CHAPTER NO. 4

Techniques of Integration

Exercise NO. 4.1

write down the indefinite integral of each of the following.

1 $\quad 0$

$= \int 0\,dx$

$=$ constant.

he

2 $\quad \sqrt{x}$

$\int \sqrt{x}\,dx$

$= \dfrac{x^{1/2+1}}{1/2+1} + c$

$= \dfrac{x^{3/2}}{3/2} + c$

$= \dfrac{2}{3} x^{3/2} + c$ Ans.

3 $\quad \dfrac{1+x}{x}$

$= \int \left(\dfrac{1}{x} + 1\right) dx$

$= \int \dfrac{1}{x}\,dx + \int 1\,dx$

$= \ln|x| + x + c$ Ans.

4 $\dfrac{x^2-1}{x^2+1}$

$$= \int \frac{x^2+1-2}{x^2+1}\,dx$$

$$= \int \left(1 - \frac{2}{x^2+1}\right) dx$$

$$= \int 1\,dx - 2\int \frac{1}{1+x^2}\,dx$$

$$= x - 2\tan^{-1}(x) + c \text{ Ans.}$$

5 $\tan^2 x$

$$= \int (\sec^2 x - 1)\,dx$$

$$= \int \sec^2 x\,dx - \int 1\,dx$$

$$= \tan x - x + c \text{ Ans.}$$

6 $\cot^2 x$

$$= \int (\operatorname{cosec}^2 x - 1)\,dx$$

$$= \int \operatorname{cosec}^2 x\,dx - \int 1\,dx$$

$$= -\cot x - x + c \text{ Ans.}$$

7 $\cos^2 x$

$$= \int \left(\frac{1+\cos 2x}{2}\right) dx$$

$$\frac{1}{2}\int 1\,dx + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin 2x}{2} + c$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c \quad \text{Ans.}$$

8 $\sin^2 x$

$$= \int\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c \quad \text{Ans.}$$

9 $\sqrt{1-\cos x}$

$$= \int\sqrt{2\sin^2 x/2}\,dx$$

$$= \int\sqrt{2}\,\sin x/2\,dx$$

$$= \sqrt{2}\int \sin x/2\,dx$$

$$= \sqrt{2}\left(\frac{-\cos x/2}{1/2}\right) + c$$

$$= -2\sqrt{2}\cos\frac{x}{2} + c \quad \text{Ans.}$$

10 $\sqrt{4-x^2}$

put $x = 2\sin\theta$ ---- (i)

$dx = 2\cos\theta\,d\theta$

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$$= \int \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta d\theta$$

$$= 2\sqrt{1 - \sin^2\theta}\ 2\cos\theta d\theta$$

$$= 4\int \cos^2\theta d\theta$$

$$= 4\int \frac{1 + \cos 2\theta}{2} d\theta$$

$$= 2\int (1 + \cos 2\theta) d\theta$$

$$= 2\left[\int 1 d\theta + \int \cos 2\theta d\theta\right]$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right] + c \quad \text{---(ii)}$$

$$(i) \Rightarrow \quad \sin\theta = \frac{x}{2}$$

$$\theta = \sin^{-1}\left(\frac{x}{2}\right)$$

put in (ii)

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) + \frac{2\sin\theta\cos\theta}{2}\right] + c$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) + \sin\theta\sqrt{1 - \sin^2\theta}\right] + c$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{1 - x^2/4}\right] + c$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) + x\sqrt{\frac{4 - x^2}{4}}\right] + c$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4 - x^2}}{2} + c \quad \text{Ans.}$$

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11. $\sqrt{4+x^2}$

put

$$x = 2\sinh\theta$$
$$dx = 2\cosh\theta\, d\theta$$

$$= \int \sqrt{4 + 4\sinh^2\theta}\; 2\cosh\theta\, d\theta$$

$$= \int 2\sqrt{1+\sinh^2\theta} \cdot 2\cosh\theta\, d\theta$$

$$= 4\int \cosh^2\theta\, d\theta$$

$$= 4\int \frac{1+\cosh 2\theta}{2}\, d\theta$$

$$= 2\int (1+\cosh 2\theta)\, d\theta$$

$$= 2\left[\theta + \frac{\sinh 2\theta}{2}\right] + c$$

$$= 2\left[\theta + \sinh\theta\cosh\theta\right] + c$$

$$= 2\left[\theta + \sinh\theta\sqrt{1+\sinh^2\theta}\right] + c$$

$$= 2\left[\sinh^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{1 + x^2/4}\right] + c$$

$$= 2\left[\sinh^{-1}\left(\frac{x}{2}\right) + x\sqrt{\frac{4+x^2}{4}}\right] + c$$

$$= 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + \frac{x}{2}\sqrt{4+x^2} + c \text{ Ans.}$$

12. $\sqrt{x^2-4}$

put

$$x = 2\cosh\theta$$

$$dx = 2\sinh\theta \, d\theta$$

$$= \int \sqrt{4\cosh^2\theta - 4} \cdot 2\sinh\theta \, d\theta$$

$$= \int \sqrt{4(\cosh^2\theta - 1)} \cdot 2\sinh\theta \, d\theta$$

$$= \int 2\sinh\theta \cdot 2\sinh d\theta$$ 1

$$= \int 4\sinh^2\theta \, d\theta$$

$$= 4\int \sinh^2\theta \, d\theta$$

$$= 4\int \frac{\cosh 2\theta - 1}{2} \, d\theta$$ 2

$$= 2\int (\cosh 2\theta - 1) \, d\theta$$

$$= 2\left[\frac{\sinh 2\theta}{2} - \theta\right] + c$$

$$= 2\{\sinh\theta\cosh\theta - \theta\} + c$$

$$= 2\{\sqrt{\cosh^2\theta - 1}\cosh\theta - \theta\} + c$$

(i) $\Rightarrow$ $x = 2\cosh\theta$

$$\cosh\theta = \frac{x}{2}$$

$$\theta = \cosh^{-1}\left(\frac{x}{2}\right)$$

$$= 2\left\{\sqrt{x^2/4 - 1}\,(x/2) - \cosh^{-1}(x/2)\right\} + c$$

$$= 2\left\{\frac{x}{2}\sqrt{\frac{x^2 - 4}{4}} - \ln \frac{x + \sqrt{x^2 - 4}}{2}\right\} + c$$

$$= \frac{x}{2}\sqrt{x^2-4} - 2\ln \frac{x+\sqrt{x^2-4}}{2} + c \text{ Ans.}$$

Table of Integrals.

1 $$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad ; n \neq -1$$

$$[f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad ; n \neq -1$$

2 $$\int \frac{1}{x} dx = \ln|x| + c.$$

$$\int [f(x)]^{-1} f'(x)\, dx = \int \frac{f'(x)}{f(x} dx$$

$$= \ln|f(x)| + c.$$

$$\therefore \int \frac{1}{ax+b} dx$$

$$\frac{1}{a}\ln|ax+b| + c$$

$\because$

$$\frac{d}{dx}(\ln x)$$

$$= \frac{1}{x} \quad ; x > 0$$

$$\frac{d}{dx}[\ln(-x)] = \frac{-1}{-x}$$

$$\int \frac{1}{x} dx = \ln x \text{ or } \ln(-x)$$