_1 - y_1 - x_2 - y_2, x_3 + y_3) \\
 &= (x_1 + x_2 + y_1 + y_2, -x_1 - x_2 - y_1 - y_2, x_3 + y_3) \\
 &= (x_1 + x_2, -x_1 - x_2, x_3) + (y_1 + y_2, -y_1 - y_2, y_3) \\
 &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
 &= T(u_1) + T(u_2)
 \end{aligned}$$

(ii) Let  $\alpha \in \mathbb{R}$  &  $u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$  then we prove

$$T(\alpha u_1) = \alpha T(u_1)$$

Now

$$\begin{aligned}
 T(\alpha u_1) &= T(\alpha(x_1, x_2, x_3)) \\
 &= T(\alpha x_1, \alpha x_2, \alpha x_3) \\
 &= (\alpha x_1 + \alpha x_2, -\alpha x_1 - \alpha x_2, \alpha x_3) \\
 &= \alpha(x_1 + x_2, -x_1 - x_2, x_3) \\
 &= \alpha T(x_1, x_2, x_3) \\
 &= \alpha T(u_1)
 \end{aligned}$$

Hence  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$

$$(ii) \quad T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& \text{ } u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \text{ then we prove}$$

$$(i) \quad T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$T(u_1 + u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$\begin{aligned}
T(u_1 + u_2) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= (x_2 + y_2, -(x_1 + y_1), -(x_3 + y_3)) \\
&= (x_2 + y_2, -x_1 - y_1, -x_3 - y_3) \\
&= (x_2, -x_1, -x_3) + (y_2, -y_1, -y_3) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}$$

(ii) Let  $\alpha \in \mathbb{R}$  &  $u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$  then we prove

$$T(\alpha u_1) = \alpha T(u_1)$$

Now

$$\begin{aligned}
T(\alpha u_1) &= T(\alpha(x_1, x_2, x_3)) \\
&= T(\alpha x_1, \alpha x_2, \alpha x_3) \\
&= (\alpha x_2, -\alpha x_1, -\alpha x_3) \\
&= \alpha(x_2, -x_1, -x_3) \\
&= \alpha T(x_1, x_2, x_3) \\
&= \alpha T(u_1)
\end{aligned}$$

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Hence  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

$$(iv) T(x_1, x_2, x_3) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$$

Sol: Given transformation is

$$T(x_1, x_2, x_3) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \text{ then we prove}$$

$$(i) T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned}
T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
&= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= ((x_1 + y_1) - 3(x_2 + y_2) - 2(x_3 + y_3), (x_2 + y_2) - 4(x_3 + y_3), x_3 + y_3) \\
&= (x_1 - 3x_2 - 2x_3 + y_1 - 3y_2 - 2y_3, x_2 - 4x_3 + y_2 - 4y_3, x_3 + y_3)
\end{aligned}$$



$$T(u_1 + u_2) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3) + (y_1 - 3y_2 - 2y_3, y_2 - 4y_3, y_3) \\ = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$

$$= T(u_1) + T(u_2)$$

(ii) Let  $\alpha \in \mathbb{R}$  &  $u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$  then we prove

$$T(\alpha u_1) = \alpha T(u_1)$$

Now

$$T(\alpha u_1) = T(\alpha(x_1, x_2, x_3))$$

$$= T(\alpha x_1, \alpha x_2, \alpha x_3)$$

$$= (3\alpha x_1 - 3\alpha x_2 - 2\alpha x_3, \alpha x_2 - 4\alpha x_3, \alpha x_3)$$

$$= \alpha(3x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$$

$$= \alpha T(x_1, x_2, x_3)$$

$$= \alpha T(u_1)$$

Hence  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

$$(V) \quad T(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_3, x_2)$$

Sol. Given Transformation is

$$T(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_3, x_2)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

$$\& \quad u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3 \quad \text{then we prove}$$

$$T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$T(u_1 + u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$= ((x_1 + y_1) + (x_3 + y_3), (x_1 + y_1) - (x_3 + y_3), x_2 + y_2)$$

$$= (x_1 + x_3 + y_1 + y_3, x_1 - x_3 + y_1 - y_3, x_2 + y_2)$$

$$= (x_1 + x_3, x_1 - x_3, x_2) + (y_1 + y_3, y_1 - y_3, y_2)$$

$$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$

$$= T(u_1) + T(u_2)$$

(ii) Let  $a \in \mathbb{R}$  &  $u_1 = (x_1, x_2, x_3) \in \mathbb{R}^3$  then we prove  
 $T(au_1) = aT(u_1)$

Now

$$\begin{aligned} T(au_1) &= T(a(x_1, x_2, x_3)) \\ &= T(ax_1, ax_2, ax_3) \\ &= (ax_1 + ax_3, ax_1 - ax_3, ax_2) \\ &= a(x_1 + x_3, x_1 - x_3, x_2) \\ &= aT(x_1, x_2, x_3) \\ &= aT(u_1) \end{aligned}$$

Hence  $T$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

Q3 Show that each of the following transformations is not linear.

