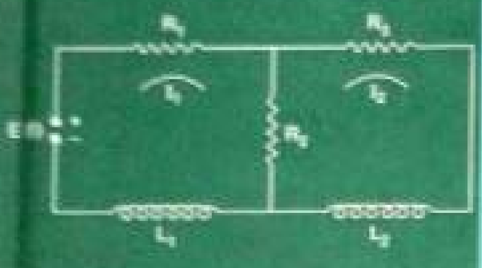
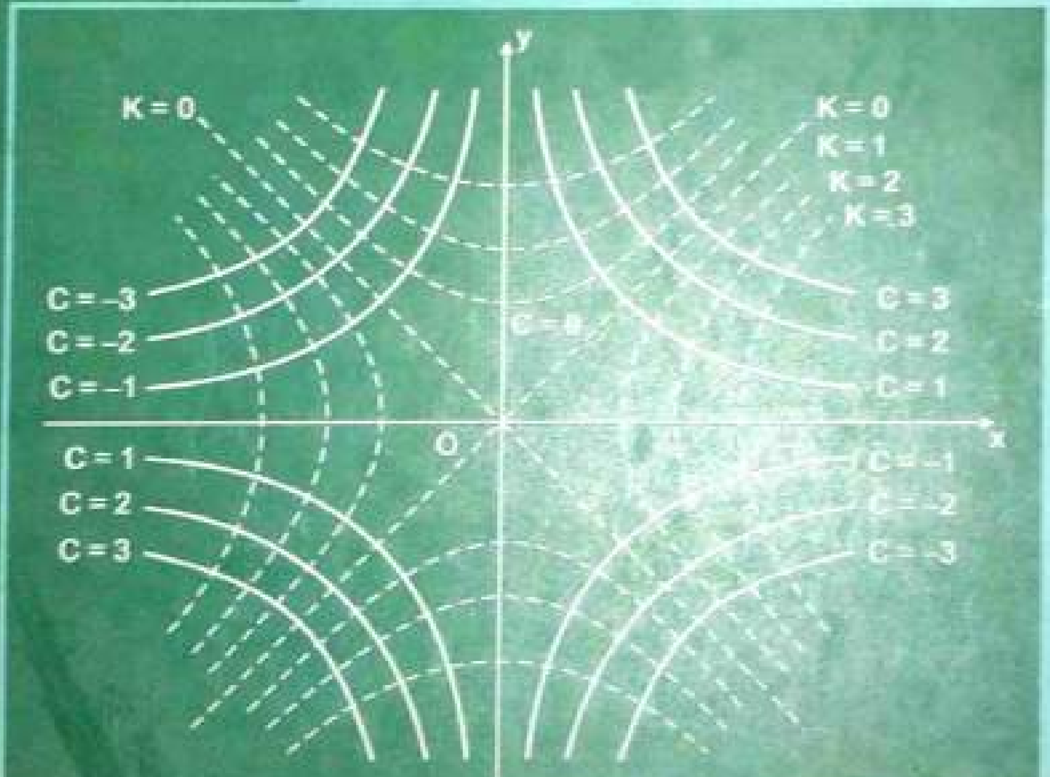


ORDINARY DIFFERENTIAL EQUATIONS

for

Scientists and Engineers



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CHAPTER 1

BASIC CONCEPTS OF DIFFERENTIAL EQUATIONS

1.1 INTRODUCTION

Differential equations are of fundamental importance in science and engineering because many physical laws and relations appear mathematically in the form of such equations . We see that whenever a physical law involves a rate of change of a function , such as velocity , acceleration , etc., it will lead to a differential equation . To give a simple example , one of the most important laws in mechanics is Newton's second law , which states that the rate of change of momentum of a body is proportional to the force applied to it . For motion in a straight line , the mathematical statement of this law is :

$$\frac{d}{dt}(mV) = F \qquad (1)$$

where m is the mass , V the speed , t the time , and F the applied force . We can write equation (1) as

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right) = F \qquad (2)$$

where x denotes the distance along the line from some origin on it . Equation (2) is called the differential equation expressing Newton's second law of motion .

Differential equations have vast applications in almost all branches of engineering , electromagnetic theory, fluid mechanics , elasticity , heat conduction , diffusion , plasma physics , quantum mechanics , biological and other social sciences .

1.2 DIFFERENTIAL EQUATION

An equation which involves the derivatives or differentials is called a **differential equation** . For example ,

(1) $\frac{dy}{dx} = x + 5$ (2) $dy = (x + 2y)\,dx$

(3) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$ (4) $x\frac{dy}{dx} + y = 3$

(5) $y''' + 2(y'')^2 + y' = \cos x$ (6) $(y'')^2 + (y')^3 + 3y = x^2$

(7) $\frac{\partial z}{\partial x} = z + x\frac{\partial z}{\partial y}$ (8) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$

1.3 ORDINARY DIFFERENTIAL EQUATION

A differential equation which involves one or more derivatives of an unknown function of a single independent variable is called an ordinary differential equation . The equation may also contain y itself as well as given functions of x and constants . For example , the equations (1) to (6) given above are the examples of the ordinary differential equations .

1.4 PARTIAL DIFFERENTIAL EQUATION

A differential equation which involves an unknown function of two or more independent variables and its partial derivatives is called a partial differential equation . For example , equations (7) and (8) given above are the examples of partial differential equations .

1.5 ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest derivative occuring in the equation . For example , equations (1) , (2) , (4) and (7) given above are of first order , while equations (3) , (6) and (8) are of second-order and equation (5) is of third order .

1.6 DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the power of the highest derivative occurring in the equation . For example , all the above equations are of degree 1 except equation (6) which is of degree 2 .

If the differential equation involves any radical sign or any power in fractions , then the degree of that equation is the degree of the highest derivative after removing the radical sign and fractional power . For example , the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2} \quad \text{can be written as}$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

The order of this equation is 2 and the degree is also 2 .

1.7 LINEAR AND NON – LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if

(i) it is of first degree in the dependent variable and all of its derivatives.

(ii) the coefficients (of the dependent variable and its derivatives) are either constants or functions only of the independent variables . A differential equation that is not linear is said to non–linear .

For example , equations (1) , (2) , (3) , (4) , (7) , and (8) given above are linear , while equations (5) , and (6) are non–linear . Also the differential equation $y'' + 4yy' + 2y = \cos x$ is non – linear because of the occurrence of the product of y and one of its derivatives .

1.8 HOMOGENEOUS AND NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

A linear differential equation is said to be homogeneous if every term in the equation contains either the dependent variable or one of its derivatives , otherwise it is said to be non–homogeneous or inhomogeneous . For example , equations (3) and (7) are homogeneous while all the other equations are non-homogeneous .

1.9 SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables , not involving any derivative, such that this relation and the derivatives obtained from it satisfy the given differential equation.

For example , $y = A\cos x + B\sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$,

because $\frac{dy}{dx} = -A\sin x + B\cos x$ and $\frac{d^2y}{dx^2} = -A\cos x - B\sin x$

so that $\frac{d^2y}{dx^2} + y = -A\cos x - B\sin x + A\cos x + B\sin x = 0$.

1.10 GENERAL SOLUTION

When the number of arbitrary constants (functions) in the solution of an ordinary (partial) differential equation is equal to the order of the differential equation , then this solution is called the **general solution or complete solution** or **complete integral** . For example , $y = A\cos x + B\sin x$ is called the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ because it involves two arbitrary constants and also the order of the differential equation is 2 .

1.11 PARTICULAR SOLUTION

A particular solution is a solution obtained from the general solution by giving particular values to the arbitrary constants (functions) . It is also called a **particular integral** . For example ,

$$y = 4\cos x + 5\sin x$$

is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

1.12 SINGULAR SOLUTION

A singular solution is a solution which exists but cannot be obtained from the general solution . For example , as can be seen by substitution that $y = Cx - C^2$ is a general solution of the differential equation $(y')^2 - xy' + y = 0$. (1)

This solution represents a family of straight lines, one line for each C. These are particular solutions. Substitution also shows that $y = \frac{x^2}{4}$ is also a solution. This is a singular solution of equation (1) since we cannot obtain it from $y = Cx - C^2$ by choosing a suitable C.

1.13 FAMILY OF CURVES

Let x and y be the independent and dependent variables respectively. Then the equation

$$f(x, y, C) = 0 \qquad (1)$$

containing one arbitrary constant C, represents a family of curves. For example, the equation

$$x^2 + y^2 = a^2$$

where a is arbitrary, represents a family of circles with centre at the origin and radius a.

Similarly, the equation

$$f(x, y, A, B) = 0 \qquad (2)$$

containing two arbitrary constants A and B also represents a family of curves. We often say that it is a two parameter family of curves. For example, the equation $y = mx + C$, where m and C are arbitrary constants, represents a two parameter family of straight lines having slope m and intercept C on the y – axis.

1.14 FORMATION OF DIFFERENTIAL EQUATIONS

To form a differential equation from the one parameter family of curves

$$f(x, y, C) = 0 \qquad (1)$$

we need two equations. One equation is given by equation (1) and the second equation is obtained by differentiating, equation (1) w.r.t. x. On eliminating C from the two equations, we obtain an equation containing x, y, and y' which is a first order differential equation.

To form a differential equation from a two – parameter family of curves

$$f(x, y, A, B) = 0 \qquad (2)$$

we need three equations. One equation is given by equation (2) and the remaining two equations are obtained by differentiating twice equation (2) w.r.t. x. On eliminating A and B from these three equations, we obtain a second order differential equation.

Similarly, if an equation contains n arbitrary constants, differentiate it successively n times to get n additional equations containing n arbitrary constants and derivatives. Now eliminate n arbitrary constants from the $(n + 1)$ equations to get a differential equation of order n.

EXAMPLE (1): **Form the differential equation of the family of straight lines passing through the origin.**

SOLUTION: The equation of all the straight lines passing through the origin is

$$y = m x \tag{1}$$

where m is the parameter.

Differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = m \tag{2}$$

Eliminating m from equations (1) and (2), we get $\frac{dy}{dx} = \frac{y}{x}$

or $\quad x\frac{dy}{dx} = y$

is the required differential equation.

EXAMPLE (2): **Form the differential equation of the family of concentric circles $x^2 + y^2 = a^2$, where a is an arbitrary constant.**

SOLUTION: Given that

$$x^2 + y^2 = a^2 \tag{1}$$

Differentiating equation (1) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

or $\quad x + y\frac{dy}{dx} = 0$

is the required differential equation.

EXAMPLE (3): **Form the differential equation of the family of parabolas $y^2 = 4ax$, where a is an arbitrary constant.**

SOLUTION: Given that

$$y^2 = 4ax \tag{1}$$

Differentiating equation (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \tag{2}$$

Now eliminate a from equations (1) and (2). From equation (1)

$$4a = \frac{y^2}{x}$$

Substituting in equation (2), we get

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

or $\quad 2x\frac{dy}{dx} - y = 0$

which is the required differential equation.

EXAMPLE (4): **Form the differential equation from the relations**

(i) $y = A\cos 2x + B\sin 2x$ **(ii)** $y = A e^{2x} + B e^{-3x}$

where A and B are arbitrary constants.

SOLUTION: (i) $y = A\cos 2x + B\sin 2x$ (1)

Differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

Differentiating again w.r.t. x, we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= -4A\cos 2x - 4B\sin 2x\\ &= -4(A\cos 2x + B\sin 2x)\\ &= -4y \qquad [\text{using equation (1)}]\end{aligned}$$

or $\frac{d^2y}{dx^2} + 4y = 0$ is the required differential equation.

(ii) $y = A e^{2x} + B e^{-3x}$ (1)

Differentiating equation (1) w.r.t. x, we get

$$\frac{dy}{dx} = 2Ae^{2x} - 3Be^{-3x} \tag{2}$$

Differentiating again equation (2), w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 9Be^{-3x} \tag{3}$$

Adding equations (2) and (3), we get

$$\begin{aligned}\frac{d^2y}{dx^2} + \frac{dy}{dx} &= 6Ae^{2x} + 6Be^{-3x}\\ &= 6(Ae^{2x} + Be^{-3x}) = 6y\end{aligned}$$

or $$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

is the required differential equation.

1.15 FIRST ORDER DIFFERENTIAL EQUATION

It $y(x)$ is a differentiable function of x, then any first order ordinary differential equation can be written in the implicit form as $F(x, y, y') = 0$.

That is, a first order ordinary differential equation must contain y' of the unknown function $y(x)$, and may contain y itself and given functions of x. For example,

$$y' - 4xy + x^2 = 0, \quad yy' + x - \cos x = 0, \quad \text{and} \quad (y')^2 + xy' - y + 2x = 0$$

are first order differential equations. Remember that a first order differential equation needs not explicitly contain either x or y. For example, $y' = 4$ and $(y')^2 = 4$ are first order differential equations in which x and y do not appear explicitly. Similarly, $y' = -y^2$ does not explicitly contain x, although x is implicitly in y, which is a function of x.

1.16 SOLUTION OF A FIRST ORDER DIFFERENTIAL EQUATION

A solution of a first order ordinary differential equation $F(x, y, y') = 0$ is a function $y = \phi(x)$ that has $y' = \phi'(x)$ and has the property that $F[x, \phi(x), \phi'(x)] = 0$ for all values of the independent variable x for which ϕ is defined.

For example, as can be seen by substitution that $y = Cx - C^2$ is a solution of the differential equation

$(y')^2 - xy' + y = 0$. This solution holds for all values of x.

A solution may be defined on the entire real line or on only part of the line called an interval. Thus we say that $y = \phi(x)$ is a solution of $F(x, y, y') = 0$ on an interval I if

$F[x, \phi(x), \phi'(x)] = 0$ for all x in I.

For example, $y = e^{2x}$ is a solution of $y' - 2y = 0$ for all x, while $y = \ln x$ is solution of $y' = \frac{1}{x}$ for $x > 0$. Note that it is important to define the interval I since the differential equation does not necessarily describe the solution for all values of the independent variable x.

IMPLICIT SOLUTION

Sometimes a solution of a differential equation will appear as an implicit function as

$H(x, y) = C \qquad (C = \text{Constant})$

and is called an **implicit solution**, in contrast to an explicit solution $y = \phi(x)$. For example,

$$x^2 + y^2 = 4 \tag{1}$$

defines implicitly y as a function of x, and this function is a solution of the first order differential equation

$$yy' = -x \tag{2}$$

In this example, equation (1) actually defines two solutions of the differential equation (2), namely $y = f_1(x) = \sqrt{4 - x^2}$ and $y = f_2(x) = -\sqrt{4 - x^2}$ both defined for $-2 \le x \le 2$. Again, the fact that both of these functions are solutions of the first order differential equation (2) can be verified by substituting them into equation (2).

1.17 SECOND ORDER DIFFERENTIAL EQUATION

If $y(x)$ is a differentiable function, then a second order ordinary differential equation can be written in the implicit form as $F(x, y, y', y'') = 0$.

For example, $x y'' + x^2 y' - 2y + 3\cos x = 0$ and $(y'')^2 + (y')^3 + 3y - x^2 = 0$ are second order differential equations. Remember that a second order differential equation must explicitly contain y'' term, while it need not explicitly contain x, y, and y'. For example, $y'' = 10$ and $(y'')^2 = 4$ are second order differential equations, even though x, y, and y' do not appear explicitly.

The differential equation may or may not be defined for all values of the independent variable x.

For example, $\dfrac{d^2 y}{dx^2} + x^2 \dfrac{dy}{dx} - 2xy = \sin x$

is defined for all values of x, while the differential equation

$$y'' + \frac{1}{x} y = \frac{1}{x^2 - 4}$$

is defined only on intervals not containing 0, 2, or -2, where the coefficient $\frac{1}{x}$ and the term $\frac{1}{x^2-4}$ are not defined.

1.18 SOLUTION OF A SECOND ORDER DIFFERENTIAL EQUATION

A solution of a second order ordinary differential equation

$$F(x, y, y', y'') = 0$$

on an interval I is a function $y = \phi(x)$ that has $y' = \phi'(x)$ and $y'' = \phi''(x)$ and has the property that $f[x, \phi(x), \phi'(x), \phi''(x)] = 0$

for all x in that interval I. For example, as can be seen by substitution that $y = \cos 2x$ is a solution of the differential equation $y'' + 4y = 0$ for all x, while $y = \sqrt{\frac{2}{\pi x}} \sin x$ is a solution of the differential equation $y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0$ on the interval $x > 0$ i.e. $(0, \infty)$.

TRIVIAL SOLUTION

If $y \equiv 0$ (i.e. y identically equal to zero) is a solution to a differential equation on an interval I then $y \equiv 0$ is called the **trivial** solution to that differential equation on I.

EXAMPLE (5): **Show that $y = A x + \frac{B}{x}$ is a solution of the differential equation**

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} - y = 0 .$$

SOLUTION: The given differential equation is

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} - y = 0 \qquad (1)$$

The given function is $y = A x + \frac{B}{x}$, so that $\frac{d y}{d x} = A - \frac{B}{x^2}$ and $\frac{d^2 y}{d x^2} = \frac{2 B}{x^3}$

Substituting the values of y, $\frac{d y}{d x}$ and $\frac{d^2 y}{d x^2}$ in equation (1), we get

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} - y = x^2\left(\frac{2 B}{x^3}\right) + \left(A - \frac{B}{x^2}\right) - \left(A x + \frac{B}{x}\right)$$

$$= \frac{2 B}{x} + A x - \frac{B}{x} - A x - \frac{B}{x} = 0$$

Since the given differential equation is satisfied, therefore the given function y is its solution.

1.19 nTH ORDER DIFFERENTIAL EQUATION

If $y(x)$ is a differentiable function, then an ordinary differential equation of order n can be written in the implicit form as

$$F\left(x, y, y', \ldots\ldots, y^{(n)}\right) = 0 \qquad \left(y^{(n)} = \frac{d^n y}{d x^n}\right)$$

For example, $x^2 \frac{d^3 y}{d x^3} - 3 \frac{d^2 y}{d x^2} + 4 x \frac{d y}{d x} - 2 y - e^x = 0$ and $\frac{d^4 y}{d x^4} - 2 \frac{d^2 y}{d x^2} + y - \cos x = 0$

are third and fourth order differential equations respectively.

Remember that an nth order linear differential equation must explicitly contain $y^{(n)}$ term, but $x, y, y', \ldots\ldots, y^{(n-1)}$ may or may not be explicitly present.

For example, $\frac{d^4 y}{d x^4} - 3 \frac{d y}{d x} + 2 y = 0$

is a fourth order differential equation. Here y, y', and necessarily $y^{(iv)}$ appear explicitly, while x, y'', and y''' do not.

1.20 SOLUTION OF AN nTH ORDER DIFFERENTIAL EQUATION

A solution of an nth order ordinary differential equation $F\left(x, y, y', \ldots\ldots, y^{(n)}\right) = 0$ on an interval I is a function $y = \phi(x)$ that has $y' = \phi'(x)$, $y'' = \phi''(x)$, ..., $y^{(n)} = \phi^{(n)}(x)$

and has the property that

$$F\left[x, \phi(x), \phi'(x), \ldots\ldots, \phi^{(n)}(x)\right] = 0$$

for all x in that interval I.

EXAMPLE (6): **Show that $y = x e^{2x}$ is a solution of the differential equation**

$$\frac{d^3 y}{d x^3} - 6\frac{d^2 y}{d x^2} + 12\frac{d y}{d x} - 8 y = 0$$

SOLUTION: The given equation is

$$\frac{d^3 y}{d x^3} - 6\frac{d^2 y}{d x^2} + 12\frac{d y}{d x} - 8 y = 0 \qquad (1)$$

Since $y = x e^{2x}$, therefore

$$\frac{d y}{d x} = 2 x e^{2x} + e^{2x} = (2x+1)e^{2x}$$

$$\frac{d^2 y}{d x^2} = 2(2x+1)e^{2x} + 2e^{2x} = 4(x+1)e^{2x}$$

$$\frac{d^3 y}{d x^3} = 8(x+1)e^{2x} + 4e^{2x} = 8xe^{2x} + 12e^{2x}$$

Substituting the values of y, and its derivatives in equation (1), we get

$$\frac{d^3 y}{d x^3} - 6\frac{d^2 y}{d x^2} + 12\frac{d y}{d x} - 8 y = 8xe^{2x} + 12e^{2x} - 24(x+1)e^{2x} + 12(2x+1)e^{2x} - 8xe^{2x}$$

$$= (8x - 24x + 24x - 8x)e^{2x} + (12 - 24 + 12)e^{2x} = 0$$

Since the differential equation (1) is satisfied, therefore the given function y is its solution.

1.21 INITIAL CONDITIONS

FOR FIRST ORDER DIFFERENTIAL EQUATION

Since the general solution of a first order differential equation $F(x, y, y') = 0$ contains one arbitrary constant, therefore it will represent an infinite number of solutions. However, for a physical problem it would not make sense to have an infinite number of solutions. To overcome this difficulty we require an additional information which will assign a unique value to the arbitrary constant. We give this information by specifying the value of y for one particular value x_0 of the independent variable say $y(x_0) = A$. This value is called an **initial condition** and it will give a unique solution to the problem.

FOR SECOND ORDER DIFFERENTIAL EQUATION

The general solution of a second order differential equation $F(x, y, y', y'') = 0$ will contain two arbitrary constants, therefore we require two additional pieces of information to obtain a unique solution of the problem. We give this information by specifying values of the unknown function

(i.e. dependent variable) and its derivative (slope of the curve) at the same given value x_0 of the independent variable in the interval considered . Thus the conditions

$$y(x_0) = A \quad \text{and} \quad y'(x_0) = B$$

where x_0, A , and B are given constants , are called the **initial** conditions .

FOR nTH ORDER DIFFERENTIAL EQUATION

The general solution of an nth order differential equation $F(x, y, y', \ldots\ldots y^{(n)}) = 0$ will contain n arbitrary constants , therefore we require n additional pieces of information to obtain a unique solution of the problem . We give this information by specifying values of the unknown function (i.e. dependent variable) and its derivatives upto order (n – 1) at the same given value x_0 of the independent variable in the interval considered . Thus the conditions

$$y(x_0) = C_1, \quad y'(x_0) = C_2, \ldots\ldots, y^{(n-1)}(x_0) = C_n$$

where $x_0, C_1, C_2, \ldots\ldots$, and C_n are given constants , are called the **initial** conditions .

1.22 FIRST ORDER INITIAL – VALUE PROBLEM

A first order differential equation $F(x, y, y') = 0$ together with an initial condition

$$y(x_0) = A$$

is called a first order initial – value problem . The initial condition is used to determine a unique value of C in the general solution . For example

$$y' = xy; \quad y(0) = 1$$

and $\quad x(y')^2 + 2xy' - y = 0; \quad y(0) = 2$

are first order initial – value problem .

1.23 SECOND ORDER INITIAL – VALUE PROBLEM

A second order differential equation $F(x, y, y', y'') = 0$ together with two initial conditions

$$y(x_0) = A \quad \text{and} \quad y'(x_0) = B$$

is called a second order initial – value problem .

The initial condition are used to determine the unique values of the constants in the general solution .

For example

$$y'' - 4y' + 4y = e^x; \quad y(0) = 3, \quad y'(0) = 1$$

is a second order initial – value problems .

1.24 nTH ORDER INITIAL – VALUE PROBLEM

An nth order differential equation $F\left(x, y, y', \ldots\ldots, y^{(n)}\right) = 0$ together with n initial conditions $y(x_0) = C_1$, $y'(x_0) = C_2, \ldots\ldots, y^{(n-1)}(x_0) = C_n$

is called an nth order initial – value problem. The initial condition are used to determine the unique values of the constants in the general solution. For example

$$y''' - y'' - y' + y = 2; \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 0$$

is a third order initial – value problem.

1.25 BOUNDARY CONDITIONS

If the conditions are given at more than one value of the independent variable, the conditions are called the boundary conditions. Thus

$$y(P_1) = A \quad \text{and} \quad y(P_2) = B$$

where A and B are constants and P_1 and P_2 refer to the end (or boundary) points P_1 and P_2 of an internal I on which the second order differential equation is considered, are the **boundary conditions**.

1.26 SECOND ORDER BOUNDARY – VALUE PROBLEM

A second order differential equation

$$F(x, y, y', y'') = 0$$

together with the boundary conditions

$$y(P_1) = A \quad \text{and} \quad y(P_2) = B$$

is called a second order boundary – value problem.

For example, $y'' + 3y' + 2y = x^2; \quad y(0) = 1, \quad y(1) = 1$

is a second order boundary – value problem, because the two boundary conditions are given at two different values of the variable x.

1.27 LINEAR COMBINATION

If $y_1(x)$ and $y_2(x)$ are any two functions and C_1 and C_2 are any two constants, then an expression of the form $C_1 y_1(x) + C_2 y_2(x)$

is called a linear combination of the functions $y_1(x)$ and $y_2(x)$.

Similarly, if $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ are any n functions and $C_1, C_2, \ldots\ldots, C_n$ any n constants, then an expression of the form $C_1 y_1(x) + C_2 y_2(x) + \ldots\ldots + C_n y_n$

is called a linear combination of the functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$.

1.28 LINEAR DEPENDENCE AND INDEPENDENCE OF FUNCTIONS

LINEAR DEPENDENCE OF TWO FUNCTIONS

Two functions $y_1(x)$ and $y_2(x)$ are said to be linearly dependent on an internal I, if there exist two constants C_1 and C_2 (not both zero), such that

$$C_1 y_1(x) + C_2 y_2(x) = 0 \text{ identically.}$$

There is, however, an easier way to see that two functions $y_1(x)$ and $y_2(x)$ are linearly dependent.

If $C_1 y_1(x) + C_2 y_2(x) = 0$ (where C_1 and C_2 are not both zero), we may suppose that $C_1 \neq 0$. Then dividing the above equation by C_1, we get

$$y_1(x) + \frac{C_2}{C_1} y_2(x) = 0 \quad \text{or} \quad y_1(x) = -\frac{C_2}{C_1} y_1(x) = C y_2(x)$$

Thus two functions are linearly dependent on an interval I if and only if one of the these functions is a constant multiple of the other for all x on I.

For example, the functions $3e^{2x}$ and $5e^{2x}$ are linearly dependent because we can find C_1 and C_2 not both zero such that

$$C_1(3e^{2x}) + C_2(5e^{2x}) = 0 \text{ identically. For example, } C_1 = 5, \ C_2 = -3.$$

LINEAR INDEPENDENCE OF TWO FUNCTIONS

Two functions that are not linearly dependent are called linearly independent. In other words, two functions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if for any two constants C_1 and C_2, the equation $C_1 y_1(x) + C_2 y_2(x) = 0$ holds only when $C_1 = 0$, $C_2 = 0$

Linear independence of $y_1(x)$ and $y_2(x)$ on I means that one function cannot be written as a constant multiple of the other for all x in the relevant interval.

For example, the functions e^x and xe^x are linearly independent since $C_1 e^x + C_2 x e^x = 0$, identically if and only if $C_1 = 0$, $C_2 = 0$.

LINEAR DEPENDENCE OF n FUNCTIONS

The n functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ are said to be linearly dependent if there exist n constants $C_1, C_2, \ldots\ldots, C_n$, (not all zero), such that

$$C_1 y_1(x) + C_2 y_2(x) + \ldots\ldots + C_n y_n(x) = 0 \text{, identically.}$$

For example, the functions $2e^{3x}, 5e^{3x}, e^{-4x}$ are linearly dependent because we can find C_1, C_2, C_3 not all zero such that $C_1(2e^{3x}) + C_2(5e^{3x}) + C_3(e^{-4x}) = 0$, identically, for instance $C_1 = -5$, $C_2 = 2$, $C_3 = 0$.

LINEAR INDEPENDENCE OF n FUNCTIONS

The n functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ are said to be linear independent if the equation $C_1 y_1(x) + C_2 y_2(x) + \ldots\ldots + C_n y_n(x) = 0$, where the C's are constants, holds only when $C_1 = C_2 = C_3 = \ldots\ldots = C_n = 0$.

For example, the functions $1, x, x^2$ are linearly independent since $C_1 + C_2 x + C_3 x^2 = 0$, identically if and only if $C_1 = 0, C_2 = 0, C_3 = 0$.

1.29 THE WRONSKIAN

The Wronskian determinant or simply the Wronskian of two functions $y_1(x)$ and $y_2(x)$ denoted by W is defined as the second order determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

The Wronskian W of n functions $y_1(x), y_2(x), \ldots\ldots y_n(x)$

is defined as the nth order determinant

$$W(y_1, \ldots\ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_n \\ y_1' & y_2' & \cdots\cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots\cdots & y_n^{(n-1)} \end{vmatrix}$$

The Wronskian was defined by a Polish mathematician I.M.H. Wronski (1778 – 1853).

EXAMPLE (7): **Find the Wronskian of the functions :**

(i) $\cos\theta, \sin\theta$ **(ii)** x, x^2, x^3

SOLUTION: (i) The Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2 + \sin^2\theta = 1$$

(ii) The Wronskian is given by

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

This example shows that the Wronskian is in general a non constant function .

1.30 TEST FOR LINEAR DEPENDENCE AND INDEPENDENCE OF SOLUTIONS

Let the nth order homogeneous linear differential equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+P_1\frac{dy}{dx}+P_n y=0 \qquad (1)$$

have continuous coefficients defined over the interval $a \le x \le b$.

Let $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ be the solutions of equation (1). Then a necessary and sufficient condition that they will be linearly independent is that their Wronskian is non – zero for $a \le x \le b$. If, however, the Wronskian vanishes identically, the solutions are linearly dependent.

EXAMPLE (8): **Determine whether the following sets of functions are linearly dependent or linearly independent :**

(i) x^2, x^3, x^{-2} (ii) $e^x, 2e^x, e^{-x}$

SOLUTION: (i) The Wronskian of the functions x^2, x^3, x^{-2} is given by

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}=\begin{vmatrix} x^2 & x^3 & x^{-2} \\ 2x & 3x^2 & -2x^{-3} \\ 2 & 6x & 6x^{-4} \end{vmatrix}=20\ne 0$$

Since the Wronskian is non – zero, therefore the given set of functions is linearly independent.

(ii) The Wronskian of the functions $e^x, 2e^x, e^{-x}$ is given by

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}=\begin{vmatrix} e^x & 2e^x & e^{-x} \\ e^x & 2e^x & -e^{-x} \\ e^x & 2e^x & e^{-x} \end{vmatrix}$$

$$=e^x(2+2)-2e^x(1+1)+e^{-x}(2e^{2x}-2e^{2x})$$

$$=4e^x-4e^x+0=0$$

Since the Wronskian is zero, therefore the given set of functions is linearly dependent.

EXAMPLE (9): **Show that $y_1(x)=\sin 2x$ and $y_2(x)=\cos 2x$ are two linearly independent solutions of the differential equation $\frac{d^2y}{dx^2}+4y=0$.**

SOLUTION: Given that

$$\frac{d^2y}{dx^2}+4y=0 \qquad (1)$$

Since $y_1(x)=\sin 2x$, therefore $\frac{dy_1}{dx}=2\cos 2x$, and $\frac{d^2y_1}{dx^2}=-4\sin 2x$

Now $\frac{d^2 y_1}{d x^2} + 4 y_1 = -4 \sin 2x + 4 \sin 2x = 0$

which shows that $y_1(x) = \sin 2x$ is a solution of the differential equation (1).

Similarly, we can show that $y_2(x) = \cos 2x$ is also a solution of differential equation (1).

Now the Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -4\sin^2 2x - 4\cos^2 2x = -4 \neq 0$$

Since the Wronskian is non-zero, therefore $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of equation (1).

1.31 SOLVED PROBLEMS

SOLUTIONS OF DIFFERENTIAL EQUATIONS

PROBLEM (1): **Show that each of the following expressions is a solution of the corresponding differential equation :**

(i) $(y-C)^2 = Cx; \quad 4x\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$

(ii) $y = C_1 x + C_2 e^x; \quad (x-1)y'' - xy' + y = 0$

SOLUTION: (i) Given that $(y-C)^2 = Cx$ (1)

Then $2(y-C)\frac{dy}{dx} = C$ or $\frac{dy}{dx} = \frac{C}{2(y-C)}$

Substituting the value of this derivative into the given differential equation , we get

$$4x\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 4x\frac{C^2}{4(y-C)^2} + 2x\frac{C}{2(y-C)} - y$$

$$= \frac{xC^2}{(y-C)^2} + \frac{xC}{y-C} - y$$

$$= \frac{xC^2 + xC(y-C) - y(y-C)^2}{(y-C)^2}$$

$$= \frac{xCy - y(y-C)^2}{(y-C)^2}$$

$$= \frac{y\left[xC - (y-C)^2\right]}{(y-C)^2}$$

$$= \frac{y(0)}{(y-C)^2} = 0 \quad \text{[using equation (1)]}$$

Since the differential equation is satisfied , therefore the given expression is a solution of the given differential equation .

(ii) Given that $y = C_1 x + C_2 e^x$

Then $y' = C_1 + C_2 e^x$ and $y'' = C_2 e^x$

Substituting the values of these derivatives into the given differential equation , we get

$$(x-1)y'' - xy' + y = (x-1)C_2 e^x - x(C_1 + C_2 e^x) + C_1 x + C_2 e^x$$

$$= xC_2 e^x - C_2 e^x - xC_1 - xC_2 e^x + C_1 x + C_2 e^x$$

$$= 0$$

Since the differential equation is satisfied , therefore the given expression is a solution of the given differential equation .

FORMATION OF DIFFERENTIAL EQUATIONS

PROBLEM (2): **Find the differential equation of all the circles having radius r.**

SOLUTION: The equation of all the circles of radius r and centre at (h, k) is

$$(x-h)^2+(y-k)^2 = r^2 \tag{1}$$

where h and k are arbitrary constants.

Differentiating equation (1) w.r.t. x, we get

$$2(x-h)+2(y-k)\frac{dy}{dx} = 0$$

or $$(x-h)+(y-k)\frac{dy}{dx} = 0 \tag{2}$$

Differentiating equation (2) w.r.t. x, we get

$$1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \tag{3}$$

From equation (3), we have

$$y-k = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \tag{4}$$

Substituting the value of $(y-k)$ in equation (2), we get

$$(x-h)-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\frac{dy}{dx} = 0$$

or $$x-h = \frac{\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \tag{5}$$

From equations (1), (4), and (5), we get

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2} = r^2$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

PROBLEM (3): **Form the differential equation of all the circles passing through the origin having centres on the x – axis .**

SOLUTION: The equation of any circle passing through the origin and having centre on the x – axis is

$$x^2 + y^2 + 2gx = 0 \tag{1}$$

Differentiating equation (1) w.r.t. x , we get

$$2x + 2y\frac{dy}{dx} + 2g = 0 \tag{2}$$

Now we have to eliminate g from equations (1) and (2) . From equation (2) , we have

$$2g = -2x - 2y\frac{dy}{dx}$$

Substituting in equation (1) , we get

$$x^2 + y^2 + \left(-2x - 2y\frac{dy}{dx}\right)x = 0$$

or $$-x^2 + y^2 - 2xy\frac{dy}{dx} = 0$$

or $$2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

which is the required differential equation .

PROBLEM (4): **Form the differential equation of all the ellipses with semi – major axis a and semi – minor axis b .**

SOLUTION: The equation of an ellipse with semi – major axis a and semi – minor axis b and centre at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are arbitrary constants .

From the above equation , we can write

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \tag{1}$$

Differentiating equation (1) w.r.t. x , we get

$$2y\frac{dy}{dx} = -\frac{2b^2}{a^2}x$$

or $$y\frac{dy}{dx} = -\frac{b^2}{a^2}x \tag{2}$$

Differentiating again equation (2) w.r.t. x , we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{b^2}{a^2} \tag{3}$$

From equation (2), we have

$$\frac{y}{x}\frac{dy}{dx} = -\frac{b^2}{a^2} \qquad (4)$$

From equations (3) and (4), we get

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = \frac{y}{x}\frac{dy}{dx}$$

or
$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

PROBLEM (5): Form the differential equation from the relation

$$x^2+y^2+2ax+2by+C = 0$$

where a, b, and c are arbitrary constants

SOLUTION: Given that

$$x^2+y^2+2ax+2by+C = 0 \qquad (1)$$

Differentiating equation (1) w.r.t. x, we get

$$2x+2y\frac{dy}{dx}+2a+2b\frac{dy}{dx} = 0$$

or
$$x+y\frac{dy}{dx}+a+b\frac{dy}{dx} = 0 \qquad (2)$$

Differentiating again equation (2) w.r.t. x, we get

$$1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+b\frac{d^2y}{dx^2} = 0 \qquad (3)$$

ifferentiating equation (3) w.r.t. x, we get

$$y\frac{d^3y}{dx^3}+\frac{dy}{dx}\frac{d^2y}{dx^2}+2\frac{dy}{dx}\frac{d^2y}{dx^2}+b\frac{d^3y}{dx^3} = 0$$

$$y\frac{d^3y}{dx^3}+3\frac{dy}{dx}\frac{d^2y}{dx^2}+b\frac{d^3y}{dx^3} = 0 \qquad (4)$$

Now we have to eliminate b from equations (3) and (4).

Multiply equation (3) by $\frac{d^3y}{dx^3}$ and equation (4) by $\frac{d^2y}{dx^2}$, we get

$$\frac{d^3y}{dx^3}+y\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}+b\frac{d^2y}{dx^2}\frac{d^3y}{dx^3} = 0 \qquad (5)$$

$$y\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}+3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2+b\frac{d^2y}{dx^3}\frac{d^3y}{dx^3} = 0 \qquad (6)$$

Subtracting equation (6) from equation (5), we get

$$\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$$

or $$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$$

which is the required differential equation .

PROBLEM (6): **Form the differential equation from the relation**

$$y = e^x(A\cos x + B\sin x)$$

where A and B are arbitrary constants .

SOLUTION: The given equation is

$$y = e^x(A\cos x + B\sin x) \tag{1}$$

Differentiating equation (1) w.r.t. x , we get

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

$$= e^x(-A\sin x + B\cos x) + y \quad [\text{ using equation (1) }] \tag{2}$$

Differentiating again w.r.t. x , we get

$$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x) + \frac{dy}{dx}$$

$$= -e^x(A\cos x + B\sin x) + \left(\frac{dy}{dx} - y\right) + \frac{dy}{dx} \quad [\text{ using equation (2) }]$$

$$= -y + \frac{dy}{dx} - y + \frac{dy}{dx} \quad [\text{ using equation (1) }]$$

or $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation .

PROBLEM (7): **Form the differential equation from the relation**

$$y = a e^x + b e^{2x} + c e^{-3x}$$

where a , b , and c are arbitrary constants .

SOLUTION: $$y = a e^x + b e^{2x} + c e^{-3x} \tag{1}$$

Differentiating equation (1) w.r.t. x , we get

$$\frac{dy}{dx} = a e^x + 2 b e^{2x} - 3 c e^{-3x}$$

$$= (a e^x + b e^{2x} + c e^{-3x}) + b e^{2x} - 4 c e^{-3x}$$

$$= y + b e^{2x} - 4 c e^{-3x} \quad [\text{ using equation (1) }] \tag{2}$$

Differentiating again w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2 b e^{2x} + 12 c e^{-3x}$$

$$= \frac{dy}{dx} + 2(b e^{2x} - 4 c e^{-3x}) + 20 c e^{-3x}$$

$$= \frac{dy}{dx} + 2\left(\frac{dy}{dx} - y\right) + 20 c e^{-3x} \qquad [\text{ using equation (2) }]$$

or $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20 c e^{-3x} \qquad (3)$$

Differentiating again w.r.t. x, we get

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = -60 c e^{-3x}$$

$$= -3(20 c e^{-3x})$$

$$= -3\left(\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y\right) \qquad [\text{ using equation (3) }]$$

or $$\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$$

which is the required differential equation .

LINEAR DEPENDENCE AND INDEPENDENCE

PROBLEM (8): **Determine whether the following sets of functions are linearly dependent or linearly independent ?**

(i) $\cos(\ln x^2), \ \sin(\ln x^2)$

(ii) $\ln x, -\ln x^2, \ln x^3$

(iii) $e^{-2x}, e^{-x}, e^{x}, e^{2x}$

SOLUTION: (i) $\cos(\ln x^2), \ \sin(\ln x^2)$

The Wronskian of the given functions is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln x^2) & \sin(\ln x^2) \\ -\frac{2}{x}\sin(\ln x^2) & \frac{2}{x}\cos(\ln x^2) \end{vmatrix}$$

$$= \frac{2}{x}\cos^2(\ln x^2) + \frac{2}{x}\sin^2(\ln x^2)$$

$$= \frac{2}{x}\left[\cos^2(\ln x^2) + \sin^2(\ln x^2)\right] = \frac{2}{x} \neq 0$$

Since the Wronskian is non - zero , therefore the given functions are linearly independent on any interval not containing the origin .

(ii) $\ln x, -\ln x^2, \ln x^3$

The Wronskian of the given functions is given by

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} \ln x & -\ln x^2 & \ln x^3 \\ \frac{1}{x} & -\frac{2}{x} & \frac{3}{x} \\ -\frac{1}{x^2} & \frac{2}{x^2} & -\frac{3}{x^2} \end{vmatrix} = \begin{vmatrix} \ln x & -2\ln x & 3\ln x \\ \frac{1}{x} & -\frac{2}{x} & \frac{3}{x} \\ -\frac{1}{x^2} & \frac{2}{x^2} & -\frac{3}{x^2} \end{vmatrix}$$

Multiplying column (1) by 2 and adding it in column (2), we get

$$= \begin{vmatrix} \ln x & 0 & 3\ln x \\ \frac{1}{x} & 0 & \frac{3}{x} \\ -\frac{1}{x^2} & 0 & -\frac{3}{x} \end{vmatrix} = 0$$

Since the Wronskian is zero, therefore the given set of functions is linearly dependent.

(iii) The Wronskian is given by

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix}$$

$$= e^{-2x} \cdot e^{-x} \cdot e^{x} \cdot e^{2x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix}$$

Subtracting column 1 from columns 2, 3, 4, we get

$$W = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -3 & 0 \\ -8 & 7 & 9 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ -3 & -3 & 0 \\ 7 & 9 & 16 \end{vmatrix}$$

Subtract column 1 from column 2, we get

$$W = \begin{vmatrix} 1 & 2 & 4 \\ -3 & 0 & 0 \\ 7 & 2 & 16 \end{vmatrix} = 3(32 - 8) = 72 \neq 0$$

Since the Wronskian is non – zero, therefore the set of given functions is linearly independent.

PROBLEM (9): **Show that $y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x$ are two linearly independent solutions of the differential equation $\dfrac{d^2 y}{d x^2} - 2\dfrac{d y}{d x} + 2 y = 0$.**

SOLUTION: Given that

$$\frac{d^2 y}{d x^2} - 2\frac{d y}{d x} + 2 y = 0 \qquad (1)$$

Since $y_1(x) = e^x \sin x$, therefore $\dfrac{d y_1}{d x} = e^x(\cos x + \sin x)$,

and $\dfrac{d^2 y_1}{d x^2} = e^x(-\sin x + \cos x) + e^x(\cos x + \sin x) = 2 e^x \cos x$

Now $\dfrac{d^2 y_1}{d x^2} - 2\dfrac{d y_1}{d x} + 2 y_1 = 2 e^x \cos x - 2 e^x(\cos x + \sin x) + 2 e^x \sin x = 0$

showing that $y_1(x) = e^x \sin x$ is a solution of the differential equation (1).

Similarly, we can show that $y_2(x) = e^x \cos x$ is also a solution of differential equation (1).

Now the Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x(\sin x + \cos x) & e^x(-\sin x + \cos x) \end{vmatrix}$$

$$= e^{2x}(\sin x \cos x - \sin^2 x) - e^{2x}(\sin x \cos x + \cos^2 x)$$

$$= -e^{2x}(\sin^2 x + \cos^2 x) = -e^{-2x} \neq 0$$

Since the Wronskian is non – zero, therefore $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of equation (1).

PROBLEM (10): **Show that $\dfrac{d^3 y}{d x^3} - 3\dfrac{d y}{d x} + 2 y = 0$ has only two linearly independent solutions of the form $y = e^{mx}$.**

SOLUTION: Since $y = e^{mx}$, $\dfrac{d y}{d x} = m e^{mx}$, $\dfrac{d^2 y}{d x^2} = m^2 e^{mx}$, and $\dfrac{d^3 y}{d x^3} = m^3 e^{mx}$.

Substituting y and its derivatives in the given differential equation, we get

$$e^{mx}(m^3 - 3m + 2) = 0$$

Since $e^{mx} \neq 0$, therefore $m^3 - 3m + 2 = 0$

or $(m-1)^2(m+2) = 0$ which implies $m = 1, 1, -2$.

Since $\begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} \neq 0$, but $\begin{vmatrix} e^x & e^x & e^{-2x} \\ e^x & e^x & -2e^{-2x} \\ e^x & e^x & 4e^{-2x} \end{vmatrix} = 0$,

the linearly independent solutions are $y_1 = e^x$ and $y_2 = e^{-2x}$.

PROBLEM (11): **Verify that $y_1(x) = e^x$, $y_2(x) = xe^x$, and $y_3(x) = e^{-2x}$ are three linearly independent solutions of the differential equation**

$$\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0.$$

SOLUTION: Given that $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$ (1)

Since $y_1(x) = e^x$, therefore $\frac{dy_1}{dx} = e^x$, $\frac{d^2y_1}{dx^2} = e^x$, $\frac{d^3y_1}{dx^3} = e^x$.

Now $\frac{d^3y_1}{dx^3} - 3\frac{dy_1}{dx} + 2y_1 = e^x - 3e^x + 2e^x = 0$

Also, since $y_2 = xe^x$, therefore $\frac{dy_2}{dx} = e^x(x+1)$, $\frac{d^2y_2}{dx^2} = e^x(x+2)$, $\frac{d^3y_2}{dx^3} = e^x(x+3)$

Now $\frac{d^3y_2}{dx^3} - 3\frac{dy_2}{dx} + 2y_2 = e^x(x+3) - 3e^x(x+1) + 2xe^x = 0$

Finally, since $y_3(x) = e^{-2x}$, therefore $\frac{dy_3}{dx} = -2e^{-2x}$, $\frac{d^2y_3}{dx^2} = 4e^{-2x}$, $\frac{d^3y_3}{dx^3} = -8e^{2x}$

and $\frac{d^3y_3}{dx^3} - 3\frac{dy_3}{dx} + 2y_3 = -8e^{-2x} - 3(-2e^{-2x}) + 2e^{-2x} = -8e^{-2x} + 8e^{-2x} = 0$

showing that $y_1(x)$, $y_2(x)$, and $y_3(x)$ are the solutions of differential equation (1).

The Wronskian is given by

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & xe^x & e^{-2x} \\ e^x & xe^x + e^x & -2e^{-2x} \\ e^x & xe^x + 2e^x & 4e^{-2x} \end{vmatrix}$$

$$= e^x e^x e^{-2x} \begin{vmatrix} 1 & x & 1 \\ 1 & x+1 & -2 \\ 1 & x+2 & 4 \end{vmatrix}$$

Subtracting row 1 from rows 2 and 3, we get

$$W = \begin{vmatrix} 1 & x & 1 \\ 0 & 1 & -3 \\ 0 & 2 & 3 \end{vmatrix} = 3 + 6 = 9 \neq 0$$

Since the Wronskian W is non-zero, therefore the given solutions are linearly independent.

1.32 EXERCISE

ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Find the order and degree of each of the following differential equations :

(1) $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = e^x$

(2) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 = 4$

(3) $\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$

(4) $y + \left(\frac{dy}{dx}\right)^2 = 5y^2$

(5) $\frac{d^3y}{dx^3} = \sqrt{\frac{dy}{dx}}$

(6) $\frac{d^2y}{dx^2} = \sqrt[4]{y + \left(\frac{dy}{dx}\right)^2}$

(7) $\left(\frac{d^3y}{dx^3}\right)^2 - \left(\frac{d^2y}{dx^2}\right)^4 + xy = 0$

(8) $\frac{d^4y}{dx^4} + 3\left(\frac{d^2y}{dx^2}\right)^3 + 5y = 0$

SOLUTIONS OF DIFFERENTIAL EQUATIONS

Show that the given function is a solution of the given differential equation :

(9) $\frac{dy}{dx} + y = 1;\quad y = 1 + Ce^{-x}$

(10) $\frac{dy}{dx} = \frac{2xy}{2 - x^2};\quad y = \frac{C}{x^2 - 2}$

(11) $\frac{dy}{dx} - y\tan x = \sec^3 x;\quad y = \sec x \tan x$

(12) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0,\quad y = e^{-x}\cos 3x$

(13) $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0,\quad x^2 + 4y = 0$

(14) $(1 + x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0;\quad y = \frac{1}{1 + x^2}$

(15) $\frac{d^2y}{dx^2} + y = 2\sin x;\quad y = A\sin x + B\cos x - x\cos x$

(16) $\frac{d^2y}{dx^2} - y = 4 - x;\quad y = Ae^x + Be^{-x} + x - 4$

FORMULATION OF DIFFERENTIAL EQUATIONS

Form the differential equations from the following relations : (A , B , C constants)

(17) $y = A\sin x$

(18) $y = \sin(x + A)$

(19) $y = Ae^{2x} + Be^{-2x}$

(20) $y = A\cos(x + B)$

(21) $y = A\sin x + B\cos x + x\sin x$

(22) $y = A\cosh 4x + B\sinh 4x$

(23) $y^2 = Ax + B$

(24) $y = Ax^2 + Bx$

(25) $y = A\cos x^2 + B\sin x^2$

(26) $y = Ae^x + Be^{-x} + x^2$

(27) $y = Ax^2 + Bx + C$

(28) $y = Ae^{2x} + Be^x + C$

PROBLEM (29): Form the differential equation of the family of straight lines with slope and y – intercept equal .

PROBLEM (30): Form the differential equation of all the circles passing through the origin having centres on the y – axis.

[Hint: Use $x^2 + y^2 + 2fy = 0$, f being arbitrary constant].

PROBLEM (31): Form the differential equation of the family of circles of fixed radius a with centres on the y – axis.

[Hint : Use $x^2 + (y - k)^2 = a^2$, K being arbitrary constant]

PROBLEM (32): Form the differential equation of the family of circles of variable radius r with centres on the x – axis.

[Hint: Use $(x - A)^2 + y^2 = r^2$, A and r being arbitrary constants]

PROBLEM (33): Form the differential equation of the family of parabolas having latus – rectum 4 a and axes parallel to the x – axis.

[Hint: Use $(y - k)^2 = 4a(x - h)$, h and k are arbitrary]

PROBLEM (34): Form the differential equation of the family of hyperbolas in standard form.

$\left[\text{Hint: Use } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{, a and b are arbitrary constants} \right]$

LINEAR DEPENDENCE AND INDEPENDENCE OF FUNCTIONS

Determine whether the following sets of functions are linearly dependent or linearly independent ?

(35) e^x, xe^x

(36) $x^2, x^2 \ln x$

(37) $3e^{2x}, 5e^{2x}$

(38) e^x, e^{2x}, e^{-2x}

(39) $1 + x, 1 + 2x, x^2$

(40) $1 - x, 1 + x, 1 - 3x$

(41) $e^x, e^{-x}, \sin ax$

(42) $\sin x, \cos x, \sin 2x$

(43) $1, \sin^2\theta, \cos^2\theta \quad 0 \le \theta \le 2\pi$

(44) $\sin x, 2\cos x, 3\sin x + \cos x$

(45) $e^{2x}, xe^{2x}, x^2e^{2x}$

(46) $\ln x, x\ln x, x^2 \ln x$

PROBLEM (47): Show that $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two linearly independent solutions of the differential equation $\frac{d^2y}{dx^2} + 9y = 0$.

PROBLEM (48): Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are two linearly independent solutions of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

PROBLEM (49): Show that $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = x^2$ are three linearly independent solutions of the differential equation $\frac{d^3y}{dx^3} = 0$.

PROBLEM (50): Show that $y_1(x) = e^x$, $y_2(x) = e^{-x}$, $y_3(x) = e^{2x}$ are three linearly independent solutions of the differential equation $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0$.

CHAPTER 2

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

2.1 INTRODUCTION

We know that the implicit form of the first order differential equation is

$$F(x, y, y') = 0 \qquad (1)$$

Further if the differential equation (1) is of first degree, then it can be written in the explicit form as

$$\frac{dy}{dx} = f(x, y) \qquad (2)$$

In this chapter, we will discuss the solution of differential equations of the first order and first degree. Furthermore, the differential equation reducible to these forms will also be discussed.

2.2 TYPES OF EQUATIONS

The following are the four types of the differential equations of first order and first degree.

(i) Separable differential equations

(ii) Homogeneous differential equations

(iii) Exact differential equations

(iv) Linear differential equations

2.3 SEPARABLE DIFFERENTIAL EQUATIONS

A differential equation $\frac{dy}{dx} = f(x, y)$ is called separable if it can be written in the form $g(y)\frac{dy}{dx} = f(x)$

Writing in differential notation, we get $\quad g(y)\,dy = f(x)\,dx \qquad (1)$

Since the L.H.S. of this equation involves only y and the R.H.S. involves only x, we say that the variables have been separated. Integrating both sides of equation (1), we get

$$\int g(y)\,dy = \int f(x)\,dx + C$$

Assuming that we can carry these integrations, we obtain an equation which explicitly or implicitly defines the general solution of the differential equation.

EXAMPLE (1): **Solve the following differential equations:**

(i) $\dfrac{dy}{dx} = 8x^3y^2$ (ii) $\dfrac{dy}{dx} = 1 + y^2$

SOLUTION: (i) $\dfrac{dy}{dx} = 8x^3y^2$

Separating the variables, we get $\dfrac{1}{y^2}dy = 8x^3 dx$

Integrating both sides, we get

$$\int \frac{1}{y^2}dy = \int 8x^3 dx$$

or $\quad -\dfrac{1}{y} = 2x^4 + C \quad$ or $\quad y = \dfrac{-1}{2x^4 + C}$

is the general solution of the differential equation.

(ii) $\dfrac{dy}{dx} = 1 + y^2$

Separating the variables, we obtain $\dfrac{dy}{1+y^2} = dx$

Integrating both sides, we get $\displaystyle\int \frac{dy}{1+y^2} = \int dx$

or $\quad \tan^{-1} y = x + C \quad$ or $\quad y = \tan(x + C)$

is the general solution of the differential equation. Note that it is important to write the constant of integration immediately when the integration is performed. ($y = \tan x + C$, with $C \neq 0$ would not be a solution).

EXAMPLE (2): **Solve the following initial – value problems :**

(i) $\dfrac{dy}{dx} = 2y;\quad y(0) = 2$ (ii) $\dfrac{dy}{dx} = x\sqrt{1-y^2};\quad y(0) = \dfrac{1}{2}$

SOLUTION: (i) $\dfrac{dy}{dx} = 2y;\quad y(0) = 2$

Separating the variables, write the equation as $\dfrac{dy}{y} = 2\,dx$

Integrating both sides, we get $\displaystyle\int \frac{dy}{y} = \int 2\,dx$

or $\quad \ln|y| = 2x + C_1$

Hence, $|y| = e^{2x + C_1} = e^{2x}.e^{C_1}$

or $\quad y = Ce^{2x}$ where $C = \pm e^{C_1}$

That is , the given equation has an infinite number of solutions , one for each real constant C . Using the initial condition $y(0) = 2$, we get $2 = C e^{2(0)} = C e^{0} = C$ so that $y = 2e^{2x}$ is the unique solution to the initial value problem .

(ii) $\dfrac{dy}{dx} = x\sqrt{1-y^2}$; $y(0) = \dfrac{1}{2}$

Separating the variables , write the equation as $\dfrac{dy}{\sqrt{1-y^2}} = x\,dx$

Integrating both sides , $\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int x\,dx$

or $\sin^{-1} y = \dfrac{x^2}{2} + C$ or $y = \sin\left(\dfrac{x^2}{2} + C\right)$ is the general solution .

Using the initial condition $y(0) = \dfrac{1}{2}$, we get $\dfrac{1}{2} = \sin C$ and $C = \sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$

Thus the unique solution is $y = \sin\left(\dfrac{x^2}{2} + \dfrac{\pi}{6}\right)$

2.4 EQUATIONS REDUCIBLE TO SEPARABLE FORM

Certain first order differential equations are not separable but can be made separable by a simple change of variables . A differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c) \tag{1}$$

where a , b , and c are constants , can be reduced to separable form by the substitution

$$ax + by + c = t$$

Now $\quad a + b\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $\quad \dfrac{dy}{dx} = \dfrac{1}{b}\left(\dfrac{dt}{dx} - a\right)$

Therefore equation (1) becomes $\dfrac{1}{b}\left(\dfrac{dt}{dx} - a\right) = f(t)$

or $\quad \dfrac{dt}{dx} = a + b f(t)$

Separating the variables , we get

or $\quad \dfrac{dt}{a + b f(t)} = dx$

Integrating both sides , we get

$$\int \frac{dt}{a + b f(t)} = x + A \tag{2}$$

where A is the constant of integration . Finally , replace t by $ax + by + c$ to get the solution in the original variables .

EXAMPLE (3): **Solve the differential equation** $\dfrac{dy}{dx} = (4x - y + 1)^2$.

SOLUTION: Let $4x - y + 1 = t$, then $4 - \dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = 4 - \dfrac{dt}{dx}$

Thus the given equation becomes

$$4 - \frac{dt}{dx} = t^2 \quad \text{or} \quad \frac{dt}{dx} = 4 - t^2$$

Separating the variables, we get $\dfrac{dt}{4 - t^2} = dx$

Integrating both sides, we get

$$\int \frac{dt}{4 - t^2} = x + c$$

or $$\int \left[\frac{1}{4(2+t)} + \frac{1}{4(2-t)} \right] dt = x + A$$

$$\frac{1}{4} \ln(2+t) - \frac{1}{4} \ln(2-t) = x + A$$

or $$\frac{1}{4} \ln \frac{2+t}{2-t} = x + A \quad \text{or} \quad \ln \frac{2+t}{2-t} = 4x + 4A$$

or $$\frac{2+t}{2-t} = e^{4x} . e^{4A} = C e^{4x} \quad (\text{where } C = e^{4A})$$

or $$2 + t = (2 - t) C e^{4x}$$

or $$4x - y + 3 = (1 - 4x + y) C e^{4x}$$

2.5 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A first order differential equation is said to be homogeneous if it can be put into the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad (1)$$

Here f is any differentiable function of $\dfrac{y}{x}$. For example,

$\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{y^3}{x^3}$, $\dfrac{dy}{dx} = \dfrac{y^2}{x^2} + \cos\left(\dfrac{y}{x}\right)$, and $\dfrac{dy}{dx} = \dfrac{y}{x} + e^{y/x}$ are homogeneous equations.

To solve equation (1), let $\dfrac{y}{x} = u$ or $y = ux$ then $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

and equation (1) becomes $u + x\dfrac{du}{dx} = f(u)$ or $x\dfrac{du}{dx} = f(u) - u$

or $$\frac{du}{f(u) - u} = \frac{dx}{x} \qquad (2)$$

Thus the variables have been separated in equation (2) and after integrating, u is replaced by $\dfrac{y}{x}$ to obtain the general solution in original variables.

EXAMPLE (4): **Solve the differential equation**

$$x\frac{dy}{dx} = \frac{y^2}{x} + y \quad (x \neq 0)$$

SOLUTION: Write the equation as $\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$ (1)

which is homogeneous .

Let $\frac{y}{x} = u$ or $y = ux$ then $\frac{dy}{dx} = u + x\frac{du}{dx}$

and so equation (1) becomes

$$u + x\frac{du}{dx} = u^2 + u \quad \text{or} \quad x\frac{du}{dx} = u^2$$

Separating the variables , we have $\frac{1}{u^2}du = \frac{dx}{x}$

Integrating both sides , we get $\int \frac{1}{u^2}du = \int \frac{1}{x}dx$ or $-\frac{1}{u} = \ln|x| + C$

or $u = \frac{-1}{\ln|x| + C}$

Since $u = \frac{y}{x}$, the general solution of the original differential equation in terms of x and y is :

$$y = \frac{-x}{\ln|x| + C}$$

2.6 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

We now consider the differential equations of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \tag{1}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants . There are two possible cases :

CASE (1): If $a_1b_2 - a_2b_1 \neq 0$, then equation (1) can be reduced to the homogeneous form by the substitution $x = X + h, \; y = Y + k$ (2)

where h and k are constants to be determined . Now $dx = dX$, $dy = dY$, therefore equation (1) becomes $\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$

$$= \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \tag{3}$$

Now choose h and k so that

$$\left.\begin{aligned} a_1h + b_1k + c_1 &= 0 \\ a_2h + b_2k + c_2 &= 0 \end{aligned}\right\} \tag{4}$$

Then equation (3) reduces to $\frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$ (5)

This equation is homogeneous in X and Y and therefore, can be solved by the substitution $Y = uX$. To get the result in terms of x and y, we use equations (2) where h and k have the values found by solving equations (4).

CASE (2): If $a_1 b_2 - a_2 b_1 = 0$ i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then the values of h and k obtained from equations (4) will not be finite, so the above method fails. In this case, we proceed as follows:

$$\text{Let } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{m} \text{ (say) so that } a_2 = a_1 m,\ b_2 = b_1 m \text{, then equation (1).}$$

becomes $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{m(a_1 x + b_1 y) + c_2}$ (6)

Let $a_1 x + b_1 y = u$, then

$$a_1 + b_1 \frac{dy}{dx} = \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b_1}\left(\frac{du}{dx} - a_1\right)$$

therefore, equation (6) becomes $\frac{1}{b_1}\left(\frac{du}{dx} - a_1\right) = \frac{u + c_1}{mu + c_2}$

which can be solved by separating the variables.

EXAMPLE (5): **Solve the following differential equation :**

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$$

SOLUTION: Here $a_1 b_2 - a_2 b_1 = 1(1) - 2(2) = 1 - 4 = -3 \neq 0$

so let $x = X + h,\ y = Y + k$ (1)

therefore $dx = dX,\ dy = dY$

Substituting in the given differential equation, we get

$$\frac{dY}{dX} = \frac{X + h + 2(Y + k) - 3}{2(X + h) + (Y + k) - 3}$$

$$= \frac{X + 2Y + (h + 2k - 3)}{2X + y + (2h + k - 3)} \quad (2)$$

Choose h and k so that

$$h + 2k - 3 = 0$$

$$2h + k - 3 = 0$$

Solving these equations we find that $h = 1$ and $k = 1$. Thus equation (2) becomes

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} = \frac{1 + 2(Y/X)}{2 + (Y/X)} \quad (3)$$

which is a homogeneous equation.

Now put $Y = uX$ therefore

$$\frac{dY}{dX} = u + X\frac{du}{dX}$$

so that from equation (3) becomes

$$u + X\frac{du}{dX} = \frac{1+2u}{2+u}$$

or $$X\frac{du}{dX} = \frac{1+2u}{2+u} - u = \frac{1-u^2}{2+u}$$

Separating the variables , we get

$$\frac{2+u}{1-u^2}\,du = \frac{dX}{X}$$

Resolving the L.H.S. into partial fractions , we get

$$\left[\frac{1}{2(1+u)} + \frac{3}{2(1-u)}\right]du = \frac{dX}{X}$$

Integrating both sides , we get

$$\int\left[\frac{1}{2(1+u)} + \frac{3}{2(1-u)}\right]du = \int\frac{dX}{X}$$

or $$\frac{1}{2}\ln(1+u) - \frac{3}{2}\ln(1-u) = \ln X + \ln C$$

or $$\ln(1+u) - 3\ln(1-u) = 2\ln X + 2\ln C$$

or $$\ln(1+u) - \ln(1-u)^3 = \ln X^2 + \ln C^2$$

or $$\ln\frac{1+u}{(1-u)^3} = \ln C^2X^2$$

or $$\frac{1+u}{(1-u)^3} = C^2X^2$$

Replacing u by Y/X, we get $\dfrac{1+(Y/X)}{[1-(Y/X)]^3} = C^2X^2$

or $$X^2\frac{X+Y}{(X-Y)^3} = C^2X^2$$

or $$X + Y = C^2(X-Y)^3$$

Since from equations (1) , we have

$X = x - 1$, $Y = y - 1$, we obtain

$$x - 1 + y - 1 = C^2(x - 1 - y + 1)^3$$

or $$x + y - 2 = C^2(x-y)^3$$

as the general solution of the given differential equation .

EXAMPLE (6): **Solve the differential equation**

$$\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3}$$

SOLUTION: Here $a_1 b_2 - a_2 b_1 = 3(-4) - 3(-4) = 0$,

therefore let $3x - 4y = u$ or $y = \frac{3}{4}x - \frac{1}{4}u$

and $\frac{dy}{dx} = \frac{3}{4} - \frac{1}{4}\frac{du}{dx}$

thus the given differential equation becomes

$$\frac{3}{4} - \frac{1}{4}\frac{du}{dx} = \frac{u-2}{u-3}$$

or $$-\frac{1}{4}\frac{du}{dx} = \frac{u-2}{u-3} - \frac{3}{4} = \frac{u+1}{4(u-3)}$$

Separating the variables, we get $dx + \frac{u-3}{u+1}du = 0$

or $$dx + \left(1 - \frac{4}{u+1}\right)du = 0$$

Integrating, we get $x + u - 4\ln(u+1) = C_1$

Replacing u by $3x - 4y$, we have

$$x + 3x - 4y - 4\ln(3x - 4y + 1) = C_1$$

or $$4x - 4y - 4\ln(3x - 4y + 1) = C_1$$

or $$x - y - \ln(3x - 4y + 1) = \frac{1}{4}C_1$$

or $$x - y + C = \ln(3x - 4y + 1),\ C = -\frac{1}{4}C_1$$

is the general solution of the given differential equation.

2.7 EXACT DIFFERENTIAL EQUATIONS

If $f(x, y) = -\frac{M(x, y)}{N(x, y)}$, then the first order ordinary differential equation

$\frac{dy}{dx} = f(x, y)$ can be written in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

The differential equation (1) is said to be exact if the L.H.S. of equation (1) is an exact (or total) differential du of some function $u = u(x, y)$. Then the differential equation (1) becomes $du = 0$. Integrating, we get $u(x, y) = C$ which is the general solution of equation (1).

TEST FOR EXACTNESS

THEOREM (2.1): **Prove that the necessary and sufficient condition for $M(x, y)\,dx + N(x, y)\,dy = 0$ to be exact is**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

PROOF: If the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1}$$

is exact, then by definition $u(x,y)$ exists such that

$$du = M\,dx + N\,dy \tag{2}$$

We know from calculus that the exact differential of $u = u(x,y)$ is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{3}$$

Comparing equations (2) and (3), we get

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

These two equations lead to

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

If these partial derivatives are continuous, then $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

and we have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (4)

Thus for differential equation (1) to be exact, it is necessary that equation (4) be satisfied.

Similarly, we can prove that if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then $M\,dx + N\,dy$ can be written as the differential of u i.e. du, often called an exact differential.

METHOD OF FINDING THE GENERAL SOLUTION

To find the function $u(x,y)$ we proceed as follows:

Since $M\,dx + N\,dy = du$ is an exact differential and $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$, we have

$$\frac{\partial u}{\partial x} = M(x,y) \tag{1}$$

and

$$\frac{\partial u}{\partial y} = N(x,y) \tag{2}$$

Integrating equation (1) partially w.r.t. x (keeping y constant), we have

$$u(x,y) = \int M(x,y)\,dx + \phi(y) \tag{3}$$

where $\phi(y)$ is the constant of integration which may depend on y.

Differentiating equation (3) partially w.r.t. y we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + \phi'(y) = N(x,y)$$

from which $\phi'(y)$ and hence $\phi(y)$ can be found.

EXAMPLE (7): Solve the differential equation $(x^3+3xy^2)\,dx+(3x^2y+y^3)\,dy=0$.

SOLUTION: Here $M = x^3+3xy^2$ and $N = 3x^2y+y^3$

so that $\dfrac{\partial M}{\partial y} = 6xy = \dfrac{\partial N}{\partial x}$ and the equation is exact.

To find its general solution, we can use any of the following methods.

METHOD (1): DIRECT METHOD

Since $(x^3+3xy^2)\,dx+(3x^2y+y^3)\,dy = du$ is an exact differential and

$du = \dfrac{\partial u}{\partial x}dx+\dfrac{\partial u}{\partial y}dy$, we must have

$$\frac{\partial u}{\partial x} = x^3+3xy^2 \tag{1}$$

and $$\frac{\partial u}{\partial y} = 3x^2y+y^3 \tag{2}$$

Integrating partially equation (1) w.r.t. x keeping y constant, we have

$$u(x,y) = \int(x^3+3xy^2)\,dx+\phi(y)$$

$$= \frac{x^4}{4}+\frac{3}{2}x^2y^2+\phi(y) \tag{3}$$

where $\phi(y)$ is the function of integration. To find $\phi(y)$ we differentiate equation (3) w.r.t. y and use equation (2) to obtain

$$\frac{\partial u}{\partial y} = 3x^2y+\phi'(y) = 3x^2y+y^3$$

or $$\phi'(y) = y^3 \text{ so that } \phi(y) = \frac{y^4}{4}$$

Then from equation (3), we get

$$u(x,y) = \frac{x^4}{4}+\frac{3}{2}x^2y^2+\frac{y^4}{4}$$

and the general solution is given by

$$\frac{x^4}{4}+\frac{3}{2}x^2y^2+\frac{y^4}{4} = C_1$$

or $$x^4+6x^2y^2+y^4 = C \quad \text{where} \quad C = 4C_1.$$

METHOD (2): USING FORMULA

The general solution of an exact differential equation is given by the formula

$$\int M(x,y)\,dx+\int[\text{terms in } N(x,y) \text{ without } x]\,dy = C$$

$$\int(x^3+3xy^2)\,dx+\int y^3\,dy = C_1$$

or $\quad \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = C_1$

or $\quad x^4 + 6x^2y^2 + y^4 = C$

METHOD (3): GROUPING OF TERMS BY INSPECTION

Write the equation as

$$x^3\,dx + (3xy^2\,dx + 3x^2y\,dy) + y^3\,dy = 0$$

or $\quad d\left(\frac{x^4}{4}\right) + d\left(\frac{3}{2}x^2y^2\right) + d\left(\frac{y^4}{4}\right) = 0$

or $\quad d\left(\frac{x^4}{4} + \frac{3}{2}x^2y^2 + \frac{y^4}{4}\right) = 0$

Then on integrating, $\frac{x^4}{4} + \frac{3}{2}x^2y^2 + \frac{y^4}{4} = C_1$

or $\quad x^4 + 6x^2y^2 + y^4 = C \qquad (\text{where } C = 4C_1)$

is the general solution of the given differential equation.

2.8 EQUATIONS REDUCIBLE TO EXACT FORM

INTEGRATING FACTORS

Sometimes a given differential equation is not exact, but can be made exact by multiplying it by some suitable factor, called the integrating factor (I.F.). For example, consider the differential equation

$$-y\,dx + x\,dy = 0$$

Here $\quad M = -y$ and $N = x$ so that $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$.

Hence the equation is not exact. But if we multiply it by $\frac{1}{x^2}$, we get

$$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0 \qquad (1)$$

Now $\quad M = -\frac{y}{x^2}$ and $N = \frac{1}{x}$, therefore

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x}$$

Thus equation (1) becomes exact. Write this equation as

$$d\left(\frac{y}{x}\right) = 0 \qquad (2)$$

Integrating equation (2), we get $\frac{y}{x} = C$ as the general solution of the given equation. Thus we see that an integrating factor is a factor which changes a non – exact differential equation into an exact differential equation.

RULES FOR FINDING INTEGRATING FACTORS

RULE (I): If $M(x,y)\,dx + N(x,y)\,dy = 0$ (1)

is not exact and $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone,

then $e^{\int f(x)\,dx}$ is an integrating factor of differential equation (1).

PROOF: Let μ be an integrating factor of equation (1), then by hypothesis $\mu M\,dx + \mu N\,dy = 0$ is an exact differential equation. Thus $\dfrac{\partial}{\partial y}(\mu M) = \dfrac{\partial}{\partial x}(\mu N)$

or $\quad \mu\dfrac{\partial M}{\partial y} + M\dfrac{\partial \mu}{\partial y} = \mu\dfrac{\partial N}{\partial x} + N\dfrac{\partial \mu}{\partial x}$

or $\quad \mu\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = N\dfrac{\partial \mu}{\partial x} - M\dfrac{\partial \mu}{\partial y}$

Since μ is a function of x alone, $\dfrac{\partial \mu}{\partial y} = 0$, therefore $\mu\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = N\dfrac{\partial \mu}{\partial x}$

or $\quad \mu\left(\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}\right) = \dfrac{\partial \mu}{\partial x}$ or $\mu f(x) = \dfrac{d\mu}{dx}$ or $\dfrac{d\mu}{\mu} = f(x)\,dx$

and so $\ln \mu = \int f(x)\,dx$ or $\mu = e^{\int f(x)\,dx}$ is the required integrating factor.

EXAMPLE (8): Solve the differential equation

$$(3xy^2 + 2y)\,dx + (2x^2y + x)\,dy = 0$$

SOLUTION: Here $M = 3xy^2 + 2y$ and $N = 2x^2y + x$

so that $\dfrac{\partial M}{\partial y} = 6xy + 2$, $\dfrac{\partial N}{\partial x} = 4xy + 1$

since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore the given equation is not exact. However,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{6xy + 2 - 4xy - 1}{2x^2y + x} = \frac{2xy + 1}{2x^2y + x} = \frac{1}{x} = f(x)$$

and $e^{\int f(x)\,dx} = e^{\int 1/x\,dx} = e^{\ln x} = x$

is an integrating factor. Multiplying the given equation by x, we get

$$(3x^2y^2 + 2xy)\,dx + (2x^3y + x^2)\,dy = 0 \qquad (1)$$

from which $M = 3x^2y^2 + 2xy$ and $N = 2x^3y + x^2$.

Then $\dfrac{\partial M}{\partial y} = 6x^2y + 2x = \dfrac{\partial N}{\partial x}$.

Hence equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx + \int [\text{ terms in } N(x,y) \text{ without } x\,]\,dy = C$$

or $$\int (3x^2y^2 + 2xy)\,dx + \int 0\,dy = C$$

or $$x^3y^2 + x^2y = C$$

RULE (2): If $M(x,y)\,dx + N(x,y)\,dy = 0$ (1)

is not exact and if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$, a function of y alone,

then $e^{\int g(y)\,dy}$ is an integrating factor of differential equation (1).

PROOF: Let μ be an integrating factor of differential equation (1), then by hypothesis

$\mu M\,dx + \mu N\,dy = 0$ is an exact differential equation. Thus $\dfrac{\partial}{\partial y}(\mu M) = \dfrac{\partial}{\partial x}(\mu N)$

or $$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

or $$\mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial y}$$

Since μ is a function of y alone, $\dfrac{\partial \mu}{\partial x} = 0$

therefore $\mu \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right) = M \dfrac{\partial \mu}{\partial y}$ or $\mu \dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \dfrac{\partial \mu}{\partial y}$

or $\mu\, g(y) = \dfrac{d\mu}{dy}$ or $\dfrac{d\mu}{\mu} = g(y)\,dy$

and so $\ln \mu = \int g(y)\,dy$ or $\mu = e^{\int g(y)\,dy}$

is the required integrating factor.

EXAMPLE (9): **Solve the differential equation**

$$(2xy^2 - y)\,dx + (2x - x^2y)\,dy = 0$$

SOLUTION: Here $M = 2xy^2 - y$ and $N = 2x - x^2y$

so that $\dfrac{\partial M}{\partial y} = 4xy - 1$, $\dfrac{\partial N}{\partial x} = 2 - 2xy$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore the given equation is not exact. However

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{2-2xy-4xy+1}{2xy^2-y}=\frac{3-6xy}{2xy^2-y}=\frac{-3(2xy-1)}{y(2xy-1)}=-\frac{3}{y}=g(y)$$

and $e^{\int g(y)\,dy}=e^{-\int 3/y\,dy}=e^{-3\ln y}=e^{\ln y^{-3}}=\frac{1}{y^3}$

is an integrating factor . Multiplying the given equation by $\frac{1}{y^3}$, we get

$$\left(2\frac{x}{y}-\frac{1}{y^2}\right)dx+\left(2\frac{x}{y^3}-\frac{x^2}{y^2}\right)dy=0 \qquad (1)$$

from which $M=2\frac{x}{y}-\frac{1}{y^2}$ and $N=2\frac{x}{y^3}-\frac{x^2}{y^2}$.

Then $\frac{\partial M}{\partial y}=-2\frac{x}{y^2}+\frac{2}{y^3}=\frac{\partial N}{\partial x}$

Hence equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx+\int[\text{terms in } N(x,y) \text{ without } x]\,dy=C$$

or $\int\left(2\frac{x}{y}-\frac{1}{y^2}\right)dx+\int 0\,dy=C$

or $\frac{x^2}{y}-\frac{x}{y^2}=C$ or $x^2y-x=Cy^2$

RULE (3): If $M(x,y)\,dx+N(x,y)\,dy=0$ (1)

is homogenous and $Mx+Ny\neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor of differential equation (1).

PROOF: If $\frac{1}{Mx+Ny}$ is an integrating factor of equation (1), then we are to show that

$$\frac{M}{Mx+Ny}dx+\frac{N}{Mx+Ny}dy=0$$

is an exact equation i.e. $\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)=\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)$

Now
$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)=\frac{(Mx+Ny)\frac{\partial M}{\partial y}-M\left(x\frac{\partial M}{\partial y}+N+y\frac{\partial N}{\partial y}\right)}{(Mx+Ny)^2}$$

$$=\frac{Ny\frac{\partial M}{\partial y}-MN-My\frac{\partial N}{\partial y}}{(Mx+Ny)^2} \qquad (2)$$

and
$$\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)=\frac{(Mx+Ny)\frac{\partial N}{\partial x}-N\left(x\frac{\partial M}{\partial x}+M+y\frac{\partial N}{\partial x}\right)}{(Mx+Ny)^2}$$

$$=\frac{Mx\frac{\partial N}{\partial x}-MN-Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2} \qquad (3)$$

Subtracting equation (3) from equation (2),

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)-\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)=\frac{N\left(x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}\right)-M\left(x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}\right)}{(Mx+Ny)^2}$$

$$=\frac{N(nM)-M(nN)}{(Mx+Ny)^2}=0$$

using Euler's theorem on homogeneous function $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$.

NOTE: If $Mx+Ny=0$ identically, then $\frac{M}{N}=-\frac{y}{x}$ and the differential equation (1) becomes

$\frac{dy}{dx}=-\frac{M}{N}=\frac{y}{x}$ or $y\,dx-x\,dy=0$ for which $\frac{1}{xy}$ is an integrating factor.

EXAMPLE (10): **Solve the differential equation $(x^4+y^4)\,dx-xy^3\,dy=0$.**

SOLUTION: The given equation is homogeneous since it can be written as

$$\frac{dy}{dx}=\frac{x^4+y^4}{xy^3}=\frac{1+\left(\frac{y}{x}\right)^4}{\left(\frac{y}{x}\right)^3}=f\left(\frac{y}{x}\right)$$

Also $\quad Mx+Ny=x^5+xy^4-xy^4=x^5\neq 0$

so $\frac{1}{x^5}$ is an integrating factor of the given differential equation. Multiplying the given equation by $\frac{1}{x^5}$,

we get $\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}dy=0$ (1)

Here $\quad M=\frac{1}{x}+\frac{y^4}{x^5}$ and $N=-\frac{y^3}{x^4}$ so that $\frac{\partial M}{\partial y}=\frac{4y^3}{x^5}=\frac{\partial N}{\partial x}$

Hence equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx+\int[\text{terms in } N(x,y) \text{ with out } x]\,dy=C$$

or $\quad \int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx+\int 0\,dy=C$ or $\ln x-\frac{1}{4}\frac{y^4}{x^4}=C$

or $\quad y^4=4x^4\ln x+Cx^4$

RULE (4): If $M(x,y)\,dx+N(x,y)\,dy=0$ (1)

is not exact and can be written in the form $\quad y\,f(xy)\,dx+x\,g(xy)\,dy=0$ (2)

and if $Mx-Ny\neq 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor of differential equation (2).

PROOF: From equation (2), we have $M = y f(xy)$ and $N = x g(xy)$

so that $Mx - Ny = xy[f(xy) - g(xy)]$

If this is an integrating factor of equation (2), then we are to show that

$$\frac{y f(xy)}{xy[f(xy)-g(xy)]}dx + \frac{x g(xy)}{xy[f(xy)-g(xy)]}dy = 0$$

or $$\frac{f(xy)}{x[f(xy)-g(xy)]}dx + \frac{g(xy)}{y[f(xy)-g(xy)]}dy = 0$$

is an exact equation i.e.

$$\frac{\partial}{\partial y}\left(\frac{f(xy)}{x[f(xy)-g(xy)]}\right) = \frac{\partial}{\partial x}\left(\frac{g(xy)}{y[f(xy)-g(xy)]}\right)$$

Now $$\frac{\partial}{\partial y}\left[\frac{f}{x(f-g)}\right] = \frac{x(f-g)\frac{\partial f}{\partial y} - fx\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}\right)}{x^2(f-g)^2}$$

$$= \frac{f\frac{\partial g}{\partial y} - g\frac{\partial f}{\partial y}}{x(f-g)^2} \qquad (3)$$

and $$\frac{\partial}{\partial x}\left[\frac{g}{y(f-g)}\right] = \frac{y(f-g)\frac{\partial g}{\partial x} - gy\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right)}{y^2(f-g)^2}$$

$$= \frac{f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x}}{y(f-g)^2} \qquad (4)$$

Subtracting equation (4) from equation (3), we get

$$\frac{\partial}{\partial y}\left[\frac{f}{x(f-g)}\right] - \frac{\partial}{\partial x}\left[\frac{g}{y(f-g)}\right] = \frac{f\frac{\partial g}{\partial y} - g\frac{\partial f}{\partial y}}{x(f-g)^2} - \frac{f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x}}{y(f-g)^2}$$

$$= \frac{f\left(y\frac{\partial g}{\partial y} - x\frac{\partial g}{\partial x}\right) + g\left(x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right)}{xy(f-g)^2} = 0$$

since $x\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial y}$ and $x\frac{\partial g}{\partial x} = y\frac{\partial g}{\partial y}$

To show this let $u = xy$, then

$$x\frac{\partial}{\partial x}f(u) = x\frac{df}{du}\cdot\frac{\partial u}{\partial x} = xy\frac{df}{du} \text{ and}$$

$$y\frac{\partial}{\partial y}f(u) = y\frac{df}{du}\frac{\partial u}{\partial y} = xy\frac{df}{du}$$

which shows that $x\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial y}$. Similarly, are can show that $x\frac{\partial g}{\partial x} = y\frac{\partial g}{\partial y}$.

EXAMPLE (11): **Solve the differential equation:**

$$y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$$

SOLUTION: The equation is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$.

Here $M = y(x^2y^2+2)$, $N = x(2-2x^2y^2)$

and $Mx - Ny = x^3y^3 + 2xy - 2xy + 2x^3y^3 = 3x^3y^3 \neq 0$

so $\dfrac{1}{Mx-Ny} = \dfrac{1}{3x^3y^3}$ is an integrating factor.

Multiplying the given equation by $\dfrac{1}{3x^3y^3}$

we get $\dfrac{x^2y^2+2}{3x^3y^2}\,dx + \dfrac{2-2x^2y^2}{3x^2y^3}\,dy = 0$

or $\left(\dfrac{1}{3x} + \dfrac{2}{3x^3y^2}\right)dx + \left(\dfrac{2}{3x^2y^3} - \dfrac{2}{3y}\right)dy = 0$ (1)

Here $M = \dfrac{1}{3x} + \dfrac{2}{3x^3y^2}$, $N = \dfrac{2}{3x^2y^3} - \dfrac{2}{3y}$

so that $\dfrac{\partial M}{\partial y} = -\dfrac{4}{3x^3y^3} = \dfrac{\partial N}{\partial x}$

Thus equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx + \int [\text{ terms in } N(x,y) \text{ without } x\,]\,dy = C_1$$

or $\displaystyle\int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx - \frac{2}{3}\int \frac{1}{y}\,dy = C_1$

or $\dfrac{1}{3}\ln x - \dfrac{1}{3x^2y^2} - \dfrac{2}{3}\ln y = C_1$

or $\ln x - \dfrac{1}{x^2y^2} - 2\ln y = 3C_1$

or $\ln\dfrac{x}{y^2} = \dfrac{1}{x^2y^2} + C$ where $C = 3C_1$

RULE (5): If the differential equation is of the form

$$x^ay^b(m\,y\,dx + n\,x\,dy) + x^cy^d(p\,y\,dx + q\,x\,dy) = 0$$

where a, b, c, d, m, n, p, q are all constants, then the integrating factor is $x^\alpha y^\beta$, where α and β are so chosen that after multiplying the equation by $x^\alpha y^\beta$, the equation becomes exact.

EXAMPLE (12): **Solve the differential equation :**

$$(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$$

SOLUTION: Here $M = y^3 - 2yx^2$ and $N = 2xy^2 - x^3$

$\dfrac{\partial M}{\partial y} = 3y^2 - 2x^2$, $\dfrac{\partial N}{\partial x} = 2y^2 - 3x^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore the equation is not exact.

The given equation can be written

$$y^3 dx + 2xy^2 dy - 2yx^2 dx - x^3 dy = 0$$

or $$x^0 y^2 (y dx + 2x dy) + x^2 y^0 (-2y dx - x dy) = 0$$

which is of the form

$$x^a y^b (my dx + nx dy) + x^c y^d (py dx + qx dy) = 0$$

Let $x^\alpha y^\beta$ be an integrating factor. Multiplying the given equation by $x^\alpha y^\beta$, we get

$$(x^\alpha y^{\beta+3} - 2y^{\beta+1} x^{\alpha+2}) dx + (2x^{\alpha+1} y^{\beta+2} - x^{\alpha+3} y^\beta) dy = 0 \qquad (1)$$

Equation (1) will be exact if

$$\frac{\partial}{\partial y}(x^\alpha y^{\beta+3} - 2y^{\beta+1} x^{\alpha+2}) = \frac{\partial}{\partial x}(2x^{\alpha+1} y^{\beta+2} - x^{\alpha+3} y^\beta)$$

or $$(\beta+3) x^\alpha y^{\beta+2} - 2(\beta+1) y^\beta x^{\alpha+2} = 2(\alpha+1) x^\alpha y^{\beta+2} - (\alpha+3) x^{\alpha+2} y^\beta$$

or $$[(\beta+3) - 2(\alpha+1)] x^\alpha y^{\beta+2} + [-2(\beta+1) + (\alpha+3)] x^{\alpha+2} y^\beta = 0$$

This implies $(\beta+3) - 2(\alpha+1) = 0$

and $-2(\beta+1) + (\alpha+3) = 0$

or $-2\alpha + \beta = -1$

$\alpha - 2\beta = -1$

Solving these equations, we get $\alpha = 1, \quad \beta = 1$

Thus xy is an integrating factor.

Multiplying the given equation by xy, we get

$$(xy^4 - 2y^2 x^3) dx + (2x^2 y^3 - x^4 y) dy = 0 \qquad (2)$$

Now since $\frac{\partial M}{\partial y} = 4xy^3 - 4yx^3 = \frac{\partial N}{\partial x}$

Therefore equation (2) is exact.

Thus its solution is given by

$$\int (xy^4 - 2y^2 x^3) dx + \int 0 \, dy = C_1$$

or $$\frac{1}{2} x^2 y^4 - \frac{1}{2} y^2 x^4 = C_1$$

or $$x^2 y^4 - y^2 x^4 = C \quad (\text{where } C = 2C_1)$$

2.9 LINEAR DIFFERENTIAL EQUATIONS

A first order differential equation is said to be linear if it can be written in the form

$$\frac{dy}{dx} + Py = Q \qquad (1)$$

where P and Q are functions of x (or constants) . For example , the differential equations

(i) $\frac{dy}{dx} + 2y = 6e^x$ (ii) $\frac{dy}{dx} + \frac{2}{x}y = 9$

(iii) $\frac{dy}{dx} + y = 5$ (iv) $\frac{dy}{dx} + 3xy = \sin x$

are linear while $\frac{dy}{dx} + 3xy^2 = \sin x$ is not .

If in equation (1), $Q = 0$ for all x in the interval in which we consider the equation (written $Q \equiv 0$) , the equation is said to be **homogeneous** ; otherwise it is said to be **non – homogeneous** or **inhomogeneous**.

Let us find a formula for the general solution of equation (1) in some interval I , assuming that P and Q are continuous in I . First consider the homogeneous equation $\frac{dy}{dx} + Py = 0$

By separating the variables , we have $\frac{dy}{y} = -P\,dx$

Integrating , $\ln|y| = -\int P\,dx + C_1$

or $|y| = e^{-\int P\,dx + C_1} = e^{-\int P\,dx} \cdot e^{C_1}$

or $y = Ce^{-\int P\,dx}$ where $C = \pm e^{C_1}$ when $y \lessgtr 0$;

If we take $C = 0$, we obtain the trivial solution $y = 0$.

Next , consider the non homogeneous equation (1) . It will be seen that it has the property of possessing an integrating factor depending only on x . We first write equation (1) as

$$[P(x)y - Q(x)]\,dx + dy = 0$$

Then $M = P(x)y - Q(x)$, $N = 1$

$$\frac{\partial M}{\partial y} = P(x), \quad \frac{\partial N}{\partial x} = 0$$

since $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{P(x) - 0}{1} = P(x)$

depends only on x , we see that $e^{\int P(x)\,dx}$ is an integrating factor .

Multiplying by this factor , equation (1) becomes $e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) \quad e^{\int P\,dx}$

which can be written as $\frac{d}{dx}\left(e^{\int P\,dx}y\right) = Q\,e^{\int P\,dx}$

Then on integrating , we have

$$e^{\int P\,dx}y = \int Q\,e^{\int P\,dx}\,dx + C$$

which is the general solution of equation (1) . We may put it as

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$$

NOTE: The choice of the value of the constant of integration in $\int P\,dx$ does not matter so that we may choose it to be zero .

EXAMPLE (13): **Solve the differential equation** $\frac{dy}{dx} + 2xy = 2e^{-x^2}$.

