6. If $u = \arcsin\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

7. If $u = \ln\left(\dfrac{x^2 + y^2}{x + y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$$

8. If $u = f(x, y)$ is a homogeneous function of degree n, prove that

$$x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = n(n-1)f.$$

9. If $u = f(r)$, where $r = \sqrt{x^2 + y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

10. If $V = \rho^m$, where $\rho^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{dz^2} = m(m+1)\rho^{m-2}.$$

Chapter 9

FUNCTIONS OF SEVERAL VARIABLES

Exercise Set 9.1 (Page 411)

1. Verify Euler's Theorem for

(a) $u = \arcsin\left(\frac{x}{y}\right) + \arctan\left(\frac{y}{x}\right)$

(b) $u = x^n \ln\left(\frac{y}{x}\right)$ (c) $u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Sol.

(a) Here u is a homogeneous function of zero degree. Therefore, by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \qquad (1)$$

Now $u = \arcsin\left(\frac{x}{y}\right) + \text{arctna}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{-x}{y^2} + \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{x} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} = 0$$

(b) Here u is a homogeneous function of degree n. Therefore, by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad (1)$$

To verify this, we have

$$\frac{\partial u}{\partial x} = nx^{n-1}\ln\frac{y}{x} + x^n\cdot\frac{1}{y/x}\cdot\frac{-y}{x^2} = nx^{n-1}\ln\frac{y}{x} - x^{n-1}$$

and $$\frac{\partial u}{\partial y} = x^n\cdot\frac{1}{y/x}\cdot\frac{1}{x} = \frac{x^n}{y}$$

Hence $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\ln\frac{y}{x} - x^n + y\frac{x^n}{y}$$

$$= nx^n \ln\frac{y}{x} - x^n + x^n = nx^n \ln\frac{y}{x} = nu.$$

(c) Here $u = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \dfrac{x^{1/4}\left[1+\left(\frac{y}{x}\right)^{1/4}\right]}{x^{1/5}\left[1+\left(\frac{y}{x}\right)^{1/5}\right]}$

$$= x^{\frac{1}{4}-\frac{1}{5}}\left[\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right] = x^{\frac{1}{20}}\,\frac{1+\left(\frac{y}{x}\right)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}$$

Thus u is a homogeneous function of degree $\frac{1}{20}$ and hence by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}u = \frac{1}{20}u$$

To verify this, we have

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\cdot\left(\frac{1}{4}x^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\cdot\left(\frac{1}{4}y^{-3/4}\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{-4/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}+\frac{1}{4}y^{1/4}\right) - [x^{1/4}+y^{1/4})\left(\frac{1}{5}y^{1/5}+\frac{1}{5}x^{1/5}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})\left(\frac{1}{20}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{1}{20}\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \frac{1}{20}u$$

2. If $u = f\left(\frac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Sol. We have $\dfrac{\partial u}{\partial y} = \dfrac{-y}{x^2}f'\left(\dfrac{y}{x}\right)$

and $\dfrac{\partial u}{\partial y} = f'\left(\dfrac{y}{x}\right)\dfrac{1}{x} = \dfrac{1}{x}f'\left(\dfrac{y}{x}\right)$

Thus $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{-y}{x}f'\left(\dfrac{y}{x}\right) + \dfrac{y}{x}f'\left(\dfrac{y}{x}\right) = 0.$

3. If $u = xyf\left(\frac{x}{y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$.

Sol. We have $\dfrac{\partial u}{\partial x} = y\left[f\left(\frac{x}{y}\right) + xf'\left(\frac{x}{y}\right)\cdot\frac{1}{y}\right]$, $\dfrac{\partial u}{\partial y} = x\left[f\left(\frac{x}{y}\right) - yf'\left(\frac{x}{y}\right)\cdot\frac{x}{y^2}\right]$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = xy\left[f\left(\frac{x}{y}\right) + \frac{x}{y}f'\left(\frac{x}{y}\right) + f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)\right]$

$$= 2xyf\left(\frac{y}{x}\right) = 2u.$$

4. If $z = \arctan\left(\frac{y}{x}\right)$, verify that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 2u$

Sol. Differentiating partially w.r.t. x, we have

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = (-y)(x^2+y^2)^{-1}$$

$$\frac{\partial^2 z}{\partial x^2} = (-1)(y)(x^2+y^2)^{-2}(-2x) = \frac{2xy}{(x^2+y^2)^2} \qquad (1)$$

Again, $$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$\frac{\partial^2 z}{\partial y^2} = (-1)(x)(x^2+y^2)^{-2}(2y) = \frac{-2xy}{(x^2+y^2)^2} \qquad (2)$$

Adding (1) and (2), we get $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

5. If $u = \arcsin\left(\dfrac{x^2+y^2}{x+y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

Sol. Writing tx, ty for x, y in the R.H.S. of the given equation, we have

$$\arcsin\frac{t(x^2+y^2)}{x+y} \neq t\arcsin\frac{x^2+y^2}{x+y}$$

Hence u is not a homogeneous function.

Let $z = \sin u = \dfrac{x^2+y^2}{x+y}$.

Then z is a homogeneous function of degree 1.

Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.z \quad\text{or}\quad x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = z$$

i.e., $x.\cos u\cdot\dfrac{\partial u}{\partial x} + y.\cos u\cdot\dfrac{\partial u}{\partial y} = z = \sin u$

Dividing by $\cos u$, we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\sin u}{\cos u} = \tan u$

6. If $u = \arcsin\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{y}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Sol. Proceeding as in Q. 5, it is easy of see that u is not a homogeneous function.

Let $z = \sin u = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

This shows that z is a homogeneous function of zero degree. Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \cdot z = 0 \quad \text{or} \quad x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = 0$$

or $\quad x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = 0.$

i.e., $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

7. If $u = \ln\left(\dfrac{x^2+y^2}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

Sol. Here u is a homogeneous function (verify !)

Let $z = e^u = \dfrac{x^2+y^2}{x+y}$. Then z is a homogeneous function of degree 1. By Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1 \cdot z$$

or $\quad x\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x} + y\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y} = z = e^u$

or $\quad x\cdot e^u\cdot\dfrac{\partial u}{\partial x} + y\cdot e^u\cdot\dfrac{\partial u}{\partial y} = e^u$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

8. If $u = f(x, y)$ is a homogeneous function of degree n, prove that

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1) f.$$

Sol. Since f is a homogeneous function of degree n, we have

$$xf_x + yf_y = nf \qquad (1)$$

Differentiating (1) w.r.t. x and y respectively, we get

$$xf_{xx} + f_x + yf_{yx} = nf_x \qquad (2)$$

and $\quad xf_{xy} + yf_{yy} + f_y = nf_y \qquad (3)$

Assuming $f_{xy} = f_{yx}$ and multiplying (2) by x and (3) by y and adding the results, we have

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} + xf_x + yf_y = n(xf_x + yf_y)$$

or $\quad x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = (n-1)(xf_x + yf_y) = n(n-1)f$, using (1).

9. If $u = f(r)$, where $r = \sqrt{x^2+y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

Sol. We have $\dfrac{\partial u}{\partial x} = f'(r)\cdot\dfrac{\partial r}{\partial x} = f'(r)\cdot\dfrac{1}{2}(x^2+y^2)^{-1/2}\,2x = f'(r)\cdot\dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\cdot\frac{\partial r}{\partial x}\cdot\frac{x}{\sqrt{x^2+y^2}} + f'(r)\frac{\sqrt{x^2+y^2}\cdot 1 - x\cdot\frac{1}{2}(x^2+y^2)^{-1/2}\,2x}{(\sqrt{x^2+y^2})^2}$$

$$= f''(r)\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + f'(r)\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= f''(r)\cdot\frac{x^2}{x^2+y^2} + f'(r)\frac{y^2}{(x^2+y^2)^{3/2}} \qquad (1)$$

By symmetry, we have

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{x^2+y^2} + f'(r)\cdot\frac{x^2}{(x^2+y^2)^{3/2}} \qquad (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2+y^2}{x^2+y^2}\right] + f'(r)\left[\frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right]$$

$$= f''(r) + f'(r)\cdot\frac{r^2}{r^3} = f''(r) + \frac{1}{r}f'(r).$$

10. If $V = \rho^m$, where $\rho^2 = x^2+y^2+z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)\rho^{m-2}$$

Sol. We have $\dfrac{\partial V}{\partial x} = m\rho^{m-1}\dfrac{\partial \rho}{\partial x} \qquad (1)$

Now, $\quad \rho^2 = x^2+y^2+z^2$

Differentiating both the sides w.r.t. x, we have

$$2\rho\cdot\frac{\partial \rho}{\partial x} = 2x \quad \text{or} \quad \frac{\partial \rho}{dx} = \frac{x}{\rho}$$

Putting this value of $\dfrac{\partial \rho}{\partial x}$ into (1), we get

$$\frac{\partial V}{\partial x} = m\rho^{m-1}\cdot\frac{x}{\rho} = m\rho^{m-2}\cdot x$$

Differentiating w.r.t. x, we obtain

$$\frac{\partial^2 V}{\partial x^2} = m\left[\rho^{m-2}\cdot 1 + (m-2)\rho^{m-3}\cdot\frac{\partial \rho}{\partial x}\cdot x\right]$$

$$= m\left[\rho^{m-2} + (m-2)\rho^{m-3}\left(\frac{x}{\rho}\right)x\right]$$

$$= m\,[\rho^{m-2} + (m-2)\rho^{m-4} \,.\, x^2] \qquad (2)$$

By symmetry, we get

$$\frac{\partial^2 V}{\partial y^2} = m\,[\rho^{m-2} + (m-2)\,\rho^{m-4} \,.\, y^2] \qquad (3)$$

and $$\frac{\partial^2 V}{\partial z^2} = m\,[\rho^{m-2} + (m-2)\,\rho^{m-4} \,.\, z^2] \qquad (4)$$

Adding (2), (3) and (4), we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m\,[3\rho^{m-2} + (m-2)\,(\rho^{m-4})\,(x^2 + y^2 + z^2)]$$

$$= m\,[3\rho^{m-2} + (m-2)\,\rho^{m-4} \,.\, \rho^2]$$

$$= m\,[3\rho^{m-2} + (m-2)\,\rho^{m-2}]$$

$$= m\,(m+1)\,\rho^{m-2}, \text{ as required.}$$