LINEARLY DEPENDENT & LINEARLY INDEPENDENT :-

Let, V be a vector space over field F. The vectors $V_1, V_2, V_3, \ldots, V_n \in V$ are said to be linearly dependent over field F if

$$a_1V_1 + a_2V_2 + a_3V_3 + \ldots + a_nV_n = 0$$

and not all $a_i = 0$.

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and for all $a_i = 0$.

EXAMPLE :-

Show that the vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1) \in R^3$ are linearly independent.

Solution :-

Let, $u = (1,0,0)$, $v = (0,1,0)$, $w = (0,0,1)$

By definition of linearly independent :-

$$au + bv + cw = 0$$

$a(1,0,0) + b(0,1,0) + c(0,0,1) = (0,0,0)$

$(a,0,0) + (0,b,0) + (0,0,c) = (0,0,0)$

$(a+0+0, 0+b+0, 0+0+c) = (0,0,0)$

$(a, b, c) = (0,0,0)$

$a = 0$, $b = 0$, $c = 0$

Since $a = b = c = 0$, so the given vectors are linearly independent.

EXAMPLE :-

Show that the vectors $(3,0,-3)$, $(-1,1,2)$, $(4,2,-2)$ and $(2,1,1) \in R^3$ are linearly dependent.

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Solution :-

Let, $u_1 = (3,0,-3)$, $u_2 = (-1,1,2)$,
$u_3 = (4,2,-2)$, $u_4 = (2,1,1)$.

By definition of linearly dependent :-

$$a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 = 0$$

$$a_1(3,0,-3) + a_2(-1,1,2) + a_3(4,2,-2) + a_4(2,1,1) = (0,0,0)$$

$$(3a_1, 0, -3a_1) + (-a_2, a_2, 2a_2) + (4a_3, 2a_3, -2a_3) + (2a_4, a_4, a_4) = (0,0,0)$$

$$(3a_1 - a_2 + 4a_3 + 2a_4,\ a_2 + 2a_3 + a_4,\ -3a_1 + 2a_2 - 2a_3 + a_4) = (0,0,0)$$

Comparing :-

$$3a_1 - a_2 + 4a_3 + 2a_4 = 0 \rightarrow (1)$$

$$a_2 + 2a_3 + a_4 = 0 \rightarrow (2)$$

$$-3a_1 + 2a_2 - 2a_3 + a_4 = 0 \rightarrow (3)$$

Now,

$$\begin{bmatrix} 3 & -1 & 4 & 2 \\ 0 & 1 & 2 & 1 \\ -3 & 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{R_1}{3} \begin{bmatrix} 1 & -1/3 & 4/3 & 2/3 \\ 0 & 1 & 2 & 1 \\ -3 & 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + 3R_1 \begin{bmatrix} 1 & -1/3 & 4/3 & 2/3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_1 + 1/3 R_2 \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (*)$$

From (*) :-

$$2a_4 = 0 \Rightarrow a_4 = 0$$

$$* \quad a_2 + 2a_3 + \cancel{a_4}^{0} = 0$$

$$a_2 + 2a_3 = 0$$

$$a_2 = -2a_3$$

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$$a_1 + 2a_3 + \cancel{a_4}^{0} = 0$$
$$a_1 + 2a_3 = 0$$
$$a_1 = -2a_3$$

Since a_1, a_2, a_3 are not equal to 0, so the given vectors are linearly dependent.

EXAMPLE :-

Determine whether the given vectors are linearly independent or linearly dependent.
$(1,-2,4,1), (2,1,0,-3), (1,-6,1,4)$.

Solution :-

Let, $u_1 = (1,-2,4,1)$, $u_2 = (2,1,0,-3)$,
$u_3 = (1,-6,1,4)$.

By definition :-

$$a_1u_1 + a_2u_2 + a_3u_3 = 0$$
$$a_1(1,-2,4,1) + a_2(2,1,0,-3) + a_3(1,-6,1,4) = (0,0,0,0)$$
$$(a_1, -2a_1, 4a_1, a_1) + (2a_2, a_2, 0, -3a_2) + (a_3, -6a_3, a_3, 4a_3) = (0,0,0,0)$$
$$(a_1 + 2a_2 + a_3,\ -2a_1 + a_2 - 6a_3,\ 4a_1 + a_3,\ a_1 - 3a_2 + 4a_3) = (0,0,0,0)$$

Comparing :-

$$a_1 + 2a_2 + a_3 = 0 \quad \rightarrow ①$$
$$-2a_1 + a_2 - 6a_3 = 0 \quad \rightarrow ②$$
$$4a_1 + a_3 = 0 \quad \rightarrow ③$$
$$a_1 - 3a_2 + 4a_3 = 0 \quad \rightarrow ④$$

Now,

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -6 \\ 4 & 0 & 1 \\ 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$R_2+2R_1,\ R_3+4R_1,\ R_4-R_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & -4 \\ 0 & -8 & -3 \\ 0 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{R_2}{5} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -4/5 \\ 0 & -8 & -3 \\ 0 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1-2R_2,\ R_3+8R_2,\ R_4+5R_2 \quad \begin{bmatrix} 1 & 0 & 13/5 \\ 0 & 1 & -4/5 \\ 0 & 0 & -47/5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \circledast$$

From $\circledast$:-

$$-a_3 = 0 \Rightarrow a_3 = 0$$

$$-47/5\, a_3 = 0 \Rightarrow a_3 = 0$$

$$* \quad a_2 - 4/5\, \cancel{a_3}^{\,0} = 0 \Rightarrow a_2 = 0$$

$$* \quad a_1 + 13/5\, \cancel{a_3}^{\,0} = 0 \Rightarrow a_1 = 0$$

Since $a_1 = a_2 = a_3 = 0$, so the given vectors are linearly independent.