SOLUTION: Here $P(x) = 2x$ and $Q(x) = 2e^{-x^2}$, therefore the integrating factor is

$$\text{I.F.} = e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Hence the general solution of the given equation can be written as

$$y\,(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + C$$

or $$y\,(e^{x^2}) = \int 2e^{-x^2}.e^{x^2}\,dx + C$$

$$= 2\int dx + C = 2x + C \quad \text{or} \quad y = (2x + C)\,e^{-x^2}$$

EXAMPLE (14): **Solve the initial–value problem:**

$$\frac{dy}{dx} + y\tan x = \sin 2x\,;\; y(0) = 1\,.$$

SOLUTION: Here $P(x) = \tan x$ and $Q(x) = \sin 2x$, therefore the integrating factor is

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x$$

Thus the general solution of the given differential equation can be written as

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$$

or $$y\,(\sec x) = \int \sin 2x\,(\sec x)\,dx + C$$

$$= \int 2\sin x\cos x\left(\frac{1}{\cos x}\right)dx + C$$

$$= \int 2\sin x\,dx + C = -2\cos x + C$$

or $$y = C\cos x - 2\cos^2 x$$

Using the initial condition $y(0) = 1$, we get $1 = C - 2$ or $C = 3$
and the solution of the given initial value problem is $y = 3\cos x - 2\cos^2 x$.

EXAMPLE (15): **Solve the differential equation $(2x - 10y^3)\dfrac{dy}{dx} + y = 0$.**

SOLUTION: This equation is not linear as it stands, but if we multiply by $\dfrac{dx}{dy}$, we get

$$2x - 10y^3 + y\frac{dx}{dy} = 0$$

or $$\frac{dx}{dy} + \frac{2}{y}x = 10y^2 \qquad (1)$$

which is linear considering y as independent variable and $P = \dfrac{2}{y}$, $Q = 10y^2$

Therefore $\text{I.F.} = e^{\int P\,dy} = e^{\int (2/y)\,dy} = e^{2\ln y} = e^{\ln y^2} = y^2$

Thus the general solution is given by

$$x.(\text{I.F}) = \int Q.(\text{I.F})\,dy + C$$

or $$x.y^2 = \int 10y^2.y^2\,dy + C = 2y^5 + C$$

or $$x = 2y^3 + Cy^{-2}$$

2.10 EQUATIONS REDUCIBLE TO LINEAR FORM

BERNOULI'S EQUATION

Certain nonlinear differential equations can be reduced to linear form . The practically most famous of these is the Bernoulli's equation which has the form

$$\frac{dy}{dx} + Py = Qy^n, \text{ where } n \neq 0, 1 \qquad (1)$$

and P and Q are functions of x alone or constants .

To solve equation (1) , divide both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad (2)$$

Now put $y^{1-n} = z$, then $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

Thus equation (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$

or $$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

which is a linear differential equation in z, and can , therefore be solved by the previous method .

EXAMPLE (16): Solve the differential equation $x\frac{dy}{dx}+y = x y^3$.

SOLUTION: Write the equation as $\frac{dy}{dx}+\frac{1}{x}y = y^3$ (1)

which is a Bernoulli's equation with $P(x) = \frac{1}{x}$, $Q(x) = 1$, and $n = 3$

Divide equation (1) by y^3, we get

$$y^{-3}\frac{dy}{dx}+\frac{1}{x}y^{-2} = 1 \qquad (2)$$

Let $y^{-2} = z$, then $-2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$ or $y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$

Thus equation (2) becomes $-\frac{1}{2}\frac{dz}{dx}+\frac{1}{x}z = 1$

or $$\frac{dz}{dx}-\frac{2}{x}z = -2 \qquad (3)$$

which is a linear differential equation in z, whose integrating factor is given by

$$\text{I.F.} = e^{-\int 2/x\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

Hence the general solution of equation (3) is

$$z\left(\frac{1}{x^2}\right) = \int -2\left(\frac{1}{x^2}\right)dx + C = \frac{2}{x}+C$$

or $$z = 2x + Cx^2$$

Since $z = y^{-2}$, therefore the solution of the given differential equation becomes

$$y^{-2} = 2x + Cx^2$$

or $$y^2 = \frac{1}{2x+Cx^2}$$

2.11 RICCATI EQUATION

A differential equation of the form

$$\frac{dy}{dx} = Py^2 + Qy + R \qquad (1)$$

where P, Q, and R are functions of x (or constants), is called a **Riccati equation** after the Italian mathematician J.F. Riccati (1676 – 1754). A Riccati equation is exactly linear when P is identically zero. If $R = 0$, then the Riccati equation becomes the Bernoulli's equation.

If we can somehow (by inspection, trial and error, or any other means) find one particular solution $Y(x)$ of a Riccati equation, then the change of variables

$$y = Y(x) + \frac{1}{u}$$

transforms the Riccati equation to a linear equation. The method is to find the general solution of this linear equation and from it produce the general solution of the original Riccati equation. This method of solution was discovered by L. Euler (1707 – 1783) in 1760.

EXAMPLE (17): **Solve the differential equation :**

$$\frac{dy}{dx} = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}$$

SOLUTION: By inspection $Y(x) = 1$ is one particular solution. Define a new variable u by putting

$$y = 1 + \frac{1}{u}. \text{ Then } \frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

Substituting these into the Riccati equation, we get

$$-\frac{1}{u^2}\frac{du}{dx} = \frac{1}{x}\left(1 + \frac{1}{u}\right)^2 + \frac{1}{x}\left(1 + \frac{1}{u}\right) - \frac{2}{x}$$

or $$-\frac{1}{u^2}\frac{du}{dx} = \frac{1}{x}\left(1 + \frac{1}{u^2} + \frac{2}{u}\right) + \frac{1}{x} + \frac{1}{xu} - \frac{2}{x}$$

$$= \frac{1}{xu^2} + \frac{3}{ux}$$

or $$\frac{du}{dx} = -\frac{1}{x} - \frac{3u}{x}$$

or $$\frac{du}{dx} + \frac{3}{x}u = -\frac{1}{x} \qquad (1)$$

which is a linear equation. To solve this equation, we find

$$\text{I.F.} = e^{\int \frac{3}{x} dx} = e^{3\ln x} = x^3$$

The general solution of equation (1) is

$$(\text{I.F.})\, u = \int (\text{I.F.})\left(-\frac{1}{x}\right) dx + C$$

or $$x^3 u = \int x^3\left(-\frac{1}{x}\right) dx + C$$

$$= -\frac{x^3}{3} + C$$

or $$u = -\frac{1}{3} + \frac{C}{x^3} \qquad (2)$$

Thus the general solution of the Riccati equation is given by

$$y = 1 + \frac{1}{u} = 1 + \frac{1}{-\frac{1}{3} + \frac{C}{x^3}}$$

2.12 GEOMETRICAL INTERPRETATION OF $\frac{dy}{dx} = f(x, y)$

Let $\frac{dy}{dx} = f(x, y)$ (1)

be a differential equation of first order and first degree. We know that $\frac{dy}{dx}$ is the slope of tangent to the curve at any point (x, y).

Let $P_0(x_0, y_0)$ be any point in the plane. Substituting the coordinates of this point in equation (1), we get $\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = m_0$, the slope of the tangent to the curve at P_0.

Take a neighbouring point $P_1(x_1, y_1)$ such that the slope of P_0P_1 is m_0.

Let $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_1$ be the slope of the tangent to the curve at P_1 determined from equation (1).

Take a neighbouring point $P_2(x_2, y_2)$ such that the slope of P_1P_2 is m_1. Continuing like this, we get a succession of points as shown in figure (2.1).

If the points are taken sufficiently close to each other, they approximate a smooth curve C defined by $y = \phi(x)$ which is a solution of equation (1) corresponding to the initial point $P_0(x_0, y_0)$. Any point on the curve C and the slope of the tangent at that point satisfy equation (1).

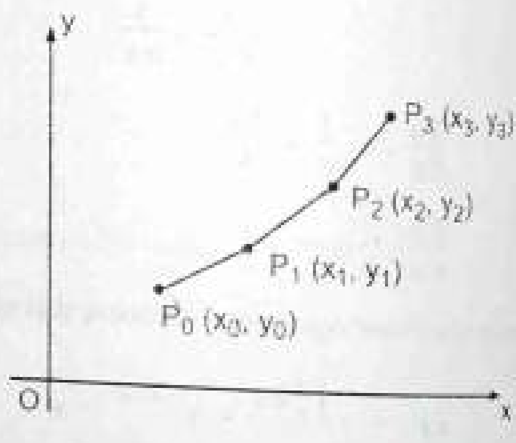


Figure (2.1)

If the initial point $P_0(x_0, y_0)$ is any other point in the plane not lying on C, the moving point $P(x, y)$ will describe another curve.

The coordinates of all points on this curve will also satisfy the equation (1). Thus through every point in the plane passes a particular curve, each point of which satisfies the given differential equation (1). The equation of each of these curves is a particular solution of differential equation (1). The equation of the system of these curves gives the general solution of the differential equation (1). Thus equation (1) represents a family of curves having a common property.

2.13 SOLVED PROBLEMS

SEPARABLE DIFFERENTIAL EQUATIONS

PROBLEM (1): **Solve the differential equations**

(i) $$e^y(1+x^2)\frac{dy}{dx} - 2x(1+e^y) = 0$$

(ii) $$x^2(y+1)\,dx + y^2(x-1)\,dy = 0$$

(iii) $$(xy^2+x)\,dx + (yx^2+y)\,dy = 0$$

(iv) $$(2y+xy)\frac{dy}{dx} = 1+\frac{4}{x}+\frac{4}{x^2}$$

SOLUTION: (i) $$e^y(1+x^2)\frac{dy}{dx} - 2x(1+e^y) = 0$$

Separating the variables , write the equation as

$$\frac{e^y}{1+e^y}\,dy = \frac{2x}{1+x^2}\,dx$$

Integrating both sides , we get

$$\int\frac{e^y}{1+e^y}\,dy = \int\frac{2x}{1+x^2}\,dx$$

or $$\ln(1+e^y) = \ln(1+x^2) + \ln C$$

$$= \ln C(1+x^2)$$

Taking the antilogarithm , we get

or $$1+e^y = C(1+x^2)$$

(ii) $$x^2(y+1)\,dx + y^2(x-1)\,dy = 0$$

Write the equation as

$$\frac{x^2}{x-1}\,dx + \frac{y^2}{y+1}\,dy = 0$$

or $$\left(x+1+\frac{1}{x-1}\right)dx + \left(y-1+\frac{1}{y+1}\right)dy = 0$$

Integrating , we get

$$\int\left(x+1+\frac{1}{x-1}\right)dx + \int\left(y-1+\frac{1}{y+1}\right)dy = C$$

or $$\frac{x^2}{2}+x+\ln(x-1)+\frac{1}{2}y^2-y+\ln(y+1) = C$$

or $$\frac{1}{2}(x^2+y^2)+(x-y)+\ln\left[(x-1)(y+1)\right] = C$$

(iii) $(xy^2+x)\,dx+(yx^2+y)\,dy=0$

Write the equation as

$$y(1+x^2)\,dy=-x(1+y^2)\,dx$$

or $$\frac{y}{1+y^2}\,dy=-\frac{x}{1+x^2}\,dx$$

Integrating both sides, we get

$$\int\frac{y}{1+y^2}\,dy=-\int\frac{x}{1+x^2}\,dx$$

or $$\frac{1}{2}\ln(1+y^2)=-\frac{1}{2}\ln(1+x^2)+C_1$$

or $$\ln\left[(1+y^2)(1+x^2)\right]=2C_1$$

Taking the antilogarithm, we get

$$(1+y^2)(1+x^2)=C\quad(\text{where } C=e^{2C_1})$$

(iv) $(2y+xy)\dfrac{dy}{dx}=1+\dfrac{4}{x}+\dfrac{4}{x^2}$

Write the equation as

$$y(2+x)\frac{dy}{dx}=\left(1+\frac{2}{x}\right)^2$$

Separating the variables, we get

$$y\,dy=\left(1+\frac{2}{x}\right)^2\left(\frac{1}{2+x}\right)dx$$
$$=\frac{(x+2)^2}{x^2}\left(\frac{1}{2+x}\right)dx=\frac{x+2}{x^2}=\frac{1}{x}+\frac{2}{x^2}$$

Integrating, we get

$$\int y\,dy=\int\left(\frac{1}{x}+\frac{2}{x^2}\right)dx$$

or $$\frac{y^2}{2}=\ln|x|-\frac{2}{x}+C$$

PROBLEM (2): **Solve the following initial value problems :**

(i) $e^x\dfrac{dy}{dx}=2(x+1)y^2;\quad y(0)=\dfrac{1}{6}$

(ii) $2y\dfrac{dy}{dx}=e^{x-y^2};\quad y(4)=2$

SOLUTION: (i) $e^x\dfrac{dy}{dx}=2(x+1)y^2;\quad y(0)=\dfrac{1}{6}$

Separating the variables, write the equation as

$$\frac{dy}{y^2} = 2(x+1)e^{-x}$$

Integrating both sides, we get

$$\int \frac{dy}{y^2} = 2\int (x+1)e^{-x}\,dx$$

or $$-\frac{1}{y} = 2[-(x+1)e^{-x} + \int e^{-x}.1\,dx] + C$$

or $$-\frac{1}{y} = 2[-(x+1)e^{-x} - e^{-x}] + C$$

or $$-\frac{1}{y} = 2[-(x+2)e^{-x}] + C$$

Using the initial condition $y(0) = \frac{1}{6}$, we get

$$-6 = 2(-2) + C \quad \text{or} \quad C = -2$$

Thus the unique solution of the given problem is

$$-\frac{1}{y} = -2(x+2)e^{-x} - 2$$

or $$y = \frac{1}{2 + (2x+4)e^{-x}}$$

(ii) $$2y\frac{dy}{dx} = e^{x-y^2}; \quad y(4) = 2$$

Write the equation as

$$2y\frac{dy}{dx} = e^{x}.e^{-y^2}$$

Separating the variables, we get

or $$2ye^{y^2}\,dy = e^{x}\,dx$$

Integrating both sides, we get

$$\int e^{y^2}.2y\,dy = \int e^{x}\,dx$$

or $$e^{y^2} = e^{x} + C$$

Using the initial condition $y(4) = 2$, we get

$$e^4 = e^4 + C$$

or $$C = 0$$

Thus the unique solution of the given problem is

$$e^{y^2} = e^{x}$$

or $$y^2 = x$$

HOMOGENEOUS DIFFERENTIAL EQUATIONS

PROBLEM (3): **Solve the following differential equations :**

(i) $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}$ (ii) $\dfrac{dy}{dx} = \dfrac{y^2}{x^2} - \dfrac{y}{x}$

(iii) $x y' = y + \dfrac{x^5 e^x}{4y^3}$ (iv) $(2x^3 + y^3)\,dx - 3xy^2\,dy = 0$

SOLUTION: (i) $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}$

Write the equation as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \qquad (1)$$

Put $y = ux$ therefore $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

and equation (1) becomes

$$u + x\frac{du}{dx} = \frac{ux + \sqrt{x^2 + u^2x^2}}{x}$$

or $$u + x\frac{du}{dx} = u + \sqrt{1 + u^2}$$

or $$x\frac{du}{dx} = \sqrt{1 + u^2}$$

Separating the variables , we have

$$\frac{du}{\sqrt{1 + u^2}} = \frac{dx}{x}$$

Integrating , we get $\displaystyle\int \frac{du}{\sqrt{1 + u^2}} = \int \frac{dx}{x}$

$$\ln\left(u + \sqrt{1 + u^2}\right) = \ln x + \ln C$$

or $$u + \sqrt{1 + u^2} = Cx$$

Replacing u by $\dfrac{y}{x}$, we get

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = Cx$$

or $$y + \sqrt{x^2 + y^2} = Cx^2$$

(ii) $$\frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x} \qquad (1)$$

The equation is homogeneous . Let $\dfrac{y}{x} = u$ or $y = ux$, then $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

and so equation (1) becomes

$$u + x\frac{du}{dx} = u^2 - u$$

or $$x\frac{du}{dx} = u^2 - 2u$$

Separating the variables , we get

or $$\frac{du}{u(u-2)} = \frac{dx}{x}$$

Integrating both sides , we get

$$\int \frac{du}{u(u-2)} = \int \frac{dx}{x}$$

or $$\frac{1}{2}\int\left(\frac{1}{u-2} - \frac{1}{u}\right) du = \int \frac{dx}{x}$$

or $$\frac{1}{2}\left[\ln(u-2) - \ln u\right] = \ln x + \ln C_1$$

or $$\frac{1}{2}\ln\left(\frac{u-2}{u}\right) = \ln C_1 x$$

or $$\ln\left(\frac{u-2}{u}\right) = 2\ln C_1 x = \ln(C_1 x)^2$$

or $$\frac{u-2}{u} = C_1^2 x^2 = C x^2 \qquad (\text{where } C = C_1^2)$$

or $$1 - \frac{2}{u} = C x^2$$

Since $u = \frac{y}{x}$, the general solution of the given differential equation in terms of x and y is :

$$1 - \frac{2x}{y} = C x^2$$

or $$y - 2x = C x^2 y$$

(iii) $$x y' = y + \frac{x^5 e^x}{4y^3}$$

Write the equation as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x^4 e^x}{4y^3} = \frac{y}{x} + \frac{x e^x}{4\frac{y^3}{x^3}} \qquad (1)$$

which is homogeneous . Let $u = \frac{y}{x}$ or $y = ux$. Then

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

so equation (1) becomes

$$u + x\frac{du}{dx} = u + \frac{x e^x}{4u^3}$$

or $$\frac{du}{dx} = \frac{e^x}{4u^3}$$

Separating the variables, we get

$$4u^3\,du = e^x\,dx$$

Integrating both sides, we get

$$\int 4u^3\,du = \int e^x\,dx$$

or $$u^4 = e^x + C$$

Since $u = \frac{y}{x}$, the general solution of the given differential equation is

$$\frac{y^4}{x^4} = e^x + C$$

or $$y^4 = x^4(e^x + C)$$

(iv) $$(2x^3 + y^3)\,dx - 3xy^2\,dy = 0$$

Write the equation as

$$\frac{dy}{dx} = \frac{2x^3 + y^3}{3xy^2} = \frac{2 + (y^3/x^3)}{3y^2/x^2} \qquad (1)$$

Let $\frac{y}{x} = u$ or $y = ux$ so that $\frac{dy}{dx} = u + x\frac{du}{dx}$

Thus equation (1) becomes

$$u + x\frac{du}{dx} = \frac{2 + u^3}{3u^2}$$

or $$x\frac{du}{dx} = \frac{2 + u^3}{3u^2} - u$$

or $$x\frac{du}{dx} = \frac{2 - 2u^3}{3u^2}$$

Separating the variables, we have

$$\frac{3u^2}{u^3 - 1}\,du = -2\frac{dx}{x}$$

Integrating, we get

$$\int \frac{3u^2}{u^3 - 1}\,du = -2\int \frac{dx}{x}$$

$$\ln(u^3 - 1) = -2\ln x + \ln C$$

or $$\ln\left[(u^3 - 1)x^2\right] = \ln C$$

or $$(u^3 - 1)x^2 = C$$

and the required solution on letting $u = \frac{y}{x}$

$$y^3 - x^3 = Cx$$

PROBLEM (4): **Solve the following initial value problems :**

(i) $$x\,y' = (y-x)^3 + y\,; \quad y(1) = \frac{3}{2}$$

(ii) $$\frac{dy}{dx} = \frac{y}{x} + \frac{2x^3\cos x^2}{y}\,; \quad y\left(\sqrt{\pi}\right) = 0$$

SOLUTION: (i) $x\,y' = (y-x)^3 + y\,; \quad y(1) = \frac{3}{2}$

Write the equation as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{(y-x)^3}{x}$$

$$= \frac{y}{x} + \frac{x^3\left(\frac{y}{x} - 1\right)^3}{x} = \frac{y}{x} + x^2\left(\frac{y}{x} - 1\right)^3 \qquad (1)$$

The equation is homogeneous . Let $\frac{y}{x} = u$ or $y = u\,x$.

Then $\frac{dy}{dx} = u + x\frac{du}{dx}$.

Therefore equation (1) becomes

$$u + x\frac{du}{dx} = u + x^2(u-1)^3$$

or $$x\frac{du}{dx} = x^2(u-1)^3$$

Separating the variables , we get

$$\frac{du}{(u-1)^3} = x\,dx$$

Integrating both sides , we get

$$\int \frac{du}{(u-1)^3} = \int x\,dx$$

$$-\frac{1}{2(u-1)^2} = \frac{x^2}{2} + C \qquad (2)$$

Since $u = \frac{y}{x}$, therefore equation (2) becomes

$$-\frac{1}{2\left(\frac{y}{x} - 1\right)^2} = \frac{x^2}{2} + C$$

$$-\frac{1}{2}\frac{x^2}{(y-x)^2} = \frac{x^2}{2} + C \qquad (3)$$

Using the initial condition $y(1) = \frac{3}{2}$, we get

$$-\frac{1}{2}\frac{1}{\left(\frac{3}{2}-1\right)^2} = \frac{1}{2}+C$$

or $$-\frac{1}{2}(4) = \frac{1}{2}+C$$

or $$-2 = \frac{1}{2}+C \quad \text{or} \quad C = -\frac{5}{2}$$

Thus the solution of the given initial – value problem is

$$-\frac{1}{2}\frac{x^2}{(y-x)^2} = \frac{x^2}{2} - \frac{5}{2}$$

or $$\frac{x^2}{(y-x)^2} = 5 - x^2$$

(ii) $$\frac{dy}{dx} = \frac{y}{x} + \frac{2x^3\cos x^2}{y}; \quad y(\sqrt{\pi}) = 0$$

Write the equation as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{2x^2\cos x^2}{y/x} \qquad (1)$$

Let $\frac{y}{x} = u$ or $y = ux$, then $\frac{dy}{dx} = u + x\frac{du}{dx}$

Thus equation (1) becomes

$$u + x\frac{du}{dx} = u + \frac{2x^2\cos x^2}{u}$$

or $$x\frac{du}{dx} = \frac{2x^2\cos x^2}{u}$$

Separating the variables, we get

$$u\,du = 2x\cos x^2$$

Integrating both sides, we get

$$\int u\,du = \int \cos x^2(2x)\,du + C$$

or $$\frac{u^2}{2} = \sin x^2 + C \qquad (2)$$

Since $u = \frac{y}{x}$, therefore equation (2) takes the form

$$\frac{y^2}{2x^2} = \sin x^2 + C$$

or $$y^2 = 2x^2\sin x^2 + C \qquad (3)$$

Using the initial condition $y(\sqrt{\pi}) = 0$, we get

$$0 = 2\pi\sin\pi + C \quad \text{or} \quad C = 0$$

Thus from equation (3), we get

$$y^2 = 2x^2 \sin x^2$$

or $\quad y = x\sqrt{2\sin x^2}$ is solution of the given initial – value problem.

PROBLEM (5): **Solve the following differential equation :**

(i) $\dfrac{dy}{dx} = \dfrac{2x-5y+3}{2x+4y-6}$ (ii) $\dfrac{dy}{dx} = \dfrac{1-2y-4x}{1+y+2x}$

SOLUTION: (i) $\dfrac{dy}{dx} = \dfrac{2x-5y+3}{2x+4y-6}$

Here $a_1 b_2 - a_2 b_1 = (2)(4) - (-5)(2) \neq 0$, so let

$$x = X+h,\ y = Y+k \tag{1}$$

therefore $dx = dX$ and $dy = dY$

and $\quad \dfrac{dY}{dX} = \dfrac{2(X+h)-5(Y+k)+3}{2(X+h)+4(Y+k)-6}$

$$= \frac{2X-5Y+(2h-5k+3)}{2X+4Y+(2h+4k-6)} \tag{2}$$

Choose h and k so that

$$2h-5k+3 = 0$$

$$2h+4k-6 = 0$$

Solving these equations simultaneously we find that $h = 1$ and $k = 1$. Thus equation (2) becomes

$$\frac{dY}{dX} = \frac{2X-5Y}{2X+4Y} \tag{3}$$

which is a homogeneous equation. Now put $Y = uX$ therefore $\dfrac{dY}{dX} = u + X\dfrac{du}{dX}$

so that from equation (3)

$$u + X\frac{du}{dX} = \frac{2X-5uX}{2X+4uX} = \frac{2-5u}{2+4u}$$

or $\quad X\dfrac{du}{dX} = \dfrac{2-5u}{2+4u} - u = \dfrac{2-7u-4u^2}{2+4u}$

Separating the variables, $\dfrac{dX}{X} = \dfrac{2+4u}{2-7u-4u^2}\,du$

or $\quad \dfrac{dX}{X} + \dfrac{2+4u}{4u^2+7u-2}\,du = 0$

$$\frac{dX}{X} + \frac{2+4u}{(4u-1)(u+2)}\,du = 0$$

Resolving the second term in partial fractions

$$\frac{dX}{X} + \frac{4}{3}\,\frac{du}{4u-1} + \frac{2}{3}\,\frac{du}{u+2} = 0$$

Integrating , we get

$$\int\left[\frac{dX}{X}+\frac{4}{3}\frac{du}{4u-1}+\frac{2}{3}\frac{du}{u+2}\right]=0$$

or $\quad \ln X+\frac{1}{3}\ln(4u-1)+\frac{2}{3}\ln(u+2)=\ln C_1$

or $\quad 3\ln X+\ln(4u-1)+2\ln(u+2)=3\ln C_1$

or $\quad \ln X^3(4u-1)(u+2)^2=\ln C_1^3=\ln C$

or $\quad X^3(4u-1)(u+2)^2=C$

Replacing u by $\frac{Y}{X}$, we get $X^3\left(4\frac{Y}{X}-1\right)\left(\frac{Y}{X}+2\right)^2=C$

or $\quad (4Y-X)(Y+2X)^2=C$

Since from equations (1) , we have

$X=x-1$ and $Y=y-1$, we obtain

$$(4y-x-3)(y+2x-3)^2=C$$

as the general solution of the given differential equation .

(ii) $\quad \frac{dy}{dx}=\frac{1-2y-4x}{1+y+2x}$

Write the equation as

$$\frac{dy}{dx}=\frac{1-2(y+2x)}{1+(y+2x)} \tag{1}$$

Let $y+2x=z$, $y=-2x+z$ and $\frac{dy}{dx}=-2+\frac{dz}{dx}$

Thus equation (1) becomes

$$-2+\frac{dz}{dx}=\frac{1-2z}{1+z}$$

or $\quad \frac{dz}{dx}=\frac{1-2z}{1+z}+2=\frac{3}{1+z}$

Separating the variables , we get

$$(1+z)\,dz=3\,dx$$

Integrating both sides , we get

$$\int(1+z)\,dz=\int 3\,dx$$

$$z+\frac{z^2}{2}=3x+C_1$$

or $\quad z^2+2z-6x=C \quad (\text{when } C=2C_1)$

Replacing $z = y + 2x$, we get

$$(y+2x)^2 + 2(y+2x) - 6x = C$$

or $$(y+2x)^2 + 2(y-x) = C$$

EXACT EQUATIONS

PROBLEM (6): **Solve the differential equation**

(i) $$(3x^2 + y\cos x)\,dx + (\sin x - 4y^3)\,dy = 0$$

(ii) $$(2xe^y + e^x)\,dx + (x^2+1)e^y\,dy = 0$$

SOLUTION: (i) Here $M = 3x^2 + y\cos x$, $N = \sin x - 4y^3$

so that $\dfrac{\partial M}{\partial y} = \cos x = \dfrac{\partial N}{\partial x}$ and the equation is exact.

To find its general solution, we can use any of the following methods.

METHOD (1): DIRECT METHOD

Since $M\,dx + N\,dy = du$ an exact differential and

$du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$, we must have

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N$$

or $$\frac{\partial u}{\partial x} = 3x^2 + y\cos x \qquad (1)$$

$$\frac{\partial u}{\partial y} = \sin x - 4y^3 \qquad (2)$$

Integrating equation (1) partially w.r.t. x (keeping y constant), we have

$$u(x,y) = x^3 + y\sin x + \phi(y) \qquad (3)$$

where $\phi(y)$ is the constant of integration which may depend on y.

Now $\dfrac{\partial u}{\partial y} = \sin x + \phi'(y) = \sin x - 4y^3$

or $\phi'(y) = -4y^3$ so that $\phi(y) = -y^4$. Then from equation (3)

$u(x,y) = x^3 + y\sin x - y^4$ and the general solution is given by

$$x^3 + y\sin x - y^4 = C$$

METHOD (2): USING FORMULA

The general solution of an exact differential equation is given by the formula

$$\int M(x,y)\,dx + \int [\text{terms in } N(x,y) \text{ without } x]\,dy = C$$

i.e. $$\int (3x^2 + y\cos x)\,dx - \int 4y^3\,dy = C$$

or $$x^3 + y\sin x - y^4 = C$$

METHOD (3): GROUPING OF TERMS BY INSPECTION

Write the equation as

$$3x^2\,dx + (y\cos x\,dx + \sin x\,dy) - 4y^3\,dy = 0$$

i.e. $\quad d(x^3) + d(y\sin x) - d(y^4) = 0$

or $\quad d(x^3 + y\sin x - y^4) = 0$

Then on integrating $x^3 + y\sin x - y^4 = C$ is the general solution.

(ii) $\quad (2xe^y + e^x)\,dx + (x^2+1)e^y\,dy = 0$

Here $M = 2xe^y + e^x$ and $N = (x^2+1)e^y$

so that $\dfrac{\partial M}{\partial y} = 2xe^y = \dfrac{\partial N}{\partial x}$

Thus the equation is exact and its general solution is given by the formula

$$\int M(x,y)\,dx + \int [\text{ terms in } N(x,y) \text{ without } x\,]\,dy = C$$

or $\quad \displaystyle\int (2xe^y + e^x)\,dx + \int e^y\,dy = C$

or $\quad x^2e^y + e^x + e^y = C \quad$ or $\quad e^x + e^y(x^2+1) = C$

PROBLEM (7): **Solve the following differential equations :**

(i) $\quad (6x^2 + 4y^3 + 12y)\,dx + (3x + 3xy^2)\,dy = 0$

(ii) $\quad (2x^2y^2 + e^xy)\,dx - (e^x + y^3)\,dy = 0$

(iii) $\quad (x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

(iv) $\quad y(2xy+1)\,dx + x(1 + 2xy - x^3y^3)\,dy = 0$

(v) $\quad (2y + 6xy^2)\,dx + (3x + 8x^2y)\,dy = 0$

SOLUTION: (i) $(6x^2 + 4y^3 + 12y)\,dx + (3x + 3xy^2)\,dy = 0$

Here $M = 6x^2 + 4y^3 + 12$ and $N = 3x + 3xy^2$

so that $\dfrac{\partial M}{\partial y} = 12y^2 + 12, \quad \dfrac{\partial N}{\partial x} = 3 + 3y^2$

since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore the given equation is not exact. However,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{12y^2 + 12 - 3 - 3y^2}{3x + 3xy^2} = \frac{9y^2 + 9}{3x + 3xy^2} = \frac{9(y^2+1)}{3x(1+y^2)} = \frac{3}{x} = f(x)$$

and $e^{\int f(x)\,dx} = e^{\int 3/x\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

is an integrating factor. Multiplying the given equation by x^3, we get

$$(6x^5 + 4x^3y^3 + 12x^3y)\,dx + (3x^4 + 3x^4y^2)\,dy = 0 \qquad (1)$$

from which $M = 6x^5 + 4x^3y^3 + 12x^3y$ and $N = 3x^4 + 3x^4y^2$. Then

$$\frac{\partial M}{\partial y} = 12x^3y^2 + 12x^3 = \frac{\partial N}{\partial x}.$$

Hence equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx + \int [\text{ terms in } N(x,y) \text{ without } x\,]\,dy = C$$

or $$\int (6x^5 + 4x^3y^3 + 12x^3y)\,dx + \int 0\,dy = C$$

or $$x^6 + x^4y^3 + 3x^4y = C$$

(ii) $$(2x^2y^2 + e^xy)\,dx - (e^x + y^3)\,dy = 0$$

Here $M = 2x^2y^2 + e^xy$, $N = -(e^x + y^3)$

so that $\dfrac{\partial M}{\partial y} = 4x^2y + e^x$ and $\dfrac{\partial N}{\partial x} = -e^x$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore the given equation is not exact. However

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-e^x - 4x^2y - e^x}{2x^2y^2 + e^xy} = \frac{-2e^x - 4x^2y}{2x^2y^2 + e^xy}$$

$$= \frac{-2(e^x + 2x^2y)}{y(e^x + 2x^2y)} = -\frac{2}{y} = g(y)$$

and $e^{\int g(y)\,dy} = e^{-\int (2/y)\,dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \dfrac{1}{y^2}$

is an integrating factor. Multiplying the given equation by $\dfrac{1}{y^2}$, we get

$$\left(2x^2 + \frac{e^x}{y}\right)dx - \left(\frac{e^x}{y^2} + y\right)dy = 0 \qquad (1)$$

from which $M = 2x^2 + \dfrac{e^x}{y}$ and $N = -\dfrac{e^x}{y^2} + y$. Then

$$\frac{\partial M}{\partial y} = -\frac{e^x}{y^2} = \frac{\partial N}{\partial x}$$

Hence equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx + \int [\text{ terms in } N(x,y) \text{ without } x\,]\,dy = C$$

or $$\int \left(2x^2 + \frac{e^x}{y}\right)dx - \int y\,dy = C$$

or $$\frac{2}{3}x^3 + \frac{e^x}{y} - \frac{y^2}{2} = C$$

(iii) $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0$

The equation is homogeneous since it can be written as

$$\frac{dy}{dx}=\frac{x^2y-2xy^2}{x^3-3x^2y}=\frac{\frac{y}{x}-\frac{2y^2}{x^2}}{1-\frac{3y}{x}}=f\left(\frac{y}{x}\right)$$

Also $\quad Mx+Ny=x^3y-2x^2y^2-x^3y+3x^2y^2=x^2y^2\neq 0$

and so $\frac{1}{x^2y^2}$ is an integrating factor of the given differential equation .

Multiplying the given equation by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0 \qquad (1)$$

Here $\quad M=\frac{1}{y}-\frac{2}{x}$ and $N=-\frac{x}{y^2}+\frac{3}{y}$

so that $\frac{\partial M}{\partial y}=-\frac{1}{y^2}=\frac{\partial N}{\partial x}$

Hence equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx+\int[\text{terms in } N(x,y) \text{ with out } x]\,dy=C$$

or $$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=C$$

or $$\frac{x}{y}-2\ln x+3\ln y=C$$

(iv) $y(2xy+1)\,dx+x(1+2xy-x^3y^3)\,dy=0$

The equation is of the form $yf(xy)\,dx+xg(xy)\,dy=0$

Here $M=y(2xy+1)$, $N=x(1+2xy-x^3y^3)$

and $\quad Mx-Ny=2x^2y^2+xy-xy-2x^2y^2+x^4y^4=x^4y^4\neq 0$

so $\quad \frac{1}{Mx-Ny}=\frac{1}{x^4y^4}$ is an integrating factor . Multiplying the given equation by $\frac{1}{x^4y^4}$, we get

$$\frac{2xy^2+y}{x^4y^4}dx+\frac{x+2x^2y^2-x^4y^3}{x^4y^4}dy=0$$

or $$\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\left(\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}\right)dy=0 \qquad (1)$$

Now $\quad M=\frac{2}{x^3y^2}+\frac{1}{x^4y^3}$, $\quad N=\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}$

so that $\frac{\partial M}{\partial y}=-\frac{4}{x^3y^3}-\frac{3}{x^4y^4}=\frac{\partial N}{\partial x}$

Thus equation (1) is exact and its general solution is given by the formula

$$\int M(x,y)\,dx + \int [\text{ terms in } N(x,y) \text{ without } x\,]\,dy = 0$$

or $$\int\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx + \int\left(-\frac{1}{y}\right)dy = C$$

or $$-\frac{1}{x^2y^2}-\frac{1}{3x^3y^3}-\ln y = C_1$$

or $$\frac{1}{x^2y^2}+\frac{1}{3x^3y^3}+\ln y = C \quad \text{where} \quad C = -C_1$$

(v) $$(2y+6xy^2)\,dx+(3x+8x^2y)\,dy = 0$$

Here $M = 2y+6xy^2, \quad N = 3x+8x^2y$

and $$\frac{\partial M}{\partial y} = 2+12xy, \quad \frac{\partial N}{\partial x} = 3+16xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore the equation is not exact. The given equation can be written

$$(6xy^2dx+8x^2ydy)+(2ydxdx+3xdy) = 0$$

or $$xy(6ydx+8xdy)+(2ydx+3xdy) = 0$$

which is of the form

$$x^ay^b(mydx+nxdy)+x^cy^d(pydx+qxdy) = 0$$

Let $x^\alpha y^\beta$ be an integrating factor. Multiplying the given equation by $x^\alpha y^\beta$, we get

$$(2x^\alpha y^{\beta+1}+6x^{\alpha+1}y^{\beta+2})\,dx+(3x^{\alpha+1}y^\beta+8x^{\alpha+2}y^{\beta+1})\,dy = 0 \quad (1)$$

Equation (1) will be exact if

$$\frac{\partial}{\partial y}(2x^\alpha y^{\beta+1}+6x^{\alpha+1}y^{\beta+2}) = \frac{\partial}{\partial x}(3x^{\alpha+1}y^\beta+8x^{\alpha+2}y^{\beta+1})$$

or $$2(\beta+1)x^\alpha y^\beta+6(\beta+2)x^{\alpha+1}y^{\beta+1} = 3(\alpha+1)x^\alpha y^\beta+8(\alpha+2)x^{\alpha+1}y^{\beta+1}$$

or $$[2(\beta+1)-3(\alpha+1)]x^\alpha y^\beta+[6(\beta+2)-8(\alpha+2)]x^{\alpha+1}y^{\beta+1} = 0$$

This implies $2(\beta+1)-3(\alpha+1) = 0$

or $-3\alpha+2\beta = 1$

and $6(\beta+2)-8(\alpha+2) = 0$

or $-8\alpha+6\beta = 4$

Solving these equations, we get $\alpha = 1, \quad \beta = 2$

Thus xy^2 is an integrating factor.

Multiplying the given equation by xy^2, we get

$$(2xy^3+6x^2y^4)\,dx+(3x^2y^2+8x^3y^3)\,dy = 0 \quad (2)$$

Now since $\frac{\partial M}{\partial y} = 6xy^2 + 24x^2y^3 = \frac{\partial N}{\partial x}$

Therefore equation (2) is exact. Thus its solution is given by

$$\int (2xy^3 + 6x^2y^4)\,dx + \int 0\,dy = C$$

$$x^2y^3 + 2x^3y^4 = C$$

LINEAR DIFFERENTIAL EQUATIONS

PROBLEM (8): **Solve the following differential equations :**

(i) $(x-1)^3 \frac{dy}{dx} + 4(x-1)^2 y = x+1$

(ii) $y \ln y\,dx + (x - \ln y)\,dy = 0$

SOLUTION: (i) $(x-1)^3 \frac{dy}{dx} + 4(x-1)^2 y = x+1$

Write the equation as $\frac{dy}{dx} + \frac{4}{x-1}y = \frac{x+1}{(x-1)^3}$

Here $P(x) = \frac{4}{x-1}$, $Q(x) = \frac{x+1}{(x-1)^3}$, therefore the integrating factor is

$$\text{I.F.} = e^{\int 4/(x-1)\,dx} = e^{4\ln(x-1)} = e^{\ln(x-1)^4} = (x-1)^4$$

Hence the general solution of given differential equation is

$$y(\text{I.F.}) = \int Q(\text{I.F})\,dx + C$$

or $$y(x-1)^4 = \int \frac{x+1}{(x-1)^3}(x-1)^4\,dx + C$$

or $$y(x-1)^4 = \int (x+1)(x-1)\,dx + C$$

or $$y(x-1)^4 = \int (x^2-1)\,dx + C$$

or $$y(x-1)^4 = \frac{x^3}{3} - x + C$$

(ii) $y \ln y\,dx + (x - \ln y)\,dy = 0$

Treating x as dependent variable, write the equation as

$$\frac{dx}{dy} + \frac{1}{y\ln y}x = \frac{1}{y}$$

Here $P(y) = \frac{1}{y\ln y}$, $Q = \frac{1}{y}$, therefore the integrating factor is

$$\text{I.F.} = e^{\int P(y)\,dy} = e^{\int 1/(y\ln y)\,dy}$$

$$= e^{\int (1/y)/\ln y\,dy} = e^{\ln \ln y} = \ln y$$

Hence the general solution of the given differential equation is

$$x\,.\,(\text{I.F.}) = \int Q(y)(\text{I.F.})\,d y + C_1$$

or $$x(\ln y) = \int \frac{1}{y}.\ln y\,d y + C = \frac{1}{2}(\ln y)^2 + C_1$$

or $$2 x \ln y = \ln^2 y + C \qquad (\text{where } C = 2 C_1)$$

PROBLEM (9): **Solve the following initial – value problems :**

(i) $$\frac{d y}{d x} + 4 x^3 y = x^3; \quad y(0) = -1$$

(ii) $$\frac{d y}{d x} + 2 y \cot x = \operatorname{cosec} x; \quad y\left(\frac{\pi}{2}\right) = 1$$

SOLUTION: (i) $$\frac{d y}{d x} + 4 x^3 y = x^3; \quad y(0) = -1$$

Here $P(x) = 4x^3$, $Q(x) = x^3$. The integrating factor is

$$\text{I.F.} = e^{\int P(x) d x} = e^{\int 4 x^3 d x} = e^{x^4}$$

The general solution of the given differential equation is

$$y\,.\,(\text{I.F.}) = \int Q(x).(\text{I.F.})\,d x + C$$

or $$y(e^{x^4}) = \int x^3 . e^{x^4} d x + C$$

or $$y e^{x^4} = \frac{1}{4}\int e^{x^4} 4 x^3 d x + C = \frac{1}{4} e^{x^4} + C$$

or $$y = \frac{1}{4} + C e^{-x^4}$$

Using the initial condition $y(0) = -1$, we get $-1 = \frac{1}{4} + C$ or $C = -\frac{5}{4}$

Thus the solution of the initial – value problem is

$$y = \frac{1}{4} - \frac{5}{4} e^{-x^4} = \frac{1}{4}(1 - 5 e^{-x^4})$$

(ii) $$\frac{d y}{d x} + 2 y \cot x = \operatorname{cosec} x; \quad y\left(\frac{\pi}{2}\right) = 1$$

Here $P(x) = 2 \cot x$, $Q(x) = \operatorname{cosec} x$

The integrating factor is

$$\text{I.F.} = e^{\int P(x) d x} = e^{\int 2 \cot x\, d x}$$

$$= e^{\int 2 \cos x / \sin x\, d x} = e^{2 \ln \sin x} = e^{\ln (\sin x)^2} = \sin^2 x$$

The general solution of the given differential equation is

$$y(\text{I.F.}) = \int Q(x).(\text{I.F.})\,dx + C$$

or $$y \sin^2 x = \int \operatorname{cosec} x \,.\, \sin^2 x \,dx + C$$

or $$y \sin^2 x = \int \sin x \,dx + C$$

or $$y \sin^2 x = -\cos x + C$$

Using the initial condition $y\left(\frac{\pi}{2}\right) = 1$, we get

$$1 = 0 + C \quad \text{or} \quad C = 1$$

Thus the solution of the initial – value problem is

$$y \sin^2 x = -\cos x + 1$$

or $$y = (1 - \cos x)\operatorname{cosec}^2 x$$

PROBLEM (10): **Solve the following differential equations :**

(i) $x\dfrac{dy}{dx} + y = y^2 \ln x$ (ii) $2\dfrac{dx}{dy} - \dfrac{x}{y} + x^3 \cos y = 0$

SOLUTION: (i) $x\dfrac{dy}{dx} + y = y^2 \ln x$

Write the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = y^2 \frac{\ln x}{x} \qquad (1)$$

which is a Bernoulli's equation with

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{\ln x}{x}, \quad \text{and } n = 2$$

Dividing equation (1) by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{\ln x}{x} \qquad (2)$$

Let $\dfrac{1}{y} = z$, then $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ or $\dfrac{1}{y^2}\dfrac{dy}{dx} = -\dfrac{dz}{dx}$

Thus equation (2) becomes

$$-\frac{dz}{dx} + \frac{1}{x}z = \frac{\ln x}{x}$$

or $$\frac{dz}{dx} - \frac{1}{x}z = -\frac{\ln x}{x} \qquad (3)$$

which is a linear differential equation in z, whose integrating factor is given by

$$\text{I.F.} = e^{\int -1/x\,dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$$

Hence the general solution of equation (3) is given by

$$z \cdot \frac{1}{x} = -\int \frac{\ln x}{x} \cdot \frac{1}{x}\, dx + C$$

$$= -\int \ln x \cdot \left(\frac{1}{x^2}\right) dx + C$$

$$= -\left[\ln x \cdot \left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right) \cdot \frac{1}{x}\, dx\right] + C$$

$$= -\left[-\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx\right] + C$$

$$= -\left[-\frac{\ln x}{x} - \frac{1}{x}\right] + C = \frac{\ln x}{x} + \frac{1}{x} + C$$

Since $z = \frac{1}{y}$, therefore the solution of the given differential equation is

$$\frac{1}{y} \cdot \frac{1}{x} = \frac{\ln x}{x} + \frac{1}{x} + C \quad \text{or} \quad \frac{1}{y} = \ln x + C x + 1$$

or $$y = \frac{1}{\ln x + C x + 1}$$

(ii) $$2\frac{dx}{dy} - \frac{x}{y} + x^3 \cos y = 0$$

Write the equation as

$$\frac{dx}{dy} - \frac{1}{2y} x = -\frac{1}{2} x^3 \cos y \qquad (1)$$

which is a Bernoulli's equation with x as dependent variable and

$$P(y) = -\frac{1}{2y}, \quad Q(y) = -\frac{1}{2} \cos y \text{ and } n = 3.$$

Dividing equation (1) by x^3, we get

$$\frac{1}{x^3} \frac{dx}{dy} - \frac{1}{2y} \cdot \frac{1}{x^2} = -\frac{1}{2} \cos y \qquad (2)$$

Let $-\frac{1}{x^2} = z$, then $\frac{2}{x^3} \frac{dx}{dy} = \frac{dz}{dy}$ or $\frac{1}{x^3} \frac{dx}{dy} = \frac{1}{2} \frac{dz}{dy}$

Thus equation (2) becomes

$$\frac{1}{2} \frac{dz}{dy} + \frac{1}{2y} z = -\frac{1}{2} \cos y$$

or $$\frac{dz}{dy} + \frac{1}{y} z = -\cos y \qquad (3)$$

which is a linear differential equation in z, whose integrating factor is given by

$$\text{I.F.} = e^{\int (1/y)\, dy} = e^{\ln y} = y$$

Hence the general solution of equation (3) is given by

$$z \cdot y = \int -\cos y \cdot (y)\, dy + C_1$$
$$= -\int y \cos y\, dy + C_1$$
$$= -\left[y \sin y - \int \sin y \cdot 1\, dy \right] + C_1 = -\left[y \sin y + \cos y \right] + C_1$$

Since $z = -\frac{1}{y^2}$, therefore the solution of the given differential equation is

$$-\frac{1}{x^2} \cdot y = -\left[y \sin y + \cos y \right] + C_1$$

or $$\frac{y}{x^2} = \cos y + y \sin y + C \quad \text{where } C_1 = -C_1$$

RICCATI EQUATIONS

PROBLEM (11): **Solve the following differential equations :**

(i) $\frac{dy}{dx} = y^2 - (2x-1)y + x^2 - x + 1$; **$Y(x) = x$ is one particular solution**

(ii) $\frac{dy}{dx} = 2e^{-x}y^2 + 3y - 4e^x$; **$Y(x) = e^x$ is one particular solution**

SOLUTION: (i) $\frac{dy}{dx} = y^2 - (2x-1)y + x^2 - x + 1$

Since $Y(x) = x$ is one particular solution, therefore define a new variable u by putting

$$y = x + \frac{1}{u}. \text{ Then } \frac{dy}{dx} = 1 - \frac{1}{u^2}\frac{du}{dx}$$

Substituting these into the Riccati equation, we get

$$1 - \frac{1}{u^2}\frac{du}{dx} = x^2 - (2x-1)x + x^2 - x + 1$$

$$1 - \frac{1}{u^2}\frac{du}{dx} = \left(x + \frac{1}{u}\right)^2 - (2x-1)\left(x + \frac{1}{u}\right) + x^2 - x + 1$$

$$= x^2 + \frac{2x}{u} + \frac{1}{u^2} - 2x^2 + x - \frac{2x}{u} + \frac{1}{u} + x^2 - x + 1$$

or $$-\frac{1}{u^2}\frac{du}{dx} = \frac{1}{u^2} + \frac{1}{u}$$

or $$\frac{du}{dx} + u = -1 \tag{1}$$

which is a linear equation. To solve this equation, we find

$$\text{I.F.} = e^{\int 1\, dx} = e^x$$

The general solution of equation (1) is

$$e^x \cdot u = \int e^x(-1)\, dx + C = -e^x + C$$

or $$u = -1 + Ce^{-x} \tag{2}$$

The general solution of the given equation is

$$y = 1+\frac{1}{u} = 1+\frac{1}{Ce^{-x}-1}$$

(ii) $\dfrac{dy}{dx} = 2e^{-x}y^2+3y-4e^x$

Since $Y(x) = e^x$ is one particular solution, therefore define a new variable u by putting

$$y = e^x+\frac{1}{u}. \text{ Then } \frac{dy}{dx} = e^x-\frac{1}{u^2}\frac{du}{dx}$$

Substituting these into the Riccati equation, we get

$$\begin{aligned} e^x-\frac{1}{u^2}\frac{du}{dx} &= 2e^{-x}\left(e^x+\frac{1}{u}\right)^2+3\left(e^x+\frac{1}{u}\right)-4e^x \\ &= 2e^x+\frac{4}{u}+2\frac{e^{-x}}{u^2}+3e^x+\frac{3}{u}-4e^x \end{aligned}$$

or $\quad -\dfrac{1}{u^2}\dfrac{du}{dx} = \dfrac{7}{u}+2\dfrac{e^{-x}}{u^2}$ (1)

or $\quad \dfrac{du}{dx}+7u = -2e^{-x}$

which is a linear differential equation. To solve this equation, we find

$$\text{I.F.} = e^{\int 7\,dx} = e^{7x}$$

The general solution of equation (1) is

$$\begin{aligned} e^{7x}.u &= \int e^{7x}(-2e^{-x})\,dx+C \\ &= \int -2e^{6x}\,dx+C = -\frac{1}{3}e^{6x}+C \end{aligned}$$

or $\quad u = -\dfrac{1}{3}e^{-x}+Ce^{-7x}$

The general solution of the given equation is

$$y = e^x+\frac{1}{u} = e^x+\frac{1}{Ce^{-7x}-\frac{1}{3}e^{-x}}$$

2.14 EXERCISE

SEPARABLE DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(1) $\frac{dy}{dx} = y(\cos x + \sin x)$

(2) $\frac{dy}{dx} = (1+x)(1+y^2)$

(3) $\frac{dy}{dx} = x^2y^2 - 2y^2 + x^2 - 2$

(4) $\frac{dy}{dx} = \frac{y \ln x}{x}$

(5) $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

(6) $\frac{dy}{dx} = \frac{e^y x}{e^y + x^2 e^y}$

(7) $\frac{dy}{dx} + y = y(\sin x + 1)$

(8) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

(9) $\frac{dy}{dx} = \frac{x^2 - xy - x + y}{xy - y^2}$

(10) $\frac{dy}{dx} = x^2 e^{3y}$

(11) $x^2 \frac{dy}{dx} = y(x-1)$

(12) $\frac{dy}{dx} = -\frac{(1+x)(1+y^2)}{(1+y)(1+x^2)}$

(13) $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

(14) $(e^y + 1)\cos x \, dx + e^y \sin x \, dy = 0$

(15) $3e^x \tan y \, dx + (1 - e^x)\sec^2 y \, dy = 0$

(16) $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

Solve the following initial – value problems :

(17) $\frac{dy}{dx} = y^2(1+x^2)$; $y(0) = -1$

(18) $y^3 \frac{dy}{dx} + x^3 = 0$; $y(0) = 1$

(19) $\frac{dy}{dx} = y(1 + \sin x)$; $y(0) = 1$

(20) $\frac{dy}{dx} + y = y(xe^{x^2} + 1)$; $y(0) = 1$

EQUATIONS REDUCIBLE TO SEPARABLE FORM

Solve the following differential equations :

(21) $\frac{dy}{dx} = (4x + y)^2$

(22) $(x+y)^2 \frac{dy}{dx} = a^2$

(23) $\frac{dy}{dx} = \cos(x+y)$

(24) $\frac{dy}{dx} = \frac{1}{x+y+1}$

HOMOGENEOUS DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(25) $x\frac{dy}{dx} = x + y$

(26) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

(27) $\frac{dy}{dx} = \frac{xy - y^2}{x^2}$

(28) $\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$

(29) $x\frac{dy}{dx} = y^2 + y$

(30) $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$

(31) $\frac{dy}{dx} = \frac{x^2+y^2}{2x^2}$ (32) $\frac{dy}{dx} = \frac{y-x}{x+y}$

(33) $x^2\frac{dy}{dx} = y^2+xy+x^2$ (34) $x^2y' = y^2+5xy+4x^2$

(35) $\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$ (36) $\frac{dy}{dx} = \frac{x^3+y^3}{3xy^2}$

(37) $\frac{dy}{dx} = \frac{x^3+y^3}{x^2y+xy^2}$ (38) $xy\frac{dy}{dx} = y^2+\frac{x^3}{x^2+1}$

(39) $(xy'-y)\cos\left(\frac{2y}{x}\right) = -3x^4$ (40) $xy'-y = x^2\tan\left(\frac{y}{x}\right)$

Solve the following initial – value problems :

(41) $x\frac{dy}{dx} = y+3x^4\cos^2\left(\frac{y}{x}\right);\quad y(1)=0$ (42) $xy' = y+x^2\sec\left(\frac{y}{x}\right);\quad y(1)=\pi$

(43) $xy\frac{dy}{dx} = 2y^2+4x^2;\quad y(2)=4$ (44) $yy' = x^3+\frac{y^2}{x};\quad y(2)=6$

EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

Solve the following differential equations:

(45) $\frac{dy}{dx} = \frac{y-x+1}{y+x-5}$ (46) $\frac{dy}{dx} = \frac{2x-y+1}{x-2y+1}$

(47) $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$ (48) $\frac{dy}{dx} = \frac{x-2y+3}{2x-4y+5}$

EXACT DIFFERENTIAL EQUATIONS

Solve the following differential equations:

(49) $(2x+e^y)\,dx+xe^y\,dy = 0$

(50) $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy = 0$

(51) $y\sin 2x\,dx-(1-y^2+\cos^2 x)\,dy = 0$

(52) $(\sin x\cos y+e^{2x})\,dx+(\cos x\sin y+\tan y)\,dy = 0$

EQUATIONS REDUCIBLE TO EXACT FORM

Solve the following differential equations:

RULE (1)

(53) $(4y-x^2)\,dx+x\,dy = 0$

(54) $(x^2+y^2+x)\,dx+xy\,dy = 0$

(55) $(x^2-2x+2y^2)\,dx+2xy\,dy = 0$

(56) $(5x^3+12x^2+6y^2)\,dx+6xy\,dy = 0$

RULE (2)

(57) $(xy^2+y)\,dx - x\,dy = 0$

(58) $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy = 0$

(59) $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy = 0$

(60) $(2xy^2+ye^x)\,dx-e^x\,dy = 0$

RULE (3)

(61) $(y^2-xy)\,dx+x^2\,dy = 0$

(62) $(x^2y-2xy^2)\,dx+(3x^2y-x^3)\,dy = 0$

(63) $(x^2-3xy+2y^2)\,dx+(3x^2-2xy)\,dy = 0$

(64) $x^2y\,dx-(x^3+y^3)\,dy = 0$

RULE (4)

(65) $(y-xy^2)\,dx-(x+x^2y)\,dy = 0$

(66) $y(xy+1)\,dx+x(1+xy+x^2y^2)\,dy = 0$

(67) $y(1-xy+x^2y^2)\,dx+x(x^2y^2-xy)\,dy = 0$

(68) $y(xy+2x^2y^2)\,dx+x(xy-x^2y^2)\,dy = 0$

RULE (5)

(69) $(3y\,dx+4x\,dy)+xy^2(-2y\,dx-3x\,dy) = 0$

(70) $x(4y\,dx+2x\,dy)+y^3(3y\,dx+5x\,dy) = 0$

(71) $(8y\,dx+8x\,dy)+x^2y^3(4y\,dx+5x\,dy) = 0$

(72) $(2y\,dx+3x\,dy)+2xy(3y\,dx+4x\,dy) = 0$

LINEAR DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(73) $2\dfrac{dy}{dx}-y = e^{x/2}$

(74) $x\dfrac{dy}{dx}-2y = x^3e^x$

(75) $(x-2)\dfrac{dy}{dx} = y+2(x-2)^3$

(76) $x\dfrac{dy}{dx}-2y = x^3\cos 4x$

(77) $x\ln x\dfrac{dy}{dx}+y = 2\ln x$

(78) $x\dfrac{dy}{dx}+3y = \dfrac{\sin x}{x^2}$

(79) $\cos^3 x\dfrac{dy}{dx}+y\cos x = \sin x$

(80) $(x+2y^3)\dfrac{dy}{dx} = y$

(Hint: Treat x as dependent variable)

Solve the following initial – value problems :

(81) $\dfrac{dy}{dx}+2xy = 4x; \quad y(0) = 3$

(82) $x^2\dfrac{dy}{dx}+xy = x^3+1; \quad y(1) = 0$

(83) $\dfrac{dy}{dx}+y = \sin x; \quad y(\pi) = 1$

(84) $\dfrac{dy}{dx}+y\cot x = 5e^{\cos x}; \quad y\left(\dfrac{\pi}{2}\right) = -4$

EQUATIONS REDUCIBLE TO LINEAR FORM

Solve the following differential equations :

(85) $\frac{dy}{dx}+\frac{y}{x}=x^3y^4$ (86) $\frac{dy}{dx}-\frac{3}{x}y=x^4y^{1/3}$

(87) $x^2y-x^3\frac{dy}{dx}=y^4\cos x$ (88) $2x\frac{dy}{dx}-y+2xy^5=0$

(89) $x\frac{dy}{dx}+y=x^3y^2$ (90) $\frac{dy}{dx}=\frac{2y}{x}-\frac{y^2}{x^2}$

(91) $\frac{dy}{dx}+y=y^2e^x$ (92) $\frac{dy}{dx}+2xy+xy^4=0$

RICCATI EQUATIONS

Solve the following differential equations , where $y(x)$ is a particular solution of the given equation :

(93) $\frac{dy}{dx}=4xy^2+(1-8x)y+4x-1;\quad Y(x)=1$

(94) $\frac{dy}{dx}=\frac{1}{2x}y^2-\frac{1}{x}y-\frac{4}{x};\quad Y(x)=4$

(95) $\frac{dy}{dx}=\frac{1}{x^2}y^2-\frac{1}{x}y+1;\quad Y(x)=x$

(96) $\frac{dy}{dx}=4x^2(y-x)^2+\frac{y}{x};\quad Y(x)=x$

(97) $\frac{dy}{dx}=3y^2-(1+6x)y+3x^2+x+1;\quad Y(x)=x$

(98) $\frac{dy}{dx}=\frac{1}{2}y^2-\frac{2}{x}y+\frac{1}{2x^2};\quad Y(x)=\frac{1}{x}$

(99) $\frac{dy}{dx}=xy^2-\frac{2}{x}y-\frac{1}{x^3};\quad Y(x)=\frac{1}{x^2}$

(100) $\frac{dy}{dx}=-e^{-x}y^2+y+e^x;\quad Y(x)=e^x$

CHAPTER 3

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

3.1 INTRODUCTION

In chapter (2), we have discussed differential equations of first order and first degree. In this chapter, we shall consider the differential equations of first order but of higher degree. First, we will give the different types of these equations and their methods of solution. Then the Clairaut differential equation and equations reducible to Clairaut form are given. Finally, the Lagrange differential equation and its solution will be discussed.

3.2 EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

We know that the general form of the first order differential equation is

$$F(x, y, y') = 0 \tag{1}$$

For convenience, we write $y' = \dfrac{dy}{dx} = p$, so that equation (1) takes the form

$$F(x, y, p) = 0 \tag{2}$$

The general form of the differential equation of first order and nth degree is

$$\left(\frac{dy}{dx}\right)^n + a_1\left(\frac{dy}{dx}\right)^{n-1} + a_2\left(\frac{dy}{dx}\right)^{n-2} + \dots\dots + a^{n-1}\frac{dy}{dx} + a_n = 0$$

or $$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots\dots + a_{n-1} p + a_n = 0 \tag{3}$$

where $n \geq 2$ and $a_1, a_2, \dots\dots, a_n$ are functions of x and y.

This equation in its most general form cannot be solved. We discuss the cases where a solution of this equation exists.

3.3 TYPES OF EQUATIONS

Differential equations of the first order but not of first degree can be of the following types :

(i) Equations solvable for p

(ii) Equations solvable for y

(iii) Equations solvable for x

In each case, the problem is reduced to that of solving one or more equations of the first order and first degree.

3.4 EQUATIONS SOLVABLE FOR p

In this case, since equation (3) is a polynomial of degree n in p, therefore it can be resolved into n linear factors of the first degree. Suppose the differential (3) can be written in the form

$$[p-f_1(x,y)][p-f_2(x,y)]\ldots\ldots[p-f_n(x,y)]=0$$

Equating each factor to zero, we get n differential equations of the first order and first degree,

i.e. $\quad p-f_1(x,y)=0, \quad p-f_2(x,y)=0, \ldots\ldots, p-f_n(x,y)=0$

or $\quad \dfrac{dy}{dx}=f_1(x,y), \quad \dfrac{dy}{dx}=f_2(x,y), \ldots\ldots, \dfrac{dy}{dx}=f_n(x,y)$

Each of these equations can be solved by the methods already known.

Let their solutions be

$$\phi_1(x,y,C_1)=0, \quad \phi_2(x,y,C_2)=0, \ldots\ldots, \phi_n(x,y,C_n)=0 \qquad (4)$$

Since the given equation is of first order, its general solution must have only one arbitrary constant, therefore we can take

$$C_1=C_2=\ldots\ldots=C_n=C\,(\text{say})$$

Therefore, the solution of equation (3) is the product of solutions in equation (4) and can be written in the form $\quad \phi_1(x,y,C)\phi_2(x,y,C)\ldots\ldots\phi_n(x,y,C)=0$

EXAMPLE (1): **Solve the following differential equations :**

(i) $\quad p^2+p-2=0$

(ii) $\quad p^2+px+py+xy=0$

SOLUTION: (i) $p^2+p-2=0$

or $\quad (p+2)(p-1)=0$

This implies that $p=1$ and $p=-2$

When $p=1$, then $\dfrac{dy}{dx}=1$, therefore

$$y=x+C_1 \qquad (1)$$

When $p=2$, then $\dfrac{dy}{dx}=-2$, therefore

$$y=-2x+C_2 \qquad (2)$$

Therefore the complete solution is given by

$$(y-x-C_1)(y+2x-C_2)=0$$

This equation contains two arbitrary constants. Since the equation is of first order, it must have only one arbitrary constant. Thus the complete solution is

$$(y-x-C)(y+2x-C)=0$$

(ii) $p^2 + px + py + xy = 0$

or $p(p+x) + y(p+x) = 0$

or $(p+x)(p+y) = 0$

This implies that $p = -x$ and $p = -y$

When $p = -x$, then $\dfrac{dy}{dx} = -x$, therefore

$$y = -\frac{x^2}{2} + C_1 \qquad (1)$$

When $p = -y$, then $\dfrac{dy}{dx} = -y$

or $\dfrac{dy}{y} = -dx$, therefore

$$\ln y = -x + C_2$$

or $x + \ln y - C_2 = 0$ (2)

Thus the solution is given by

$$\left(y + \frac{x^2}{2} - C_1\right)(x + \ln y - C_2) = 0$$

This equation contains two arbitrary constants. Since the order of the equation is 1, it must have only one arbitrary constant. Therefore, the complete solution is

$$\left(y + \frac{x^2}{2} - C\right)(x + \ln y - C) = 0$$

3.5 EQUATIONS SOLVABLE FOR y

If the differential equation $F(x, y, p) = 0$ is solvable for y, then we can write y explicitly in terms of x and p as

$$y = f(x, p) \qquad (1)$$

Differentiating this equation w.r.t. x, we get

$$p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx} = F\left(x, p, \frac{dp}{dx}\right) = 0$$

which is an equation of the first order and first degree in two variables x and p, which will give rise to a solution of the form

$$\phi(x, p, C) = 0 \qquad (2)$$

The elimination of p between equations (1) and (2) gives a relation between x, y and C, which is the required solution of the given equation. When the elimination of p between these equations is not possible, the values of x and y in terms of p from these equations can be found i.e. $x = \phi_1(p, C)$ and $y = \phi_2(p, C)$, where C is an arbitrary constant and p is a parameter. These two equation together will constitute the required solution.

EXAMPLE (2): **Solve the following differential equations :**

(i) $y = x + p^3$ (ii) $3p^5 - py + 1 = 0$

SOLUTION: (i) $y = x + p^3$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 + 3p^2 \frac{dp}{dx}$$

or $$p = 1 + 3p^2 \frac{dp}{dx} \qquad \left(\text{since } \frac{dy}{dx} = p\right)$$

or $$p - 1 = 3p^2 \frac{dp}{dx}$$

or $$\frac{dx}{dp} = \frac{3p^2}{p-1}$$

Separating the variables, we get

$$dx = \frac{3p^2}{p-1}\, dp$$

$$dx = \frac{3p^2 - 3 + 3}{p-1} = 3(p+1) + \frac{3}{p-1}$$

Integrating both sides, we get

$$x = 3\left(\frac{p^2}{2} + p\right) + 3\ln(p-1) + C$$

Thus $$x = \frac{3}{2}p^2 + 3p + 3\ln(p-1) + C \tag{1}$$

and $$y = x + p^3 = p^3 + \frac{3}{2}p^2 + 3p + 3\ln(p-1) + C \tag{2}$$

Equations (1) and (2) constitute the complete solution of the given equation.

(ii) $\mathbf{3p^5 - py + 1 = 0}$

Solving this equation for y, we get

$$y = 3p^4 + \frac{1}{p}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 12p^3 \frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$$

or $$p = 12p^3 \frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$$

$$= \left(12p^3 - \frac{1}{p^2}\right)\frac{dp}{dx}$$

Separating the variables, we get

$$dx = \left(12p^3 - \frac{1}{p^2}\right)\frac{dp}{p} = \left(12p^2 - \frac{1}{p^3}\right)dp$$

Integrating both sides, we get

$$x = 4p^3 - \frac{1}{2p^2} + C \tag{1}$$

and $$y = 3p^4 + \frac{1}{p} \tag{2}$$

Equations (1) and (2) constitute the required solution of the given problem.

3.6 EQUATIONS SOLVABLE FOR x

When the differential equation (2) is solvable for x, then we have

$$x = f(y, p) \tag{1}$$

Differentiating equation (1) w.r.t. y gives

$$\frac{1}{p} = \frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy} = F\left(y, p, \frac{dp}{dy}\right)$$

which is an equation of first order and first degree in two variables y and p and can be solved to obtain.

$$\phi(y, p, C) = 0 \text{ (say)} \tag{2}$$

Obtain the solution by eliminating p from equations (1) and (2), whenever possible.

If it is not possible to eliminate p, then relations (1) and (2) together constitute the required solution.

EXAMPLE (3): **Solve the following differential equations :**

(i) $x = 4p + 4p^3$ (ii) $p^2 - 2xp + 1 = 0$

SOLUTION: (i) $x = 4p + 4p^3$ (1)

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = 4\frac{dp}{dy} + 12p^2\frac{dp}{dy}$$

$$\frac{1}{p} = (4 + 12p^2)\frac{dp}{dy} \qquad \left(\text{since } \frac{dx}{dy} = \frac{1}{p}\right)$$

Separating the variables, we get

$$dy = (4p + 12p^3)\,dp$$

Integrating both sides, we get

$$y = 2p^2 + 3p^4 + C \tag{2}$$

Equations (1) and (2) constitute the complete solution of the given differential equation.

(ii) $p^2 - 2xp + 1 = 0$

Solving for x, we get $x = \frac{p^2+1}{2p} = \frac{1}{2}\left(p + \frac{1}{p}\right)$ (1)

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{dp}{dy} - \frac{1}{p^2}\frac{dp}{dy}\right)$$

or $\quad \dfrac{1}{p} = \dfrac{1}{2}\left(\dfrac{dp}{dy} - \dfrac{1}{p^2}\dfrac{dp}{dy}\right) \quad \left(\text{since } \dfrac{dx}{dy} = \dfrac{1}{p}\right)$

or $\quad \dfrac{1}{p} = \dfrac{1}{2}\left(1 - \dfrac{1}{p^2}\right)\dfrac{dp}{dy}$

Separating the variables, we get $dy = \dfrac{1}{2}\left(p - \dfrac{1}{p}\right)dp$

Integrating both sides, we get

$$y = \frac{1}{4}p^2 - \frac{1}{2}\ln p + C \tag{2}$$

Equations (1) and (2) constitute the complete solution of the given differential equation.

3.7 CLAIRAUT EQUATION

A differential equation of the form

$$y = px + f(p) \tag{1}$$

is called **Clairaut equation** named after the French mathematician A.C. Clairaut (1713 – 1765).

This equation is solvable for y. To solve it, we differentiate w.r.t. x to obtain

$$p = \frac{dy}{dx} = p + xp' + f'(p)p' = [x + f'(p)]p' + p$$

or $\quad [x + f'(p)]p' = 0$

Rejecting the factor $x + f'(p)$ which does not involve $\dfrac{dp}{dx}$, we have

$$p' = \frac{dp}{dx} = 0$$

or $\quad p = C = \text{constant} \tag{2}$

Eliminating p between equations (1) and (2), we get

$$y = Cx + f(c) \tag{3}$$

which is the required solution of the Clairaut equation.

Thus it appears that to solve Clairaut equation, it is necessary only to replace p by C.

NOTE: If we eliminate p between $x + f'(p) = 0$ and equation (1), we get a solution which does not contain any arbitrary constant, and hence, is not a particular solution of equation (3). Such a solution is called **singular solution**.

EXAMPLE (4): **Solve the following differential equations :**

(i) $\quad y = px + \sqrt{a^2p^2 + b^2} \qquad$ (ii) $\quad y = p(x - b) + \dfrac{a}{p}$

SOLUTION: (i) $y = px + \sqrt{a^2p^2 + b^2}$

This equation is in standard form of Clairaut equation. Its solution can be written by replacing p by C.

Thus $\quad y = Cx + \sqrt{a^2C^2 + b^2}$ is the solution of the given equation.

(ii) $y = p(x-b) + \frac{a}{p}$

This equation can be written as $y = px - pb + \frac{a}{p}$

which is the general form of Clairaut equation .

Its solution can be written by replacing p by C. Thus

$y = Cx - Cb + \frac{a}{C}$ is the solution of the given equation .

3.8 EQUATIONS REDUCIBLE TO CLAIRAUT FORM

The equation of the form

$$y^2 = pxy + f\left(\frac{py}{x}\right) \qquad (1)$$

can be reduced to Clairaut form by making the substitutions $u = x^2$, and $v = y^2$.

Put $u = x^2$ then $\frac{du}{dx} = 2x$, $v = y^2$ therefore $\frac{dv}{dx} = 2y\frac{dy}{dx} = 2yp$

Now $\frac{dv}{du} = \frac{dv}{dx}\frac{dx}{du} = \frac{py}{x}$

Then equation (1) becomes

$$v = u\frac{dv}{du} + f\left(\frac{dv}{du}\right)$$

which is obviously the Clairaut equation .

EXAMPLE (5): **Solve the differential equation :**

$$x^2y^2 = px^3y + p^2y^2 + pxy + 9x^2.$$

SOLUTION: The differential equation can be written as

$$y^2 = pxy + \frac{p^2y^2}{x^2} + \frac{py}{x} + 9 \qquad (1)$$

Let $x^2 = u$ and $y^2 = v$, then $2x\,dx = du$ and $2y\,dy = dv$

or $\frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\frac{dy}{dx} = \frac{py}{x}$

Substituting in equation (1) , we get

$$v = \frac{x}{y}\left(\frac{dv}{du}\right)xy + \left(\frac{dv}{du}\right)^2 + \frac{dv}{du} + 9$$

$$= x^2\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 + \frac{dv}{du} + 9$$

$$= u\frac{dv}{dx} + \left(\frac{dv}{du}\right)^2 + \frac{dv}{du} + 9 \qquad (2)$$

Let $\frac{dv}{dx} = P$, then equation (2) becomes

$$v = up + (P^2 + P + 9)$$

which is of Clairaut form. Thus its general solution is given by

$$v = uC + (C^2 + C + 9)$$

or $$y^2 = x^2 C + (C^2 + C + 9)$$
$$= C(x^2 + C + 1) + 9$$

3.9 EQUATIONS REDUCIBLE TO CLAIRAUT FORM BY TRANSFORMATION

EXAMPLE (6): **Solve the differential equation :**

$$\mathbf{y p^2 + x^3 p - x^2 y = 0.}$$

SOLUTION: $$y p^2 + x^3 p - x^2 y = 0 \qquad (1)$$

Let $x^2 = u$, and $y^2 = v$, then

$$2x\,dx = du \quad \text{and} \quad 2y\,dy = dv$$

$$\frac{dv}{du} = \frac{y}{x}\frac{dy}{dx} = \frac{yp}{x} = \frac{\sqrt{v}}{\sqrt{u}}p$$

Substituting in equation (1), we get

$$\sqrt{v}\left(\frac{u}{v}\right)\left(\frac{dv}{du}\right)^2 + u^{3/2}\left(\frac{\sqrt{u}}{\sqrt{v}}\right)\left(\frac{dv}{du}\right) - u\sqrt{v} = 0$$

or $$\frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2 + \frac{u^2}{\sqrt{v}}\frac{dv}{du} - u\sqrt{v}$$

or $$\left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} - v = 0$$

Letting $\frac{dv}{du} = P$, then the above equation becomes

$$P^2 + uP - v = 0$$

or $$v = uP + P^2$$

which is of Clairaut equation. Thus its general solution is

$$v = uC + C^2$$

or $$y^2 = x^2 C + C^2$$

EXAMPLE (7): **Solve the differential equation :**

$$\mathbf{y = 3xp + 6y^2p^2}$$

SOLUTION: Multiply the given equation by y^2, we get

$$y^3 = 3xy^2p + 6y^4p^2 \qquad (1)$$

Using the transformation $y^3 = u$, then $3y^2\frac{dy}{dx} = \frac{dx}{dx}$

or $\quad 3y^2 p = \frac{du}{dx} \qquad \left(\text{since } \frac{dy}{dx} = p\right)$

Thus equation (1) becomes

$$u = x\frac{du}{dx} + 6\left(\frac{1}{9}\right)\left(\frac{du}{dx}\right)^2$$

$$= x\frac{du}{dx} + \frac{2}{3}\left(\frac{du}{dx}\right)^2$$

which is of Clairaut form. Thus the solution is $\quad u = Cx + \frac{2}{3}C^2$

or $\quad y^3 = Cx + \frac{2}{3}C^2$

3.10 LAGRANGE EQUATION

A differential equation of the form

$$y = x f_1(p) + f_2(p) \tag{1}$$

is called **Lagrange equation** named after the French mathematician J.L. Lagrange (1736 – 1813).

If $f_1(p) = p$ and $f_2(p) = f(p)$, equation (1) reduces to Clairaut equation.

Equation (1) is solvable for y. To solve this equation, we differentiate w.r.t. x to obtain

$$p = \frac{dy}{dx} = x f_1'(p)\frac{dp}{dx} + f_1(p) + f_2'(p)\frac{dp}{dx}$$

or $\quad p - f_1(p) = \left[x f_1'(p) + f_2'(p)\right]\frac{dp}{dx}$

or $\quad \left[p - f_1(p)\right]\frac{dx}{dp} - f_1'(p)x = f_2'(p)$

$$\frac{dx}{dp} - \frac{f_1'(p)}{p - f_1(p)}x = \frac{f_2'(p)}{p - f_1(p)} \qquad [p \neq f_1(p)] \tag{2}$$

which is a first order linear equation with x as dependent variable and p as independent variable.

Note that since we have divided by $p - f_1(p)$, we must restrict $f_1(p)$ so that $f_1(p) \neq p$.

Equation (2) can be solved for x in the terms of p. Let the solution be

$$x = \phi(p, C)$$

With this value of x, equation (1) becomes

$$y = \phi(p, C) f_1(p) + f_2(p) \tag{3}$$

Eliminating p from equations (1) and (3), we get the required solution. If it is not possible to eliminate p, then the values of x and y in terms of p in equations (1) and (3) constitute the required solution of the Lagrange equation in parametric form.

EXAMPLE (8): **Solve the differential equation :**

$$y = 2\,x\,p + 3\,p\,.$$

SOLUTION: Comparing this equation with the general form of Lagrange equation , we have

$$f_1(p) = 2\,p \quad \text{and} \quad f_2(p) = 3\,p$$

Thus the corresponding linear equation

$$\frac{dx}{dp} - \frac{f_1'(p)}{p - f_1(p)}\,x = \frac{f_2'(p)}{p - f_1(p)}$$

takes the form

$$\frac{dx}{dp} + \frac{2}{p}\,x = -\frac{3}{p} \tag{1}$$

To solve equation (1) , we have

$$\text{I.F.} = e^{\int \frac{2}{x} dp} = e^{2 \ln p} = e^{\ln p^2} = p^2$$

Thus the solution of equation (1) is

$$(\text{I.F.})\,x = \int (\text{I.F.})\left(-\frac{3}{p}\right) dp + C$$

or $$p^2 x = \int p^2\left(-\frac{3}{p}\right) dp + C$$

$$= -3 \int p\; dp + C = -\frac{3}{2} p^2 + C$$

or $$x = -\frac{3}{2} + \frac{C}{p^2} \tag{2}$$

Substitute this value of x in the given differential equation , we get

$$y = 2\left[\left(-\frac{3}{2}\right) + \frac{C}{p^2}\right] p + 3\,p$$

$$= -3\,p + \frac{2\,C}{p} + 3\,p = \frac{2\,C}{p} \tag{3}$$

Equations (2) and (3) constitute the general solution of the given differential equation in parametric form .

Eliminating p between equations (2) and (3) , we get

$$y = \frac{2\,x + 3}{p} = (2\,x + 3)\,\frac{2\,C}{y}$$

or $$y^2 = \sqrt{2\,C}\,\sqrt{2\,x + 3}$$

or $$y = A\sqrt{2\,x + 3} \quad \left(\text{where } A = \sqrt{2\,C}\right)$$

3.11 SOLVED PROBLEMS

EQUATIONS SOLVABLE FOR p

PROBLEM (1): **Solve the following differential equations :**

(i) $p^2 + 2xp - 3x^2 = 0$ (ii) $x + yp^2 = p(1 + xy)$

(iii) $p^2 - 2p\cosh x + 1 = 0$ (iv) $(p - x)(p - y)(p + x + y) = 0$

SOLUTION: (i) $p^2 + 2xp - 3x^2 = 0$

Solving this equation for p , we get

$$p = \frac{-2x \pm \sqrt{4x^2 + 12x^2}}{2} = \frac{-2x \pm 4x}{2} = -x \pm 2x$$

This implies that $p = x$ and $p = -3x$

When $p = x$, then $\frac{dy}{dx} = x$ or $dy = x\,dx$, therefore

$$y = \frac{x^2}{2} + C_1 \qquad (1)$$

When $p = -3x$ then $\frac{dy}{dx} = -3x$ or $dy = -3x\,dx$, therefore

$$y = -3\frac{x^2}{2} + C_2 \qquad (2)$$

Thus the complete solution is given by

$$\left(y - \frac{x^2}{2} - C\right)\left(y + 3\frac{x^2}{2} - C\right) = 0$$

Since the given equation is of first order , therefore it will have only one arbitrary constant .

(ii) $x + yp^2 = p(1 + xy)$

or $yp^2 - pxy - p + x = 0$

or $yp(p - x) - 1(p - x) = 0$

or $(p - x)(yp - 1) = 0$

This implies that $p = x$, $p = \frac{1}{y}$

When $p = x$, then $\frac{dy}{dx} = x$ or $dy = x\,dx$, therefore

$$y = \frac{x^2}{2} + C_1 \qquad (1)$$

When $p = \frac{1}{y}$, then $\frac{dy}{dx} = \frac{1}{y}$ or $y\,dy = dx$, therefore

$$\frac{y^2}{2} = x + C_2 \qquad (2)$$

From equations (1) and (2), the complete solution is

$$\left(y-\frac{x^2}{2}-C_1\right)\left(x-\frac{y^2}{2}+C_2\right)=0$$

Since the given equation is of order 1, therefore, it will contain only one arbitrary constant.

Thus the complete solution is

$$\left(y-\frac{x^2}{2}-C\right)\left(x-\frac{y^2}{2}+C\right)=0$$

(iii) $\mathbf{p^2-2p\cosh x+1=0}$

Since $\cosh x=\frac{e^x+e^{-x}}{2}$, therefore

$$p^2-2p\left(\frac{e^x+e^{-x}}{2}\right)+1=0$$

$$p^2-p(e^x+e^{-x})+1=0$$

$$p^2-pe^x-pe^{-x}+1=0$$

$$p(p-e^x)-e^{-x}(p-e^x)=0$$

$$(p-e^x)(p-e^{-x})=0$$

This implies that $p=e^x$ and $p=e^{-x}$

When $p=e^x$, then $\frac{dy}{dx}=e^x$ or $dy=e^x\,dx$, therefore

$$y=e^x+C_1 \tag{1}$$

When $p=e^{-x}$, then $\frac{dy}{dx}=e^{-x}$ or $dy=e^{-x}\,dx$, therefore

$$y=-e^{-x}+C_2 \tag{2}$$

From equations (1) and (2), the solution is given by

$$(y-e^x-C_1)(y+e^{-x}-C_2)=0$$

Since the given equation is of order 1, therefore, it will contain only one arbitrary constant.

Thus the complete solution is

$$(y-e^x-C)(y+e^{-x}-C)=0$$

(iv) $(p-x)(p-y)(p+x+y)=0$

This implies that $p=x,\ p=y,\ p=-x-y$

When $p=x$, then $\frac{dy}{dx}=x$ or $dy=x\,dx$, therefore

$$y=\frac{x^2}{2}+C_1$$

or $$2y=x^2+C_1 \tag{1}$$

When $p = y$, then $\frac{dy}{dx} = y$ or $\frac{dy}{y} = dx$, therefore

$$\ln y = x + C_2 \tag{2}$$

When $p = -x - y$ therefore $\frac{dy}{dx} = -x - y$

or $\frac{dy}{dx} + y = -x$

This a first order linear ordinary differential equation , therefore

$$\text{I.F.} = e^{\int p\,dx} = e^{\int dx} = e^{x}$$

Thus the solution of this equation is given by

$$(\text{I.F.})\,y = \int (\text{I.F.})(-x)\,dx + C_3$$

or $$e^{x}.y = \int e^{x}(-x)\,dx + C_3 = -x e^{x} + e^{x} + C_3$$

or $$y = -x + 1 + C_3 e^{-x} \tag{3}$$

From equations (1), (2) and (3) the solution is given by

$$(2y - x^2 - C_1)(\ln y - x - C_2)(y + x - 1 - C_3 e^{-x}) = 0$$

Now this equation contains 3 arbitrary constants . Since the given equation is of order 1 . It will have only one arbitrary constant .

Therefore , the complete solution is given by

$$(2y - x^2 - C)(\ln y - x - C)(y + x - 1 - C e^{-x}) = 0$$

EQUATIONS SOLVABLE FOR y

PROBLEM (2): Solve the following differential equations :

(i) $y = p^2 x + p$ (ii) $2y + p^2 + 2p = 2x(p+1)$

(iii) $e^{p-y} = p^2 - 1$ (iv) $y = p\tan p + \ln\cos p$

SOLUTION: (i) $y = p^2 x + p$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = p^2 + 2xp\frac{dp}{dx} + \frac{dp}{dx}$$

$$p = p^2 + 2xp\frac{dp}{dx} + \frac{dp}{dx} \qquad \left(\text{since } \frac{dy}{dx} = p\right)$$

$$(p - p^2) = (2xp + 1)\frac{dp}{dx}$$

or $$\frac{dx}{dp} = \frac{2xp + 1}{p(1-p)}$$

or $\quad \frac{dx}{dp} = \frac{2x}{1-p} + \frac{1}{p(1-p)}$

or $\quad \frac{dx}{dp} - \frac{2x}{1-p} = \frac{1}{p(1-p)}$

This is a first order linear differential equation . To solve this equation , we have

$$\text{I.F.} = e^{\int P\,dp} = e^{\int -\frac{2}{1-p}dp} = e^{2\ln(1-p)} = (1-p)^2$$

Thus the solution of equation is given by

$$(\text{I.F})\,x = \int (\text{I.F.}) \frac{1}{p(1-p)}\,dp + C$$

or $\quad (1-p)^2 x = \int (1-p^2) \frac{1}{p(1-p)}\,dp + C = \int \frac{1-p}{p}\,dp + C$

or $\quad (1-p)^2 x = \ln p - p + C$

Thus $\quad x = \frac{\ln p - p + C}{(1-p)^2} \qquad (1)$

and $\quad y = p^2 x + p = p^2 \frac{(\ln p - p + C)}{(1-p)^2} + p$

Equations (1) and (2) constitute the complete solution of the given equation .

(ii) $\quad 2y + p^2 + 2p = 2x(p+1)$

$$y = xp + x - \frac{p^2}{2} - p$$

Thus $\quad \frac{dy}{dx} = p + x\frac{dp}{dx} + 1 - p\frac{dp}{dx} - \frac{dp}{dx}$

$$p = p + x\frac{dp}{dx} - p\frac{dp}{dx} - \frac{dp}{dx} + 1$$

or $\quad p\frac{dp}{dx} + \frac{dp}{dx} - x\frac{dp}{dx} = 1$

$$(p + 1 - x)\frac{dp}{dx} = 1$$

Thus $\quad \frac{dx}{dp} = p + 1 - x$

or $\quad \frac{dx}{dp} + x = 1 + p$

This is a first order linear differential equation with p as independent variable . To solve this equation, we have

$$\text{I.F.} = e^{\int dp} = e^p$$

Thus the solution is

$$(\text{I.F.})\,x = \int (\text{I.F.})(1+p)\,dp + C$$

$$e^{p}.x = \int e^{p}.(1+p)\,dp + C$$

$$= (1+p)e^{p} - e^{p} + C$$

or $$x = (1+p) - 1 + Ce^{-p}$$

$$= p + Ce^{-p} \qquad (1)$$

Now $$y = xp + x - \frac{p^2}{2} - p$$

$$= (p + Ce^{-p})p + (p + Ce^{-p}) - \frac{p^2}{2} - p$$

$$= p^2 + pCe^{-p} + p + Ce^{-p} - \frac{p^2}{2} - p$$

$$y = \frac{p^2}{2} + C(p+1)e^{-p} \qquad (2)$$

Equations (1) and (2) constitute the complete solution of the given equation .

(iii) $e^{p-y} = p^2 - 1$

Here $p - y = \ln(p^2 - 1)$

or $y = p - \ln(p^2 - 1)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{dp}{dx} - \frac{2p}{p^2-1}\frac{dp}{dx}$$

or $$p = \frac{dp}{dx} - \frac{2p}{p^2-1}\frac{dp}{dx} = \left(1 - \frac{2p}{p^2-1}\right)\frac{dp}{dx}$$

or $$\frac{dx}{dp} = \frac{1}{p} - \frac{2}{p^2-1} = \frac{1}{p} + \left[\frac{1}{p+1} - \frac{1}{p-1}\right]$$

or $$dx = \left(\frac{1}{p} + \frac{1}{p+1} - \frac{1}{p-1}\right)dp$$

Integrating , we get

$$x = \ln p + \ln(p+1) - \ln(p-1) + C \qquad (1)$$

and $$y = p - \ln(p^2 - 1) \qquad (2)$$

Equations (1) and (2) constitute the required solution of the given differential equation .

(iv) $y = p \tan p + \ln \cos p$

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = p \sec^2 p \frac{dp}{dx} + \tan p \frac{dp}{dx} - \frac{\sin p}{\cos p}\frac{dp}{dx}$$

or $$p = p \sec^2 p \frac{dp}{dx} + \tan p \frac{dp}{dx} - \tan p \frac{dp}{dx} \qquad \left(\text{since } \frac{dy}{dx} = p\right)$$

Thus $$p = p \sec^2 p \frac{dp}{dx}$$

or $$\frac{dx}{dp} = \sec^2 p$$

Separating the variables , we get

$$dx = \sec^2 p \ dp$$

Integrating both sides , we get

Now $x = \tan p + C$ (1)

and $y = p \tan p + \ln \cos p$ (2)

Equations (1) and (2) constitute the required solution of the given differential equation .

EQUATIONS SOLVABLE FOR x

PROBLEM (3): **Solve the following differential equations :**

(i) $p^3 + x = p(y+3)$ (ii) $y = 2px + y^2 p^3$

(iii) $p = \tan\left(x - \frac{p}{1+p^2}\right)$ (iv) $y^2 \ln y = xyp + p^2$

SOLUTION: (i) $p^3 + x = p(y+3)$

Solving the equation for x , we get

$$x = py + 3p - p^3 \qquad (1)$$

Differentiating w.r.t. y , we get

$$\frac{dx}{dy} = p + y\frac{dp}{dy} + 3\frac{dp}{dy} - 3p^2\frac{dp}{dy}$$

or $$\frac{1}{p} = p + (y + 3 - 3p^2)\frac{dp}{dy} \qquad \left(\text{since } \frac{dx}{dy} = \frac{1}{p}\right)$$

or $$\left(\frac{1-p^2}{p}\right)\frac{dy}{dp} = y + 3 - 3p^2 = y + 3(1-p^2)$$

or $$\frac{dy}{dp} + \frac{p}{p^2-1}y = 3p \qquad (2)$$

which is a linear differential equation . To solve this equation , we have

$$\text{I.F.} = e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\ln(p^2-1)} = e^{\ln\sqrt{p^2-1}} = \sqrt{p^2-1}$$

Thus the solution of equation (2) is

$$\sqrt{p^2-1}\,y = \int \sqrt{p^2-1}\,.\,3\,p\,d\,p + C = (p^2-1)^{3/2} + C$$

or $$y = p^2 - 1 + \frac{C}{\sqrt{p^2-1}} \tag{3}$$

From equation (1), we get

$$x = p\left(p^2 - 1 + \frac{C}{\sqrt{p^2-1}}\right) + 3\,p - p^3$$

$$= 2\,p + \frac{C\,p}{\sqrt{p^2-1}} \tag{4}$$

Equations (3) and (4) constitute the complete solution of the given differential equation in parametric form.