(i)  $T: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $T(x_1, x_2) = x_1 x_2$

Sol. Given transformation is

$$T(x_1, x_2) = x_1 x_2$$

$$\text{Let } u_1 = (x_1, x_2)$$

$$\& u_2 = (y_1, y_2) \in \mathbb{R}^2 \text{ then we prove}$$

$$(i) T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2) + (y_1, y_2)) \\ &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_1 + y_1)(x_2 + y_2) \quad \text{--- (1)} \end{aligned}$$

&

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= x_1 x_2 + y_1 y_2 \quad \text{--- (2)} \end{aligned}$$

from (1) & (2)  $T(u_1 + u_2) \neq T(u_1) + T(u_2)$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}$

(ii)  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  defined by  $T(x_1, x_2) = (x_1 + 1, 2x_2, x_1 + x_2)$

Sol. Given transformation is

$$T(x_1, x_2) = (x_1 + 1, 2x_2, x_1 + x_2)$$

Let  $u_1 = (x_1, x_2)$

&  $u_2 = (y_1, y_2) \in \mathbb{R}^2$  then we prove

(i)  $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2) + (y_1, y_2)) \\ &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_1 + y_1 + 1, 2(x_2 + y_2), (x_1 + y_1) + (x_2 + y_2)) \\ &= (x_1 + y_1 + 1, 2x_2 + 2y_2, x_1 + x_2 + y_1 + y_2) \quad \text{--- ①} \end{aligned}$$

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= (x_1 + 1, 2x_2, x_1 + x_2) + (y_1 + 1, 2y_2, y_1 + y_2) \\ &= (x_1 + y_1 + 2, 2x_2 + 2y_2, x_1 + x_2 + y_1 + y_2) \quad \text{--- ②} \end{aligned}$$

from ① & ②

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ .

(iii)  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  $T(x_1, x_2, x_3) = (|x_1|, 0)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (|x_1|, 0)$$

Let  $u_1 = (x_1, x_2, x_3)$

&  $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$  then we prove

(i)  $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \end{aligned}$$

$$T(u_1 + u_2) = (|x_1 + y_1|, 0) \quad \text{————— ①}$$

4

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= (|x_1|, 0) + (|y_1|, 0) \\ &= (|x_1| + |y_1|, 0) \quad \text{————— ②} \end{aligned}$$

From ① & ②

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

(iv)  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(x_1, x_2) = (x_1^2, x_2^2)$ .

Sol. Given transformation is

$$T(x_1, x_2) = (x_1^2, x_2^2)$$

Let  $u_1 = (x_1, x_2)$

&  $u_2 = (y_1, y_2) \in \mathbb{R}^2$  then we prove

(i)  $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2) + (y_1, y_2)) \\ &= T(x_1 + y_1, x_2 + y_2) \\ &= ((x_1 + y_1)^2, (x_2 + y_2)^2) \quad \text{—————} \end{aligned}$$

4

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= (x_1^2, x_2^2) + (y_1^2, y_2^2) \\ &= (x_1^2 + y_1^2, x_2^2 + y_2^2) \quad \text{————— ②} \end{aligned}$$

from ① & ②

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

(v)  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by  $T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1, 1, 1)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1, 1, 1)$$

$$\text{Let } u_1 = (x_1, x_2, x_3)$$

&  $u_2 = (y_1, y_2, y_3) \in \mathbb{R}^3$  then we prove

$$(i) T(u_1 + u_2) = T(u_1) + T(u_2)$$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\ &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) + (1, 1, 1) \\ &= (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1) \quad \text{--- (1)} \end{aligned}$$

&

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= (x_1, x_2, x_3) + (1, 1, 1) + (y_1, y_2, y_3) + (1, 1, 1) \\ &= (x_1 + 1, x_2 + 1, x_3 + 1) + (y_1 + 1, y_2 + 1, y_3 + 1) \\ &= (x_1 + y_1 + 2, x_2 + y_2 + 2, x_3 + y_3 + 2) \quad \text{--- (2)} \end{aligned}$$

for (1) & (2)

$$T(u_1 + u_2) \neq T(u_1) + T(u_2)$$

Hence  $T$  is not a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

Q3 Determine which of the following transformations are linear:

(a)  $T: M_{22} \rightarrow \mathbb{R}$  defined by

$$(i) T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

Sol. Given transformation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

Let  $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

4 &  $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{22}$  then we prove

(i)  $T(A_1 + A_2) = T(A_1) + T(A_2)$

Now

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right)$$

$$= a_1 + a_2 + d_1 + d_2 \quad \text{————— (1)}$$

4

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1 + d_1 + a_2 + d_2$$

$$= a_1 + a_2 + d_1 + d_2 \quad \text{————— (2)}$$

from (1) & (2)

$$T(A_1 + A_2) = T(A_1) + T(A_2)$$

(ii) Let  $a \in R$  &  $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_{22}$  then we prove

$$T(aA_1) = aT(A_1)$$

Now  $T(aA_1) = T\left(a \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} aa_1 & ab_1 \\ ac_1 & ad_1 \end{bmatrix}\right)$$

$$= aa_1 + ad_1$$

$$= a(a_1 + d_1)$$

$$= aT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= aT(A_1)$$

Hence  $T$  is a linear transformation from  $M_{22}$  to  $R$

(ii)  $T: M_{22} \rightarrow R$  defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Sol. Given transformation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Let  $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

&  $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

then we prove

$$T(A_1 + A_2) = T(A_1) + T(A_2)$$

Now

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right)$$

$$= (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)(c_1 + c_2) \quad \text{--- (1)}$$

Now

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1 d_1 - b_1 c_1 + a_2 d_2 - b_2 c_2 \quad \text{--- (2)}$$

from (1) & (2)

$$T(A_1 + A_2) \neq T(A_1) + T(A_2)$$

Hence  $T$  is a linear transformation from  $M_{22}$  to  $R$ .

(b)  $T: P_2(x) \rightarrow P_2(x)$  defined by

(i)  $T(a + bx + cx^2) = a + (b+c)x + (2a-3b)x^2$

Sol. Given transformation is

$$T(a + bx + cx^2) = a + (b+c)x + (2a-3b)x^2$$

Let  $U = a + bx + cx^2$

&  $V = p + qx + rx^2 \in P_2(x)$  then we prove

(i)  $T(U+V) = T(U) + T(V)$

Now

$$\begin{aligned}
 T(U+V) &= T((a+bx+cx^2) + (p+qx+rx^2)) \\
 &= T((a+p) + (b+q)x + (c+r)x^2) \\
 &= (a+p) + (b+q+c+r)x + (2a+2p-3b-3q)x^2 \\
 &= (a+p) + (b+c+q+r)x + (2a-3b+2p-3q)x^2 \\
 &= (a+(b+c)x + (2a-3b)x^2) + (p+(q+r)x + (2p-3q)x^2) \\
 &= T(a+bx+cx^2) + T(p+qx+rx^2) \\
 &= T(U) + T(V)
 \end{aligned}$$

(ii) Let  $K \in \mathbb{R}$  &  $U = a+bx+cx^2$  then we prove

$T(KU) = KT(U)$

Now

$$\begin{aligned}
 T(KU) &= T(K(a+bx+cx^2)) \\
 &= T(Ka + Kb x + Kc x^2) \\
 &= Ka + (Kb + Kc)x + (2Ka - 3Kb)x^2 \\
 &= K(a + (b+c)x + (2a-3b)x^2) \\
 &= KT(a+bx+cx^2) \\
 &= KT(U)
 \end{aligned}$$

Hence  $T$  is a linear transformation from  $P_2(x)$  to  $P_2(x)$ .

(ii)  $T: P_2(x) \longrightarrow P_2(x)$  defined by

$T(a+bx+cx^2) = (a+1) + bx + cx^2$

Sol: Given transformation is

$T(a+bx+cx^2) = (a+1) + bx + cx^2$

Let  $U = a+bx+cx^2$

&  $V = p+qx+rx^2 \in P_2(x)$  then we prove



$$(i) \quad T(u+v) = T(u) + T(v)$$

Now

$$\begin{aligned} T(u+v) &= T((a+bx+cx^2) + (b+vx+lx^2)) \\ &= T((a+b) + (b+v)x + (c+l)x^2) \\ &= (a+b+1) + (b+v)x + (c+l)x^2 \quad \text{--- (1)} \end{aligned}$$

+

$$\begin{aligned} T(u) + T(v) &= T(a+bx+cx^2) + T(b+vx+lx^2) \\ &= (a+1) + bx + cx^2 + (b+1) + vx + lx^2 \\ &= (a+b+2) + (b+v)x + (c+l)x^2 \quad \text{--- (2)} \end{aligned}$$

From (1) & (2)

$$T(u+v) \neq T(u) + T(v)$$

Hence  $T$  is not a linear transformation from  $P_2(x)$  to  $P_2(x)$ .

Q5. If  $A$  is an  $m \times n$  matrix, show that  $T(x) = Ax$  is a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Soln.

Given transformation is

$$T(x) = Ax$$

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

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