VECTOR SPACE :-

A vector space V over a field F consists of a set on which two operations (called addition and scalar multiplication) are defined so that for each pair of elements x, y in V there is a unique element $x+y$ in V and for each $\alpha \in F$ and each element x in V, there is a unique element αx in V, such that the following conditions hold:-

i) For all $x, y \in V$
$$x+y = y+x$$

ii) For all $x, y, z \in V$
$$(x+y)+z = x+(y+z)$$

iii) There exists an element in V denoted by 0, such that
$$x+0 = x \quad \text{for each } x \text{ in } V$$

iv) For each element x in V, there exists an element y in V, such that
$$x+y = 0$$

v) For each element x in V
$$1.x = x$$

vi) For each pair of elements α, β in F and x in V
$$(\alpha . \beta)x = \alpha(\beta x)$$

vii) For each element α in F and each pair of elements x, y in V
$$\alpha(x+y) = \alpha x + \alpha y$$

viii) For each pair of elements α, β in F and each element x in V
$$(\alpha+\beta)x = \alpha x + \beta x$$

EXAMPLE :-

Show that the set $\mathbb{R}^3$ is a vector space over $(F=\mathbb{R})$ with operation of coordinate wise addition and scalar multiplication, i.e $x=(x_1,x_2,x_3)$; $y=(y_1,y_2,y_3)$ in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Thus,

$$x+y=(x_1,x_2,x_3)+(y_1,y_2,y_3)$$
$$=(x_1+y_1,x_2+y_2,x_3+y_3)$$

Also,

$$\alpha x=\alpha(x_1,x_2,x_3)$$
$$=(\alpha x_1,\alpha x_2,\alpha x_3)$$

Solution :-

i) Let $x,y \in V$ such that $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ then to show that $x+y=y+x$

L.H.S $\Rightarrow$

$$x+y=(x_1,x_2,x_3)+(y_1,y_2,y_3)$$
$$=(x_1+y_1,x_2+y_2,x_3+y_3)$$
$$=(y_1+x_1,y_2+x_2,y_3+x_3)$$
$$=(y_1,y_2,y_3)+(x_1,x_2,x_3)$$
$$x+y=y+x$$

ii) Let $x,y,z \in V$ such that $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$, $z=(z_1,z_2,z_3)$ then to show that $(x+y)+z=x+(y+z)$

L.H.S $\Rightarrow$

$$(x+y)+z=[(x_1,x_2,x_3)+(y_1,y_2,y_3)]+(z_1,z_2,z_3)$$
$$=[x_1+y_1,x_2+y_2,x_3+y_3]+(z_1,z_2,z_3)$$
$$=(x_1+y_1+z_1,x_2+y_2+z_2,x_3+y_3+z_3)$$
$$=(x_1,x_2,x_3)+[y_1+z_1,y_2+z_2,y_3+z_3]$$

$= (x_1, x_2, x_3) + [(y_1, y_2, y_3) + (z_1, z_2, z_3)]$

$(x+y)+z = x+(y+z)$

iii) There exists an element in V denoted by 0, such that

$x + 0 = x$

L.H.S ⇒

$x + 0 = (x_1, x_2, x_3) + (0, 0, 0)$

$= (x_1 + 0, x_2 + 0, x_3 + 0)$

$= (x_1, x_2, x_3)$

$= x$

iv) For each element x in V, there exists an element x' in V, such that

$x + x' = 0$

L.H.S ⇒

$x + x' = (x_1, x_2, x_3) + (-x_1, -x_2, -x_3)$

$= (x_1 - x_1, x_2 - x_2, x_3 - x_3)$

$= (0, 0, 0)$

$= 0$

v) For each element x in V such that

$1.x = x$

L.H.S ⇒

$1.x = 1.(x_1, x_2, x_3)$

$= (1.x_1, 1.x_2, 1.x_3)$

$= (x_1, x_2, x_3)$

$= x$

vi) For each pair of elements $\alpha, \beta \in F = R$ and x in V, such that

$(\alpha \beta) x = \alpha (\beta x)$

L.H.S ⇒

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$(\alpha.\beta)x = (\alpha.\beta)(x_1, x_2, x_3)$

$= [(\alpha.\beta)x_1, (\alpha.\beta)x_2, (\alpha.\beta)x_3]$

$= [\alpha(\beta x_1), \alpha(\beta x_2), \alpha(\beta x_3)]$

$= \alpha(\beta x)$

vii) For each element α in F and each pair of elements x, y in V, such that

$\alpha(x+y) = \alpha x + \alpha y$

L.H.S $\Rightarrow$

$\alpha(x+y) = \alpha[(x_1, x_2, x_3) + (y_1, y_2, y_3)]$

$= \alpha[x_1+y_1, x_2+y_2, x_3+y_3]$

$= [\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \alpha x_3 + \alpha y_3]$

$= (\alpha x_1, \alpha x_2, \alpha x_3) + (\alpha y_1, \alpha y_2, \alpha y_3)$

$= \alpha x + \alpha y$

ix) For each pair of elements α, β in $F = R$ and each element x in V, such that

$(\alpha+\beta)x = \alpha x + \beta x$

L.H.S $\Rightarrow$

$(\alpha+\beta)x = (\alpha+\beta)(x_1, x_2, x_3)$

$= [(\alpha+\beta)x_1, (\alpha+\beta)x_2, (\alpha+\beta)x_3]$

$= [\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \alpha x_3 + \beta x_3]$

$= (\alpha x_1, \alpha x_2, \alpha x_3) + (\beta x_1, \beta x_2, \beta x_3)$

$= \alpha(x_1, x_2, x_3) + \beta(x_1, x_2, x_3)$

$= \alpha x + \beta x$