(ii) $$y = 2\,p\,x + y^2\,p^3 \tag{1}$$

Solving the equation for x, we get

$$2\,x = \frac{y}{p} - y^2\,p^2$$

Differentiating this equation w.r.t. y, we get

$$2\frac{d\,x}{d\,y} = \frac{1}{p} - \frac{y}{p^2}\frac{d\,p}{d\,y} - 2\,y^2\,p\frac{d\,p}{d\,y} - 2\,y\,p^2$$

or $$\frac{2}{p} - \frac{1}{p} + \frac{y}{p^2}\frac{d\,p}{d\,y} + 2\,y^2\,p\frac{d\,p}{d\,y} + 2\,y\,p^2 = 0 \qquad \left(\text{since } \frac{d\,x}{d\,y} = \frac{1}{p}\right)$$

or $$\frac{1}{p} + \frac{y}{p^2}\frac{d\,p}{d\,y} + 2\,y\,p\left(p + y\frac{d\,p}{d\,y}\right) = 0$$

or $$\frac{1}{p^2}\left(p + y\frac{d\,p}{d\,y}\right) + 2\,y\,p\left(p + y\frac{d\,p}{d\,y}\right) = 0$$

or $$\left(p + y\frac{d\,p}{d\,y}\right)\left(\frac{1}{p^2} + 2\,y\,p\right) = 0$$

The factor $\frac{1}{p^2} + 2\,y\,p$ is rejected since it does not involve $\frac{d\,p}{d\,y}$, therefore

$$p + y\frac{d\,p}{d\,y} = 0$$

Separating the variables, we get

$$\frac{d\,y}{y} + \frac{d\,p}{p} = 0$$

Integrating, we get

$$\ln y + \ln p = \ln C$$

or $$\ln y\,p = \ln C$$

or $\quad yp = C$

or $\quad p = \dfrac{C}{y}$

Substituting this value in equation (1), we get

$$y = 2\frac{C}{y}x + y^2\frac{C^3}{y^3} = \frac{2C}{y}x + \frac{C^3}{y}$$

or $\quad y^2 = 2Cx + C^3$

which is the general solution of the given differential equation.

(iii) $\quad p = \tan\left(x - \dfrac{p}{1+p^2}\right)$

Write the equation as

$$x = \tan^{-1}p + \frac{p}{1+p^2} \tag{1}$$

Differentiating both sides of equation (1) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1-p^2}{(1+p^2)^2}\frac{dp}{dy}$$

or $$\frac{1}{p} = \left[\frac{1}{1+p^2} + \frac{1-p^2}{(1+p^2)^2}\right]\frac{dp}{dy} \qquad \left(\text{since } \frac{dx}{dy} = \frac{1}{p}\right)$$

$$= \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

Separating the variables, we get

$$dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating both sides, we get

$$y = -\frac{1}{1+p^2} + C \tag{2}$$

Equations (1) and (2) constitute the complete solution of the given differential equation.

(iv) $\quad y^2 \ln y = xyp + p^2 \qquad (1)$

Solving the equation for x, we get

$$x = \frac{1}{p}y\ln y - \frac{p}{y}$$

Differentiating this equation w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{p}y \cdot \frac{1}{y} + \frac{1}{p}\ln y - \frac{1}{p^2}\frac{dp}{dy}y\ln y - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

or $$\frac{1}{p} = \frac{1}{p} + \frac{1}{p}\ln y - \frac{y\ln y}{p^2}\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

or $\quad \frac{1}{p}\ln y+\frac{p}{y^2}-\frac{y\ln y}{p^2}\frac{dp}{dy}-\frac{1}{y}\frac{dp}{dy}=0$

or $\quad \frac{p}{y^2}\left(\frac{y^2\ln y}{p^2}+1\right)-\frac{1}{y}\left(\frac{y^2\ln y}{p^2}+1\right)\frac{dp}{dy}=0$

or $\quad \left(\frac{y^2\ln y}{p^2}+1\right)\left(\frac{p}{y^2}-\frac{1}{y}\frac{dp}{dy}\right)=0$

Rejecting the first factor since it does not involve $\frac{dp}{dy}$, we get

$$\frac{p}{y^2}-\frac{1}{y}\frac{dp}{dy}=0$$

or $\quad p-y\frac{dp}{dy}=0$

or $\quad \frac{dp}{p}=\frac{dy}{y}$

Integrating , we get

$$\ln p=\ln y+\ln C$$

or $\quad p=Cy$

Substituting this value of p in equation (1) , we get

$$y^2\ln y=xy(Cy)+C^2y^2$$

or $\quad y^2\ln y=Cxy^2+C^2y^2$

or $\quad \ln y=Cx+C^2$

is the general solution of the given differential equation .

CLAIRAUT EQUATION

PROBLEM (4): Solve the following differential equations :

(i) $\sin px\cos y=\cos px\sin y+p$

(ii) $(y-px)^2=1+p^2$

(iii) $p^2(x^2-a^2)-2pxy+y^2-b^2=0$

(iv) $p^2x(x-2)+p(2y-2xy-x+2)+y^2+y=0$

SOLUTION: (i) $\sin px\cos y=\cos px\sin y+p$

or $\quad \sin px\cos y-\cos px\sin y=p$

or $\quad \sin(px-y)=p$

or $\quad px-y=\sin^{-1}p$

or $\quad y=px-\sin^{-1}p$

which is of Clairaut equation . Hence its general solution is

$$y=Cx-\sin^{-1}C$$

(ii) $(y-px)^2 = 1+p^2$

This equation can be written as

$$y = px \pm \sqrt{1+p^2}$$

Both the component equations are of Clairaut form. Thus the general solution of this equation is

$$y = Cx \pm \sqrt{1+C^2}$$

or $(y-Cx)^2 = 1+C^2$

(iii) $p^2(x^2-a^2)-2pxy+y^2-b^2 = 0$

or $y^2-2pxy+p^2x^2 = a^2p^2+b^2$

or $(y-px)^2 = a^2p^2+b^2$

or $y-px = \pm\sqrt{a^2p^2+b^2}$

or $y = px \pm \sqrt{a^2p^2+b^2}$

Both the components equations are of Clairaut form. Thus the general solution of this equation is

$$y = Cx \pm \sqrt{a^2c^2+b^2}$$

or $(y-Cx)^2 = a^2c^2+b^2$

(iv) $p^2x(x-2)+p(2y-2xy-x+2)+y^2+y = 0$

or $(y^2-2pxy+p^2x^2)+2p(y-px)+(y-px)+2p = 0$

or $(y-px)^2+(2p+1)(y-px)+2p = 0$

or $(y-px+2p)(y-px+1) = 0$

Both the component equations are of Clairaut form. Thus the general solution is

$$(y-Cx+2C)(y-Cx+1) = 0$$

EQUATIONS REDUCIBLE TO CLAIRAUT FORM

PROBLEM (5): **Solve the differential equation :**

$$x^2(y-px) = yp^2$$

by reducing it to Clairaut form.

SOLUTION: $x^2(y-px) = yp^2$ (1)

Multiplying equation (1) by y and re-arranging

$$x^2y^2-px^3y = y^2p^2$$

or $y^2 = pxy+\left(\frac{yp}{x}\right)^2$

which is of the form $y^2 = pxy+f\left(\frac{yp}{x}\right)$

Let $x^2 = u$ and $y^2 = v$, then $2x\,dx = du$ and $2y\,dy = dv$

so that $\frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}\frac{dy}{dx} = \frac{\sqrt{v}}{\sqrt{u}}p$

Substituting in equation (1), we get

$$u\left(\sqrt{v}-\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{u}\right)=\sqrt{v}\frac{u}{v}\left(\frac{dv}{du}\right)^2$$

or $$u\sqrt{v}-\frac{u^2}{\sqrt{v}}\frac{dv}{du}=\frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2$$

or $$v-u\frac{dv}{du}=\left(\frac{dv}{du}\right)^2$$

or $$v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$$

Letting $\frac{dv}{du}=P$, then the above equation becomes

$$v=uP+P^2$$

which is of Clairaut form. Thus the solution is given by

$$v=uC+C^2$$

or $$y^2=Cx^2+C^2$$

PROBLEM (6): **Solve the differential equation :**

$$(px-y)(py+x)=2p$$

SOLUTION: Let $x^2=u$ and $y^2=v$, then $2x\,dx=du$ and $2y\,dy=dv$

or $$\frac{dv}{du}=\frac{y}{x}\frac{dy}{dx}=\frac{\sqrt{v}}{\sqrt{u}}p$$

Substituting in the given equation, we get

$$\left(\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{u}-\sqrt{v}\right)\left(\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{v}+\sqrt{u}\right)=2\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$$

$$\left(u\frac{dv}{du}-v\right)\left(\sqrt{u}\frac{dv}{du}+\sqrt{u}\right)=2\sqrt{u}\frac{dv}{du}$$

or $$\left(u\frac{dv}{du}-v\right)\left(\frac{dv}{dx}+1\right)=2\frac{dv}{dx}$$

or $$u\left(\frac{dv}{du}\right)^2+(u-v-2)\frac{dv}{dx}-v=0$$

Letting $\frac{dv}{du}=P$, then

$$uP^2+(u-v-2)P-v=0$$

or $$v=uP^2+(u-v-2)P$$

or $$(1+P)v=uP^2+uP-2P=uP(1+P)-2P$$

or $$v=uP-\frac{2P}{1+P}$$

which is of Clairaut form and has the solution

$$v = uC - \frac{2C}{1+C}$$

Replacing u by x^2 and v by y^2, we get

$$y^2 = Cx^2 - \frac{2C}{1+C}$$

as the required solution .

PROBLEM (7): **Solve the following differential equation :**

$$\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0$$

by reducing it to Clairaut form using the transformation $\sin x = u$, $\sin y = v$.

SOLUTION: $$\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0 \qquad (1)$$

Since $\sin x = u$, $\sin y = v$, therefore

$$\cos x\, dx = du \quad \text{and} \quad \cos y\, dy = dv$$

Then $$\frac{dv}{du} = \frac{\cos y\, dy}{\cos x\, dx} = \frac{\cos y}{\cos x}\frac{dy}{dx} = \frac{\cos y}{\cos x}p$$

Thus equation (1) becomes

$$\cos^2 y \frac{\cos^2 x}{\cos^2 y}\left(\frac{dv}{du}\right)^2 + u \cos x \cos y \frac{\cos x}{\cos y}\frac{dv}{du} - v\cos^2 x = 0$$

$$\cos^2 x\left(\frac{dv}{du}\right)^2 + u\cos^2 x\left(\frac{dv}{du}\right) - v\cos^2 x = 0$$

or $$\left(\frac{dv}{du}\right)^2 + u\frac{dv}{du} - v = 0$$

or $$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

Letting $\frac{dv}{du} = P$, the above equation becomes

$$v = uP + P^2$$

which is of Clairaut form . The general solution is given by

$$v = uC + C^2$$

or $$\sin y = C\sin x + C^2$$

PROBLEM (8): **Solve the following differential equation :**

$$e^{4x}(p-1) + e^{2y}p^2 = 0$$

by reducing it to Clairaut form using the transformation

$e^{2x} = u$ and $e^{2y} = v$.

SOLUTION: $$e^{4x}(p-1)+e^{2y}p^2 = 0 \qquad (1)$$

Since $e^{2x} = u$ and $e^{2y} = v$, therefore

$$2e^{2x}dx = du \quad\text{and}\quad 2e^{2y}dy = dv$$

Then $$\frac{dv}{du} = \frac{e^{2y}}{2e^{2x}}\frac{dy}{dx} = \frac{v}{u}p$$

Substituting these in equation (1), we get

$$u^2\left(\frac{u}{v}\frac{dv}{du}-1\right)+v\frac{u^2}{v^2}\left(\frac{dv}{du}\right)^2 = 0$$

or $$\frac{u^3}{v}\frac{dv}{du}-u^2+\frac{u^2}{v}\left(\frac{dv}{du}\right)^2 = 0$$

$$\frac{u}{v}\frac{dv}{du}-1+\frac{1}{v}\left(\frac{dv}{du}\right)^2 = 0$$

or $$u\frac{dv}{du}-v+\left(\frac{dv}{du}\right)^2 = 0$$

or $$v = u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$$

Letting $\frac{dv}{du} = P$, the above equation becomes

$$v = uP+P^2$$

which is of Clairaut form. The general solution is given by

$$v = uC+C^2$$

or $$e^{2y} = Ce^{2x}+C^2$$

LAGRANGE EQUATION

PROBLEM (9): **Solve the following differential equation :**

$$y = xp^2+p^4.$$

SOLUTION: Comparing this equation with the general form of Lagrange equation, we have

$$f_1(p) = p^2 \quad\text{and}\quad f_2(p) = p^4$$

Thus the corresponding linear equation

$$\frac{dx}{dp}-\frac{f_1'(p)}{p-f_1(p)}x = \frac{f_2'(p)}{p-f_1(p)}$$

takes the form

$$\frac{dx}{dp}-\frac{2p}{p-p^2}x = \frac{4p^3}{p-p^2}$$

$$\frac{dx}{dp}+\frac{2}{p-1}x=\frac{4p^2}{1-p} \qquad (1)$$

which is a linear differential equation. To solve equation (1), we have

$$\text{I.F.}=e^{\int\frac{2}{p-1}dp}=e^{2\ln(p-1)}=(p-1)^2$$

Thus the solution of equation (1) is

$$\begin{aligned}(p-1)^2x &= \int (p-1)^2\frac{4p^2}{1-p}\,dp+C\\ &= \int (1-p)4p^2\,dp+C\\ &= \int (4p^2-4p^3)\,dp+C\\ &= \frac{4}{3}p^3-p^4+C\end{aligned}$$

$$x=\left(\frac{4}{3}p^3-p^4\right)(p-1)^{-2}+C(p-1)^{-2} \qquad (2)$$

Substituting this value of x in the given differential equation, we get

$$\begin{aligned}y &= p^2\left[\left(\frac{4}{3}p^3-p^4\right)(p-1)^{-2}+C(p-1)^{-2}\right]+p^4\\ &= Cp^2(p-1)^{-2}+p^5\left(\frac{4}{3}-p\right)(p-1)^{-2}+p^4 \qquad (3)\end{aligned}$$

Equations (2) and (3) constitute the general solution of the given differential equation in parametric form.

3.12 EXERCISE

EQUATIONS SOLVABLE FOR p

Solve the following differential equations :

(1) $p^2+p-6=0$

(2) $(xp+x+y)(yp+x)=0$

(3) $p(p-1)(p-x)(p-2y)=0$

(4) $p^2=x^5$

(5) $x^2p^2+xyp-6y^2=0$

(6) $xp^2+(y-x)p-y=0$

(7) $xyp^2-(x^2+y^2)p+xy=0$

(8) $(p-xy)(p-x^2)(p-y^2)=0$

EQUATIONS SOLVABLE FOR y

Solve the following differential equations :

(9) $p^2-xy+y=0$

(10) $xp^2-yp-y=0$

(11) $xp^2=2y(p+2)$

(12) $xp^2+2px-y=0$

(13) $y=p\sin p+\cos p$

(14) $p^3+p=e^y$

(15) $y=(2+p)x+p^2$

(16) $xp^2-2yp+4x=0$

EQUATIONS SOLVABLE FOR x

Solve the following differential equations :

(17) $x=y+p^2$

(18) $x=y+a\ln p$

(19) $p^2-xp+1=0$

(20) $p^2x=2yp-3$

(21) $y=2xp+4p^2y$

(22) $y=2xp+y^2p^3$

(23) $yp^2-2xp+y=0$

(24) $p^3-4xyp+8y^2=0$

CLAIRAUT EQUATION

Solve the following differential equations :

(25) $y=px+p^2$

(26) $y=px+\cos p$

(27) $y=px+a\tan^{-1}p$

(28) $y=px+e^p$

(29) $(y-px)(p-1)=p$

(30) $xp^2-yp+a=0$

(31) $y^2+x^2p^2-2xyp=\dfrac{4}{p^2}$

(32) $(x-a)p^2+(x-y)p-y=0$

EQUATIONS REDUCIBLE TO CLAIRAUT FORM

PROBLEM (33): Solve the following differential equation :

$$xyp^2+(x^2-y^2-1)p-xy=0$$

by reducing it to Clairaut form using the transformation $x^2=u$ and $y^2=v$.

PROBLEM (34): Solve the following differential equation :

$$y = 2px + 4yp^2$$

by reducing it to Clairaut form using the transformation $x = u$ and $y^2 = v$.

PROBLEM (35): Solve the following differential equation :

$$y = 2px + y^2p^3$$

by reducing it to Clairaut form using the transformation $x = u$ and $y^2 = v$.

PROBLEM (36): Solve the following differential equation :

$$e^{3x}(p-1) + e^{2y}p^3 = 0$$

by reducing it to Clairaut form using the transformation $e^x = u$ and $e^y = v$.

PROBLEM (37): Solve the following differential equation :

$$y^2(y - xp) = x^4p^2$$

by reducing it to Clairaut form using the transformation $x = \frac{1}{u}$ and $y^2 = \frac{1}{v}$.

PROBLEM (38): Solve the following differential equation :

$$(y + xp)^2 = x^2p$$

by reducing it to Clairaut form using the transformation $xy = u$.

LAGRANGE EQUATIONS

Solve the following differential equations :

(39) $y = -xp + p^2$

(40) $y = 2xp - p^2$

(41) $y = x(1+p) + p^2$

(42) $y = 3xp + 4p^3$

CHAPTER 4

SECOND AND HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

4.1 INTRODUCTION

The ordinary differential equations may be divided into two large classes , namely **linear** equations and **non – linear** equations . Non – linear equations (of second and higher order) are difficult in general , while linear equations are much simpler because there are standard methods for solving many practically important linear differential equations .

So far , we have discussed the first order linear differential equations in chapter 1 . In this chapter, we will discuss the methods of solution of second and higher order linear differential equations . We first discuss second order linear differential equations for two main reasons . First , they have important applications in mechanics and in electric circuit theory . Second , their theory is similar to that of linear differentials equations of any order n , so that extension to higher order n can be easily done .

Whereas second order linear differential equations have various applications , higher order differential equations occur very rarely in applications of science and engineering . For example , a fourth – order linear differential equation governs the bending of an elastic beam , such as a wooden or iron girder in a building or a bridge .

4.2 SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

The general second order linear differential equation has the form

$$\left.\begin{aligned} &\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = Q \\ \text{or}\quad &y''+P_1y'+P_2y = Q \end{aligned}\right] \qquad (1)$$

where P_1, P_2, and Q are functions of x alone (or constants) and y is an unknown function . The functions P_1 and P_2 in equation (1) are called the **coefficients** of the differential equation and Q is called a **forcing** function .

Note that the coefficient of $\frac{d^2y}{dx^2}$ is 1 . We call equation (1) the **standard form**. If the first term of the differential equation is say $P_0(x)\frac{d^2y}{dx^2}$, we have to divide by $P_0(x)$ to get the standard form (1) with $\frac{d^2y}{dx^2}$ as the first term .

If $Q \neq 0$, then equation (1) is called **non – homogeneous** or **complete** equation. If $Q = 0$ (i.e. $Q(x) = 0$ for all x considered), then equation (1) becomes simply

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \tag{2}$$

and is called **homogeneous, reduced**, or **complementary** equation. For example,

$$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 2xy = \sin x \tag{3}$$

is a second order non – homogeneous linear differential equation, while the equation

$$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 2xy = 0$$

is called the corresponding second order homogeneous linear differential equation.

If P_1 and P_2 are both constants, equation (1) is said to have **constant** coefficients, otherwise it is said to have **variable** coefficients. For example, equation (3) is a second order non – homogeneous linear differential equation with variable coefficients, while

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

is a second order non – homogeneous linear differential equation with constant coefficients.

A differential equation which cannot be written in the form of equation (1) is called **nonlinear**. For example, $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + x^2y = e^{-x}$ and $y'' - yy' + 6xy = 0$ are second order nonlinear differential equations.

4.3 SOLUTION OF SECOND ORDER LINEAR DIFFERENTIAL EQUATION

A solution of a second order linear differential equation

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = Q$$

on an interval I is a function $y = \phi(x)$ that has $y' = \phi'(x)$ and $y'' = \phi''(x)$ and has the property $\phi''(x) + P_1\phi'(x) + P_2\phi(x) = Q$ for all x in that interval.

EXAMPLE (1): **Show that $y = C_1\cos 2x + C_2\sin 2x + x$ is a solution of the differential equation $y'' + 4y = 4x$.**

SOLUTION: Given that $y'' + 4y = 4x$ (1)

Since $y = C_1\cos 2x + C_2\sin 2x + x$, therefore

$$y' = -2C_1\sin 2x + 2C_2\cos 2x + 1, \quad y'' = -4C_1\cos 2x - 4C_2\sin 2x$$

Substituting these values in equation (1), we get

$$y'' + 4y = -4C_1 \cos 2x - 4C_2 \sin 2x + 4C_1 \cos 2x + 4C_2 \sin 2x + 4x = 4x$$

Since the differential equation (1) is satisfied, therefore the given function y is its solution.

4.4 SECOND ORDER LINEAR INITIAL – VALUE PROBLEM

A second order linear differential equation

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = Q$$

together with two initial conditions

$$y(x_0) = A \quad \text{and} \quad y'(x_0) = B$$

is called a second order linear initial – value problem. The initial conditions are used to determine the unique values of the constants in the general solution. For example,

$$y'' + 4y = 4x; \quad y(0) = 1, \quad y'(0) = 7$$

is a second order linear initial – value problem.

4.5 SOLUTION OF A SECOND ORDER LINEAR INITIAL–VALUE PROBLEM

The solution of a second order linear initial – value problem is a function that is twice differentiable, satisfies the given differential equation, and satisfies the given initial conditions.

For example, consider the above mentioned initial – value problem

$$y'' + 4y = 4x; \quad y(0) = 1, \quad y'(0) = 7$$

We have shown in example (1) that

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + x \qquad (1)$$

is a solution of the differential equation for any real constants C_1 and C_2. We must choose the constants to satisfy the initial conditions.

Using the initial condition $y(0) = 1$, we get from equation (1) $C_1 = 1$.

Also $y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x + 1 \qquad (2)$

Using the second initial condition $y'(0) = 7$, we get from equation (2) that

$$7 = 2C_2 + 1 \quad \text{or} \quad C_2 = 3$$

Substituting the values of C_1 and C_2 in equation (1), the solution of the initial–value problem is given by

$$y = \cos 2x + 3 \sin 2x + x$$

4.6 SECOND ORDER LINEAR BOUNDARY – VALUE PROBLEM

A second order linear differential equation

$$y'' + P_1 y' + P_2 y = Q \qquad (1)$$

together with the boundary conditions

$$y(P_1) = A \quad \text{and} \quad y(P_2) = B \tag{2}$$

is called a second order linear boundary – value problem. For example,

$$y'' + 2y' = e^x; \; y(0) = 1, \; y(1) = 1$$

is a second order boundary – value problem, because the two boundary conditions are given at two different values of the variable x.

4.7 SOLUTION OF A SECOND ORDER BOUNDARY – VALUE PROBLEM

The solution of a second order boundary – value problem is a function that is twice differentiable, satisfies the given differential equation, and satisfies the given boundary conditions.

For example, consider the boundary – value problem

$$y'' + y = 0; \quad y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = -3$$

It can be checked by substitution into the differential equation that

$$y(x) = C_1 \cos x + C_2 \sin x \tag{1}$$

is a solution of the differential equation for any real constants C_1 and C_2. We must choose the constants to satisfy the boundary conditions.

Using the boundary condition $y(0) = 2$, we get from equation (1) that $C_1 = 2$.

Using the second boundary condition $y\left(\frac{\pi}{2}\right) = -3$, we get from equation (1) that $C_2 = -3$.

Substituting the values of C_1 and C_2 in equation (1), the solution of the boundary – value problem is

$$y(x) = 2\cos x - 3\sin x$$

4.8 OPERATOR NOTATION

Writing $\frac{d}{dx} = D$, we have $\frac{dy}{dx} = Dy$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D.Dy = D^2y$

The second order linear differential equation $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = Q$

can be written as $(D^2 + P_1D + P_2)y = Q$

or briefly $F(D)y = Q$

where $F(D) = D^2 + P_1D + P_2$ is called an operator polynomial in D.

For example, $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 2xy = \sin x$

can be written as $(D^2 + x^2D - 2x)\,y = \sin x$.

4.9 LINEAR OPERATOR

An operator L is called a linear operator if for any constants C_1 and C_2 and any functions $y_1(x)$, $y_2(x)$, we have

$$L(C_1 y_1 + C_2 y_2) = C_1 L(y_1) + C_2 L(y_2).$$

THEOREM (4.1): **Prove that**

(i) **D and D^2 are linear operators.**

(ii) **$F(D) = D^2 + P_1 D + P_2$ is a linear operator.**

PROOF: (i) We have

$$D(C_1 y_1 + C_2 y_2) = \frac{d}{dx}(C_1 y_1 + C_2 y_2)$$

$$= C_1 \frac{dy_1}{dx} + C_2 \frac{dy_2}{dx} = C_1 D y_1 + C_2 D y_2$$

and so D is a linear operator. Also

$$D^2(C_1 y_1 + C_2 y_2) = \frac{d^2}{dx^2}(C_1 y_1 + C_2 y_2)$$

$$= C_1 \frac{d^2 y_1}{dx^2} + C_2 \frac{d^2 y_2}{dx^2} = C_1 D^2 y_1 + C_2 D^2 y_2$$

and so D^2 is a linear operator.

(ii) $$F(D)(C_1 y_1 + C_2 y_2) = (D^2 + P_1 D + P_2)(C_1 y_1 + C_2 y_2)$$

$$= D^2(C_1 y_1 + C_2 y_2) + P_1 D(C_1 y_1 + C_2 y_2) + P_2(C_1 y_1 + C_2 y_2)$$

$$= (C_1 D^2 y_1 + C_2 D^2 y_2) + (P_1 C_1 D y_1 + P_1 C_2 D y_2) + (P_2 C_1 y_1 + P_2 C_2 y_2)$$

$$= (C_1 D^2 y_1 + P_1 C_1 D y_1 + P_2 C_1 y_1) + (C_2 D^2 y_2 + P_1 C_2 D y_2 + P_2 C_2 y_2)$$

$$= C_1 (D^2 + P_1 D + P_2) y_1 + C_2 (D^2 + P_1 D + P_2) y_2$$

$$= C_1 F(D) y_1 + C_2 F(D) y_2$$

and so $F(D)$ is a linear operator.

4.10 PRINCIPLE OF SUPERPOSITION

THEOREM (4.2): **Prove that if $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the second order linear differential equation $F(D)y = 0$, then $y = C_1 y_1(x) + C_2 y_2(x)$ where C_1 and C_2 are arbitrary constants, is the general solution of $F(D)y = 0$.**

PROOF: Since $y_1(x)$ and $y_2(x)$ are solutions of the equation $F(D)y = 0$, we have $F(D)y_1 = 0$ and $F(D)y_2 = 0$. Then

$$F(D)[C_1 y_1(x) + C_2 y_2(x)] = C_1 F(D) y_1 + C_2 F(D) y_2 = 0 + 0 = 0$$

and so $C_1 y_1 + C_2 y_2$ is a solution.

Since it contains two arbitrary constants, it is the general solution of $F(D)y = 0$.

4.11 SECOND ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the second order homogeneous linear differential equation

$$\frac{d^2 y}{dx^2} + A\frac{dy}{dx} + By = 0 \qquad (1)$$

where A and B are constants.

These equations have important applications, especially in connection with mechanical and electrical vibrations. To solve equation (1) we remember that a first order linear differential equation $\frac{dy}{dx} + Ky = 0$ with constants coefficient K has an exponential function as solution i.e. $y = Ce^{-kx}$. This suggests that we may try to see whether $y = e^{mx}$ (2)

is a solution of differential equation (1) for some m?

Now from equation (2), we have

$$\frac{dy}{dx} = m e^{mx} \quad \text{and} \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}$$

Substituting these in equation (1), we obtain

$$m^2 e^{mx} + A m e^{mx} + B e^{mx} = 0$$

or $$(m^2 + Am + B) e^{mx} = 0$$

Since $e^{mx} \neq 0$, this implies that

$$m^2 + Am + B = 0 \qquad (3)$$

Thus equation (2) is a solution of equation (1) if m is a solution of the quadratic equation (3).

Equation (3) is called the **characteristic** equation or **auxiliary** equation of equation (1). The roots of equation (3) are called the characteristic roots. The roots are

$$m_1 = \frac{-A + \sqrt{A^2 - 4B}}{2} \quad \text{and} \quad m_2 = \frac{-A - \sqrt{A^2 - 4B}}{2} \qquad (4)$$

This shows that the functions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are solutions of equation (1).

NATURE OF CHARACTERISTIC ROOTS

There are three different cases to be considered depending on the sign of the discriminant $A^2 - 4B$.

(i) Roots real and distinct if $A^2 - 4B > 0$.

(ii) Roots real and repeated if $A^2 - 4B = 0$.

(iii) Complex conjugate roots if $A^2 - 4B < 0$.

CASE (1): ROOTS REAL AND DISTINCT

If the roots m_1 and m_2 given by equation (4) are real and distinct i.e. $m_1 \neq m_2$, then $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are two linearly independent solutions of equation (1). Hence, by the principle of superposition

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} \tag{5}$$

is the general solution of equation (1) with two arbitrary constants.

As a special case, if the two roots given by equation (4) are $m_1 = m$ and $m_2 = -m$, then solution (5) can be written as

$$y = K_1 e^{mx} + K_2 e^{-mx}$$

Let $K_1 = \dfrac{C_1 + C_2}{2}$ and $K_2 = \dfrac{C_1 - C_2}{2}$, then

$$y = \left(\frac{C_1 + C_2}{2}\right) e^{mx} + \left(\frac{C_1 - C_2}{2}\right) e^{-mx}$$

$$= C_1\left(\frac{e^{mx} + e^{-mx}}{2}\right) + C_2\left(\frac{e^{mx} - e^{-mx}}{2}\right)$$

$$= C_1 \cosh mx + C_2 \sinh mx \tag{6}$$

EXAMPLE (2): Solve the differential equation $y'' - y = 0$.

SOLUTION: Comparing with the general equation (1), we see that $A = 0$ and $B = -1$. The characteristic equation (3) becomes $m^2 - 1 = 0$. Its roots are $m_1 = 1$ and $m_2 = -1$. Hence the general solution of the given differential equation can be written as

$$y = C_1 e^{x} + C_2 e^{-x}$$

CASE (2): ROOTS REAL AND REPEATED

When the discriminant $A^2 - 4B = 0$, then the roots m_1 and m_2 given by equation (4) are real and repeated i.e. $m_1 = m_2 = m = -\dfrac{A}{2}$ is the only one root and we get only one solution as

$$y_1 = e^{-Ax/2}$$

To find the second linearly independent solution, let $y_2 = u y_1$ and try to find the function u such that y_2 becomes a solution of differential equation (1). For this we substitute $y_2 = u y_1$ and its derivatives

$$y_2' = u y_1' + u' y_1$$

and $$y_2'' = u y_1'' + u' y_1' + u' y_1' + u'' y_1$$
$$= u'' y_1 + 2 u' y_1' + u y_1''$$

into equation (1). This gives

$$(u'' y_1 + 2 u' y_1' + u y_1'') + A(u' y_1 + u y_1') + B u y_1 = 0$$

Collecting terms, we get

$$u'' y_1 + u'(2 y_1' + A y_1) + u(y_1'' + A y_1' + B y_1) = 0 \qquad (7)$$

The expression in the last parentheses $y_1'' + A y_1' + B y_1 = 0$, since y_1 is a solution of differential equation (1). The expression in the first parentheses is zero, too, since

$$2 y_1' + A y_1 = 2\left(-\frac{A}{2}\right) e^{-Ax/2} + A e^{-Ax/2} = 0$$

Thus equation (7) reduces to $u'' y_1 = 0$

Since $y_1 \neq 0$, therefore $u'' = 0$.

Integrating twice, we get $u = C_1 x + C_2$.

To get a second independent solution $y_2 = u y_1$, we can simply take $u = x$. Then $y_2 = x y_1$.

Hence in the case of a real repeated root of equation (3) the two linearly independent solutions are

$$e^{-Ax/2} \text{ and } x e^{-Ax/2}.$$

The corresponding general solution of differential equation (1) is given by

$$y = (C_1 + C_2 x) e^{-Ax/2}$$

where C_1 and C_2 are arbitrary constants.

EXAMPLE (3): **Solve the differential equation $y'' + 8 y' + 16 y = 0$.**

SOLUTION: Comparing with the general equation (1), we see that $A = 8$, $B = 16$.

The characteristic equation (3) becomes

$$m^2 + 8m + 16 = 0$$

or $$(m+4)^2 = 0$$

Its roots are $m_1 = -4$ and $m_2 = -4$. i.e. the roots are real and repeated.

Hence, the general solution of the given differential equation is

$$y = (C_1 + C_2 x) e^{-4x}$$

CASE (3): COMPLEX CONJUGATE ROOTS

When the discriminant $A^2 - 4B < 0$, then the roots m_1 and m_2 given by equation (4) are complex conjugate, say $m_1 = a + ib$ and $m_2 = a - ib$

where $a = -\frac{A}{2}$ and $b = \frac{\sqrt{4B - A^2}}{2}$.

Remember that if the coefficients A and B in auxiliary equation (3) are real, then whenever $a + ib$ is a complex root of equation (3), so also is $a - ib$. Thus in this case, the two linearly independent solutions of equation (1) are

$$y_1 = e^{(a+ib)x} \quad \text{and} \quad y_2 = e^{(a-ib)x}$$

Hence, by the principle of superposition, the general solution of equation (1) are given by

$$y = K_1 e^{(a+ib)x} + K_2 e^{(a-ib)x} = e^{ax}(K_1 e^{ibx} + K_2 e^{-ibx})$$

Using the Euler's formula, we get

$$y = e^{ax}\left[K_1(\cos bx + i\sin bx) + K_2(\cos bx - i\sin bx)\right]$$

$$y = e^{ax}\left[(K_1 + K_2)\cos bx + i(K_1 - K_2)\sin bx\right]$$

or $$y = e^{ax}(C_1 \cos bx + C_2 \sin bx) \tag{8}$$

where $C_1 = K_1 + K_2$ and $C_2 = i(K_1 - K_2)$ are arbitrary constants.

EXAMPLE (4): **Solve the differential equation $y'' + 2y' + 6y = 0$.**

SOLUTIION: Comparing with the general equation (1), we see that $A = 2$, $B = 6$.

The characteristic equation (3) becomes

$$m^2 + 2m + 6 = 0$$

This implies $m = \frac{-2 \pm \sqrt{4 - 24}}{2} = \frac{-2 \pm 2\sqrt{5}\,i}{2} = -1 \pm \sqrt{5}\,i$

Thus the two complex conjugate roots are $m_1 = -1 + \sqrt{5}\,i$ and $m_2 = -1 - \sqrt{5}\,i$

Hence the general solution of the given differential equation is

$$y = e^{-x}\left(C_1 \cos\sqrt{5}\,x + C_2 \sin\sqrt{5}\,x\right)$$

where C_1 and C_2 are arbitrary constants.

4.12 LINEAR DIFFERENTIAL EQUATION OF ORDER n

The general linear differential equation of order n has the form

$$\left.\begin{array}{l} \dfrac{d^n y}{dx^n} + P_1 \dfrac{d^{n-1} y}{dx^{n-1}} + P_2 \dfrac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_{n-1}\dfrac{dy}{dx} + P_n y = Q \\ \text{or} \\ y^{(n)}(x) + P_1 y^{(n-1)}(x) + P_2 y^{(n-2)}(x) \ldots\ldots + P_{n-1} y'(x) + P_n y = Q \end{array}\right] \tag{1}$$

where the coefficients $P_1, \ldots, P_n$, and Q are functions of x alone or constants, and y is an unknown function. Note that the coefficient of $y^{(n)}$ is 1. We call equation (1) the **standard form**. If the first term of the differential equation is say $P_0(x)\,y^{(n)}$, we have to divide by $P_0(x)$ to get the standard form (1) with $y^{(n)}$ as the first term.

If $Q \neq 0$, then equation (1) is called **non – homogeneous** or **complete** equation. If $Q = 0$ (i.e. $Q(x) = 0$ for all x considered), then equation (1), becomes

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + P_1 \frac{dy}{dx} + P_n y = 0 \tag{2}$$

and is called **homogeneous, reduced** or **complementary** equation. For example,

$$\frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + 3\frac{dy}{dx} - 2xy = \sin x \tag{3}$$

is a third order non – homogeneous linear differential equation, while

$$\frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + 3\frac{dy}{dx} - 2xy = 0 \tag{4}$$

is called the corresponding third order homogeneous linear differential equation.

If $P_1, P_2, \ldots\ldots, P_n$ are all constants, equation (1) is said to have **constant** coefficients, otherwise it is said to have **variable** coefficients. For example, equation (3) is a third order non – homogeneous linear differential equation with variable coefficients, while

$$\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

is a third order non – homogeneous linear differential equation with constant coefficients.

A differential equation which cannot be written in the form of equation (1) is called **nonlinear**. For example,

$$y''' + 3x^2 (y'')^2 + 6x(y')^3 - 6y = \sin x \quad \text{and} \quad y''' - y^2 y'' + 3xyy' + 6xy = e^{2x}$$

are third order non – linear differential equations.

4.13 SOLUTION OF nTH ORDER LINEAR DIFFERENTIAL EQUATION

A solution of an nth order linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

on an interval I is a function $y = \phi(x)$ that has

$y' = \phi'(x),\ y'' = \phi''(x), \ldots\ldots, y^{(n)}(x) = \phi^{(n)}(x)$ and has the property

$\phi^{(n)}(x) + P_1 \phi^{(n-1)}(x) + \ldots\ldots + P_{n-1}\phi'(x) + P_n \phi(x) = Q$ for all x in that interval

EXAMPLE (5): **Show that $y = C_1 + C_2 e^{2x} + C_3 e^{-2x} - \frac{1}{8}x^2$ is a solution of the differential equation $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = x$**

SOLUTION: Given that

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = x \tag{1}$$

Since $y = C_1 + C_2 e^{2x} + C_3 e^{-2x} - \frac{1}{8}x^2$, therefore

$$\frac{dy}{dx} = 2C_2 e^{2x} - 2C_3 e^{-2x} - \frac{1}{4}x$$

$$\frac{d^2y}{dx^2} = 4C_2 e^{2x} + 4C_3 e^{-2x} - \frac{1}{4}$$

$$\frac{d^3y}{dx^3} = 8C_2 e^{2x} - 8C_3 e^{-2x}$$

Substituting the values of these derivatives in equation (1), we get

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 8C_2 e^{2x} - 8C_3 e^{-2x} - 8C_2 e^{2x} + 8C_3 e^{-2x} + x = x$$

Since the differential equation (1) is satisfied, therefore the given function y is its solution.

4.14 nTH ORDER LINEAR INITIAL – VALUE PROBLEM

An nth order linear differential equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

together with a set of n initial conditions

$$y(x_0) = C_1, y'(x_0) = C_2, \ldots\ldots, y^{(n-1)}(x_0) = C_n$$

is called an nth order linear initial – value problem. The initial conditions are used to determine the unique values of the constants in the general solution.

For example, $(D^3 + 3D^2 + 3D + 1)y = 8e^x$; $y(0) = -1$, $y'(0) = -3$, $y''(0) = 5$ is a third order initial – value problem.

4.15 SOLUTION OF AN nTH ORDER LINEAR INITIAL – VALUE PROBLEM

The solution of an nth order initial – value problem is a function that is n times differentiable, satisfies the given differential equation, and satisfies the given initial conditions.

For example, consider the initial – value problem

$$(D^3 - D^2 - D + 1)y = 0; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = 0$$

We can check by substitution into the differential equation that

$$y(x) = C_1 e^{-x} + (C_2 + C_3 x)e^x \tag{1}$$

is a solution of the differential equation for any real constants C_1, C_2, and C_3. We must choose the constants to satisfy the initial conditions.

From equation (1), we have

$$y'(x) = -C_1 e^{-x} + (C_2 + C_3 x) e^{x} + C_3 e^{x} \tag{2}$$

and $$y''(x) = C_1 e^{-x} + (C_2 + C_3 x) e^{x} + 2 C_3 e^{x} \tag{3}$$

Using the initial condition $y(0) = 2$, we get from equation (1)

$$C_1 + C_2 = 2 \tag{4}$$

Using the initial condition $y'(0) = 1$, we get from equation (2)

$$-C_1 + C_2 + C_3 = 1 \tag{5}$$

Using the initial condition $y''(0) = 0$, we get from equation (3)

$$C_1 + C_2 + 2 C_3 = 0 \tag{6}$$

Solving equations (4), (5), and (6), we get $C_1 = 0$, $C_2 = 2$, $C_3 = -1$

Substituting the values of these constants in equation (1), the solution of the initial – value problem is given by $y(x) = (2 - x) e^{x}$.

4.16 OPERATOR NOTATION

Since $\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = D.Dy = D^2 y, \ldots\ldots$, and $\dfrac{d^n y}{dx^n} = D^n y$,

therefore nth order linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

can be written as $(D^n + P_1 D^{n-1} + \ldots\ldots + P_{n-1} D + P_n) y = Q$

or briefly $F(D) y = Q$

where $F(D) = D^n + P_1 D^{n-1} + \ldots\ldots + P_{n-1} D + P_n$ is called an operator polynomial in D.

For example, the differential equation

$$\frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 2xy = \sin x$$

can be written as $(D^3 + x^2 D^2 + 3D - 2x) y = \sin x$

4.17 LINEAR OPERATOR

An operator L is called a linear operator if for any constants $C_1, C_2, \ldots\ldots, C_n$ and any functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$, we have

$$L(C_1 y_1 + C_2 y_2 + \ldots\ldots + C_n y_n) = C_1 L(y_1) + C_2 L(y_2) + \ldots\ldots + C_n L(y_n)$$

THEOREM (4.3): **Prove that the operator**

$$F(D) = D^n + P_1 D^{n-1} + \dots\dots + P_{n-1} D + P_n \text{ is a linear operator .}$$

PROOF: We have

$$F(D)(C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n)$$

$$= (D^n + P_1 D^{n-1} + \dots\dots + P_{n-1} D + P_n)(C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n)$$

$$= D^n (C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n) + P_1 D^{n-1}(C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n)$$

$$+ \dots\dots + P_{n-1} D (C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n) + P_n (C_1 y_1 + C_2 y_2 + \dots\dots + C_n y_n)$$

$$= (C_1 D^n y_1 + C_2 D^n y_2 + \dots\dots + C_n D^n y_n)$$

$$+ (P_1 C_1 D^{n-1} y_1 + P_1 C_2 D^{n-1} y_2 + \dots\dots + P_1 C_n D^{n-1} y_n)$$

$$+ \dots\dots + (P_{n-1} C_1 D y_1 + P_{n-1} C_2 D y_2 + \dots\dots + P_{n-1} C_n D y_n)$$

$$+ (P_n C_1 y_1 + P_n C_2 y_2 + \dots\dots + P_n C_n y_n)$$

$$= (C_1 D^n y_1 + P_1 C_1 D^{n-1} y_1 + \dots\dots + P_{n-1} C_1 D y_1 + P_n C_1 y_1)$$

$$+ (C_2 D^n y_2 + P_1 C_2 D^{n-1} y_2 + \dots\dots + P_{n-1} C_2 D y_2 + P_n C_2 y_2)$$

$$+ \dots\dots + (C_n D^n y_n + P_1 C_n D^{n-1} y_n + \dots\dots + P_{n-1} C_n D y_n + P_n C_n y_n)$$

$$= C_1 (D^n + P_1 D^{n-1} + \dots\dots + P_{n-1} D + P_n) y_1$$

$$+ C_2 (D^n + P_1 D^{n-1} + \dots\dots + P_{n-1} D + P_n) y_2$$

$$+ \dots\dots + C_n (D^n + P_1 D^{n-1} + \dots\dots + P_{n-1} D + P_n) y_n$$

$$= C_1 F(D) y_1 + C_2 F(D) y_2 + \dots\dots + C_n F(D) y_n$$

and so $F(D)$ is a linear operator .

4.18 PRINCIPLE OF SUPERPOSITION

THEOREM (4.4): **Prove that if $y_1(x), y_2(x), \dots\dots, y_n(x)$ are n linearly independent solutions of the nth order linear differential equation $F(D)y = 0$, then**

$$y = C_1 y_1(x) + C_2 y_2(x) + \dots\dots + C_n y_n(x)$$

where $C_1, C_2, \dots\dots, C_n$ are arbitrary constants , is the general solution of $F(D)y = 0$.

PROOF: Since $y_1, y_2, \dots\dots, y_n$ are solutions of $F(D)y = 0$, we have

$$F(D)y_1 = 0, \quad F(D)y_2 = 0, \dots\dots, \text{ and } F(D)y_n = 0$$

Then $F(D)\left[C_1 y_1 + C_2 y_2 + \ldots\ldots + C_n y_n\right]$

$$= C_1 F(D) y_1 + C_2 F(D) y_2 + \ldots\ldots + C_n F(D) y_n$$

$$= 0 + 0 + \ldots\ldots + 0 = 0$$

and so $C_1 y_1 + C_2 y_2 + \ldots\ldots C_n y_n$ is a solution. And since it contains n arbitrary constants, it is the general solution of $F(D) y = 0$.

4.19 HIGHER ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the nth order homogeneous linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \qquad (1)$$

where the coefficients $P_1, \ldots, P_n$ are constants.

Equation (1) can be solved by the methods for second – order homogeneous linear differential equations with constant coefficients. As discussed already that $y = e^{mx}$ is a solution of the first and second order linear differential equations with constant coefficients.

This suggests that we may try to see whether $y = e^{mx}$ is a solution of equation (1) for some m.

Letting $y = e^{mx}$, m = constant, in equation (1), we get

$$(m^n + P_1 m^{n-1} + \ldots\ldots + P_{n-1} m + P_n) e^{mx} = 0$$

If m is chosen so that it satisfies the equation

$$m^n + P_1 m^{n-1} + \ldots\ldots + P_{n-1} m + P_n = 0 \qquad (2)$$

then $y = e^{mx}$ will be a solution of equation (1).

Equation (2) is called the **auxiliary** equation or the **characteristic** equation of equation (1). This can be factored as $(m - m_1)(m - m_2) \ldots\ldots (m - m_n) = 0$ which has roots $m_1, m_2, \ldots\ldots, m_n$ called the characteristic roots.

NOTE: Equation (2) is the same as $D^n + P_1 D^{n-1} + \ldots\ldots + P_{n-1} D + P_n = 0$

NATURE OF CHARACTERISTIC ROOTS

There are three different cases to be considered depending upon the nature of the roots of the auxiliary equation.

(i) Roots real and distinct

(ii) Roots real and repeated

(iii) Complex roots

We now discuss these cases one by one.

CASE (1): ROOTS REAL AND DISTINCT

Let $m_1 \neq m_2 \neq m_3 \neq \ldots\ldots \neq m_n$. Then $e^{m_1 x}, e^{m_2 x}, , \ldots\ldots, e^{m_n x}$ are linearly independent solutions of equation (1). Hence by the principle of superposition

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots + C_n e^{m_n x} \qquad (3)$$

is the general solution of equation (1) with n arbitrary constants.

EXAMPLE (6): **Solve the differential equation** $\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

SOLUTION: Write the equation as $(D^3 - D^2 - 4D + 4)y = 0$

or $(D-1)(D-2)(D+2)y = 0$

The characteristic roots are $1, 2, -2$ and the general solution of the given equation is

$$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-2x}$$

CASE (2): ROOTS REAL AND REPEATED

When two roots of the auxiliary equation are equal, say $m_1 = m_2$, the general solution (3) becomes $y = (C_1 + C_2)e^{m_1 x} + C_3 e^{m_3 x} + \ldots\ldots + C_n e^{m_n x}$

which has only $(n-1)$ arbitrary constants, as $C_1 + C_2$ is one arbitrary constant, while we require n arbitrary constants for the general solution. Hence this method fails. So we proceed as follows:

When the two roots of the auxiliary equation are m_1, the auxiliary equation will be $(D - m_1)^2 = 0$ and the original differential equation will then be

$$(D - m_1)^2 y = 0$$

To solve this equation write it as

$$(D - m_1)(D - m_1)y = 0 \qquad (4)$$

Let $(D - m_1)y = z \qquad (5)$

Then equation (4) becomes $(D - m_1)z = 0$

or $\frac{dz}{dx} - m_1 z = 0$

Separating the variables, we get

$$\frac{dz}{z} = m_1\, dx$$

Integrating both sides, we get $\ln z = m_1 x + K$

or $z = C_2 e^{m_1 x}$ (where $C_2 = e^K$)

Substituting this value in equation (5), we have

$$(D - m_1)y = C_2 e^{m_1 x}$$

or $\frac{dy}{dx} - m_1 y = C_2 e^{m_1 x}$

which is a linear differential equation whose solution can be found to be

$$y = e^{m_1 x}(C_1 + C_2 x)$$

Thus the part of the general solution corresponding to two equal roots m_1 is $(C_1 + C_2 x)e^{m_1 x}$.

Similarly, we can prove that if the auxiliary equation has a root m, which repeats three times, then the solution corresponding to that root is $(C_1 + C_2 x + C_3 x^2)e^{m_1 x}$.

In general, we can prove, that if the auxiliary equation has a root m_1 of multiplicity r, then the solution corresponding to that root is

$$e^{m_1 x}(C_1 + C_2 x + \dots\dots\dots + C_r x^{r-1})$$

EXAMPLE (7): **Solve the differential equation** $\dfrac{d^3 y}{d x^3} - 2\dfrac{d^2 y}{d x^2} - 4\dfrac{d y}{d x} + 8y = 0$.

SOLUTION: Write the equation as $(D^3 - 2D^2 - 4D + 8)y = 0$

The auxiliary equation is

$$D^3 - 2D^2 - 4D + 8 = 0 \quad \text{or} \quad (D-2)^2(D+2) = 0$$

The characteristic roots are $2, 2, -2$. The general solution of the given differential equation is

$$y = (C_1 + C_2 x)e^{2x} + C_3 e^{-2x}$$

CASE (3): COMPLEX ROOTS

If $P_1, P_2, \dots\dots, P_n$ are real, then when $a + ib$ is a complex root of equation (1) so also is $a - ib$ (where a, b are real). The corresponding part of the general solution is

$$\begin{aligned} A e^{(a+ib)x} + B e^{(a-ib)x} &= e^{ax}(A e^{ibx} + B e^{-ibx}) \\ &= e^{ax}\left[A(\cos bx + i \sin bx) + B(\cos bx - i \sin bx)\right] \end{aligned}$$

(using Euler's Formula)

$$\begin{aligned} &= e^{ax}\left[(A+B)\cos bx + i(A-B)\sin bx\right] \\ &= e^{ax}\left[C_1 \cos bx + C_2 \sin bx\right] \end{aligned}$$

where $C_1 = A + B$ and $C_2 = i(A - B)$ are arbitrary constants.

EXAMPLE (8): **Solve the differential equation** $\dfrac{d^3 y}{d x^3} + 8y = 0$.

SOLUTION: Write the equation as $(D^3 + 8)y = 0$

The auxiliary equation is $D^3 + 8 = 0$ or $(D+2)(D^2 - 2D + 4) = 0$.

The characteristic roots are $-2, 1 \pm \sqrt{3}\, i$.

The general solution of the given differential equation is

$$y = C_1 e^{-2x} + e^{x}(C_2 \cos\sqrt{3}\,x + C_3 \sin\sqrt{3}\,x)$$

REPEATED COMPLEX ROOTS

Similarly, if a pair of complex roots occurs twice, say $a \pm i\,b$, $a \pm i\,b$, then the corresponding solution is

$$(A+Bx)e^{(a+ib)x}+(C+Dx)e^{(a-ib)x}$$

$$= e^{ax}\left[(A+Bx)e^{ibx}+(C+Dx)e^{-ibx}\right]$$

$$= e^{ax}\left[(A+Bx)(\cos bx+i\sin bx)+(C+Dx)(\cos bx-i\sin bx)\right]$$

$$= e^{ax}\left[\{(A+C)+(B+D)x\}\cos bx+i\{(A-C)+(B-D)x\}\sin bx\right]$$

$$= e^{ax}\left[(C_1+C_2x)\cos bx+(C_3+C_4x)\sin bx\right]$$

where $C_1 = A+C$, $C_2 = B+D$, $C_3 = i(A-C)$, $C_4 = i(B-D)$ are arbitrary constants.

EXAMPLE (9): **Solve the differential equation $(D^2-2D+4)^2 y = 0$.**

SOLUTION: The auxiliary equation is $(D^2-2D+4)^2 = 0$

which has the characteristic roots as $1 \pm i\sqrt{3}$, $1 \pm i\sqrt{3}$, which are repeated complex roots.

The general solution of the given differential is

$$y = e^{x}\left[(C_1+C_2x)\cos(\sqrt{3}\,x)+(C_3+C_4x)\sin(\sqrt{3}\,x)\right]$$

4.20 INVERSE OPERATOR

Since D denotes the differential operator, the symbol $\frac{1}{D}$ stands for the operator of integration.

Also, if $F(D)\,y = Q$, we will symbolically express $y = \frac{1}{F(D)}Q$

and call $\frac{1}{F(D)}$ as the inverse operator of $F(D)$ i.e. they cancel each other's effect.

THEOREM (4.5): **Prove that $\frac{1}{D}Q = \int Q\,dx$**

PROOF: Let $\frac{1}{D}Q = z$.

Operating on both sides by D, we get $D\frac{1}{D}Q = Dz$

or $Q = \frac{dz}{dx}$ (by definition)

Integrating both sides w.r.t. x, we get

$$z = \int Q\,dx$$

or $$\frac{1}{D}Q = \int Q\,dx$$

THEOREM (4.6): **Prove that** $\dfrac{1}{D-m}Q = e^{mx}\int e^{-mx}Q\,dx$.

PROOF: Let $\dfrac{1}{D-m}Q = z$

Operating on both sides by $(D-m)$, we get

$$(D-m)\frac{1}{D-m}Q = (D-m)z$$

or $\quad Q = (D-m)z \qquad$ (by definition)

or $\quad \dfrac{dz}{dx} - mz = Q$

which is a first order linear differential equation in z. The integrating factor is given by

I.F. $= e^{-\int m\,dx} = e^{-mx}$ and the solution of this equation is

$z(\text{I.F.}) = \int (\text{I.F.})\,Q\,dx \qquad$ (omitting constant of integration)

or $\quad ze^{-mx} = \int e^{-mx}\,Q\,dx$

or $\quad z = e^{mx}\int e^{-mx}Q\,dx$

i.e. $\quad \dfrac{1}{D-m}Q = e^{mx}\int e^{-mx}Q\,dx$

4.21 COMPLEMENTARY FUNCTION

Let $y = y_c(x)$ be the general solution of the complementary or reduced equation $F(D)y = 0$, then $y_c(x)$ is said to be the complementary function (C.F.) of the equation $F(D)y = Q$.

4.22 PARTICULAR INTEGRAL

THEOREM (4.7): **A particular integral (P.I.) of the differential equation** $F(D)y = Q$ **is**

$$y = \frac{1}{F(D)}Q.$$

PROOF: Given that $F(D)y = Q \qquad (1)$

For, when $y = \dfrac{1}{F(D)}Q$, we have

L.H.S. of equation (1) $= F(D)\dfrac{1}{F(D)}Q = Q =$ R.H.S. of equation (1)

Thus $\quad y = \dfrac{1}{F(D)}Q$ satisfies equation (1).

Hence it is a particular integral of differential equation (1).

4.23 METHODS FOR FINDING THE PARTICULAR INTEGRALS

We now discuss the methods for finding the particular integrals of the differential equation

$$F(D)y = Q$$

depending on the different forms of the function Q.

DIFFERENT FORMS OF THE FUNCTION Q

(1) $Q = e^{ax}$ (2) $Q = \sin ax$ or $Q = \cos ax$

(3) $Q = x^m$, m is a positive integer (4) $Q = e^{ax}V$, where V is any function of x

(5) $Q = xV$, where V is any function of x.

CASE (1): PARTICULAR INTEGRAL WHEN $Q = e^{ax}$

THEOREM (4.8): **Prove that** $\frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(a)}$ if $F(a) \neq 0$.

PROOF: If $F(D) = D^2 + P_1 D + P_2$, then

$$\begin{aligned} F(D)e^{ax} &= (D^2 + P_1 D + P_2)e^{ax} \\ &= (a^2 + P_1 a + P_2)e^{ax} \\ &= F(a)e^{ax} \end{aligned}$$

Thus if $F(a) \neq 0$, then $\frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(a)}$.

Similarly, if $F(D) = D^n + P_1 D^{n-1} + \ldots\ldots + P_{n-1}D + P_n$, then

$$\begin{aligned} F(D)e^{ax} &= (D^n + P_1 D^{n-1} + \ldots\ldots + P_{n-1}D + P_n)e^{ax} \\ &= (a^n + P_1 a^{n-1} + \ldots\ldots + P_{n-1}a + P_n)e^{ax} \\ &= F(a)e^{ax} \end{aligned}$$

Thus if $F(a) \neq 0$, then $\frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(a)}$.

EXAMPLE (10): **Find the particular integrals of the following differential equations :**

(i) $(D^2 - 3D + 2)y = e^{5x}$ (ii) $(D^3 - 5D^2 + 8D - 4)y = 3e^{-x}$.

SOLUTION: (i) Here $a = 5$ and $F(D) = D^2 - 3D + 2$ so that

$$F(5) = 5^2 - 3(5) + 2 = 25 - 15 + 2 = 12 \neq 0.$$

Thus the particular integral is given by

$$P.I = \frac{1}{D^2 - 3D + 2}e^{5x} = \frac{e^{5x}}{12}$$

(ii) Here $a = -1$ and $F(D) = D^3 - 5D^2 + 8D - 4$ so that

$$F(-1) = (-1)^3 - 5(-1)^2 + 8(-1) - 4 = -1 - 5 - 8 - 4 = -18 \neq 0$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3 - 5D^2 + 8D - 4} 3e^{-x} = -\frac{3}{18} e^{-x} = -\frac{1}{6} e^{-x}$$

CASE OF FAILURE

When $F(a) = 0$, the rule in theorem (4.8) for evaluating the particular integral fails. In this case, we have the following rule.

THEOREM (4.9): Prove that

$$\frac{1}{F(D)} e^{ax} = x \frac{1}{\frac{d}{dD}[F(D)]} e^{ax} \quad \text{if} \quad F(a) = 0$$

PROOF: If $F(a) = 0$, then by the factor theorem, $(D - a)$ is a factor of $F(D)$.

Let $\quad F(D) = (D - a)\phi(D)$ (1)

$$\begin{aligned}
\text{Therefore } \frac{1}{F(D)} e^{ax} &= \frac{1}{(D-a)\phi(D)} e^{ax} \\
&= \frac{1}{(D-a)} \left[\frac{1}{\phi(D)} e^{ax} \right] \\
&= \frac{1}{D-a} \left[\frac{1}{\phi(a)} e^{ax} \right] \qquad [\text{assuming } \phi(a) \neq 0] \\
&= \frac{1}{\phi(a)} \frac{1}{D-a} e^{ax} \\
&= \frac{1}{\phi(a)} e^{ax} \int e^{-ax} . e^{ax} \, dx \qquad [\text{using theorem (4.6)}] \\
&= \frac{1}{\phi(a)} e^{ax} \int 1 \, dx \\
&= \frac{x e^{ax}}{\phi(a)} \qquad (2)
\end{aligned}$$

Differentiating equation (1) w.r.t. D, we get

$$F'(D) = \phi(D) + (D - a)\phi'(D)$$

Therefore $F'(a) = \phi(a)$

Thus from equation (2), we get

$$\frac{1}{F(D)} e^{ax} = \frac{x e^{ax}}{F'(a)} = x \left[\frac{1}{F'(a)} e^{ax} \right] = x \frac{1}{F'(D)} e^{ax} = x \frac{1}{\frac{d}{dD}[F(D)]} e^{ax}$$

The rule may be repeated as long as we have a case of failure, each time placing a factor x before the symbol of operation and differentiating the denominator of this symbol w.r.t. D.

EXAMPLE (11): **Find the particular integral of the differential equation :**

$$(D-3)^3 y = e^{3x}.$$

SOLUTION: Here $a = 3$ and $F(D) = (D-3)^3$ so that $F(3) = 0$

The particular integral is given by

$$y_p(x) = \frac{1}{(D-3)^3} e^{3x}$$

$$= x \frac{1}{\frac{d}{dD}(D-3)^3} e^{3x} = x \frac{1}{3(D-3)^2} e^{3x}$$

$$= \frac{1}{3} x^2 \frac{1}{\frac{d}{dD}(D-3)^2} e^{3x} = \frac{1}{3} x^2 \frac{1}{2(D-3)} e^{3x}$$

$$= \frac{1}{6} x^3 \frac{1}{\frac{d}{dD}(D-3)} e^{3x} = \frac{1}{6} x^3 e^{3x}$$

CASE (2): PARTICULAR INTEGRAL WHEN $Q = \sin ax$ or $Q = \cos ax$

THEOREM (4.10): **Prove that**

(i) $$\frac{1}{F(D^2)} \sin ax = \frac{\sin ax}{F(-a^2)}$$

(ii) $$\frac{1}{F(D^2)} \cos ax = \frac{\cos ax}{F(-a^2)}$$

if $F(-a^2) \neq 0$

PROOF: (i) We know that

$$D^2(\sin ax) = -a^2 \sin ax$$

$$(D^2)^2 \sin ax = (-a^2)^2 \sin ax$$

$$(D^2)^n \sin ax = (-a^2)^n \sin ax$$

If $F(D) = D^2 + P_1 D + P_2$, then

$$F(D^2) = (D^2)^2 + P_1 D^2 + P_2$$

and
$$F(D^2) \sin ax = \left[(D^2)^2 + P_1(D^2) + P_2\right] \sin ax$$

$$= \left[(-a^2)^2 + P_1(-a^2) + P_2\right] \sin ax$$

$$= F(-a^2) \sin ax$$

Thus if $F(-a^2) \neq 0$, then $\frac{1}{F(D^2)} \sin ax = \frac{\sin ax}{F(-a^2)}$

Similarly, if $F(D) = D^n + P_1 D^{n-1} + \ldots\ldots + P_n$, then

$$F(D^2) = (D^2)^n + P_1(D^2)^{n-1} + \ldots\ldots + P_n$$

and $F(D^2)\sin ax = [(D^2)^n + P_1(D^2)^{n-1} + \ldots\ldots + P_n]\sin ax$

$$= [(-a^2)^n + P_1(-a^2)^{n-1} + \ldots\ldots + P_n]\sin ax$$

$$= F(-a^2)\sin ax$$

Thus if $F(-a^2) \neq 0$, then $\frac{1}{F(D^2)}\sin ax = \frac{\sin ax}{F(-a^2)}$

(ii) Proceed as in (1) and get the result.

In general, when $F(D)$ contains different odd powers of D, we first express every odd power of D, higher than the first, as the product of D and an even power of D. For example, D^3 is written as $D^2 . D$, D^5 as $D^4 . D$ etc. Then we replace D^2 by $-a^2$ in all even powers of D. Thus $F(D)$ becomes linear in D and then we proceed as explained above.

EXAMPLE (12): Find the particular integrals of the following differential equations :

(i) $(D^2+4)y = \sin 3x$ **(ii)** $(D^2+1)^2 y = \sin 2x$

(iii) $(D^2+2D+3)y = \cos 2x$ **(iv)** $(D^3+D^2+2D-1)y = \cos 2x$

SOLUTION: (i) $(D^2+4)y = \sin 3x$

The particular integral is given by

$$P.I = \frac{1}{D^2+4}\sin 3x$$

$$= \frac{1}{(-3^2)+4}\sin 3x = \frac{1}{-9+4}\sin 3x = -\frac{\sin 3x}{5}$$

(ii) $(D^2+1)^2 y = \sin 2x$

The particular integral is given by

$$y_p(x) = \frac{1}{(D^2+1)^2}\sin 2x = \frac{1}{(-4+1)^2}\sin 2x = \frac{\sin 2x}{9}$$

(iii) $(D^2+2D+3)y = \cos 2x$

The particular integral is given by

$$P.I = \frac{1}{D^2+2D+3}\cos 2x$$

$$= \frac{1}{(-2^2)+2D+3}\cos 2x = \frac{1}{2D-1}\cos 2x$$

$$= \frac{2D+1}{4D^2-1}\cos 2x = \frac{(2D+1)}{4(-2^2)-1}\cos 2x$$

$$= \frac{2D+1}{-16-1}\cos 2x = -\frac{1}{17}(2D+1)\cos 2x$$

$$= -\frac{1}{17}(2D\cos 2x + \cos 2x)$$

$$= -\frac{1}{17}(-4\sin 2x + \cos 2x) = \frac{1}{17}(4\sin 2x - \cos 2x)$$

(iv) $(D^3+D^2+2D-1)y = \cos 2x$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3+D^2+2D-1}\cos 2x$$

$$= \frac{1}{D(-4)-4+2D-1}\cos 2x$$

$$= -\frac{1}{2D+5}\cos 2x = -\frac{2D-5}{4D^2-25}\cos 2x$$

$$= -\frac{(2D-5)}{4(-4)-25}\cos 2x = \frac{1}{41}(2D-5)\cos 2x$$

$$= \frac{1}{41}(-4\sin 2x - 5\cos 2x) = -\frac{1}{41}(4\sin 2x + 5\cos 2x)$$

CASE OF FAILURE

When $F(-a^2) = 0$, the rules in theorem (4.10) for evaluating particular integral fails. For this case we have the following rule.

THEOREM (4.11): Prove that

$$\left.\begin{aligned} &\text{(i)} \quad \frac{1}{F(D^2)}\sin ax = x\frac{1}{\frac{d}{dD}[F(D^2)]}\sin ax \\ &\text{(ii)} \quad \frac{1}{F(D^2)}\cos ax = x\frac{1}{\frac{d}{dD}[F(D^2)]}\cos ax \end{aligned}\right] \quad \text{if } F(-a^2) = 0$$

PROOF: We know that

$$\frac{1}{F(D^2)}(\cos ax + i\sin ax) = \frac{1}{F(D^2)}e^{iax}$$

$$= x\frac{1}{\frac{d}{dD}[F(D^2)]}e^{iax} \qquad [\text{Using theorem (4.9)}]$$

$$= x\frac{1}{\frac{d}{dD}[F(D^2)]}(\cos ax + i\sin ax)$$

Equating imaginary and real parts, we get

$$\frac{1}{F(D^2)}\sin ax = x\frac{1}{\frac{d}{dD}[F(D^2)]}\sin ax$$

$$\frac{1}{F(D^2)}\cos ax = x\frac{1}{\frac{d}{dD}[F(D^2)]}\cos ax$$

The rule may be repeated as long as we have a case of failure, each time placing a factor x before the symbol of operation and differentiating the denominator of this symbol w.r.t. D.

EXAMPLE (13): **Find the particular integrals of the differential equations :**

(i) $(D^2+4)y = \cos 2x$ (ii) $(D^3+D^2+D+1)y = \sin x$

SOLUTION: (i) Here $a = 2$ and $F(D^2) = D^2+4$ so that

$$F(-2^2) = -2^2+4 = -4+4 = 0$$

Thus the particular integral is given by

$$\begin{aligned} P.I &= \frac{1}{D^2+4}\cos 2x \\ &= x\frac{1}{\frac{d}{dD}(D^2+4)}\cos 2x \\ &= x\frac{1}{2D}\cos 2x = \frac{1}{2}x\left(\frac{1}{D}\cos 2x\right) \\ &= \frac{1}{2}x\left(\frac{\sin 2x}{2}\right) = \frac{1}{4}x\sin 2x \end{aligned}$$

(ii) $(D^3+D^2+D+1)y = \sin x$

Here $a = 1$, $F(D^2) = D^3+D^2+D+1 = D^2.D+D^2+D+1$

so that $F(-1^2) = (-1)D-1+D+1 = 0$

The particular integral is given by

$$\begin{aligned} y_p(x) &= \frac{1}{D^3+D^2+D+1}\sin x \\ &= x\frac{1}{\frac{d}{dD}(D^3+D^2+D+1)}\sin x \qquad \left[\text{since } F(-a^2) = 0\right] \\ &= x\frac{1}{3D^2+2D+1}\sin x = x\frac{1}{-3+2D+1}\sin x \\ &= \frac{1}{2}x\frac{1}{D-1}\sin x = \frac{1}{2}x\frac{D+1}{D^2-1}\sin x \\ &= \frac{1}{2}x\left(-\frac{1}{2}\right)(D+1)\sin x = -\frac{1}{4}x(\cos x+\sin x) \end{aligned}$$

CASE (3): PARTICULAR INTEGRAL WHEN $Q = x^m$, m IS A POSITIVE INTEGER

To evaluate $\frac{1}{F(D)}x^m$, expand $\frac{1}{F(D)}$ in ascending powers of D by the Binomial theorem, taking the expansion upto the term D^m, since $D^n x^m = 0$ when $n > m$ and then operate each term of the expansion on x^m.

EXAMPLE (14): **Find the particular integral of the differential equations :**

(i) $(D^2-1)y = x^2$ (ii) $(D^3-3D-2)y = x^2$

SOLUTION: (i) $(D^2 - 1)y = x^2$

The particular integral is given by

$$P.I = \frac{1}{D^2 - 1}x^2$$

$$= -\frac{1}{1 - D^2}x^2 = -(1 - D^2)^{-1}x^2$$

$$= -(1 + D^2)x^2 = -x^2 - D^2x^2 = -x^2 - 2$$

(ii) The particular integral is given by

$$y_p(x) = \frac{1}{D^2 - 3D - 2}x^2 = -\frac{1}{2\left(1 + \frac{3}{2}D - \frac{D^2}{2}\right)}x^2$$

$$= -\frac{1}{2}\frac{1}{1 + \left(\frac{3}{2}D - \frac{D^2}{2}\right)}x^2 = -\frac{1}{2}\left[1 + \left(\frac{3}{2}D - \frac{D^2}{2}\right)\right]^{-1}x^2$$

$$= -\frac{1}{2}\left[1 - \left(\frac{3}{2}D - \frac{D^2}{3}\right) + \left(\frac{3}{2}D - \frac{D^2}{2}\right)^2\right]x^2$$

$$= -\frac{1}{2}\left(1 - \frac{3}{2}D + \frac{9}{4}D^2\right)x^2 = -\frac{1}{2}\left(x^2 - 3x + \frac{9}{2}\right)$$

CASE (4): PARTICULAR INTEGRAL WHEN $Q = e^{ax}V$, WHERE V IS ANY FUNCTION OF x

THEOREM (4.12): Prove that $\frac{1}{F(D)}(e^{ax}V) = e^{ax}\frac{1}{F(D+a)}V$.

PROOF: We have $D(e^{ax}V) = e^{ax}DV + ae^{ax}V = e^{ax}(D+a)V$

and
$$D^2(e^{ax}V) = D\left[e^{ax}(D+a)V\right]$$

$$= ae^{ax}(D+a)V + e^{ax}D(D+a)V$$

$$= e^{ax}(D^2 + 2aD + a^2)V$$

$$= e^{ax}(D+a)^2V$$

$$D^n(e^{ax}V) = e^{ax}(D+a)^nV$$

Since $F(D) = D^n + P_1D^{n-1} + \ldots\ldots + P_n$, therefore

$$F(D+a) = (D+a)^n + P_1(D+a)^{n-1} + \ldots\ldots + P_n \tag{1}$$

Now
$$F(D)e^{ax}V = (D^n + P_1D^{n-1} + \ldots\ldots + P_n)e^{ax}V$$

$$= D^n(e^{ax}V) + P_1D^{n-1}(e^{ax}V) + \ldots\ldots + P_n(e^{ax}V)$$

$$= e^{ax}(D+a)^n V + P_1 e^{ax}(D+a)^{n-1}V + \ldots\ldots + P_n e^{ax} V$$

$$= e^{ax}\left[(D+a)^n + P_1(D+a)^{n-1} + \ldots\ldots + P_n\right] V$$

$$= e^{ax} F(D+a) V \qquad \left[\text{using equation (1)}\right] \tag{2}$$

Let $\quad F(D+a)V = Z$ so that $V = \dfrac{1}{F(D+a)} Z$

Then from equation (2) $\quad F(D) e^{ax} \dfrac{1}{F(D+a)} Z = e^{ax} Z \qquad (3)$

Operating on both sides of equation (3) by $\dfrac{1}{F(D)}$, we get

$$\frac{1}{F(D)} e^{ax} Z = \frac{1}{F(D)}\left\{F(D) e^{ax} \frac{1}{F(D+a)} Z\right\} = e^{ax} \frac{1}{F(D+a)} Z$$

or $\quad \dfrac{1}{F(D)} e^{ax} V = e^{ax} \dfrac{1}{F(D+a)} V \qquad$ (replacing Z by V)

EXAMPLE (15): **Find the particular integrals of the following differential equations :**

(i) $(D^2 + 2D + 4) y = e^x \sin 2x$ (ii) $(D^2 + D - 2) y = x^2 e^{2x}$

(iii) $(D^3 + 1) y = x e^x$ (iv) $(D^3 + D) y = e^{-2x} \cos 2x$

SOLUTION: (i) Here $a = 1$, therefore the particular integral is given by

$$P.I = \frac{1}{D^2 + 2D + 4} e^x \sin 2x$$

$$= e^x \frac{1}{(D+1)^2 + 2(D+1) + 4} \sin 2x$$

$$= e^x \frac{1}{D^2 + 4D + 7} \sin 2x = e^x \frac{1}{4D + 3} \sin 2x$$

$$= e^x \frac{4D - 3}{16D^2 - 9} \sin 2x = -\frac{e^x}{73}(4D - 3) \sin 2x$$

$$= -\frac{e^x}{73}(8\cos 2x - 3\sin 2x)$$

(ii) $\quad (D^2 + D - 2) y = x^2 e^{2x}$

The particular integral is given by

$$P.I. = \frac{1}{D^2 + D - 2} x^2 e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 + (D+2) - 2} x^2$$

$$= e^{2x} \frac{1}{D^2 + 5D + 4} x^2$$

$$= e^{2x} \frac{1}{4\left[1 + \left(\frac{5}{4}D + \frac{D^2}{4}\right)\right]} x^2$$

$$= \frac{e^{2x}}{4}\left[1+\left(\frac{5}{4}D+\frac{D^2}{4}\right)\right]^{-1} x^2$$

$$= \frac{e^{2x}}{4}\left[1-\left(\frac{5}{4}D+\frac{D^2}{4}\right)+\left(\frac{5}{4}D+\frac{D^2}{4}\right)^2+\ldots\ldots\right]x^2$$

$$= \frac{e^{2x}}{4}\left[1-\frac{5}{4}D-\frac{D^2}{4}+\frac{25}{16}D^2\right]x^2$$

$$= \frac{e^{2x}}{4}\left(1-\frac{5}{4}D+\frac{21}{16}D^2\right)x^2$$

$$= \frac{e^{2x}}{4}\left(x^2-\frac{5}{2}x+\frac{21}{8}\right)$$

$$= \frac{e^{2x}}{32}(8x^2-20x+21)$$

(iii) $(D^3+1)y = xe^x$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3+1}xe^x$$

$$= e^x\frac{1}{(D+1)^3+1}x$$

$$= e^x\frac{1}{D^3+3D^2+3D+1+1}x$$

$$= e^x\frac{1}{2+(3D+3D^2+D^3)}x$$

$$= e^x\left(\frac{1}{2}\right)\frac{1}{1+\left(\frac{3}{2}D+\frac{3}{2}D^2+\frac{D^3}{2}\right)}x$$

$$= \frac{1}{2}e^x\left[1+\left(\frac{3}{2}D+\frac{3}{2}D^2+\frac{D^3}{2}\right)\right]^{-1}x$$

$$= \frac{1}{2}e^x\left[1-\left(\frac{3}{2}D+\frac{3D^2}{2}+\frac{D^3}{2}\right)\right]x$$

$$= \frac{1}{2}e^x\left(1-\frac{3}{2}D\right)x = \frac{1}{2}e^x\left(x-\frac{3}{2}\right)$$

(iv) $(D^3+D)y = e^{-2x}\cos 2x$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3+D}e^{-2x}\cos 2x$$

$$= e^{-2x}\frac{1}{(D-2)^3+(D-2)}\cos 2x$$

$$= e^{-2x}\frac{1}{D^3-6D^2+13D-10}\cos 2x$$

$$= e^{-2x}\frac{1}{(-4)D-6(-4)+13D-10}\cos 2x$$

$$= e^{-2x}\frac{1}{9D+14}\cos 2x$$

$$= e^{-2x}\frac{9D-14}{81D^2-196}\cos 2x$$

$$= e^{-2x}\frac{9D-14}{81(-4)-196}\cos 2x$$

$$= -\frac{e^{-2x}}{520}(9D-14)\cos 2x$$

$$= -\frac{e^{-2x}}{520}(-18\sin 2x-14\cos 2x)$$

$$= \frac{e^{-2x}}{260}(9\sin 2x+7\cos 2x)$$

CASE (5): PARTICULAR INTEGRAL WHEN $Q = xV$, WHERE V IS ANY FUNCTION OF x

THEOREM (4.13): Prove that

$$\frac{1}{F(D)}(xV) = x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V$$

PROOF: Let $y = xU$, then $Dy = D(xU) = xDU + U$

and $D^2y = D(Dy) = D(xDU+U)$

$$= xD^2U + DU + DU = xD^2U + 2DU$$

$$D^n y = xD^nU + nD^{n-1}U$$

Now
$$\begin{aligned}
F(D)xU &= (D^n + P_1D^{n-1} + \dots\dots + P_{n-1}D + P_n)xU \\
&= D^n(xU) + P_1D^{n-1}(xU) + \dots\dots + P_{n-1}D(xU) + P_n(xU) \\
&= (xD^nU + nD^{n-1}U) + P_1\left[xD^{n-1}U + (n-1)D^{n-2}U\right] + \dots\dots \\
&\quad + P_{n-1}(xDU+U) + P_n(xU) \\
&= x(D^n + P_1D^{n-1} + \dots\dots + P_{n-1}D + P_n)U \\
&\quad + (nD^{n-1} + P_1(n-1)D^{n-2} + \dots\dots + P_{n-1})U \\
&= xF(D)U + F'(D)U \qquad (1)
\end{aligned}$$

Let $V = F(D)U$ so that $U = \frac{1}{F(D)}V$, the equation (1) becomes

$$F(D)\left(x\frac{1}{F(D)}V\right) = xF(D)\left(\frac{1}{F(D)}V\right) + F'(D)\left(\frac{1}{F(D)}V\right)$$

$$= xV + F'(D)\frac{1}{F(D)}V$$

or $$xV = F(D)x\frac{1}{F(D)}V - F'(D)\frac{1}{F(D)}V$$

Operating on both sides by $\frac{1}{F(D)}$, we get

or $$\frac{1}{F(D)}xV = x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V$$

EXAMPLE (16): Find the particular integrals of the following differential equations :

(i) $\frac{d^2y}{dx^2} + 4y = x\sin x$ (ii) $\frac{d^2y}{dx^2} - 4y = x\cos 2x$

SOLUTION: (i) $\frac{d^2y}{dx^2} + 4y = x\sin x$

Write the equation as $(D^2+4)y = x\sin x$

The particular integral is given by

$$\text{P.I.} = \frac{1}{D^2+4}x\sin x$$

$$= x\frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}\sin x \qquad \text{[Using theorem (4.13)]}$$

$$= x\frac{1}{-1+4}\sin x - \frac{2D}{(-1+4)^2}\sin x$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}D(\sin x)$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}\cos x$$

(ii) $\frac{d^2y}{dx^2} - 4y = x\cos 2x$

Write the equation as $(D^2-4)y = x\cos 2x$

The particular integral is given by

$$\text{P.I.} = \frac{1}{D^2-4}x\cos 2x$$

$$= x\frac{1}{D^2-4}\cos 2x - \frac{2D}{(D^2-4)^2}\cos 2x$$

$$= x\frac{1}{-4-4}\cos 2x - \frac{2D}{(-4-4)^2}\cos 2x$$

$$= -\frac{1}{8}x\cos 2x - \frac{1}{32}D(\cos 2x)$$

$$= -\frac{1}{8}x\cos 2x - \frac{1}{32}(-2\sin 2x)$$

$$= -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x$$

4.24 GENERAL SOLUTION OF DIFFERENTIAL EQUATION $F(D)y = Q$

THEOREM (4.14): **Prove that the general solution of the differential equation**

$$F(D)y = Q$$

is $y = y_c(x) + y_p(x)$

where $y_c(x)$ **is the complementary function and** $y_p(x)$ **is the particular integral of the given equation.**

PROOF: Given differential equation is $F(D)y = Q$ (1)

Since $y_c(x)$ is the complementary function of equation (1), therefore

$$F(D)y_c = 0 \qquad (2)$$

Also $y_p(x)$ is a particular integral of equation (1), therefore

i.e. $y_p(x) = \frac{1}{F(D)}Q$ so $F(D)y_p(x) = Q$ (3)

Now if $y = y_c(x) + y_p(x)$, then

$$\begin{aligned} F(D)y &= F(D)\left[y_c(x) + y_p(x)\right] \\ &= F(D)y_c(x) + F(D)y_p(x) \\ &= 0 + Q = Q \qquad [\text{using equations (2) and (3)}] \end{aligned}$$

Hence $y = y_c(x) + y_p(x)$ is a solution of equation (1). And, since it contains two arbitrary constants, it is the general solution of equation (1).

EXAMPLE (17): **Solve the following differential equations :**

(i) $(D^2 + 1)y = x^3 + e^x \cos x$ (ii) $(D^2 - 2D + 1)y = x\sin x$

(iii) $(D^3 + 1)y = \cos x$ (iv) $(D^4 + 4D^2)y = 96x^2$

SOLUTION: (i) $(D^2 + 1)y = x^3 + e^x \cos x$

The auxiliary equation is $D^2 + 1 = 0$ whose characteristic roots are $\pm i$.

The complementary function is

$$y_c(x) = C_1 \cos x + C_2 \sin x \qquad (1)$$

The particular integral is given by

$$\begin{aligned}
y_p(x) &= \frac{1}{D^2+1}(x^3+e^x\cos x) \\
&= \frac{1}{D^2+1}x^3+\frac{1}{D^2+1}e^x\cos x \\
&= (1+D^2)^{-1}x^3+e^x\frac{1}{(D+1)^2+1}\cos x \\
&= (1-D^2+D^4)x^3+e^x\frac{1}{D^2+2D+2}\cos x \\
&= (x^3-6x)+e^x\frac{1}{-1+2D+2}\cos x = x^3-6x+e^x\frac{1}{2D+1}\cos x \\
&= x^3-6x+e^x\frac{2D-1}{4D^2-1}\cos x = x^3-6x+e^x\frac{2D-1}{4(-1)-1}\cos x \\
&= x^3-6x-\frac{1}{5}e^x(2D-1)\cos x \\
&= x^3-6x-\frac{1}{5}e^x(-2\sin x-\cos x) \\
&= x^3-6x+\frac{1}{5}e^x(2\sin x+\cos x) \qquad (2)
\end{aligned}$$

From equations (1) and (2) the general solution of the given differential equation is

$$\begin{aligned}
y &= y_c(x)+y_p(x) \\
&= C_1\cos x+C_2\sin x+x^3-6x+\frac{1}{5}e^x(2\sin x+\cos x)
\end{aligned}$$

(ii) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y = x\sin x$

Write the equation as $(D^2-2D+1)y = x\sin x$

The auxiliary equation is $D^2-2D+1=0$

or $(D-1)^2=0$ whose roots are $1, 1$.

The complementary function is

$$y_c(x) = (C_1+C_2x)e^x \qquad (1)$$

The particular integral is given by

$$\begin{aligned}
y_p(x) &= \frac{1}{D^2-2D+1}x\sin x \\
&= x\frac{1}{D^2-2D+1}\sin x-\frac{2D-2}{(D^2-2D+1)^2}\sin x \\
&= x\frac{1}{-1-2D+1}\sin x-\frac{2D-2}{(-1-2D+1)^2}\sin x \\
&= x\left(-\frac{1}{2D}\right)\sin x-\frac{2D-2}{4D^2}\sin x
\end{aligned}$$

$$= -\frac{1}{2}x\left(\frac{1}{D}\sin x\right) - \frac{2D-2}{4(-1)}\sin x = \frac{1}{2}x\cos x + \frac{1}{2}(D-1)\sin x$$

$$= \frac{1}{2}x\cos x + \frac{1}{2}(\cos x - \sin x) \qquad (2)$$

From equations (1) and (2) the general solution is given by

$$y = y_c(x) + y_p(x)$$

$$= (C_1 + C_2 x)e^x + \frac{1}{2}x\cos x + \frac{1}{2}(\cos x - \sin x)$$

(iii) $(D^3 + 1)y = \cos x$

The auxiliary equation is $D^3 + 1 = 0$ or $(D+1)(D^2 - D + 1) = 0$

The characteristic roots are $D = -1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

The complementary function is

$$y_c(x) = C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right) \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3+1}\cos x = \frac{1}{D^2 . D + 1}\cos x$$

$$= \frac{1}{-D+1}\cos x = \frac{1+D}{1-D^2}\cos x$$

$$= \frac{1+D}{1-(-1)}\cos x = \frac{1}{2}(1+D)\cos x = \frac{1}{2}(\cos x - \sin x) \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y = y_p(x) + y_c(x)$$

$$= C_1 e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{2}(\cos x - \sin x)$$

(iv) $(D^4 + 4D^2)y = 96x^2$

The auxiliary equation is $D^4 + 4D^2 = 0$ or $D^2(D^2 + 4) = 0$

having the characteristic roots $D = 0, 0,\ D = \pm 2i$.

The complementary function is

$$y_c(x) = (C_1 + C_2 x) + (C_3\cos 2x + C_4\sin 2x)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^4 + 4D^2}96x^2 = 96\frac{1}{D^2}\frac{1}{D^2+4}x^2$$

$$= 24\frac{1}{D^2}\frac{1}{\left(1+\frac{D^2}{4}\right)}x^2 = 24\frac{1}{D^2}\left(1+\frac{D^2}{4}\right)^{-1}x^2$$

$$= 24\frac{1}{D^2}\left(1-\frac{D^2}{4}\right)x^2 = 24\left(\frac{1}{D^2}-\frac{1}{4}\right)x^2 = 2x^4-6x^2$$

The general solution of the given differential equation is

$$y(x) = y_c(x)+y_p(x)$$

$$= C_1 + C_2 x + C_3\cos 2x + C_4\sin 2x + 2x^4 - 6x^2$$

4.25 METHOD OF VARIATION OF PARAMETERS

This method (also called variation of constants) due to French mathematician J.L. Lagrange (1736 – 1813) is a general method for finding the solution of the nth order non – homogeneous linear differential equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots+P_{n-1}\frac{dy}{dx}+P_n y = Q$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants and Q is given function x.

FOR SECOND ORDER DIFFERENTIAL EQUATIONS

For simplicity, we discuss the method for the second order differential equation.

$$y'' + P_1 y' + P_2 y = Q \qquad (1)$$

To use this method it is necessary to know the general solution of the corresponding homogeneous equation

$$y'' + P_1 y' + P_2 y = 0 \qquad (2)$$

Suppose that the complementary function of equation (1) [i.e. the general solution of equation (2)] is

$$y_c = C_1 y_1(x) + C_2 y_2(x) \qquad (3)$$

where C_1 and C_2 are arbitrary constants and y_1 and y_2 are two linearly independent solutions of homogeneous differential equation (2).

The method of variation of parameters involves in replacing C_1 and C_2 by the two unknown functions $u_1(x)$ and $u_2(x)$ to be determined such that the resulting function

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \qquad (4)$$

becomes a particular integral of equation (1). This replacement of constants or parameters by variables gives the method its name. Differentiating equation (4), we get

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2 \qquad (5)$$

Now equation (4) contains two unknown functions u_1 and u_2, we must have two conditions for determining them. However, one of these conditions is that the differential equation must be satisfied. Thus we are at liberty to impose arbitrarily the second condition. We choose this condition to be the one which simplifies equation (5) most, namely

$$u_1' y_1 + u_2' y_2 = 0 \qquad (6)$$

Then equation (5) reduces to

$$y_p' = u_1 y_1' + u_2 y_2' \qquad (7)$$

Differentiating equation (7), we get

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' \qquad (8)$$

Since y_p is a particular integral of equation (1), therefore substituting the values of y_p, y_p', and y_p'' from equations (4), (7) and (8) into equation (1) and collecting terms containing u_1 and terms containing u_2, we obtain

$$u_1\left(y_1'' + P_1 y_1' + P_2 y_1\right) + u_2\left(y_2'' + P_1 y_2' + P_2 y_2\right) + u_1' y_1' + u_2' y_2' = Q$$

The two bracketed terms are zero, since y_1 and y_2 are solutions of the homogeneous equation (2), thus this equation reduces to

$$u_1' y_1' + u_2' y_2' = Q \qquad (9)$$

This gives a second equation relating $u_1'(x)$ and $u_2'(x)$. Equations (6) and (9)

i.e. $\quad u_1' y_1 + u_2' y_2 = 0$

$\quad\quad u_1' y_1' + u_2' y_2' = Q$

is a system of two linear algebraic equations for the unknown functions u_1' and u_2'. The solution of this system can be obtained using Cramer's rule as

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ Q & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 Q}{W}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 Q}{W} \qquad (10)$$

where $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian.

The unknown functions $u_1(x)$ and $u_2(x)$ can then be obtained by integrating u_1' and u_2' respectively,

i.e.
$$\left.\begin{aligned} u_1(x) &= -\int \frac{y_2 Q}{W}\,dx \\ u_2(x) &= \int \frac{y_1 Q}{W}\,dx \end{aligned}\right\} \qquad (11)$$

Substituting $u_1(x)$ and $u_2(x)$ from equations (11) we obtain the particular integral from equation (4), and finally the general solution of equation (1) is given by

$$y = y_c + y_p$$

NOTE (1): If the constants of integration in equations (11) are left arbitrary, then equation (4) represents the general solution of equation (1).

NOTE (2): In applying the method of variation of parameters to solve equation (1), we can work directly with the equations in (10) or (11). It is not necessary to rederive them. The steps are as follows :

(1) Find the complementary function y_c and write the functions y_1 and y_2.

(2) Calculate W and u_1 and u_2 from equations (11).

(3) Substitute u_1 and u_2 in equation (4) to find the particular integral y_p and then write the general solution of equation (1) as $y = y_c + y_p$. Alternatively, if we write the constants of integration C_1 and C_2 in equations (11), then equation (4) represents the general solution of differential equation (1).

FOR THIRD ORDER DIFFERENTIAL EQUATION

We now discuss the method for the third order differential equation

$$\frac{d^3y}{dx^3} + P_1\frac{d^2y}{dx^2} + P_2\frac{dy}{dx} + P_3 y = Q$$

i.e. $$y''' + P_1 y'' + P_2 y' + P_3 y = Q \tag{1A}$$

The method works if we know the complementary function of equation (1A) i.e. the general solution of the corresponding homogeneous equation

$$y''' + P_1 y'' + P_2 y' + P_3 y = 0 \tag{2A}$$

Suppose that the complementary function of equation (1A) [i.e. the general solution of equation (2A)] is

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) \tag{3A}$$

where C_1, C_2, and C_3 are arbitrary constants and $y_1(x)$, $y_2(x)$, and $y_3(x)$ are three linearly independent solutions of equation (2A).

The method of variation of parameters involves in replacing C_1, C_2, and C_3 by the three unknown functions $u_1(x)$, $u_2(x)$, and $u_3(x)$ to be determined such that the resulting function

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x) \tag{4A}$$

becomes a particular integral of equation (1A). Differentiating equation (4A), we get

$$y_p' = u_1 y_1' + u_2 y_2' + u_3 y_3' + u_1' y_1 + u_2' y_2 + u_3' y_3 \tag{5A}$$

Now equation (4A) contains three unknown functions u_1, u_2, and u_3, we must have three conditions for determining them. However, one of these conditions is that the differential equation must be satisfied. Thus we are at liberty to impose arbitrarily the remaining two conditions. We choose one of the two conditions to be the one which simplifies equation (5A) most, namely

$$u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \tag{6A}$$

Then equation (5A) reduces to

$$y_p' = u_1 y_1' + u_2 y_2' + u_3 y_3' \tag{7A}$$

By differentiating equation (7A), we get

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_3 y_3'' + u_1' y_1' + u_2' y_2' + u_3' y_3' \tag{8A}$$

Thus in order to avoid the second order derivatives of u, we impose the other condition so that

$$u_1' y_1' + u_2' y_2' + u_3' y_3' = 0 \tag{9A}$$

Thus equation (8A) reduces to

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_3 y_3'' \tag{10A}$$

Differentiating equation (10A) again, we get

$$y_p''' = u_1 y_1''' + u_2 y_2''' + u_3 y_3''' + u_1' y_1'' + u_2' y_2'' + u_3' y_3'' \tag{11A}$$

Since y_p is a particular integral of equation (1A), therefore substituting the values of y_p, y_p', y_p'', and y_p''' from equations (4A), (7A), (10A), and (11A) into equation (1A), we get

$$(u_1 y_1''' + u_2 y_2''' + u_3 y_3''' + u_1' y_1'' + u_2' y_2'' + u_3' y_3'') + P_1 (u_1 y_1'' + u_2 y_2'' + u_3 y_3'')$$
$$+ P_2 (u_1 y_1' + u_2 y_2' + u_3 y_3') + P_3 (u_1 y_1 + u_2 y_2 + u_3 y_3) = Q$$

or $$u_1 (y_1''' + P_1 y_1'' + P_2 y_1' + P_3 y_1) + u_2 (y_2''' + P_1 y_2'' + P_2 y_2' + P_3 y_2)$$
$$+ u_3 (y_3''' + P_1 y_3'' + P_2 y_3' + P_3 y_3) + u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = Q$$

The three bracketed terms are zero, since y_1, y_2, and y_3 are solutions of the homogeneous equation (2A). Thus this equation reduces to

$$u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = Q \tag{12A}$$

This gives the third equation relating u_1', u_2', and u_3'. Equations (6A), (9A), and (12A)

i.e. $$\left.\begin{aligned} u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= Q \end{aligned}\right] \tag{13A}$$

is a system of three linear algebraic equations for the unknown functions u_1', u_2', and u_3'. The solution of this system can be obtained using Cramer's rule.

The determinant of the coefficients of this system is the Wronskian and is given by

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

The unknown u_1', u_2', and u_3' are given by

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ Q & y_2'' & y_3'' \end{vmatrix}}{W} = \frac{\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} Q}{W} = \frac{(y_2 y_3' - y_3 y_2')\,Q}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & Q & y_3'' \end{vmatrix}}{W} = -\frac{\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} Q}{W} = \frac{(y_3 y_1' - y_1 y_3')\,Q}{W}$$

$$u_3' = \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & Q \end{vmatrix}}{W} = \frac{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} Q}{W} = \frac{(y_1 y_2' - y_2 y_1')\,Q}{W}$$

The unknown functions $u_1(x)$, $u_2(x)$, and $u_3(x)$ can then be obtained by integrating u_1', u_2', and u_3' respectively. Thus

i.e.
$$\left.\begin{aligned} u_1(x) &= \int \frac{(y_2 y_3' - y_3 y_2')\,Q}{W}\,dx \\ u_2(x) &= \int \frac{(y_3 y_1' - y_1 y_3')\,Q}{W}\,dx \\ u_3(x) &= \int \frac{(y_1 y_2' - y_2 y_1')\,Q}{W}\,dx \end{aligned}\right\} \qquad (14A)$$

Substituting $u_1(x)$, $u_2(x)$, and $u_3(x)$ from equations (14A) we obtain the particular integral from equation (4A), and finally the general solution of equation (1A) is given by

$y(x) = y_c + y_p$.

NOTE (1): If the constants of integration in equations (14A) are left arbitrary, then equation (4A) represents the general solution of equation (1A).

NOTE (2): In applying the method of variation of parameters to solve equation (1A), we can work directly with the equation in (14A). It is not necessary to rederive them. The steps are as follows:

(1) Find the complementary function y_c and write the functions y_1, y_2, and y_3.

(2) Calculate Wronskian W and u_1, u_2, and u_3 from equations (14A).

(3) Substitute u_1, u_2, and u_3 in equation (4A) to find the particular integral y_p and then write the general solution of equation (1A) as $y = y_c + y_p$. Alternatively, if we write the constants of integration C_1, C_2, and C_3 in equations (14A), then equation (4A) represents the general solution of differential equation (1A).

EXAMPLE (18): Solve the following differential equations :

(i) $y'' + y = \tan x\ , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ (ii) $(D^3 + D)y = \sec x$

SOLUTION: (i) Write the equation as

$$(D^2 + 1)y = \tan x$$

The auxiliary equation is $D^2 + 1 = 0$ having roots $D = \pm i$.

The complementary function of the given equation is

$$y_c = C_1 \cos x + C_2 \sin x \tag{1}$$

Here $y_1(x) = \cos x$ and $y_2(x) = \sin x$

This gives $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Now from equations (10)

$$\begin{aligned} u_1(x) &= -\int \frac{y_2 Q}{W}\, dx = -\int \frac{\sin x \tan x\, dx}{1} \\ &= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx \\ &= -\int (\sec x - \cos x)\, dx \\ &= -\ln|\sec x + \tan x| + \sin x \end{aligned}$$

and $u_2(x) = \int \frac{y_1 Q}{W}\, dx = \int \frac{\cos x \tan x}{1}\, dx = \int \sin x\, dx = -\cos x$

Thus the particular integral is given by

$$\begin{aligned} y_p &= u_1(x) y_1(x) + u_2(x) y_2(x) \\ &= \left[-\ln|\sec x + \tan x| + \sin x \right] \cos x - \cos x \sin x \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned} \tag{2}$$

From equations (1) and (2) the general solution of the given differential equation is given by

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x| \end{aligned}$$

(ii) $(D^3 + D)y = \sec x$

The auxiliary equation is $D^3 + D = 0$

or $D(D^2 + 1) = 0$ having characteristic roots $D = 0, \pm i$.

The complementary function is

$$y_c(x) = C_1 + C_2 \cos x + C_3 \sin x \tag{1}$$

Here $y_1(x) = 1, \quad y_2(x) = \cos x, \quad y_3(x) = \sin x$

This gives

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

Now
$$u_1(x) = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W} dx$$

$$= \int \frac{[\cos x \cos x - \sin x(-\sin x)] \sec x}{1} dx$$

$$= \int (\cos^2 x + \sin^2 x) \sec x \; dx$$

$$= \int \sec x \; dx = \ln|\sec x + \tan x|$$

$$u_2(x) = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W} dx$$

$$= \int \frac{(-\cos x)\sec x}{1} dx = -\int dx = -x$$

and
$$u_3(x) = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W} dx$$

$$= \int \frac{(-\sin x)\sec x}{1} dx = -\int \tan x \, dx = \ln \cos x$$

Thus the particular integral is given by

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + u_3(x) y_3(x)$$

$$= \ln|\sec x + \tan x| - x \cos x + \sin x \ln(\cos x) \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is given by

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1 + C_2 \cos x + C_3 \sin x + \ln|\sec x + \tan x| - x\cos x + \sin x \ln(\cos x)$$

4.26 METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is also the general method for finding the particular integral of the nth order non – homogeneous linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots + P_{n-1} \frac{dy}{dx} + P_n y = Q(x) \qquad (1)$$

where $P_1, \ldots\ldots, P_n$ are constants, and $Q(x)$ may be expontential functions, polynomials, cosines, sines, or sums or products of such functions as given in the following table.

TABLE

Terms in $Q(x)$	Choice for y_p
$A e^{ax}$	$K e^{ax}$
$A x^n \,(n = 0, 1, 2, \ldots\ldots)$	$K_n x^n + K_{n-1} x^{n-1} + \ldots\ldots + K_1 x + K_0$
$A \cos bx$, $A \sin bx$	$K_1 \cos bx + K_2 \sin bx$
$A e^{ax} \cos bx$, $A e^{ax} \sin bx$	$e^{ax}(K_1 \cos bx + K_2 \sin bx)$
$A e^{ax} x^n \,(n = 0, 1, 2, \ldots\ldots)$	$e^{ax}(K_n x^n + K_{n-1} x^{n-1} + \ldots\ldots + K_1 x + K_0)$
$A x^n \sin bx$ or $A x^n \cos bx$, $n = 0, 1, 2, \ldots\ldots$	$(K_n x^n + K_{n-1} x^{n-1} + \ldots\ldots + K_1 x + K_0) \cos bx$ $+ (L_n x^n + L_{n-1} x^{n-1} + \ldots\ldots + L_1 x + L_0) \sin bx$

RULES FOR THE METHOD OF SOLUTION

(i) BASIC RULE

If $Q(x)$ in equation (1) is one of the functions in the first column in the table, choose the corresponding function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into equation (1).

(ii) MODIFIED RULE

If a term in the choice for y_p happens to be in the complementary function, then multiply the choice for y_p by x (or by x^2 if this solution corresponds to a double root of the auxiliary equation).

(iii) SUM RULE

If $Q(x)$ is a sum of several functions in first column, then choose for y_p the sum of functions of the second column.

EXAMPLE (19): **Solve the following differential equations :**

(i) $y'' + 4y = 8x^2$ (ii) $y'' - 3y' + 2y = e^x$

(iii) $y'' + 2y' + y = e^{-x}$ (iv) $y'' + 2y' + 2y = x^2 + \sin x$

SOLUTION: (i) $y'' + 4y = 8x^2$

Write the equation as $(D^2 + 4)y = 8x^2$

The auxiliary equation is $D^2+4=0$ having characteristic roots $D=\pm 2i$.

Therefore the complementary function is given by

$$y_c(x) = C_1 \cos 2x + C_2 \sin 2x \tag{1}$$

Since $Q(x) = 8x^2$, therefore from the table we choose the particular integral

$$y_p = K_2 x^2 + K_1 x + K_0 \tag{2}$$

Then $y_p' = 2K_2 x + K_1$, $y_p'' = 2K_2$

Substituting the values of y_p and y_p'' in the given differential equation, we get

$$2K_2 + 4(K_2 x^2 + K_1 x + K_0) = 8x^2$$

or $$4K_2 x^2 + 4K_1 x + (2K_2 + 4K_0) = 8x^2$$

Equating the coefficients of x^2, x, and constant term on both sides, we get

$$4K_2 = 8, \quad 4K_1 = 0, \quad 2K_2 + 4K_0 = 0$$

or $$K_2 = 2, \quad K_1 = 0, \quad K_0 = -1$$

Thus equation (2) becomes

$$y_p = 2x^2 - 1 \tag{3}$$

From equations (1) and (3), we get the general solution as

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= C_1 \cos 2x + C_2 \sin 2x + 2x^2 - 1 \end{aligned}$$

(ii) $y'' - 3y' + 2y = e^x$

Write the equation as $(D^2 - 3D + 2)y = e^x$

The auxiliary equation is $D^2 - 3D + 2 = 0$

or $(D-1)(D-2) = 0$ having characteristic roots $D = 1, 2$.

Thus the complementary function is given by

$$y_c(x) = C_1 e^x + C_2 e^{2x} \tag{1}$$

Now we cannot choose $y_p = Ke^x$ since e^x appears in the complementary function. Thus we choose

$$y_p = Kxe^x \tag{2}$$

therefore $y_p' = K(xe^x + e^x)$, $y_p'' = K(xe^x + e^x + e^x) = K(xe^x + 2e^x)$

Substituting in the given differential equation, we get

$$K(xe^x + 2e^x) - 3K(xe^x + e^x) + 2Kxe^x = e^x$$

or $-Ke^x = e^x$ or $K = -1$.

Thus from equation (2), we get

$$y_p = -xe^x \tag{3}$$

Hence from equations (1) and (3), we get the general solution as

$$y(x) = y_c(x) + y_p(x)$$
$$= C_1 e^x + C_2 e^{2x} - x e^x$$

(iii) $y'' + 2y' + y = e^{-x}$

Write the equation as $(D^2 + 2D + 1)y = e^{-x}$

The auxiliary equation is $D^2 + 2D + 1 = 0$

or $(D+1)^2 = 0$ having characteristic roots $D = -1, -1$.

Thus the complementary function is given by

$$y_c(x) = (C_1 + C_2 x)e^{-x} \tag{1}$$

Now we cannot choose the particular integral as $y_p = K e^{-x}$. Since e^{-x} and $x e^{-x}$ are the solutions in the complementary function, therefore we choose

$$y_p = K x^2 e^{-x} \tag{2}$$

therefore $y_p' = K(-x^2 e^{-x} + 2x e^{-x})$

$$y_p'' = K(x^2 e^{-x} - 2x e^{-x} + 2e^{-x} - 2x e^{-x})$$
$$= K(x^2 e^{-x} - 4x e^{-x} + 2e^{-x})$$

Substituting y_p, y_p', and y_p'' in the given differential equation, we get

$$K(x^2 e^{-x} - 4x e^{-x} + 2e^{-x}) + 2K(-x^2 e^{-x} + 2x e^{-x}) + K x^2 e^{-x} = e^{-x}$$

$$2K e^{-x} = e^{-x} \quad \text{or} \quad K = \frac{1}{2}$$

Thus from equation (2) we get $y_p = \frac{1}{2} x^2 e^{-x}$ (3)

From equations (1) and (3), we get the general solution as

$$y(x) = y_c(x) + y_p(x)$$
$$= (C_1 + C_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$$

(iv) $y'' + 2y' + 2y = x^2 + \sin x$

Write the differential equation as

$$(D^2 + 2D + 2)y = x^2 + \sin x \tag{1}$$

The auxiliary equation is

$$D^2 + 2D + 2 = 0$$

having characteristic roots $D = -1 \pm i$.

The complementary function is

$$y_c(x) = e^{-x}(C_1 \cos x + C_2 \sin x) \tag{2}$$

Since the R.H.S. is the sum of two functions, therefore as a particular integral, we take

$$y_p(x) = K_2 x^2 + K_1 x + K_0 + K_3 \cos x + K_4 \sin x \tag{3}$$

$$y_p'(x) = 2K_2 x + K_1 - K_3 \sin x + K_4 \cos x$$

$$y_p''(x) = 2K_2 - K_3 \cos x - K_4 \sin x$$

Substituting in equation (1), we get

$$2K_2 - K_3 \cos x - K_4 \sin x + 2(2K_2 x + K_1 - K_3 \sin x + K_4 \cos x)$$
$$+ 2(K_2 x^2 + K_1 x + K_0 + K_3 \cos x + K_4 \sin x) = x^2 + \sin x$$

or
$$2K_2 x^2 + (2K_1 + 4K_2)x + (2K_2 + 2K_1 + 2K_0) + (K_3 + 2K_4)\cos x$$
$$+ (K_4 - 2K_3)\sin x = x^2 + \sin x$$

Comparing the coefficients of like powers on both sides, we get

$$2K_2 = 1,\ 2K_1 + 4K_2 = 0,\ 2K_2 + 2K_1 + 2K_0 = 0,\ K_3 + 2K_4 = 0,\ K_4 - 2K_3 = 1$$

This implies $K_2 = \frac{1}{2},\ K_1 = -1,\ K_0 = \frac{1}{2},\ K_3 = -\frac{2}{5},\ K_4 = \frac{1}{5}$

Substituting these values in equation (3), the particular integral is given by

$$y_p(x) = \frac{1}{2}x^2 - x + \frac{1}{2} - \frac{2}{5}\cos x + \frac{1}{5}\sin x$$

$$= \frac{1}{2}(x-1)^2 + \frac{1}{5}(\sin x - 2\cos x) \tag{4}$$

From equations (2) and (4) the general solution of equation (1) is

$$y = y_c(x) + y_p(x)$$

$$= e^{-x}(C_1 \cos x + C_2 \sin x) + \frac{1}{2}(x-1)^2 + \frac{1}{5}(\sin x - 2\cos x)$$

EXAMPLE (20): **Solve the following differential equations using the method of undetermined coefficients**

(i) $(D^3 + 2D^2 - D - 2)y = x^2 + e^x$

(ii) $(D^3 + 4D)y = 16 \sin 2x$

SOLUTION: (i) $(D^3 + 2D^2 - D - 2)y = x^2 + e^x$ (1)

The auxiliary equation is $D^3 + 2D^2 - D - 2 = 0$

or $(D^2 - 1)(D + 2) = 0$ having characteristic roots $D = \pm 1, -2$.

The complementary function is

$$y_c(x) = C_1 e^x + C_2 e^{-x} + C_3 e^{-2x} \tag{2}$$

Since $Q(x) = x^2 + e^x$, therefore corresponding to the first term x^2, we choose the terms in the particular integral as $K_2 x^2 + K_1 x + K_0$. Now for the second term e^x, we cannot choose the term in the particular integral as $K e^x$, since e^x appears in the complementary function. Therefore, we take the term $K_3 x e^x$.

Thus we choose the particular integral as

$$y_p(x) = K_2 x^2 + K_1 x + K_0 + K_3 x e^x \quad (3)$$

Now $y_p'(x) = 2K_2 x + K_1 + K_3 x e^x + K_3 e^x$

$$y_p''(x) = 2K_2 + K_3 x e^x + 2K_3 e^x$$

$$y_p'''(x) = K_3 x e^x + 3K_3 e^x$$

Substituting the values of $y, y_p'(x), y_p''(x), y_p'''(x)$ in the given differential equation, we get

$$K_3 x e^x + 3K_3 e^x + 4K_2 + 2K_3 x e^x + 4K_3 e^x - 2K_2 x - K_1 - K_3 x e^x$$
$$- K_3 e^x - 2K_2 x^2 - 2K_1 x - 2K_0 - 2K_3 x e^x = x^2 + e^x$$

or $6K_3 e^x - 2K_2 x^2 - (2K_1 + 2K_2)x + (4K_2 - K_1 - 2K_0) = x^2 + e^x$

Comparing the coefficients of like powers on both sides, we get

$$6K_3 = 1, \quad -2K_2 = 1, \quad -(2K_1 + 2K_2) = 0, \quad 4K_2 - K_1 - 2K_0 = 0$$

This implies $K_3 = \frac{1}{6}, \quad K_2 = -\frac{1}{2}, \quad K_1 = \frac{1}{2}, \quad K_0 = -\frac{5}{4}$

Substituting these values in equation (3), the particular integral is given by

$$y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4} + \frac{1}{6}x e^x \quad (4)$$

From equations (2) and (4) the general solution of equation (1) is given by

$$y(x) = y_c(x) + y_p(x)$$
$$= C_1 e^x + C_2 e^{-x} + C_3 e^{-2x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{5}{4} + \frac{1}{6}x e^x$$

(ii) $(D^3 + 4D)y = 16 \sin 2x$

The auxiliary equation is $D^3 + 4D = 0$ (1)

or $D(D^2 + 4) = 0$ having characteristic roots $D = 0, \pm 2i$.

The complementary function is

$$y_c(x) = C_1 + C_2 \cos 2x + C_3 \sin 2x \quad (2)$$

Since $Q(x) = 16 \sin 2x$, therefore we choose the terms $K_1 x \cos 2x + K_2 x \sin 2x$ in the particular integral, since $\sin 2x$ appears in the complementary function. Thus we take

$$y_p(x) = K_1 x \cos 2x + K_2 x \sin 2x \quad (3)$$

Now $y_p'(x) = -2K_1 x \sin 2x + K_1 \cos 2x + 2K_2 x \cos 2x + K_2 \sin 2x$

$$y_p''(x) = -4K_1 x \cos 2x - 2K_1 \sin 2x - 2K_1 \sin 2x$$
$$- 4K_2 x \sin 2x + 2K_2 \cos 2x + 2K_2 \cos 2x$$
$$= -4K_1 x \cos 2x - 4K_1 \sin 2x - 4K_2 x \sin 2x + 4K_2 \cos 2x$$

and $\quad y_p'''(x) = 8K_1 x\sin 2x - 4K_1\cos 2x - 8K_1\cos 2x$

$$-8K_2 x\cos 2x - 4K_2\sin 2x - 8K_2\sin 2x$$

$$= 8K_1 x\sin 2x - 12K_1\cos 2x - 8K_2 x\cos 2x - 12K_2\sin 2x$$

Substituting the values of $y_p'(x)$, and $y_p'''(x)$ in the given differential equation, we get

$$8K_1 x\sin 2x - 12K_1\cos 2x - 8K_2 x\cos 2x - 12K_2\sin 2x$$

$$-8K_1 x\sin 2x + 4K_1\cos 2x + 8K_2 x\cos 2x + 4K_2\sin 2x = 16\sin 2x$$

or $\quad -8K_1\cos 2x - 8K_2\sin 2x = 16\sin 2x$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ on both sides, we get

$$-8K_1 = 0 \quad \text{and} \quad -8K_2 = 16$$

or $\quad K_1 = 0 \quad \text{and} \quad K_2 = -2$

Substituting these values in equation (3), the particular integral is given by

$$y_p(x) = -2x\sin 2x \qquad (4)$$

From equations (2) and (4) the general solution of equation (1) is given by

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1 + C_2\cos 2x + C_3\sin 2x - 2x\sin 2x$$

4.27 SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

A second order linear differential equation with variable coefficients has the form

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = Q \qquad (1)$$

where P_1, P_2, and Q are functions of x only.

For most of the differential equations of type (1) it is impossible to find solutions in compact form in terms of elementary functions. In most cases it is necessary to use techniques such as power series method to obtain information about solutions. However, there is one class of such equations that arise in applications for which the closed form solutions can be obtained. We discuss this class now.

4.28 EULER – CAUCHY DIFFERENTIAL EQUATION

A second order linear differential equation of the form

$$x^2y'' + Axy' + By = Q, \quad x \neq 0 \qquad (2)$$

where A and B are constants, is called an Euler – Cauchy differential equation.

Equation (2) can be written as

$$y'' + A\frac{1}{x}y' + B\frac{1}{x^2}y = \frac{1}{x^2}Q$$

which is not defined for $x = 0$. This is why we put the restriction that $x \neq 0$.

HOMOGENEOUS EULER – CAUCHY DIFFERENTIAL EQUATION

We begin by solving the homogeneous Euler – Cauchy equation

$$x^2 y'' + A x y' + B y = 0 , \; x > 0 \tag{3}$$

There are two approaches commonly used to solve differential equation (3). In both approaches, Euler – Cauchy equation (3) is transformed into a second order linear differential equation with constant coefficients, which we know how to solve.

METHOD (1)

To solve equation (3), let $y = x^m$, where m is some number. Then

$y' = m x^{m-1}$ and $y'' = m(m-1)x^{m-2}$, so that equation (3) becomes

$$x^2 m(m-1)x^{m-2} + A x m x^{m-1} + B x^m = 0$$

or $$m(m-1)x^m + A m x^m + B x^m = 0$$

or $$\left[m^2 + (A-1)m + B \right] x^m = 0$$

Since $x^m \neq 0$, we obtain the auxiliary equation

$$m^2 + (A-1)m + B = 0 \tag{4}$$

Now we consider three cases depending on the roots of equation (4).

(1) Roots real and distinct

(2) Roots real and repeated

(3) Complex conjugate roots.

CASE (1): ROOTS REAL AND DISTINCT

If the roots m_1 and m_2 of the auxiliary equation (4) are real and distinct, then $y_1(x) = x^{m_1}$ and $y_2(x) = x^{m_2}$ are two linearly independent solutions of differential equation (3) for all x for which these functions are defined. The corresponding general solution is :

$$y = C_1 x^{m_1} + C_2 x^{m_2} \tag{5}$$

where C_1 and C_2 are arbitrary constants.

EXAMPLE (21): **Solve the Euler – Cauchy equation** $x^2 y'' - \frac{5}{2} x y' - 2y = 0 , \; x > 0$.

SOLUTION: Here $A = -\frac{5}{2}$ and $B = -2$. Then the auxiliary equation (4) becomes

$$m^2 - \frac{7}{2} m - 2 = 0 \quad \text{or} \quad 2m^2 - 7m - 4 = 0 \quad \text{or} \quad (2m+1)(m-4) = 0$$

Its roots are $m_1 = -\frac{1}{2}$ and $m_2 = 4$. Hence the two linearly independent solutions for all positive x are $y = x^{-1/2}$ and $y_2 = x^4$, and the corresponding general solution of the given differential equation is :

$$y = C_1 x^{-1/2} + C_2 x^4$$

CASE (2): ROOTS REAL AND REPEATED

If the roots m_1 and m_2 of the auxiliary equation (4) are real and repeated, then

$m_1 = m_2 = m = \frac{1}{2}(1-A)$ is the only one root and we get only one solution

$$y_1 = x^m = x^{(1-A)/2}$$

To find the second linearly independent solution, we use the method of reduction of order. That is, we set $y_2 = u\,y_1$ and try to find the function u such that y_2 becomes a solution of the Euler – Cauchy equation (3). For this we substitute

$y_2 = u\,y_1$ and its derivatives

$$y_2' = u\,y_1' + u'\,y_1$$

$$y_2'' = u\,y_1'' + u'\,y_1' + u'\,y_1' + u''\,y_1 = u''\,y_1 + 2\,u'\,y_1' + u\,y_1''$$

into equation (3). This gives

$$x^2\left(u''\,y_1 + 2\,u'\,y_1' + u\,y_1''\right) + A\,x\left(u\,y_1' + u'\,y_1\right) + B\,u\,y_1 = 0$$

Collecting terms, we get

$$u''\,x^2 y_1 + u'\,x\left(2\,x\,y_1' + A\,y_1\right) + u\left(x^2 y_1'' + A\,x\,y_1' + B\,y_1\right) = 0 \tag{6}$$

The expression in the last parentheses

$x^2 y_1'' + A\,x\,y_1' + B\,y_1 = 0$ since y_1 is a solution of equation (3).

The expression in the first parentheses becomes

$$\begin{aligned} 2\,x\,y_1' + A\,y_1 &= 2\,x\,\frac{1}{2}(1-A)\,x^{-(1+A)/2} + A\,x^{(1-A)/2} \\ &= (1-A)\,x^{(1-A)/2} + A\,x^{(1-A)/2} \\ &= x^{(1-A)/2} = y_1 \end{aligned}$$

Thus equation (6) reduces to $u''\,x^2\,y_1 + u'\,x\,y_1 = 0$

or $(u''\,x + u')\,x\,y_1 = 0$

Since $x > 0$ and $y_1 \neq 0$, therefore $x\,y_1 \neq 0$ and thus $u''\,x + u' = 0$

Separating variables, $\dfrac{u''}{u'} = -\dfrac{1}{x}$

Integrating, $\ln|u'| = -\ln x = \ln\dfrac{1}{x}$

or $u' = \dfrac{1}{x}$ and $u = \ln x$

Thus $y_2 = y_1 \ln x$ is a second solution. Hence, in this case, the two linearly independent solutions of differential equation (3) are

$$y_1 = x^m \text{ and } y_2 = x^m \ln x \text{, where } m = \frac{1}{2}(1-A).$$

Thus the general solution of differential equation (3) is given by

$$y = C_1 x^m + C_2 x^m \ln x = (C_1 + C_2 \ln x) x^m \quad (7)$$

where C_1 and C_2 are arbitrary constants.

EXAMPLE (22): **Solve the Euler – Cauchy differential equation**

$$x^2 y'' - 3 x y' + 4 y = 0 , x > 0 .$$

SOLUTION: Here $A = -3$ and $B = 4$, then the auxiliary equation (4) becomes

$$m^2 - 4m + 4 = 0 \text{ or } (m-2)^2 = 0$$

Its roots are $m_1 = m_2 = m = 2$. The two linearly independent solutions for all $x > 0$ are x^2 and $x^2 \ln x$. Hence, the general solution of the given differential equation is :

$$y = (C_1 + C_2 \ln x) x^2$$

CASE (3): COMPLEX CONJUGATE ROOTS

If the roots m_1 and m_2 of the auxiliary equation (4) are complex conjugates, say $m_1 = a + i b$ and $m_2 = a - i b$, then

$y_1(x) = x^{a+ib}$ and $y_2(x) = x^{a-ib}$ are two linearly independent solutions of differential equation (3).

The corresponding general solution can be written as

$$y = c x^{a+ib} + d x^{a-ib} \quad (8)$$

Now using the fact that $x^a = e^{a \ln x}$ for $x > 0$ and the Euler formula $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$, we have

$$x^{a+ib} = x^a . x^{ib} = x^a . e^{ib(\ln x)} = x^a \left[\cos(b \ln x) + i \sin(b \ln x) \right]$$

Replacing b by $-b$, we get

$$x^{a-ib} = x^a \left[\cos(b \ln x) - i \sin(b \ln x) \right]$$

Thus equation (8) becomes

$$\begin{aligned} y &= x^a \left[c \cos(b \ln x) + i c \sin(b \ln x) + d \cos(b \ln x) - i d \sin(b \ln x) \right] \\ &= x^a \left[(c+d) \cos(b \ln x) + i (c-d) \sin(b \ln x) \right] \\ &= x^a \left[C_1 \cos(b \ln x) + C_2 \sin(b \ln x) \right] \quad (9) \end{aligned}$$

where $C_1 = c + d$ and $C_2 = i(c-d)$ are arbitrary constants. Thus if the roots of the auxiliary equation (4) are complex conjugates, then the general solution of Euler – Cauchy equation (3) is given by equation (9).

EXAMPLE (23): **Solve the Euler – Cauchy equation** $x^2 y'' + 5 x y' + 13 y = 0, \quad x > 0.$

SOLUTION: Here $A = 5$ and $B = 13$, then the auxiliary equation (4) becomes

$m^2 + 4m + 13 = 0$, whose roots are

and $m = \dfrac{-4 \pm \sqrt{16-52}}{2} = \dfrac{-4 \pm \sqrt{-36}}{2} = \dfrac{-4 \pm 6i}{2} = -2 \pm 3i$

Thus from equation (9), the general solution of the given Euler – Cauchy equation is given by

$$y = x^{-2}\left[C_1 \cos(3 \ln x) + C_2 \sin(3 \ln x) \right]$$

ALTERNATIVE METHOD OF SOLUTION

The Euler – Cauchy differential equation (3) can also be solved using the substitution

$x = e^t$ or $t = \ln x$, $x > 0$.

Now $\quad y' = \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{1}{x}\dfrac{dy}{dt}$

or $\quad x y' = \dfrac{dy}{dt} \qquad (10)$

and $\quad y'' = \dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\dfrac{dt}{dx}$

$$= \frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \left(e^{-t}\frac{d^2 y}{dt^2} - e^{-t}\frac{dy}{dt}\right)\frac{1}{x} = \left(\frac{1}{x}\frac{d^2 y}{dt^2} - \frac{1}{x}\frac{dy}{dt}\right)\frac{1}{x}$$

$$= \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)\frac{1}{x^2}$$

or $\quad x^2 y'' = \dfrac{d^2 y}{dt^2} - \dfrac{dy}{dt} \qquad (11)$

Using equations (10) and (11), Euler – Cauchy equation (3) takes the form

$$\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) + A\frac{dy}{dt} + B y = 0$$

or $\quad \dfrac{d^2 y}{dt^2} + (A-1)\dfrac{dy}{dt} + B y = 0 \qquad (12)$

This is a second order homogeneous linear differential equation with constant coefficients, whose auxiliary equation is $m^2 + (A-1)m + B = 0$ which is the same equation (4) of method (1). The general solution of the original Euler – Cauchy equation (3) is then obtained by substituting

$t = \ln x$.

Note that we can solve the Euler – Cauchy differential equation (3) for $x < 0$, by using the transformation $t = \ln(-x)$. The details are similar to those for $x > 0$ and we will not discuss them.

EXAMPLE (24): **Solve the following Euler – Cauchy differential equations :**

(i) $x^2 y'' - 2 x y' + 2 y = 0$

(ii) $x^2 y'' - 3 x y' + 4 y = 0$

(iii) $x^2 y'' - x y' + 10 y = 0, \ x > 0$

SOLUTION: (i) $x^2 y'' - 2 x y' + 2 y = 0$

Let $x = e^t$ or $t = \ln x, \ x > 0,$

Since $A = -2$ and $B = 2$, the transformed equation (12) becomes

$$\frac{d^2 y}{d t^2} - 3 \frac{d y}{d t} + 2 y = 0 \qquad \text{(i)}$$

Its auxiliary equation is $m^2 - 3 m + 2 = 0$

or $(m - 1)(m - 2) = 0$

whose roots are $m_1 = 1$ and $m_2 = 2$.

The general solution of the transformed equation (i) is given by

$$y(t) = C_1 e^t + C_2 e^{2t} \qquad \text{(ii)}$$

Let $t = \ln t$ in equation (ii) to get the general solution of the given differential equation as

$$y(x) = C_1 x + C_2 x^2$$

(ii) $x^2 y'' - 3 x y' + 4 y = 0$

Let $x = e^t$ or $t = \ln x, \ x > 0,$

Since $A = -3$ and $B = 4$, the transformed equation (12) becomes

$$\frac{d^2 y}{d t^2} - 4 \frac{d y}{d t} + 4 y = 0 \qquad \text{(i)}$$

Its auxiliary equation is $m^2 - 4 m + 4 = 0$ or $(m - 2)^2 = 0$

whose roots are $m_1 = m_2 = m = 2$.

The general solution of the transformed equation (i) is given by

$$y(t) = (C_1 + C_2 t) e^{2t} \qquad \text{(ii)}$$

Let $t = \ln x$ in equation (ii) to get the general solution of the given differential equation as

$$y(x) = (C_1 + C_2 \ln x) e^{2 \ln x} = (C_1 + C_2 \ln x) x^2$$

(iii) $x^2 y'' - x y' + 10 y = 0, \ x > 0$

Let $x = e^t$ or $t = \ln x$ for $x > 0$

Then since $A = -1$ and $B = 10$, we have from the transformed equation (12)

$$\frac{d^2 y}{d t^2} - 2 \frac{d y}{d t} + 10 y = 0 \qquad \text{(i)}$$

Its auxiliary equation is $m^2 - 2m + 10 = 0$ whose roots are $m = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$

Thus the general solution of equation (i) is given by

$$y(t) = e^t (C_1 \cos 3t + C_2 \sin 3t)$$

Let $t = \ln x$ to get the general solution of the given equation as

$$y(x) = e^{\ln x}\left[C_1 \cos(3\ln x) + C_2 \sin(3\ln x)\right]$$

$$= x\left[C_1 \cos(3\ln x) + C_2 \sin(3\ln x)\right]$$

NON – HOMOGENEOUS EULER – CAUCHY DIFFERENTIAL EQUATIONS

We now solve the non – homogeneous Euler – Cauchy differential equation (2)

i.e. $$x^2 y'' + A x y' + B y = Q, \quad x \neq 0 \tag{13}$$

Let $x = e^t$ or $t = \ln x$, $x > 0$. Then $x y' = \frac{dy}{dt}$ and $x^2 y'' = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$.

Thus Euler – Cauchy differential equation (13) takes the form

$$\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) + A\frac{dy}{dt} + B y = Q(e^t)$$

or $$\frac{d^2 y}{dt^2} + (A-1)\frac{dy}{dt} + B y = Q(e^t) \tag{14}$$

This is a second order non – homogeneous linear differential equation with constant coefficients and can be solved by the methods already known. The general solution of the original Euler – Cauchy equation is then obtained by substituting $t = \ln x$.

EXAMPLE (25): **Solve the following differential equations:**

(i) $x^2 y'' + x y' - 4y = x^2$ (ii) $x^2 y'' - 3xy + 5y = \sin(\ln x)$

(iii) $x^2 y'' - x y' + y = \ln x$ (iv) $x^2 y'' - 3 x y' + 4y = x^2 \ln x$

SOLUTION: (i) $x^2 y'' + x y' - 4y = x^2$

Comparing this equation with the general form of Euler – Cauchy differential equation, we see that

$$A = 1, \quad B = -4, \quad \text{and} \quad Q(x) = x^2$$

Let $x = e^t$ so that $t = \ln x$, $x > 0$. Then the differential equation (14) becomes

$$\frac{d^2 y}{dt^2} - 4y = e^{2t}$$

or $$(D_1^2 - 4)y = e^{2t} \qquad \left(\frac{d}{dt} = D_1\right) \tag{1}$$

The auxiliary equation is $D_1^2 - 4 = 0$ having characteristic roots $D_1 = \pm 2$.

The complementary function is

$$y_c(t) = C_1 e^{2t} + C_2 e^{-2t} \tag{2}$$

The particular integral is

$$y_p(t) = \frac{1}{D_1^2 - 4} e^{2t}$$

$$= t \frac{1}{\frac{d}{dD_1}(D_1^2 - 4)} e^{2t}$$

$$= t \frac{1}{2D_1} e^{2t} = \frac{1}{4} t e^{2t} \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is given by

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1 e^{2t} + C_2 e^{-2t} + \frac{1}{4} t e^{2t} \tag{4}$$

Since $t = \ln x$, the general solution (4) takes the form

$$y(x) = C_1 x^2 + C_2 x^{-2} + \frac{1}{4} x^2 \ln x$$

(ii) $x^2 y'' - 3xy + 5y = \sin(\ln x)$

Comparing this equation with the general form of Euler – Cauchy differential equation, we see that

$$A = -3, \quad B = 5, \quad Q(x) = \sin(\ln x)$$

Let $x = e^t$ so that $t = \ln x$, $x > 0$. Then the differential equation (14) becomes

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = \sin t$$

or $\quad (D_1^2 - 4D_1 + 5) y = \sin t \qquad \left(\frac{d}{dt} = D_1\right) \qquad (1)$

The auxiliary equation is $D_1^2 - 4D_1 + 5 = 0$ having characteristic roots $D = 2 \pm i$.

The complementary function is

$$y_c(t) = e^{2t}(C_1 \cos t + C_2 \sin t) \tag{2}$$

The particular integral is

$$y_p(t) = \frac{1}{D_1^2 - 4D_1 + 5} \sin t$$

$$= \frac{1}{-1 - 4D_1 + 5} \sin t = \frac{1}{4} \frac{1}{1 - D_1} \sin t$$

$$= \frac{1}{4} \frac{1 + D_1}{1 - D_1^2} \sin t = \frac{1}{8}(1 + D_1) \sin t$$

$$= \frac{1}{8}(\sin t + \cos t) \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is given by

$$y(t) = y_c(t) + y_p(t)$$
$$= e^{2t}(C_1 \cos t + C_2 \sin t) + \frac{1}{8}(\sin t + \cos t) \quad (4)$$

Since $t = \ln x$, the general solution (4) takes the form

$$y(x) = x^2[C_1 \cos(\ln x) + C_2 \sin(\ln x)] + \frac{1}{8}[\sin(\ln x) + \cos(\ln x)]$$

(iii) $x^2 y'' - x y' + y = \ln x$

Comparing this equation with the general form of Euler – Cauchy differential equation, we see that

$$A = -1, \quad B = 1, \quad Q(x) = \ln x$$

Let $x = e^t$ so that $t = \ln x$, $x > 0$. Then the differential equation (14) becomes

$$\frac{d^2 y}{dt^2} - 2\frac{dy}{dt} + y = t$$

or $$(D_1^2 - 2D_1 + 1)y = t \qquad \left(\frac{d}{dt} = D_1\right) \quad (1)$$

The auxiliary equation is $D_1^2 - 2D_1 + 1 = 0$

or $(D_1 - 1)^2 = 0$ having characteristic roots $D_1 = 1, 1$.

The complementary function is

$$y_c(t) = (C_1 + C_2 t)e^t \quad (2)$$

The particular integral is

$$y_p(t) = \frac{1}{D_1^2 - 2D_1 + 1} t$$
$$= \frac{1}{1 - (2D_1 - D_1^2)} t = [1 - (2D_1 - D_1^2)]^{-1} t$$
$$= [1 + (2D_1 - D_1^2) + \dots\dots] t$$
$$= (1 + 2D_1)t = t + 2 \quad (3)$$

From equations (2) and (3) the general solution of equation (1) is given by

$$y(t) = y_c(t) + y_p(t)$$
$$= (C_1 + C_2 t)e^t + t + 2 \quad (4)$$

Since $t = \ln x$, the general solution (4) takes the form

$$y(x) = (C_1 + C_2 \ln x)x + \ln x + 2$$

(iv) $x^2 y'' - 3xy' + 4y = x^2 \ln x$

Comparing this equation with the general form of Euler – Cauchy differential equation, we see that

$$A = -3, \quad B = 4, \quad Q(x) = x^2 \ln x$$

Let $x = e^t$ so that $t = \ln x$, $x > 0$. Then the differential equation (14) becomes

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}t$$

or $$\left(D_1^2 - 4D_1 + 4\right)y = e^{2t}t \qquad \left(\frac{d}{dt} = D_1\right) \tag{1}$$

The auxiliary equation is $D_1^2 - 4D_1 + 4 = 0$

or $\quad (D_1 - 2)^2 = 0$

having characteristic roots $D = 2, 2$.

The complementary function is

$$y_c(t) = (C_1 + C_2 t)e^{2t} \tag{2}$$

The particular integral is

$$\begin{aligned} y_p(t) &= \frac{1}{D_1^2 - 4D_1 + 4}e^{2t}t \\ &= e^{2t}\frac{1}{(D_1+2)^2 - 4(D_1+2) + 4}t \\ &= e^{2t}\frac{1}{D_1^2}t = \frac{1}{6}e^{2t}t^3 \end{aligned} \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is given by

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= (C_1 + C_2 t)e^{2t} + \frac{1}{6}e^{2t}t^3 \end{aligned} \tag{4}$$

Since $t = \ln x$, the general solution (4) takes the form

$$y(x) = (C_1 + C_2 \ln x)x^2 + \frac{1}{6}x^2 \ln^3 x$$

4.29 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

An nth order linear differential equation with variable coefficients has the form

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots + P_{n-1}\frac{dy}{dx} + P_n y = Q(x) \tag{1}$$

where $P_1, P_2, \dots\dots, P_n$, and Q are functions of x only.

For most of the differential equations of type (1) it is impossible to find solutions in compact form in terms of elementary functions. In most cases it is necessary to use techniques such as power series method to obtain information about solutions. However, there is one class of such equations that arise in applications for which the closed form solutions can be obtained. We discuss this class now.

4.30 EULER – CAUCHY LINEAR DIFFERENTIAL EQUATION

An nth order linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + A_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + A_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + A_{n-1} x \frac{dy}{dx} + A_n y = Q(x),\ x \neq 0 \tag{1}$$

where $A_1, A_2, \dots\dots, A_n$ are constants, is called an nth order Euler – Cauchy linear differential equation. Equation (1) can be written as

$$\frac{d^n y}{dx^n} + A_1 \frac{1}{x} \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{1}{x^2} \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + A_{n-1} \frac{1}{x^{n-1}} \frac{dy}{dx} + A_n \frac{1}{x^n} y = \frac{1}{x^n} Q$$

which is not defined for $x = 0$. This is why we put the restriction that $x \neq 0$.

In operator notation, equation (1) can be written as

$$(x^n D^n + A_1 x^{n-1} D^{n-1} + A_2 x^{n-2} D^{n-2} + \dots\dots\dots + A_{n-1} x D + A_n) y = Q \tag{2}$$

SOLUTION OF EULER – CAUCHY DIFFERENTIAL EQUATION

The method for solving second order Euler – Cauchy differential equations can be extended to solve an nth order Euler – Cauchy differential equation (1) or (2).

Equation (1) or (2) can be solved using the substitution $x = e^t$ or $t = \ln x,\ x > 0$.

Then if D_1 is defined by $D_1 = \frac{d}{dt}$, we have

$$Dy = \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

or $$xDy = D_1 y \tag{3}$$

$$D^2 y = \frac{d^2 y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)$$

or $$x^2 D^2 y = (D_1^2 - D_1) y = D_1 (D_1 - 1) y \tag{4}$$

Now
$$\begin{aligned}
D^3 y = \frac{d^3 y}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = \frac{d}{dt}\left(\frac{d^2 y}{dx^2}\right)\frac{dt}{dx} \\
&= \frac{d}{dt}\left[\frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)\right]\frac{dt}{dx} \\
&= \frac{d}{dt}\left[e^{-2t}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)\right]\frac{dt}{dx} \\
&= \left[e^{-2t}\left(\frac{d^3 y}{dt^3} - \frac{d^2 y}{dt^2}\right) - 2e^{-2t}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)\right]\frac{1}{x} \\
&= \left[\frac{1}{x^2}\left(\frac{d^3 y}{dt^3} - \frac{d^2 y}{dt^2}\right) - \frac{2}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)\right]\frac{1}{x}
\end{aligned}$$

$$= \left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)\frac{1}{x^3}$$

or $\quad x^3D^3y = (D_1^3 - 3D_1^2 + 2D_1)y = D_1(D_1-1)(D_1-2)y \qquad (5)$

Similarly, we can prove that

$$x^4D^4y = D_1(D_1-1)(D_1-2)(D_1-3)y$$

and in general, we get

$$x^nD^ny = [D_1(D_1-1)(D_1-2)(D_1-3)\ldots\ldots(D_1-n+1)]y$$

Substituting the values of $xDy, x^2D^2y, x^3D^3y, \ldots\ldots, x^nD^ny$ in equation (2) we get

$$[\{D_1(D_1-1)(D_1-2)\ldots\ldots(D_1-n+1)\} + A_1\{D_1(D_1-1)(D_1-2)\ldots\ldots(D_1-n+2)\}$$

$$+ \ldots\ldots + A_{n-2}\{D_1(D_1-1)\} + A_{n-1}D_1 + A_n]y = Q(e^t) \qquad (6)$$

which is an nth order linear differential equation with constant coefficients and can be solved by the methods already known. The general solution of the original Euler – Cauchy equation (1) or (2) is then obtained by substituting $t = \ln x$.

EXAMPLE (26): **Solve the following differential equations :**

(i) $\quad (x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$

(ii) $\quad (x^3D^3 + xD - 1)y = 3x^4$

SOLUTION: (i) $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$

Let $x = e^t$ or $t = \ln x, \ x > 0$. Then

$$xD = D_1, \quad x^2D^2 = D_1^2 - D_1, \quad x^3D^3 = D_1^3 - 3D_1^2 + 2D_1$$

Substituting into the given differential equation, we get

$$[D_1^3 - 3D_1^2 + 2D_1 + 3(D_1^2 - D_1) - 2D_1 + 2]y = 0$$

or $\quad (D_1^3 - 3D_1 + 2)y = 0 \qquad (1)$

The auxiliary equation is $\quad D_1^3 - 3D_1 + 2 = 0$

or $\quad (D_1-1)(D_1^2 + D_1 - 2) = 0$

or $\quad (D_1-1)(D_1+2)(D_1-1) = 0$

or $\quad (D_1-1)^2(D_1+2) = 0$ having characteristic roots $D_1 = 1, 1, -2$.

The general solution of the transformed equation (1) is given by

$$y(t) = (C_1 + C_2t)e^t + C_3e^{-2t} \qquad (2)$$

Let $t = \ln x$ in equation (2) to get the general solution of the given differential equation as

$$y(x) = (C_1 + C_2\ln x)x + C_3x^{-2}$$

(ii) $(x^3D^3+xD-1)y = 3x^4$

Let $x = e^t$ or $t = \ln x$, $x > 0$. Then

$$xD = D_1, \quad x^3D^3 = D_1^3-3D_1^2+2D_1$$

Substituting into the given differential equation, we get

$$(D_1^3-3D_1^2+2D_1+D_1-1)y = 3e^{4t}$$

or $$(D_1^3-3D_1^2+3D_1-1)y = 3e^{4t}$$

or $$(D_1-1)^3 = 3e^{4t} \tag{1}$$

The auxiliary equation is $(D_1-1)^3 = 0$ having characteristic roots $D = 1, 1, 1$.

The complementary function is given by

$$y_c(t) = (C_1+C_2t+C_3t^2)e^t \tag{2}$$

The particular integral is given by

$$y_p(t) = \frac{1}{(D_1-1)^3}3e^{4t} = 3\frac{1}{27}e^{4t} = \frac{1}{9}e^{4t} \tag{3}$$

From equations (2) and (3), the general solution of equation (1) is given by

$$\begin{aligned} y(t) &= y_c(t)+y_p(t) \\ &= (C_1+C_2t+C_3t^2)e^t+\frac{1}{9}e^{4t} \end{aligned} \tag{4}$$

Since $t = \ln x$, the general solution (4) takes the form

$$y(x) = (C_1+C_2\ln x+C_3\ln^2 x)x+\frac{1}{9}x^4$$

4.31 SECOND ORDER LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

A second order linear differential equation

$$(ax+b)^2\frac{d^2y}{dx^2}+A_1(ax+b)\frac{dy}{dx}+A_2y = Q(x) \tag{1}$$

where A_1, A_2, and a and b are constants, is called the Legendre's linear differential equation.

Note that Euler – Cauchy equation is a special case of equation (1) with $a = 1$ and $b = 0$. Equation (1) can be reduced to linear differential equation with constant coefficients using the transformation

$$ax+b = e^t \quad \text{or} \quad t = \ln(ax+b)$$

Then $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a}{ax+b}\frac{dy}{dt}$$

or $$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt} \tag{2}$$

and $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{a}{ax+b}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d}{dt}\left(a e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \left(a e^{-t}\frac{d^2y}{dt^2} - a e^{-t}\frac{dy}{dt}\right)\left(\frac{a}{ax+b}\right)$$

$$= \left(\frac{a}{ax+b}\frac{d^2y}{dx^2} - \frac{a}{ax+b}\frac{dy}{dt}\right)\left(\frac{a}{ax+b}\right)$$

$$= \frac{a^2}{(ax+b)^2}\frac{d^2y}{dx^2} - \frac{a^2}{(ax+b)^2}\frac{dy}{dt}$$

or $$(ax+b)^2 y'' = a^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \qquad (3)$$

Using equations (2) and (3), the Legendre's linear differential equation takes the form

$$a^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + A_1 a\frac{dy}{dt} + A_2 y = Q\left(\frac{e^t - b}{a}\right) \qquad (4)$$

If we let $\frac{d}{dt} = D_1$, then equation (4) can be written as

$$\left[a^2 D_1(D_1 - 1) + A_1 a D_1 + A_2\right]y = Q\left(\frac{e^t - b}{a}\right) \qquad (5)$$

which is a second order linear differential equation with constant coefficients and can be solved by the methods already known.

EXAMPLE (27): **Solve the following differential equations :**

(i) $$(2x+5)^2\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 0$$

(ii) $$(x+2)^2\frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$$

SOLUTION: (i) $(2x+5)^2\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 0$

Comparing with the general form of Legendre's linear differential equation, we see that

$$a = 2,\ b = 5,\ A_1 = -6,\ A_2 = 8, \text{ and } Q(x) = 0$$

Let $2x+5 = e^t$ or $t = \ln(2x+5)$.

Then the differential equation

$$\left[a^2 D_1(D_1 - 1) + A_1 a D_1 + A_2\right]y = Q\left(\frac{e^t - b}{a}\right)$$

takes the form $\left[4D_1(D_1 - 1) - 12D_1 + 8\right]y = 0$

or $$\left[4D_1^2 - 4D_1 - 12D_1 + 8\right]y = 0$$

or $$\left(4D_1^2 - 16D_1 + 8\right)y = 0$$

or $$\left(D_1^2 - 4D_1 + 2\right)y = 0 \qquad (1)$$

which is a second order linear differential equation with constant coefficients.

The auxiliary equation is $D_1^2 - 4D_1 + 2 = 0$

having characteristic roots as $D = 2 \pm \sqrt{2}$.

The general solution of equation (1) is

$$\begin{aligned} y(t) &= C_1 e^{(2+\sqrt{2})t} + C_2 e^{(2-\sqrt{2})t} \\ &= C_1 e^{(2+\sqrt{2})\ln(2x+5)} + C_2 e^{(2-\sqrt{2})\ln(2x+5)} \\ &= C_1 (2x+5)^{(2+\sqrt{2})} + C_2 (2x+5)^{(2-\sqrt{2})} \qquad (2) \end{aligned}$$

(ii) $(x+2)^2 \dfrac{d^2 y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x+4$

Comparing with the general form of Legendre linear differential equation, we see that

$$a = 1,\ b = 2,\ A_1 = -1,\ A_2 = 1, \text{ and } Q = 3x+4$$

Let $x+2 = e^t$ so that $t = \ln(x+2)$. Also $x = e^t - 2$.

Then the differential equation

$$\left[a^2 D_1 (D_1 - 1) + A_1 a D_1 + A_2\right] y = Q\left(\frac{e^t - b}{a}\right)$$

takes the form $\left[D_1(D_1 - 1) - D_1 + 1\right] y = 3e^t - 2$

or $\quad (D_1^2 - 2D_1 + 1) y = 3e^t - 2 \qquad (1)$

which is a second order linear differential equation with constant coefficients.

The auxiliary equation is $D_1^2 - 2D_1 + 1 = 0$

or $\quad (D_1 - 1)^2 y = 0$ having characteristic roots $D = 1, 1$.

The complementary function is

$$y_c(t) = (C_1 + C_2 t)e^t \qquad (2)$$

The particular integral is given by

$$\begin{aligned} y_p(t) &= \frac{1}{(D_1 - 1)^2}(3e^t - 2) \\ &= 3\frac{1}{(D_1 - 1)^2} e^t - \frac{2}{(D_1 - 1)^2} \\ &= 3t \frac{1}{\frac{d}{dD_1}(D_1 - 1)^2} e^t - \frac{2}{(0-1)^2} \\ &= 3t\frac{1}{2(D_1 - 1)} e^t - 2 = \frac{3}{2}t^2 \frac{1}{\frac{d}{dD_1}(D_1 - 1)} e^t - 2 \\ &= \frac{3}{2}t^2 e^t - 2 \qquad (3) \end{aligned}$$

From equations (2) and (3) the general solution of equation (1) is

$$\begin{aligned} y(t) &= y_c(t)+y_p(t) \\ &= (C_1+C_2 t)e^t+\frac{3}{2}t^2e^t-2 \end{aligned} \tag{4}$$

Since $t = \ln(x+2)$, therefore the general solution (4) takes the form

$$\begin{aligned} y(x) &= \left[C_1+C_2\ln(x+2)\right]e^{\ln(x+2)}+\frac{3}{2}\ln^2(x+2)e^{\ln(x+2)}-2 \\ &= \left[C_1+C_2\ln(x+2)\right](x+2)+\frac{3}{2}(x+2)\ln^2(x+2)-2 \end{aligned}$$

4.32 LEGENDRE'S LINEAR DIFFERENTIAL EQUATION OF ORDER n

An nth order linear differential equation of the form

$$(ax+b)^n\frac{d^ny}{dx^n}+A_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+A_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}}$$
$$+\dots\dots+A_{n-1}(ax+b)\frac{dy}{dx}+A_ny = Q(x) \tag{1}$$

where $A_1, A_2, \dots\dots, A_n$, and a and b are constants, is called an nth order Legendre's linear differential equation.

Note that Euler – Cauchy equation of order n is a special case of equation (1) with $a = 1$ and $b = 0$.

In operator notation, equation (1) can be written as

$$\left[(ax+b)^nD^n+A_1(ax+b)^{n-1}D^{n-1}+A_2(ax+b)^{n-2}D^{n-2}\right.$$
$$\left.+\dots\dots+A_{n-1}(ax+b)D+A_n\right]y = Q(x) \tag{2}$$

SOLUTION OF LEGENDRE'S DIFFERENTIAL EQUATION

The method for solving second order Legendre's differential equation can be extended to solve an nth order Legendre's differential equation (1) or (2).

Equation (1) or (2) can be solved using the substitution $ax+b = e^t$ or $t = \ln(ax+b)$.

Then if D_1 is defined by $D_1 = \frac{d}{dt}$, we have

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt}$$

or $$(ax+b)Dy = aD_1y \tag{3}$$

and $$(ax+b)^2\frac{d^2y}{dx^2} = a^2\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

or $$(ax+b)^2D^2y = a^2(D_1^2-D_1)y = a^2D_1(D_1-1)y \tag{4}$$

Now $$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left[\frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]\frac{dt}{dx}$$

$$= \frac{d}{dt}\left[(a^2.e^{-2t})\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]\frac{dt}{dx}$$

$$= \left[a^2e^{-2t}\left(\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}\right)-2a^2e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]\frac{a}{(ax+b)}$$

$$= \left[\frac{a^2}{(ax+b)^2}\left(\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}\right)-\frac{2a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right]\frac{a}{(ax+b)}$$

$$= \frac{a^3}{(ax+b)^3}\left(\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}\right)-\frac{2a^3}{(ax+b)^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

or $$(ax+b)^3\frac{d^3y}{dt^3} = a^3\left(\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right)$$

or $$(ax+b)^3D^3y = a^3(D_1^3-3D_1^2+2D_1)y$$

$$= a^3D_1(D_1-1)(D_1-2)y \qquad (5)$$

Similarly, we can prove that

$$(ax+b)^4D^4y = a^4D_1(D_1-1)(D_1-2)(D_1-3)y$$

and in general, we get

$$(ax+b)^nD^ny = a^n[D_1(D_1-1)(D_1-2)(D_1-3)\ldots\ldots\ldots D_1-n+1]y$$

Substituting the values of $(ax+b)Dy$, $(ax+b)^2D^2y$, ..., $(ax+b)^nD^ny$ in equation (2), we get

$$[a^n\{D_1(D_1-1)(D_1-2)\ldots\ldots\ldots(D_1-n+1)\}$$
$$+A_1a^{n-1}\{D_1(D_1-1)(D_1-2)\ldots\ldots\ldots(D-n+2)\}$$
$$+\ldots\ldots\ldots+A_{n-2}a^2D_1(D_1-1)+A_{n-1}aD_1+A_n]y = Q\left(\frac{e^t-b}{a}\right) \qquad (6)$$

which is an nth order linear differential equation with constant coefficients and can be solved using the methods already known. The general solution of the original Legendre's differential equation is then obtained by substituting $t = \ln(ax+b)$.

EXAMPLE (28): Solve the differential equation

$$[(x+2)^3D^3+(x+2)^2D^2+(x+2)D-1]y = 3x^2$$

SOLUTION: Comparing this equation with the general form of Legendre's linear differential equation, we see that

$$a = 1, \quad b = 2, \quad Q(x) = 3x^2$$

Let $x+2 = e^t$ or $t = \ln(x+2)$. Also $x = e^t-2$

Then $\quad (x+2)Dy = D_1 y, \quad (x+2)^2 D^2 y = D_1(D_1-1)y,$

$$(x+2)^3 D^3 y = D(D_1-1)(D-2)y$$

Thus the given differential equation becomes

$$[D_1(D_1-1)(D_1-2)+D_1(D_1-1)+D_1-1]y = 3(e^t-2)^2$$

$$(D_1^3-3D_1^2+2D_1+D_1^2-D_1+D_1-1)y = 3(e^{2t}-4e^t+4)$$

or $$(D_1^3-2D_1^2+2D_1-1)y = 3(e^{2t}-4e^t+4) \tag{1}$$

The auxiliary equation is $D_1^3-2D_1^2+2D_1-1=0$

or $\quad (D_1-1)(D_1^2-D_1+1)y = 0$ having characteristic roots $D = 1, \dfrac{1\pm\sqrt{3}\,i}{2}$.

The complementary function is

$$y_c(t) = C_1 e^t + e^{t/2}\left(C_2\cos\frac{\sqrt{3}}{2}t + C_3\sin\frac{\sqrt{3}}{2}t\right) \tag{2}$$

The particular integral is given by

$$y_p(t) = \frac{1}{D_1^3-2D_1^2+2D_1-1}3(e^{2t}-4e^t+4)$$

$$= 3\frac{1}{D_1^3-2D_1^2+2D_1-1}e^{2t} - 12\frac{1}{D_1^3-2D_1^2+2D_1-1}e^t$$

$$+ 12\frac{1}{D_1^3-2D_1^2+2D_1-1}e^0$$

$$= e^{2t} - 12t\frac{1}{\frac{d}{dD_1}(D_1^3-2D_1^2+2D_1-1)}e^t - 12$$

$$= e^{2t} - 12t\frac{1}{3D_1^2-4D_1+2}e^t - 12$$

$$= e^{2t} - 12te^t - 12 \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is given by

$$y(t) = y_c(t)+y_p(t)$$

$$= C_1e^t + e^{t/2}\left(C_2\cos\frac{\sqrt{3}}{2}t + C_3\sin\frac{\sqrt{3}}{2}t\right) + e^{2t} - 12te^t - 12 \tag{4}$$

Since $t = \ln(x+2)$, the general solution (4) takes the form

$$y(x) = C_1(x+2) + (x+2)^{1/2}\left[C_2\cos\left\{\frac{\sqrt{3}}{2}\ln(x+2)\right\} + C_3\sin\left\{\frac{\sqrt{3}}{2}\ln(x+2)\right\}\right]$$
$$+(x+2)^2 - 12(x+2)\ln(x+2) - 12$$

4.33 SOLVED PROBLEMS

HOMOGENEOUS DIFFERENTIAL EQUATIONS

PROBLEM (1): **Solve the following initial value problem :**

(i) $y'' + y' - 2y = 0; \quad y(0) = 4, \; y'(0) = -5$

(ii) $y'' - 4y' + 4y = 0; \quad y(0) = 3, \; y'(0) = 1$

(iii) $y'' + 2y' + 3y = 0; \quad y(0) = 2, \; y'(0) = -3$

SOLUTION: (i) $y'' + y' - 2y = 0; \quad y(0) = 4, \; y'(0) = -5$

Comparing with the general form of second order homogeneous linear differential equation , we see that

$$A = 1, \; B = -2$$

The characteristic equation becomes

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0$$

Its roots are $m_1 = 1$ and $m_2 = -2$, so that we obtain the general solution as

$$y = C_1 e^x + C_2 e^{-2x} \tag{1}$$

where the arbitrary constants C_1 and C_2 are to be determined .

Using the first initial condition $y(0) = 4$, we obtain

$$C_1 + C_2 = 4 \tag{2}$$

Also from equation (1) $y'(x) = C_1 e^x - 2C_2 e^{-2x}$ (3)

Using the second initial condition $y'(0) = -5$, we obtain from equation (3) that

$$C_1 - 2C_2 = -5 \tag{4}$$

From equations (2) and (4) , we get $C_1 = 1$ and $C_2 = 3$.

Substituting the values of C_1 and C_2 in equation (1) , we obtain the required solution as

$$y = e^x + 3e^{-2x}$$

(ii) $y'' - 4y' + 4y = 0; \quad y(0) = 3, \; y'(0) = 1$

Comparing with the general form of second order homogeneous linear differential equation , we see that

$$A = -4, \; B = 4$$

The characteristic equation becomes

$$m^2 - 4m + 4 = 0$$

or $(m-2)^2 = 0$

which has repeated roots $m_1 = m_2 = 2$.

The general solution of the given differential equation becomes

$$y(x) = (C_1 + C_2 x) e^{2x} \tag{1}$$

where C_1 and C_2 are arbitrary constants .

Using the initial condition $y(0) = 3$, we get $C_1 = 3$

Now from equation (1)

$$y'(x) = 2(C_1 + C_2 x)e^{2x} + C_2 e^{2x} \qquad (2)$$

Using the second initial condition $y'(0) = 1$, equation (2) gives

$$2C_1 + C_2 = 1 \quad \text{or} \quad C_2 = -5$$

Substituting the values of C_1 and C_2 in equation (1) we obtain the required solution as

$$y(x) = (3 - 5x)e^{2x}$$

(iii) $y'' + 2y' + 3y = 0;\ y(0) = 2,\ y'(0) = -3$

Comparing with the general form of second order homogeneous linear differential equation, we see that

$$A = 2 \text{ and } B = 3$$

The characteristic equation becomes

$$m^2 + 2m + 3 = 0$$

Solving this equation, we get $m = \dfrac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \sqrt{2}\,i$

i.e. $m_1 = -1 + \sqrt{2}\,i$ and $m_2 = -1 - \sqrt{2}\,i$ are the two complex conjugate roots.

The general solution of the given differential equation is:

$$y(x) = e^{-x}\left(C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x\right) \qquad (1)$$

where the arbitrary constants C_1 and C_2 are to be determined.

Using the initial condition $y(0) = 2$, equation (i) gives $C_1 = 2$. Now from equation (1).

$$y'(x) = \sqrt{2}\,e^{-x}\left(-C_1 \sin\sqrt{2}\,x + C_2 \cos\sqrt{2}\,x\right) - e^{-x}\left(C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x\right) \quad (2)$$

Using the second initial condition $y'(0) = -3$, we get $\sqrt{2}\,C_2 - C_1 = -3$

or $\quad \sqrt{2}\,C_2 - 2 = -3 \quad$ or $\quad C_2 = -\dfrac{1}{\sqrt{2}}$

Substituting the values of C_1 and C_2 in equation (1), we obtain the required solution as

$$y(x) = e^{-x}\left(2\cos\sqrt{2}\,x - \frac{1}{\sqrt{2}}\sin\sqrt{2}\,x\right)$$

PROBLEM (2): **Solve the following differential equations :**

(i) $(D^3 + 2D^2 - 5D - 6)y = 0$ (ii) $(D^4 + 5D^2 - 36)y = 0$

(iii) $(D^4 + 4)y = 0$ (iv) $(D^2 - 2D + 5)^2 y = 0$

SOLUTION: (i) $(D^3 + 2D^2 - 5D - 6)y = 0$

The auxiliary equation is $D^3 + 2D^2 - 5D - 6 = 0$

or $\quad (D+1)(D^2 + D - 6) = 0$

or $\quad (D+1)(D-2)(D+3) = 0$

The characteristic roots are $\quad D = -1, 2, -3$

The general solution of the given differential equation is

$$y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{-3x}$$

(ii) $\quad (D^4 + 5D^2 - 36)y = 0$

The auxiliary equation is $\quad D^4 + 5D^2 - 36 = 0$

or $\quad (D^2 - 4)(D^2 + 9) = 0$

The characteristic roots are $\quad \pm 2, \pm 3i$.

The general solution of the given differential equation is

$$y = C_1 e^{2x} + C_2 e^{-2x} + (C_3 \cos 3x + C_4 \sin 3x)$$

(iii) $\quad (D^4 + 4)y = 0$

The auxiliary equation is $\quad D^4 + 4 = 0$

or $\quad D^4 + 4D^2 + 4 - 4D^2 = 0$

$$(D^2 + 2)^2 - (2D)^2 = 0$$

or $\quad (D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

which has characteristic roots as $\quad D = -1 \pm i, \quad 1 \pm i$.

The general solution of the given differential equation is

$$y(x) = e^{-x}(C_1 \cos x + C_2 \sin x) + e^{x}(C_3 \cos x + C_4 \sin x)$$

(iv) $\quad (D^2 - 2D + 5)^2 y = 0$

The auxiliary equation is $\quad (D^2 - 2D + 5)^2 = 0$

The characteristic roots are $\quad 1 \pm 2i, \; 1 \pm 2i$, which are repeated complex roots.

The general solution of the given differential equation is given by

$$y = e^{x}\left[(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x\right]$$

PROBLEM (3): **Solve the following differential equations :**

(i) $\quad (D^5 + 2D^3 + D)y = 0$ $\qquad$ (ii) $\quad (D^6 - 64)y = 0$

(iii) $\quad (D^2 + 1)^3 y = 0$ $\qquad$ (iv) $\quad (D^3 + 64)^2 y = 0$

SOLUTION: (i) $\quad (D^5 + 2D^3 + D)y = 0$

The auxiliary equation is $\quad D^5 + 2D^3 + D = 0$

or $\quad D(D^4 + 2D^2 + 1) = 0$

or $\quad D(D^2 + 1)^2 = 0$ having characteristic roots $\quad D = 0, \pm i, \pm i$.

The general solution of the given equation is

$$y(x) = C_1 + (C_2 + C_3 x)\cos x + (C_4 + C_5 x)\sin x$$

(ii) $(D^6 - 64)y = 0$

The auxiliary equation is $D^6 - 64 = 0$

or $(D^3 + 8)(D^3 - 8) = 0$

or $(D+2)(D^2 - 2D + 4)(D-2)(D^2 + 2D + 4) = 0$

The characteristic roots are $D = \pm 2,\ 1 \pm \sqrt{3}\,i,\ -1 \pm \sqrt{3}\,i$.

The general solution of the given equation is

$$y(x) = C_1 e^{2x} + C_2 e^{-2x} + e^{x}\left(C_3 \cos\sqrt{3}\,x + C_4 \sin\sqrt{3}\,x\right) + e^{-x}\left(C_5 \cos\sqrt{3}\,x + C_4 \sin\sqrt{3}\,x\right)$$

(iii) $(D^2 + 1)^3 y = 0$

The auxiliary equation is $(D^2 + 1)^3 = 0$

having characteristic roots $D = \pm i,\ \pm i,\ \pm i$ which are repeated complex roots.

The general solution of the given differential equation is

$$y(x) = (C_1 + C_2 x + C_3 x^2)\cos x + (C_4 + C_5 x + C_6 x^2)\sin x$$

(iv) $(D^3 + 64)^2 y = 0$

The auxiliary equation is $(D^3 + 64)^2 = 0$

or $\left[(D+4)(D^2 - 4D + 16)\right]^2 = 0$

The characteristic roots are

$$D = -4, -4, 2 \pm 2\sqrt{3}\,i,\ 2 \pm 2\sqrt{3}\,i$$

The general solution of the given differential equation is

$$y(x) = (C_1 + C_2 x)e^{-4x} + e^{2x}\left[(C_3 + C_4 x)\cos 2\sqrt{3}\,x + (C_5 + C_6 x)\sin 2\sqrt{3}\,x\right]$$

NON – HOMOGENEOUS DIFFERENTIAL EQUATIONS

PROBLEM (4): **Solve the following differential equations :**

(i) $(D^2 + 3D - 18)y = 9\sinh 3x$

(ii) $(D^3 - 6D^2 + 12D - 8)y = e^{2x}$

SOLUTION: (i) $(D^2 + 3D - 18)y = 9\sinh 3x$

The auxiliary equation is $D^2 + 3D - 18 = 0$ having characteristic roots $D = -\frac{3}{2} \pm \frac{9}{2}i$.

The complementary function is

$$y_c(x) = e^{-3x/2}\left(C_1 \cos\frac{9}{2}x + C_2 \sin\frac{9}{2}x\right) \tag{1}$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2+3D-18} 9\sinh 3x$$

$$= 9\frac{1}{D^2+3D-18}\left(\frac{e^{3x}-e^{-3x}}{2}\right)$$

$$= \frac{9}{2}\frac{1}{D^2+3D-18}e^{3x} - \frac{9}{2}\frac{1}{D^2+3D-18}e^{-3x}$$

$$= \frac{9}{2}x\frac{1}{\frac{d}{dD}(D^2+3D-18)}e^{3x} - \frac{9}{2}\left(-\frac{1}{18}\right)e^{-3x}$$

$$= \frac{9}{2}x\frac{1}{2D+3}e^{3x} + \frac{1}{4}e^{-3x} = \frac{9}{2}x\left(\frac{1}{9}\right)e^{3x} + \frac{1}{4}e^{-3x}$$

$$= \frac{1}{2}x e^{3x} + \frac{1}{4}e^{-3x} \quad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y = y_c(x) + y_p(x)$$

$$= e^{-3x/2}\left(C_1\cos\frac{9}{2}x + C_2\sin\frac{9}{2}x\right) + \frac{1}{2}x e^{3x} + \frac{1}{4}e^{-3x}$$

(ii) $(D^3 - 6D^2 + 12D - 8)y = e^{2x}$

The auxiliary equation is $D^3 - 6D^2 + 12D - 8 = 0$

or $(D-2)^3 = 0$ having characteristic roots $D = 2, 2, 2$.

The complementary function is

$$y_c(x) = (C_1 + C_2 x + C_3 x^2)e^{2x} \quad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3 - 6D^2 + 12D - 8}e^{2x}$$

$$= x\frac{1}{\frac{d}{dD}(D^3 - 6D^2 + 12D - 8)}e^{2x} \qquad [\text{since } F(2) = 0]$$

$$= x\frac{1}{3D^2 - 12D + 12}e^{2x}$$

$$= x^2\frac{1}{\frac{d}{dD}(3D^2 - 12D + 12)}e^{2x} \qquad [\text{since } F(2) = 0]$$

$$= x^2\frac{1}{6D-12}e^{2x}$$

$$= x^3\frac{1}{\frac{d}{dD}(6D-12)}e^{2x} \qquad [\text{since } F(2) = 0]$$

$$= x^3\frac{1}{6}e^{2x} = \frac{1}{6}x^3 e^{2x} \quad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x) + y_p(x)$$
$$= (C_1 + C_2 x + C_3 x^2) e^{2x} + \frac{1}{6} x^3 e^{2x}$$

CASE 2

PROBLEM (5): **Solve the following differential equations :**

(i) $(D^2 + 9) y = \cos 2x + \sin 3x$

(ii) $(D^3 + D^2 + D + 1) y = \sin 2x + \cos 3x$

SOLUTION: (i) $(D^2 + 9) y = \cos 2x + \sin 3x$

The auxiliary equation is $D^2 + 9 = 0$ having characteristic roots $D = \pm 3i$.

The complementary function is

$$y_c(x) = C_1 \cos 3x + C_2 \sin 3x \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2 + 9}(\cos 2x + \sin 3x)$$
$$= \frac{1}{D^2 + 9} \cos 2x + \frac{1}{D^2 + 9} \sin 3x$$
$$= \frac{1}{-4 + 9} \cos 2x + x \frac{1}{\frac{d}{dD}(D^2 + 9)} \sin 3x$$
$$= \frac{1}{5} \cos 2x + x \left(\frac{1}{2D}\right) \sin 3x$$
$$= \frac{1}{5} \cos 2x - \frac{1}{6} x \cos 3x \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x) + y_p(x)$$
$$= C_1 \cos 3x + C_2 \sin 3x + \frac{1}{5} \cos 2x - \frac{1}{6} x \cos 3x$$

(ii) $(D^3 + D^2 + D + 1) y = \sin 2x + \cos 3x$

The auxiliary equation is $D^3 + D^2 + D + 1 = 0$

or $(D + 1)(D^2 + 1) = 0$ having characteristic roots $D = -1, \pm i$.

The complementary function is

$$y_c(x) = C_1 e^{-x} + C_2 \cos x + C_3 \sin x \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3 + D^2 + D + 1} \sin 2x + \frac{1}{D^3 + D^2 + D + 1} \cos 3x$$

$$= \frac{1}{-4D-4+D+1}\sin 2x + \frac{1}{-9D-9+D+1}\cos 3x$$

$$= \frac{1}{-3D-3}\sin 2x + \frac{1}{-8D-8}\cos 3x$$

$$= -\frac{1}{3}\frac{1}{D+1}\sin 2x - \frac{1}{8}\frac{1}{D+1}\cos 3x$$

$$= -\frac{1}{3}\frac{D-1}{D^2-1}\sin 2x - \frac{1}{8}\frac{D-1}{D^2-1}\cos 3x$$

$$= -\frac{1}{3}\left(-\frac{1}{5}\right)(D-1)\sin 2x - \frac{1}{8}\left(-\frac{1}{10}\right)(D-1)\cos 3x$$

$$= \frac{1}{15}(2\cos 2x - \sin 2x) + \frac{1}{80}(-3\sin 3x - \cos 3x)$$

$$= \frac{1}{15}(2\cos 2x - \sin 2x) - \frac{1}{80}(3\sin 3x + \cos 3x) \qquad (2)$$

From equations (1) and (2), the general solution of the given differential equation is

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1 e^{-x} + C_2\cos x + C_3\sin x + \frac{1}{15}(2\cos 2x - \sin 2x) - \frac{1}{80}(3\sin 3x + \cos 3x)$$

CASE (3)

PROBLEM (6): **Solve the following differential equations :**

(i) $(D^2 - D + 1)y = 12x^2 + 6x^3 - x^4$

(ii) $(D^3 + 3D^2)y = 180x^3 + 24x$

(iii) $D^4(D^2 - 1)y = x^2$

SOLUTION: (i) $(D^2 - D + 1)y = 12x^2 + 6x^3 - x^4$

The auxiliary equation is $D^2 - D + 1 = 0$ having characteristic roots $D = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

The complementary function is given by

$$y_c(x) = e^{-x/2}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right) \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2 - D + 1}(12x^2 + 6x^3 - x^4)$$

$$= \frac{1}{1-(D-D^2)}(12x^2 + 6x^3 - x^4)$$

$$= \left[1-(D-D^2)\right]^{-1}(12x^2 + 6x^3 - x^4)$$

$$= \left[1+(D-D^2)+(D-D^2)^2+(D-D^2)^3+(D-D^2)^4\right](12x^2 + 6x^3 - x^4)$$

$$= (1 + D - D^2 + D^2 - 2D^3 + D^4 + D^3 - 3D^4 + D^4)(12x^2 + 6x^3 - x^4)$$
$$= (1 + D - D^3 - D^4)(12x^2 + 6x^3 - x^4)$$
$$= (12x^2 + 6x^3 - x^4) + (24x + 18x^2 - 4x^3) - (36 - 24x) + 24$$
$$= -12 + 48x + 30x^2 + 2x^3 - x^4 \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x) + y_p(x)$$
$$= e^{-x/2}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right) - 12 + 48x + 30x^2 + 2x^3 - x^4$$

(ii) $(D^3 + 3D^2)y = 180x^3 + 24x$

The auxiliary equation is $D^3 + 3D^2 = 0$ having characteristic roots as $D = 0, 0, -3$.

The complementary function is given by

$$y_c(x) = C_1 + C_2 x + C_3 e^{-3x}$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2(D+3)}(180x^3 + 24x)$$
$$= \frac{1}{D^2}\frac{1}{3\left(1 + \frac{D}{3}\right)}(180x^3 + 24x)$$
$$= \frac{1}{3D^2}\left(1 + \frac{D}{3}\right)^{-1}(180x^3 + 24x)$$
$$= \frac{1}{3D^2}\left(1 - \frac{D}{3} + \frac{D^2}{9} - \frac{D^3}{27}\right)(180x^3 + 24x)$$
$$= \left(\frac{1}{3D^2} - \frac{1}{9D} + \frac{1}{27} - \frac{D}{81}\right)(180x^3 + 24x)$$
$$= \frac{1}{3}\left(180\frac{x^5}{20} + 24\frac{x^3}{6}\right) - \frac{1}{9}\left(180\frac{x^4}{4} + 24\frac{x^2}{2}\right)$$
$$+ \frac{1}{27}(180x^3 + 24x) - \frac{1}{81}(540x^2 + 24)$$
$$= 3x^5 - 5x^4 + 8x^3 - 8x^2 + \frac{8}{9}x - \frac{8}{27}$$

Hence the general solution of the given differential equation is

$$y(x) = y_c(x) + y_p(x)$$
$$= C_1 + C_2 x + C_3 e^{-3x} + 3x^5 - 5x^4 + 8x^3 - 8x^2 + \frac{8}{9}x - \frac{8}{27}$$

(iii) $D^4(D^2 - 1)y = x^2$

The auxiliary equation is $D^4(D^2 - 1) = 0$

The characteristic roots are $D = 0, 0, 0, 0, \pm 1$

The complementary function is given by

$$y_c(x) = (C_1 + C_2 x + C_3 x^2 + C_4 x^3) + C_5 e^{x} + C_6 e^{-x} \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^4(D^2-1)} x^2 = -\frac{1}{D^4}\frac{1}{(1-D^2)} x^2$$

$$= -\frac{1}{D^4}(1-D^2)^{-1} x^2 = -\frac{1}{D^4}(1+D^2)x^2$$

$$= -\frac{1}{D^4}(x^2+2) = -\frac{1}{D^3}\left(\frac{x^3}{3}+2x\right)$$

$$= -\frac{1}{D^2}\left(\frac{x^4}{12}+x^2\right) = -\frac{1}{D}\left(\frac{x^5}{60}+\frac{x^3}{3}\right)$$

$$= -\left(\frac{x^6}{360}+\frac{x^4}{12}\right) \qquad (2)$$

From equations (1) and (2) the general solution of the given differential is given by

$$y = y_c(x) + y_p(x)$$

$$= (C_1 + C_2 x + C_3 x^2 + C_4 x^3) + C_5 e^{x} + C_6 e^{-x} - \left(\frac{x^6}{360}+\frac{x^4}{12}\right)$$

CASE (4)

PROBLEM (7): **Solve the following differential equations :**

(i) $(D^2 - 4D + 4)y = e^{2x}\cos 2x$

(ii) $(D^2 - 5D + 6)y = x^3 e^{2x}$

(iii) $(D^3 - 3D - 2)y = 540\, x^3 . e^{-x}$

(iv) $(D^3 - D^2 + 4D - 4)y = 68\, e^{x}\sin 2x$

SOLUTION: (i) $(D^2 - 4D + 4)y = e^{2x}\cos 2x$

The auxiliary equation is $D^2 - 4D + 4 = 0$ or $(D-2)^2 = 0$ having characteristic roots $D = 2, 2$.

The complementary function is

$$y_c(x) = (C_1 + C_2 x)e^{2x} \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2-4D+4} e^{2x}\cos x$$

$$= e^{2x}\frac{1}{(D+2)^2 - 4(D+2)+4}\cos 2x$$

$$= e^{2x}\frac{1}{D^2}\cos 2x = e^{2x}\left(-\frac{\cos 2x}{4}\right)$$

$$= -\frac{1}{4}e^{2x}\cos 2x \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x)+y_p(x)$$
$$= (C_1+C_2x)e^{2x}-\frac{1}{4}e^{2x}\cos 2x$$

(ii) $(D^2-5D+6)y = x^3e^{2x}$

The auxiliary equation is $D^2-5D+6=0$

or $(D-2)(D-3)=0$ having characteristic roots are $D=2,3$.

The complementary function is

$$y_c(x) = C_1e^{2x}+C_2e^{3x} \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2-5D+6}x^3e^{2x}$$
$$= e^{2x}\frac{1}{(D+2)^2-5(D+2)+6}x^3$$
$$= e^{2x}\frac{1}{D^2-D}x^3 = e^{2x}\frac{1}{D}\frac{1}{D-1}x^3$$
$$= -e^{2x}\frac{1}{D}(1-D)^{-1}x^3$$
$$= -e^{2x}\frac{1}{D}(1+D+D^2+D^3)x^3$$
$$= -e^{2x}\left(D^2+D+1+\frac{1}{D}\right)x^3$$
$$= -e^{2x}\left(6x+3x^2+x^3+\frac{x^4}{4}\right)$$
$$= -\frac{1}{4}e^{2x}(x^4+4x^3+12x^2+24) \qquad (2)$$

From equations (1) and (2), the general solution is given by

$$y = y_c(x)+y_p(x)$$
$$= C_1e^{2x}+C_2e^{3x}-\frac{1}{4}e^{2x}(x^4+4x^3+12x^2+24)$$

(iii) $(D^3-3D-2)y = 540x^3e^{-x}$

The auxiliary equation is $D^3-3D-2=0$

or $(D+1)^2(D-2)=0$ having characteristic roots as $D=-1,-1,2$.

The complementary function is

$$y_c(x) = (C_1 + C_2 x) e^{-x} + C_3 e^{2x} \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3 - 3D - 2} 540 x^3 e^{-x}$$

$$= 540 e^{-x} \frac{1}{(D-1)^3 - 3(D-1) - 2} x^3$$

$$= 540 e^{-x} \frac{1}{D^3 - 3D^2 + 3D - 1 - 3D + 3 - 2}$$

$$= 540 e^{-x} \frac{1}{D^3 - 3D^2} x^3 = 540 e^{-x} \frac{1}{D^2} \cdot \frac{1}{D-3} x^3$$

$$= 540 e^{-x} \frac{1}{D^2} \frac{1}{-3\left(1 - \frac{D}{3}\right)} x^3$$

$$= -180 e^{-x} \frac{1}{D^2} \left(1 - \frac{D}{3}\right)^{-1} x^3$$

$$= -180 e^{-x} \frac{1}{D^2} \left(1 + \frac{D}{3} + \frac{D^2}{9} + \frac{D^3}{27}\right) x^3$$

$$= -180 e^{-x} \left(\frac{1}{D^2} + \frac{1}{3D} + \frac{1}{9} + \frac{D}{27}\right) x^3$$

$$= -180 e^{-x} \left(\frac{x^5}{20} + \frac{x^4}{12} + \frac{x^3}{9} + \frac{x^2}{9}\right)$$

$$= -e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2) \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x) + y_p(x)$$

$$= (C_1 + C_2 x) e^{-x} + C_3 e^{2x} - e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2)$$

(iv) $(D^3 - D^2 + 4D - 4) y = 68 e^x \sin 2x$

The auxiliary equation is $D^3 - D^2 + 4D - 4 = 0$

or $(D-1)(D^2+4) = 0$ having characteristic roots as $D = 1, \pm 2i$.

The complementary function is

$$y_c(x) = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^3 - D^2 + 4D - 4} 68 e^x \sin 2x$$

$$= 68 e^x \frac{1}{(D+1)^3 - (D+1)^2 + 4(D+1) - 4} \sin 2x$$

$$= 68\,e^{x}\frac{1}{D^3+3D^2+3D+1-D^2-2D-1+4D+4-4}\sin 2x$$

$$= 68\,e^{x}\frac{1}{D^3+2D^2+5D}\sin 2x$$

$$= 68\,e^{x}\frac{1}{D}\cdot\frac{1}{D^2+2D+5}\sin 2x$$

$$= 68\,e^{x}\frac{1}{D}\,\frac{1}{-4+2D+5}\sin 2x$$

$$= 68\,e^{x}\frac{1}{D}\,\frac{1}{2D+1}\sin 2x = 68\,e^{x}\frac{1}{D}\,\frac{2D-1}{4D^2-1}\sin 2x$$

$$= 68\,e^{x}\frac{1}{D}\left(-\frac{1}{17}\right)(2D-1)\sin 2x$$

$$= -4\,e^{x}\left(2-\frac{1}{D}\right)\sin 2x = -4\,e^{x}\left(2\sin 2x+\frac{\cos 2x}{2}\right)$$

$$= -e^{x}(8\sin 2x+2\cos 2x) \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x)+y_p(x)$$

$$= C_1 e^{x}+C_2\cos 2x+C_3\sin 2x-e^{x}(8\sin 2x+2\cos 2x)$$

CASE (5)

PROBLEM (8): **Solve the following differential equations :**

(i) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y = x\,e^{x}\sin x$

(iii) $(D^2+1)^2 y = 24\,x\cos x$

SOLUTION: (i) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y = x\,e^{x}\sin x$

Write the equation as $(D^2-2D+1)y = x\,e^{x}\sin x$

The auxiliary equation is $D^2-2D+1 = 0$

or $(D-1)^2 = 0$ having characteristic roots as 1, 1.

The complementary function is

$$y_c(x) = (C_1+C_2x)e^{x} \qquad (1)$$

The particular integral is given by

$$y_p(x) = \frac{1}{D^2-2D+1}x\,e^{x}\sin x$$

$$= \frac{1}{(D-1)^2}e^{x}(x\sin x)$$

$$= e^{x}\frac{1}{(D+1-1)^2}x\sin x = e^{x}\frac{1}{D^2}x\sin x$$

$$= e^{x}\left[x\frac{1}{D^2}\sin x - \frac{2D}{D^4}\sin x\right]$$

$$= e^{x}\left[x(-\sin x) - \frac{2}{D^3}\sin x\right]$$

$$= e^{x}(-x\sin x - 2\cos x)$$

$$= -e^{x}(x\sin x + 2\cos x) \qquad (2)$$

From equations (1) and (2) the general solution is given by

$$y = y_c(x) + y_p(x)$$

$$= (C_1 + C_2 x)e^{x} - e^{x}(x\sin x + 2\cos x)$$

(ii) The auxiliary equation is $(D^2+1)^2 = 0$

having characteristic roots $D = \pm i, \pm i$ i.e. $D = i, i, -i, -i$.

The complementary function is

$$y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x \qquad (1)$$

The particular integral can be found as follows :

$$y_p(x) = 24\frac{1}{(D^2+1)^2}x\cos x$$

$$= \text{Real part of}\left[24\frac{1}{(D^2+1)^2}x e^{ix}\right]$$

$$= \text{R.P. of}\left[24e^{ix}\frac{1}{\{(D+i)^2+1\}^2}x\right]$$

$$= \text{R.P. of}\left[24e^{ix}\frac{1}{\{D^2+2Di-1+1\}^2}x\right]$$

$$= \text{R.P. of}\left[24e^{ix}\frac{1}{(D^2+2Di)^2}x\right]$$

$$= \text{R.P. of}\left[24e^{ix}\frac{1}{D^2}\frac{1}{(2i+D)^2}x\right]$$

$$= \text{R.P. of}\left[24e^{ix}\frac{1}{D^2}\left(-\frac{1}{4}\right)\frac{1}{\left(1+\frac{D}{2i}\right)^2}x\right]$$

$$= \text{R.P. of}\left[-6e^{ix}\frac{1}{D^2}\left(1+\frac{D}{2i}\right)^{-2}x\right]$$

$$= \text{R.P. of}\left[-6e^{ix}\frac{1}{D^2}\left(1-\frac{D}{i}\right)x\right]$$

$$= \text{R.P. of}\left[-6e^{ix}\left(\frac{1}{D^2}-\frac{1}{Di}\right)x\right]$$

$$= \text{R.P. of}\left[-6e^{ix}\left(\frac{x^3}{6}-\frac{x^2}{2i}\right)\right]$$

$$= \text{R.P. of}\left[-6e^{ix}\left(\frac{x^3}{6}+\frac{ix^2}{2}\right)\right]$$

$$= \text{R.P. of}\left[-6(\cos x+i\sin x)\left(\frac{x^3}{6}+i\frac{x^2}{2}\right)\right]$$

$$= \text{R.P. of}\left[(-x^3\cos x+3x^2\sin x)+i(-x^3\sin x-3x^2\cos x)\right]$$

$$= 3x^2\sin x - x^3\cos x \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is

$$y(x) = y_c(x)+y_p(x)$$

$$= (C_1+C_2x)\cos x+(C_3+C_4x)\sin x+3x^2\sin x-x^3\cos x$$

METHOD OF VARIATION OF PARAMETERS

PROBLEM (9): **Solve the following differential equations using the method of variation of parameters.**

(i) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = e^{-x}$ (ii) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y = 10$

(iii) $(D^3+D)y = \sec^2 x$

SOLUTION: (i) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = e^{-x}$

Write the equation as $(D^2+2D+1)y = e^{-x}$

The auxiliary equation is $D^2+2D+1 = 0$

or $(D+1)^2 = 0$ having characteristic roots $D = -1, -1$.

The complementary function is

$$y_c(x) = (C_1+C_2x)e^{-x} \qquad (1)$$

Here $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$

Then $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} \end{vmatrix}$

$$= -xe^{-2x}+e^{-2x}+xe^{-2x} = e^{-2x}$$

Now $u_1(x) = -\int \frac{y_2 Q}{W}\,dx$

$$= -\int \frac{x e^{-x} \cdot e^{-x}}{e^{-2x}} dx = -\int x\, dx = -\frac{x^2}{2}$$

and $$u_2(x) = \int \frac{y_1 Q}{W} dx = \int \frac{e^{-x} \cdot e^{-x}}{e^{-2x}} dx = \int dx = x$$

Thus the particular integral is given by

$$\begin{aligned} y_p &= u_1(x) y_1(x) + u_2(x) y_2(x) \\ &= -\frac{x^2}{2} e^{-x} + x(x e^{-x}) \\ &= \left(-\frac{x^2}{2} + x^2\right) e^{-x} = \frac{x^2}{2} e^{-x} \end{aligned} \tag{2}$$

From equations (1) and (2) the general solution of the given differential equation is

$$\begin{aligned} y &= y_c + y_p \\ &= (C_1 + C_2 x) e^{-x} + \frac{x^2}{2} e^{-x} \end{aligned}$$

(ii) $$\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 5y = 10$$

Write the equation as $(D^2 + 4D + 5) y = 10$

The auxiliary equation is $D^2 + 4D + 5 = 0$ having characteristic roots as $D = -2 \pm i$.

The complementary function is

$$y_c(x) = e^{-2x}(C_1 \cos x + C_2 \sin x) \tag{1}$$

Here $y_1(x) = e^{-2x} \cos x$, $y_2(x) = e^{-2x} \sin x$

Then $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} \cos x & e^{-2x} \sin x \\ -e^{-2x} \sin x - 2e^{-2x} \cos x & e^{-2x} \cos x - 2e^{-2x} \sin x \end{vmatrix}$$

$$= e^{-4x} \cos^2 x - 2e^{-4x} \sin x \cos x + e^{-4x} \sin^2 x + 2e^{-4x} \sin x \cos x$$

$$= e^{-4x}(\cos^2 x + \sin^2 x) = e^{-4x}$$

Now $$u_1(x) = -\int \frac{y_2 Q}{W} dx$$

$$= -10 \int \frac{e^{-2x} \sin x}{e^{-4x}} dx = -10 \int e^{2x} \sin x\, dx$$

$$= -10\left[\frac{e^{2x}(2 \sin x - \cos x)}{5}\right] = e^{2x}(2 \cos x - 4 \sin x)$$

and $$u_2(x) = \int \frac{y_1 Q}{W} dx$$

$$= 10\int \frac{e^{-2x}\,4\cos x}{e^{-4x}}\,dx = 10\int e^{2x}\cos x\,dx$$

$$= 10\left[\frac{e^{2x}(\sin x + 2\cos x)}{5}\right] = e^{2x}(2\sin x + 4\cos x)$$

Thus the particular integral is given by

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$= e^{2x}(2\cos x - 4\sin x)(e^{-2x}\cos x) + e^{2x}(2\sin x + 4\cos x)(e^{-2x}\sin x)$$

$$= 2\cos^2 x - 4\sin x\cos x + 2\sin^2 x + 4\sin x\cos x$$

$$= 2(\cos^2 x + \sin^2 x) = 2 \tag{2}$$

From equations (1) and (2) the general solution of the given differential equation is

$$y = y_c(x) + y_p(x)$$

$$= e^{-2x}(C_1\cos x + C_2\sin x) + 2$$

NOTE: In this problem the integrals have been written using the formulas

(i) $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

(ii) $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

(iii) $(D^3 + D)y = \sec^2 x$

The auxiliary equation is $D^3 + D = 0$

or $D(D^2+1) = 0$ having characteristic roots $D = 0, \pm i$.

The complementary function is

$$y_c(x) = C_1 + C_2\cos x + C_3\sin x \tag{1}$$

Here $y_1(x) = 1$, $y_2(x) = \cos x$, $y_3(x) = \sin x$. This gives

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

Now $$u_1(x) = \int \frac{(y_2y_3' - y_3y_2')Q}{W}\,dx$$

$$= \int \frac{[\cos x \cdot \cos x - \sin x(-\sin x)]\sec^2 x}{1}\,dx$$

$$= \int \frac{(\cos^2 x + \sin^2 x)\sec^2 x}{1}\, dx$$

$$= \int \sec^2 x \, dx = \tan x$$

$$u_2(x) = \int \frac{(y_3 y_1' - y_1 y_3')\, Q}{W}\, dx$$

$$= \int \frac{(-\cos x)\sec^2 x}{1}\, dx = -\int \sec x \, dx = -\ln|\sec x + \tan x|$$

and
$$u_3(x) = \int \frac{(y_1 y_2' - y_2 y_1')\, Q}{W}\, dx$$

$$= \int \frac{(-\sin x)(\sec^2 x)}{1}\, dx = -\int \sec x \tan x \, dx = -\sec x$$

Thus the particular integral is given by

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + u_3(x) y_3(x)$$

$$= \tan x - \cos x \ln|\sec x + \tan x| - \sec x \sin x$$

$$= \tan x - \cos x \ln|\sec x + \tan x| - \tan x$$

$$= -\cos x \ln|\sec x + \tan x| \qquad (2)$$

From equations (1) and (2) the general solution of the given differential equation is given by

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1 + C_2 \cos x + C_3 \sin x - \cos x \ln|\sec x + \tan x|$$

METHOD OF UNDETERMINED COEFFICIENTS

PROBLEM (10): **Solve the following differential equations :**

(i) $y'' - 2y' + 3y = x^3 + \sin x$ (ii) $y'' + 4y' + 4y = 3x e^{-2x}$

(iii) $y'' - 4y' + 13y = 3e^{2x} \sin 3x$

SOLUTION: (i) $y'' - 2y' + 3y = x^3 + \sin x$

Write the differential equation as

$$(D^2 - 2D + 3)y = x^3 + \sin x \qquad (1)$$

The auxiliary equation is

$D^2 - 2D + 3 = 0$ having characteristic roots $D = 1 \pm \sqrt{2}\, i$.

The complementary function is

$$y_c(x) = e^x (C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x) \qquad (2)$$

Since no term of $Q(x)$ appears in the complementary function, therefore the particular integral is taken as

$$y_p(x) = K_3x^3 + K_2x^2 + K_1x + K_0 + K_4\cos x + K_5\sin x \tag{3}$$

Thus $\quad y_p' = 3K_3x^2 + 2K_2x + K_1 - K_4\sin x + K_5\cos x$

$$y_p'' = 6K_3x + 2K_2 - K_4\cos x - K_5\sin x$$

Substituting the values of y_p'', y_p', and y_p in equation (1), we get

$$(6K_3x + 2K_2 - K_4\cos x - K_5\sin x) - 2(3K_3x^2 + 2K_2x + K_1 - K_4\sin x + K_5\cos x)$$
$$+ 3(K_3x^3 + K_2x^2 + K_1x + K_0 + K_4\cos x + K_5\sin x) = x^3 + \sin x$$

or $\quad 3K_3x^3 + (3K_2 - 6K_3)x^2 + (3K_1 - 4K_2 + 6K_3)x + (2K_2 - 2K_1 + 3K_0)$

$$+ (2K_4 - 2K_5)\cos x + (2K_5 + 2K_4)\sin x = x^3 + \sin x$$

Equating the coefficients of like terms on both sides, we get

$$3K_3 = 1, \quad 3K_2 - 6K_3 = 0, \quad 3K_1 - 4K_2 + 6K_3 = 0$$
$$2K_2 - 2K_1 + 3K_0 = 0, \quad 2K_4 - 2K_5 = 0, \quad 2K_5 + 2K_4 = 1$$

This implies $K_3 = \frac{1}{3}$, $K_2 = \frac{2}{3}$, $K_1 = \frac{2}{9}$, $K_0 = -\frac{8}{27}$, $K_4 = \frac{1}{4}$, $K_5 = \frac{1}{4}$

Substituting these values in equation (3), the particular integral is given by

$$y_p(x) = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}\cos x + \frac{1}{4}\sin x \tag{4}$$

From equations (2) and (4) the general solution of equation (1) is

$$y(x) = y_c(x) + y_p(x)$$
$$= e^x\left(C_1\cos x\sqrt{2}x + C_2\sin\sqrt{2}x\right) + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}\cos x + \frac{1}{4}\sin x$$

(ii) $\quad y'' + 4y' + 4y = 3xe^{-2x}$

Write the differential equation as

$$(D^2 + 4D + 4)y = 3xe^{-2x} \tag{1}$$

The auxiliary equation is

$$D^2 + 4D + 4 = 0$$

or $\quad (D+2)^2 = 0$ having characteristic roots $D = -2, -2$.

The complementary function is

$$y_c(x) = (C_1 + C_2x)e^{-2x} \tag{2}$$

Since the complementary function contains the terms e^{-2x} and xe^{-2x}, and as $Q(x)$ has the term xe^{-2x}, therefore we take the particular integral as

$$y_p(x) = K_3x^3e^{-2x} + K_2x^2e^{-2x}$$
$$= e^{-2x}(K_3x^3 + K_2x^2) \tag{3}$$

Thus $y_p' = e^{-2x}(3K_3x^2+2K_2x)-2e^{-2x}(K_3x^3+K_2x^2)$

$$= e^{-2x}[-2K_3x^3+(3K_3-2K_2)x^2+2K_2x]$$

and $y_p'' = e^{-2x}[-6K_3x^2+(6K_3-4K_2)x+2K_2]$

$$+e^{-2x}[4K_3x^3+(-6K_3+4K_2)x^2-4K_2x]$$

$$= e^{-2x}[4K_3x^3+(4K_2-12K_3)x^2+(6K_3-8K_2)x+2K_2]$$

Substituting the values of these derivatives in equation (1), we get

$$e^{-2x}[4K_3x^3+(4K_2-12K_3)x^2+(6K_3-8K_2)x+2K_2-8K_3x^3$$
$$+(12K_3-8K_2)x^2+8K_2x+4K_3x^3+4K_2x^2] = 3xe^{-2x}$$

or $e^{-2x}(6K_3x+2K_2) = 3xe^{-2x}$

or $6K_3 = 3$ and $2K_2 = 0$

or $K_3 = \frac{1}{2}$ and $K_2 = 0$

With these values of K_2 and K_3, the particular integral in equation (3) becomes

$$y_p(x) = \frac{1}{2}x^3e^{-2x} \tag{4}$$

From equations (2) and (4) the general solution of equation (1) is

$$y(x) = y_c(x)+y_p(x)$$
$$= (C_1+C_2x)e^{-2x}+\frac{1}{2}x^3e^{-2x}$$

(iii) $y''-4y'+13y = 3e^{2x}\sin 3x$

Write the differential equation as

$$(D^2-4D+13)y = 3e^{2x}\sin 3x \tag{1}$$

The auxiliary equation is

$D^2-4D+13 = 0$ having characteristic roots $D = 2\pm 3i$.

The complementary function is

$$y_c(x) = e^{2x}(C_1\cos 3x+C_2\sin 3x) \tag{2}$$

Since $e^{2x}\sin 3x$ appears in the complementary function, therefore we take the particular integral as

$$y_p(x) = xe^{2x}(K_1\cos 3x+K_2\sin 3x)$$
$$= K_1xe^{2x}\cos 3x+K_2xe^{2x}\sin 3x$$

Thus $y_p' = K_1(-3xe^{2x}\sin 3x+2xe^{2x}\cos 3x+e^{2x}\cos 3x)$

$$+K_2(3xe^{2x}\cos 3x+2xe^{2x}\sin 3x+e^{2x}\sin 3x)$$

$$y_p'' = K_1[(-9xe^{2x}\cos 3x - 6xe^{2x}\sin 3x - 3e^{2x}\sin 3x)$$
$$+(-6xe^{2x}\sin 3x + 4xe^{2x}\cos 3x + 2e^{2x}\cos 3x) + (-3e^{2x}\sin 3x + 2e^{2x}\cos 3x)]$$
$$+K_2[(-9xe^{2x}\sin 3x + 6xe^{2x}\cos 3x + 3e^{2x}\cos 3x)$$
$$+(6xe^{2x}\cos 3x + 4xe^{2x}\sin 3x + 2e^{2x}\sin 3x) + (3e^{2x}\cos 3x + 2e^{2x}\sin 3x)]$$
$$= K_1[-5xe^{2x}\cos 3x - 12xe^{2x}\sin 2x - 6e^{2x}\sin 3x + 4e^{2x}\cos 3x]$$
$$+K_2[-5xe^{2x}\sin 3x + 12xe^{2x}\cos 3x + 6e^{2x}\cos 3x + 4e^{2x}\sin 3x]$$

Substituting the values of these derivatives in equation (1), we get after simplification

$$-6K_1e^{2x}\sin 3x + 6K_2e^{2x}\cos 3x = 3e^{2x}\sin 3x$$

This implies that $-6K_1 = 3$ and $6K_2 = 0$

or $K_1 = -\frac{1}{2}$ and $K_2 = 0$

With these values of K_1 and K_2, the particular integral becomes

$$y_p(x) = -\frac{1}{2}xe^{2x}\cos 3x \tag{3}$$

From equations (2) and (3) the general solution of equation is

$$y(x) = y_c(x) + y_p(x)$$
$$= e^{2x}(C_1\cos 3x + C_2\sin 3x) - \frac{1}{2}xe^{2x}\cos 3x$$

PROBLEM (11): **Solve the following differential equations :**

(i) $y'' + 4y = x^2\sin 2x$

(ii) $y'' - 4y' + 4y = x^3e^{2x} + xe^{2x}$

(iii) $(D^4 + 8D^2 + 16)y = 4\sin 2x$

SOLUTION: (i) $y'' + 4y = x^2\sin 2x$

Write the differential equation as

$$(D^2+4)y = x^2\sin 2x \tag{1}$$

The auxiliary equation is

$D^2 + 4 = 0$ having characteristic roots $D = \pm 2i$.

The complementary function is

$$y_c(x) = C_1\cos 2x + C_2\sin 2x \tag{2}$$

As an initial guess, take the particular integral as

$$y_p(x) = (Ax^2 + Bx + C)\cos 2x + (Ex^2 + Fx + G)\sin 2x$$

Since the terms $\cos 2x$ and $\sin 2x$ of the particular integral are included in the complementary function, therefore the particular integral is taken as

$$y_p(x) = (Ax^3+Bx^2+Cx)\cos 2x + (Ex^3+Fx^2+Gx)\sin 2x$$

Then $$y_p'(x) = -2(Ax^3+Bx^2+Cx)\sin 2x + (3Ax^2+2Bx+C)\cos 2x$$
$$+2(Ex^3+Fx^2+Gx)\cos 2x + (3Ex^2+2Fx+G)\sin 2x$$

$$y_p''(x) = -4(Ax^3+Bx^2+Cx)\cos 2x - 2(3Ax^2+2Bx+C)\sin 2x$$
$$-2(3Ax^2+2Bx+C)\sin 2x + (6Ax+2B)\cos 2x$$
$$-4(Ex^3+Fx^2+Gx)\sin 2x + 2(3Ex^2+2Fx+G)\cos 2x$$
$$+2(3Ex^2+2Fx+G)\cos 2x + (6Ex+2F)\sin 2x$$

Substituting the values of these derivatives in the given differential equation, we get after simplification

$$-12Ax^2\sin 2x + 12Ex^2\cos 2x + (-8B+6E)x\sin 2x$$
$$+(8F+6A)x\cos 2x + (-4C+2F)\sin 2x + (4G+2B)\cos 2x = x^2\sin 2x$$

Comparing the coefficients of like powers on both sides, we get

$$-12A = 1,\quad 12E = 0,\quad -8B+6E = 0,\quad 8F+6A = 0$$
$$-4C+2F = 0,\quad 4G+2B = 0$$

From these equations, we get

$$A = -\frac{1}{12},\quad E = 0,\quad B = 0,\quad G = 0,\quad F = \frac{1}{16},\quad C = \frac{1}{32}$$

With these values of the constants, the particular integral is given by

$$y_p(x) = -\frac{1}{12}x^3\cos x + \frac{1}{32}x\cos 2x + \frac{1}{16}x^2\sin 2x \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is

$$y(x) = y_c(x) + y_p(x)$$
$$= C_1\cos 2x + C_2\sin 2x - \frac{1}{12}x^3\cos x + \frac{1}{16}x^2\sin 2x + \frac{1}{32}x\cos x$$

(ii) $y'' - 4y' + 4y = x^3e^{2x} + xe^{2x}$

Write the differential equation as

$$(D^2-4D+4)y = x^3e^{2x} + xe^{2x} \tag{1}$$

The auxiliary equation is

$$D^2-4D+4 = 0$$

or $(D-2)^2 = 0$ having characteristic roots $D = 2, 2$.

The complementary function is

$$y_c(x) = (C_1 + C_2x)e^{2x} \tag{2}$$

An initial guess, take the particular integral as

$$y_p(x) = (Ax^3 + Bx^2 + Cx + E)e^{2x}$$

Since the terms e^{2x} and xe^{2x} of the particular integral are included in the complementary function, therefore the particular integral is taken as

$$y_p(x) = (Ax^5 + Bx^4 + Cx^3 + Ex^2)e^{2x} \tag{3}$$

Thus $y_p'(x) = 2(Ax^5 + Bx^4 + Cx^3 + Ex^2)e^{2x} + (5Ax^4 + 4Bx^3 + 3Cx^2 + 2Ex)e^{2x}$

$$y_p''(x) = 4(Ax^5 + Bx^4 + Cx^3 + Ex^2)e^{2x} + 2(5Ax^4 + 4Bx^3 + 3Cx^2 + 2Ex)e^{2x}$$
$$+ 2(5Ax^4 + 4Bx^3 + 3Cx^2 + 2Ex)e^{2x} + (20Ax^3 + 12Bx^2 + 6Cx + 2E)e^{2x}$$

Substituting the values of these derivatives in the given differential equation, and after simplification, we get $(20Ax^3 + 12Bx^2 + 6Cx + 2E)e^{2x} = x^3e^{2x} + xe^{2x}$

Comparing the coefficients of like powers on both sides, we get

$$20A = 1, \quad 12B = 0, \quad 6C = 1, \quad 2E = 0$$

from which $A = \frac{1}{20}$, $B = 0$, $C = \frac{1}{6}$, $E = 0$.

With these values of the constants, the particular integral is given by

$$y_p(x) = \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x} \tag{4}$$

From equations (2) and (4) the general solution of equation (1) is

$$y(x) = y_c(x) + y_p(x)$$
$$= (C_1 + C_2x)e^{2x} + \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$$

(iii) $(D^4 + 8D^2 + 16)y = 4\sin 2x$

The auxiliary equation is $D^4 + 8D^2 + 16 = 0$

or $(D^2 + 4)^2 = 0$ having characteristic roots $D = \pm 2i, \pm 2i$.

The complementary function is

$$y_c(x) = (C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x \tag{2}$$

Since $Q(x) = 4\sin 2x$, and as $\sin 2x$ and $x\sin 2x$ appear in the complementary function, therefore we take the particular integral as

$$y_p(x) = K_1x^2\cos 2x + K_2x^2\sin 2x$$

Now $y_p'(x) = -2K_1x^2\sin 2x + 2K_1x\cos 2x + 2K_2x^2\cos 2x + 2K_2x\sin 2x$

$$y_p''(x) = -4K_1x^2\cos 2x - 8K_1x\sin 2x + 2K_1\cos 2x$$
$$-4K_2x^2\sin 2x + 8K_2x\cos 2x + 2K_2\sin 2x$$

$$y_p'''(x) = 8K_1x^2\sin 2x - 24K_1x\cos 2x - 12K_1\sin 2x$$
$$-8K_2x^2\cos 2x - 24K_2x\sin 2x + 12K_2\cos 2x$$

$$y_p^{(iv)}(x) = 16K_1x^2\cos 2x + 64K_1x\sin 2x - 48K_1\cos 2x$$
$$+16K_2x^2\sin 2x - 64K_2x\cos 2x - 48K_2\sin 2x$$

Substituting the values of y, $y_p'(x)$, $y_p''(x)$, $y_p'''(x)$, and $y_p^{(iv)}(x)$ in the given differential equation, we get

$$16K_1x^2\cos 2x + 64K_1x\sin 2x - 48K_1\cos 2x + 16K_2x^2\sin 2x$$
$$-64K_2x\cos 2x - 48K_2\sin 2x - 32K_1x^2\cos 2x$$
$$-64K_1x\sin 2x + 16K_1\cos 2x - 32K_2x^2\sin 2x$$
$$+64K_2x\cos 2x + 16K_2\sin 2x + 16K_1x^2\cos 2x + 16K_2x^2\sin 2x = 4\sin 2x$$

or
$$32K_1\cos 2x - 32K_2\sin 2x = 4\sin 2x$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ on both sides, we get

$$K_1 = 0 \quad \text{and} \quad -32K_2 = 4 \quad \text{or} \quad K_2 = -\frac{1}{8}$$

Substituting the values of K_1 and K_2 in equation (3), the particular integral is given by

$$y_p(x) = -\frac{1}{8}x^2\sin 2x \qquad (4)$$

From equations (2) and (4) the general solution of equation (1) is given by

$$y(x) = y_c(x) + y_p(x)$$
$$= (C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x - \frac{1}{8}x^2\sin 2x$$

EULER – CAUCHY DIFFERENTIAL EQUATIONS

PROBLEM (12): **Solve the following differential equations :**

(i) $x^2y'' - 2xy' + 2y = x + x^3$

(ii) $x^2y'' - xy' + 4y = \cos(\ln x) + x\sin(\ln x)$

(iii) $(x^3D^3 + 3x^2D^2 + xD + 8)y = 65\cos(\ln x)$

(iv) $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \ln x)^2$

SOLUTION: (i) $x^2y'' - 2xy' + 2y = x + x^3$

Comparing this equation with the general form of Euler – Cauchy differential equation

$$x^2y'' + Axy' + By = Q$$

We see that $A = -2$, $B = 2$, $Q(x) = x + x^3$

Let $x = e^t$ so that $t = \ln x$, $x > 0$. Then the differential equation

$$\frac{d^2y}{dt^2}+(A-1)\frac{dy}{dt}+By=Q(e^t)$$

takes the form $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^t+e^{3t}$ (1)

The auxiliary equation is

$$(D_1^2-3D_1+2)y=0$$

or $(D_1-1)(D_1-2)=0$ having characteristic roots $D_1=1,2$.

The complementary function is

$$y_c(t)=C_1e^t+C_2e^{2t} \quad (2)$$

The particular integral is given by

$$\begin{aligned}
y_P(t) &= \frac{1}{D_1^2-3D_1+2}(e^t+e^{3t}) \\
&= \frac{1}{D_1^2-3D_1+2}e^t+\frac{1}{D_1^2-3D_1+2}e^{3t} \\
&= t\frac{1}{\frac{d}{dD_1}(D_1^2-3D_1+2)}e^t+\frac{1}{9-9+2}e^{3t} \\
&= t\frac{1}{2D_1-3}e^t+\frac{1}{2}e^{3t} \\
&= -te^t+\frac{1}{2}e^{3t} \quad (3)
\end{aligned}$$

From equations (2) and (3) the general solution of equation (1) is

$$\begin{aligned}
y(t) &= y_c(t)+y_P(t) \\
&= C_1e^t+C_2e^{2t}-te^t+\frac{1}{2}e^{3t} \quad (4)
\end{aligned}$$

Since $t=\ln x$, the general solution (4) takes the form

$$y(x)=C_1x+C_2x^2-x\ln x+\frac{1}{2}x^3$$

(ii) $x^2y''-xy'+4y=\cos(\ln x)+x\sin(\ln x)$

Comparing this equation with the general form of Euler – Cauchy differential equation

$$x^2y''+Axy'+By=Q$$

we see that $A=-1$, $B=4$, $Q(x)=\cos(\ln x)+x\sin(\ln x)$

Let $x=e^t$ so that $t=\ln x$, $x>0$. Then the differential equation

$$\frac{d^2y}{dt^2}+(A-1)\frac{dy}{dt}+By=Q(e^t)$$

takes the form $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+4y=\cos t+e^t\sin t$ (1)

The auxiliary equation is

$$D_1^2 - 2D_1 + 4y = 0 \qquad \left(\frac{d}{dt} = D_1\right)$$

having characteristic roots $D = 1 \pm \sqrt{3}\,i$.

The complementary function is

$$y_c(t) = e^t\left(C_1 \cos\sqrt{3}\,t + C_2 \sin\sqrt{3}\,t\right) \tag{2}$$

The particular integral is given by

$$\begin{aligned}
y_p(t) &= \frac{1}{D_1^2 - 2D_1 + 4}(\cos t + e^t \sin t) \\
&= \frac{1}{D_1^2 - 2D_1 + 4}\cos t + \frac{1}{D_1^2 - 2D_1 + 4} e^t \sin t \\
&= \frac{1}{3 - 2D_1}\cos t + e^t \frac{1}{(D_1+1)^2 - 2(D_1+1) + 4}\sin t \\
&= \frac{3 + 2D_1}{9 - 4D_1^2}\cos t + e^t \frac{1}{D_1^2 + 3}\sin t \\
&= \frac{3 + 2D_1}{9 - 4(-1)}\cos t + e^t \frac{1}{-1 + 3}\sin t \\
&= \frac{1}{13}(3 + 2D_1)\cos t + \frac{1}{2}e^t \sin t \\
&= \frac{1}{13}(3\cos t - 2\sin t) + \frac{1}{2}e^t \sin t
\end{aligned} \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= e^t\left(C_1 \cos\sqrt{3}\,t + C_2 \sin\sqrt{3}\,t\right) + \frac{1}{13}(3\cos t - 2\sin t) + \frac{1}{2}e^t \sin t
\end{aligned} \tag{4}$$

Since $t = \ln x$, the general solution (4) takes the form

$$y(x) = x\left[C_1 \cos\left(\sqrt{3}\ln x\right) + C_2 \sin\left(\sqrt{3}\ln x\right)\right] + \frac{1}{13}\left[3\cos(\ln x) - 2\sin(\ln x)\right] + \frac{1}{2}x\sin(\ln x)$$

(iii) $(x^3D^3 + 3x^2D^2 + xD + 8)y = 65\cos(\ln x)$ (1)

Comparing this equation with the general form of Euler – Cauchy differential equations, we see that

$$A_1 = 3, \quad A_2 = 1, \quad A_3 = 8, \quad Q(x) = 65\cos(\ln x)$$

Let $x = e^t$ or $t = \ln x$, $x > 0$. Then

$$xDy = D_1 y, \quad x^2D^2y = D_1(D_1 - 1)y, \quad x^3D^3y = D_1(D_1-1)(D_1-2)y$$

Thus equation (1) becomes

$$\left[D_1(D_1-1)(D_1-2) + 3D_1(D_1-1) + D_1 + 8\right]y = 65\cos t$$

or $\quad (D_1^3 - 3D_1^2 + 2D_1 + 3D_1^2 - 3D_1 + D_1 + 8)\,y = 65\cos t$

or $\quad (D_1^3 + 8)\,y = 65\cos t \qquad (2)$

The auxiliary equation is $D_1^3 + 8 = 0$

or $\quad (D_1 + 2)(D_2 - 2D_1 + 4) = 0$ having characteristic roots $D_1 = -2,\ 1 \pm \sqrt{3}\,i$.

The complementary function is

$$y_c(t) = C_1 e^{-2t} + e^t\left(C_2\cos\sqrt{3}\,t + C_3\sin\sqrt{3}\,t\right) \qquad (3)$$

The particular integral is given by

$$y_p(t) = \frac{1}{D_1^3 + 8}\,65\cos t = 65\,\frac{1}{D_1^2 \cdot D_1 + 8}\cos t$$

$$= 65\,\frac{1}{8 - D_1}\cos t = 65\,\frac{8 + D_1}{64 - D_1^2}\cos t$$

$$= 65\,\frac{1}{65}(8 + D_1)\cos t = 8\cos t - \sin t \qquad (4)$$

From equations (3) and (4) the general solution of equation (2) is given by

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1 e^{-2t} + e^t\left(C_2\cos\sqrt{3}\,t + C_3\sin\sqrt{3}\,t\right) + 8\cos t - \sin t \qquad (5)$$

Since $x = e^t$, the general solution (5) takes the form

$$y(x) = C_1 x^{-2} + x\left[C_2\cos\left(\sqrt{3}\ln x\right) + C_3\sin\left(\sqrt{3}\ln x\right)\right] + 8\cos(\ln x) - \sin(\ln x)$$

(iv) $\quad (x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\,y = (1 + \ln x)^2 \qquad (1)$

Comparing this equation with the general form of Euler – Cauchy differential equations, we see that

$$A_1 = 6,\quad A_2 = 9,\quad A_3 = 3,\quad A_4 = 1,\quad Q(x) = (1 + \ln x)^2$$

Let $x = e^t$ or $t = \ln x$, $x > 0$. Then

$$x\,Dy = D_1 y,\quad x^2D^2y = D_1(D_1 - 1)y,\quad x^3D^3y = D_1(D_1 - 1)(D_1 - 2)y$$
$$x^4D^4y = D_1(D_1 - 1)(D_1 - 2)(D_1 - 3)y.$$

Thus equation (1) becomes

$$\left[D_1(D_1 - 1)(D_1 - 2)(D_1 - 3) + 6D_1(D_1 - 1)(D_1 - 2) + 9D_1(D_1 - 1) + 3D_1 + 1\right]y = (1 + t)^2$$

or $\quad (D_1^4 - 6D_1^3 + 11D_1^2 + 6D_1 + 6D_1^3 - 18D_1^2 + 12D_1 + 9D_1^2 - 9D_1 + 3D_1 + 1)\,y = t^2 + 2t + 1$

or $\quad (D_1^4 + 2D_1^2 + 1)\,y = t^2 + 2t + 1$

or $\quad (D_1^2 + 1)^2\,y = t^2 + 2t + 1 \qquad (2)$

The auxiliary equation is $(D_1^2+1)^2 = 0$

having characteristic roots $D_1 = \pm i, \pm i$ or $D = i, i, -i, -i$.

The complementary function is

$$y_c(t) = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t \tag{3}$$

The particular integral is given by

$$y_p(t) = \frac{1}{(D_1^2+1)^2}(t^2+2t+1)$$

$$= (1+D_1^2)^{-2}(t^2+2t+1)$$

$$= (1-2D_1^2)(t^2+2t+1)$$

$$= t^2+2t+1-2(2) = t^2+2t-3 \tag{4}$$

From equations (3) and (4) the general solution of equation (2) is given by

$$y(t) = y_c(t) + y_p(t)$$

$$= (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t + t^2 + 2t - 3$$

$$= (C_1 + C_2 \ln x)\cos(\ln x) + (C_3 + C_4 \ln x)\sin(\ln x) + \ln^2 x + 2\ln x - 3$$

LEGENDGRE'S LINEAR DIFFERENTIAL EQUATIONS

PROBLEM (13): **Solve the following differential equations :**

(i) $$(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos\ln(x+1)$$

(ii) $$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$$

(iii) $$16(x+1)^4\frac{d^4y}{dx^4} + 96(x+1)^3\frac{d^3y}{dx^3} + 104(x+1)^2\frac{d^2y}{dx^2} + 8(x+1)\frac{dy}{dx} + y = x^2+4x+3$$

SOLUTION: (i) $$(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos\ln(x+1)$$

Comparing this equation with the general form of Legendre's linear differential equation , we see that

$a = 1$, $b = 1$, $A_1 = 1$, $A_2 = 1$, and $Q(x) = 4\cos\ln(x+1)$

Let $x+1 = e^t$ so that $t = \ln(x+1)$

Then the differential equation

$$\left[a^2 D_1(D_1-1) + A_1 a D_1 + A_2\right]y = Q\left(\frac{e^t - b}{a}\right) \text{ takes the form}$$

$$\left[D_1(D_1-1) + D_1 + 1\right]y = 4\cos t$$

or

$$(D_1^2+1)y = 4\cos t \tag{1}$$

The auxiliary equation is $D_1^2+1=0$ having characteristic roots $D=\pm i$.

The complementary function is

$$y_c(t) = C_1\cos t + C_2\sin t \tag{2}$$

The particular integral is given by

$$\begin{aligned} y_p(t) &= \frac{1}{D_1^2+1}4\cos t \\ &= t\frac{1}{\frac{d}{dD_1}(D_1^2+1)}4\cos t \\ &= 4t\frac{1}{2D_1}\cos t = 2t\left(\frac{1}{D_1}\cos t\right) \\ &= 2t\sin t \end{aligned} \tag{3}$$

From equations (2) and (3) the general solution of equation (1) is

$$\begin{aligned} y(t) &= y_c(t)+y_p(t) \\ &= C_1\cos t + C_2\sin t + 2t\sin t \end{aligned} \tag{4}$$

Since $t=\ln(x+1)$, therefore the general solution (4) takes the form

$$y(x) = C_1\cos\ln(x+1) + C_2\sin\ln(x+1) + 2\ln(x+1)\sin\ln(x+1)$$

(ii) $$(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y = 3x^2+4x+1$$

Comparing this equation with the general form of Legendre's linear differential equation, we see that

$$a=3,\ b=2,\ A_1=3,\ A_2=-36, \text{ and } Q(x)=3x^2+4x+1$$

Let $3x+2=e^t$ so that $t=\ln(3x+2)$. Also $x=\dfrac{e^t-2}{3}$.

Thus the differential equation

$$\left[a^2D_1(D_1-1)+A_1aD_1+A_2\right]y = Q\left(\frac{e^t-b}{a}\right)$$

takes the form $\left[9D_1(D_1-1)+9D_1-36\right]y = 3\left(\dfrac{e^t-2}{3}\right)^2+4\left(\dfrac{e^t-2}{3}\right)+1$

or $$(9D_1^2-36)y = \frac{1}{3}(e^{2t}-4e^t+4)+\frac{4}{3}(e^t-2)+1 = \frac{1}{3}e^{2t}-\frac{1}{3}$$

or $$(D_1^2-4)y = \frac{1}{27}(e^{2t}-1) \tag{1}$$

which is a second order linear differential equation with constant coefficients.

The auxiliary equation is $D_1^2-4=0$ having characteristic roots $D=\pm 2$.

The complementary function is

$$y_c(t) = C_1e^{2t}+C_2e^{-2t} \tag{2}$$

The particular integral is given by

$$y_p(t) = \frac{1}{D_1^2-4}\frac{1}{27}(e^{2t}-1)$$

$$= \frac{1}{27}\left(\frac{1}{D_1^2-4}e^{2t} - \frac{1}{D_1^2-4}\right)$$

$$= \frac{1}{27}\left[t\frac{1}{\frac{d}{dD_1}(D_1^2-4)}e^{2t} - \frac{1}{0-4}\right]$$

$$= \frac{1}{27}\left(t\frac{1}{2D_1}e^{2t} + \frac{1}{4}\right) = \frac{1}{27}\left(\frac{1}{4}te^{2t} + \frac{1}{4}\right)$$

$$= \frac{1}{108}(te^{2t}+1) \qquad (3)$$

From equations (2) and (3) the general solution of equation (1) is

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1e^{2t} + C_2e^{-2t} + \frac{1}{108}(te^{2t}+1) \qquad (4)$$

Since $t = \ln(3x+2)$, therefore the general solution (4) takes the form

$$y(x) = C_1(3x+2)^2 + \frac{C_2}{(3x+2)^2} + \frac{1}{108}[(3x+2)^2\ln(3x+2)+1]$$

(iii) $$16(x+1)^4\frac{d^4y}{dx^4} + 96(x+1)^3\frac{d^3y}{dx^3} + 104(x+1)^2\frac{d^2y}{dx^2} + 8(x+1)\frac{dy}{dx} + y = x^2+4x+3$$

Comparing this equation with the general form of Legendre's linear differential equation, we see that

$$a = 1,\ b = 1,\ A_1 = 96,\ A_2 = 104,\ A_3 = 8,\ A_4 = 1,\ Q(x) = x^2+4x+3$$

Write the equation as

$$[16(x+1)^4D^4 + 96(x+1)D^3 + 104(x+1)^2D^2 + 8(x+1)D + 1]y = x^2+4x+3 \qquad (1)$$

Let $x+1 = e^t$ or $t = \ln(x+1)$. Then

$$(x+1)Dy = D_1y,\quad (x+1)^2D^2y = D_1(D_1-1)y,$$

$$(x+1)^3D^3y = D_1(D_1-1)(D_1-2)y,$$

$$(x+1)^4D^4y = D_1(D_1-1)(D_1-2)(D_1-3)y$$

Thus equation (1) becomes

$$[16D_1(D_1-1)(D_1-2)(D_1-3) + 96D_1(D_1-1)(D-2) + 104D_1(D_1-1)$$

$$+ 8D_1 + 1]y = (e^t-1)^2 + 4(e^t-1) + 3$$

or $$(16D_1^4 - 96D_1^3 + 176D_1^2 - 96D_1 + 96D_1^3 - 288D_1^2 + 192D_1 + 104D_1^2 - 104D_1 + 8D_1 + 1)y$$

$$= e^{2t} + 2e^t$$

or $\quad (16\,D_1^4 - 8\,D_1^2 + 1) = e^{2t} + 2\,e^t$

or $\quad (4\,D_1^2 - 1)^2\, y = e^{2t} + 2\,e^t \qquad (2)$

The auxiliary equation is $(4\,D_1^2 - 1)^2 = 0$

having characteristic roots $D_1 = \pm\frac{1}{2}, \pm\frac{1}{2}$ i.e. $D_1 = \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$.

The complementary function is

$$y_c(t) = (C_1 + C_2 t)\,e^{t/2} + (C_3 + C_4 t)\,e^{-t/2} \qquad (3)$$

The particular integral is given by

$$y_p(t) = \frac{1}{(4\,D_1^2 - 1)^2}\,e^{2t} + \frac{1}{(4\,D_1^2 - 1)^2}\,2\,e^t$$

$$= \frac{1}{225}\,e^{2t} + \frac{2}{9}\,e^t \qquad (4)$$

From equations (3) and (4) the general solution of equation (2) is given by

$$y(t) = y_c(t) + y_p(t)$$

$$= (C_1 + C_2 t)\,e^{t/2} + (C_3 + C_4 t)\,e^{2t} + \frac{1}{225}\,e^{2t} + \frac{2}{9}\,e^t \qquad (5)$$

Since $t = \ln(x+1)$, the general solution (5) takes the form

$$y(x) = [C_1 + C_2 \ln(x+1)]\,e^{\frac{1}{2}\ln(x+1)} + [C_3 + C_4 \ln(x+1)]\,e^{-\frac{1}{2}\ln(x+1)}$$

$$+ \frac{1}{225}(x+1)^2 + \frac{2}{9}(x+1)$$

$$= [C_1 + C_2 \ln(x+1)]\,(x+1)^{1/2} + [C_3 + C_4 \ln(x+1)]\,(x+1)^{-1/2}$$

$$+ \frac{1}{225}(x+1)^2 + \frac{2}{9}(x+1)$$

4.34 EXERCISE

HOMOGENEOUS DIFFERENTIAL EQUATIONS

Solve the following initial – value problems :

(1) $(D^2+5D+6)y=0;\ \ y(0)=1,\ \ y'(0)=2$

(2) $(D^2-8D+16)y=0;\ \ y(0)=2,\ \ y'(0)=-1$

(3) $(D^2+4)y=0;\ \ y\left(\frac{\pi}{4}\right)=1,\ \ y'\left(\frac{\pi}{4}\right)=3$

Solve the following boundary – value problems :

(4) $y''+4y=0;\ \ y(0)=1,\ \ y\left(\frac{\pi}{4}\right)=2$

(5) $y''+2y'+y=0;\ \ y(0)=0,\ \ y(1)=2$

(6) $y''-4y=0;\ \ y(0)=6,\ \ y(\infty)=0$

Solve the following differential equations :

(7) $(D^3+2D^2-D-2)y=0$

(8) $(D^3-3D+2)y=0$

(9) $(D^3-2D^2+4D-8)y=0$

(10) $(D^4-5D^2+4)y=0$

(11) $(D^2-4)(D+3)^2y=0$

(12) $(D^4-8D^2+16)y=0$

(13) $(D-1)(D+2)^3y=0$

(14) $(D^4-16)y=0$

(15) $(D-1)^2(D^2+1)=0$

(16) $(D^4+13D^2+36)y=0$

(17) $(D^2-2D+2)^2y=0$

(18) $(D^5-2D^3+D)y=0$

(19) $(D^5-4D^3)y=0$

(20) $(D-1)^3(D+1)^2y=0$

(21) $(D-1)^2(D^2+1)^2y=0$

(22) $(D^2+1)(D^2+4)^2y=0$

NON – HOMOGENEOUS EQUATIONS

Solve the following differential equations :

CASE (1)

(23) $(D^2+2D+1)y=2e^{2x}$

(24) $(D^2-2D+5)y=e^{-x}$

(25) $(D^2+6D+9)y=2e^{-3x}$

(26) $(D^2-4D+3)y=\frac{1}{2}(e^x+e^{-x})$

(27) $(D^2-2D+1)y=(e^x+1)^2$

(28) $(D^3-3D^2+4)y=e^{3x}$

(29) $(D^3-D)y=e^x+e^{-x}$

(30) $(D^3-D^2-D+1)y=e^x$

(31) $(D^4+D^3-6D^2)y=2e^{2x}$

(32) $(D^4-1)y=e^x-e^{-x}$

(33) $(D^4-8D^2-9)y=50\sinh 2x$

(34) $(D^4-2D^2+1)y=40\cosh x$

CASE (2)

Solve the following differential equations :

(35) $(D^2-5D+6)y = \sin 3x$ (36) $(D^2-2D+1)y = \cos 3x$

(37) $(D^2+4)y = \cos 2x + \cos 4x$ (38) $(D^2-4D+4)y = e^{-4x}+5\cos 3x$

(39) $(D^3+6D^2+11D+6)y = 2\sin x$ (40) $(D^3-2D^2+3)y = \cos x$

(41) $(D^4-2D^2+1)y = \cos x$ (42) $(D^4-1)y = \sin x$

(43) $(D^4+D^3)y = \cos 4x$ (44) $(D^4+10D^2+9)y = 48(\sin 3x+\sin x)$

(45) $(D^5-D)y = 8\sin 2x$ (46) $(D^5+2D^3+D)y = \sin x+\cos x$

CASE (3)

Solve the following differential equations :

(47) $(D^2+D-6)y = x$ (48) $(D^2+D+1)y = x^2$

(49) $(D^2-4)y = 16x^3$ (50) $(D^2-3D+2)y = e^{2x}+x^2$

(51) $(D^2+4)y = 3e^x+\sin 2x+x^2$ (52) $(D^3-13D+12)y = x$

(53) $(D^3+3D^2+2D)y = x^2$ (54) $(D^3+D^2+D+1)y = 2e^{-x}+x^2$

(55) $(D^3-D^2)y = 2x^3$ (56) $(D^4-6D^3+9D^2)y = 54x+18$

(57) $(D^4-2D^3+D^2)y = x^3$ (58) $(D^5-D)y = 12e^x+8\sin x-2x$

CASE (4)

Solve the following differential equations :

(59) $(D^2-2D+4)y = e^x\cos x$ (60) $(D^2+2D+1)y = e^x\sin 2x$

(61) $(D^2+4D+4)y = x^2e^{-2x}$ (62) $(D^2-2D+5)y = e^{2x}\sin x$

(63) $(D^2+1)y = e^{-x}+\cos x+x^3+e^x\sin x$ (64) $(D^3+3D^2-4)y = xe^{-2x}$

(65) $(D^3-3D^2+3D-1)y = e^x(x+1)$ (66) $(D^3-3D+2)y = x^2e^x$

(67) $(D^3+1)y = e^x\cos x$ (68) $(D^3-D^2+3D+5)y = e^x\cos 2x$

(69) $(D^4+6D^3+11D^2+6D)y = 20e^{-2x}\sin x$ (70) $(D^4-1)y = e^x\cos x$

CASE (5)

Solve the following differential equations :

(71) $\dfrac{d^2y}{dx^2}-y = 2x\sin x$ (72) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y = x\sin x$

(73) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y = x\cos x$ (74) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y = x\sin 2x$

(75) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y = xe^x\sin x$ (76) $(D^4-1)y = x\sin x$

METHOD OF VARIATION OF PARAMETERS

Using the method of variation of parameters, solve the following differential equations.

(77) $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x$ (78) $\frac{d^2y}{dx^2}-y=e^x$

(79) $\frac{d^2y}{dx^2}+y=\sin x$ (80) $\frac{d^2y}{dx^2}+y=\sec x \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$

(81) $\frac{d^2y}{dx^2}+y=\operatorname{cosec} x$ (82) $\frac{d^2y}{dx^2}+y=\cot x$

(83) $\frac{d^2y}{dx^2}-\frac{dy}{dx}=e^x\cos x$ (84) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$

(85) $\frac{d^2y}{dx^2}+4y=4\sec^2 2x$ (86) $(D^3-D)y=x$

(87) $(D^3+D)y=\tan x$ (88) $(D^3+D)y=\operatorname{cosec} x$

(89) $(D^3-6D^2+11D-6)y=e^{-x}$ (90) $(D^3-5D^2+8D-4)y=e^{2x}$

(91) $(D^3+6D^2+12D+8)y=12e^{-2x}$ (92) $(D^3-3D^2+3D-1)y=2x^{-2}e^x$

METHOD OF UNDETERMINED COEFFICIENTS

Solve the following differential equations :

(93) $y''-y'-2y=e^{3x}$ (94) $y''+3y'+2y=\sin x$

(95) $y''+y'+y=x^2$ (96) $y''-y'=e^x\sin 2x$

(97) $y''-9y=x+e^{2x}-\sin 2x$ (98) $y''-2y'+y=xe^x$

(99) $y''-3y'+2y=2x^2+3e^{2x}$ (100) $y''+y=4x\cos x$

(101) $(D^3-3D^2+3D-1)y=e^x$ (102) $(D^3-D)y=3e^x+\sin x$

(103) $(D^3-1)y=x^3+2$ (104) $(D^3+3D^2+2D)y=x^2+4x+8$

(105) $(D^3-6D^2+11D-6)y=2xe^{-x}$ (106) $(D^3+3D^2-4)y=xe^{-2x}$

(107) $(D^4+D^3)y=e^{4x}+e^{-4x}$ (108) $(D^4-2D^2+1)y=x-\sin x$

EULER'S – CAUCHY DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(109) $x^2y''+xy'-9y=0$ (110) $x^2y''-xy'+y=0$

(111) $x^2y''+9xy'+25y=0$ (112) $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$

(113) $x^2y''-3xy'-5y=\sin(\ln x)$ (114) $x^2y''+xy'=12\ln x$

(115) $x^2y''-2xy'+2y=\ln^2 x-\ln x^2$ (116) $x^2y''-2xy'+2y=x^2\ln x$

(117) $x^2 y'' - 3xy' + 5y = x^2 \sin(\ln x)$

(118) $(x^3 D^3 - 6)y = 0$

(119) $(x^3 D^3 + 6x^2 D^2 + 4xD - 4)y = 0$

(120) $(x^3 D^3 + x^2 D^2)y = x$

(121) $(x^3 D^3 + 2x^2 D^2 - xD + 1)y = \frac{1}{x}$

(122) $(x^3 D^3 + 2x^2 D^2 + 2)y = 10\left(x + \frac{1}{x}\right)$

(123) $(x^3 D^3 + 3x^2 D^2 + xD)y = 24x^2$

(124) $(x^3 D^3 - x^2 D^2 + 2xD - 2)y = x^3 + 3x$

(125) $(x^3 D^3 + 2x^2 D^2)y = x + \sin(\ln x)$

(126) $(x^3 D^3 + 3x^2 D^2 + xD)y = x^3 \ln x$

(127) $(x^3 D^3 + 3x^2 D^2 + xD + 1)y = x \ln x$

(128) $(x^3 D^3 + 2xD - 2)y = x^2 \ln x + 3x$

(129) $(x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3xD + 1)y = 0$

(130) $(x^4 D^4 + 2x^3 D^3 + x^2 D^2 - xD + 1)y = x + \ln x$

LEGENDRE'S LINEAR DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(131) $(2x+1)^2 \frac{d^2 y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

(132) $(x+2)^2 \frac{d^2 y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4$

(133) $(x+1)^2 \frac{d^2 y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$

(134) $(x+3)^2 \frac{d^2 y}{dx^2} - 4(x+3)\frac{dy}{dx} + 6y = \ln(x+3)$

(135) $(x+1)^2 \frac{d^2 y}{dx^2} + (x+1)\frac{dy}{dx} - y = \ln(x+1)^2 + x - 1$

(136) $(x+1)^2 \frac{d^2 y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin[\ln(1+x)]$

(137) $(x+1)^2 \frac{d^2 y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = (x+1)^3$

(138) $(2x+1)^2 \frac{d^2 y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

(139) $(2x-1)^3 \frac{d^3 y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = 0$

(140) $[(3x-2)^3 D^3 + (3x-2)^2 D^2 + (3x-2)D - 4]y = \ln(3x-2)$

CHAPTER 5

SPECIAL METHODS FOR SECOND AND HIGHER ORDER DIFFERENTIAL EQUATIONS

5.1 INTRODUCTION

In chapter (4), we have discussed standard methods for the solutions of second and higher order linear differential equations. In this chapter, we will discuss some special methods for the solution of second and higher order differential equations. These methods are all grouped under the heading of reduction of order.

First, we discuss differential equations in which either the dependent variable or independent variable is absent. Then the linear differential equations are discussed in which one integral (solution) of complementary function is known. Furthermore, second order differential equations are discussed in which the dependent variable or independent variable is changed. Finally, the exact differential equations and the method of factorization of operators are discussed.

5.2 METHOD OF REDUCTION OF ORDER

SECOND ORDER DIFFERENTIAL EQUATIONS

In this method, the idea is to reduce the problem of solving a second order differential equation to one of solving one or more first order differential equations. We will discuss two cases in which this method can be used.

CASE (1): DEPENDENT VARIABLE y ABSENT

Consider the second order differential equation

$$F(x, y, y', y'') = 0 \qquad (1)$$

in which x is the independent variable and y is the dependent variable.

If the dependent variable y does not appear explicitly in the differential equation (although y' may, and y'' must), we say that the dependent variable is absent. In this case, differential equation (1) has the form

$$F(x, y', y'') = 0 \qquad (2)$$

Let $y' = p$ then $y'' = \frac{dp}{dx}$ so that equation (2) reduces to the first order differential equation in p as

$$F\left(x, p, \frac{dp}{dx}\right) = 0$$

We solve this equation for p, and then obtain the solution of the original differential equation by integrating $y(x) = \int p(x)\,dx$.

EXAMPLE (1): **Solve the differential equation** $xy'' + 2y' = 4x^3$.

SOLUTION: Since the dependent variable y does not appear explicitly, let $y' = p$ and $y'' = \frac{dp}{dx}$ in the given differential equation, we get

$$x\frac{dp}{dx} + 2p = 4x^3$$

or
$$\frac{dp}{dx} + \frac{2}{x}p = 4x^2 \tag{i}$$

which is the first order linear differential equation whose integrating factor is :

$$\text{I.F.} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Thus the solution of differential equation (i) is given by

$$(\text{I.F.})p = \int (\text{I.F.})Q\,dx + C$$

or
$$x^2 p = \int 4x^2 . x^2\,dx + C$$
$$= \int 4x^4\,dx + C = \frac{4}{5}x^5 + C_1$$

or
$$p = \frac{4}{5}x^3 + \frac{C_1}{x^2}$$

or
$$\frac{dy}{dx} = y' = \frac{4}{5}x^3 + \frac{C_1}{x^2}$$

The solution of the given differential equation is given by integrating this equation

$$y(x) = \frac{1}{5}x^4 - \frac{C_1}{x} + C_2$$

where C_1 and C_2 are arbitrary constants .

CASE (2): INDEPENDENT VARIABLE x ABSENT

If the independent variable x does not appear explicitly , then the differential equation (1) has the form $F(y, y', y'') = 0$ (3)

To solve equation (3) , let $y' = p$, then

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}y' = p\frac{dp}{dy}$$

Then equation (3) becomes $F\left(y, p, p\frac{dp}{dy}\right) = 0$

Solve this differential equation for p in terms of y , then use $y' = p$ to solve for y in terms of x.

EXAMPLE (2): **Solve the differential equation** $1 + y\,y'' + (y')^2 = 0$ **.**

SOLUTION: Since the independent variable x does not appear explicitly, let $y' = p$ and $y'' = p\dfrac{dp}{dy}$ in the given differential equation, we get

$$1 + y\,p\frac{dp}{dy} + p^2 = 0$$

or $$y\,p\frac{dp}{dy} = -(1 + p^2)$$

or $$\frac{p\,dp}{1+p^2} + \frac{dy}{y} = 0$$

Integrating, we get $\frac{1}{2}\ln(1+p^2) + \ln y = C_1$

or $\ln(1+p^2) + 2\ln y = C_2$ where $C_2 = 2C_1$

or $\ln\left[(1+p^2)y^2\right] = C_2$

from which $(1+p^2)y^2 = a^2$ where $a^2 = e^{C_2}$

and $p^2 = \dfrac{a^2 - y^2}{y^2}$ or $p = \pm\dfrac{\sqrt{a^2-y^2}}{y}$

or $$\frac{dy}{dx} = \pm\frac{\sqrt{a^2-y^2}}{y}$$

Separating the variables, we get $\pm\dfrac{y\,dy}{\sqrt{a^2-y^2}} = dx$

Integrating, $\mp\sqrt{a^2-y^2} = x + b$

Squaring both sides gives $(x+b)^2 + y^2 = a^2$

which is the general solution of the given differential equation.

5.3 HIGHER ORDER DIFFERENTIAL EQUATIONS

CASE (1): DEPENDENT VARIABLE y ABSENT

Now we discuss the higher order differential equation

$$F(x, y, y', y'', \ldots\ldots, y^{(n)}) = 0 \qquad (1)$$

If the dependent variable y is absent, then the nth order linear differential equation (1) takes of the form

$$F(x, y', y'', \ldots\ldots, y^{(n)}) = 0$$

Now the substitution $y' = p$, $y'' = \dfrac{dp}{dx}$, ... will transform equation (1) to the form

$$F\left(x, p, \frac{dp}{dx}, \ldots\ldots, \frac{d^{n-1}p}{dx^{n-1}}\right) = 0$$

i.e. the order of the differential equation is reduced by one.

If the nth order linear differential equation is of the form

$$F\left(x, y^{(k)}, y^{(k+1)}, \ldots\ldots, y^{(n)}\right) = 0 \qquad (2)$$

then the substitution $y^{(k)} = q,\ y^{(k+1)} = q' = \frac{dq}{dx}, \ldots\ldots$ will reduce equation (2) to the form

$$F\left(x, q, \frac{dq}{dx}, \ldots\ldots\ldots, \frac{d^{n-k}q}{dx^{n-k}}\right) = 0$$

i.e. that order of differential equation (2) will be reduced by k.

EXAMPLE (3): **Solve the following differential equations :**

(i) $x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$ (ii) $x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 0$

SOLUTION: (i) $x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$

Since the dependent variable y is absent, so let $\frac{dy}{dx} = p$ so that

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} \quad \text{and} \quad \frac{d^3y}{dx^3} = \frac{d^2p}{dx^2}$$

Substituting into the given differential equation, we get

$$x^2\frac{d^2p}{dx^2} - 4x\frac{dp}{dx} + 6p = 4 \qquad (1)$$

which is a second order Euler – Cauchy differential equation. To solve equation (1), let $x = e^t$, then

$$x\frac{dp}{dx} = \frac{dp}{dt} \quad \text{and} \quad x^2\frac{d^2p}{dx^2} = \frac{d^2p}{dt^2} - \frac{dp}{dt}$$

Thus equation (1) becomes

$$\frac{d^2p}{dt^2} - \frac{dp}{dt} - 4x\frac{dp}{dt} + 6p = 4$$

or

$$\frac{d^2p}{dt^2} - 5\frac{dp}{dt} + 6p = 4$$

or

$$\left(D_1^2 - 5D_1 + 6\right)p = 4 \quad \left(\text{where } D_1 = \frac{d}{dt}\right) \qquad (2)$$

which is a second order linear differential equation with constant coefficients.

The characteristic roots of the auxiliary equation are $D_1 = 2, 3$.

The complementary function is

$$p_c = C_1e^{2t} + C_2e^{3t}$$

The particular integral is

$$p_p = \frac{1}{D_1^2 - 5D_1 + 6}.4 = \frac{2}{3}$$

Thus the solution of equation (2) is

$$p = C_1 e^{2t} + C_2 e^{3t} + \frac{2}{3}$$

or $$\frac{dy}{dx} = C_1 e^{2\ln x} + C_2 e^{3\ln x} + \frac{2}{3} = C_1 x^2 + C_2 x^3 + \frac{2}{3}$$

or $$y = \frac{1}{3} C_1 x^3 + \frac{1}{4} C_2 x^4 + \frac{2}{3} x + C_3$$

(ii) $$x\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} = 0$$

Let $\frac{d^2y}{dx^2} = q$, then $\frac{d^3y}{dx^3} = \frac{dq}{dx}$ and the given equation becomes

$$x\frac{dq}{dx} - 2q = 0$$

Separating the variables $\frac{dq}{q} = 2\frac{dx}{x}$

Integrating, we get

$$\ln q = 2\ln x + \ln C_1 = \ln x^2 + \ln C_1$$

or $$q = C_1 x^2$$

or $$\frac{d^2y}{dx^2} = C_1 x^2$$

or $$\frac{dy}{dx} = \frac{1}{3} C_1 x^3 + C_2$$

and $$y = \frac{1}{12} C_1 x^4 + C_2 x + C_3$$

CASE (2): INDEPENDENT VARIABLE x ABSENT

If the independent variable x is absent, then the nth order linear differential equation (1) takes of the form $F\left(y, y', y'', \ldots\ldots, y^{(n)}\right) = 0$

Then the substitution $y' = \frac{dy}{dx} = p$, $y'' = \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

$$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dy}\left(\frac{d^2y}{dx^2}\right)\frac{dy}{dx}$$

$$= \left[\frac{d}{dy}\left(p\frac{dp}{dy}\right)\right]\frac{dy}{dx}$$

$$= \left[p\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right]p$$

$$= p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \ldots\ldots$$

will reduce the order of the equation by one.

EXAMPLE (4): **Solve the differential equation :**

$$\frac{dy}{dx}\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^2 = 2\left(\frac{d^2y}{dx^2}\right)^2$$

SOLUTION: Since the independent variable x does not appear explicitly , so let $y' = \frac{dy}{dx} = p$

Then $y'' = \frac{d^2y}{dx^2} = p\frac{dp}{dy}$ and $y''' = \frac{d^3y}{dx^3} = p^2\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2$

Substituting the values of these derivatives in the given equation , we get

$$p\left[p^2\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2\right]+p^2 = 2\left(p\frac{dp}{dy}\right)^2$$

or $$p^2\left[p\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2+1-2\left(\frac{dp}{dy}\right)^2\right] = 0$$

or $$p^2\left[p\frac{d^2p}{dy^2}-\left(\frac{dp}{dy}\right)^2+1\right] = 0$$

This implies either $p = 0$ i.e. $\frac{dy}{dx} = 0$ or $y = C$

or $$p\frac{d^2p}{dy^2}-\left(\frac{dp}{dy}\right)^2+1 = 0 \qquad (1)$$

which is a second order differential equation with the independent variable y absent .

Let $\frac{dp}{dy} = u$, then $\frac{d^2p}{dy^2} = \frac{d}{dy}\left(\frac{dp}{dy}\right) = \frac{du}{dy} = \frac{du}{dp}\frac{dp}{dy} = u\frac{du}{dp}$

and equation (1) becomes

$$pu\frac{du}{dp}-u^2+1 = 0$$

or $$pu\frac{du}{dp} = u^2-1$$

Separating the variables , we get

$$\frac{u}{u^2-1}du = \frac{dp}{p}$$

Integrating both sides , we get

$$\frac{1}{2}\ln(u^2-1) = \ln p+\ln A$$

or $$\ln\sqrt{u^2-1} = \ln Ap$$

or $$\sqrt{u^2-1} = Ap$$

or $$u^2-1 = A^2p^2$$

or $$u^2 = 1+A^2p^2$$

or $$u = \sqrt{1+A^2p^2}$$

or $\quad \dfrac{dp}{dy} = \sqrt{1 + A^2 p^2}$

or $\quad \dfrac{dp}{\sqrt{1 + A^2 p^2}} = dy$

or $\quad \dfrac{dp}{A\sqrt{\dfrac{1}{A^2} + p^2}} = dy$

Integrating both sides , we get

$$\frac{1}{A}\sinh^{-1} A p = y + K$$

or $\quad \sinh^{-1} A p = A y + B \quad$ (where $B = A K$)

or $\quad A p = \sinh (A y + B)$

or $\quad A \dfrac{dy}{dx} = \sinh (A y + B)$

or $\quad \dfrac{A\, dy}{\sinh (A y + B)} = dx$

Let $A y + B = t$, then $A\, dy = dt$

Thus $\quad \dfrac{dt}{\sinh t} = dx$

or $\quad \ln \tanh \dfrac{t}{2} = x + C$

or $\quad \ln \tanh \left(\dfrac{A y + B}{2} \right) = x + C$

5.4 LINEAR EQUATIONS WITH ONE KNOWN INTEGRAL BELONGING TO COMPLEMENTARY FUNCTION

Consider a second order linear differential equation

$$y'' + P_1 (x) y' + P_2 (x) y = Q (x) \qquad (1)$$

Suppose that we know one solution $y_1 (x)$ of the corresponding homogeneous differential equation

$$y'' + P_1 y' + P_2 y = 0 \qquad (2)$$

on some interval I.

Now we see how to use y_1 to find the general solution of equation (1) . The idea is to look for a solution of the form

$$y(x) = u (x) y_1 (x) \quad \text{or} \quad \text{briefly } y = u y_1$$

in which u is a non – constant function of x .

Now $y' = u y_1' + u' y_1$ and $y'' = u'' y_1 + 2u' y_1' + u y_1''$

Substituting into the differential equation (1), we get

$$u'' y_1 + 2u' y_1' + u y_1'' + P_1 (u y_1' + u' y_1) + P_2 u y_1 = Q(x)$$

Collecting terms, we get

$$u'' y_1 + u' (2y_1' + P_1 y_1) + u (y_1'' + P_1 y_1' + P_2 y) = Q(x)$$

Since y_1 is a solution of differential equation (2), the expression in the last parentheses is zero. The above equation reduces to

$$u'' y_1 + u' (2y_1' + P_1 y_1) = Q(x) \tag{3}$$

Remembering that y_1, y_1', P_1, and Q are known, this is a differential equation for u. Furthermore, the dependent variable u is not explicitly present. To solve equation (3), let $u' = U$. Then $u'' = U'$, and equation (3) becomes

$$U' y_1 + U (2y_1' + P_1 y_1) = Q$$

Dividing this equation by y_1, we get

$$U' + \left(2\frac{y_1'}{y_1} + P_1\right) U = \frac{Q}{y_1} \tag{4}$$

which is a first order linear differential equation for u. The integrating factor is

$$\text{I.F.} = e^{\int \left(2\frac{y_1'}{y_1} + P_1\right) dx} = e^{2\ln y_1} \cdot e^{\int P_1\, dx} = y_1^2 \cdot e^{\int P_1\, dx}$$

Thus the solution of equation (4) is

$$U \cdot y_1^2\, e^{\int P_1\, dx} = \int y_1^2\, e^{\int P_1\, dx} \cdot \frac{Q}{y_1}\, dx + C_1$$

$$= \int y_1 Q \cdot e^{\int P_1\, dx}\, dx + C_1$$

or $$U = \frac{1}{y_1^2}\, e^{-\int P_1\, dx} \int y_1 Q \cdot e^{\int P_1\, dx}\, dx + C_1 \frac{1}{y_1^2}\, e^{-\int P_1\, dx} \tag{5}$$

Since $U = u'$, so that $u = \int U\, dx$

or $$u = \int \left[\frac{1}{y_1^2}\, e^{-\int P_1\, dx} \int y_1 Q \cdot e^{\int P_1\, dx}\, dx\right] dx + C_1 \int \left(\frac{1}{y_1^2}\, e^{-\int P_1\, dx}\right) dx + C_2 \tag{6}$$

Once we know u, then $y = u y_1$ is the general solution of equation (1) on I.

EXAMPLE (5): **Solve the non – homogeneous linear differential equation**

$$x^2 y'' - x y' + y = x^3,\ x > 0 \text{ given that one solution is } y_1 = x.$$

SOLUTION: Since $y_1 = x$ is one solution of the differential equation therefore let $y = u\,y_1$ be the general solution of this equation.

Write the equation as $y'' - \frac{1}{x} y' + \frac{1}{x^2} y = x$

Here $P_1 = -\frac{1}{x}$, $P_2 = \frac{1}{x^2}$, $Q = x$. Then

$$\frac{1}{y_1^2} e^{-\int P_1 dx} = \frac{1}{x^2} e^{\int \frac{1}{x} dx} = \frac{1}{x^2} e^{\ln x} = \frac{1}{x^2} x = \frac{1}{x}$$

and $e^{\int P_1 dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

From equation (6) above, we have

$$u = \int \left[\frac{1}{x} \int x \cdot x \cdot \frac{1}{x}\, dx \right] dx + C_1 \int \frac{1}{x}\, dx + C_2$$

$$= \int \frac{1}{x} \left(\frac{x^2}{2} \right) dx + C_1 \ln x + C_2$$

$$= \frac{x^2}{6} + C_1 \ln x + C_2$$

Thus the general solution of the given differential equation is

$$y = u\,y_1 = u\,x = \frac{x^3}{6} + C_1 x \ln x + C_2 x$$

5.5 METHOD OF FINDING PARTICULAR INTEGRALS

To find a particular integral of the second order linear homogeneous differential equation with variable coefficients

$$\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \qquad (1)$$

we have the following rules.

RULE (1): Let $y = e^{mx}$ be a particular integral of equation (1), then

$$\frac{dy}{dx} = m e^{mx} \quad \text{and} \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}$$

Substituting the values of these derivatives in equation (1), we get

$$m^2 e^{mx} + P_1 m e^{mx} + P_2 e^{mx} = 0$$

$$(m^2 + P_1 m + P_2) e^{mx} = 0$$

This implies that $m^2 + P_1 m + P_2 = 0 \qquad (2)$

PARTICULAR CASES

(i) If $m = 1$, then from equation (2), e^x is a particular solution of equation (1) if
$1 + P_1 + P_2 = 0$.

(ii) If $m = -1$, then from equation (2), e^{-x} is a particular solution of equation (1)
$1 - P_1 + P_2 = 0$.

(iii) If $m = a$, then from equation (2), e^{ax} is a particular solution of equation (1) if
$a^2 + P_1 a + P_2 = 0$

or $\quad 1 + \dfrac{P_1}{a} + \dfrac{P_2}{a^2} = 0$

RULE (2): Let $y = x^m$ be a particular integral of equation (1), then

$$\frac{dy}{dx} = m x^{m-1} \quad \text{and} \quad \frac{d^2 y}{dx^2} = m(m-1)x^{m-2}$$

Substituting the values of these derivatives in equation (1), we get

$$m(m-1)x^{m-2} + P_1 m x^{m-1} + P_2 x^m = 0$$

or $$\left[m(m-1) + P_1 m x + P_2 x^2\right] x^{m-2} = 0$$

This implies that $m(m-1) + P_1 m x + P_2 x^2 = 0$ (3)

PARTICULAR CASES

(i) If $m = 1$, then from equation (3), x is a particular solution of equation (1)
if $\quad P_1 x + P_2 x^2 = 0$ or $P_1 + P_2 x = 0$

(ii) If $m = 2$ then from equation (3), x^2 is a particular solution of equation (1)
if $\quad 2 + 2P_1 x + P_2 x^2 = 0$

EXAMPLE (6): **Solve the differential equation :**

$$x^2 \frac{d^2 y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

SOLUTION: Write the equation as

$$\frac{d^2 y}{dx^2} - 2\left(\frac{1}{x} + 1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2} + \frac{1}{x}\right)y = x \qquad (1)$$

Comparing it with the general form, we have

$$P_1 = -2\left(\frac{1}{x} + 1\right), \quad P_2 = 2\left(\frac{1}{x^2} + \frac{1}{x}\right)$$

and $$P_1 + P_2 x = -2\left(\frac{1}{x} + 1\right) + 2\left(\frac{1}{x} + 1\right) = 0$$

Thus $y = x$ is a particular integral of

$$\frac{d^2y}{dx^2}-2\left(\frac{1}{x}+1\right)\frac{dy}{dx}+2\left(\frac{1}{x^2}+\frac{1}{x}\right)y = 0$$

Let $y = ux$ so that $\frac{dy}{dx} = u + x\frac{du}{dx}$

and $$\frac{d^2y}{dx^2} = \frac{du}{dx}+x\frac{d^2u}{dx^2}+\frac{du}{dx} = 2\frac{du}{dx}+x\frac{d^2u}{dx^2}$$

Substituting the values of these derivatives in the given differential equation , we get

$$2\frac{du}{dx}+x\frac{d^2u}{dx^2}-2\left(\frac{1}{x}+1\right)\left(u+x\frac{du}{dx}\right)+2\left(\frac{1}{x^2}+\frac{1}{x}\right)ux = x$$

or $$x\frac{d^2u}{dx^2}-2x\frac{du}{dx} = x$$

or $$\frac{d^2u}{dx^2}-2\frac{du}{dx} = 1$$

which is a second order differential equation with constant coefficients and with the dependent variable v absent .

Letting $\frac{du}{dx} = p$, the above equation becomes

$$\frac{dp}{dx}-2p = 1 \qquad (2)$$

which is a first order linear differential equation . To solve this equation , we have

$$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}$$

The solution of equation (2) is

$$e^{-2x}p = \int e^{-2x}1\,dx+C_1$$

$$= -\frac{1}{2}e^{-2x}+C_1$$

or $$p = -\frac{1}{2}+C_1e^{2x}$$

or $$\frac{du}{dx} = -\frac{1}{2}+C_1e^{2x}$$

or $$u = -\frac{1}{2}x+\frac{1}{2}C_1e^{2x}+C_2$$

or $$\frac{y}{x} = -\frac{1}{2}x+\frac{1}{2}C_1e^{2x}+C_2$$

or $$y = -\frac{1}{2}x^2+\frac{1}{2}C_1xe^{2x}+C_2x$$

is the solution of the given differential equation .

EXAMPLE (7): **Solve the differential equation :**

$$x\frac{d^2y}{dx^2}-(2x+1)\frac{dy}{dx}+(x+1)y=(x^2+x-1)e^{2x}$$

SOLUTION: Write the equation as

$$\frac{d^2y}{dx^2}-\left(2+\frac{1}{x}\right)\frac{dy}{dx}+\left(1+\frac{1}{x}\right)y=\left(x+1-\frac{1}{x}\right)e^{2x} \tag{1}$$

Comparing it with the general form of second order differential equation

$$\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=Q(x), \text{ we find that}$$

$$P_1=-\left(2+\frac{1}{x}\right),\quad P_2=\left(1+\frac{1}{x}\right),\quad Q=\left(x+1-\frac{1}{x}\right)e^{2x}$$

Now $\quad 1+P_1+P_2=1-\left(2+\frac{1}{x}\right)+\left(1+\frac{1}{x}\right)=0$

Therefore $y=e^x$ is an integral of the complementary function .

Thus the transformation becomes $y=e^x u$

Now $\quad \frac{dy}{dx}=e^x\frac{du}{dx}+ue^x$

$$\frac{d^2y}{dx^2}=e^x\frac{d^2u}{dx^2}+e^x\frac{du}{dx}+ue^x+e^x\frac{du}{dx}$$

$$=e^x\frac{d^2u}{dx^2}+2e^x\frac{du}{dx}+ue^x$$

Substituting the values of these derivatives in equation (1) , we get

$$e^x\frac{d^2u}{dx^2}+2e^x\frac{du}{dx}+ue^x-\left(2+\frac{1}{x}\right)\left(e^x\frac{du}{dx}+ue^x\right)+\left(1+\frac{1}{x}\right)e^xu=\left(x+1-\frac{1}{x}\right)e^{2x}$$

or $\quad e^x\frac{d^2u}{dx^2}-\frac{1}{x}e^x\frac{du}{dx}=\left(x+1-\frac{1}{x}\right)e^{2x}$

or $\quad \frac{d^2u}{dx^2}-\frac{1}{x}\frac{du}{dx}=\left(x+1-\frac{1}{x}\right)e^x$

which is a second order linear differential equation with the dependent variable u absent .

Letting $\frac{du}{dx}=p$, the above equation becomes

$$\frac{dp}{dx}-\frac{1}{x}p=\left(x+1-\frac{1}{x}\right)e^x \tag{2}$$

which is a first order linear differential equation . To solve this equation , we have

$$\text{I.F.}=e^{\int -\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x}$$

Thus the solution of equation (2) is

$$\frac{1}{x}p = \int \frac{1}{x}\left(x+1-\frac{1}{x}\right)e^x\,dx + C$$

$$= \int \left(1+\frac{1}{x}-\frac{1}{x^2}\right)e^x\,dx + C$$

$$= \int \left(e^x+\frac{e^x}{x}-\frac{e^x}{x^2}\right)dx + C$$

$$= e^x + \frac{1}{x}e^x + \int \frac{e^x}{x^2}\,dx - \int \frac{e^x}{x^2}\,dx + K$$

$$= e^x + \frac{1}{x}e^x + K$$

or $$p = xe^x + e^x + Kx$$

or $$\frac{du}{dx} = xe^x + e^x + Kx$$

or $$u = xe^x - e^x + e^x + K\frac{x^2}{2} + C_2$$

$$= xe^x + \frac{K}{2}x^2 + C_2 = xe^x + C_1x^2 + C_2 \qquad \left(\text{where } C_1 = \frac{K}{2}\right)$$

or $$\frac{y}{e^x} = xe^x + C_1x^2 + C_2$$

or $$y = xe^{2x} + C_1x^2e^x + C_2e^x$$

5.6 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ONE KNOWN INTEGRAL OF COMPLEMENTARY FUNCTION

Now consider the nth order linear differential equation with variable coefficients

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + P_{n-1}\frac{dy}{dx} + P_ny = Q(x)$$

i.e. $$y^{(n)} + P_1y^{(n-1)} + \dots\dots + P_{n-1}y' + P_ny = Q(x) \qquad (1)$$

If a particular integral $y = y_1(x)$ of the corresponding homogeneous differential equation

$$y^{(n)} + P_1y^{(n-1)} + \dots\dots + P_{n-1}y' + P_ny = 0 \qquad (2)$$

is known, then the substitution $y_2 = u\,y_1$ will transform equation (1) into an equation of the same order but with the dependent variable absent. In turn, the order of this equation may be reduced by the methods already known. Equation (2) is called the **reduced** equation of equation (1).

EXAMPLE (8): **Solve the following differential equation**

$$x^3(\sin x)\frac{d^3y}{dx^3}-(3x^2\sin x+x^3\cos x)\frac{d^2y}{dx^2}+(6x\sin x+2x^2\cos x)\frac{dy}{dx}-(6\sin x+2x\cos x)y=0$$

Given that $y = x$ is a particular integral.

SOLUTION: Since $y = x$ is a particular integral, therefore the transformation is $y = xu$.

Then $\frac{dy}{dx}=x\frac{du}{dx}+u$, $\frac{d^2y}{dx^2}=x\frac{d^2u}{dx^2}+2\frac{du}{dx}$ and $\frac{d^3y}{dx^3}=x\frac{d^3u}{dx^3}+3\frac{d^2u}{dx^2}$

Substituting the values of these derivatives in the given equation, we get

$$x^3\sin x\left(x\frac{d^3u}{dx^3}+3\frac{d^2u}{dx^2}\right)-(3x^2\sin x+x^3\cos x)\left(x\frac{d^2u}{dx^2}+2\frac{du}{dx}\right)$$
$$+(6x\sin x+2x^2\cos x)\left(x\frac{du}{dx}+u\right)-(6\sin x+2x\cos x)xu=0$$

$$x^4\sin x\frac{d^3u}{dx^3}+3x^3\sin x\frac{d^2u}{dx^2}-3x^3\sin x\frac{d^2u}{dx^2}-x^4\cos x\frac{d^2u}{dx^2}-6x^2\sin x\frac{du}{dx}$$
$$-2x^3\cos x\frac{du}{dx}+6x^2\sin x\frac{du}{dx}+2x^3\cos x\frac{du}{dx}$$
$$+6x(\sin x)u+2x^2(\cos x)u-6x(\sin x)u-2x^2(\cos x)u=0$$

or $\quad x^4\sin x\frac{d^3u}{dx^3}-x^4\cos x\frac{d^2u}{dx^2}=0$

or $\quad \sin x\frac{d^3u}{dx^3}-\cos x\frac{d^2u}{dx^2}=0 \qquad (1)$

Let $\frac{d^2u}{dx^2}=q$, then $\frac{d^3u}{dx^3}=\frac{dq}{dx}$, and equation (1) becomes

$$\sin x\frac{dq}{dx}-\cos x\, q=0$$

or $\quad \frac{dq}{dx}-\cot x\, q=0$

or $\quad \frac{dq}{q}=\cot x\, dx$

or $\quad \ln q=\ln\sin x+\ln C_1=\ln C_1\sin x$

or $\quad q=C_1\sin x$ or $\frac{d^2u}{dx^2}=C_1\sin x$

or $\quad \frac{du}{dx}=-C_1\cos x+C_2$

or $\quad u=-C_1\sin x+C_2x+C_3$

or $\quad \frac{y}{x}=-C_1\sin x+C_2x+C_3$

or $\quad y=-C_1x\sin x+C_2x^2+C_3x$

5.7 SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

The general form of the second order linear differential equation with variable coefficients is

$$\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=Q(x)$$

If the coefficients P_1 and P_2 are constants, then this equation can be solved by the methods already known, otherwise no general method is known. The following are certain methods which will be used to find the solution.

5.8 CHANGE OF DEPENDENT VARIABLE

Consider the second order linear differential equation with variable coefficients

$$\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=Q(x) \qquad (1)$$

Under the transformation $y=uv$; $u=u(x)$ and $v=v(x)$, we have

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2v}{dx^2}+\frac{dv}{dx}\frac{du}{dx}$$

$$=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

Thus equation (1) becomes

$$u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}+P_1(x)\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)+P_2(x)uv=Q(x)$$

or $$\frac{d^2v}{dx^2}+\frac{2}{u}\frac{du}{dx}\frac{dv}{dx}+\frac{v}{u}\frac{d^2u}{dx^2}+P_1(x)\frac{dv}{dx}+P_1(x)\frac{v}{u}\frac{du}{dx}+P_2(x)v=\frac{1}{u}Q(x)$$

or $$\frac{d^2v}{dx^2}+\left[\frac{2}{u}\frac{du}{dx}+P_1(x)\right]\frac{dv}{dx}+\frac{1}{u}\left[\frac{d^2u}{dx^2}+P_1(x)\frac{du}{dx}+P_2(x)u\right]v=\frac{1}{u}Q(x)$$

or $$\frac{d^2v}{dx^2}+R_1(x)\frac{dv}{dx}+R_2(x)v=Q_1(x) \qquad (2)$$

where $R_1(x)=\frac{2}{u}\frac{du}{dx}+P_1(x)$, $\quad R_2(x)=\frac{1}{u}\left[\frac{d^2u}{dx^2}+P_1(x)\frac{du}{dx}+P_2(x)u\right]$

and $Q_1(x)=\frac{1}{u}Q(x)$

CASE (1): REDUCTION TO LINEAR FORM WITH DEPENDENT VARIABLE ABSENT

If $y=u$ is a known solution of $\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0$, then $\frac{d^2u}{dx^2}+P_1(x)\frac{du}{dx}+P_2(x)u=0$ and so $R_2(x)=0$. Thus equation (2) becomes

$$\frac{d^2v}{dx^2}+R_1(x)\frac{dv}{dx}=Q_1(x) \tag{3}$$

which is a linear differential equation with dependent variable v absent .

The further substitution $\frac{dv}{dx}=p$, $\frac{d^2v}{dx^2}=\frac{dp}{dx}$ reduces equation (3) to

$$\frac{dp}{dx}+R_1(x)p=Q_1(x) \tag{4}$$

which is a first order linear differential equation .

EXAMPLE (9): **Solve the differential equation :**

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=8x^3 \quad \text{given that} \quad y=x$$

is a solution of the corresponding homogeneous equation .

SOLUTION: Write the equation as

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=8x \tag{i}$$

Since $y=x$ is a known solution , therefore the transformation becomes $y=xv$

Thus $\frac{dy}{dx}=x\frac{dv}{dx}+v$, $\frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+2\frac{dv}{dx}$

Equation (i) thus becomes

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}+\frac{1}{x}\left(x\frac{dv}{dx}+v\right)-\frac{1}{x^2}(xv)=8x$$

or $$x\frac{d^2v}{dx^2}+3\frac{dv}{dx}=8x$$

or $$\frac{d^2v}{dx^2}+\frac{3}{x}\frac{dv}{dx}=8 \tag{ii}$$

which is a second order differential equation with dependent variable v absent .

Let $\frac{dv}{dx}=p$, then $\frac{d^2v}{dx^2}=\frac{dp}{dx}$ and equation (ii) becomes

$$\frac{dp}{dx}+\frac{3}{x}p=8 \tag{iii}$$

which is a first order linear differential equation . The integrating factor is

$$\text{I.F.}=e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3$$

The solution of equation (iii) is

$$x^3\cdot p=\int 8x^3\,dx+C_1=2x^4+C_1$$

or $$p=2x+\frac{C_1}{x^3}$$

or $$\frac{dv}{dx} = 2x + \frac{C_1}{x^3}$$

or $$v = x^2 - \frac{1}{2} C_1 \frac{1}{x^2} + C_2$$

The solution of the given differential equation is given by

$$y = xv = x^3 - \frac{1}{2} C_1 \frac{1}{x} + C_2 x$$

CASE (2): REMOVAL OF FIRST DERIVATIVE

If $y = u$ is chosen so that

$$R_1(x) = \frac{2}{u}\frac{du}{dx} + P_1(x) = 0 \tag{5}$$

Then equation (2) becomes

$$\frac{d^2 v}{dx^2} + R_2(x)\, v = Q_1(x) \tag{6}$$

Now from equation (5), we have

$$\frac{du}{dx} = -\frac{1}{2} P_1(x)\, u \tag{7}$$

Separating the variables, we get

$$\frac{du}{u} = -\frac{1}{2} P_1(x)\, dx$$

Integrating, we get

$$\ln u = -\frac{1}{2} \int P_1(x)\, dx$$

or $$u = e^{-\frac{1}{2}\int P_1(x)\, dx}$$

Now differentiating equation (7), we get

$$\frac{d^2 u}{dx^2} = -\frac{1}{2} P_1(x) \frac{du}{dx} - \frac{1}{2} u \frac{dP_1}{dx}$$

so that
$$\begin{aligned}
R_2(x) &= \frac{1}{u}\left(\frac{d^2 u}{dx^2} + P_1 \frac{du}{dx} + P_2 u\right) \\
&= P_2(x) + \frac{P_1(x)}{u}\frac{du}{dx} + \frac{1}{u}\frac{d^2 u}{dx^2} \\
&= P_2(x) + \frac{P_1(x)}{u}\frac{du}{dx} - \frac{1}{2}\frac{P_1(x)}{u}\frac{du}{dx} - \frac{1}{2}\frac{dP_1}{dx} \\
&= P_2(x) + \frac{1}{2}\frac{P_1(x)}{u}\frac{du}{dx} - \frac{1}{2}\frac{dP_1}{dx} \\
&= P_2(x) - \frac{1}{4}P_1^2(x) - \frac{1}{2}\frac{dP_1}{dx} \qquad \text{[using equation (7)]}
\end{aligned}$$

and $$Q_1(x) = \frac{1}{u} Q(x)$$

(i) If $R_2(x) = P_2(x) - \frac{1}{4}P_1^2(x) - \frac{1}{2}\frac{dP_1}{dx} = A$, a constant, then equation (2) becomes

$$\frac{d^2v}{dx^2} + Av = Q_1(x) \tag{8}$$

which is a second order linear differential equation with constant coefficients. Equation (8) is said to be the **normal form** of the differential equation (1).

(ii) If $R_2(x) = \frac{A}{x^2}$, equation (2) on page (215) becomes

$$x^2\frac{d^2v}{dx^2} + Av = x^2Q_1(x) \tag{9}$$

which is a second order Euler – Cauchy differential equation and the substitution $x = e^t$ will reduce it to one with constant coefficients.

EXAMPLE (10): Solve the differential equations :

(i) $$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$$

(ii) $$4x^2\frac{d^2y}{dx^2} + 4x^3\frac{dy}{dx} + (x^4 + 2x^2 + 1)y = 0$$

SOLUTION: (i) $$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right)y = xe^x$$

Comparing this equation with the general form of second order differential equation

$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = Q(x)$, we find that

$P_1 = -\frac{2}{x}$, $P_2 = 1 + \frac{2}{x^2}$ and $Q = xe^x$

Now $u = e^{-\frac{1}{2}\int P_1 dx} = e^{-\frac{1}{2}\int -\frac{2}{x}dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

and $R_1(x) = \frac{2}{u}\frac{du}{dx} + P_1(x) = \frac{2}{x}(1) - \frac{2}{x} = 0$

Also, $P_2 - \frac{1}{4}P_1^2 - \frac{1}{2}\frac{dP_1}{dx} = 1 + \frac{2}{x^2} - \frac{1}{4}\left(-\frac{2}{x}\right)^2 - \frac{1}{2}\left(\frac{2}{x^2}\right)$

$= 1 + \frac{2}{x^2} - \frac{1}{x^2} - \frac{1}{x^2} = 1 = A$

Thus the transformation becomes $y = uv = xv$.

The normal form of the given differential equation is

$\frac{d^2v}{dx^2} + Av = Q_1(x) = \frac{1}{u}Q(x)$ becomes

$$\frac{d^2 v}{dx^2} + v = \frac{1}{x}(x e^x)$$

or $$\frac{d^2 v}{dx^2} + v = e^x$$

which is a linear differential equation with constant coefficients , whose solution is

$$v = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^x$$

or $$\frac{y}{x} = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^x$$

or $y = C_1 x \cos x + C_2 x \sin x + \frac{1}{2} x e^x$ is the required solution .

(ii) $$4x^2 \frac{d^2 y}{dx^2} + 4x^3 \frac{dy}{dx} + (x^4 + 2x^2 + 1) y = 0$$

Write the equation as

$$\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(\frac{1}{4} x^2 + \frac{1}{2} + \frac{1}{4x^2}\right) y = 0$$

Comparing it with the general form of second order differential equation

$$\frac{d^2 y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) y = Q(x)$$

we have $P_1 = x$, $P_2 = \frac{1}{4} x^2 + \frac{1}{2} + \frac{1}{4x^2}$, $Q = 0$

Now $$u = e^{-\frac{1}{2}\int P_1 \, dx} = e^{-\frac{1}{2}\int x \, dx} = e^{-x^2/4}$$

and $$R_1(x) = \frac{2}{u}\frac{du}{dx} + P_1(x) = 2 e^{x^2/4} e^{-x^2/4}\left(-\frac{x}{2}\right) + x = -x + x = 0$$

Also, $$R_2(x) = P_2 - \frac{1}{4} P_1^2 - \frac{1}{2}\frac{dP_1}{dx}$$

$$= \frac{1}{4} x^2 + \frac{1}{2} + \frac{1}{4x^2} - \frac{1}{4} x^2 - \frac{1}{2} = \frac{1}{4x^2} = \frac{A}{x^2} \quad \text{where } A = \frac{1}{4}$$

Thus the transformation becomes $y = uv = e^{-x^2/4} v$

and the reduced equation $x^2 \frac{d^2 v}{dx^2} + Av = x^2 Q_1(x) = x^2 \frac{1}{u} Q(x)$ becomes

$$x^2 \frac{d^2 v}{dx^2} + \frac{1}{4} v = 0 \qquad (1)$$

which is a second order homogeneous Euler – Cauchy differential equation .

To solve this equation , let $x = e^t$, then

$$x^2 \frac{d^2 v}{dx^2} = \frac{d^2 v}{dt^2} - \frac{dv}{dt}$$

Thus equation (1) becomes

$$\frac{d^2 v}{dt} - \frac{dv}{dt} + \frac{1}{4}v = 0$$

or $$\left(D_1^2 - D_1 + \frac{1}{4}\right)v = 0 \quad \left(\text{where } D_1 = \frac{d}{dt}\right)$$

The auxiliary equation is $\left(D_1 - \frac{1}{2}\right)^2 = 0$ having characteristic roots as $D = \frac{1}{2}, \frac{1}{2}$.

The solution becomes

$$v = (C_1 + C_2 t)e^{\frac{1}{2}t} = (C_1 + C_2 \ln x)e^{\frac{1}{2}\ln x}$$

or $$e^{x^2/4}y = (C_1 + C_2 \ln x)\sqrt{x}$$

or $$y = \sqrt{x}\,e^{-x^2/4}(C_1 + C_2 \ln x)$$

5.9 CHANGE OF INDEPENDENT VARIABLE

Consider the second order differential equation

$$\frac{d^2 y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = Q(x) \tag{1}$$

Let the transformation be $z = z(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Substituting the values of these derivatives in equation (1), we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P_1(x)\frac{dy}{dz}\frac{dz}{dx} + P_2(x)y = Q(x)$$

or $$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left[\frac{d^2z}{dx^2} + P_1(x)\frac{dz}{dx}\right]\frac{dy}{dz} + P_2(x)y = Q(x)$$

or $$\frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + P_1(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \frac{P_2(x)y}{\left(\frac{dz}{dx}\right)^2} = \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} \tag{2}$$

Let $z = z(x)$ be chosen so that $\frac{dz}{dx} = \sqrt{\frac{\pm P_2(x)}{a^2}}$, the sign being that which makes $\frac{dz}{dx}$ real and a^2 being any positive constant. (we may take $a^2 = 1$).

If now $\dfrac{\frac{d^2z}{dx^2} + P_1(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = A$ (a constant), equation (2) becomes

$$\frac{d^2y}{dz^2}+A\frac{dy}{dz}\pm a^2y=\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} \qquad (3)$$

which is a second order linear differential equation with constant coefficients .

EXAMPLE (11): **Solve the following differential equation :**

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{1}{x^4}y=\frac{2x^2+1}{x^6}$$

SOLUTION: Comparing this equation with the general form of second order linear differential equation , we find

$$P_1=\frac{2}{x},\quad P_2=\frac{1}{x^4},\quad a^2=1,\quad Q=\frac{2x^2+1}{x^6}$$

When $\frac{dz}{dx}=\sqrt{\frac{+P_2(x)}{a^2}}=\sqrt{\frac{1}{x^4}}=\frac{1}{x^2}$, then

$$\frac{\frac{d^2z}{dx^2}+P_1(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{-\frac{2}{x^3}+\frac{2}{x}\frac{1}{x^2}}{\frac{1}{x^4}}=0=A$$

Thus the transformation becomes $z=-\frac{1}{x}$, and the equation

$$\frac{d^2y}{dz^2}+A\frac{dy}{dz}\pm a^2y=\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} \text{ takes the form}$$

$$\frac{d^2y}{dz^2}+y=\frac{2x^2+1}{x^6}(x^4)$$

or $$\frac{d^2y}{dz^2}+y=2+\frac{1}{x^2}$$

or $$\frac{d^2y}{dz^2}+y=2+z^2$$

which is a second order linear differential equation with constant coefficients with a change of independent variable . The solution of this equation is

$$y=C_1\cos z+C_2\sin z+z^2 \qquad (2)$$

Replacing z by $-\frac{1}{x}$ in equation (2) , we get

$$y=C_1\cos\left(-\frac{1}{x}\right)+C_2\sin\left(-\frac{1}{x}\right)+\frac{1}{x^2}$$

$$=C_1\cos\left(\frac{1}{x}\right)-C_2\sin\left(\frac{1}{x}\right)+\frac{1}{x^2}$$

5.10 EXACT EQUATIONS

A second order differential equation

$$F(x, y, y', y'') = Q(x)$$

is said to be **exact** if it can be obtained by differentiating once the first order differential equation

$$G(x, y, y') = \int Q(x)\,dx + C$$

For example, the equation $x^2 y'' + 3xy' + y = e^{2x}$

is an exact equation since it may be obtained by differentiating once the equation

$$x^2 y' + xy = \frac{1}{2}e^{2x} + C$$

An nth order differential equation

$$F\left(x, y, y', \ldots\ldots, y^{(n)}\right) = Q(x)$$

is said to be **exact** if it can be obtained by differentiating once an equation

$$G\left(x, y, y', \ldots\ldots, y^{(n-1)}\right) = \int Q(x)\,dx + C$$

of one lower order.

For example, the equation

$$3y^2 y''' + 14yy'y'' + 4(y')^2 + 12y'y'' = 2x$$

is an exact equation since it may be obtained by differentiating once the equation

$$3y^2 y'' + 4y(y')^2 + 6(y')^2 = x^2 + C$$

5.11 TEST FOR EXACTNESS

THEOREM (5.1): Prove that the second order linear differential equation

$$\mathbf{P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x)}$$

is exact if and only if $P_2 - P_1' + P_0'' = 0$

PROOF: Let $P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x)$ (1)

be an exact differential equation. Let this equation be obtained by differentiating

$$R_0(x)y' + R_1(x)y = \int Q(x)\,dx + C \tag{2}$$

Differentiating this equation, we get

$$R_0 y'' + R_0' y' + R_1 y' + R_1' y = Q(x)$$

or $$R_0 y'' + \left(R_0' + R_1\right)y' + R_1' y = Q(x) \tag{3}$$

Now equation (1) and (3) must be the same equations, therefore equating the coefficients, we get

$$P_0 = R_0, \quad P_1 = R_0' + R_1, \quad P_2 = R_1'$$

Now $\quad P_2 - P_1' + P_0'' = R_1' - R_0'' - R_1' + R_0'' = 0$

Conversely, suppose that $P_2 - P_1' + P_0'' = 0$ (4)

where $P_0 = R_0, \quad P_1 = R_0' + R_1, \quad P_2 = R_1'$

or $\quad R_1 = P_1 - R_0' = P_1 - P_0'$

Substituting the values of R_0 and R_1 in equation (2), we get

$$P_0 y' + (P_1 - P_0')y = \int Q(x)\,dx + C \tag{5}$$

Differentiating both sides of this equation w.r.t. x, we get

$$P_0 y'' + P_0' y' + (P_1 - P_0')y' + (P_1' - P_0'')y = Q(x)$$

or $\quad P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x) \quad$ [using equation (4)]

is the required equation.

THEOREM (5.2): Prove that the third order linear differential equation

$$\mathbf{P_0(x)y''' + P_1(x)y'' + P_2(x)y' + P_3(x)y = Q(x)}$$

is exact if and only if $P_3 - P_2' + P_1'' - P_0''' = 0$

PROOF: Let the given differential equation

$$P_0(x)y''' + P_1(x)y'' + P_2(x)y' + P_3(x)y = Q(x) \tag{1}$$

be an exact equation. Let this equation be obtained by differentiating

$$R_0(x)y'' + R_1(x)y' + R_2(x)y = \int Q(x)\,dx + C_1 \tag{2}$$

Differentiating this equation, we get

$$R_0 y''' + R_0' y'' + R_1 y'' + R_1' y' + R_2 y' + R_2' y = Q(x)$$

or $\quad R_0 y''' + (R_0' + R_1)y'' + (R_1' + R_2)y' + R_2' y = Q(x)$ (3)

Now equation (1) and (3) must be the same equations, therefore equating the coefficients, we get

$$P_0 = R_0, \quad P_1 = R_0' + R_1, \quad P_2 = R_1' + R_2, \quad P_3 = R_2'$$

Now $\quad P_3 - P_2' + P_1'' - P_0''' = R_2' - R_1'' - R_2' + R_0''' + R_1'' - R_0''' = 0$

Conversely, suppose that $P_3 - P_2' + P_1'' - P_0''' = 0$ (4)

where $P_0 = R_0, \quad P_1 = R_0' + R_1, \quad P_2 = R_1' + R_2, \quad P_3 = R_2'$

This implies $R_0 = P_0$, $R_1 = P_1 - R_0' = P_1 - P_0'$

$$R_2 = P_2 - R_1' = P_2 - P_1' + P_0''$$

Substituting the values of R_0, R_1, and R_2 in equation (2), we get

$$P_0 y'' + (P_1 - P_0')y' + (P_2 - P_1' + P_0'')y = \int Q(x)\,dx + C$$

Differentiating both sides of this equation w.r.t. x, we get

$$P_0 y''' + P_0' y'' + (P_1 - P_0')y'' + (P_1' - P_0'')y' + (P_2 - P_1' + P_0'')y' + (P_2' - P_1'' + P_0''')y = Q(x)$$

or $\quad P_0(x)y''' + P_1(x)y'' + P_2(x)y' + P_3(x)y = Q(x)\quad$ [using equation (4)]

is the required equation.

GENERALIZATION

Similarly, we can prove that an nth order linear differential equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \dots\dots + P_{n-1}y' + P_n y = Q(x)$$

is exact if and only if

$$P_n - P_{n-1}' + P_{n-2}'' + \dots\dots + (-1)^n P_0^{(n)} = 0$$

EXAMPLE (12): Show that the following differential equation

(i) $\quad(\cos x)y'' + (2\sin x)y' + (3\cos x)y = \tan^2 x$

(ii) $\quad(x^3 - 2x)y''' + (8x^2 - 5)y'' + 15xy' + 5y = 0$ are exact.

SOLUTION: (i) $(\cos x)y'' + (2\sin x)y' + (3\cos x)y = \tan^2 x$

Comparing this differential equation with

$P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x)$, we get

$P_0(x) = \cos x$, $\quad P_1(x) = 2\sin x$, $\quad P_2(x) = 3\cos x$

Now $\quad P_0'' = -\cos x$, $\quad P_1' = 2\cos x$, $\quad Q(x) = \tan^2 x$

Since $\quad P_2 - P_1' + P_0'' = 3\cos x - 2\cos x - \cos x = 0$

therefore the given differential equation is exact.

(ii) $\quad(x^3 - 2x)y''' + (8x^2 - 5)y'' + 15xy' + 5y = 0$

Comparing this differential equation with

$P_0(x)y''' + P_1(x)y'' + P_2(x)y' + P_3(x)y = Q(x)$, we get

$P_0(x) = x^3 - 2x$, $\quad P_1(x) = 8x^2 - 5$, $\quad P_2(x) = 15x$, $\quad P_3(x) = 5$, $\quad Q(x) = 0$

Now $\quad P_0''' = 6$, $\quad P_1'' = 16$, $\quad P_2' = 15$

Since $P_3 - P_2' + P_1'' - P_0''' = 5 - 15 + 16 - 6 = 21 - 21 = 0$

therefore the given differential equation is exact.

EXAMPLE (13): **Solve the following differential equation :**

$$(1+x^2)y'' + 3xy' + y = 1 + 3x^2$$

SOLUTION: Comparing the given differential equation with

$P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x)$, we find that

$P_0(x) = 1 + x^2$, $P_1(x) = 3x$, $P_2(x) = 1$, $Q(x) = 1 + 3x^2$

Since $P_2 - P_1' + P_0'' = 1 - 3 + 2 = 0$, therefore the given differential equation is exact.

The first integral of this equation is

$$P_0 y' + (P_1 - P_0')y = \int Q(x)\,dx + C_1 \qquad (1)$$

or $$(1+x^2)y' + (3x - 2x)y = \int (1+3x^2)\,dx + C_1$$

or $$(1+x^2)y' + xy = (x^3 + x) + C_1$$

or $$y' + \frac{x}{1+x^2}y = x + \frac{C_1}{1+x^2} \qquad (2)$$

which is a first order linear differential equation. The integrating factor is

$$\text{I.F.} = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} = \sqrt{1+x^2}$$

The solution of equation (2) is

$$\sqrt{1+x^2}\,y = \int \sqrt{1+x^2}\,.\,x\,dx + C_1\int \frac{dx}{\sqrt{1+x^2}}$$

or $$\sqrt{1+x^2}\,y = \frac{1}{3}(1+x^2)^{3/2} + C_1 \ln\left(x + \sqrt{1+x^2}\right) + C_2$$

EXAMPLE (14): **Solve the differential equation :**

$$(1+x+x^2)y''' + (3+6x)y'' + 6y' = 0$$

SOLUTION: Comparing this equation with

$$P_0(x)y''' + P_1(x)y'' + P_2(x)y' + P_3(x)y = Q(x)$$

we have $P_0 = 1 + x + x^2$, $P_1 = 3 + 6x$, $P_2 = 6$, $P_3 = 0$, $Q(x) = 0$

Since $P_3 - P_2' + P_1'' - P_0''' = 0 - 0 + 0 - 0 = 0$, therefore the given equation is exact. Its first integral is

$$P_0 y'' + (P_1 - P_0')y' + (P_2 - P_1' + P_0'')y = \int Q(x)\,dx + C_1$$

or $$(1+x+x^2)y'' + (4x+2)y' + 2y = C_1 \qquad (1)$$

Now let us examine equation (1) for exactness. Comparing it with

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x), \text{ we have}$$

$$P_0 = 1 + x + x^2, \quad P_1 = 4x + 2, \quad P_2 = 2, \quad Q(x) = C_1$$

Since $P_2 - P_1' + P_0'' = 2 - 4 + 2 = 0$, therefore equation (1) is exact. Hence its integral is

$$P_0 y' + (P_1 - P_0')y = \int Q(x)\,dx + C_2$$

or $$(1 + x + x^2)y' + (2x + 1)y = \int C_1\,dx + C_2 = C_1 x + C_2$$

or $$y' + \frac{2x+1}{1+x+x^2}y = \frac{C_1 x + C_2}{1+x+x^2} \tag{2}$$

which is a first order linear differential equation.

Its integrating factor is

$$\text{I.F.} = e^{\int \frac{2x+1}{1+x+x^2}dx} = e^{\ln(1+x+x^2)} = 1 + x + x^2$$

The solution of equation (2) is

$$(1 + x + x^2).y = \int (1 + x + x^2)\frac{C_1 x + C_2}{1+x+x^2}\,dx + C_3$$

$$= \int (C_1 x + C_2)\,dx + C_3$$

$$= \frac{1}{2}C_1 x^2 + C_2 x + C_3$$

INTEGRATING FACTOR

Suppose that the given equation is not exact, and let the coefficients $P_0, P_1, \ldots\ldots$ are of the type $a x^q + \ldots\ldots$ Then in such cases x^m can be taken as an integrating factor. Multiply the given equation by x^m and apply the condition of exactness which will give a particular value of m. Thus the exact value of the integrating factor will be known to us. The rest of the method is the same as used above.

EXAMPLE (15): **Solve the differential equation**

$$(2x^3 + 2x^2)y'' + (7x^2 + 3x)y' - 3y = x^2$$

SOLUTION: Comparing the given equation with

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x) \tag{1}$$

we have $P_0 = 2x^3 + 2x^2, \quad P_1 = 7x^2 + 3x, \quad P_2 = -3, \quad Q(x) = x^2$

Since $P_2 - P_1' + P_0'' = -3 - 14x + 3 + 12x + 4 \neq 0$, therefore the given equation is not exact.

Multiplying the given equation by x^m, we get

$$(2x^{m+3}+2x^{m+2})y''+(7x^{m+2}+3x^{m+1})y'-3x^m y = x^{m+2} \tag{2}$$

which must be exact. Comparing it with equation (1), we find that

$$P_0 = 2x^{m+3}+2x^{m+2}, \quad P_1 = 7x^{m+2}+3x^{m+1}, \quad P_2 = -3x^m, \quad Q(x) = x^{m+2}$$

Now equation (2) is exact if and only if $P_2 - P_1' + P_0'' = 0$

or $\quad -3x^m - 7(m+2)x^{m+1} - 3(m+1)x^m$

$$+2(m+2)(m+3)x^{m+1}+2(m+1)(m+2)x^m = 0$$

or $\quad [-3-3(m+1)+2(m+1)(m+2)]x^m + [-7(m+2)+2(m+2)(m+3)]x^{m+1} = 0$

or $\quad [-3(m+2)+2(m+1)(m+2)]x^m + (m+2)[-7+2(m+3)]x^{m+1} = 0$

or $\quad (m+2)[\{-3+2(m+1)\}x^m + \{-7+2(m+3)\}x^{m+1}] = 0$

This implies that $\quad m+2 = 0 \quad$ or $\quad m = -2$

Substituting this value of m in equation (2), we get

$$(2x+2)y''+\left(7+\frac{3}{x}\right)y'-\frac{3}{x^2}y = 1 \tag{3}$$

which must be exact. Comparing equation (3) with equation (1), we have

$$P_0 = 2x+2, \quad P_1 = 7+\frac{3}{x}, \quad P_2 = -\frac{3}{x^2}, \quad Q(x) = 1$$

Since $P_2 - P_1' + P_0'' = -\frac{3}{x^2} - \frac{3}{x^2} + 0 = 0$, therefore equation (3) is exact. Its first integral is

$$P_0 y' + (P_1 - P_0')y = \int Q(x)\,dx + C_1$$

or $\quad (2x+2)y' + \left(5+\frac{3}{x}\right)y = \int 1\,dx + C_1 = x + C_1$

or $$y' + \frac{5x+3}{2x(x+1)}y = \frac{x+C_1}{2(x+1)} \tag{4}$$

which is a first order linear differential equation. Its integrating factor is

$$\text{I.F.} = e^{\int \frac{5x+3}{2x(x+1)}dx} = e^{\int\left(\frac{3}{2x}+\frac{1}{x+1}\right)dx}$$

$$= e^{\frac{3}{2}\ln x + \ln(x+1)} = e^{\ln x^{3/2}} \cdot e^{\ln(x+1)} = (x+1)x^{3/2}$$

The solution of equation (4) is

$$(x+1)x^{3/2} \cdot y = \int (x+1)x^{3/2}\frac{x+C_1}{2(x+1)}\,dx + C_2$$

$$= \frac{1}{2}\int (x^{5/2}+C_1 x^{3/2})\,dx + C_2$$

$$= \frac{1}{7}x^{7/2} + \frac{1}{5}C_1 x^{5/2} + C_2$$

or $$(x+1)y = \frac{1}{7}x^2 + \frac{1}{5}C_1 x + C_2 x^{-3/2}$$

EXACTNESS OF NON – LINEAR EQUATION

EXAMPLE (16): **Show that the following equation is exact and find its first integral :**

$$x^2\frac{d^2y}{dx^2} + 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 + 3x\frac{dy}{dx} + y = 0$$

SOLUTION: Let us write

$$\frac{du}{dx} = x^2\frac{d^2y}{dx^2} + 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 + 3x\frac{dy}{dx} + y = 0 \quad (1)$$

Take $u_1 = x^2\frac{dy}{dx}$

Then $$\frac{du_1}{dx} = x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} \quad (2)$$

Subtracting equation (2) from equation (1), we get

$$\frac{du}{dx} - \frac{du_1}{dx} = 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 + x\frac{dy}{dx} + y = 0 \quad (3)$$

Now take $u_2 = y^2\left(\frac{dy}{dx}\right)^2$

Then $$\frac{du_2}{dx} = 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 \quad (4)$$

Subtracting equation (4) from equation (3), we get

$$\frac{du}{dx} - \frac{du_1}{dx} - \frac{du_2}{dx} = x\frac{dy}{dx} + y \quad (5)$$

Next take $u_3 = xy$

Then $$\frac{du_3}{dx} = x\frac{dy}{dx} + y \quad (6)$$

From equations (5) and (6), we get

$$\frac{du}{dx} - \frac{du_1}{dx} - \frac{du_2}{dx} = \frac{du_3}{dx}$$

or $$\frac{du}{dx} = \frac{d}{dx}(u_1 + u_2 + u_3)$$

Hence equation (1) is exact and it can be written as

$$\frac{d}{dx}(u_1 + u_2 + u_3) = 0$$

Integrating, we get

$$u_1 + u_2 + u_3 = C$$

or $$x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2+xy=C$$

is the required first integral.

ALTERNATIVE METHOD (GROUPING OF TERMS)

Write the given equation as

$$\left[x^2\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)\right]+\left[2y^2\frac{dy}{dx}\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^3\right]+\left[x\frac{dy}{dx}+y\right]=0$$

or $$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)+\frac{d}{dx}\left[y^2\left(\frac{dy}{dx}\right)^2\right]+\frac{d}{dx}(xy)=0$$

Integrating, we get

$$x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2+xy=C$$

EXAMPLE (17): **Solve the differential equation :**

$$x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0$$

SOLUTION: Write the given equation as

$$x^2y\frac{d^2y}{dx^2}+x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}+y^2=0$$

or $$x^2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\left(2xy\frac{dy}{dx}-y^2\right)=0$$

or $$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-\left(\frac{2xy\frac{dy}{dx}-y^2}{x^2}\right)=0$$

or $$\frac{d}{dx}\left(y\frac{dy}{dx}\right)-\frac{d}{dx}\left(\frac{y^2}{x}\right)$$

Integrating, we get

$$y\frac{dy}{dx}-\frac{y^2}{x}=C_1 \qquad (1)$$

Let $y^2=u$, then $2y\frac{dy}{dx}=\frac{du}{dx}$. Thus equation (1) becomes

$$\frac{1}{2}\frac{du}{dx}-\frac{u}{x}=C_1$$

or $$\frac{du}{dx}-\frac{2}{x}u=2C_1 \qquad (2)$$

which is a first order linear differential equation. Its integrating factor is

$$\text{I.F.}=e^{\int-\frac{2}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=\frac{1}{x^2}$$

Thus the solution of equation (2) is

$$\frac{1}{x^2}\cdot u = \int \frac{1}{x^2}\cdot 2C_1\, dx + C_2 = -2C_1\frac{1}{x} + C_2$$

or $\quad u = -2C_1 x + C_2 x^2$

or $\quad y^2 = -2C_1 x + C_2 x^2$

5.12 FACTORIZATION OF OPERATORS

(i) CASE OF CONSTANT COEFFICIENTS

Consider first the second order homogeneous linear differential equation with constant coefficients

$$\frac{d^2 y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0$$

In operator notation, this becomes

$$(D^2 + P_1 D + P_2)y = 0$$

Suppose that the differential operator can be factored as

$$(D - m_1)(D - m_2)y = 0$$

where we understand that first y is operated on by $D - m_2$, then the result of this step is operated on by $(D - m_1)$. That is, we begin at the right and proceed to the left. The same technique can be used to solve the higher order differential equations.

Furthermore, it can be easily verified that the order of the factors is immaterial, i.e. they commute.

For example

$$\begin{aligned}(D-m_1)(D-m_2)y &= (D-m_1)(y' - m_2 y)\\ &= D(y' - m_2 y) - m_1(y' - m_2 y)\\ &= y'' - m_2 y' - m_1 y' + m_1 m_2 y\\ &= y'' - (m_1 + m_2)y' + m_1 m_2 y\end{aligned}$$

and

$$\begin{aligned}(D-m_2)(D-m_1)y &= (D-m_2)(y' - m_1 y)\\ &= D(y' - m_1 y) - m_2(y' - m_1 y)\\ &= y'' - m_1 y' - m_2 y' + m_1 m_2 y\\ &= y'' - (m_1 + m_2)y' + m_1 m_2 y\end{aligned}$$

are the same.

(ii) CASE OF VARIABLE COEFFICIENTS

Consider the second order linear differential equation with variable coefficients

$$P_0(x)\frac{d^2 y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = Q(x) \qquad (1)$$

It may be possible to separate the L.H.S. of equation (1) into two linear operators

$F_1(D)$ and $F_2(D)$ so that

$$[F_1(D).F_2(D)]y = Q(x)$$

or $$F_1(D)[F_2(D)y] = Q(x) \quad (2)$$

Setting $F_2(D)y = v$, equation (2) becomes

$$F_1(D)v = Q(x)$$

which is a first order linear differential equation .

The factors in this case are not commutative , since

$$\begin{aligned}(xD-2)(D-x)y &= (xD-2)(y'-xy)\\ &= xD(y'-xy)-2(y'-xy)\\ &= x(y''-xy'-y)-2(y'-xy)\\ &= xy''-(x^2+2)y'+xy\end{aligned}$$

and $$\begin{aligned}(D-x)(xD-2)y &= (D-x)(xy'-2y)\\ &= D(xy'-2y)-x(xy'-2y)\\ &= xy''+y'-2y'-x^2y'+2xy\\ &= xy''-(x^2+1)y'+2xy\end{aligned}$$

are not the same .

EXAMPLE (18): **Solve the differential equation :**

(i) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y = 0$ (ii) $x\dfrac{d^2y}{dx^2}-(x+2)\dfrac{dy}{dx}+2y = 0$

SOLUTION: (i) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y = 0$

Write the equation in operator notation as

$$(D^2-3D+2)y = 0$$

or $$(D-1)(D-2)y = 0$$

or $$(D-1)[(D-2)y] = 0$$

Let $$(D-2)y = u \quad (1)$$

Then $$(D-1)u = 0$$

$$\frac{du}{dx}-u = 0$$

or $$\frac{du}{u} = dx$$

Integrating $\ln u = x + \ln C_1$

or $\quad u = C_1 e^x$

Then from equation (1), we have

$$(D-2)y = C_1 e^x \tag{2}$$

or $\quad \frac{dy}{dx} - 2y = C_1 e^x$

which is a first order linear differential equation. The integrating factor is

$$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}$$

The solution of equation (2) is

$$e^{-2x} y = \int e^{-2x} . C_1 e^x \, dx + C_2 = -C_1 e^{-x} + C_2$$

or $\quad y = -C_1 e^x + C_2 e^{-2x}$

(ii) $\quad x\frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + 2y = 0$

Write the equation in operator notation as

$$\left[xD^2 - (x+2)D + 2\right]y = 0$$

or $\quad \left[xD^2 - xD - 2D + 2\right]y = 0$

or $\quad \left[xD(D-1) - 2(D-1)\right]y = 0$

or $\quad \left[(xD-2)(D-1)\right]y = 0$

or $\quad (xD-2)\left[(D-1)y\right] = 0 \qquad (1)$

Let $(D-1)y = v$, then equation (1) becomes

$$(xD-2)v = 0$$

or $\quad x\frac{dv}{dx} - 2v = 0$

Separating the variables, we get

$$\frac{dv}{v} = 2\frac{dv}{x}$$

or $\quad \ln v = 2\ln x + \ln C_1$

or $\quad \ln v = \ln C_1 x^2$

or $\quad v = C_1 x^2$

or $\quad (D-1)y = C_1 x^2$

or $\quad \frac{dy}{dx} - y = C_1 x^2 \qquad (2)$

which is a first order linear differential equation. To solve equation (2), we have

$$\text{I.F.} = e^{-\int dx} = e^{-x}$$

Thus the solution of equation (2) is

$$e^{-x}.y = \int e^{-x} C_1 x^2\, dx + C_2$$

$$= C_1(-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}) + C_2$$

or $\quad y = -C_1(x^2 + 2x + 2) + C_2 e^x$

METHOD TO FACTORIZE THE OPERATOR

Suppose that the factors of the operator on the L.H.S. of equation (1) page (230) are $pD + q$ and $rD + s$, where p, q, r, and s are function of x to be determined.

i.e. $\quad (P_0 D^2 + P_1 D + P_2)y = (pD+q)(rD+s)y$

$$= (pD+q)\left(r\frac{dy}{dx} + sy\right)$$

$$= pD\left(r\frac{dy}{dx}\right) + pD(sy) + qr\frac{dy}{dx} + qsy$$

$$= pr\frac{d^2y}{dx^2} + p\frac{dr}{dx}\frac{dy}{dx} + ps\frac{dy}{dx} + py\frac{ds}{dx} + qr\frac{dy}{dx} + qsy$$

$$= pr\frac{d^2y}{dx^2} + \left(p\frac{dr}{dx} + ps + qr\right)\frac{dy}{dx} + \left(p\frac{ds}{dx} + qs\right)y$$

or $\quad (P_0 D^2 + P_1 D + P_2)y = \left[prD^2 + \left(p\frac{dr}{dx} + ps + qr\right)D + \left(p\frac{ds}{dx} + qs\right)\right]y$

Comparing the coefficients on both sides, we get

$$pr = P_0 \qquad (1)$$

$$p\frac{dr}{dx} + ps + qr = P_1 \qquad (2)$$

and $\quad p\frac{ds}{dx} + qs = P_2 \qquad (3)$

Here p and r are two factors of P_0 which we may take in any order and determine q and s from the remaining two equations.

EXAMPLE (19): Factorize the operator of the differential equation :

$$\left[xD^2 + (x^2+1)D + 2x\right]y = 2x$$

SOLUTION: Let the factors of the operator on the L.H.S. be $(pD+q)$ and $(rD+s)$ where p, q, r, and s are functions of x to be determined.

i.e. $\quad \left[xD^2 + (x^2+1)D + 2x\right]y = (pD+q)(rD+s)y \qquad (1)$

In the present case, $P_0 = x$, $P_1 = x^2+1$, $P_2 = 2x$, $Q = 2x$.
Equations (1), (2), and (3) above take the form

$$pr = x \tag{2}$$

$$p\frac{dr}{dx}+ps+qr = x^2+1 \tag{3}$$

$$p\frac{ds}{dx}+qs = 2x \tag{4}$$

From equation (2) we may take $p = x$ and $r = 1$. Now let $q = A_1x+B_1$ and $s = A_2x+B_2$.
From equations (3) and (4), we get

$$x(0)+x(A_2x+B_2)+(A_1x+B_1)(1) = x^2+1$$

$$A_2x+(A_1x+B_1)(A_2x+B_2) = 2x$$

or $$A_2x^2+(A_1+B_2)x+B_1 = x^2+1 \tag{5}$$

and $$A_1A_2x^2+(A_2+A_1B_2+B_1A_2)x+B_1B_2 = 2x \tag{6}$$

Comparing coefficients of like powers in equations (5) and (6), we get

$$A_2 = 1,\quad A_1+B_2 = 0,\quad B_1 = 1$$

$$A_1A_2 = 0,\quad A_2+A_1B_2+B_1A_2 = 2,\quad B_1B_2 = 0$$

From these equations, we get

$$A_1 = 0,\quad B_1 = 1,\quad A_2 = 1,\quad B_2 = 0$$

Thus $q = 1$, $s = x$

and from equation (1), we get

$$\left[xD^2+(x^2+1)D+2x\right]y = (xD+1)(D+x)y$$

which are the required factors of the operator.

5.13 EQUATIONS OF THE FORM $\frac{d^2y}{dx^2} = f(x)$

Consider the equation of the form

$$\frac{d^2y}{dx^2} = f(x) \tag{1}$$

Integrating both sides w.r.t., we get

$$\frac{dy}{dx} = \int f(x)\,dx+C_1 = F(x)+C_1$$

Integrating again, we have

$$y = \int F(x)\,dx+C_2$$

as the required solution.

In general, the solution of a differential equation of the form $\frac{d^ny}{dx^n} = f(x)$ is obtained by integrating it n times successively.

EXAMPLE (20): **Solve the differential equation :**

(i) $$\frac{d^2 y}{d x^2} = x e^x$$ (ii) $$x^2 \frac{d^3 y}{d x^3} + 1 = 0$$

SOLUTION: (i) $$\frac{d^2 y}{d x^2} = x e^x$$

Integrating the given equation , we get

$$\frac{d y}{d x} = \int x e^x \, d x + C_1$$

$$= x e^x - e^x + C_1 = (x-1) e^x + C_1$$

Integrating again , we get

$$y = \int (x-1) e^x \, d x + C_1 x + C_2$$

$$= (x-1) e^x - e^x + C_1 x + C_2$$

$$= (x-2) e^x + C_1 x + C_2$$ as the required solution .

(ii) $$x^2 \frac{d^3 y}{d x^3} + 1 = 0$$

Write the equation as

$$\frac{d^3 y}{d x^3} = -\frac{1}{x^2}$$

Integrating this equation w.r.t. x , we get

$$\frac{d^2 y}{d x^2} = \frac{1}{x} + C_1$$

Integrating again , we get

$$\frac{d y}{d x} = \ln x + C_1 x + C_2$$

Integrating again , we get

$$y = x \ln x - x + \frac{1}{2} C_1 x^2 + C_2 x + C_3$$

$$= x \ln x + \frac{1}{2} C_1 x^2 + (C_2 - 1) x + C_3$$

5.14 EQUATIONS OF THE FORM $\frac{d^2 y}{d x^2} = f(y)$

Consider the equation of the form

$$\frac{d^2 y}{d x^2} = f(y) \qquad (1)$$

Multiplying both sides of equation (1) by $2\frac{d y}{d x}$, we get

$$2\frac{d y}{d x}\frac{d^2 y}{d x^2} = 2 f(y) \frac{d y}{d x}$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2f(y)\frac{dy}{dx}$$

Integrating w.r.t. x , we get

$$\left(\frac{dy}{dx}\right)^2 = \int 2f(y)\frac{dy}{dx}\,dx + C_1$$

$$= 2\int f(y)\,dy + C_1$$

or $$\frac{dy}{dx} = \sqrt{2\int f(y)\,dy + C_1}$$

Separating the variables

$$\frac{dy}{\sqrt{2\int f(y)\,dy + C_1}} = dx$$

Integrating , we get

$$\int\frac{dy}{\sqrt{2\int f(y)\,dy + C_1}} = x + C_2$$

which is the required solution as it contains two arbitrary constants C_1 and C_2.

EXAMPLE (21): **Solve the differential equation** $y^3\frac{d^2y}{dx^2} = a$, **where a is any constant.**

SOLUTION: Write the equation as

$$\frac{d^2y}{dx^2} = \frac{a}{y^3}$$

Multiplying both sides by $2\frac{dy}{dx}$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = \frac{2a}{y^3}\frac{dy}{dx}$$

Integrating , we get

$$\left(\frac{dy}{dx}\right)^2 = \int\frac{2a}{y^3}\frac{dy}{dx}\,dx + C_1 = 2a\int\frac{dy}{y^3} + C_1$$

$$= -\frac{a}{y^2} + C_1 = \frac{C_1y^2 - a}{y^2}$$

or $$\frac{dy}{dx} = \frac{\sqrt{C_1y^2 - a}}{y}$$

Separating the variables , we get

$$\frac{y}{\sqrt{C_1y^2 - a}}\,dy = dx$$

Integrating , we get

$$\sqrt{C_1 y^2 - a} = x + C_2$$

or $$C_1 y^2 - a = (x + C_2)^2$$

EXAMPLE (22): **Solve the following initial – value problem :**

$$\frac{d^2 y}{d x^2} = \sec^2 y \tan y \quad \text{given that when} \quad x = 0, \; y = 0 \; \text{ and } \; \frac{dy}{dx} = 1 .$$

SOLUTION: Multiplying the given equation by $2\frac{dy}{dx}$, we get

$$2\frac{dy}{dx}\frac{d^2 y}{dx^2} = 2\sec^2 y \tan y \frac{dy}{dx}$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\sec^2 y \tan y \frac{dy}{dx}$$

Integrating , we get

$$\left(\frac{dy}{dx}\right)^2 = \int 2\sec^2 y \tan y \frac{dy}{dx}\, dx$$

$$= \int 2\sec^2 y \tan y \; dy = \tan^2 y + C_1$$

$$\frac{dy}{dx} = \sqrt{\tan^2 y + C_1} \qquad (1)$$

Using the condition that when $y = 0$, $\frac{dy}{dx} = 1$, we get from equation (1) $C_1 = 1$

Thus equation (1) becomes

$$\frac{dy}{dx} = \sqrt{1 + \tan^2 y} = \sec y$$

Separating the variables

$$\cos y \; dy = dx$$

or $$\sin y = x + C_2 \qquad (2)$$

Using the condition that when $x = 0$, $y = 0$, we get from equation (2) that $C_2 = 0$

Thus equation (2) becomes

$$\sin y = x \quad \text{or} \quad y = \sin^{-1} x$$

is the solution of the given problem .

5.15 RICCATI EQUATION

Consider the Riccati equation

$$\frac{dy}{dx} = P(x) y^2 + Q(x) y + R(x), \quad P(x) \neq 0 \qquad (1)$$

Let the substitution be

$$y = -\frac{1}{Pu}\frac{du}{dx}, \quad \text{where } u = u(x).$$

Then $\dfrac{dy}{dx} = -\dfrac{1}{Pu}\dfrac{d^2u}{dx^2} + \dfrac{1}{Pu^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{1}{P^2u}\dfrac{dP}{dx}\dfrac{du}{dx}$

Equation (1) becomes

$$-\frac{1}{Pu}\frac{d^2u}{dx^2} + \frac{1}{Pu^2}\left(\frac{du}{dx}\right)^2 + \frac{1}{P^2u}\frac{dP}{dx}\frac{du}{dx} = \frac{1}{Pu^2}\left(\frac{du}{dx}\right)^2 - \frac{Q}{Pu}\frac{du}{dx} + R$$

or $$-\frac{1}{Pu}\frac{d^2u}{dx^2} + \frac{1}{Pu}\left(Q + \frac{1}{P}\frac{dP}{dx}\right)\frac{du}{dx} = R$$

or $$\frac{d^2u}{dx^2} - \left(Q + \frac{1}{P}\frac{dP}{dx}\right)\frac{du}{dx} + PuR = 0 \qquad (2)$$

Equation (2) is a second order linear differential equation with variable coefficients and can be solved by the methods already known .

EXAMPLE (23): **Solve the differential equation :**

$$\frac{dy}{dx} = -2y^2 - 5y - 2$$

SOLUTION: Comparing the given equation with

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

we find that $P = -2,\ Q = -5,\ R = -2$

The substitution becomes

$$y = -\frac{1}{Pu}\frac{du}{dx} = \frac{1}{2u}\frac{du}{dx} \qquad (1)$$

and the reduced equation

$$\frac{d^2u}{dx^2} - \left(Q + \frac{1}{P}\frac{dP}{dx}\right)\frac{du}{dx} + PuR = 0 \quad \text{becomes}$$

$$\frac{d^2u}{dx^2} + 5\frac{du}{dx} + 4u = 0 \qquad (2)$$

which is a second order homogeneous linear equation with constant coefficients . To solve equation (2), we have

$$(D^2 + 5D + 4)u = 0$$

The auxiliary equation is $D^2 + 5D + 4 = 0$ having characteristic roots as $D = -1, -4$.

Thus the solution of equation (2) is

$$u = C_1 e^{-x} + C_2 e^{-4x}$$

From equation (1), the solution is

$$y = \frac{1}{2u}\frac{du}{dx} = \frac{-C_1 e^{-x} - 4C_2 e^{-4x}}{2(C_1 e^{-x} + C_2 e^{-4x})}$$

$$= -\frac{1}{2}\frac{C_1 + 4C_2 e^{-3x}}{C_1 + C_2 e^{-3x}} = -\frac{1}{2}\frac{\frac{C_1}{C_2} + 4e^{-3x}}{\frac{C_1}{C_2} + e^{-3x}}$$

$$= -\frac{1}{2}\left(\frac{C + 4e^{-3x}}{C + e^{-3x}}\right) \qquad \left(\text{where } C = \frac{C_1}{C_2}\right)$$

or $\quad 2y(C + e^{-3x}) = -(C + 4e^{-3x})$

or $\quad 2y(1 + Ce^{3x}) = -(4 + Ce^{3x})$

5.16 SOLVED PROBLEMS

DEPENDENT VARIABLE ABSENT

PROBLEM (1): **Solve the following differential equations :**

(i) $x y'' - y' = -\frac{2}{x} - \ln x$ (ii) $x y'' - y' + x(y')^2 = 0$

SOLUTION: (i) $x y'' - y' = -\frac{2}{x} - \ln x$

Since the dependent variable y does not appear explicitly, so let $y' = p$ and $y'' = \frac{dp}{dx}$ in the given differential equation, we get

$$x\frac{dp}{dx} - p = -\frac{2}{x} - \ln x$$

or
$$\frac{dp}{dx} - \frac{1}{x}p = -\frac{2}{x^2} - \frac{\ln x}{x} \qquad (1)$$

which is the first order differential equation with $P = -\frac{1}{x}$ and $Q = -\frac{2}{x^2} - \frac{\ln x}{x}$.

The integrating factor is

$$\text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

Thus the solution of differential equation (1) is

$$(\text{I.F.})\,p = \int (\text{I.F.})\,Q\,dx + C_1$$

$$p\left(\frac{1}{x}\right) = \int \frac{1}{x}\left(-\frac{2}{x^2} - \frac{\ln x}{x}\right)dx + C_1$$

$$= \int\left(-\frac{2}{x^3} - \frac{\ln x}{x^2}\right)dx + C_1 = \frac{1}{x^2} + \frac{\ln x}{x} + \frac{1}{x} + C_1$$

or
$$p = \frac{1}{x} + \ln x + 1 + C_1 x$$

or
$$\frac{dy}{dx} = \frac{1}{x} + \ln x + 1 + C_1 x$$

Integrating w.r.t. x, we get

$$y = \ln x + x\ln x - x + x + \frac{1}{2}C_1 x^2 + C_2$$

$$= (x+1)\ln x + \frac{1}{2}C_1 x^2 + C_2$$

(ii) $x y'' - y' + x(y')^2 = 0$

Since the dependent variable y does not appear explicitly, so let $y' = p$ and $y'' = \frac{dp}{dx}$ in the given differential equation, we get

$$x\frac{dp}{dx}-p+xp^2=0$$

or $$\frac{dp}{dx}-\frac{1}{x}p=-p^2$$

which is a Bernoulli's equation. To solve this equation we divide by p^2 to get

$$\frac{1}{p^2}\frac{dp}{dx}-\frac{1}{xp}=-1 \qquad (1)$$

Let $-\dfrac{1}{p}=z$, then $\dfrac{1}{p^2}\dfrac{dp}{dx}=\dfrac{dz}{dx}$

Thus equation (1) becomes

$$\frac{dz}{dx}+\frac{1}{x}z=-1 \qquad (2)$$

which is a first order linear differential equation. To solve equation (2), we have the integrating factor

$$\text{I.F.}=e^{\int P\,dx}=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

Thus the solution of differential equation (2) is

$$(\text{I.F.})\,z=\int(\text{I.F.})\,Q\,dx+C_1$$

$$x.z=\int x(-1)\,dx+C=-\frac{x^2}{2}+C$$

or $$z=-\frac{x}{2}+\frac{C}{x}$$

But $z=-\dfrac{1}{p}$, therefore $-\dfrac{1}{p}=-\dfrac{x}{2}+\dfrac{C}{x}$

or $$\frac{1}{p}=\frac{x}{2}-\frac{C}{x}$$

or $$p=\frac{2x}{x^2-2C}=\frac{2x}{x^2-C_1} \quad \text{where} \quad C_1=2C$$

or $$\frac{dy}{dx}=\frac{2x}{x^2-C_1}$$

$y=\ln(x^2-C_1)+C_2$ is the required solution.

PROBLEM (2): **Solve the following initial – value problems :**

$$xy''+(y')^2=1;\quad y(2)=-\pi,\quad y'(2)=0$$

SOLUTION: Since the dependent variable y does not appear explicitly, therefore let $y'=p$, $y''=\dfrac{dp}{dx}$ in the give differential equation, we get

$$x\frac{dp}{dx}+p^2=1$$

or $$x\frac{dp}{dx}=1-p^2$$

Separating the variables, we have

$$\frac{dp}{1-p^2} = \frac{dx}{x}$$

or $$\left[\frac{1}{2(1-p)} + \frac{1}{2(1+p)}\right]dp = \frac{dx}{x}$$

Integrating both sides, we get

$$-\frac{1}{2}\ln(1-p) + \frac{1}{2}\ln(1+p) = \ln x + \ln C_1$$

or $$\frac{1}{2}\ln\frac{1+p}{1-p} = \ln C_1 x$$

or $$\ln\left(\frac{1+p}{1-p}\right)^{1/2} = \ln C_1 x$$

or $$\left(\frac{1+p}{1-p}\right)^{1/2} = C_1 x$$

or $$\frac{1+p}{1-p} = C_1^2 x^2$$

or $$1+p = C_1^2 x^2 - p C_1^2 x^2$$

or $$p(1 + C_1^2 x^2) = C_1^2 x^2 - 1$$

or $$p = \frac{C_1^2 x^2 - 1}{1 + C_1^2 x^2}$$

or $$\frac{dy}{dx} = 1 - \frac{2}{1 + C_1^2 x^2} \qquad (1)$$

Integrating again, we get

$$y = x - \frac{2}{C_1}\tan^{-1}(C_1 x) + C_2 \qquad (2)$$

Using the initial condition $y'(2) = 0$, we get from equation (1)

$$0 = 1 - \frac{2}{1 + 4C_1^2}$$

or $$2 = 1 + 4C_1^2 \quad \text{or} \quad C_1 = \frac{1}{2}$$

Thus from equation (2), we get

$$y = x - 4\tan^{-1}\left(\frac{x}{2}\right) + C_2 \qquad (3)$$

Using the initial condition $y(2) = -\pi$, we get from equation (3)

$$-\pi = 2 - 4\tan^{-1}1 + C_2$$

or $$-\pi = 2 - 4\left(\frac{\pi}{4}\right) + C_2 \quad \text{or} \quad C_2 = -2$$

Finally, from equation (3) the solution of the given initial – value problem is

$$y = x - 4\tan^{-1}\left(\frac{x}{2}\right) - 2$$

INDEPENDENT VARIABLE ABSENT

PROBLEM (3): **Solve the following differential equations :**

(i) $y y'' = 2(y')^2 - 2y'$ (ii) $y y'' + (y')^2 = 2$

SOLUTION: (i) $y y'' = 2(y')^2 - 2y'$

Since the independent variable x does not appear explicitly, so let $y' = p$ and $y'' = p\dfrac{dp}{dy}$ in the given differential equation, we get

$$y p \frac{dp}{dy} = 2p^2 - 2p$$

or $$p\left(y\frac{dp}{dy} - 2p + 2\right) = 0 \qquad (1)$$

This implies either $p = 0$ or $y\dfrac{dp}{dy} - 2p + 2 = 0$.

If $p = 0$ then $\dfrac{dy}{dx} = 0$ or $y = C_3$ is a solution.

If $y\dfrac{dp}{dy} - 2p + 2 = 0$, then $\dfrac{dp}{p-1} = 2\dfrac{dy}{y}$

or $$\ln(p-1) = 2\ln y + \ln C_1^2 = \ln C_1^2 y^2$$

or $$p - 1 = C_1^2 y^2$$

or $$\frac{dy}{dx} = 1 + C_1^2 y^2$$

or $$\frac{dy}{1 + C_1^2 y^2} = dx$$

$$\frac{1}{C_1}\tan^{-1} C_1 y = x + K$$

or $$\tan^{-1} C_1 y = C_1 x + C_2 \quad (C_2 = C_1 K)$$

or $$C_1 y = \tan(C_1 x + C_2)$$

(ii) $y y'' + (y')^2 = 2$

Since the independent variable x is absent, so let $y' = p$ and $y'' = p\dfrac{dp}{dy}$ in the given differential equation, we get

$$y p \frac{dp}{dy} + p^2 = 2$$

or $$y p \frac{dp}{dy} = 2 - p^2$$

Separating the variables, we get

$$\frac{p}{2-p^2}\,dp = \frac{dy}{y}$$

Integrating both sides, we get

$$-\frac{1}{2}\ln(2-p^2) = \ln y + \ln K_1$$

or $\quad \ln(2-p^2) = -2\ln K_1 y = \ln(K_1 y)^{-2}$

or $\quad 2-p^2 = \dfrac{1}{K_1^2 y^2}$

or $\quad p^2 = 2 - \dfrac{1}{K_1^2 y^2} = \dfrac{2K_1^2 y^2 - 1}{K_1^2 y^2}$

or $\quad p = \pm\dfrac{\sqrt{2K_1^2 y^2 - 1}}{K_1 y}$

or $\quad \dfrac{dy}{dx} = \pm\dfrac{\sqrt{2K_1^2 y^2 - 1}}{K_1 y}$

or $\quad \dfrac{K_1 y\,dy}{\sqrt{2K_1^2 y^2 - 1}} = \pm dx$

or $\quad \dfrac{1}{4K_1}\dfrac{4K_1^2 y}{\sqrt{2K_1^2 y^2 - 1}}\,dy = \pm dx$

or $\quad \dfrac{1}{2K_1}\sqrt{2K_1^2 y^2 - 1} = \pm(x + K^2)$

or $\quad \dfrac{1}{4K_1^2}(2K_1^2 y^2 - 1) = (x + K_2)^2$

or $\quad \dfrac{1}{2}y^2 - \dfrac{1}{4K_1^2} = x^2 + 2K_2 x + K_2^2$

or $\quad \dfrac{1}{2}y^2 = x^2 + 2K_2 x + \left(K_2^2 + \dfrac{1}{4K_1^2}\right)$

or $\quad y^2 = 2x^2 + 4K_2 x + \left(2K_2^2 + \dfrac{1}{2K_1^2}\right)$

$\quad = 2x^2 + C_1 x + C_2 \quad$ where $C_1 = 4K_2$ and $C_2 = 2K_2^2 + \dfrac{1}{2K_1^2}$

PROBLEM (4): **Solve the following boundary – value problem :**

$$y'' + y = 0; \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 1$$

SOLUTION: Since the independent variable x does not appear explicitly, therefore let $y' = p$, $y'' = p\dfrac{dp}{dx}$ in the give differential equation, we get

$$p\frac{dp}{dy}+y=0$$

Separating the variables, we get

$$p\,dp=-y\,dy$$

Integrating both sides, we get

$$\frac{p^2}{2}=-\frac{y^2}{2}+K_1$$

or $\quad p^2=-y^2+2K_1 \quad$ or $\quad p=\sqrt{2K_1-y^2}$

or $\quad \dfrac{dy}{dx}=\sqrt{2K_1-y^2}$

Again separating the variables, we get

$$\frac{dy}{\sqrt{2K_1-y^2}}=dx$$

Integrating again, we get

$$\sin^{-1}\frac{y}{\sqrt{2K_1}}=x+K_2$$

or
$$\begin{aligned} y &= \sqrt{2K_1}\sin(x+K_2)\\ &= \sqrt{2K_1}(\sin x\cos K_2+\cos x\sin K_2)\\ &= \sqrt{2K_1}\cos K_2\sin x+\sqrt{2K_1}\sin K_2\cos x\\ &= C_1\cos x+C_2\sin x \qquad (1)\end{aligned}$$

where $C_1=\sqrt{2K_1}\sin K_2$ and $C_2=\sqrt{2K_1}\cos K_2$.

Using the boundary condition $y(0)=1$, we get from equation (1) $\quad C_1=1$

Using the boundary condition $y\left(\dfrac{\pi}{2}\right)=1$, we get from equation (1) $\quad C_2=1$.

With these values of C_1 and C_2, the solution of the given initial – value problem is

$$y=\cos x+\sin x$$

HIGHER ORDER DIFFERENTIAL EQUATIONS

PROBLEM (5): Solve the following differential equations :

(i) $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}=x^2$ $\qquad$ (ii) $2x\dfrac{d^2y}{dx^2}\dfrac{d^3y}{dx^3}=\left(\dfrac{d^2y}{dx^2}\right)^2-1$

(iii) $\dfrac{d^4y}{dx^4}\dfrac{d^3y}{dx^3}=1$ $\qquad$ (iv) $\left(\dfrac{d^3y}{dx^3}\right)^2+x\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}=0$

SOLUTION: (i) $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}=x^2$

Since the dependent variable y does not appear explicitly, therefore let $\dfrac{d^2y}{dx^2}=q$, then $\dfrac{d^3y}{dx^3}=\dfrac{dq}{dx}$.

The given differential equation becomes

$$\frac{dq}{dx} + q = x^2 \tag{1}$$

which is a first order linear differential equation. To solve equation (1), the integrating factor is

$$\text{I.F.} = e^{\int dx} = e^x$$

The solution of equation (1) is

$$e^x . q = \int e^x . x^2 \, dx + C_1 = x^2 e^x - 2x e^x + 2e^x + C_1$$

or $\quad q = x^2 - 2x + 2 + C_1 e^{-x}$

or $\quad \dfrac{d^2y}{dx^2} = x^2 - 2x + 2 + C_1 e^{-x}$

or $\quad \dfrac{dy}{dx} = \dfrac{x^3}{3} - x^2 + 2x - C_1 e^{-x} + C_2$

or $\quad y = \dfrac{1}{12} x^4 - \dfrac{1}{3} x^3 + x^2 + C_1 e^{-x} + C_2 x + C_3$

$$= C_1 e^{-x} + C_2 x + C_3 + \frac{1}{12} x^2 (x^2 - 4x + 12)$$

(ii) $\quad 2x \dfrac{d^2y}{dx^2} \dfrac{d^3y}{dx^3} = \left(\dfrac{d^2y}{dx^2}\right)^2 - 1$

Since the dependent variable y is absent, therefore

let $\dfrac{d^2y}{dx^2} = q$, then $\dfrac{d^3y}{dx^3} = \dfrac{dq}{dx}$. The given equation becomes

$$2xq \frac{dq}{dx} = q^2 - 1$$

Separating the variables, we get

$$\frac{2q}{q^2 - 1} dq = \frac{dx}{x}$$

Integrating

$$\ln(q^2 - 1) = \ln x + \ln C$$

or $\quad \ln(q^2 - 1) = \ln C_1 x$

or $\quad q^2 - 1 = C_1 x$

or $\quad \left(\dfrac{d^2y}{dx^2}\right)^2 = C_1 x + 1$

or $\quad \dfrac{d^2y}{dx^2} = \sqrt{C_1 x + 1}$

Integrating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{3C_1} (C_1 x + 1)^{3/2} + C_2$$

and $$y = \left(\frac{2}{3C_1}\right)\left(\frac{2}{5C_1}\right)(C_1 x+1)^{5/2} + C_2 x + C_3$$

$$= \frac{4}{15C_1^2}(C_1 x+1)^{5/2} + C_2 x + C_3$$

(iii) $$\frac{d^4 y}{dx^4}\frac{d^3 y}{dx^3} = 1$$

Let $\frac{d^3 y}{dx^3} = q$, then $\frac{d^4 y}{dx^4} = \frac{dq}{dx}$ and the given equation becomes

$$q\frac{dq}{dx} = 1$$

or $$q\,dq = dx \quad \text{or} \quad \frac{q^2}{2} = x + K$$

or $$q^2 = 2x + C_1 \quad (C_1 = 2K)$$

or $$q = \pm\sqrt{2x + C_1}$$

or $$\frac{d^3 y}{dx^3} = \pm\sqrt{2x + C_1}$$

or $$\frac{d^2 y}{dx^2} = \pm\frac{1}{3}(2x + C_1)^{3/2} + C_2$$

or $$\frac{dy}{dx} = \pm\frac{1}{15}(2x + C_1)^{5/2} + C_2 x + C_3$$

or $$y = \pm\frac{1}{105}(2x + C_1)^{7/2} + C_2\frac{x^2}{2} + C_3 x + C_4$$

(iv) $$\left(\frac{d^3 y}{dx^3}\right)^2 + x\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} = 0$$

Let $\frac{d^2 y}{dx^2} = q$, then $\frac{d^3 y}{dx^3} = \frac{dq}{dx}$ and the given equation becomes

$$\left(\frac{dq}{dx}\right)^2 + x\frac{dq}{dx} - q = 0$$

or $$q = x\frac{dq}{dx} + \left(\frac{dq}{dx}\right)^2 \tag{1}$$

which is a Clairaut equation. The solution of equation (1) is given by

$$q = xC_1 + C_1^2$$

or $$\frac{d^2 y}{dx^2} = xC_1 + C_1^2$$

Integrating twice, we get

$$\frac{dy}{dx} = \frac{1}{2}C_1 x^2 + C_1^2 x + C_2$$

and $$y = \frac{1}{6}C_1 x^3 + \frac{1}{2}C_1^2 x^2 + C_2 x + C_3$$

PROBLEM (6): **Solve the following initial – value problems :**

(i) $$x\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}=0;\quad y(-1)=2,\quad y'(-1)=-3,\quad y''(-1)=10$$

(ii) $$\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}=0;\quad y(1)=1,\quad y'(1)=0,\quad y''(1)=-1$$

SOLUTION: (i) $$x\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}=0;\quad y(-1)=2,\quad y'(-1)=-3,\quad y''(-1)=10$$

Since the dependent variable y does not appear explicitly, so let

$$\frac{d^2y}{dx^2}=q,\ \text{then}\ \frac{d^3y}{dx^3}=\frac{dq}{dx}$$

Thus the given equation becomes

$$x\frac{dq}{dx}-4q=0$$

or $$\frac{dq}{q}=4\frac{dx}{x}$$

or $$\ln q=4\ln x+\ln C_1$$

or $$q=C_1x^4$$

or $$\frac{d^2y}{dx^2}=C_1x^4 \tag{1}$$

Using the initial condition $y''(-1)=10$, we get from equation (1) that

$$C_1(-1)^4=10\quad\text{or}\quad C_1=10$$

Thus equation (1) becomes

$$\frac{d^2y}{dx^2}=10x^4$$

Integrating, we get

$$\frac{dy}{dx}=2x^5+C_2 \tag{2}$$

Using the initial condition $y'(-1)=-3$, we get from equation (2) that

$$-3=2(-1)^5+C_2\quad\text{or}\quad C_2=-1$$

Equation (2) becomes

$$\frac{dy}{dx}=2x^5-1$$

Integrating again, we get

$$y=\frac{1}{3}x^6-x+C_3 \tag{3}$$

Using the initial condition $y(-1)=2$, we get from equation (3) that

$$2=\frac{1}{3}(-1)^6-(-1)+C_3$$

or $\quad 2 = \frac{1}{3} + 1 + C_3 \quad$ or $\quad C_3 = \frac{2}{3}$

Thus the final solution is

$$y = \frac{1}{3}x^6 - x + \frac{2}{3}$$

(ii) $\quad \frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} = 0 ; \quad y(1) = 1, \quad y'(1) = 0, \quad y''(1) = -1$

Since the dependent variable y does not appear explicitly , so let

$$\frac{d^2y}{dx^2} = q, \text{ then } \frac{d^3y}{dx^3} = \frac{dq}{dx}$$

Thus the given equation becomes

$$\frac{dq}{dx} - 4q = 0$$

or $\quad \frac{dq}{q} = 4\,dx$

or $\quad \ln q = 4x + \ln C_1$

or $\quad q = C_1 e^{4x}$

or $\quad \frac{d^2y}{dx^2} = C_1 e^{4x} \qquad (1)$

Using the initial condition $y''(1) = -1$, we get from equation (1) that

$$C_1 e^4 = -1 \quad \text{or} \quad C_1 = -e^{-4}$$

Thus equation (1) becomes

$$\frac{d^2y}{dx^2} = -e^{-4}e^{4x} = -e^{4(x-1)}$$

Integrating , we get

$$\frac{dy}{dx} = -\frac{1}{4}e^{4(x-1)} + C_2 \qquad (2)$$

Using the initial condition $y'(1) = 0$, we get from equation (2) that $0 = -\frac{1}{4}e^0 + C_2$ or $C_2 = \frac{1}{4}$

Thus equation (2) becomes

$$\frac{dy}{dx} = -\frac{1}{4}e^{4(x-1)} + \frac{1}{4}$$

Integrating again , we get

$$y = -\frac{1}{16}e^{4(x-1)} + \frac{1}{4}x + C_3 \qquad (3)$$

Using the initial condition $y(1) = 1$, we get from equation (3) that

$$1 = -\frac{1}{16} + \frac{1}{4} + C_3$$

or $$C_3 = 1 + \frac{1}{16} - \frac{1}{4} = \frac{13}{16}$$

Thus the final solution is

$$y = -\frac{1}{16} e^{4(x-1)} + \frac{1}{4} x + \frac{13}{16}$$

LINEAR EQUATIONS WITH ONE KNOWN INTEGRAL OF COMPLEMENTARY FUNCTION

PROBLEM (7): **Solve the following differential equation :**

$$\sin^2 x \frac{d^2 y}{d x^2} = 2 y$$

given that $y = \cot x$ is an integral of the complementary function .

SOLUTION: Let $y = u \cot x$, then

$$\frac{dy}{dx} = -u \operatorname{cosec}^2 x + \cot x \frac{du}{dx}$$

and $$\frac{d^2 y}{d x^2} = -2 u \operatorname{cosec} x (- \operatorname{cosec} x \cot x) - \operatorname{cosec}^2 x \frac{du}{dx} + \cot x \frac{d^2 u}{d x^2} - \operatorname{cosec}^2 x \frac{du}{dx}$$

$$= \cot x \frac{d^2 u}{d x^2} - 2 \operatorname{cosec}^2 x \frac{du}{dx} + 2 u \operatorname{cosec}^2 x \cot x$$

Substituting the values of these derivatives in the given differential equation , we get

$$\sin^2 x \left(\cot x \frac{d^2 u}{d x^2} - 2 \operatorname{cosec}^2 x \frac{du}{dx} + 2 u \operatorname{cosec}^2 x \cot x \right) = 2 u \cot x$$

or $$\sin x \cos x \frac{d^2 u}{d x^2} - 2 \frac{du}{dx} + 2 u \cot x = 2 u \cot x$$

or $$\frac{d^2 u}{d x^2} - \frac{2}{\sin x \cos x} \frac{du}{dx} = 0$$

which is a second order differential equation with the dependent variable u absent .

Letting $\frac{du}{dx} = z$, the above equation becomes

$$\frac{dz}{dx} - \frac{2}{\sin \cos x} z = 0$$

Separating the variables , we get

$$\frac{dz}{z} = \frac{2}{\sin x \cos x} dx = \frac{2 \sec^2 x}{\tan x} dx$$

Integrating , we get

$$\ln z = 2 \ln \tan x + \ln C_1$$

or $$z = C_1 \tan^2 x$$

or $$\frac{du}{dx} = C_1 \tan^2 x = C_1 (\sec^2 x - 1)$$

or $\quad u = C_1(\tan x - x) + C_2$

or $\quad \dfrac{y}{\cot x} = C_1(\tan x - x) + C_2$

or $\quad y = C_1(1 - x\cot x) + C_2 \cot x$

METHODS OF FINDING PARTICULAR INTEGRALS

PROBLEM (8): **Solve the following differential equations :**

(i) $\quad (2x^2+1)y'' - 4xy' + 4y = 0$

(ii) $\quad \dfrac{d^2y}{dx^2} - \dfrac{3}{x}\dfrac{dy}{dx} + \dfrac{3}{x^2}y = 2x - 1$

SOLUTION: (i) $(2x^2+1)y'' - 4xy' + 4y = 0$

Write the equation as $y'' - \dfrac{4x}{2x^2+1}y' + \dfrac{4}{2x^2+1}y = 0$ (1)

Comparing this equation with the general form $y'' + P_1 y' + P_2 y = 0$, we find that

$$P_1 = -\frac{4x}{2x^2+1},\quad P_2 = \frac{4}{2x^2+1},\quad Q = 0$$

Here $P_1 + P_2 x = 0$, therefore $y_1 = x$ is one particular integral of the complementary function.

Let $y = ux$ be the general solution of the differential equation.

From equation (6) on page (208), we have

$$u = C_1 \int\left(\frac{1}{y_1^2}e^{-\int P_1 dx}\right)dx + C_2$$

$$= C_1 \int\left(\frac{1}{x^2}.e^{\int\frac{4x}{2x^2+1}dx}\right)dx + C_2$$

$$= C_1 \int\frac{1}{x^2}.e^{\ln(2x^2+1)}dx + C_2 = C_1 \int\frac{1}{x^2}(2x^2+1)\,dx + C_2$$

$$= C_1 \int\left(2 + \frac{1}{x^2}\right)dx + C_2 = C_1\left(2x - \frac{1}{x}\right) + C_2$$

The general solution is given by

$$y = ux = C_1(2x^2 - 1) + C_2 x$$

(ii) $\quad \dfrac{d^2y}{dx^2} - \dfrac{3}{x}\dfrac{dy}{dx} + \dfrac{3}{x^2}y = 2x - 1$

Comparing the given equation with the general form of second order linear differential equation, we have

$$P_1 = -\frac{3}{x},\quad P_2 = \frac{3}{x^2},\quad Q = 2x - 1$$

Here $P_1 + P_2 x = -\dfrac{3}{x} + \dfrac{3}{x} = 0$. Therefore $y_1 = x$ is a particular integral of

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 0$$

Let $\quad y = u\,x$ so that $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

and $\quad \dfrac{d^2y}{dx^2} = \dfrac{du}{dx} + x\dfrac{d^2u}{dx^2} + \dfrac{du}{dx} = 2\dfrac{du}{dx} + x\dfrac{d^2u}{dx^2}$

Substituting the values of these derivatives in the given differential equation , we get

$$2\frac{du}{dx} + x\frac{d^2u}{dx^2} - \frac{3}{x}\left(u + x\frac{du}{dx}\right) + \frac{3}{x^2}u\,x = 2x - 1$$

or $\quad x\dfrac{d^2u}{dx^2} - \dfrac{du}{dx} = 2x - 1$

or $\quad \dfrac{d^2u}{dx^2} - \dfrac{1}{x}\dfrac{du}{dx} = 2 - \dfrac{1}{x}$

which is a second order differential equation with dependent variable u absent .

Letting $\dfrac{du}{dx} = z$, the above equation becomes

$$\frac{dz}{dx} - \frac{1}{x}z = 2 - \frac{1}{x} \qquad (1)$$

which is a first order linear differential equation . To solve this equation , we have

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Thus the solution of equation (1) is

$$\frac{1}{x}z = \int \frac{1}{x}\left(2 - \frac{1}{x}\right)dx + K$$

$$= \int\left(\frac{2}{x} - \frac{1}{x^2}\right)dx + K = 2\ln x + \frac{1}{x} + K$$

or $\quad z = 2x\ln x + 1 + Kx$

or $\quad \dfrac{du}{dx} = 2x\ln x + 1 + Kx$

or $\quad u = \displaystyle\int (2x\ln x + 1 + Kx)\,dx + C_2$

$$= x^2\ln x - \frac{x^2}{2} + x + \frac{Kx^2}{2} + C_2$$

$$= x^2\ln x + x + \frac{1}{2}(K-1)x^2 + C_2$$

$$= x^2\ln x + x + C_1x^2 + C_2$$

The general solution is given by

$$y = u\,x = (x^2\ln x + x + C_1x^2 + C_2)\,x$$

or $\quad y = x^3\ln x + x^2 + C_1x^3 + C_2x$

PROBLEM (9): **Solve the following differential equation :**

$$x(x\cos x-2\sin x)\frac{d^2y}{dx^2}+(x^2+2)\sin x\frac{dy}{dx}-2(x\sin x+\cos x)y=0$$

SOLUTION: Write the equation as

$$\frac{d^2y}{dx^2}+\frac{(x^2+2)\sin x}{x(x\cos x-2\sin x)}\frac{dy}{dx}-\frac{2(x\sin x+\cos x)}{x(x\cos x-2\sin x)}y=0$$

Comparing it with the general form of second order linear differential equation , we find that

$$P_1=\frac{(x^2+2)\sin x}{x(x\cos x-2\sin x)},\quad P_2=-\frac{2(x\sin x+\cos x)}{x(x\cos x-2\sin x)}$$

and $$2+2P_1x+P_2x^2=2+\frac{2(x^2+2)\sin x}{x\cos x-2\sin x}-\frac{2x(x\sin x+\cos x)}{x\cos x-2\sin x}$$

$$=\frac{2x\cos x-4\sin x+2x^2\sin x+4\sin x-2x^2\sin x-2x\cos x}{x\cos x-2\sin x}$$

$$=0$$

Thus $y_1=x^2$ is a particular integral of the complementary function .

Let $y=x^2u$ be the transformation , then

$$\frac{dy}{dx}=x^2\frac{du}{dx}+2xu\quad\text{and}\quad\frac{d^2y}{dx^2}=x^2\frac{d^2u}{dx^2}+4x\frac{du}{dx}+2u$$

Substituting the values of these derivatives in the given equation , we get

$$x(x\cos x-2\sin x)\left(x^2\frac{d^2u}{dx^2}+4x\frac{du}{dx}+2u\right)+(x^2+2)\sin x\left(x^2\frac{du}{dx}+2xu\right)$$
$$-2(x\sin x+\cos x)x^2u=0$$

or $$(x^4\cos x-2x^3\sin x)\frac{d^2u}{dx^2}+(4x^3\cos x-8x^2\sin x)\frac{du}{dx}+(2x^2\cos x-4x\sin x)u$$
$$+(x^4\sin x+2x^2\sin x)\frac{du}{dx}+2x^3(\sin x)u+4x(\sin x)u-2x^3(\sin x)u-2x^2(\cos x)u=0$$

or $$(x^4\cos x-2x^3\sin x)\frac{d^2u}{dx^2}+(x^4\sin x+4x^3\cos x-6x^2\sin x)\frac{du}{dx}=0$$

or $$(x^2\cos x-2x\sin x)\frac{d^2u}{dx^2}+(x^2\sin x+4x\cos x-6\sin x)\frac{du}{dx}=0$$

or $$\frac{d^2u}{dx^2}+\left(\frac{x^2\operatorname{sn}x+4x\cos x-6\sin x}{x^2\cos x-2x\sin x}\right)\frac{du}{dx}=0$$

which is a second order differential equation with the dependent variable u absent .

Letting $\frac{du}{dx}=z$, the above equation becomes

$$\frac{dz}{dx}+\frac{x^2\sin x+4x\cos x-6\sin x}{x^2\cos x-2x\sin x}z=0$$

Separating the variables , we get

or $$\frac{dz}{z} = \frac{-x^2\sin x - 4x\cos x + 6\sin x}{x^2\cos x - 2x\sin x}\,dx$$

$$= \frac{(-4x\cos x + 8\sin x) - x^2\sin x - 2\sin x}{x^2\cos x - 2x\sin x}\,dx$$

$$= \left[-\frac{4}{x} + \frac{-x^2\sin x - 2\sin x}{x^2\cos x - 2x\sin x}\right]dx$$

Integrating both sides , we get

$$\ln z = -4\ln x + \ln(x^2\cos x - 2x\sin x) + \ln C_1$$

$$= \ln\frac{C_1(x^2\cos x - 2x\sin x)}{x^4}$$

or $$z = \frac{C_1(x^2\cos x - 2x\sin x)}{x^4}$$

or $$\frac{du}{dx} = \frac{C_1(x^2\cos x - 2x\sin x)}{x^4}$$

or $$u = C_1\left[\int\frac{\cos x}{x^2}\,dx - \int\frac{2\sin x}{x^3}\,dx\right] + C_2$$

$$= C_1\left[\frac{\sin x}{x^2} + \int\frac{2\sin x}{x^3}\,dx - \int\frac{2\sin x}{x^3}\,dx\right] + C_2$$

or $$u = C_1\frac{\sin x}{x^2} + C_2$$

or $$y = ux^2 = C_1\sin x + C_2x^2$$

is the general solution of the given differential equation .

PROBLEM (10): **Solve the following differential equation :**

$$(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$$

SOLUTION: Write the equation as

$$\frac{d^2y}{dx^2} - \left(\frac{2x+5}{x+2}\right)\frac{dy}{dx} + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x \qquad (1)$$

Comparing this equation with the general form of second order differential equation , we find that

$$P_1 = -\frac{2x+5}{x+2}, \quad P_2 = \frac{2}{x+2}$$

and $$1 + \frac{P_1}{2} + \frac{P_2}{2^2} = 1 - \frac{2x+5}{2x+4} + \frac{1}{2x+4}$$

$$= \frac{2x+5}{2x+4} - \frac{2x+5}{2x+4} = 0$$

Thus $y_1 = e^{2x}$ is a particular integral of the complementary function .

Let $y = u e^{2x}$ so that $\frac{dy}{dx} = 2ue^{2x} + e^{2x}\frac{du}{dx}$

and $\frac{d^2y}{dx^2} = 4ue^{2x} + 2e^{2x}\frac{du}{dx} + e^{2x}\frac{d^2u}{dx^2} + 2e^{2x}\frac{du}{dx}$

$$= e^{2x}\frac{d^2u}{dx^2} + 4e^{2x}\frac{du}{dx} + 4ue^{2x}$$

Substituting the values of these derivatives in the given differential equation (1), we get

$$e^{2x}\frac{d^2u}{dx^2} + 4e^{2x}\frac{du}{dx} + 4ue^{2x} - \left(\frac{2x+5}{x+2}\right)\left(2ue^{2x} + e^{2x}\frac{du}{dx}\right) + \left(\frac{2}{x+2}\right)ue^{2x} = \frac{x+1}{x+2}e^{x}$$

or $$e^{2x}\frac{d^2u}{dx^2} + \left(\frac{2x+3}{x+2}\right)e^{2x}\frac{du}{dx} = \frac{x+1}{x+2}e^{x}$$

or $$\frac{d^2u}{dx^2} + \frac{2x+3}{x+2}\frac{du}{dx} = \frac{x+1}{x+2}e^{-x}$$

which is a second order differential equation with the dependent variable u absent.

Letting $\frac{du}{dx} = z$, the above equation becomes

$$\frac{dz}{dx} + \frac{2x+3}{x+2}z = \frac{x+1}{x+2}e^{-x} \qquad (2)$$

which is a first order linear differential equation. To solve this equation, we have

$$\text{I.F.} = e^{\int\frac{2x+3}{x+2}dx} = e^{\int\left(2-\frac{1}{x+2}\right)dx} = e^{2x-\ln(x+2)} = e^{2x}\cdot e^{-\ln(x+2)} = \frac{e^{2x}}{x+2}$$

Thus the solution of equation (2) is

$$\frac{e^{2x}}{x+2}z = \int\frac{e^{2x}}{x+2}\frac{x+1}{x+2}e^{-x}\,dx + C$$

$$= \int e^{x}\cdot\frac{x+1}{(x+2)^2}\,dx + C$$

$$= \int\left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right]e^{x}\,dx + C$$

$$= \int\frac{e^{x}}{x+2}\,dx - \int\frac{e^{x}}{(x+2)^2}\,dx + C$$

$$= \frac{e^{x}}{x+2} + \int\frac{e^{x}}{(x+2)^2}\,dx - \int\frac{e^{x}}{(x+2)^2}\,dx + C$$

$$= \frac{e^{x}}{x+2} + C_1$$

or $$z = e^{-x} + C_1(x+2)e^{-2x}$$

or $$\frac{du}{dx} = e^{-x} + C_1(x+2)e^{-2x}$$

or $$u = -e^{-x} + C_1\left[-\frac{1}{2}(x+2)e^{-2x} - \frac{1}{4}e^{-2x}\right] + C_2$$

or $\quad \dfrac{y}{e^{2x}} = -e^{-x} - \dfrac{1}{4}C_1(2x+5)e^{-2x} + C_2$

or $\quad y = -e^{x} - \dfrac{1}{4}C_1(2x+5) + C_2 e^{2x}$ is the general solution .

PROBLEM (11): **Solve the following differential equation :**

$$\frac{d^3y}{dx^3} - 9\frac{dy}{dx} = 0$$ **given that $y_1 = e^{-3x}$ is an integral of the complementary function .**

SOLUTION: Since $y_1 = e^{-3x}$ is an integral of the complementary function , therefore the transformation becomes $y = e^{-3x}u$

Then $\quad \dfrac{dy}{dx} = e^{-3x}\dfrac{du}{dx} - 3e^{-3x}u$

$$\frac{d^2y}{dx^2} = e^{-3x}\frac{d^2u}{dx^2} - 3e^{-3x}\frac{du}{dx} - 3e^{-3x}\frac{du}{dx} + 9e^{-3x}u$$

$$= e^{-3x}\frac{d^2u}{dx^2} - 6e^{-3x}\frac{du}{dx} + 9e^{-3x}u$$

$$\frac{d^3y}{dx^3} = e^{-3x}\frac{d^3u}{dx^3} - 3e^{-3x}\frac{d^2u}{dx^2} - 6e^{-3x}\frac{d^2u}{dx^2} + 18e^{-3x}u\frac{du}{dx} + 9e^{-3x}\frac{du}{dx} - 27e^{-3x}u$$

$$= e^{-3x}\frac{d^3u}{dx^3} - 9e^{-3x}\frac{d^2u}{dx^2} + 27e^{-3x}\frac{du}{dx} - 27e^{-3x}u$$

Substituting the values of these derivatives in the given equation , we get

$$e^{-3x}\frac{d^3u}{dx^3} - 9e^{-3x}\frac{d^2u}{dx^2} + 27e^{-3x}\frac{du}{dx} - 27e^{-3x}u - 9e^{-3x}\frac{du}{dx} + 27e^{-3x}u = 0$$

or $$\frac{d^3u}{dx^3} - 9\frac{d^2u}{dx^2} + 18\frac{du}{dx} = 0 \qquad (1)$$

which is a third order linear differential equation with the dependent variable u absent .

Let $\quad \dfrac{du}{dx} = p$, then $\dfrac{d^2u}{dx^2} = \dfrac{dp}{dx}$ and $\dfrac{d^3u}{dx^3} = \dfrac{d^2p}{dx^2}$.

Equation (1) becomes

$$\frac{d^2p}{dx^2} - 9\frac{dp}{dx} + 18p = 0$$

or $$(D^2 - 9D + 18)p = 0 \qquad (2)$$

The auxiliary equation $D^2 - 9D + 18 = 0$ or $(D-3)(D-6) = 0$

has characteristic roots as $D = 3, 6$.

The solution of equation (2) is

$$p = K_1 e^{3x} + K_2 e^{6x}$$

or $$\frac{du}{dx} = K_1 e^{3x} + K_2 e^{6x}$$

or $\quad u = \frac{1}{3}K_1 e^{3x} + \frac{1}{6}K_2 e^{6x} + C_3$

Thus the solution of the given differential equation is

$$y = e^{-3x}u = e^{-3x}\left(\frac{1}{3}K_1 e^{3x} + \frac{1}{6}e^{6x} + C_3\right)$$

$$= \frac{1}{3}K_1 + \frac{1}{6}e^{3x} + C_3 e^{-3x}$$

$$= C_1 + C_2 e^{3x} + C_3 e^{-3x}$$

PROBLEM (12): **Solve the following differential equation :**

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 8y = 0 \quad \text{given that} \quad y_1 = x^2$$

is an integral of the complementary function .

SOLUTION: Write the equation as

$$\frac{d^3y}{dx^3} + \frac{3}{x}\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{dy}{dx} - \frac{8}{x^3}y = 0 \qquad (1)$$

Since $y_1 = x^2$ is an integral of the complementary function , therefore the transformation becomes $y = x^2 u$.

Then $\quad \frac{dy}{dx} = x^2\frac{du}{dx} + 2xu$

$$\frac{d^2y}{dx^2} = x^2\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + 2x\frac{du}{dx} + 2u$$

$$= x^2\frac{d^2u}{dx^2} + 4x\frac{du}{dx} + 2u$$

$$\frac{d^3y}{dx^3} = x^2\frac{d^3u}{dx^3} + 2x\frac{d^2u}{dx^2} + 4x\frac{d^2u}{dx^2} + 4\frac{du}{dx} + 2\frac{du}{dx}$$

$$= x^2\frac{d^3u}{dx^3} + 6x\frac{d^2u}{dx^2} + 6\frac{du}{dx}$$

Substituting the values of these derivatives in equation (1) , we get

$$x^2\frac{d^3u}{dx^3} + 6x\frac{d^2u}{dx^2} + 6\frac{du}{dx} + \frac{3}{x}\left(x^2\frac{d^2u}{dx^2} + 4x\frac{du}{dx} + 2u\right) + \frac{1}{x^2}\left(x^2\frac{du}{dx} + 2xu\right) - \frac{8}{x^3}(x^2u) = 0$$

or $$x^2\frac{d^3u}{dx^3} + 9x\frac{d^2u}{dx^2} + 19\frac{du}{dx} = 0 \qquad (2)$$

which is a second order differential equation with dependent variable u absent .

Let $\quad \frac{du}{dx} = p$, then $\frac{d^2u}{dx^2} = \frac{dp}{dx}$ and $\frac{d^3u}{dx^3} = \frac{d^2p}{dx^2}$.

Thus equation (2) becomes

$$x^2\frac{d^2p}{dx^2} + 9x\frac{dp}{dx} + 19p = 0 \qquad (3)$$

which is a second order Euler – Cauchy differential equation. To solve equation (3), let $x = e^t$, then

$$x\frac{dp}{dx} = \frac{dp}{dt} \quad \text{and} \quad x^2\frac{d^2p}{dx^2} = \frac{d^2p}{dt^2} - \frac{dp}{dt}$$

Equation (3) becomes

$$\frac{d^2p}{dt^2} - \frac{dp}{dt} + 9\frac{dp}{dt} + 19\,p = 0$$

or $$\frac{d^2p}{dx^2} + 8\frac{dp}{dt} + 19\,p = 0$$

or $$\left(D_1^2 + 8\,D_1 + 19\right)p = 0 \qquad \left(\text{where } D_1 = \frac{d}{dt}\right) \tag{4}$$

The auxiliary equation is $D_1^2 + 8\,D_1 + 19 = 0$ having characteristic roots as $D_1 = -4 \pm \sqrt{3}\,i$.

The solution of equation (4) is

$$p = e^{-4t}\left(C_1 \cos\sqrt{3}\,t + C_2 \sin\sqrt{3}\,t\right)$$

or $$\frac{du}{dx} = e^{-4\ln x}\left[C_1 \cos\left(\sqrt{3}\ln x\right) + C_2 \sin\left(\sqrt{3}\ln x\right)\right]$$

$$= \frac{1}{x^4}\left[C_1 \cos\left(\sqrt{3}\ln x\right) + C_2 \sin\left(\sqrt{3}\ln x\right)\right]$$

or $$u = \int \frac{1}{x^4}\left[C_1 \cos\left(\sqrt{3}\ln x\right) + C_2 \sin\left(\sqrt{3}\ln x\right)\right] dx + C_3$$

To evaluate this integral, let $\ln x = z$, then

$$\frac{1}{x}dx = dz \quad \text{or} \quad dx = x\,dz = e^z\,dz. \text{ Thus}$$

$$u = \int e^{-4z}\left[C_1 \cos\left(\sqrt{3}\,z\right) + C_2 \sin\left(\sqrt{3}\,z\right)\right] e^z\,dz + C_3$$

$$= K_1 \int e^{-3z} \cos\left(\sqrt{3}\,z\right) dz + K_2 \int e^{-3z} \sin\left(\sqrt{3}\,z\right) dz + C_3$$

$$= K_1\left[\frac{e^{-3z}\left(-3\cos\sqrt{3}\,z + \sqrt{3}\sin\sqrt{3}\,z\right)}{9+3}\right]$$

$$+ K_2\left[\frac{e^{-3z}\left(-3\sin\sqrt{3}\,z - \sqrt{3}\cos\sqrt{3}\,z\right)}{9+3}\right] + C_3$$

$$= K_1\left[\frac{-3\cos\left(\sqrt{3}\ln x\right) + \sqrt{3}\sin\left(\sqrt{3}\ln x\right)}{12\,x^3}\right]$$

$$+ K_2\left[\frac{-3\sin\left(\sqrt{3}\ln x\right) - \sqrt{3}\cos\left(\sqrt{3}\ln x\right)}{12\,x^3}\right] + C_3$$

$$= \left(\frac{-3K_1-\sqrt{3}K_2}{12}\right)\frac{\cos\left(\sqrt{3}\ln x\right)}{x^3}+\left(\frac{\sqrt{3}K_1-3K_2}{12}\right)\frac{\sin\left(\sqrt{3}\ln x\right)}{x^3}+C_3$$

or $$u = C_1\frac{\cos\left(\sqrt{3}\ln x\right)}{x^3}+C_2\frac{\sin\left(\sqrt{3}\ln x\right)}{x^3}+C_3$$

Thus the solution of the given differential equation is

$$y = x^2 u = C_1\frac{\cos\left(\sqrt{3}\ln x\right)}{x}+C_2\frac{\sin\left(\sqrt{3}\ln x\right)}{x}+C_3 x^2$$

CHANGE OF DEPENDENT VARIABLE

PROBLEM (13): **Solve the following differential equations :**

(i) $$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y = -3e^{x^2}\sin 2x$$

(ii) $$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y = 2x^3-x^2$$

SOLUTION: (i) $$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y = -3e^{x^2}\sin 2x$$

Comparing it with the general form of second order linear differential equation , we get

$$P_1 = -4x,\quad P_2 = 4x^2-1,\quad Q = -3e^{x^2}\sin 2x$$

Now $$u = e^{-\frac{1}{2}\int P_1\,dx} = e^{-\frac{1}{2}\int(-4x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

and $$R_1(x) = \frac{2}{u}\frac{du}{dx}+P_1(x) = 2e^{-x^2}(e^{x^2}.2x)-4x = 4x-4x = 0$$

Also , $$R_2(x) = P_2-\frac{1}{4}P_1^2-\frac{1}{2}\frac{dP_1}{dx} = 4x^2-1-\frac{1}{4}(16x^2)-\frac{1}{2}(-4)$$

$$= 4x^2-1-4x^2+2 = 1 = A$$

$$Q_1(x) = \frac{1}{u}Q(x) = e^{-x^2}\left(-3e^{x^2}\sin 2x\right) = -3\sin 2x$$

Thus the transformation becomes $y = uv = e^{x^2}v$ and the normal form of the differential equation

$\frac{d^2v}{dx^2}+Av = Q_1(x)$ takes the form

$$\frac{d^2v}{dx^2}+v = -3\sin 2x \qquad (1)$$

which is a second order linear differential equation with constant coefficients and with the removal of first derivative . The complementary function of equation (1) is

$$v_c = C_1\cos x+C_2\sin x$$

The particular integral of equation (1) is

$$v_p = \frac{1}{D^2+1}(-3\sin 2x) = -3\frac{1}{-4+1}\sin 2x = \sin 2x$$

The general solution of equation (1) is

$$v = v_c + v_p = C_1 \cos x + C_2 \sin x + \sin 2x$$

Thus the solution of given equation is

$$y = uv = e^{x^2}(C_1 \cos x + C_2 \sin x + \sin 2x)$$

(ii) $$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 2x^3 - x^2$$

Write the equation as $\dfrac{d^2y}{dx^2} - \dfrac{3}{x}\dfrac{dy}{dx} + \dfrac{3}{x^2}y = 2x - 1$

Comparing this equation with the general form of second order differential equation

we have $P_1 = -\dfrac{3}{x}$, $P_2 = \dfrac{3}{x^2}$, $Q = 2x - 1$

Now $$u = e^{-\frac{1}{2}\int P_1\,dx} = e^{-\frac{1}{2}\int -\frac{3}{x}dx} = e^{\int \frac{3}{2x}dx} = e^{\frac{3}{2}\ln x} = x^{3/2}$$

and $$R_1(x) = \frac{2}{u}\frac{du}{dx} + P_1(x) = \frac{2}{x^{3/2}}\left(\frac{3}{2}x^{1/2}\right) - \frac{3}{x} = \frac{3}{x} - \frac{3}{x} = 0$$

Also, $$R_2(x) = P_2 - \frac{1}{4}P_1^2 - \frac{1}{2}\frac{dP_1}{dx}$$

$$= \frac{3}{x^2} - \frac{1}{4}\left(\frac{9}{x^2}\right) - \frac{1}{2}\left(\frac{3}{x^2}\right) = \frac{3}{x^2} - \frac{9}{4x^2} - \frac{3}{2x^2} = -\frac{3}{4x^2} = \frac{A}{x^2} \quad \text{where } A = -\frac{3}{4}$$

Thus the transformation becomes $y = uv = x^{3/2}v$

and the reduced equation $x^2\dfrac{d^2v}{dx^2} + Av = x^2 Q_1(x) = x^2 \dfrac{1}{u} Q(x)$ becomes

$$x^2\frac{d^2v}{dx^2} - \frac{3}{4}v = x^2\frac{1}{x^{3/2}}(2x-1)$$

or $$x^2\frac{d^2v}{dx^2} - \frac{3}{4}v = 2x^{3/2} - x^{1/2} \qquad (1)$$

which is a second order Euler – Cauchy differential equation.

To solve this equation, let $x = e^t$, then $x^2\dfrac{d^2v}{dx^2} = \dfrac{d^2v}{dt^2} - \dfrac{dv}{dt}$

Thus equation (1) becomes

$$\frac{d^2v}{dt^2} - \frac{dv}{dt} - \frac{3}{4}v = 2e^{3t/2} - e^{t/2}$$

or $$\left(D_1^2 - D_1 - \frac{3}{4}\right)v = 2e^{3t/2} - e^{t/2} \qquad \left(\text{where } D_1 = \frac{d}{dt}\right)$$

The auxiliary equation is $D_1^2 - D_1 - \dfrac{3}{4} = 0$ or $4D_1^2 - 4D_1 - 3 = 0$ or $(2D_1 + 1)(2D_1 - 3) = 0$

having characteristic roots as $D_1 = -\dfrac{1}{2}, \dfrac{3}{2}$.

The complementary function is

$$v_c = C_1 e^{-\frac{1}{2}t} = C_2 e^{\frac{3}{2}t}$$

The particular integral is

$$v_p = \frac{1}{D_1^2 - D_1 - \frac{3}{4}}\left(2e^{3t/2} - e^{t/2}\right)$$

$$= \frac{2}{D_1^2 - D_1 - \frac{3}{4}} e^{3t/2} - \frac{1}{D_1^2 - D_1 - \frac{3}{4}} e^{t/2}$$

$$= 2t\frac{1}{2D_1 - 1} e^{3t/2} - e^{t/2}$$

$$= 2t\left(\frac{1}{2}\right)e^{3t/2} - e^{t/2} = te^{3t/2} - e^{t/2}$$

The general solution of equation (1) is

$$v = v_c + v_p = C_1 e^{-t/2} + C_2 e^{3t/2} + te^{3t/2} - e^{t/2}$$

$$= C_1 e^{-\frac{1}{2}\ln x} + C_2 e^{\frac{3}{2}\ln x} + \ln x \, . \, e^{\frac{3}{2}\ln x} - e^{\frac{1}{2}\ln x}$$

or $$\frac{y}{x^{3/2}} = C_1 x^{-1/2} + C_2 x^{3/2} + x^{3/2}\ln x - x^{t/2}$$

or $$y = C_1 x + C_2 x^3 + x^3 \ln x - x^2$$

CHANGE OF INDEPENDENT VARIABLE

PROBLEM (14): **Solve the following differential equations :**

(i) $$x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + y = \frac{1}{x^2}$$

(ii) $$\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2(\cos^3 x)y = 2\cos^5 x$$

(iii) $$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (\sin^2 x)y = \cos x - \cos^3 x$$

(iv) $$\frac{d^2y}{dx^2} - \frac{1}{\sqrt{x}}\frac{dy}{dx} + \frac{1}{4x^2}\left(-8 + \sqrt{x} + x\right)y = 0$$

SOLUTION: (i) $$x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + y = \frac{1}{x^2}$$

Write the equation as $$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{1}{x^6}y = \frac{1}{x^8}$$

Comparing it with the general form of second order linear differential equation , we have

$$P_1 = \frac{3}{x}, \quad P_2 = \frac{1}{x^6}, \quad a^2 = 1, \quad Q = \frac{1}{x^8}$$

when $\quad \dfrac{dz}{dx} = \sqrt{\dfrac{+P_2(x)}{a^2}} = \sqrt{\dfrac{1}{x^6}} = \dfrac{1}{x^3}$, then

$$\frac{\dfrac{d^2z}{dx^2} + P_1(x)\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{-\dfrac{3}{x^4} + \left(\dfrac{3}{x}\right)\left(\dfrac{1}{x^3}\right)}{\left(\dfrac{1}{x^3}\right)^2} = 0 = A$$

Thus the transformation becomes $z = -\dfrac{1}{2x^2}$, and the equation

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} \pm a^2 y = \frac{Q(x)}{\left(\dfrac{dz}{dx}\right)^2} \text{ takes the form}$$

$$\frac{d^2y}{dz^2} + y = \frac{1}{x^8} \cdot x^6 = \frac{1}{x^2}$$

or $\quad \dfrac{d^2y}{dz^2} + y = -2z \qquad \left(\text{since } z = -\dfrac{1}{2x^2}\right) \qquad (1)$

which is a second order linear differential equation with constant coefficients with a change of independent variable . The complementary function of equation (1) is

$$y_c = C_1 \cos z + C_2 \sin z$$

The particular integral is

$$\begin{aligned} y_p &= \frac{1}{D^2+1}(-2z) \\ &= -2(1+D^2)^{-1} z \\ &= -2(1-D^2)z = -2z \end{aligned}$$

The solution of equation (1) is

$$y = y_c + y_p = C_1 \cos z + C_2 \sin z - 2z$$

Replacing z by $-\dfrac{1}{2x^2}$, we get

$$\begin{aligned} y &= C_1 \cos\left(-\frac{1}{2x^2}\right) + C_2 \sin\left(-\frac{1}{2x^2}\right) - 2\left(-\frac{1}{2x^2}\right) \\ &= C_1 \cos\left(\frac{1}{2x^2}\right) - C_2 \sin\left(\frac{1}{2x^2}\right) + \frac{1}{x^2} \end{aligned}$$

(ii) $\quad \cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2(\cos^3 x)y = 2\cos^5 x$

Write the equation as

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2(\cos^2 x)y = 2\cos^4 x$$

Comparing it with the general form of second order linear differential equation, we have

$$P_1 = \tan x, \quad P_2 = -2\cos^2 x, \quad a^2 = 1, \quad Q = 2\cos^4 x$$

when $\frac{dz}{dx} = \sqrt{\frac{-P_2(x)}{a^2}} = \sqrt{2\cos^2 x} = \sqrt{2}\cos x$, then

$$\frac{\frac{d^2 z}{dx^2} + P_1(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\sqrt{2}\sin x + \tan x\left(\sqrt{2}\cos x\right)}{2\cos^2 x}$$

$$= \frac{-\sqrt{2}\sin x + \sqrt{2}\sin x}{2\cos^2 x} = 0 = A$$

Thus the transformation becomes $z = \sqrt{2}\sin x$, and the equation

$$\frac{d^2 y}{dz^2} + A\frac{dy}{dz} \pm a^2 y = \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} \text{ takes the form}$$

$$\frac{d^2 y}{dz^2} - y = \frac{2\cos^4 x}{2\cos^2 x} = \cos^2 x$$

or $$\frac{d^2 y}{dz^2} - y = 1 - \sin^2 x$$

or $$\frac{d^2 y}{dz^2} - y = 1 - \frac{z^2}{2}$$

or $$\left(D_1^2 - 1\right)y = 1 - \frac{z^2}{2} \qquad \left(\text{where } D_1 = \frac{d}{dz}\right) \tag{1}$$

which is a second order linear differential equation with constant coefficients with a change of independent variable. The complementary function of equation (1) is

$$y_c = C_1 e^z + C_2 e^{-z}$$

The particular integral is given by

$$y_c = \frac{1}{\left(D_1^2 - 1\right)}\left(1 - \frac{z^2}{2}\right) = -\left(1 - D_1^2\right)^{-1}\left(1 - \frac{z^2}{2}\right)$$

$$= -\left(1 + D_1^2\right)\left(1 - \frac{z^2}{2}\right) = -1 + \frac{z^2}{2} + 1 = \frac{z^2}{2}$$

The solution of equation (1) is

$$y = y_c + y_p = C_1 e^z + C_2 e^{-z} + \frac{z^2}{2}$$

Replacing z by $\sqrt{2}\sin x$, we get

$$y = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x} + \sin^2 x$$

(iii) $$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (\sin^2 x)y = \cos x - \cos^3 x$$

Comparing this equation with the general form of second order linear differential equation, we have

$$P_1 = -\cot x, \quad P_2 = -\sin^2 x, \quad a^2 = 1$$

When $$\frac{dz}{dx} = \sqrt{\frac{-P_2(x)}{a^2}} = \sqrt{\frac{\sin^2 x}{1}} = \sin x$$

then $$\frac{\frac{d^2z}{dx^2} + P_1(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cot x \sin x}{\sin^2 x} = 0 = A$$

Thus the transformation becomes $z = -\cos x$, and the equation

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} \pm a^2 y = \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}$$ takes the form

$$\frac{d^2y}{dz^2} - y = \frac{\cos x - \cos^3 x}{\sin^2 x}$$

or $$\frac{d^2y}{dz^2} - y = \frac{\cos x(1 - \cos^2 x)}{\sin^2 x}$$

or $$\frac{d^2y}{dz^2} - y = \cos x$$

or $$\frac{d^2y}{dz^2} - y = -z \qquad (\text{since } z = -\cos x)$$

which is a second order linear differential equation with constant coefficients with a change of independent variable. The solution of this equation is

$$y = C_1 e^z + C_2 e^{-z} + z \tag{2}$$

Replacing $z = -\cos x$ in equation (3), we get

$$y = C_1 e^{-\cos x} + C_2 e^{\cos x} - \cos x$$

as the general solution of the given **differential equation**.

(iv) $$\frac{d^2y}{dx^2} - \frac{1}{\sqrt{x}}\frac{dy}{dx} + \frac{1}{4x^2}(-8 + \sqrt{x} + x)y = 0$$

Comparing this equation with the general form of second order differential equation

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = Q(x)$$

we find $P_1 = -\frac{1}{\sqrt{x}}$, $P_2 = \frac{1}{4x^2}(-8 + \sqrt{x} + x)$, $Q = 0$

Now $$u = e^{-\frac{1}{2}\int P_1(x)\,dx} = e^{-\frac{1}{2}\int -\frac{1}{\sqrt{x}}dx} = e^{\int \frac{1}{2\sqrt{x}}dx} = e^{\sqrt{x}}$$

and $\quad R_1(x) = \frac{2}{u}\frac{du}{dx} + P_1(x) = 2e^{-\sqrt{x}}\, e^{\sqrt{x}}\, \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}} = 0$

Also, $\quad R_2(x) = P_2 - \frac{1}{4}P_1^2 - \frac{1}{2}\frac{dP_1}{dx}$

$$= \frac{1}{4x^2}\left(-8+\sqrt{x}+x\right) - \frac{1}{4}\left(-\frac{1}{\sqrt{x}}\right)^2 - \frac{1}{2}\left(\frac{1}{2x^{3/2}}\right)$$

$$= -\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x} - \frac{1}{4x} - \frac{1}{4x^{3/2}} = -\frac{2}{x^2} = \frac{A}{x^2}, \quad \text{where } A = -2$$

$$Q_1(x) = \frac{1}{u}Q(x) = e^{-\sqrt{x}}(0) = 0$$

Thus the transformation becomes $y = uv = e^{\sqrt{x}}v$, and the reduced differential equation

$$x^2\frac{d^2v}{dx^2} + Av = x^2 Q_1(x) = x^2(0) = 0$$

takes the form $\quad x^2\frac{d^2v}{dx^2} - v = 0 \qquad (1)$

which is a second order Euler – Cauchy differential equation .

To solve this equation , let $x = e^t$, then $x^2\frac{d^2v}{dx^2} = \frac{d^2v}{dt^2} - \frac{dv}{dt}$

Thus equation (1) becomes

$$\frac{d^2v}{dt^2} - \frac{dv}{dt} - 2v = 0$$

or $\quad (D_1^2 - D_1 - 2)v = 0 \qquad \left(\text{where } D_1 = \frac{d}{dt}\right)$

The auxiliary equation is $D_1^2 - D_1 - 2 = 0$ or $(D_1+1)(D_1-2) = 0$

having characteristic roots as $D = -1, 2$.

Thus the solution of equation (1) is

$$v = C_1 e^{-t} + C_2 e^{2t}$$

Replacing t by $\ln x$, we get

$$v = C_1 e^{-\ln x} + C_2 e^{2\ln x} = C_1 x^{-1} + C_2 x^2$$

Hence the final solution is

$$y = uv = e^{\sqrt{x}}(C_1 x^{-1} + C_2 x^2)$$

EXACT EQUATIONS

PROBLEM (15): **Solve the differential equation :**

$$xy''' + (x^2+x+3)y'' + (4x+2)y' + 2y = 0$$

SOLUTION: Comparing this equation with the general form

$$P_0(x)y''' + P_1(x)y'' + P_2(x)y' + P_3(x)y = Q(x)$$

we find that $P_0 = x$, $P_1 = x^2+x+3$, $P_2 = 4x+2$, $P_3 = 2$, $Q(x) = 0$

Since $P_3 - P_2' + P_1'' - P_0''' = 2 - 4 + 2 + 0 = 0$, therefore the given equation is exact. Its first integral is

$$P_0 y'' + (P_1 - P_0')y' + (P_2 - P_1' + P_0'')y = \int Q(x)\,dx + C_1$$

or $$x y'' + (x^2 + x + 2)y' + (2x + 1)y = C_1 \qquad (1)$$

Now let us examine equation (1) for exactness. Comparing it with

$P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x)$, we have

$P_0 = x,\quad P_1 = x^2 + x + 2,\quad P_2 = 2x + 1,\quad Q(x) = C_1$

Since $P_2 - P_1' + P_0'' = 2x + 1 - 2x - 1 + 0 = 0$, therefore equation (1) is exact. Hence its integral is

$$P_0 y' + (P_1 - P_0')y = \int Q(x)\,dx + C_2$$

or $$x y' + (x^2 + x + 1)y = \int C_1\,dx + C_2 = C_1 x + C_2$$

or $$y' + \left(x + 1 + \frac{1}{x}\right)y = C_1 + \frac{C_2}{x} \qquad (2)$$

which is a first order linear differential equation.

The integrating factor is

$$\text{I.F.} = e^{\int\left(x + 1 + \frac{1}{x}\right)dx} = e^{\frac{1}{2}x^2 + x + \ln x} = e^{\ln x} \cdot e^{\frac{1}{2}x^2 + x} = x e^{\frac{1}{2}x^2 + x}$$

Thus the complete solution is

$$x\left(e^{\frac{1}{2}x^2 + x}\right)y = \int x e^{\frac{1}{2}x^2 + x}\left(C_1 + \frac{C_2}{x}\right)dx + C_3$$

$$= C_1 \int x e^{\frac{1}{2}x^2 + x}\,dx + C_2 \int e^{\frac{1}{2}x^2 + x}\,dx + C_3$$

NOTE: Sometimes when the integral cannot be evaluated by known standard methods, the final answer is given in terms of an integral as shown above.

PROBLEM (16): **Solve the differential equation**

$$x^5 y'' + 3x^3 y' + (3x^2 - 6x^3)y = x^4 + 2x^3 - 5$$

SOLUTION: Comparing the given equation with

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = Q(x) \qquad (1)$$

we have $P_0 = x^5,\quad P_1 = 3x^3,\quad P_2 = 3x^2 - 6x^3,\quad Q(x) = x^4 + 2x^3 - 5$

Since $P_2 - P_1' + P_0'' = 3x^2 - 6x^3 - 9x^2 + 20x^3 \neq 0$, therefore the given equation is not exact.

Multiplying the given equation by x^m, we get

$$x^{m+5}y'' + 3x^{m+3}y' + (3x^{m+2} - 6x^{m+3})y = x^{m+4} + 2x^{m+3} - 5x^m \qquad (2)$$

which must be exact. Comparing it with equation (1), we find that

$$P_0 = x^{m+5},\ P_1 = 3x^{m+3},\ P_2 = 3x^{m+2} - 6x^{m+3},\ Q(x) = x^{m+4} + 2x^{m+3} - 5x^m$$

Now equation (2) must be exact if and only if $P_2 - P_1' + P_0'' = 0$

$$3x^{m+2} - 6x^{m+3} - 3(m+3)x^{m+2} + (m+4)(m+5)x^{m+3} = 0$$

or
$$[3 - 3(m+3)]x^{m+2} + [(m+4)(m+5) - 6]x^{m+3} = 0$$

or
$$-3(m+2)x^{m+2} + (m^2 + 9m + 14)x^{m+3} = 0$$

or
$$-3(m+2)x^{m+2} + (m+2)(m+7)x^{m+3} = 0$$

or
$$(m+2)[-3x^{m+2} + (m+7)x^{m+3}] = 0$$

This implies that $m + 2 = 0$ or $m = -2$ for all x.

Substituting this value of m in equation (2), we get

$$x^3 y'' + 3xy' + (3 - 6x)y = x^2 + 2x - \frac{5}{x^2} \tag{3}$$

which must be exact. Comparing equation (3) with equation (1), we have

$$P_0 = x^3,\quad P_1 = 3x,\quad P_2 = 3 - 6x,\quad Q(x) = x^2 + 2x - \frac{5}{x^2}$$

Since $P_2 - P_1' + P_0'' = 3 - 6x - 3 + 6x = 0$, therefore equation (3) is exact. Its first integral is

$$P_0 y' + (P_1 - P_0')y = \int Q(x)\,dx + C_1$$

or
$$x^3 y' + (3x - 3x^2)y = \int \left(x^2 + 2x - \frac{5}{x^2}\right) dx + C_1$$
$$= \frac{x^3}{3} + x^2 + \frac{5}{x} + C_1$$

or
$$y' + \frac{3(1-x)}{x^2} y = \frac{1}{3} + \frac{1}{x} + \frac{5}{x^4} + \frac{C_1}{x^3} \tag{4}$$

which is a first order linear differential equation. Its integrating factor is

$$\text{I.F.} = e^{\int \frac{3(1-x)}{x^2} dx} = e^{-\frac{3}{x} - 3\ln x} = e^{-\frac{3}{x}} \cdot e^{-3\ln x} = e^{-\frac{3}{x}} \cdot e^{\ln x^{-3}} = \frac{1}{x^3} e^{-3/x}$$

The solution of equation (4) is

$$\frac{1}{x^3} e^{-3/2} \cdot y = \int \left(\frac{1}{3} + \frac{1}{x} + \frac{5}{x^4} + \frac{C_1}{x^3}\right) \frac{1}{x^3} e^{-3/x}\, dx + C_2$$

PROBLEM (17): Solve the following differential equation :

$$x^2 y \frac{d^2 y}{dx^2} + x^2 \left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} - 2y^2 = 0$$

SOLUTION: Let us write

$$\frac{du}{dx} = x^2 y \frac{d^2 y}{dx^2} + x^2 \left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} - 2y^2 = 0 \tag{1}$$

Take $u_1 = x^2 y \dfrac{dy}{dx}$

Then $\dfrac{du_1}{dx} = x^2 y \dfrac{d^2y}{dx^2} + x^2\left(\dfrac{dy}{dx}\right)^2 + 2xy\dfrac{dy}{dx}$ (2)

Subtracting equation (2) from equation (1), we get

$$\frac{du}{dx} - \frac{du_1}{dx} = -4xy\frac{dy}{dx} - 2y^2 = 0 \quad (3)$$

Now take $u_2 = -2xy^2$

Then $\dfrac{du_2}{dx} = -4xy\dfrac{dy}{dx} - 2y^2$ (4)

Subtracting equations (3) and (4), we get

$$\frac{du}{dx} - \frac{du_1}{dx} = \frac{du_2}{dx}$$

or $$\frac{du}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} = \frac{d}{dx}(u_1 + u_2)$$

Hence equation (1) is exact and it can be written as

$$\frac{d}{dx}(u_1 + u_2) = 0$$

Integrating, we get

$$u_1 + u_2 = C_1$$

or $$x^2 y\frac{dy}{dx} - 2xy^2 = C_1$$

or $$y\frac{dy}{dx} - \frac{2}{x}y^2 = \frac{C_1}{x^2} \quad (5)$$

Taking $y^2 = v$, so that $2y\dfrac{dy}{dx} = \dfrac{dv}{dx}$. Thus equation (5) becomes

$$\frac{1}{2}\frac{dv}{dx} - \frac{2}{x}v = \frac{C_1}{x^2}$$

$$\frac{dv}{dx} - \frac{4}{x}v = \frac{2C_1}{x^2} \quad (6)$$

which is a first order linear differential equation. Its integrating factor is

$$\text{I.F.} = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = \frac{1}{x^4}$$

The solution of equation (6) is

$$\frac{1}{x^4}v = \int \frac{1}{x^4}\frac{2C_1}{x^2}dx + C_2 = 2C_1\left(-\frac{1}{5x^5}\right) + C_2$$

or $$\frac{y^2}{x^4} = -\frac{2C_1}{5x^5} + C_2$$

or $$xy^2 = C_2x^5 - \frac{2C_1}{5}$$

FACTORIZATION OF OPERATOR

PROBLEM (18): **Solve the differential equation by using the method of factorization of operator**

$$\frac{d^3 y}{d x^3} - 3\frac{d^2 y}{d x^2} + 4 y = 0$$

SOLUTION: Write the given equation in operator notation as

$$(D^3 - 3 D^2 + 4) y = 0$$

Factorizing the operator on the L.H.S. , we get

$$(D + 1)(D - 2)(D - 2) y = 0 \qquad (1)$$

We begin by setting

$$(D - 2)(D - 2) y = u \qquad (2)$$

Equation (1) becomes

$$(D + 1) u = 0$$

or $\frac{d u}{d x} + u = 0$

or $\frac{d u}{u} = - d x$

or $\ln u = - x + \ln C_1$

or $u = C_1 e^{-x} \qquad (3)$

From equations (2) and (3) , we get

$$(D - 2)(D - 2) y = C_1 e^{-x} \qquad (4)$$

Now let $(D - 2) y = v \qquad (5)$

so that equation (4) becomes

$$(D - 2) v = C_1 e^{-x}$$

or $\frac{d v}{d x} - 2 v = C_1 e^{-x} \qquad (6)$

which is a first order linear differential equation . To solve this equation , we have

$$\text{I.F.} = e^{\int -2\, dx} = e^{-2x}$$

The solution of equation (6) becomes

$$e^{-2x} v = \int e^{-2x} C_1 e^{-x}\, d x + C_2$$

$$= -\frac{1}{3} C_1 e^{-3x} + C_2$$

or $v = -\frac{1}{3} C_1 e^{-x} + C_2 e^{2x} \qquad (7)$

Now from equations (5) and (7) , we get

$$(D - 2) y = -\frac{1}{3} C_1 e^{-x} + C_2 e^{2x}$$

or $$\frac{dy}{dx}-2y=-\frac{1}{3}C_1e^{-x}+C_2e^{2x} \tag{8}$$

which is again a first order linear differential equation. The integrating factor is

$$\text{I.F.}=e^{\int -2\,dx}=e^{-2x}$$

and the solution of equation (8) is

$$\begin{aligned}e^{-2x}.y &= \int e^{-2x}\left(-\frac{1}{3}C_1e^{-x}+C_2e^{2x}\right)dx+C_3\\ &= \int\left(-\frac{1}{3}C_1e^{-3x}+C_2\right)dx+C_3\\ &= \frac{1}{9}C_1e^{-3x}+C_2x+C_3\end{aligned}$$

or $$\begin{aligned}y &= \frac{1}{9}C_1e^{-x}+C_2xe^{2x}+C_3e^{2x}\\ &= \frac{1}{9}C_1e^{-x}+(C_2x+C_3)e^{2x}\end{aligned}$$

PROBLEM (19): **Solve the following differential equation by factoring :**

$$\left[(x+3)D^2-(2x+7)D+2\right]y=(x+3)^2e^x$$

SOLUTION: Write the equation as

$$\left[(x+3)D^2-(2x+6)D-D+2\right]y=(x+3)^2e^x$$

or $$\left[(x+3)D(D-2)-(D-2)\right]y=(x+3)^2e^x$$

or $$\left[(x+3)D-1\right](D-2)y=(x+3)^2e^x \tag{1}$$

Let $(D-2)y=v$, then equation (1) becomes

$$\left[(x+3)D-1\right]v=(x+3)^2e^x$$

or $$(x+3)\frac{dv}{dx}-v=(x+3)^2e^x$$

or $$\frac{dv}{dx}-\frac{1}{x+3}v=(x+3)e^x \tag{2}$$

which is a first order linear differential equation. To solve equation (2), we have

$$\text{I.F.}=e^{\int -\frac{1}{x+3}dx}=e^{-\ln(x+3)}=\frac{1}{x+3}$$

Thus the solution of equation (2) is

$$\frac{1}{x+3}v=\int\frac{1}{x+3}(x+3)e^x\,dx+C_1=e^x+C_1$$

or $$v=e^x(x+3)+C_1(x+3)$$

or $\quad (D-2)y = e^{x}(x+3) + C_1(x+3)$

or $\quad \dfrac{dy}{dx} - 2y = e^{x}(x+3) + C_1(x+3) \qquad (3)$

which is again a first order linear differential equation. The integrating factor is

$$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}$$

Thus the solution of equation (3) is

$$\begin{aligned}
e^{-2x}y &= \int e^{-2x}\left[e^{x}(x+3) + C_1(x+3)\right]dx + C_2 \\
&= \int \left[(x+3)e^{-x} + C_1(x+3)e^{-2x}\right]dx + C_2 \\
&= \left[-(x+3)e^{-x} - e^{-x}\right] + C_1\left[-\frac{1}{2}(x+3)e^{-2x} - \frac{1}{4}e^{-2x}\right] + C_2 \\
&= -xe^{-x} - 4e^{-x} + C_1\left[\left(-\frac{1}{2}xe^{-2x}\right) - \frac{7}{6}e^{-2x}\right] + C_2
\end{aligned}$$

or $\quad y = -xe^{x} - 4e^{x} + C_1\left(-\dfrac{1}{2}x - \dfrac{7}{6}\right) + C_2e^{2x}$

PROBLEM (20): **Solve the following initial – value problem :**

$$\frac{d^2y}{dx^2} = 2(y^3 + y); \quad y(0) = 0, \quad y'(0) = 1$$

SOLUTION: Multiplying the given equation by $2\dfrac{dy}{dx}$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4(y^3 + y)\frac{dy}{dx}$$

or $\quad \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)^2 = 4(y^3 + y)\dfrac{dy}{dx}$

Integrating both sides, we get

$$\begin{aligned}
\left(\frac{dy}{dx}\right)^2 &= \int 4(y^3 + y)\frac{dy}{dx}\,dx \\
&= \int 4(y^3 + y)\,dy \\
&= y^4 + 2y^2 + C_1 \qquad (1)
\end{aligned}$$

Using the condition that $\dfrac{dy}{dx} = 1$ for $y = 0$, we get $C_1 = 1$

Thus equation (1) becomes

$$\left(\frac{dy}{dx}\right)^2 = y^4 + 2y^2 + 1 = (y^2 + 1)^2$$

or $\quad \dfrac{dy}{dx} = y^2 + 1$

Separating the variables

$$\frac{dy}{y^2+1} = dx$$

Integrating , we get

$$\tan^{-1} y = x + C_2 \tag{2}$$

Using the condition that $y(0) = 0$, we get $C_2 = 0$.

Thus equation (2) becomes

$$\tan^{-1} y = x$$

or $$y = \tan x$$

is the required solution .

RICCATI EQUATION

PROBLEM (21): Solve the Riccati equation :

$$\frac{dy}{dx} = -y^2 + \frac{2}{x}y - \frac{2}{x^2}$$

by reducing it to second order linear form .

SOLUTION: Comparing the given equation with

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

we have $P = -1$, $Q = \frac{2}{x}$, $R = -\frac{2}{x^2}$

The substitution becomes

$$y = -\frac{1}{Pu}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} \tag{1}$$

and the reduced equation

$$\frac{d^2u}{dx^2} - \left(Q + \frac{1}{P}\frac{dP}{dx}\right)\frac{dy}{dx} + PuR = 0 \quad \text{becomes}$$

$$\frac{d^2u}{dx^2} - \frac{2}{x}\frac{du}{dx} + \frac{2}{x^2}u = 0$$

or $$x^2\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + 2u = 0 \tag{2}$$

which is a second order homogeneous Euler – Cauchy equation . To solve equation (2) , set

$$x = e^t, \text{ then } x\frac{du}{dx} = \frac{du}{dt}$$

$$x^2\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2} - \frac{du}{dt}$$

Thus equation (2) becomes

$$\frac{d^2u}{dt^2} - \frac{du}{dt} - 2\frac{du}{dt} + 2u = 0$$

or $\qquad \frac{d^2u}{dt^2} - 3\frac{du}{dt} + 2u = 0$

$$(D_1^2 - 3D_1 + 2)u = 0$$

The auxiliary equation is $D_1^2 - 3D_1 + 2 = 0$ or $(D_1 - 1)(D_1 - 2) = 0$ having characteristic roots as $D_1 = 1, 2$. The complementary function is

$$u = C_1 e^t + C_2 e^{2t} = C_1 x + C_2 x^2$$

The substitution (1) becomes

$$y = \frac{1}{u}\frac{du}{dx} = \frac{C_1 + 2C_2 x}{C_1 x + C_2 x^2}$$

$$= \frac{\frac{C_1}{C_2} + 2x}{\frac{C_1}{C_2}x + x^2} = \frac{C + 2x}{Cx + x^2} \qquad \left(\text{where } C = \frac{C_1}{C_2}\right)$$

or $\qquad (Cx + x^2)y = (C + 2x)$

5.17 EXERCISE

DEPENDENT VARIABLE ABSENT

Solve the following differential equations :

(1) $y'' = 1 + (y')^2$

(2) $y'' + (y')^2 = 0$

(3) $xy'' + 2y' = x$

(4) $xy'' - y' = 3x^2$

(5) $(1 - x^2)y'' - xy' = 2$

(6) $(1 + x^2)y'' + 2xy' = \dfrac{2}{x^3}$

(7) $y'' - y' = e^x$

(8) $xy'' + y' = 4x$

(9) $(1 + x^2)y'' + xy' = 0$

(10) $xy'' - 3y' = x^2$

Solve the following initial – value problems :

(11) $y'' - y' = \sin x; \quad y(0) = 1, \quad y'(0) = 1$

(12) $y'' \sin x + 2y' \cos x = \cos x; \quad y\left(\dfrac{\pi}{4}\right) = 0, \quad y'\left(\dfrac{\pi}{4}\right) = 0$

INDEPENDENT VARIABLE ABSENT

Solve the following differential equations :

(13) $y'' + (y')^2 = 0$

(14) $yy'' + 2(y')^2 = 0$

(15) $y^2y'' + (y')^3 = 0$

(16) $y'' = (y')^3 + y'$

(17) $yy'' + (y + 1)(y')^2 = 0$

(18) $y'' = 2y(y')^3$

(19) $y'' + e^y(y')^3 = 0$

(20) $yy'' = (y')^2$

(21) $3yy'y'' = (y')^3 - 1$ Hint: use $p^3 = u$

(22) $yy'' + (y')^2 = y'$

Solve the following initial – value problems :

(23) $2yy'' - (y')^2 = 0; \quad y(-6) = 2, \quad y'(-6) = 4$

(24) $yy'' + (y')^2 + 1 = 0; \quad y(0) = 1, \quad y'(0) = 0$

HIGHER ORDER DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(25) $x\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 12x$

(26) $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} = e^x$

(27) $\dfrac{d^4y}{dx^4} - a^2\dfrac{d^2y}{dx^2} = 0$

(28) $\dfrac{d^4y}{dx^4} - \cot x\dfrac{d^3y}{dx^3} = 0$

(29) $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 8\cos 3x$

(30) $\left(\dfrac{d^4y}{dx^4}\right)^2 = 4\dfrac{d^3y}{dx^3}$

LINEAR EQUATIONS WITH ONE KNOWN INTEGRAL OF COMPLEMENTARY FUNCTION

Solve the following differential equations :

(31) $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0;\quad y_1=e^x$

(32) $x\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+2y=0;\quad y_1=e^x$

(33) $y''-\frac{2x}{1+x^2}y'+\frac{2}{1+x^2}y=0;\quad y_1=x$

(34) $x^2y''-3xy'+4y=0;\quad y_1=x^2$ (35) $x^2y''-5xy'+9y=0;\quad y_1=x^3$

(36) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-9y=0,\quad y_1=x^3$ (37) $x^2y''+3xy'+y=0;\quad y_1=\frac{1}{x}$

(38) $y''+4y=0;\quad y_1=\cos 2x$ (39) $xy''+2y'+xy=0;\quad y_1=\frac{\sin x}{x}$

(40) $x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0;\quad y_1=\frac{\cos x}{\sqrt{x}}$

USE OF PARTICULAR INTEGRALS

Solve the following differential equations :

(41) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=8x^3$

(42) $(x-1)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=(x-1)^2$

(43) $y''-\frac{2}{x}y'+\frac{2}{x^2}y=x\ln x$

(44) $x^2\frac{d^2y}{dx^2}-(x^2+2x)\frac{dy}{dx}+(x+2)y=x^3e^x$

(45) $(x+1)\frac{d^2y}{dx^2}-2(x+3)\frac{dy}{dx}+(x+5)y=e^x$

(46) $x\frac{d^2y}{dx^2}-2(x+1)\frac{dy}{dx}+(x+2)y=(x-2)e^{2x}$

(47) $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)y=e^x\sin x$

(48) $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+4x^2y=xe^{x^2}$

REDUCTION OF ORDER IN HIGHER ORDER DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(49) $\frac{d^4y}{dx^4}-y=0;\quad y_1=e^x$ (50) $x^3\frac{d^3y}{dx^3}-x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}=0;\quad y_1=x^4$

CHANGE OF DEPENDENT VARIABLE

Solve the following differential equations :

(51) $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$

(52) $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + (x^2 + 5)y = xe^{-x^2/2}$

(53) $\dfrac{d^2y}{dx^2} - 2\tan x\dfrac{dy}{dx} + 5y = e^x \sec x$

(54) $\cot x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + (\cot x + 2\tan x)y = \sec x$

(55) $(1 - x^2)\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} - (1 + x^2)y = x$

(56) $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + y = \dfrac{\sin 2x}{x}$

CHANGE OF INDEPENDENT VARIABLE

Solve the following differential equations :

(57) $(1 + x^2)^2\dfrac{d^2y}{dx^2} + 2x(1 + x^2)\dfrac{dy}{dx} + 4y = 0$

(58) $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4x^3y = x^5$

(59) $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3y = 8x^3\sin x^2$

(60) $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3y = 2x^3$

(61) $\dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + (\cos^2 x)y = 0$

(62) $\dfrac{d^2y}{dx^2} + \cot x\dfrac{dy}{dx} + 4(\operatorname{cosec}^2 x)y = 0$ Hint : $z = 2\ln\tan\dfrac{x}{2}$

EXACT DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(63) $2x^2y'' + 5xy' + y = 0$

(64) $x^2y'' + 3xy' + y = 0$

(65) $xy'' + (x + 2)y' + y = 0$

(66) $(1 - x^2)y'' - 3xy' - y = 1$

(67) $x^2y'' + 3xy' + y = \dfrac{1}{(1 - x)^2}$

(68) $(1 + x^2)y'' + 4xy' + 2y = \sec x$

(69) $(x^2 + 1)y'' + 4xy' + 2y = 2\cos x - 2x$

(70) $y'' = -2(\operatorname{cosec}^2 x)y = 0$ [Hint: Multiply both sides by $\cot x$]

(71) $(x + \sin x)y''' + 3(1 + \cos x)y'' - 3(\sin x)y' - (\cos x)y = -\sin x$

(72) $(x^3 - 4x)y''' + (9x^2 - 12)y'' + 18xy' + 6y = 6$

FACTORIZATION OF OPERATORS

Solve the following differential equations :

(73) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ (74) $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-\frac{dy}{dx}+2y=0$

(75) $x\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}-y=0$ Hint: $(D+1)y=v$

(76) $x\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}-y=x^2$ Hint : $(D+1)y=v$

(77) $(x+1)\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}-2y=0$ Hint : $(D+1)y=v$

(78) $[xD^2-(x+2)D+2]y=x^3$

Hint: L.H.S. $=(xD-2)(D-1)y$

(79) $(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x$

Hint: L.H.S. $=\{(x+2)D-1\}(D-2)y$ and let $(D-2)y=v$

(80) $[(x+1)D^2-(3x+4)D+3]y=(3x+2)e^{3x}$

EQUATIONS OF THE FORM $\frac{d^ny}{dx^n}=f(x)$

Solve the following differential equations :

(81) $y''\cos^2 x=1$ (82) $\frac{d^2y}{dx^2}=x^2\sin x$

(83) $\frac{d^2y}{dx^2}=x+\ln x$ (84) $\frac{d^3y}{dx^3}=xe^x$

EQUATIONS OF THE FORM $\frac{d^2y}{dx^2}=f(y)$

Solve the following initial – value problems :

(85) $\frac{d^2y}{dx^2}=-e^{-2y}$; $y=0$, $\frac{dy}{dx}=1$ when $x=3$

(86) $2\frac{d^2y}{dx^2}=\sin 2y$; $y=\frac{\pi}{2}$, $\frac{dy}{dx}=1$ when $x=0$

(87) $\frac{d^2y}{dx^2}=3\sqrt{y}$, $y=1$, $\frac{dy}{dx}=2$ when $x=0$

(88) $\frac{d^2y}{dx^2}=-\cot y\,\mathrm{cosec}^2 y$, $y=0$, $\frac{dy}{dx}=1$ when $x=0$

RICCATI EQUATIONS

Solve the following differential equations :

(89) $\dfrac{dy}{dx} = 1 + y^2$

(90) $\dfrac{dy}{dx} = -y^2 + \dfrac{2}{x^2}$

CHAPTER 6

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

6.1 INTRODUCTION

So far, we have discussed the solutions of linear differential equations having one dependent variable and one independent variable. In many problems, we come across linear differential equations having two or more dependent variables and one independent variable. Such differential equations constitute a system of simultaneous linear differential equations. In this chapter, we shall discuss the methods to solve such systems. The complete solutions of these systems can only be obtained if the number of differential equations is equal to the number of dependent variables. We shall restrict ourselves to the solutions of systems having constant coefficients. We shall see in the next chapters, how these systems of differential equations occur in various physical applications.

6.2 SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

SYSTEM OF TWO EQUATIONS IN TWO UNKNOWNS

Let x and y be the dependent variables and t the independent variable. Then the simplest system of differential equations is the system of two first order linear differential equations in two unknowns x and y, which has the form

$$\frac{dx}{dt} = a_{11}x + a_{12}y + f_1(t) \qquad \text{or} \qquad (D - a_{11})x - a_{12}y = f_1(t)$$

$$\frac{dy}{dt} = a_{12}x + a_{22}y + f_2(t) \qquad\qquad -a_{12}x + (D - a_{22})y = f_2(t)$$

where all the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ are constants. If both $f_1(t)$ and $f_2(t)$ are zero, the system is called **homogeneous** otherwise **non – homogeneous**.

For example, the system of differential equations

$$\frac{dx}{dt} = x - 2y + e^t \qquad \text{or} \qquad (D-1)x + 2y = e^t$$

$$\frac{dy}{dt} = 4x + 3y - t \qquad\qquad -4x + (D-3)y = -t \qquad \left(D = \frac{d}{dt}\right)$$

is the first order non – homogeneous system in two unknowns x and y.

The general system of two linear differential equations in two unknowns x and y with constant coefficients has the form

$$f_{11}(D)x + f_{12}(D)y = g_1(t)$$

$$f_{21}(D)x + f_{22}(D)y = g_2(t) \qquad \left(D = \frac{d}{dt}\right)$$

where the coefficients $f_{11}(D)$, $f_{12}(D)$, $f_{21}(D)$, $f_{22}(D)$ are all polynomial operators in D with constant coefficients.

For example, the system of differential equations

$$(D^2 + D + 1)x + (D^2 + 1)y = e^t$$

$$(D^2 + D)x + D^2 y = e^{-t}$$

is the non – homogeneous system in two unknowns x and y.

SYSTEM OF n EQUATIONS IN n UNKNOWNS

The system of n first order linear differential equations in n unknowns $y_1, y_2, \ldots\ldots, y_n$ with constant coefficients has the form

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + \ldots\ldots\ldots a_{1n}y_n + f_1(t)$$

$$\frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + \ldots\ldots\ldots a_{2n}y_n + f_2(t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dy_n}{dt} = a_{n1}y_1 + a_{n2}y_2 + \ldots\ldots\ldots a_{nn}y_n + f_n(t)$$

or

$$\left.\begin{array}{l} (D - a_{11})y_1 - a_{12}y_2 - \ldots\ldots - a_{1n}y_n = f_1(t) \\ -a_{21}y_1 + (D - a_{22})y_2 - \ldots\ldots - a_{2n}y_n = f_2(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -a_{n1}y_1 - a_{n2}y_2 - \ldots\ldots + (D - a_{nn})y_n = f_n(t) \end{array}\right]$$

If all of $f_1, f_2, \ldots\ldots, f_n$ are zero, the system is called **homogeneous** otherwise **non – homogeneous**.

For example, the system of differential equations

$$\frac{dx}{dt} = -x - z + 1 \qquad \text{or} \qquad (D+1)x + z = 1$$

$$\frac{dy}{dt} = -x - 2y + z - t \qquad\qquad x + (D+2)y - z = -t$$

$$\frac{dz}{dt} = x + y - 3z + e^t \qquad\qquad -x - y + (D+3)z = e^t$$

is the first order non – homogeneous system in three unknowns x, y, and z.

The general system of n linear differential equations in n unknowns $y_1, y_2, \dots, y_n$ has the form

$$f_{11}(D)y_1 + f_{12}(D)y_2 + \dots + f_{1n}(D)y_n = g_1(t)$$

$$f_{21}(D)y_1 + f_{22}(D)y_2 + \dots + f_{2n}(D)y_n = g_2(t)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f_{n1}(D)y_1 + f_{n2}(D)y_2 + \dots + f_{nn}(D)y_n = g_n(t)$$

where the coefficients $f_{11}(D), \dots, f_{nn}(D)$ are all polynomial operators in D with constant coefficients. For example, the system of differential equations

$$(D-1)x + (D+2)y = 1 + e^t$$

$$(D+2)y + (D+1)z = 2 + e^t$$

$$(D-1)x + (D+1)z = 3 + e^t$$

is the non – homogeneous system in three unknowns x, y, and z.

6.3 METHODS OF SOLUTION

Consider the system

$$f_{11}(D)x + f_{12}(D)y = g_1(t) \tag{1}$$

$$f_{21}(D)x + f_{22}(D)y = g_2(t) \tag{2}$$

There are two methods to solve this system.

METHOD (1): Use of Differentiation

Sometimes, x or y can be conveniently eliminated if we differentiate equation (1) or (2). From the resulting equation after eliminating one dependent variable (x or y) we can solve for the second variable and then the value of the resulting variable can be found. This method will be used when found very necessary. It can be difficult in general.

METHOD (2): Use of Operator D

Since the polynomial operators obey all the rules of algebra, we shall solve this system by the method of elimination similar to that used in solving an algebraic system of simultaneous equations.

Operating on both sides of equation (1) with $f_{22}(D)$ and operating on both sides of equation (2) by $f_{12}(D)$, we have

$$f_{11}(D)f_{22}(D)x + f_{12}(D)f_{22}(D)y = f_{22}(D)g_1(t) \tag{3}$$

$$f_{21}(D)f_{12}(D)x + f_{22}(D)f_{12}(D)y = f_{12}(D)g_2(t) \tag{4}$$

Subtracting equation (4) from equation (3), we have

$$\left[f_{11}(D)f_{22}(D) - f_{21}(D)f_{12}(D)\right]x = f_{22}(D)g_1(t) - f_{12}(D)g_2(D) \tag{5}$$

which is a linear differential equation in x and t and can be solved to give the value of x. Substituting this value of x in either equation (1) or (2), the value of y can be obtained.

NOTE: (1) We can also eliminate x to get a linear differential equation in y and t which can be solved for y and x can be obtained from equation (1) or (2) after putting the value of y.

(2) Since $f_{11}(D)$ and $f_{22}(D)$ are functions with constant coefficients, therefore
$f_{11}(D)f_{22}(D) = f_{22}(D)f_{11}(D)$

(3) In almost all problems, we shall use method (2).

6.4 NUMBER OF ARBITRARY CONSTANTS

The number of arbitrary independent constants in the general solution of the system

$$f_{11}(D)x + f_{12}(D)y = g_1(t)$$
$$f_{21}(D)x + f_{22}(D)y = g_2(t)$$

is equal to the degree in D of the determinant

$$\Delta = \begin{vmatrix} f_{11}(D) & f_{12}(D) \\ f_{21}(D) & f_{22}(D) \end{vmatrix}$$

provided Δ does not vanish identically. If $\Delta = 0$, the system is dependent, such systems will not be considered here.

Similarly, the number of arbitrary independent constants in the general solution of the system

$$f_{11}(D)x + f_{12}(D)y + f_{13}(D)z = g_1(t)$$
$$f_{21}(D)x + f_{22}(D)y + f_{23}(D)z = g_2(t)$$
$$f_{31}(D)x + f_{32}(D)y + f_{33}(D)z = g_3(t)$$

is equal to the degree in D of the determinant $\Delta = \begin{vmatrix} f_{11}(D) & f_{12}(D) & f_{13}(D) \\ f_{21}(D) & f_{22}(D) & f_{23}(D) \\ f_{31}(D) & f_{32}(D) & f_{33}(D) \end{vmatrix}$

provided Δ does not vanish identically.

EXAMPLE (1): **Solve the system of differential equations :**

$$2\frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^t \qquad (1)$$

$$\frac{dx}{dt} + 3x + y = 0 \qquad (2)$$

SOLUTION: **METHOD (1): Use of Differentiation**

Differentiating equation (2) w.r.t. t, we get

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + \frac{dy}{dt} = 0 \qquad (3)$$

Multiplying equation (1) by -1, equation (2) by -1, and equation (3) by 1, and adding, we get

$$\frac{d^2x}{dt^2} + x = -e^t \qquad (4)$$

which is a second order linear differential equation. To solve equation (4),

$$\text{C.F.} = C_1 \cos t + C_2 \sin t$$

and $$\text{P.I.} = -\frac{1}{D^2+1} e^t = -\frac{1}{2} e^t$$

Thus the solution is

$$x = C_1 \cos t + C_2 \sin t - \frac{1}{2} e^t \qquad (5)$$

To find y in a similar manner, we differentiate equation (1) to obtain

$$2\frac{d^2 x}{dt^2} + \frac{d^2 y}{dt^2} - 4\frac{dx}{dt} - \frac{dy}{dt} = e^t \qquad (6)$$

Now eliminate x and its derivatives from equations (1), (2), (3) and (6).

Multiplying equation (1) by 3, equation (2) by 4, equation (3) by -2, and equation (6) by 1 and adding, we get

$$\frac{d^2 y}{dt^2} + y = 4 e^t \qquad (7)$$

The solution of this equation is

$$y = C_3 \cos t + C_4 \sin t + 2 e^t \qquad (8)$$

The solution given by equations (5) and (8) contains 4 arbitrary constants.

Since $$\begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix} = -(D^2+1)$$

is of degree 2 in D, therefore the solution must have two arbitrary constants.

To reduce the number of arbitrary constants, substitute equations (5) and (8) in equation (2), we get

$$\frac{d}{dt}\left(C_1 \cos t + C_2 \sin t - \frac{1}{2} e^t\right) + 3\left(C_1 \cos t + C_2 \sin t - \frac{1}{2} e^t\right) + (C_3 \cos t + C_4 \sin t + 2 e^t) = 0$$

or $$-C_1 \sin t + C_2 \cos t - \frac{1}{2} e^t + 3 C_1 \cos t + 3 C_2 \sin t - \frac{3}{2} e^t + C_3 \cos t + C_4 \sin t + 2 e^t = 0$$

or $$(3 C_1 + C_2 + C_3) \cos t + (-C_1 + 3 C_2 + C_4) \sin t = 0$$

This implies that $C_3 = -(3 C_1 + C_2)$ and $C_4 = C_1 - 3 C_2$

Thus equation (8) becomes

$$y = (C_1 - 3 C_2) \sin t - (3 C_1 + C_2) \cos t + 2 e^t \qquad (9)$$

Equations (5) and (9) constitute the complete solution of the given system.

i.e. $$x = C_1 \cos t + C_2 \sin t - \frac{1}{2} e^t$$

$$y = (C_1 - 3 C_2) \sin t - (3 C_1 + C_2) \cos t + 2 e^t$$

NOTE: In this particular problem, to find the value of y, it is simpler to proceed as follows:

From equation (2), we have

$$y = -\frac{dx}{dt} - 3x$$

$$= -\left(-C_1 \sin t + C_2 \cos t - \frac{1}{2}e^t\right) - 3\left(C_1 \cos t + C_2 \sin t - \frac{1}{2}e^t\right)$$

$$= (C_1 - 3C_2)\sin t - (3C_1 + C_2)\cos t + 2e^t$$

which is the same as equation (9) above.

METHOD (2): Use of Operator D

Write the equations in D operator notation

$$2(D-2)x + (D-1)y = e^t \qquad (1)$$

$$(D+3)x + y = 0 \qquad (2)$$

We multiply equation (2) by $D-1$. Actually, we operate on equation (2) with $D - 1 = \left(\frac{d}{dt} - 1\right)$, to get

$$(D-1)(D+3)x + (D-1)y = 0 \qquad (3)$$

Subtracting equation (3) from equation (1), we get

$$2(D-2)x - (D-1)(D+3)x = e^t$$

or $\quad (D^2+1)x = -e^t$

This is equation (4) of method (1). The general solution of this equation is

$$x = C_1 \cos t + C_2 \sin t - \frac{1}{2}e^t \qquad (4)$$

To find the value of y, we use equation (2), from which, we have

$$y = -(D+3)x = -(D+3)\left(C_1 \cos t + C_2 \sin t - \frac{1}{2}e^t\right)$$

$$= C_1 \sin t - 3C_1 \cos t - C_2 \cos t - 3C_2 \sin t + \frac{1}{2}e^t + \frac{3}{2}e^t$$

$$= (C_1 - 3C_2)\sin t - (3C_1 + C_2)\cos t + 2e^t$$

EXAMPLE (2): **Solve the system of differential equations:**

$$(D+1)^2 x + 2Dy + 3Dz = 1 \qquad (1)$$

$$Dx + z = 0 \qquad (2)$$

$$x - Dy - Dz = 0 \qquad (3)$$

SOLUTION: First operating on equation (2) with D, we get

$$D^2 x + Dz = 0 \qquad (4)$$

Multiplying equation (3) with 2 and adding to equation (1), we get

$$(D+1)^2 x + 2x + Dz = 1 \tag{5}$$

Subtracting equation (4) from equation (5)

$$2Dx + x + 2x = 1$$

or $$(2D+3)x = 1$$

or $$\frac{dx}{dt} + \frac{3}{2}x = \frac{1}{2}$$

which is a linear differential equation, whose solution is

$$x \cdot e^{3t/2} = \int \frac{1}{2} \cdot e^{3t/2}\, dt + C_1$$

or $$x = \frac{1}{3} + C_1 e^{-3t/2} \tag{6}$$

From equations (2) and (6), we have

$$z = -Dx = \frac{3}{2} C_1 e^{-3t/2} \tag{7}$$

From equation (3), we get

$$Dy = x - Dz = \frac{1}{3} + C_1 e^{-3t/2} + \frac{9}{4} C_1 e^{-3t/2}$$

or $$\frac{dy}{dt} = \frac{1}{3} + \frac{13}{4} C_1 e^{-3t/2}$$

or $$y = \frac{1}{3}t - \frac{13}{6} C_1 e^{-3t/2} + C_2 \tag{8}$$

Since $$\begin{vmatrix} (D+1)^2 & 2D & 3D \\ D & 0 & 1 \\ 1 & -D & -D \end{vmatrix} = 2D^2 + 3D$$

is of degree 2 in D, there are two arbitrary constants and the general solution is

$$x = \frac{1}{3} + C_1 e^{-3t/2}, \quad y = \frac{1}{3}t - \frac{13}{6} C_1 e^{-3t/2} + C_2, \quad z = \frac{3}{2} C_1 e^{-3t/2}$$

6.5 SOLVED PROBLEMS

PROBLEM (1): **Solve the system of differential equations :**

$$\frac{dx}{dt} - 7x + y = 0 \tag{1}$$

$$\frac{dy}{dt} - 2x - 5y = 0 \tag{2}$$

SOLUTION: Writing D for $\frac{d}{dt}$, the given system can be written as

$$(D-7)x + y = 0 \tag{3}$$

$$-2x + (D-5)y = 0 \tag{4}$$

Operating on equation (3) with $D-5$ and subtracting equation (4), we get

$$(D-7)(D-5)x + 2x = 0$$

or $$(D^2 - 12D + 35)x + 2x = 0$$

or $$(D^2 - 12D + 37)x = 0 \tag{5}$$

The roots of the auxiliary equation are

$$D = \frac{12 \pm \sqrt{144-148}}{2} = \frac{12 \pm 2i}{2} = 6 \pm i$$

Thus the solution of equation (5) is

$$x = e^{6t}(C_1 \cos t + C_2 \sin t) \tag{6}$$

To find y, we have from equation (3)

$$y = -(D-7)x = -(D-7)\left[e^{6t}(C_1 \cos t + C_2 \sin t)\right]$$

$$= -\left[e^{6t}(-C_1 \sin t + C_2 \cos t) + 6e^{6t}(C_1 \cos t + C_2 \sin t)\right]$$

$$+ 7e^{6t}C_1 \cos t + 7e^{6t}C_2 \sin t$$

$$= e^{6t}(-C_2 - 6C_1 + 7C_1)\cos t + e^{6t}(C_1 - 6C_2 + 7C_2)\sin t$$

$$= e^{6t}\left[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t\right] \tag{7}$$

Hence the complete solution is given by equations (6) and (7)

$$x = e^{6t}(C_1 \cos t + C_2 \sin t)$$

$$y = e^{6t}\left[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t\right]$$

PROBLEM (2): **Solve the system of differential equations :**

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t \tag{1}$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t \tag{2}$$

SOLUTION: Writing D for $\frac{d}{dt}$, the given system can be written as

$$Dx+(D-2)y = 2\cos t-7\sin t \quad (3)$$

$$(D+2)x-Dy = 4\cos t-3\sin t \quad (4)$$

Operating on equation (3) with D and on equation (4) with $(D-2)$, we get

$$D^2x+D(D-2)y = -2\sin t-7\cos t \quad (5)$$

$$(D^2-4)x-D(D-2)y = -4\sin t-3\cos t-8\cos t+6\sin t \quad (6)$$

Adding equations (5) and (6), we get

$$D^2x+(D^2-4)x = -18\cos t$$

or $$(D^2-2)x = -9\cos t \quad (7)$$

The roots of auxiliary equation are $D = \pm\sqrt{2}$. Thus

$$C.F. = C_1e^{\sqrt{2}t}+C_2e^{-\sqrt{2}t}$$

and $$P.I. = \frac{1}{D^2-2}(-9\cos t) = -9\left(\frac{1}{-1-2}\right)\cos t = 3\cos t$$

Thus the solution of equation (7) is

$$x = C_1e^{\sqrt{2}t}+C_2e^{-\sqrt{2}t}+3\cos t \quad (8)$$

To find y, add equations (3) and (4) to get

$$2Dx+2x-2y = 6\cos t-10\sin t$$

or $$Dx+x-y = 3\cos t-5\sin t$$

or $$y = Dx+x-3\cos t+5\sin t$$

$$= D\left(C_1e^{\sqrt{2}t}+C_2e^{-\sqrt{2}t}+3\cos t\right)+C_1e^{\sqrt{2}t}+C_2e^{-\sqrt{2}t}+3\cos t-3\cos t+5\sin t$$

$$= \sqrt{2}C_1e^{\sqrt{2}t}-\sqrt{2}C_2e^{-\sqrt{2}t}-3\sin t+C_1e^{\sqrt{2}t}+C_2e^{-\sqrt{2}t}+5\sin t$$

$$= (1+\sqrt{2})C_1e^{\sqrt{2}t}+(1-\sqrt{2})C_2e^{-\sqrt{2}t}+2\sin t \quad (9)$$

Thus the complete solution is given by equations (8) and (9).

PROBLEM (3): **Solve the system of differential equations :**

$$(D+1)x+(D-1)y = e^t \quad (1)$$

$$(D^2+D+1)x+(D^2-D+1)y = t^2 \quad (2)$$

SOLUTION: Operating on equation (1) with D^2+D+1 and on equation (2) with $D+1$ and subtracting, we get

$$\left[(D^2+D+1)(D-1)-(D+1)(D^2-D+1)\right]y = (D^2+D+1)e^t-(D+1)t^2$$

or $$-2y = 3e^t-2t-t^2$$

or $\quad y = \frac{1}{2}t^2 + t - \frac{3}{2}e^t \qquad (3)$

Next, operating on equation (1) with $D^2 - D + 1$ and on equation (2) with $D - 1$ and subtracting, we get

$$[(D+1)(D^2-D+1)-(D-1)(D^2+D+1)]x = (D^2-D+1)e^t-(D-1)t^2$$

or $\quad 2x = e^t - 2t + t^2$

or $\quad x = \frac{1}{2}t^2 - t + \frac{1}{2}e^t \qquad (4)$

Equations (3) and (4) is the solution of the given system.

Since $\begin{vmatrix} D+1 & D-1 \\ D^2+D+1 & D^2-D+1 \end{vmatrix} = 2$ is of degree 0 in D, therefore there are no arbitrary constants in the solution.

PROBLEM (4): **Solve the system of differential equations :**

$$(D^2+4)x - 3Dy = 0 \qquad (1)$$

$$3Dx + (D^2+4)y = 0 \qquad (2)$$

SOLUTION: Operating on equation (1) with $D^2 + 4$ and equation (2) with 3 D, and adding, we get

$$(D^2+4)^2x + 9D^2x = 0$$

or $\quad [(D^2+4)^2+9D^2]x = 0$

or $\quad (D^4+8D^2+16+9D^2)x = 0$

or $\quad (D^4+17D^2+16)x = 0$

or $\quad (D^2+16)(D^2+1)x = 0$

The solution of this equation is

$$x = C_1\cos 4t + C_2\sin 4t + C_3\cos t + C_4\sin t \qquad (3)$$

Next, operating on equation (1) with 3 D and equation (2) with $D^2 + 4$, and subtracting, we get

$$-9D^2y - (D^2+4)^2y = 0$$

or $\quad [(D^2+4)^2+9D^2]y = 0$

or $\quad (D^2+16)(D^2+1)y = 0$

The solution of this equation is

$$y = C_5\cos 4t + C_6\sin 4t + C_7\cos t + C_8\sin t \qquad (4)$$

Here the solution given by equations (3) and (4) contains 8 arbitrary constants.

Since $\Delta = \begin{vmatrix} D^2+4 & -3D \\ 3D & D^2+4 \end{vmatrix} = D^4 + 17D^2 + 16$

is of degree 4 in D, therefore the solution must have 4 arbitrary independent constants. To reduce the number of arbitrary constants, we substitute equations (3) and (4) in equation (1).

Thus $$(D^2+4)(C_1\cos 4t + C_2 \sin 4t + C_3 \cos t + C_4 \sin t)$$
$$-3D(C_5 \cos 4t + C_6 \sin 4t + C_7 \cos t + C_8 \sin t) = 0$$

or $$-16C_1\cos 4t + 4C_1\cos 4t - 16C_2 \sin 4t + 4C_2 \sin 4t - C_3\cos t + 4C_3 \cos t - C_4 \sin t$$
$$+4C_4 \sin t + 12C_5 \sin 4t - 12C_6 \cos 4t + 3C_7 \sin t - 3C_8 \cos t = 0$$

or $$-12C_1\cos 4t - 12C_2 \sin 4t + 3C_3 \cos t + 3C_4 \sin t + 12C_5 \sin 4t$$
$$-12C_6 \cos 4t + 3C_7 \sin t - 3C_8 \cos t = 0$$

or $$(-12C_1 - 12C_6)\cos 4t + (-12C_2 + 12C_5)\sin 4t$$
$$+(3C_3 - 3C_8)\cos t + (3C_4 + 3C_7)\sin t = 0$$

This implies $C_5 = C_2,\ C_6 = -C_1,\ C_7 = -C_4,\ C_8 = C_3$.

Thus the complete solution is

$$x = C_1\cos 4t + C_2\sin 4t + C_3\cos t + C_4 \sin t$$
$$y = C_2\cos 4t - C_1 \sin 4t - C_4 \cos t + C_3 \sin t$$

PROBLEM (5): **Solve the system of differential equations :**

$$(2D^2-4)x - Dy = 2t \qquad (1)$$
$$2Dx + (4D-3)y = 0 \qquad (2)$$

SOLUTION: Operating on equation (1) with $(4D-3)$ and on equation (2) with D, and adding, we get

$$\left[(2D^2-4)(4D-3)+2D^2\right]x = (4D-3)2t$$

or $$(8D^3-4D^2-16D+12)x = 8-6x$$

or $$(2D^3-D^2-4D+3)x = 2-\frac{3}{2}x \qquad (3)$$

To solve this equation, the auxiliary equation is

$$2D^3-D^2-4D+3 = 0$$

or $$(D-1)^2(2D+3) = 0$$

The roots are $D = 1, 1, -\frac{3}{2}$ and the complementary function is

$$\text{C.F.} = (C_1+C_2 t)e^t + C_3 e^{-3t/2}$$

$$\text{P.I.} = \frac{1}{3-4D-D^2+2D^3}\left(2-\frac{3}{2}t\right)$$

$$= \frac{1}{3}\,\frac{1}{\left(1-\frac{4}{3}D-\frac{D^2}{3}+\frac{2}{3}D^3\right)}\left(2-\frac{3}{2}t\right)$$

$$= \frac{1}{3}\left(1-\frac{4}{3}D-\frac{D^2}{3}+\frac{2}{3}D^3\right)^{-1}\left(2-\frac{3}{2}t\right)$$

$$= \frac{1}{3}\left(1+\frac{4}{3}D\right)\left(2-\frac{3}{2}t\right)$$

$$= \frac{1}{3}\left(2-\frac{3}{2}t-2\right) = -\frac{1}{2}t$$

Thus the solution of equation (3) is

$$x = (C_1 + C_2 t)e^t + C_3 e^{-3t/2} - \frac{1}{2}t \tag{4}$$

Now multiplying equation (1) by 4 and adding it to equation (2), we get

$$(8D^2 - 16)x + 2Dx - 3y = 8t$$

or $$3y = 8D^2x + 2Dx - 16x - 8t \tag{5}$$

From equation (4), we get

$$Dx = (C_1 + C_2 t)e^t + C_2 e^t - \frac{3}{2}C_3 e^{-3t/2} - \frac{1}{2}$$

and $$D^2x = (C_1 + C_2 t)e^t + 2C_2 e^t + \frac{9}{4}C_3 e^{-3t/2}$$

Substituting the values of Dx and D^2x in equation (5), we get

$$\begin{aligned} 3y &= 8(C_1 + C_2 t)e^t + 16C_2 e^t + 18C_3 e^{-3t/2} + 2(C_1 + C_2 t)e^t \\ &\quad + 2C_2 e^t - 3C_3 e^{-3t/2} - 1 - 16(C_1 + C_2 t)e^t - 16C_3 e^{-3t/2} + 8t - 8t \\ &= -6(C_1 + C_2 t)e^t + 18C_2 e^t - C_3 e^{-3t/2} - 1 \end{aligned}$$

or $$y = -2(C_1 + C_2 t)e^t + 6C_2 e^t - \frac{1}{3}C_3 e^{-3t/2} - \frac{1}{3}.$$

$$= -2(C_1 + C_2 t - 3C_2)e^t - \frac{1}{3}C_3 e^{-3t/2} - \frac{1}{3} \tag{6}$$

Equations (4) and (6) constitute the complete solution of the given system.

PROBLEM (6): **Solve the system of differential equations :**

$$Dx + (D+1)y = 1 \tag{1}$$

$$(D+2)x - (D-1)z = 1 \tag{2}$$

$$(D+1)y + (D+2)z = 0 \tag{3}$$

SOLUTION: Subtracting equation (3) from equation (1), we get

$$Dx - (D+2)z = 1 \tag{4}$$

Operating on equation (2) with D and on equation (4) with $(D+2)$ and subtracting, we get

$$-D(D-1)z + (D+2)^2 z = 0 - (D+2)1$$

or $$(5D+4)z = -2$$

or $$5\frac{dz}{dt} + 4z = -2$$

which is a first order linear differential equation , whose solution is

$$z = -\frac{1}{2} + C_1 e^{-4t/5} \tag{5}$$

Substituting for z in equation (3) , we get

$$\begin{aligned}(D+1)y = -(D+2)z &= -(D+2)\left(-\frac{1}{2} + C_1 e^{-4t/5}\right)\\ &= \frac{4}{5}C_1 e^{-4t/5} + 1 - 2C_1 e^{-4t/5}\\ &= 1 - \frac{6}{5}C_1 e^{-4t/5}\end{aligned}$$

or $$\frac{dy}{dt} + y = 1 - \frac{6}{5}C_1 e^{-4t/5}$$

which is again a linear differential equation in y and t .

The solution of this equation is

$$\begin{aligned}y e^t &= \int\left(1 - \frac{6}{5}C_1 e^{-4t/5}\right) e^t\, dt + C_2\\ &= \int\left(e^t - \frac{6}{5}C_1 e^{t/5}\right) dt + C_2\\ &= e^t - 6C_1 e^{t/5} + C_2\end{aligned}$$

or $$y = 1 - 6C_1 e^{-4t/5} + C_2 e^{-t} \tag{6}$$

Finally , substituting for y in equation (1) , we get

$$\begin{aligned}Dx = 1 - (D+1)y &= 1 - (D+1)(1 - 6C_1 e^{-4t/5} + C_2 e^{-t})\\ &= 1 - \left(1 - \frac{6}{5}C_1 e^{-4t/5}\right)\end{aligned}$$

or $$\frac{dx}{dt} = \frac{6}{5}C_1 e^{-4t/5}$$

or $$x = -\frac{3}{2}C_1 e^{-4t/5} + C_3 \tag{7}$$

Since $$\begin{vmatrix} D & D+1 & 0 \\ D+2 & 0 & -(D-1) \\ 0 & D+1 & D+2 \end{vmatrix} = -(5D^2 + 9D + 4)$$

is of degree 2 in D , but there are three arbitrary constants in the general solution .

Substituting for x and z in equation (2) , we get

$$-\frac{3}{2}\left(-\frac{4}{5}\right)C_1 e^{-4t/5} - 3C_1 e^{-4t/5} + 2C_3 - \left(-\frac{4}{5}C_1 e^{-4t/5} + \frac{1}{2} - C_1 e^{-4t/5}\right) = 1$$

or $$\frac{6}{5}C_1 e^{-4t/5} - 3C_1 e^{-4t/5} + \frac{4}{5}C_1 e^{-4t/5} + C_1 e^{-4t/5} + 2C_3 = \frac{3}{2}$$

or $$2C_1e^{-4t/5} - 2C_1e^{-4t/5} + 2C_3 = \frac{3}{2}$$

or $$C_3 = \frac{3}{4}$$

Thus the complete solution is given by

$$x = \frac{3}{4} - \frac{3}{2}C_1e^{-4t/5}, \quad y = 1 - 6C_1e^{-4t/5} + C_2e^{-t}, \quad z = -\frac{1}{2} + C_1e^{-4t/5}$$

PROBLEM (7): **Solve the system of differential equations :**

$$t\frac{dx}{dt} + y = 0 \tag{1}$$

$$t\frac{dy}{dt} + x = 0 \tag{2}$$

SOLUTION: Let $t = e^u$, then $\frac{dx}{dt} = \frac{dx}{du}\frac{du}{dt} = \frac{1}{t}\frac{dx}{du}$

or $t\frac{dx}{dt} = \frac{dx}{du}$ and similarly $t\frac{dy}{dt} = \frac{dy}{du}$

Thus equations (1) and (2) , become

$$\frac{dx}{du} + y = 0$$

$$\frac{dy}{du} + x = 0$$

In operator notation , these equations take the form

$$D_1x + y = 0 \tag{3}$$

$$x + D_1y = 0 \qquad \left(D_1 = \frac{d}{du}\right) \tag{4}$$

Operating on equation (3) with D_1 and subtracting equation (4) from equation (3) , we get

$$D_1^2x - x = 0$$

or $$(D_1^2 - 1)x = 0 \tag{5}$$

The solution of this equation is

$$x = C_1e^u + C_2e^{-u}$$

From equation (3) , we have

$$y = -D_1x = -C_1e^u + C_2e^{-u} \tag{6}$$

Since $t = e^u$, equations (5) and (6) become

$$x = C_1t + C_2t^{-1} \tag{7}$$

$$y = -C_1t + C_2t^{-1} \tag{8}$$

Equations (7) and (8) constitute the complete solution of the given system .

PROBLEM (8): **Solve the system of differential equations :**

$$t\frac{dx}{dt} = t - 2x \qquad (1)$$

$$t\frac{dy}{dt} = tx + ty + 2x - t \qquad (2)$$

SOLUTION: From equation (1), we have

$$\frac{dx}{dt} + \frac{2}{t}x = 1 \qquad (3)$$

which is a first order linear differential equation .

To solve this equation , we have

$$\therefore \text{I.F.} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

Thus the solution of equation (3) is

$$x t^2 = \int 1 \cdot t^2 dt + C_1 = \frac{1}{3}t^3 + C_1$$

or $$x = \frac{1}{3}t + \frac{C_1}{t^2} \qquad (4)$$

Adding equations (1) and (2) , we get

$$t\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = t(x+y)$$

or $$\frac{dx}{dt} + \frac{dy}{dt} = x + y$$

or $$\frac{dx + dy}{x+y} = dt$$

Integrating , we get

$$\ln(x+y) = t + \ln C_2$$

or $$x + y = C_2 e^t$$

or $$y = C_2 e^t - x$$

Substituting the value of x from equation (4) , we get

$$y = C_2 e^t - \frac{C_1}{t^2} - \frac{1}{3}t \qquad (5)$$

Equations (4) and (5) constitute the complete solution of the given system .

PROBLEM (9): **Solve the system of differential equations :**

$$t\frac{dx}{dt} + 2(x - y) = t \qquad (1)$$

$$t\frac{dy}{dt} + x + 5y = t^2 \qquad (2)$$

SOLUTION: Let $t = e^u$, then $\frac{dx}{dt} = \frac{dx}{du}\frac{du}{dt} = \frac{1}{t}\frac{dx}{du}$

or $t\frac{dx}{dt} = \frac{dx}{du}$ and similarly $t\frac{dy}{dt} = \frac{dy}{du}$

Thus equations (1) and (2) become

$$\frac{dx}{du} + 2(x-y) = e^u$$

$$\frac{dy}{du} + x + 5y = e^{2u}$$

In operator notation, the above equations take the form

$$(D_1 + 2)x - 2y = e^u \tag{3}$$

$$x + (D_1 + 5)y = e^{2u} \qquad \left(D_1 = \frac{d}{du}\right) \tag{4}$$

Operating on equation (3) with $D_1 + 5$, and multiplying equation (4) with 2 and adding, we get

$$(D_1+2)(D_1+5)x + 2x = (D_1+5)e^u + 2e^{2u}$$

or $$(D_1^2 + 7D_1 + 12)x = 6e^u + 2e^{2u} \tag{5}$$

To solve equation (5), the auxiliary equation is

$$D_1^2 + 7D_1 + 12 = 0$$

The roots of this equation $D_1 = -3, -4$, therefore

$$\text{C.F.} = C_1 e^{-3u} + C_2 e^{-4u}$$

$$\text{P.I.} = \frac{1}{(D_1^2 + 7D_1 + 12)}(6e^u + 2e^{2u}) = \frac{3}{10}e^u + \frac{1}{15}e^{2u}$$

Thus the solution of equation (5) is

$$x = C_1 e^{-3u} + C_2 e^{-4u} + \frac{3}{10}e^u + \frac{1}{15}e^{2u} \tag{6}$$

From equation (3), we have

$$\begin{aligned}
y &= \frac{1}{2}(D_1+2)x - \frac{1}{2}e^u \\
&= \frac{1}{2}D_1 x + x - \frac{1}{2}e^u \\
&= \frac{1}{2}\left(-3C_1e^{-3u} - 4C_2e^{-4u} + \frac{3}{10}e^u + \frac{2}{15}e^{2u}\right) \\
&\quad + C_1e^{-3u} + C_2e^{-4u} + \frac{3}{10}e^u + \frac{1}{15}e^{2u} - \frac{1}{2}e^u \\
&= -\frac{1}{2}C_1e^{-3u} - C_2e^{-4u} - \frac{1}{20}e^u + \frac{2}{15}e^{2u} \qquad (7)
\end{aligned}$$

Since $t = e^u$, equation (6) and (7) become

$$x = C_1 t^{-3} + C_2 t^{-4} + \frac{3}{10} t + \frac{1}{15} t^2$$

$$y = -\frac{1}{2} C_1 t^{-3} - C_2 t^{-4} - \frac{1}{20} t + \frac{2}{15} t^2$$

which is the general solution of the given system .

PROBLEM (10): **Solve the system of differential equations**

$$t^2 \frac{d^2 x}{dt^2} + t \frac{dx}{dt} + 2y = 0 \qquad (1)$$

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} - 2x = 0 \qquad (2)$$

SOLUTION: Let $t = e^u$, then $\frac{dx}{dt} = \frac{dx}{du}\frac{du}{dt} = \frac{1}{t}\frac{dx}{du}$

or $t\frac{dx}{dt} = \frac{dx}{du}$ and similarly $t\frac{dy}{dt} = \frac{dy}{du}$

Also, $\frac{d^2 x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{du}\left(\frac{dx}{dt}\right)\frac{du}{dt} = \frac{d}{du}\left(\frac{1}{t}\frac{dx}{du}\right)\frac{du}{dt}$

$$= \left[\frac{d}{du}\left(e^{-u}\frac{dx}{du}\right)\right]\frac{du}{dt}$$

$$= \left(e^{-u}\frac{d^2 x}{du^2} - e^{-u}\frac{dx}{du}\right)\frac{1}{t}$$

$$= \left(e^{-u}\frac{d^2 x}{dt^2} - e^{-u}\frac{dx}{du}\right)e^{-u}$$

$$= \left(\frac{d^2 x}{du^2} - \frac{dx}{du}\right)e^{-2u} = \left(\frac{d^2 x}{du^2} - \frac{dx}{du}\right)\frac{1}{t^2}$$

or $t^2\frac{d^2 x}{dt^2} = \frac{d^2 x}{du^2} - \frac{dx}{du}$

Similarly, $t^2\frac{d^2 y}{dt^2} = \frac{d^2 y}{du^2} - \frac{dy}{du}$

Thus equations (1) and (2) become

$$\frac{d^2 x}{du^2} - \frac{dx}{du} + \frac{dx}{du} + 2y = 0$$

$$\frac{d^2 y}{du^2} - \frac{dy}{du} + \frac{dy}{du} - 2x = 0$$

or $\frac{d^2 x}{du^2} + 2y = 0$

$$\frac{d^2 y}{du^2} - 2x = 0$$

In operator notation, these equations take the form

$$D_1^2 x + 2y = 0 \tag{3}$$

$$-2x + D_1^2 y = 0 \qquad \left(D_1 = \frac{d}{du}\right) \tag{4}$$

Operating on equation (3) with D_1^2 and multiplying equation (4) with 2 and subtracting, we get

$$(D_1^4 + 4)x = 0$$

The auxiliary equation is $D_1^4 + 4 = 0$

or $\qquad (D_1^2 + 2)^2 - 4D_1^2 = 0$

or $\qquad (D_1^2 + 2D_1 + 2)(D_1^2 - 2D_1 + 2) = 0$

The roots of this equation are $D_1 = 1 \pm i, \quad -1 \pm i$.

Thus the solution is

$$x = e^u(C_1 \cos u + C_2 \sin u) + e^{-u}(C_3 \cos u + C_4 \sin u) \tag{5}$$

Now $\quad D_1 x = e^u(-C_1 \sin u + C_2 \cos u) + e^u(C_1 \cos u + C_2 \sin u)$

$\qquad + e^{-u}(-C_3 \sin u + C_4 \cos u) - e^{-u}(C_3 \cos u + C_4 \sin u)$

$\qquad = e^u\left[(C_1 + C_2)\cos u + (C_2 - C_1)\sin u\right] + e^{-u}\left[(C_4 - C_3)\cos u - (C_3 + C_4)\sin u\right]$

and $\quad D_1^2 x = e^u\left[-(C_1 + C_2)\sin u + (C_2 - C_1)\cos u\right]$

$\qquad + e^u\left[(C_1 + C_2)\cos u + (C_2 - C_1)\sin u\right]$

$\qquad + e^{-u}\left[-(C_4 - C_3)\sin u - (C_3 + C_4)\cos u\right.$

$\qquad \left. - e^{-u}\left[(C_4 - C_3)\cos u - (C_3 + C_4)\sin u\right]\right.$

$\qquad = e^u(2C_2 \cos u - 2C_1 \sin u) + e^{-u}(-2C_4 \cos u + 2C_3 \sin u)$

From equation (3), we have

$$y = -\frac{1}{2}D_1^2 x = e^u(C_1 \sin u - C_2 \cos u) + e^{-u}(C_4 \cos u - C_3 \sin u) \tag{6}$$

Since $t = e^u$ and $u = \ln t$, equation (5) and (6) become

$$x = t\left[C_1 \cos(\ln t) + C_2 \sin(\ln t)\right] + t^{-1}\left[C_3 \cos(\ln t) + C_4 \sin(\ln t)\right]$$

$$y = t\left[C_1 \sin(\ln t) - C_2 \cos(\ln t)\right] + t^{-1}\left[C_4 \cos(\ln t) - C_3 \sin(\ln t)\right]$$

which is the complete solution of the given system.

6.6 EXERCISE

Solve the following systems of linear differential equations :

(1) $Dx + wy = 0$
$-wx + Dy = 0$

(2) $Dx - y = t$
$x + Dy = 1$

(3) $Dx + 2y = -\sin t$
$-2x + Dy = \cos t$

(4) $(D-1)x - y = 0$
$-4x + (D+2)y = 0$

(5) $(D-1)x + Dy = 0$
$(D+1)x + (2D+2)y = 0$

(6) $(D+1)x - y = e^t$
$-x + (D+1)y = e^t$

(7) $(D+2)x + 3y = 0$
$3x + (D+2)y = 2e^{2t}$

(8) $Dx - (D+1)y = -e^t$
$x + (D-1)y = e^{2t}$

(9) $(5D+4)x - (2D+1)y = e^{-t}$
$(D+8)x - 3y = 5e^{-t}$

(10) $(D-1)x + Dy = 2t+1$
$(2D+1)x + 2Dy = t$

(11) $Dx + (D+1)y = t$
$(D^2+1)x + (D^2+D+1)y = t^2$

(12) $(D^2+D+1)x + (D^2+1)y = e^t$
$(D^2+D)x + D^2y = e^{-t}$

(13) $(D^2-3)x - 4y = 0$
$x + (D^2+1)y = 0$

(14) $(D^2-4D+4)x - y = 0$
$-25x + (D^2+4D+4)y = 16e^t$

(15) $(D^2+16)x-6Dy=0$

$6Dx+(D^2+16)y=0$

(16) $\dfrac{dx}{dt}=2t, \quad \dfrac{dy}{dt}=3x+2t, \quad \dfrac{dz}{dt}=x+4y+t$

(17) $(D-1)x+(D+2)y=1+e^t$

$(D+2)y+(D+1)z=2+e^t$

$(D-1)x+(D+1)z=3+e^t$

INITIAL – VALUE PROBLEMS

(18) $(D-2)x-y=0$

$-3x+Dy=2 \qquad x(0)=1, \quad y(0)=-1$

(19) $(D+14)x-10y=0$

$5x+(D-1)y=0 \qquad x(0)=-1, \quad y(0)=1$

(20) $(D+1)x+z=1$

$x+(D+2)y-z=-1 \qquad x(0)=-1, \quad y(0)=x(0)=0$

$-x-y+(D+3)z=3$

CHAPTER 7

SIMULTANEOUS AND TOTAL DIFFERENTIAL EQUATIONS

7.1 INTRODUCTION

In this chapter, we will discuss the methods to solve the simultaneous linear differential equations in three variables x, y, and z. Furthermore, we will discuss the total differential equation in three variables x, y, and z. Exact total differential equations and homogeneous total differential equations will also be discussed. Finally, the methods to solve systems of total differential equations will be presented.

7.2 SIMULTANEOUS DIFFERENTIAL EQUATIONS

The equations of the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (1)$$

are called simultaneous differential equations, where P, Q, and R are functions of x, y, and z.

7.3 METHODS OF SOLUTION

The following are the methods of solution of equations (1).

METHOD 1

First take any two members of equations (1), $\frac{dx}{P} = \frac{dy}{Q}$ (say)

Integrating this equation, suppose we get a solution of the form $u_1(x, y, z) = C_1$. Sometimes, such an equation is obtained after cancellation of some factor from the chosen two members of equations (1).

Again, take the other two members of equations (1), $\frac{dy}{Q} = \frac{dz}{R}$ (say)

Integrating this equation, suppose we get another solution of the form $u_2(x, y, z) = C_2$. These two solutions so obtained form the complete solution. Here, u_1 and u_2 are two independent solutions of the given equations. The solutions u_1 and u_2 are said to be independent if $\frac{u_1}{u_2}$ is not merely constant. For example, $u_1 = x^2 + y^2 + z^2$ and $u_2 = x + y + z$ are independent solutions whereas $u_1 = x + y + z$ and $u_2 = 2(x + y + z)$ are not independent.

EXAMPLE (1): **Solve the following simultaneous differential equations :**

(i) $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$ (ii) $$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

SOLUTION: (i) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Taking the first two members , we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating , we get

$$\ln x = \ln y + \ln C_1$$

or $$x = C_1 y \qquad (1)$$

Taking the last two members , we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating , we get

$$\ln y = \ln z + \ln C_2$$

or $$y = C_2 z \qquad (2)$$

Equation (1) and (2) constitute the complete solution .

(ii) $$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Taking the first two members , we get

$$\frac{dx}{yz} = \frac{dy}{zx}$$

or $$\frac{dx}{y} = \frac{dy}{x}$$

or $$x\,dx = y\,dy$$

Integrating , we get

or $$x^2 - y^2 = C_1 \quad (\text{where } C_1 = 2C) \qquad (1)$$

Taking the first and last members , we have

$$\frac{dy}{zx} = \frac{dz}{xy}$$

or $$\frac{dy}{z} = \frac{dz}{y}$$

or $$y\,dy = z\,dz$$

Integrating , we get

$$y^2 - z^2 = C_2 \quad (\text{where } C_2 = 2C) \qquad (2)$$

Equations (1) and (2) constitute the complete solution .

METHOD 2: SECOND INTEGRAL FOUND WITH THE HELP OF THE FIRST

Suppose that only one solution $u_1(x, y, z) = C_1$ can be found using method (1). Then this solution can be used to obtain an equation in two variables. The solution of this resulting equation will give us a second solution which will involve the arbitrary constant C_1. To find the final form of the second solution, the arbitrary constant C_1 must be eliminated with help of the first solution $u_1(x, y, z) = C_1$.

EXAMPLE (2): **Solve the simultaneous differential equations :**

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y+2x)}$$

SOLUTION: Take $\frac{dx}{1} = \frac{dy}{-2}$ or $dy = -2\,dx$

Integrating, we get $y = -2x + C_1$

or $\quad y + 2x = C_1 \qquad (1)$

Using this relation, the given differential equation becomes

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin C_1}$$

Take $\frac{dx}{1} = \frac{dz}{3x^2 \sin C_1}$

or $\quad dz = 3x^2 \sin C_1 \, dx$

Integrating, we get $z = x^3 \sin C_1 + C_2$

or $\quad z - x^3 \sin C_1 = C_2$

or $\quad z - x^3 \sin(y + 2x) = C_2$ [using equation (1)] $\qquad (2)$

Equations (1) and (2) form the complete solution.

METHOD 3: SOLUTION USING MULTIPLIERS

Consider the simultaneous differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (1)$$

Let ℓ, m, n be functions of x, y, z. Then, by a well– known principle of algebra, we know that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\ell\,dx + m\,dy + n\,dz}{\ell P + m Q + n R} \qquad (2)$$

Now there are two cases to be considered.

CASE (1)

If the denominator $\ell P + mQ + nR = 0$, in equation (2), then the last fraction combined with any of the first three fractions in equation (2) implies that the numerator is also zero, i.e. $\ell\,dx + m\,dy + n\,dz = 0$, which can be integrated to give the solution $u_1(x, y, z) = C_1$.

Thus this method is used to obtain a zero – denominator and a numerator that is an exact differential. This method may be repeated to get another solution $u_2(x, y, z) = C_2$. Here ℓ, m, n are called **multipliers**. As a special case, these can be constants also. Sometimes, only one solution is possible by the use of multipliers. In such cases, second solution should be obtained by using either of the above methods.

CASE (2)

If $\ell P + m Q + n R \neq 0$ in equation (2), then the numerator in equation (2) is an exact differential of the denominator. The last fraction in equation (2) can then be combined with a suitable fraction to give a solution. However, in some problems, another set of multipliers ℓ_1, m_1, n_1 are so chosen that

$$\text{Each of the fractions } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\ell_1 dx + m_1 dy + n_1 dz}{\ell_1 P + m_1 Q + n_1 R} \qquad (3)$$

where the numerator is an exact differential of the denominator. The last fractions of equations (2) and (3) are then combined to give a solution. This method may also be repeated in some problems to get another solution. Sometimes, only one solution is possible by the use of multipliers. In such cases, second solution should be obtained by using either of the above methods.

EXAMPLE (3): **Solve the following differential equations :**

(i) $$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$$

(ii) $$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$$

SOLUTION: (i) Using $x, -y, -z$ as multipliers, we get

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} = \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2+y^2)}$$

$$= \frac{x\,dx - y\,dy - z\,dz}{0}$$

or $$x\,dx - y\,dy - z\,dz = 0$$

or $$d\left(\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2\right) = 0$$

Integrating, we get

$$x^2 - y^2 - z^2 = C_1 \qquad (1)$$

Similarly, taking $y, x, -z$ as multipliers, we get

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} = \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)}$$

or $$y\,dx + x\,dy - z\,dz = 0$$

or $$d(xy)-d\left(\frac{1}{2}z^2\right)=0$$

Integrating, we get $xy-\frac{z^2}{2}=K$

or $$2xy-z^2=C_2 \quad (C_2=2K) \tag{2}$$

Equations (1) and (2) constitute the complete solution.

(ii) $$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$$

Taking $1, 1, 0$ as multipliers, we get

$$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}=\frac{dx+dy}{2+x+y}$$

or $$\frac{dz}{z}=\frac{dx+dy}{2+x+y}$$

Integrating, we get

$$\ln z=\ln(2+x+y)+\ln C_1$$

or $$z=C_1(2+x+y) \tag{1}$$

Now taking $1, -1, 0$ as multipliers, we get

$$\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}=\frac{dx-dy}{y-x}$$

or $$\frac{dz}{z}=\frac{dx-dy}{y-x}$$

Integrating, we get

$$\ln z=-\ln(x-y)+\ln C_2$$

or $$z=\frac{C_2}{x-y} \tag{2}$$

Equations (1) and (2) constitute the complete solution.

7.4 GEOMETRICAL INTERPRETATION OF $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

From three – dimensional coordinate geometry, it is known that the direction cosines of the tangent to a curve at any point (x, y, z) are proportional to dx, dy, dz. The given differential equations, therefore express the fact that the direction cosines of the tangent to the curve at that point are proportional to P, Q, R. Suppose that the complete solution of the given equations is given by the equations $u_1(x,y,z)=C_1$ and $u_2(x,y,z)=C_2$.

Then we observe that the solution represents the curves of intersection of the two families of surfaces $u_1(x,y,z)=C_1$ and $u_2(x,y,z)=C_2$. Since C_1 and C_2 can take infinite number of values, we get a doubly infinite number of such curves.

7.5 TOTAL DIFFERENTIAL EQUATION

A differential equation of the form

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0 \qquad (1)$$

is called a total differential equation in three variables x, y, and z.

For example, the differential equations

$$(3x^2y^2 + e^x z)\,dx + (2x^3y + \sin z)\,dy + (y\cos z - e^x)\,dz = 0$$

$$(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$$

$$y\,dx + dy + dz = 0$$

are the total differential equations.

METHOD OF SOLUTION

Equation (1) can be solved directly if there exists a function $u(x,y,z)$ whose differential du is equal to the left hand side of equation (1). In other cases, equation (1) may or may not be integrable.

EXAMPLE (4): **Solve the following differential equations :**

(i) $(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$

(ii) $z(y\,dx - x\,dy) = y^2\,dz$

SOLUTION: (i) $(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$

This equation can be written as

$$(x\,dy + y\,dx) + (y\,dz + z\,dy) + (x\,dz + z\,dx) = 0$$

or $$d(xy) + d(yz) + d(xz) = 0$$

Integrating, we get

$$xy + yz + xz = C$$

(ii) $z(y\,dx - x\,dy) = y^2\,dz$

Write the equation as

$$\frac{y\,dx - x\,dy}{y^2} = \frac{dz}{z}$$

or $$d\left(\frac{x}{y}\right) = \frac{dz}{z}$$

Integrating, we get

$$\frac{x}{y} = \ln z + C$$

or $$x = y\ln z + Cy$$

7.6 NECESSARY CONDITION FOR INTEGRABILITY OF TOTAL DIFFERENTIAL EQUATION

Let the differential equation be

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

Let the solution of equation (1) be

$$\phi(x, y, z) = C \qquad (2)$$

Differential equation (2), we get

$$d\phi = 0$$

or $$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad (3)$$

Equations (1) and (3) are identical since equation (2) is a solution of equation (1). Since equations (1) and (3) are identical, therefore the coefficients P, Q, R must be proportional to $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ respectively.

Thus $$\frac{\partial \phi/\partial x}{P} = \frac{\partial \phi/\partial y}{Q} = \frac{\partial \phi/\partial z}{R} = \lambda \text{ (say)}$$

or $$\frac{\partial \phi}{\partial x} = \lambda P, \quad \frac{\partial \phi}{\partial y} = \lambda Q, \quad \frac{\partial \phi}{\partial z} = \lambda R$$

Now $$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial y}(\lambda P) = \lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y}$$

and $$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q) = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$$

If ϕ and its first partial derivatives $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are continuous, then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

or $$\lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$$

or $$\lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial x} = 0 \qquad (1)$$

Also, $$\frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial z}(\lambda Q) = \lambda\frac{\partial Q}{\partial z} + Q\frac{\partial \lambda}{\partial z}$$

and $$\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial}{\partial y}(\lambda R) = \lambda\frac{\partial R}{\partial y} + R\frac{\partial \lambda}{\partial y}$$

If ϕ and its partial derivatives $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$ are continuous, then

$$\frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z}$$

or $\qquad \lambda\dfrac{\partial Q}{\partial z}+Q\dfrac{\partial\lambda}{\partial z}=\lambda\dfrac{\partial R}{\partial y}+R\dfrac{\partial\lambda}{\partial y}$

or $\qquad \lambda\left(\dfrac{\partial Q}{\partial z}-\dfrac{\partial R}{\partial y}\right)+Q\dfrac{\partial\lambda}{\partial z}-R\dfrac{\partial\lambda}{\partial y}=0 \qquad (2)$

Finally, $\dfrac{\partial^2\phi}{\partial x\partial z}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial\phi}{\partial z}\right)=\dfrac{\partial}{\partial x}(\lambda R)=\lambda\dfrac{\partial R}{\partial x}+R\dfrac{\partial\lambda}{\partial x}$

and $\qquad \dfrac{\partial^2\phi}{\partial z\partial x}=\dfrac{\partial}{\partial z}\left(\dfrac{\partial\phi}{\partial x}\right)=\dfrac{\partial}{\partial z}(\lambda P)=\lambda\dfrac{\partial P}{\partial z}+P\dfrac{\partial\lambda}{\partial z}$

If ϕ and its first partial derivatives $\dfrac{\partial\phi}{\partial x}$ and $\dfrac{\partial\phi}{\partial z}$ are continuous, we have

$$\frac{\partial^2\phi}{\partial x\partial z}=\frac{\partial^2\phi}{\partial z\partial x}$$

or $\qquad \lambda\dfrac{\partial R}{\partial x}+R\dfrac{\partial\lambda}{\partial x}=\lambda\dfrac{\partial P}{\partial z}+P\dfrac{\partial\lambda}{\partial z}$

or $\qquad \lambda\left(\dfrac{\partial R}{\partial x}-\dfrac{\partial P}{\partial z}\right)+R\dfrac{\partial\lambda}{\partial x}-P\dfrac{\partial\lambda}{\partial z}=0 \qquad (3)$

Multiplying equations (1), (2), (3) by R, P, Q respectively and adding, we get

$$\lambda\left[R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)+P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\right]=0$$

Since $\lambda \neq 0$, therefore

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad (4)$$

which is the required condition of integrability.

EXAMPLE (5): Test the integrability of the following differential equations :

(i) $(3xz+2y)\,dx+x\,dy+x^2\,dz=0$

(ii) $y\,dx+dy+dz=0$

SOLUTION: The condition for integrability of the total differential equation

$$P\,dx+Q\,dy+R\,dz=0 \qquad (1)$$

is $\qquad P\left(\dfrac{\partial Q}{\partial z}-\dfrac{\partial R}{\partial y}\right)+Q\left(\dfrac{\partial R}{\partial x}-\dfrac{\partial P}{\partial z}\right)+R\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)=0 \qquad (2)$

(i) $\qquad (3xz+2y)\,dx+x\,dy+x^2\,dz=0$

Comparing this equation with the general form (1), we get

$$P=3xz+2y, \quad Q=x, \quad R=x^2$$

Then $\quad \dfrac{\partial P}{\partial y}=2, \quad \dfrac{\partial P}{\partial z}=3x, \quad \dfrac{\partial Q}{\partial x}=1, \quad \dfrac{\partial Q}{\partial z}=0, \quad \dfrac{\partial R}{\partial x}=2x, \quad \dfrac{\partial R}{\partial y}=0$

Substituting the values of these derivatives in equation (2), we get

$$(3xz+2y)(0-0)+x(2x-3x)+x^2(2-1)=0$$

or $$0-x^2+x^2=0$$

Since condition (2) is satisfied, therefore the given equation is integrable.

(ii) $y\,dx+dy+dz=0$

Comparing this equation with the general form (1), we get

$$P=y, \quad Q=1, \quad R=1$$

Then $$\frac{\partial P}{\partial y}=1, \quad \frac{\partial P}{\partial z}=0, \quad \frac{\partial Q}{\partial x}=0, \quad \frac{\partial Q}{\partial z}=0, \quad \frac{\partial R}{\partial x}=0, \quad \frac{\partial R}{\partial y}=0$$

Substituting the values of these derivatives in equation (2), we get

$$y(0-0)+1(0-0)+1(1-0)=0+0+1\neq 0$$

Since condition (2) is not satisfied, therefore the given equation is not integrable.

7.7 SOLUTION OF INTEGRABLE TOTAL DIFFERENTIAL EQUATION

Consider the integrable total differential equation

$$P\,dx+Q\,dy+R\,dz=0 \qquad (1)$$

SOLUTION BY REARRANGING TERMS

Sometimes, rearranging the terms of the given differential equation and / or by dividing by a suitable function of x, y, z, the equation thus obtained will contain several parts which are exact differentials. The following list will help to re-write the given differential equation. Each formula of the list is quite general, i.e., we can replace x by y (or z) and so on as may be necessary in a particular problem :

(i) $x\,dy+y\,dx=d(xy)$

(ii) $2(x\,dx+y\,dy)=d(x^2+y^2)$

(iii) $2xy\,dy+y^2\,dx=d(xy^2)$

(iv) $\dfrac{x\,dy-y\,dx}{x^2}=d\left(\dfrac{y}{x}\right)$

(v) $-\dfrac{x\,dy+y\,dx}{x^2y^2}=d\left(\dfrac{1}{xy}\right)$

(vi) $\dfrac{x\,dy-y\,dx}{xy}=d\left(\ln\dfrac{y}{x}\right)$

(vii) $\dfrac{x\,dy+y\,dx}{xy}=d\left[\ln(xy)\right]$

(viii) $\dfrac{x\,dx+y\,dy}{x^2+y^2}=d\left[\dfrac{1}{2}\ln(x^2+y^2)\right]$

(ix) $\dfrac{x\,dy-y\,dx}{x^2+y^2}=d\left(\tan^{-1}\dfrac{y}{x}\right)$

(x) $\dfrac{2x^2y\,dy-2xy^2\,dx}{x^4}=d\left(\dfrac{y^2}{x^2}\right)$

(xi) $\dfrac{2xy\,dy-y^2\,dx}{x^2}=d\left(\dfrac{y^2}{x}\right)$

(xii) $d(xyz)=xy\,dz+xz\,dy+yz\,dx$

(xiii) $2(x\,dx+y\,dy+z\,dz)=d(x^2+y^2+z^2)$

(xiv) $3x^2y\,dx+x^3\,dy=d(x^3y)$

EXAMPLE (6): **Solve the following total differential equations :**

(i) $(yz+2x)\,dx+(zx-2z)\,dy+(xy-2y)\,dz=0$

(ii) $(e^x y+\cos x)\,dx+(e^x+e^y z)\,dy+e^y\,dz=0$

SOLUTION: (i) $(yz+2x)\,dx+(zx-2z)\,dy+(xy-2y)\,dz=0$

Comparing this equation with $P\,dx+Q\,dy+R\,dz=0$, we have

$$P=yz+2x,\quad Q=zx-2z,\quad R=xy-2y$$

Now $\dfrac{\partial P}{\partial y}=z,\quad \dfrac{\partial P}{\partial z}=y,\quad \dfrac{\partial Q}{\partial x}=z,\quad \dfrac{\partial Q}{\partial z}=x-2,\quad \dfrac{\partial R}{\partial x}=y,\quad \dfrac{\partial R}{\partial y}=x-2$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

or $(yz+2x)(x-2-x+2)+(zx-2z)(y-y)+(xy-2y)(z-z)=0$

or $(yz+2x)(0)+(zx-2z)(0)+(xy-2y)(0)=0$

or $0=0$

which shows that the given equation is integrable.

Rearranging the terms, we get

$$(yz\,dx+zx\,dy+xy\,dz)+2x\,dx-2(z\,dy+y\,dz)=0$$

or $d(xyz)+d(x^2)-2\,d(yz)=0$

Integrating, we get

$$xyz+x^2-2yz=C$$

(ii) $(e^x y+\cos x)\,dx+(e^x+e^y z)\,dy+e^y\,dz=0$

Comparing this equation with $P\,dx+Q\,dy+R\,dz=0$, we have

$$P=e^x y+\cos x,\quad Q=e^x+e^y z,\quad R=e^y$$

Now $\dfrac{\partial P}{\partial y}=e^x,\quad \dfrac{\partial P}{\partial z}=0,\quad \dfrac{\partial Q}{\partial x}=e^x,\quad \dfrac{\partial Q}{\partial z}=e^y,\quad \dfrac{\partial R}{\partial x}=0,\quad \dfrac{\partial R}{\partial y}=e^y$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

or $(e^x y+\cos x)(e^y-e^y)+(e^x+e^y z)(0-0)+e^z(e^x-e^x)=0$

or $0=0$

which shows that the given equation is integrable.

Rearranging the terms, we have

$$(e^x y\,dx+e^x\,dy)+\cos x\,dx+(e^y z\,dy+e^y\,dz)=0$$

or $d(ye^x)+d(\sin x)+d(ze^y)=0$

Integrating, we get

$$ye^x+\sin x+ze^y=C$$

7.8 METHOD OF SOLUTION TREATING ONE VARIABLE AS CONSTANT

Consider the total differential equation

$$P\,dx+Q\,dy+R\,dz=0 \qquad (1)$$

STEP (1): Verify that the condition of integrability is satisfied.

STEP (2): Treat one of the variable, say z, as a constant so that $dz=0$. Then equation (1) reduces to

$$P\,dx+Q\,dy=0 \qquad (2)$$

We should select a proper variable to be constant so that the resulting equation in the remaining variables is easily integrable. Thus this selection will vary from problem to problem.

STEP (3): Let the solution of equation (2) be $u(x,y)=\phi(z)$ (3)

where $\phi(z)$ is an arbitrary constant of integration. Remember that in place of taking an absolute constant C, we have taken $\phi(z)$. This is possible because the arbitrary function $\phi(z)$ is constant w.r.t. x and y. This is in keeping with our starting assumption, namely $z=$ constant.

STEP (4): Take the total differential of equation (3) w.r.t. all the variables. Then compare the coefficients of its differentials with those of the given differential equation (1). After comparison, we get an equation in two variables ϕ and z. If the coefficients of $d\phi$ or dz involve functions of x and y, it will always be possible to eliminate them by using equation (3).

STEP (5): Solve the resulting equation in step (4) and find the function $\phi(z)$. Putting this value of ϕ in equation (3), we shall get the required solution of the given differential equation.

EXAMPLE (7): **Solve the total differential equation :**

$$yz\,dx+2zx\,dy-3xy\,dz=0$$

SOLUTION: Comparing the given equation with $P\,dx+Q\,dy+R\,dz=0$, we have

$$P=yz, \quad Q=2zx, \quad R=-3xy$$

Now $\frac{\partial P}{\partial y}=z, \quad \frac{\partial P}{\partial z}=y, \quad \frac{\partial Q}{\partial x}=2z, \quad \frac{\partial Q}{\partial z}=2x, \quad \frac{\partial R}{\partial x}=-3y, \quad \frac{\partial R}{\partial y}=-3x$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

or

$$yz(2x+3x)+2zx(-3y-y)-3xy(z-2z)=0$$

or $\quad 5xyz - 8xyz + 3xyz = 0$

or $\quad 0 = 0$

which shows that the given equation is integrable .

Considering z as a constant so that $dz = 0$. The given equation reduces to

$$yz\,dx + 2zx\,dy = 0$$

or $\quad y\,dx + 2x\,dy = 0$

or $\quad \frac{dx}{x} + 2\frac{dy}{y} = 0$

Integrating , we get $\ln x + 2\ln y = \ln K$

or $\quad xy^2 = K \qquad (1)$

As this solution is obtained by supposing the variable z to be constant , it is expected that the solution of the original equation can be obtained by replacing the constant K by some function of z say $K = \phi(z)$, giving $\quad xy^2 = \phi(z) \qquad (2)$

Differentiating this equation w.r.t. all the variables , we get

$$y^2\,dx + 2xy\,dy - \frac{d\phi}{dz}dz = 0$$

This is identical with the given equation if

$$\frac{y^2}{yz} = \frac{2xy}{2zx} = \frac{-d\phi/dz}{-3xy}$$

or $\quad \frac{y}{z} = \frac{d\phi/dz}{3xy}$

or $\quad \frac{d\phi}{dz} = \frac{3xy^2}{z} = \frac{3\phi(z)}{z}$

or $\quad \frac{d\phi}{\phi(z)} = \frac{3\,dz}{z}$

Integrating , we get

$$\ln\phi(z) = 3\ln z + \ln C$$

or $\quad \phi(z) = Cz^3 \qquad (3)$

From equations (2) and (3) , we get the solution as

$$xy^2 = Cz^3$$

7.9 CONDITION FOR EXACTNESS OF $P\,dx + Q\,dy + R\,dz = 0$

The total differential equation is said to be exact if the following three conditions are satisfied

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad (1)$$

Note that when the conditions in equation (1) are satisfied , the condition for integrability of the differential equation $P\,dx + Q\,dy + R\,dz = 0$, namely .

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \quad (2)$$

is also satisfied, since each term of equation (2) vanishes identically. If equation (1) is exact, the solution can be obtained by rearranging the terms.

EXAMPLE (8): **Solve the differential equation :**

$$(x-y)\,dx - x\,dy + z\,dz = 0$$

SOLUTION: Comparing with the general form (1), we have

$$P = x-y, \quad Q = -x, \quad R = z$$

and $\frac{\partial P}{\partial y} = -1, \quad \frac{\partial P}{\partial z} = 0, \quad \frac{\partial Q}{\partial x} = -1, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = 0, \quad \frac{\partial R}{\partial y} = 0$

Since $\frac{\partial P}{\partial y} = -1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = 0 = \frac{\partial P}{\partial z}$, therefore the equation is exact.

Consequently, it is also integrable.

Rearranging terms of the differential equation, we get

$$x\,dx - (x\,dy + y\,dx) + z\,dz = 0$$

or $$x\,dx - d(xy) + z\,dz = 0$$

Integrating, we have

$$\frac{1}{2}x^2 - xy + \frac{1}{2}z^2 = C_1$$

or $$x^2 - 2xy + z^2 = C \quad (\text{where } C = 2C_1)$$

7.10 HOMOGENEOUS TOTAL DIFFERENTIAL EQUATION

The total differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \quad (1)$$

is called homogeneous if P, Q, and R are homogeneous functions of x, y, z of the same degree.

For example, the differential equations.

$$(2xz - yz)\,dx + (2yz - zx)\,dy - (x^2 - xy + y^2)\,dz = 0$$

$$2(y+z)\,dx - (x+z)\,dy + (2y - x + z)\,dz = 0$$

are the homogeneous differential equations.

7.11 SOLUTION OF HOMOGENEOUS DIFFERENTIAL EQUATION

There are two methods to solve such equations.

METHOD (1): SOLUTION BY USE OF INTEGRATING FACTOR

STEP (1): Verify that the given equation is integrable.

STEP (2): If $Px + Qy + Rz \neq 0$, then $\frac{1}{Px+Qy+Rz}$ is the integrating factor of the given differential equation.

STEP (3): Multiply the given equation by the integrating factor. Then regrouping the terms and integrating, the required solution can be obtained.

EXAMPLE (9): **Solve the homogeneous total differential equation :**

$$yz\,dx - z^2\,dy - xy\,dz = 0$$

SOLUTION: Comparing the given equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$$P = yz, \quad Q = -z^2, \quad R = -xy$$

Now $\frac{\partial P}{\partial y} = z, \quad \frac{\partial P}{\partial z} = y, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = -2z, \quad \frac{\partial R}{\partial x} = -y, \quad \frac{\partial R}{\partial y} = -x$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

or $\quad yz(-2z+x) - z^2(-2y) - xy(z) = 0$

or $\quad -2yz^2 + xyz + 2yz^2 - xyz = 0$

or $\quad 0 = 0$

which shows that the given equation is integrable.

Also, the given equation is homogeneous of degree 2, and

$$Px + Qy + Rz = xyz - yz^2 - xyz = -yz^2 \neq 0.$$

Thus $\frac{1}{Px+Qy+Rz} = -\frac{1}{yz^2}$ is an integrating factor.

Multiplying the given equation by $-\frac{1}{yz^2}$, we get

$$-\frac{1}{z}dx + \frac{1}{y}dy + \frac{x}{z^2}dz = 0$$

or $\quad \frac{dy}{y} + \frac{x\,dz - z\,dx}{z^2} = 0$

or $\quad \frac{dy}{y} - \frac{z\,dx - x\,dz}{z^2} = 0$

or $\quad \frac{dy}{y} - d\left(\frac{x}{z}\right) = 0$

Integrating, we get

$$\ln y - \frac{x}{z} = \ln C$$

or $\quad \ln y = \frac{x}{z} + \ln C$

or $\quad y = C e^{x/z}$

METHOD (2):

If $Px+Qy+Rz=0$, then the above method fails. In such cases, we apply the following method which is applicable to all homogeneous equations.

STEP (1): Verify that the given equation is integrable.

STEP (2): Put $x=uz$, $y=vz$ so that $dx=u\,dz+z\,du$ and $dy=v\,dz+z\,dv$.

In this way, one variable, say z, can be separated from the others. Then regrouping the terms and integrating the required solution can be obtained.

Finally, u and v are replaced by $\frac{x}{z}$ and $\frac{y}{z}$ respectively so as to get the required solution in x, y, and z.

EXAMPLE (10): Solve the homogeneous total differential equation :

(i) $z^2\,dx+(z^2-2yz)\,dy+(2y^2-yz-xz)\,dz=0$

(ii) $2(y+z)\,dx-(x+z)\,dy+(2y-x+z)\,dz=0$

SOLUTION: (i) $z^2\,dx+(z^2-2yz)\,dy+(2y^2-yz-xz)\,dz=0$

Comparing this equation with $P\,dx+Q\,dy+R\,dz=0$, we have

$$P=z^2,\quad Q=z^2-2yz,\quad R=2y^2-yz-xz$$

Now $\frac{\partial P}{\partial y}=0,\quad \frac{\partial P}{\partial z}=2z,\quad \frac{\partial Q}{\partial x}=0,\quad \frac{\partial Q}{\partial z}=2z-2y,\quad \frac{\partial R}{\partial x}=-z,\quad \frac{\partial R}{\partial y}=4y-z$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

$$z^2(2z-2y-4x+z)+(z^2-2yz)(-z-2z)+(2y^2-yz-xz)(0-0)=0$$

or $$z^2(3z-6y)+(z^2-2yz)(-3z)=0$$

or $$3z^3-6yz^2-3z^3+6yz^2=0$$

or $$0=0$$

which shows that the given equation is integrable.

Also, the given equation is homogeneous of degree 2, and

$$Px+Qy+Rz=z^2x+z^2y-2y^2z+2y^2z-yz^2-xz^2=0$$

Therefore, let $x=uz$, $y=vz$. Then $dx=u\,dz+z\,du$, $dy=v\,dz+z\,dv$.

Substituting in the given differential equation, we get

$$z^2(u\,dz+z\,du)+(z^2-2vz^2)(v\,dz+z\,dv)+(2v^2z^2-vz^2-uz^2)\,dz=0$$

or $$z^3\,du+z^3(1-2v)\,dv+(uz^2+vz^2-2v^2z^2+2v^2z^2-vz^2-uz^2)\,dz=0$$

or $$z^3\,du+z^3(1-2v)\,dv=0$$

or $$du+(1-2v)\,dv=0$$

Integrating, we get

$$u+v-v^2 = C$$

Replacing u by $\frac{x}{z}$ and v by $\frac{y}{z}$, we get

$$\frac{x}{z}+\frac{y}{z}-\frac{y^2}{z^2} = C$$

or $$xz+yz-y^2 = Cz^2$$

or $$(x+y)z-y^2 = Cz^2$$

(ii) $$2(y+z)dx-(x+z)dy+(2y-x+z)dz = 0$$

Comparing this equation with $P\,dx+Q\,dy+R\,dz = 0$, we have

$$P = 2(y+z), \quad Q = -(x+z), \quad R = 2y-x+z$$

and $$\frac{\partial P}{\partial y} = 2, \quad \frac{\partial P}{\partial z} = 2, \quad \frac{\partial Q}{\partial x} = -1, \quad \frac{\partial Q}{\partial z} = -1, \quad \frac{\partial R}{\partial x} = -1, \quad \frac{\partial R}{\partial y} = 2$$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right) = 0$$

$$2(y+z)(-1-2)-(x+z)(-1-2)+(2y-x+z)(2+1) = 0$$

$$= -6(y+z)+3(x+z)+3(2y-x+z) = 0$$

or $$0 = 0$$

which shows that the given equation is integrable.

Also, the given equation is homogeneous of degree 1. Let $x = uz$, $y = vz$.

Then $dx = u\,dz+z\,du$ and $dy = v\,dz+z\,dv$.

Substituting in the given differential equation, we get

$$2(vz+z)(u\,dz+z\,du)-(uz+z)(v\,dz+z\,dv)+(2vz-uz+z)dz = 0$$

or $$2z(v+1)(u\,dz+z\,du)-z(u+1)(v\,dz+z\,dv)+z(2v-u+1)dz = 0$$

or $$2(v+1)(u\,dz+z\,du)-(u+1)(v\,dz+z\,dv)+(2v-u+1)dz = 0$$

Rearranging, we get

$$2z(v+1)du-z(u+1)dv+(uv+u+v+1)dz = 0$$

or $$2z(v+1)du-z(u+1)dv+(u+1)(v+1)dz = 0$$

Dividing by $z(u+1)(v+1)$, we get

$$\frac{2}{u+1}du-\frac{1}{v+1}dv+\frac{1}{z}dz = 0$$

Integrating, we get

$$2\ln(u+1)-\ln(v+1)+\ln z = \ln C$$

$$\ln \frac{z(u+1)^2}{(v+1)} = \ln C$$

or
$$z(u+1)^2 = C(v+1)$$

or

Replacing u by $\frac{x}{z}$ and v by $\frac{y}{z}$, we get

$$z\left(\frac{x+z}{z}\right)^2 = C\left(\frac{y+z}{z}\right)$$

or
$$(x+z)^2 = C(y+z)$$

7.12 EQUATION EXACT AND HOMOGENEOUS

THEOREM (7.1): **Prove that $xP + yQ + zR = C$ is the solution of $P\,dx + Q\,dy + R\,dz = 0$, when the equation is exact and homogeneous of degree $n \neq -1$.**

SOLUTION: Given solution is

$$xP + yQ + zR = C$$

Differentiating this equation, we get

$$\left(P + x\frac{\partial P}{\partial x} + y\frac{\partial Q}{\partial x} + z\frac{\partial R}{\partial x}\right)dx + \left(x\frac{\partial P}{\partial y} + Q + y\frac{\partial Q}{\partial y} + z\frac{\partial R}{\partial y}\right)dy$$
$$+\left(x\frac{\partial P}{\partial z} + y\frac{\partial Q}{\partial z} + R + z\frac{\partial R}{\partial z}\right)dz = 0 \qquad (1)$$

Since $P\,dx + Q\,dy + R\,dz = 0$ is exact, therefore

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

Substituting these in equation (1), we get

$$\left(P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z}\right)dx + \left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right)dy$$
$$+\left(R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right)dz = 0 \qquad (2)$$

Since the given equation is homogeneous of degree n, it follows that P, Q, and R are all homogeneous function of degree n. Using Euler's theorem on homogeneous function

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = nP$$

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z} = nQ$$

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} = nR$$

Using these relations, equation (2) becomes

$$(P+nP)dx+(Q+nQ)dy+(R+nR)dz=0$$

$$(n+1)(Pdx+Qdy+Rdz)=0$$

Since $n \neq -1$, therefore $Pdx+Qdy+Rdz=0$.

This complete the proof of the theorem.

EXAMPLE (11): **Solve the differential equation :**

$$(x-3y-z)dz+(2y-3x)dy+(z-x)dz=0$$

SOLUTION: Comparing the given differential equation with $Pdx+Qdy+Rdz=0$, we have

$$P=x-3y-z, \quad Q=2y-3x, \quad R=z-x$$

The given differential equation in homogeneous of degree $n=1 \neq -1$.

Also, $\frac{\partial P}{\partial y}=-3=\frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z}=0=\frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x}=-1=\frac{\partial P}{\partial z}$

Therefore, the given differential equation is exact. Consequently, it is also integrable.

Hence the solution of the differential equation is

$$xP+yQ+zR=0$$

or $$x(x-3y-z)+y(2y-3x)+z(z-x)=0$$

or $$x^2+2y^2+z^2-6xy-2xz=C$$

where C is an arbitrary constant.

7.13 SYSTEM OF TOTAL DIFFERENTIAL EQUATIONS

The differential equations

$$P_1dx+Q_1dy+R_1dz=0 \tag{1}$$

$$P_2dx+Q_2dy+R_2dz=0 \tag{2}$$

is called a system of total differential equations in three variables x, y, and z.

CASE (1)

If both equations (1) and (2) are integrable and have solutions

$$u_1(x,y,z)=C_1 \text{ and } u_2(x,y,z)=C_2$$

respectively, then these solutions taken together form the complete solution of the given system.

EXAMPLE (12): **Solve the system of total differential equations :**

$$(y+z)dx+(z+x)dy+(x+y)dz=0 \tag{1}$$

$$(x+z)dx+ydy+xdz=0 \tag{2}$$

SOLUTION: We know that the condition of integrability of the total differential equation

$$Pdx+Qdy+Rdz=0 \tag{3}$$

is
$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \quad (4)$$

Comparing equation (1) with the general form (3), we have

$$P = y+z, \quad Q = z+x, \quad R = x+y$$

and
$$\frac{\partial P}{\partial y}=1, \quad \frac{\partial P}{\partial z}=1, \quad \frac{\partial Q}{\partial x}=1, \quad \frac{\partial Q}{\partial z}=1, \quad \frac{\partial R}{\partial x}=1, \quad \frac{\partial R}{\partial y}=1$$

Substituting these derivatives in equation (4), we get

$$(y+z)(1-1)+(z+x)(1-1)+(x+y)(1-1)=0$$
$$(y+z)(0)+(z+x)(0)+(x+y)(0)=0$$

Thus equation (1) is integrable.

Next, comparing equation (2) with the general form (3), we have

$$P = x+z, \quad Q = y, \quad R = x$$

and
$$\frac{\partial P}{\partial y}=0, \quad \frac{\partial P}{\partial z}=1, \quad \frac{\partial Q}{\partial x}=0, \quad \frac{\partial Q}{\partial z}=0, \quad \frac{\partial R}{\partial x}=1, \quad \frac{\partial R}{\partial y}=0$$

Substituting these derivatives in equation (4), we get

$$(x+z)(0-0)+y(1-1)+x(0-0)=0$$
$$(x+z)(0)+y(0)+x(0)=0$$

Therefore, equation (2) is integrable.

The first equation is also exact, since

$$\frac{\partial P}{\partial y}=1=\frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z}=1=\frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x}=1=\frac{\partial P}{\partial z}$$

Therefore, equation (1) may be written as

$$(y\,dx+x\,dy)+(z\,dy+y\,dz)+(x\,dz+z\,dx)=0$$

whose solution is

$$xy+yz+zx=C_1 \quad (5)$$

The second equation is also exact, since

$$\frac{\partial P}{\partial y}=0=\frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z}=0=\frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x}=1=\frac{\partial P}{\partial z}$$

Therefore, equation (2) may be written as

$$x\,dx+y\,dy+(z\,dx+x\,dz)=0$$

or
$$\frac{1}{2}d(x^2+y^2)+d(xz)=0$$

whose solution is

$$x^2+y^2+2xz=C_2 \quad (6)$$

Equations (5) and (6) constitute the general solution.

CASE (2)

Consider $$P_1\,dx + Q_1\,dy + R_1\,dz = 0 \tag{1}$$

$$P_2\,dx + Q_2\,dy + R_2\,dz = 0 \tag{2}$$

If one equation is integrable [say equation (1)], but equation (2) is not integrable, then let

$$u_1(x, y, z) = C_1 \tag{3}$$

be the solution of equation (1). To obtain the solution of equation (2), we use equations (1), (2), and (3) to eliminate one variable and its differential, and integrate the resulting equation.

EXAMPLE (13): **Solve the following systems of total differential equations :**

$$\mathbf{dx + 2\,dy - (x + 2y)\,dz = 0} \tag{1}$$

$$\mathbf{2\,dx + dy + (x - y)\,dz = 0} \tag{2}$$

SOLUTION: We know that the condition of integrability of the total differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \tag{3}$$

is $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \tag{4}$$

Comparing equation (1) with the general form (3), we have

$$P = 1, \quad Q = 2, \quad R = -(x + 2y)$$

and $$\frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = 0, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = -1, \quad \frac{\partial R}{\partial y} = -2$$

Substituting these derivatives in equation (4), we get

$$1(0 + 2) + 2(-1 - 0) - (x + 2y)(0 - 0) = 0$$

or $$0 = 0$$

Thus equation (1) is integrable.

Next, comparing equation (2) with the general form (3), we have

$$P = 2, \quad Q = 1, \quad R = x - y$$

and $$\frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = 0, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = 1, \quad \frac{\partial R}{\partial y} = -1$$

Substituting these derivatives in equation (4), we get

$$2(0 + 1) + 1(1 - 0) + (x - y)(0 - 0) = 2 + 1 = 3 \neq 0$$

Thus equation (2) is not integrable.

Equation (1) can be written as

$$\frac{dx + 2\,dy}{x + 2y} - dz = 0$$

Integrating, we get

$$\ln(x + 2y) - z = \ln C_1$$

or $$x + 2y = C_1 e^z \tag{5}$$

Next, multiplying equation (2) by 2 and subtracting it from equation (1), we get

$$-3\,dx - 3x\,dz = 0$$

or

$$dx + x\,dz = 0$$

or

$$\frac{dx}{x} + dz = 0$$

Integrating, we get

$$\ln x + z = \ln C_2$$

or

$$x = C_2 e^{-z} \qquad (6)$$

Equations (5) and (6) constitute the complete solution of the given system.

CASE (3)

If both equations of the system are not integrable, then write them in simultaneous form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

and solve them by the methods already discussed.

NOTE: The system of total differential equations can be written in the form

$$P_1\frac{dx}{dz} + Q_1\frac{dy}{dz} + R_1 = 0$$

$$P_2\frac{dx}{dz} + Q_2\frac{dy}{dz} + R_2 = 0$$

Solving these equations for $\frac{dx}{dz}$ and $\frac{dy}{dz}$, we get

$$\frac{dx/dz}{Q_1R_2 - Q_2R_1} = \frac{dy/dz}{R_1P_2 - R_2P_1} = \frac{1}{P_1Q_2 - P_2Q_1}$$

or

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

which is of the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where P, Q, and R are functions of x, y, and z. Thus the system of total differential equations can always be put in the form of simultaneous differential equations.

7.14 GEOMETRICAL INTERPRETATION OF $P\,dx + Q\,dy + R\,dz = 0$

The given differential equation expresses that the tangent to a curve is perpendicular to a certain line, the direction cosines of this tangent line and another line being proportional to (dx, dy, dz) and (P, Q, R) respectively.

Suppose that the total differential equation .

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

is integrable and that its solution is

$$f(x, y, z) = C \qquad (2)$$

Since equation (2) has one arbitrary constant , it represents an infinite number of surfaces .

7.15 ORTHOGONALITY OF THE FAMILY OF INTEGRAL SURFACES OF $P\,dx + Q\,dy + R\,dz = 0$ AND THE FAMILY OF INTEGRAL CURVES OF $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

We know that the total differential equation

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

means , geometrically , that a straight line whose direction cosines are proportional to (dx, dy, dz) is perpendicular to a line whose direction cosines are proportional to (P, Q, R). As a result , a point which satisfies equation (1) must move in a direction at right angles to a line whose direction cosines are proportional to P, Q, R.

On the other hand , the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (2)$$

means , geometrically , that a straight line whose direction cosines are proportional to dx, dy, dz is parallel to a line whose direction cosines are proportional to P, Q, R. As a result , a point which satisfies equation (2) must move in a direction parallel to the line whose direction cosines are proportional to P, Q, R.

From the above discussion , it follows that the curves traced out by the points that are moving according to the condition (1) are orthogonal to the curves traced out by the points that are moving according to the conditions (2) . The former curves are any of the curves upon the surfaces given by equation (1) . Thus , geometrically , the curves represented by equations (2) are orthogonal to the surfaces represented by equation (1) .

In case equation (1) is not integrable , there cannot exist a family of surfaces which is orthogonal to the curves given by equations (2) .

EXAMPLE (14): **Show that the locus of $dx + dy + dz = 0$ is orthogonal to the locus of**

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}.$$

SOLUTION: The total differential equation $dx + dy + dz = 0$ has the solution

$$x + y + z = C \qquad (1)$$

Since C is an arbitrary constant , equation (1) represents a family of parallel planes .

The simultaneous equations

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$$

has solutions $x = z + C_1$ and $y = z + C_2$.

These are the equations of two planes intersecting in the line

$$\frac{x - C_1}{1} = \frac{y - C_2}{1} = \frac{z}{1} \tag{2}$$

The planes given by equation (1) are orthogonal to the lines given by equations (2) .

EXAMPLE (15): **Show that the locus of $z\,dx - x\,dz = 0$ is orthogonal to the locus of**

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

SOLUTION: The equation $z\,dx - x\,dz = 0$ can be written as $\frac{dx}{x} = \frac{dz}{z}$

whose solution is $x = C_1 z$ (1)

Equation (1) represents a family of planes passing through the axis of y .

The simultaneous equations

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

have solutions $x^2 + z^2 = C_1$ (2)

and $y = C_2$ (3)

Equation (2) represents a right circular cylinder and equation (3) represents a plane that cuts the cylinder in a circle . The differential equations therefore represents a system of circles , whose centres all lie on the y - axis and whose planes are all perpendicular to this axis . Thus the planes determined by equation (1) are orthogonal to the circles determined by equations (2) and (3) .

7.16 SOLVED PROBLEMS

PROBLEM (1): **Solve the following differential equations :**

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

SOLUTION: Taking the first two members , we get

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

or $\quad x^2\,dx = y^2\,dz$

Integrating , we get

$$\frac{x^3}{3} = \frac{y^3}{3} + C$$

or $\quad x^3 - y^3 = C_1 \quad (\text{where } C_1 = 3C) \qquad (1)$

Taking the first and last members , we get

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

or $\quad x\,dx = z\,dz$

Integrating , we get

$$\frac{x^2}{2} = \frac{z^2}{2} + C$$

or $\quad x^2 - z^2 = C_2 \quad (\text{where } C_2 = 2C) \qquad (2)$

Equations (1) and (2) constitute the complete solution .

SECOND INTEGRAL FOUND WITH THE HELP OF THE FIRST

PROBLEM (2): **Solve the differential equations :**

(i) $$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

(ii) $$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4}$$

SOLUTION: (i) $$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

Taking the first two members , we get

$$\frac{dx}{xy} = \frac{dy}{y^2} \quad \text{or} \quad \frac{dx}{x} = \frac{dy}{y}$$

Integrating , we get

$$\ln x = \ln y + \ln C_1$$

or $\quad x = C_1 y \qquad (1)$

Now, taking the last two members, we have

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

or $$\frac{dy}{y^2} = \frac{dz}{C_1 y^2 z - 2C_1^2 y^2} \quad \text{[using equation (1)]}$$

or $$\frac{dy}{1} = \frac{dz}{C_1 z - 2C_1^2}$$

or $$C_1\, dy = \frac{dz}{z - 2C_1}$$

Integrating, we get

$$C_1 y = \ln(z - 2C_1) + C_2$$

or $$x = \ln\left(z - \frac{2x}{y}\right) + C_2 \qquad (2)$$

Equation (1) and (2) constitute the general solution.

(ii) $$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4}$$

Taking the first two members, we get

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)}$$

or $$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating, we get

$$\ln x = -\ln y + \ln C_1$$

or $$\ln x + \ln y = \ln C_1$$

or $$xy = C_1 \qquad (1)$$

Next, taking the first and third members, we have

$$\frac{dx}{xz(z^2 + xy)} = \frac{dz}{x^4}$$

$$\frac{dx}{xz(z^2 + C_1)} = \frac{dz}{x^4} \quad \text{[using equation (1)]}$$

or $$x^3\, dx = (z^2 + C_1)\, z\, dz$$

Integrating, we get

$$\frac{x^4}{4} = \frac{1}{4}(z^2 + C_1)^2 + C_2$$

or $$x^4 = (z^2 + C_1)^2 + 4C_2$$

or $$x^4 = (z^2 + xy)^2 + 4C_2 \qquad (2)$$

Equation (1) and (2) constitute the complete solution.

METHOD 3: CASE (1)

PROBLEM (3): **Solve the differential equations :**

(i) $$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

(ii) $$\frac{dx}{mz-ny} = \frac{dy}{nx-\ell z} = \frac{dz}{\ell y-mx}$$

(iii) $$\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9z(x^3-y^3)}$$

(iv) $$\frac{dx}{x(2y^4-z^4)} = \frac{dy}{y(z^4-2x^4)} = \frac{dz}{z(x^4-y^4)}$$

SOLUTION: (i) $$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

Taking x, y, z as multipliers, we get

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)-y^2(z^2+x^2)+z^2(x^2+y^2)}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{0}$$

or $$x\,dx+y\,dy+z\,dz = 0$$

Integrating, we get

$$x^2+y^2+z^2 = C_1 \tag{1}$$

Again, taking $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers, we get

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

$$= \frac{\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}}{y^2-z^2+z^2+x^2-x^2-y^2} = \frac{\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}}{0}$$

or $$\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z} = 0$$

Integrating, we get

$$\ln x-\ln y-\ln z = \ln C_2$$

$$\ln\frac{x}{yz} = \ln C_2$$

or $$x = C_2\,yz \tag{2}$$

Equations (1) and (2) constitute the complete solution.

(ii) $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-\ell z} = \dfrac{dz}{\ell y-mx}$

Choosing ℓ, m, n as multipliers, we get

$$\frac{dx}{mz-ny} = \frac{dy}{nx-\ell z} = \frac{dz}{\ell y-mx}$$

$$= \frac{\ell dx+mdy+ndz}{\ell mz-\ell ny+mnx-\ell mz+n\ell y-mnx} = \frac{\ell dx+mdy+ndz}{0}$$

or $\quad \ell dx+mdy+ndz = 0$

or $\quad \ell x+my+nz = C_1 \qquad (1)$

Again, choosing x, y, z as multipliers, we get

$$\frac{dx}{mz-ny} = \frac{dy}{nx-\ell z} = \frac{dz}{\ell y-mx}$$

$$= \frac{xdx+ydy+zdz}{mxz-nxy+nxy-\ell yz+\ell yz-mxz} = \frac{xdx+ydy+zdz}{0}$$

or $\quad xdx+ydy+zdz = 0$

Integrating, we get

$$x^2+y^2+z^2 = C_2 \qquad (2)$$

Equations (1) and (2) constitute the complete solution.

(iii) $\dfrac{dx}{y^3x-2x^4} = \dfrac{dy}{2y^4-x^3y} = \dfrac{dz}{9z(x^3-y^3)}$

Taking $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{3z}$ as multipliers, we get

$$\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9z(x^3-y^3)}$$

$$= \frac{dx/x+dy/y+dz/3z}{(y^3-2x^3)+(2y^3-x^3)+3(x^3-y^3)}$$

$$= \frac{dx/x+dy/y+dz/3z}{0}$$

or $\quad \dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{3z} = 0$

Integrating, we get

$$\ln x+\ln y+\frac{1}{3}\ln z = \ln C_1$$

or $\quad \ln(xyz^{1/3}) = \ln C_1$

or $\quad xyz^{1/3} = C_1 \qquad (1)$

Now from the first two members, we get

$$(2y^4-x^3y)\,dx = (y^3x-2x^4)\,dy$$

Dividing both sides by $x^3 y^3$, we get

$$\left(\frac{2y}{x^3} - \frac{1}{y^2}\right) dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right) dy$$

$$\left(\frac{1}{x^2}\,dy - \frac{2y}{x^3}\,dx\right) + \left(\frac{1}{y^2}\,dx - \frac{2x}{y^3}\,dy\right) = 0$$

$$d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$$

Integrating, we get

$$\frac{y}{x^2} + \frac{x}{y^2} = C_2$$

or $\quad x^3 + y^3 = C_2 x^2 y^2 \qquad (2)$

Equations (1) and (2) constitute the complete solution.

(iv) $$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$ as multipliers, we get

$$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$$

$$= \frac{dx/x + dy/y + 2\,dz/z}{(2y^4 - z^4) + (z^4 - 2x^4) + 2(x^4 - y^4)}$$

$$= \frac{dx/x + dy/y + 2\,dz/z}{0}$$

or $$\frac{dx}{x} + \frac{dy}{y} + 2\frac{dz}{z} = 0$$

Integrating, we get

$$\ln x + \ln y + 2\ln z = \ln C_1$$

or $\quad \ln(xyz^2) = \ln C_1$

or $\quad xyz^2 = C_1 \qquad (1)$

Again choosing x^3, y^3, z^3 as multipliers, we get

$$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)}$$

$$= \frac{x^3\,dx + y^3\,dy + z^3\,dz}{x^4(2y^4 - z^4) + y^4(z^4 - 2x^4) + z^4(x^4 - y^4)}$$

$$= \frac{x^3\,dx + y^3\,dy + z^3\,dz}{0}$$

or $\quad x^3\,dx + y^3\,dy + z^3\,dz = 0$

Integrating, we get

$$x^4 + y^4 + z^4 = C_2 \qquad (2)$$

Equations (1) and (2) constitute the complete solution.

METHOD 3: CASE (2)

PROBLEM (4): **Solve the following differential equations :**

(i) $$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

(ii) $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

(iii) $$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz} = \frac{dz}{z(x+y)}$$

(iv) $$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

SOLUTION: (i) $$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

From the last two members, we get

$$\frac{dy}{-2xy} = \frac{dz}{-2xz}$$

or $$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\ln y = \ln z + \ln C_1$$

or $$y = C_1 z \qquad (1)$$

Next, choosing x, y, z as multipliers, we get

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} = \frac{x\,dx+y\,dy+z\,dz}{x(y^2+z^2-x^2)-2xy^2-2xz^2}$$

or $$\frac{dz}{-2xz} = \frac{x\,dx+y\,dy+z\,dz}{-xy^2-xz^2-x^3}$$

or $$\frac{dz}{z} = \frac{2(x\,dx+y\,dy+z\,dz)}{x^2+y^2+z^2}$$

Integrating, we get

$$\ln z = \ln(x^2+y^2+z^2) + \ln C_2$$

or $$z = C_2(x^2+y^2+z^2) \qquad (2)$$

Equations (1) and (2) constitute the complete solution.

(ii) $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

Choosing $1, -1, 0$ and $0, 1, -1$, and $-1, 0, 1$ as multipliers, we get

$$\text{each fraction} = \frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)} = \frac{dz-dx}{(z^2-x^2)+y(z-x)}$$

or $$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

or $$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

Taking the first two members, we get

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrating, we get

$$\ln(x-y) = \ln(y-z) + \ln C_1$$

or $$\ln\frac{x-y}{y-z} = \ln C_1$$

or $$\frac{x-y}{y-z} = C_1 \qquad (1)$$

Taking the last two members, we get

$$\frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

Integrating, we get

$$\ln(y-z) = \ln(z-x) + \ln C_2$$

or $$\ln\frac{y-z}{z-x} = \ln C_2$$

or $$\frac{y-z}{z-x} = C_2 \qquad (2)$$

Equations (1) and (2) constitute the complete solution.

(iii) $$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz} = \frac{dz}{z(x+y)}$$

Choosing $1, -1, 0$ as multipliers, we get

$$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz} = \frac{dz}{z(x+y)} = \frac{dx-dy}{x^2+y^2+yz-x^2-y^2+xz}$$

$$= \frac{dx-dy}{z(x+y)} \qquad (1)$$

Now $$\frac{dz}{z(x+y)} = \frac{dx-dy}{z(x+y)}$$

or $$dz-dx+dy = 0$$

or $$d(z-x+y) = 0$$

Integrating, we get

$$z-x+y = C_1 \qquad (1)$$

Next, taking $x, y, 0$ as multipliers, we get

$$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-xz} = \frac{dz}{z(x+y)} = \frac{x\,dx+y\,dy}{x(x^2+y^2+yz)+y(x^2+y^2-xz)}$$

$$= \frac{x\,dx+y\,dy}{x(x^2+y^2)+y(x^2+y^2)}$$

$$= \frac{x\,dx+y\,dy}{(x+y)(x^2+y^2)}$$

or $$\frac{dz}{z(x+y)} = \frac{x\,dx+y\,dy}{(x+y)(x^2+y^2)}$$

or $$\frac{dz}{z} = \frac{x\,dx+y\,dy}{x^2+y^2}$$

or $$\frac{2\,dz}{z} = \frac{2x\,dx+2y\,dy}{x^2+y^2}$$

or $$2\ln z = \ln(x^2+y^2)+\ln C_2$$

or $$z^2 = C_2(x^2+y^2) \qquad (2)$$

Equations (1) and (2) constitute the complete solution.

(iv) $$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

From the first two members, we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating, we get

$$\ln x = -\ln y+\ln C_1$$

or $$xy = C_1 \qquad (1)$$

Choosing $1, 1, 0$ as multipliers, we get

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)} = \frac{dx+dy}{x^2-y^2}$$

$$= \frac{dx+dy}{(x-y)(x+y)} \qquad (2)$$

Again choosing $1, 1, 1$ as multipliers, we get

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}$$

$$= \frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$= \frac{dx+dy+dz}{(x^2-y^2)-(x-y)(2x+2y+z)}$$

$$= \frac{dx+dy+dz}{(x-y)\left[x+y-2x-2y-z\right]}$$

$$= \frac{dx+dy+dz}{(x-y)(-x-y-z)}$$

$$= -\frac{dx+dy+dz}{(x-y)(x+y+z)} \tag{3}$$

From equations (2) and (3), we get

$$\frac{dx+dy}{(x-y)(x+y)} = -\frac{dx+dy+dz}{(x-y)(x+y+z)}$$

or $$\frac{dx+dy}{x+y} + \frac{dx+dy+dz}{x+y+z} = 0$$

Integrating, we get

$$\ln(x+y) + \ln(x+y+z) = \ln C_2$$

or $$(x+y)(x+y+z) = C_2 \tag{4}$$

Equations (1) and (4) constitute the complete solution.

TOTAL DIFFERENTIAL EQUATIONS

PROBLEM (5): **Solve the following total differential equations :**

(i) $x(y^2-a^2)\,dx + y(x^2-z^2)\,dy - z(y^2-a^2)\,dz = 0$

(ii) $(y^2+z^2-x^2)\,dx - 2xy\,dy - 2xz\,dz = 0$

SOLUTION: (i) $x(y^2-a^2)\,dx + y(x^2-z^2)\,dy - z(y^2-a^2)\,dz = 0$

Comparing this equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$$P = x(y^2-a^2), \quad Q = y(x^2-z^2), \quad R = -z(y^2-a^2)$$

Now $\dfrac{\partial P}{\partial y} = 2xy, \quad \dfrac{\partial P}{\partial z} = 0, \quad \dfrac{\partial Q}{\partial x} = 2xy, \quad \dfrac{\partial Q}{\partial z} = -2yz, \quad \dfrac{\partial R}{\partial x} = 0, \quad \dfrac{\partial R}{\partial y} = -2yz$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

or $$x(y^2-a^2)(-2yz+2yz) + y(x^2-z^2)(0-0) - z(y^2-a^2)(2xy-2xy) = 0$$

or $$x(y^2-a^2)(0) + y(x^2-z^2)(0) - z(y^2-a^2)(0) = 0$$

or $$0 = 0$$

which shows that the given equation is integrable.

Dividing each term of the given equation by $(y^2-a^2)(x^2-z^2)$, we get

$$\frac{x\,dx}{x^2-z^2} + \frac{y}{y^2-a^2}\,dy - \frac{z}{(x^2-z^2)}\,dz = 0$$

$$\frac{x\,dx - z\,dz}{x^2 - z^2} + \frac{y}{y^2 - a^2}dy = 0$$

or

$$\frac{2x\,dx - 2z\,dz}{x^2 - z^2} + \frac{2y}{y^2 - a^2} = 0$$

or

Integrating, we get

$$\ln(x^2 - z^2) + \ln(y^2 - a^2) = \ln C$$

or $$\ln[(x^2 - z^2)(y^2 - a^2)] = \ln C$$

or $$(x^2 - z^2)(y^2 - a^2) = C$$

(ii) $$(y^2 + z^2 - x^2)\,dx - 2xy\,dy - 2xz\,dz = 0$$

Comparing this equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$$P = y^2 + z^2 - x^2, \quad Q = -2xy, \quad R = -2xz$$

Now $$\frac{\partial P}{\partial y} = 2y, \quad \frac{\partial P}{\partial z} = 2z, \quad \frac{\partial Q}{\partial x} = -2y, \quad \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial R}{\partial x} = -2z, \quad \frac{\partial R}{\partial y} = 0$$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

or $$(y^2 + z^2 - x^2)(0 - 0) - 2xy(-2z - 2z) - 2xz(2y + 2y) = 0$$

or $$0 + 8xyz - 8xyz = 0$$

or $$0 = 0$$

which shows that the given equation is integrable.

The given equation can be written as

$$2xy\,dy + 2xz\,dz - (y^2 + z^2 - x^2)\,dx = 0$$

Dividing throughout by x^2, we get

$$\frac{2xy\,dy + 2xz\,dz - (y^2 + z^2)\,dx}{x^2} + dx = 0$$

or $$\frac{x(2y\,dy + 2z\,dz) - (y^2 + z^2)\,dx}{x^2} = -dx$$

or $$d\left(\frac{y^2 + z^2}{x}\right) = -dx$$

Integrating both sides, we get

$$\frac{y^2 + z^2}{x} = -x + C$$

or $$y^2 + z^2 + x^2 = Cx$$

SOLUTIONS TREATING ONE VARIABLE AS CONSTANT

PROBLEM (6): **Solve the total differential equation :**

$$(2x^2+2xy+2xz^2+1)\,dx+dy+2z\,dz=0$$

SOLUTION: Comparing the given equation with $P\,dx+Q\,dy+R\,dz=0$, we have

$$P=2x^2+2xy+2xz^2+1, \quad Q=1, \quad R=2z$$

Now $\dfrac{\partial P}{\partial y}=2x, \quad \dfrac{\partial P}{\partial z}=4xz, \quad \dfrac{\partial Q}{\partial x}=0, \quad \dfrac{\partial Q}{\partial z}=0, \quad \dfrac{\partial R}{\partial x}=0, \quad \dfrac{\partial R}{\partial y}=0$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

or $\quad (2x^2+2xy+2xz^2+1)(0-0)+1(0-4xz)+2z(2x-0)$

or $\quad -4xz+4xz=0$

or $\quad 0=0$

which shows that the given equation is integrable.

Considering x as a constant so that $dx=0$. The given equation reduces to

$$dy+2z\,dz=0$$

Integrating, we get

$$y+z^2=\phi(x) \tag{1}$$

where $\phi(x)$ is the arbitrary constant of integration, because x has been treated as constant.

Differentiating equation (1) w.r.t. all the variables, we get

$$dy+2z\,dz=\frac{d\phi}{dx}dx$$

or $$\frac{d\phi}{dx}dx-dy-2z\,dz=0 \tag{2}$$

Comparing equation (2) with the given equation, we get

$$\frac{d\phi/dx}{2x^2+2xy+2xz^2+1}=\frac{-1}{1}=\frac{-2z}{2z}$$

or $$\frac{d\phi}{dx}=-(2x^2+2xy+2xz^2+1)$$

or $$\frac{d\phi}{dx}=-2x(y+z^2)-(1+2x^2)$$

$$=-2x\phi(x)-(1+2x^2) \quad [\text{using equation (1)}]$$

or $$\frac{d\phi}{dx}+2x\phi(x)=-(1+2x^2) \tag{3}$$

which is a first order linear differential equation. To solve this equation, the integrating factor is

$$\text{I.F.}=e^{\int 2x\,dx}=e^{x^2}$$

The solution of equation (3) is

$$e^{x^2}.\phi = -\int e^{x^2}.(1+2x^2)\,dx + C$$

or

$$e^{x^2}.\phi = -\int e^{x^2}\,dx - \int e^{x^2}.2x^2\,dx + C \qquad (4)$$

To evaluate the integrals in equation (4), let $x^2 = t$. Then $2x\,dx = dt$

or $\quad dx = \dfrac{dt}{2x} = \dfrac{dt}{2\sqrt{t}}$

Thus $\quad \displaystyle\int e^{x^2}\,dx = \int e^t \frac{dt}{2\sqrt{t}}$

and $\quad \displaystyle\int e^{x^2}.2x^2\,dx = \int e^t.2t\frac{1}{2\sqrt{t}}dt = \int e^t\sqrt{t}\,dt = \sqrt{t}\,e^t - \int e^t\frac{1}{2\sqrt{t}}dt$

Equation (4) then becomes

$$e^{x^2}.\phi(x) = -\int e^t\frac{1}{2\sqrt{t}}dt - \sqrt{t}\,e^t + \int e^t\frac{1}{2\sqrt{t}}\,dt + C$$

$$= -\sqrt{t}\,e^t + C = -x\,e^{x^2} + C$$

or $\quad \phi(x) = -x + Ce^{-x^2} \qquad (5)$

From equation (1) and (5), we get

$$y + z^2 = -x + Ce^{-x^2}$$

or $\quad x + y + z^2 = Ce^{-x^2}$

which is the required solution.

PROBLEM (7): **Solve the total differential equation :**

$$(e^x y + e^z)\,dx + (e^y z + e^x)\,dy + (e^y - e^x y - e^y z)\,dz = 0$$

SOLUTION: Comparing the given equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$$P = e^x y + e^z, \quad Q = e^y z + e^x, \quad R = e^y - e^x y - e^y z$$

Now $\quad \dfrac{\partial P}{\partial y} = e^x, \quad \dfrac{\partial P}{\partial z} = e^z, \quad \dfrac{\partial Q}{\partial x} = e^x, \quad \dfrac{\partial Q}{\partial z} = e^y, \quad \dfrac{\partial R}{\partial x} = -e^x y, \quad \dfrac{\partial R}{\partial y} = e^y - e^x - e^y z$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

or $\quad (e^x y + e^z)(e^y - e^y + e^x + e^y z) + (e^y z + e^x)(-e^x y - e^z) + (e^y - e^x y - e^y z)(e^x - e^x) = 0$

or $\quad (e^x y + e^z)(e^x + e^y z) - (e^y z + e^x)(e^x y + e^z) = 0$

or $\quad 0 = 0$

which shows that the given equation is integrable.

Considering z as a constant so that $dz = 0$.

The given equation reduces to

$$(e^x y + e^z)\,dx + (e^y z + e^x)\,dy = 0$$

or $$e^x y\,dx + e^x\,dy + e^z\,dx + e^y z\,dy = 0$$

or $$d(e^x y) + d(e^z x) + d(e^y z) = 0 \qquad (\text{since } z \text{ is constant})$$

Integrating, we get

$$e^x y + e^z x + e^y z = \phi(z) \tag{1}$$

where $\phi(z)$ is the arbitrary constant of integration, because z has been treated as constant.

Differentiating equation (1) w.r.t. all the variables, we get

$$e^x\,dy + e^x y\,dx + e^z\,dx + e^z x\,dz + e^y\,dz + e^y z\,dy = \frac{d\phi}{dx}\,dz$$

or $$(e^x y + e^z)\,dx + (e^y z + e^x)\,dy + \left(e^z x + e^y - \frac{d\phi}{dz}\right)dz = 0$$

Comparing this equation with the given equation, we get

$$\frac{e^x y + e^z}{e^x y + e^z} = \frac{e^y z + e^x}{e^y z + e^x} = \frac{e^z x + e^y - (d\phi/dz)}{e^y - e^x y - e^y z}$$

or $$1 = 1 = \frac{e^z x + e^y - (d\phi/dz)}{e^y - e^x y - e^y z}$$

or $$e^z x + e^y - \frac{d\phi}{dz} = e^y - e^x y - e^y z$$

or $$\frac{d\phi}{dz} = e^z x + e^y - e^y + e^x y + e^y z$$

$$= e^z x + e^x y + e^y z$$

$$= \phi(z) \quad [\text{using equation (1)}]$$

or $$\frac{d\phi}{\phi(z)} = dz$$

Integrating, we get

$$\ln\phi(z) = z + \ln C$$

or $$\phi(z) = Ce^z \tag{2}$$

From equations (1) and (2), we get

$$e^x y + e^z x + e^y z = Ce^z$$

as the required solution.

PROBLEM (8): **Solve the total differential equation :**

$$yz\ln z\,dx - zx\ln z\,dy + xy\,dz = 0$$

SOLUTION: Comparing the given equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$$P = yz\ln z, \quad Q = -zx\ln z, \quad R = xy$$

Now $\frac{\partial P}{\partial y} = z\ln z$, $\frac{\partial P}{\partial z} = y + y\ln z$, $\frac{\partial Q}{\partial x} = -z\ln z$, $\frac{\partial Q}{\partial z} = -x - x\ln z$, $\frac{\partial R}{\partial x} = y$, $\frac{\partial R}{\partial y} = x$

Substituting the values of these derivatives in the condition of integrability, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

or $$yz\ln z(-2x - x\ln z) - zx\ln z(-y\ln z) + xy(2z\ln z) = 0$$

or $$-2xyz\ln z - xyz(\ln z)^2 + xyz(\ln z)^2 + 2xyz\ln z = 0$$

or $$0 = 0$$

which shows that the given equation is integrable.

Considering z as a constant so that $dz = 0$.

The given equation reduces to

$$yz\ln z\,dx - zx\ln z\,dy = 0$$

or $$y\,dx - x\,dy = 0$$

or $$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\ln x = \ln y + \ln K$$

or $$\frac{x}{y} = K$$

As this solution is obtained by supposing the variable z to be a constant, it is expected that the solution of the original equation can be obtained by replacing the constant K by some function of z say $K = \phi(z)$,

giving $$\frac{x}{y} = \phi(z) \qquad (1)$$

Differentiating this equation w.r.t. all the variables, we get

$$\frac{y\,dx - x\,dy}{y^2} = \frac{d\phi}{dz}dz$$

or $$y\,dx - x\,dy - y^2\frac{d\phi}{dz}dz = 0$$

Comparing this equation with the given equation, we get

$$\frac{y}{yz\ln z} = \frac{-x}{-zx\ln z} = \frac{-y^2(d\phi/dz)}{xy}$$

or $$\frac{1}{z\ln z} = -\frac{y}{x}\frac{d\phi}{dz}$$

or $$\frac{1}{z\ln z} = -\frac{1}{\phi(z)}\frac{d\phi}{dz} \quad \text{[using equation (1)]}$$

or $$\frac{d\phi}{\phi(z)} = -\frac{1}{z\ln z}dz = -\frac{1/z}{\ln z}dz$$

Integrating , we get

$$\ln\phi(z) = -\ln(\ln z) + \ln C$$

or $$\phi(z) = \frac{C}{\ln z} \qquad (2)$$

From equations (1) and (2) , the required solution is given by

$$\frac{x}{y} = \frac{C}{\ln z}$$

or $$Cy = x\ln z$$

EXACT TOTAL DIFFERENTIAL EQUATION

PROBLEM (9): **Solve the following differential equation :**

$$(yz+2x)dx+(zx+2y)dy+(xy+2z)dz = 0$$

SOLUTION: Comparing the given equation with $P\,dx+Q\,dy+R\,dz = 0$, we have

$$P = yz+2x, \quad Q = zx+2y, \quad R = xy+2z$$

Now $$\frac{\partial P}{\partial y} = z, \quad \frac{\partial P}{\partial z} = y, \quad \frac{\partial Q}{\partial x} = z, \quad \frac{\partial Q}{\partial z} = x, \quad \frac{\partial R}{\partial x} = y, \quad \frac{\partial R}{\partial y} = x$$

Since $$\frac{\partial P}{\partial y} = z = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = x = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = y = \frac{\partial P}{\partial z}$$

Therefore the given equation is exact . Consequently , it is also integrable .

Rearranging the terms , we get

$$(yz\,dx+zx\,dy+xy\,dz)+2x\,dx+2y\,dy+2z\,dz) = 0$$

or $$d(xyz)+d(x^2+y^2+z^2) = 0$$

Integrating , we get

$$xyz+x^2+y^2+z^2 = C$$

which is the required solution .

HOMOGENEOUS TOTAL DIFFERENTIAL EQUATIONS

PROBLEM (10): **Solve the homogeneous total differential equation :**

$$(y^2+yz)dx+(zx+z^2)dy+(y^2-xy)dz = 0$$

SOLUTION: Comparing the given equation with $P\,dx+Q\,dy+R\,dz = 0$, we have

$$P = y^2+yz, \quad Q = zx+z^2, \quad R = y^2-xy$$

Now $$\frac{\partial P}{\partial y} = 2y+z, \quad \frac{\partial P}{\partial z} = y, \quad \frac{\partial Q}{\partial x} = z, \quad \frac{\partial Q}{\partial z} = x+2z, \quad \frac{\partial R}{\partial x} = -y, \quad \frac{\partial R}{\partial y} = 2y-x$$

Substituting the values of these derivatives in the condition of integrability , we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

or $\quad (y^2+yz)(2x-2y+2z)+(zx+z^2)(-2y)+(y^2-xy)(2y)=0$

or $\quad 2xy^2+2xyz-2y^3-2y^2z+2y^2z+2yz^2-2xyz-2yz^2+2y^3-2xy^2=0$

or $\quad 0=0$

which shows that the given equation is integrable .

Also , the given equation is homogeneous of degree 2 , and

$$\begin{aligned} Px+Qy+Rz &= (y^2+yz)x+(zx+z^2)y+(y^2-xy)z \\ &= y(xy+xz+zx+z^2+yz-xz) \\ &= y(xy+zx+z^2+yz) \\ &= y(x+z)(y+z) \neq 0 \end{aligned}$$

Thus $\dfrac{1}{Px+Qy+Rz}=\dfrac{1}{y(x+z)(y+z)}$ is an integrating factor . Multiplying the given equation by the integrating factor , we get

$$\frac{1}{x+z}dx+\frac{z}{y(y+z)}dy+\frac{y-x}{(x+z)(y+z)}dz=0$$

or $\quad \dfrac{1}{x+z}dx+\left(\dfrac{1}{y}-\dfrac{1}{y+z}\right)dy+\left(\dfrac{1}{x+z}-\dfrac{1}{y+z}\right)dz=0 \qquad$ (resolving into partial fraction)

or $\quad \dfrac{dx+dz}{x+z}+\dfrac{dy}{y}-\dfrac{dy+dz}{y+z}=0$

Integrating , we get

$$\ln(x+z)+\ln y-\ln(y+z)=\ln C$$

or $\quad \ln\left[\dfrac{y(x+z)}{y+z}\right]=\ln C$

or $\quad y(x+z)=C(y+z)$

PROBLEM (11): **Solve the following differential equation :**

$$(mz-ny)dx+(nx-\ell z)dy+(\ell y-mx)dz=0$$

SOLUTION: Comparing the given equation with $Pdx+Qdy+Rdz=0$, we have

$$P=mz-ny, \quad Q=nx-\ell z, \quad R=\ell y-mx$$

Now $\dfrac{\partial P}{\partial y}=-n, \quad \dfrac{\partial P}{\partial z}=m, \quad \dfrac{\partial Q}{\partial x}=n, \quad \dfrac{\partial Q}{\partial z}=-\ell, \quad \dfrac{\partial R}{\partial x}=-m, \quad \dfrac{\partial R}{\partial y}=\ell$

Substituting the values of these derivatives in the condition of integrability , we get

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

$$(mz-ny)(-\ell-\ell)+(nx-\ell z)(-m-m)+(\ell y-mx)(-n-n)=0$$

or $$-2\ell mz+2\ell ny-2mnx+2m\ell z-2\ell ny+2mnx=0$$

or $$0=0$$

which shows that the given equation is integrable .

Also , the given equation is homogeneous of degree 1 .

Since $Px+Qy+Rz=mzx-nxy+nxy-\ell zy+\ell yz-mxz=0$, therefore the method of integrating factor fails .

Let $x=uz$ and $y=vz$, therefore $dx=udz+zdu$ and $dv=vdz+zdv$

The given equation becomes

$$(mz-nvz)(udz+zdu)+(nuz-\ell z)(vdz+zdv)+(\ell vz-muz)dz=0$$

or $$(mz-nvz)zdu+(nuz-\ell z)zdv+\left[muz-nuvz+nuvz-\ell vz+\ell vz-muz\right]dz=0$$

or $$(m-nv)du+(nu-\ell)dv=0$$

or $$(m-nv)du=-(nu-\ell)dv$$

or $$\frac{du}{nu-\ell}=\frac{dv}{nv-m}$$

Integrating both sides , we get

$$\frac{1}{n}\ln(nu-\ell)=\frac{1}{n}\ln(nv-m)+\ln C_1$$

or $$\ln(nu-\ell)^{1/n}=\ln(nv-m)^{1/n}+\ln C_1$$

or $$(nu-\ell)^{1/n}=C_1(nv-m)^{1/n}$$

or $$\left(n\frac{x}{z}-\ell\right)=(C_1)^n\left(n\frac{y}{z}-m\right)$$

or $$nx-\ell z=C(ny-mz) \quad \text{where } C=(C_1)^n$$

SYSTEM OF TOTAL DIFFERENTIAL EQUATIONS

PROBLEM (12): **Solve the following systems of total differential equations :**

$$yz\,dx+xz\,dy+xy\,dz=0 \quad (1)$$

$$z^2dx+z^2dy+(xz+yz-xy)dz=0 \quad (2)$$

SOLUTION: We know that the condition of integrability of the total differential equation

$$Pdx+Qdy+Rdz=0 \quad (3)$$

is $$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \quad (4)$$

Comparing equation (1) with the general form (3) , we have

$$P = yz, \quad Q = xz, \quad R = xy$$

and $\frac{\partial P}{\partial y} = z, \quad \frac{\partial P}{\partial z} = y, \quad \frac{\partial Q}{\partial x} = z, \quad \frac{\partial Q}{\partial z} = x, \quad \frac{\partial R}{\partial x} = y, \quad \frac{\partial R}{\partial y} = x$

Substituting these derivatives in equation (4), we get

$$yz(x-x) + xz(y-y) + xy(z-z) = 0$$

or $\quad 0 = 0$

Thus equation (1) is integrable.

Also, since each term of equation (4) vanishes identically, therefore it is exact.

Equation (1) can be written as

$$d(xyz) = 0$$

whose solution is $\quad xyz = C_1 \qquad (5)$

Comparing equation (2) with the general form (3), we have

$$P = z^2, \quad Q = z^2, \quad R = xz + yz - xy$$

and $\frac{\partial P}{\partial y} = 0, \quad \frac{\partial P}{\partial z} = 2z, \quad \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = 2z, \quad \frac{\partial R}{\partial x} = z - y, \quad \frac{\partial R}{\partial y} = z - x$

Substituting these derivatives in equation (4), we get

$$z^2(2z - z + x) + z^2(z - y - 2z) + (xz + yz - xy)(0 - 0)$$

$$= z^2(z + x) - z^2(y + z) + 0 = z^2(x - y) \neq 0$$

Therefore equation (2) is not integrable.

Multiplying equation (1) by z and equation (2) by y, we get

$$yz^2\,dx + xz^2\,dy + xyz\,dz = 0 \qquad (6)$$

$$yz^2\,dx + yz^2\,dy + (xyz + y^2z - xy^2)\,dz = 0 \qquad (7)$$

Subtracting equation (6) from equation (7), we have

$$z^2(y - x)\,dy + y^2(z - x)\,dz = 0$$

Multiplying this equation by yz, we have

$$z^2(y^2z - xyz)\,dy + y^2(yz^2 - xyz)\,dz = 0$$

Substituting $xyz = C_1$ from equation (5), we get

$$z^2(y^2z - C_1)\,dy + y^2(yz^2 - C_1)\,dz = 0$$

Dividing by y^2z^2, we have

$$\left(z - \frac{C_1}{y^2}\right)dy + \left(y - \frac{C_1}{z^2}\right)dz = 0$$

or $\quad (z\,dy + y\,dz) - C_1\left(\frac{dy}{y^2} + \frac{dz}{z^2}\right) = 0$

or $\quad d(yz) - C_1\left(\frac{dy}{y^2} + \frac{dz}{z^2}\right) = 0$

Integrating, we get

$$yz + C_1\left(\frac{1}{y}+\frac{1}{z}\right) = C_2$$

or $$yz + C_1\left(\frac{y+z}{yz}\right) = C_2$$

or $$yz + xyz\left(\frac{y+z}{yz}\right) = C_2 \quad \left[\text{using equation (5)}\right]$$

or $$xy + yz + zx = C_2 \qquad (6)$$

Equations (5) and (6) constitute the complete solution of the given system.

7.17 EXERCISE

SIMULTANEOUS EQUATIONS

Solve the following differential equations :

(1) $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{\sin x}$ (2) $\dfrac{dx}{z^2 x} = \dfrac{dy}{z^2 x} = \dfrac{dz}{y^2 x}$

(3) $\dfrac{dx}{y^2} = \dfrac{dy}{x^2} = \dfrac{dz}{x^2 y^2 z^2}$ (4) $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

A SECOND INTEGRAL FOUND WITH THE HELP OF FIRST

Solve the following differential equations :

(5) $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z + \tan(y - 3x)}$ (6) $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2 + (x+y)^2}$

(7) $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{z/(x+y)}$ (8) $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2(x^2 - y^2)}$

(9) $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{xy}$ (10) $\dfrac{dx}{-xy^2} = \dfrac{dy}{y^3} = \dfrac{dz}{axz}$

METHOD 3: CASE (1)

Solve the following differential equations :

(11) $\dfrac{dx}{z-y} = \dfrac{dy}{x-z} = \dfrac{dz}{y-x}$ (Hint: Take $1, 1, 1$ and x, y, z as multipliers)

(12) $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ (Hint: Take $1, 1, 1, \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers)

(13) $\dfrac{yz\,dx}{y-z} = \dfrac{zx\,dy}{z-x} = \dfrac{xy\,dz}{x-y}$ (Hint: Take $1, 1, 1$ and $\dfrac{1}{yz}, \dfrac{1}{zx}, \dfrac{1}{xy}$ as multipliers)

(14) $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$ (Hint: Take x, y, z and $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers)

(15) $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{2x - 3y}$ (Hint: Take $3, 2, 1$ as multipliers)

(16) $\dfrac{dx}{zx} = \dfrac{dy}{-zy} = \dfrac{dz}{y^2 - x^2}$ (Hint: Take x, y, z as multipliers)

(17) $\dfrac{dx}{3y - 2z} = \dfrac{dy}{z - 3x} = \dfrac{dz}{2x - y}$ (Hint: Take $1, 2, 3$ and x, y, z as multipliers)

(18) $\dfrac{dx}{y - zx} = \dfrac{dy}{x + yz} = \dfrac{dz}{x^2 + y^2}$ (Hint: Take $x, -y, z$ and $y, x, -1$ as multipliers)

METHOD 3: CASE (2)

Solve the following differential equations :

(19) $\dfrac{dx}{y}=\dfrac{dy}{x}=\dfrac{dz}{z}$ (Hint: Take $1, 1, 0$ as multipliers)

(20) $\dfrac{dx}{1}=\dfrac{dy}{z}=\dfrac{dz}{y}$ (Hint: Take $0, 1, 1$ as multipliers)

(21) $\dfrac{dx}{1}=\dfrac{dy}{1}=\dfrac{dz}{x+y+z}$ (Hint: Take $1, 1, 1$ as multipliers)

(22) $\dfrac{dx}{xz}=\dfrac{dy}{yz}=\dfrac{dz}{xy}$ $\left(\text{Hint: Take } \dfrac{1}{x}, \dfrac{1}{y}, 0 \text{ as multipliers}\right)$

(23) $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}=\dfrac{dz}{z(x+y)}$ (Hint: Take $1, -1, 0$ as multipliers)

(24) $\dfrac{dx}{y^2(x-y)}=\dfrac{dy}{-x^2(x-y)}=\dfrac{dz}{z(x^2+y^2)}$ (Hint: Take $1, -1, 0$ as multipliers)

(25) $\dfrac{x^2\,dx}{y^3}=\dfrac{y^2\,dy}{x^3}=\dfrac{dz}{z}$ (Hint: Take $1, 1, 0$ as multipliers)

(26) $\dfrac{dx}{y(x+y)+az}=\dfrac{dy}{x(x+y-az)}=\dfrac{dz}{z(x+y)}$

(Hint: Take $1, 1, 0$ and $x, -y, 0$ as multipliers)

TOTAL DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(27) $(y+z)\,dx+dy+dz=0$

(28) $dx+dy+(x+y+z+1)\,dz=0$

(29) $(x-y)\,dx-x\,dy+z\,dz=0$

(30) $2yz\,dx-3zx\,dy-4xy\,dz=0$

(31) $(a+z)y\,dx+(a+z)x\,dy-xy\,dz=0$ [Hint: Divide by $xy(a+z)$]

(32) $(yz+xyz)\,dx+(zx+xyz)\,dy+(xy+xyz)\,dz=0$ (Hint: Divide by xyz)

(33) $2yz\,dx+zx\,dy-xy(1+z)\,dz=0$ (Hint: Divide by xyz)

(34) $(x^2z-y^3)\,dx+3xy^2\,dy+x^3\,dz=0$ (Hint: Divide by x^2)

TREATING ONE VARIABLE AS CONSTANT

Solve the following differential equation :

(35) $2yz\,dx+zx\,dy+xy(1+z)\,dz=0$ (Hint: Treat z as constant)

(36) $3x^2\,dx+3y^2\,dy-(x^3+y^3+e^{2x})\,dz=0$ (Hint: Treat z as constant)

(37) $(x^2+y^2+z^2)\,dx-2xy\,dy-2xz\,dz=0$ (Hint: Treat x as constant)

(38) $xz^3dx-z\,dy+2y\,dz=0$ (Hint: Treat x as constant)

(39) $(z+z^3)\cos x\,dx-(z+z^3)\,dy+(1-z^2)(y-\sin x)\,dz=0$ (Hint: Treat z as constant)

(40) $x\,dy-y\,dx-2x^2z\,dz=0$ (Hint: Treat z as constant)

HOMOGENEOUS TOTAL DIFFERENTIAL EQUATIONS

Solve the following differential equations :

(41) $(yz+z^2)\,dx-xz\,dy+xy\,dz=0$

(42) $z(z-y)\,dx+z(z+x)\,dy+x(x+y)\,dz=0$

(43) $yz^2(x^2-yz)+x^2z(y^2-xz)+xy^2(z^2-xy)=0$ (Hint: $x=uz$, $y=vz$)

(44) $(2xz-yz)\,dx+(2yz-xz)\,dy-(x^2-xy+y^2)\,dz=0$ (Hint: $x=uz$, $y=vz$)

EXACT AND HOMOGENEOUS EQUATIONS

Solve the following differential equations :

(45) $yz\,dx+zx\,dy+xy\,dz=0$

(46) $(2y-z)\,dx+2(x-z)\,dy-(x+2y)\,dz=0$

(47) $(y+3z)\,dx+(x+2z)\,dy+(3x+2y)\,dz=0$

(48) $(2xy+z^2)\,dx+(x^2+2yz)\,dy+(y^2+2xz)\,dz=0$

SYSTEMS OF TOTAL DIFFERENTIAL EQUATIONS

Solve the following systems of total differential equations :

(49) $dx+dy+(x+y)\,dz=0$

$z(dx+dy)+(x+y)\,dz=0$

(50) $2yz\,dx+x(z\,dy+y\,dz)=0$

$y\,dx-x^2z\,dy+y\,dz=0$

CHAPTER 8

APPLICATIONS OF DIFFERENTIAL EQUATIONS

8.1 INTRODUCTION

Differential equations are of fundamental importance in science and engineering because many physical laws and relations appear mathematically in the form of such equations. We shall discuss the derivation of differential equations from given physical (or other) situations. This transition from the given physical problem to a corresponding mathematical problem is called **modeling**. This is of great practical importance to the engineer, physicist, and computer scientist.

In this chapter, we shall show how some differential equations arise and how their solutions are obtained. We will discuss the applications of differential equations to various geometrical and physical problems. Furthermore, applications to mechanics and engineering systems will also be discussed.

8.2 APPLICATIONS TO GEOMETRY

Differential equations can be applied in solving a great variety of geometrical problems. For convenience, the following results from geometry which involve the derivative, are given.

(a) CARTESIAN COORDINATES

Let $P(x, y)$ be any point on the curve AB whose equation is $y = f(x)$ as shown in figure (8.1). Let PT be the tangent and PN the normal to the curve at the point P. Let TR and RN be the subtangent and subnormal respectively. Let θ be the angle between the tangent PT and the x – axis. Then

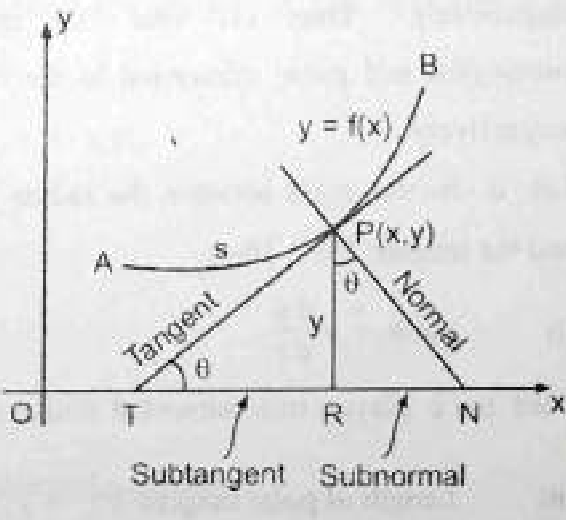


Figure (8.1)

(i) Slope of the tangent to the curve at P is

$$m = \frac{dy}{dx} = \tan\theta$$

(ii) Slope of the normal to the curve at $P = -\frac{dx}{dy}$

(iii) Equation of the tangent to the curve at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$ where (x_1, y_1) are the coordinates of any point on the line PT.

(iv) Equation of the normal to the curve at P is $y - y_1 = -\frac{dx}{dy}(x - x_1)$

where (x_1, y_1) are the coordinates of any point on the normal PN.

(v) Length of the tangent $PT = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$

(vi) Length of the normal $PN = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(vii) Length of the subtangent $TR = y\frac{dx}{dy}$

(viii) Length of the subnormal $RN = y\frac{dy}{dx}$

(ix) If $AP = s$, is the length of arc of the curve, then

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

(b) POLAR COORDINATES

Let $P(r, \theta)$ be any point on the curve AB whose polar equation is $r = f(\theta)$ as shown in figure (8.2). Let TON be a straight line perpendicular to the radius vector OP. Let the tangent and normal to the curve at P meet the straight line TON at the points T and N respectively. Then OT and ON are the polar subtangent and polar subnormal to the curve at P respectively.

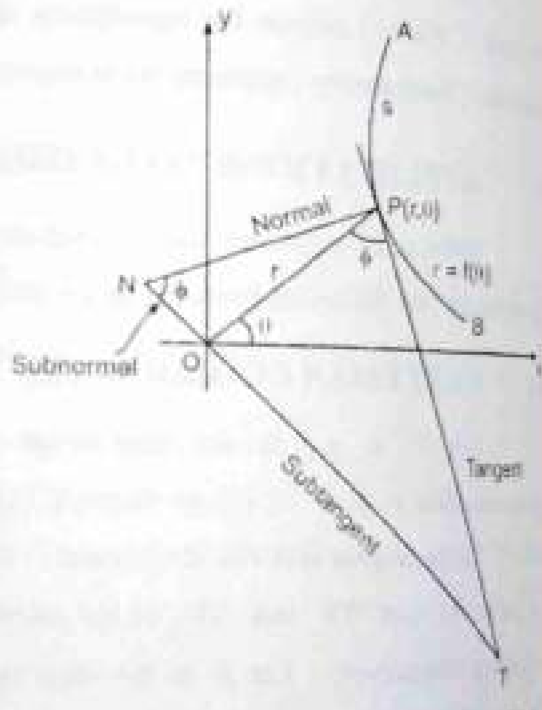


Figure (8.2)

Let ϕ be the angle between the radius vector OP and the tangent PT. Then

(i) $\tan\phi = r\frac{d\theta}{dr}$

Here $\tan\phi$ plays a role somewhat similar to that of the slope of the tangent in rectangular coordinates.

(ii) Length of polar tangent $PT = r\sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$

(iii) Length of polar normal $PN = r\sqrt{1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2}$

(iv) Length of the polar subtangent $OT = r^2\frac{d\theta}{dr}$.

(v) Length of the polar subnormal $ON = \frac{dr}{d\theta}$.

(vi) Length of the perpendicular from the pole on the tangent at P is $r\sin\phi = r^2\frac{d\theta}{ds}$.

(vii) If $AP = s$ is the length of arc of the curve, then

$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,dr$$

EXAMPLE (1): Find the equation of the curve for which the Cartesian subtangent is constant.

SOLUTION: Let $P(x, y)$ be any point on the curve $y = f(x)$.

Then the length of subtangent $= y\frac{dx}{dy}$.

According to the given condition $y\frac{dx}{dy} = a$ (a = constant)

which is a first order differential equation. Separating the variables, we get

$$\frac{dy}{y} = \frac{1}{a}dx$$

Integrating both sides, we get

$$\ln y = \frac{1}{a}x + C$$

or $y = e^{(x/a)+C} = e^{x/a}.e^{C} = Ae^{x/a}$ (where $A = e^C$)

Thus $y = Ae^{x/a}$ is the required equation of the curve.

EXAMPLE (2): Find the equation of the curve for which the length of the subnormal is proportional to the square of the abscissa.

SOLUTION: Let $P(x, y)$ be any point on the curve $y = f(x)$.

Then the length of the subnormal $= y\frac{dy}{dx}$.

According to the given condition $y\frac{dy}{dx} \propto x^2$

or $y\frac{dy}{dx} = Kx^2$

where K is the constant of proportionality. It is a first order differential equation.

Separating the variables, we get

$$y\,dy = Kx^2\,dx$$

Integrating, we get

$$\frac{y^2}{2} = K\frac{x^3}{3} + C$$

or $\quad 3y^2 = 2Kx^3 +$

or $\quad 3y^2 = 2Kx^3$ $\quad$ (where $A = 6C$)

is the required equation of the curve .

EXAMPLE (3): Find the equation of the curve for which the polar subtangent is constant .

SOLUTION: Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

Then the length of polar subtangent $= r^2 \dfrac{d\theta}{dr}$.

According to the given condition

$$r^2 \frac{d\theta}{dr} = a \quad (a = \text{constant})$$

which is a first order differential equation . Separating the variables , we get

$$a \frac{dr}{r^2} = d\theta$$

Integrating , we get

$$-\frac{a}{r} = \theta + C$$

or $\quad r(\theta + C) + a = 0$

is the required equation of the curve .

EXAMPLE (4): Find the equation of the curve for which the polar subtangent is equal to the polar subnormal .

SOLUTION: Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

Then the length of the polar subtangent $= r^2 \dfrac{d\theta}{dr}$.

Also the length of the subnormal $= \dfrac{dr}{d\theta}$.

According to the given condition

$$r^2 \frac{d\theta}{dr} = \frac{dr}{d\theta}$$

or $\quad \left(\dfrac{dr}{d\theta}\right)^2 = r^2$

or $\quad \dfrac{dr}{d\theta} = r \quad (\text{since } r \geq 0)$

which is first order differential equation . Separating the variables .

$$\frac{dr}{r} = d\theta$$

Integrating, we get

$$\ln r = \theta + A$$

or $$r = e^{\theta + A} = e^{\theta} \cdot e^{A}$$

or $$r = C e^{\theta} \quad (\text{where } C = e^{A})$$

8.3 CURVATURE AND RADIUS OF CURVATURE

Let A be a fixed point on the curve. Let arc $AP = s$, arc $PQ = \Delta s$. If the tangents at P and Q make angles ψ and $\psi + \Delta\psi$ with the positive direction of x – axis, then the angle between the tangents at P and Q is $\Delta\psi$ as shown in figure (8.3).

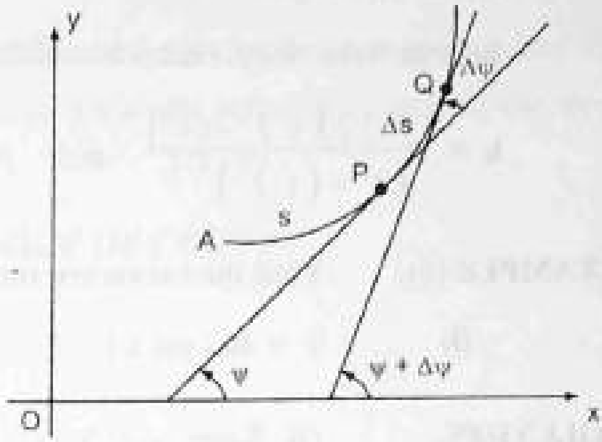


Figure (8.3)

Thus the average curvature between the points P and Q

$$= \frac{\Delta\psi}{\Delta s} \text{ rad / unit length}$$

The limiting value of this ratio as $\Delta s \to 0$ is called the **curvature** of the curve at P and is denoted by k.

Thus $$k = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds} \qquad (1)$$

This leads us to the following definition.

The curvature k of the curve at any point P is the rate of change of the angle of inclination ψ w.r.t. the arc length s. (measured in radians per unit of length).

RADIUS OF CURVATURE

The reciprocal of the curvature at P (provided $k \neq 0$) is called the **radius of curvature** of the curve at P and is denoted by ρ.

Thus $$\rho = \frac{1}{k} = \frac{ds}{d\psi} \qquad (2)$$

But k and ρ are sometimes defined so as to be positive i.e. as the absolute value of that given by formulas (1) and (2).

With this later definitions

$$k = \left|\frac{d\psi}{ds}\right| \quad \text{and} \quad \rho = \left|\frac{ds}{d\psi}\right|$$

FORMULAE FOR RADIUS OF CURVATURE

When the equation of the curve is $y = f(x)$, then

$$k = \frac{\left|\frac{d^2 y}{d x^2}\right|}{\left[1+\left(\frac{d y}{d x}\right)^2\right]^{3/2}} \quad \text{and} \quad \rho = \frac{1}{k} = \frac{\left[1+\left(\frac{d y}{d x}\right)^2\right]^{3/2}}{\left|\frac{d^2 y}{d x^2}\right|}$$

POLAR FORM

In polar form , the formulas for curvature and radius of curvature are given by

$$k = \frac{\left|r^2+2(r')^2-r r''\right|}{\left[r^2+(r')^2\right]^{3/2}} \quad \text{and} \quad \rho = \frac{1}{k} = \frac{\left[r^2+(r')^2\right]^{3/2}}{\left|r^2+2(r')^2-r r''\right|}$$

EXAMPLE (5): **Find the curvature and radius of curvature of the following curves :**

(i) $y = \ln(\cos x)$ (ii) $r = \sin\theta$

SOLUTION: (i) Since $y = \ln(\cos x)$, therefore $\frac{d y}{d x} = -\tan x$, $\frac{d^2 y}{d x^2} = -\sec^2 x$

Thus the curvature of the curve is given by

$$k = \frac{\left|\frac{d^2 y}{d x^2}\right|}{\left[1+\left(\frac{d y}{d x}\right)^2\right]^{3/2}}$$

$$= \frac{\sec^2 x}{(1+\tan^2 x)^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \frac{1}{\sec x} = \cos x$$

and the radius of curvature of the curve is given by

$$\rho = \frac{1}{k} = \frac{1}{\cos x} = \sec x$$

(ii) Since $r = \sin\theta$, therefore $r' = \cos\theta$, $r'' = -\sin\theta$

Thus the curvature of the curve is given by

$$k = \frac{\left|r^2+2(r')^2-r r''\right|}{\left[r^2+(r')^2\right]^{3/2}}$$

$$= \frac{\sin^2\theta+2\cos^2\theta+\sin^2\theta}{(\sin^2\theta+\cos^2\theta)^{3/2}} = \frac{2(\sin^2\theta+\cos^2\theta)}{1} = 2$$

and the radius of curvature of the curve is given by

$$\rho = \frac{1}{k} = \frac{1}{2}$$

8.4 FAMILY OF CURVES

The equation $f(x, y, C) = 0$ where C is a parameter, represents a family of curves, because for different values of C we get different curves, having the same property. Also, for a given value of C, the equation $f(x, y, C) = 0$ represents a unique curve in the plane.

For example, the equation $f(x, y, C) = x + y + C = 0$ represents a family of parallel straight lines, while the equation $f(x, y, C) = x^2 + y^2 - C^2 = 0$ represents a family of concentric circles of radius C with centre at the origin.

Remember that the general solution of a first order differential equation involves a parameter C and therefore represents a family of curves. We have discussed a single parameter family of curves, however there may be multiple parameter families also.

DIFFERENTIAL EQUATION OF THE FAMILY OF CURVES

The equation of the family of curves is

$$f(x, y, C) = 0 \tag{1}$$

Differentiating equation (1) w.r.t. x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \tag{2}$$

Eliminating C from equations (1) and (2), we get a relation of the form

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \tag{3}$$

Since equation (3) is free from C, it is the differential equation of the family of given curves.

If the differential equation (3) obtained by differentiating equation (1) still contains the parameter C, then we have to eliminate C from this equation by using equation (1).

EXAMPLE (6): **Find the differential equation of each of the following families of curves.**

(i) $x + y + C = 0$ (ii) $x^2 + y^2 = C^2$ (iii) $y = Cx^2$

SOLUTION: (i) $x + y + C = 0$

Differentiating this equation w.r.t. x, we get

$$1 + \frac{dy}{dx} = 0$$

which is the differential equation of the given family of curves.

(ii) $x^2 + y^2 - C^2 = 0$

Differentiating this equation w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{x}{y}$$

which is the differential equation of the given family of curves .

(iii) $y = Cx^2$ (1)

Differentiating this equation w.r.t. x , we get

$$\frac{dy}{dx} = 2Cx \qquad (2)$$

We now eliminate the parameter C from this equation .

From equation (1) we have $C = \frac{y}{x^2}$ and by substituting this into equation (2) , we find

$$\frac{dy}{dx} = 2\frac{y}{x}$$

which is the differential equation of the given family of curves .

8.5 ORTHOGONAL TRAJECTORIES

A curve which intersects every member of a given family of curves according to a given law is called a **trajectory** of the family .

A curve which cuts every member of a given family of curves at right angles is called an **orthogonal** trajectory of the family .

Two families of curves are said to be **orthogonal** , if every member of either family intersects every member of the other family at right angles as shown in figure (8.4) .

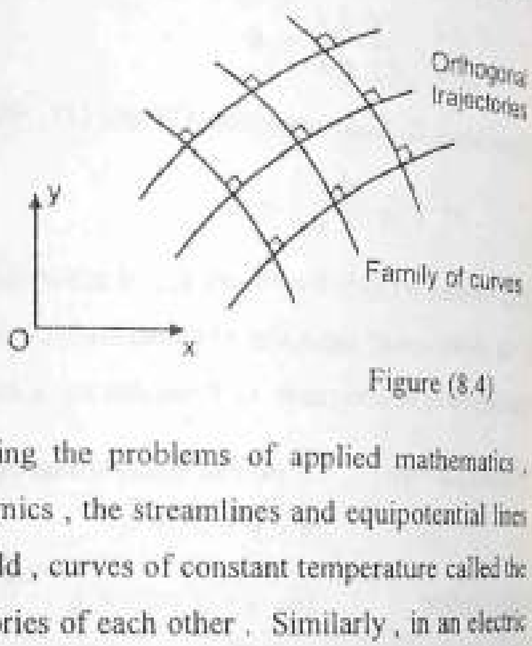


Figure (8.4)

Orthogonal trajectories are of great help in analyzing the problems of applied mathematics , especially in the field problems . For example , in fluid dynamics , the streamlines and equipotential lines are orthogonal trajectories of each other . In a temperature field , curves of constant temperature called the isothermals and the curves of heat flow are orthogonal trajectories of each other . Similarly , in an electric field , the lines of current flow and equipotential curves are orthogonal trajectories of each other . Many other such phenomena of orthogonal trajectories can be found in various other fields such as elasticity , etc.

8.6 ORTHOGONAL TRAJECTORIES IN CARTESIAN COORDINATES

Let C_1 be a member of the family of given curves . Let C_2 be a member of the family of orthogonal trajectories intersecting C_1 at the point $P(x, y)$. Let ψ_1 and ψ_2 be the angles of inclination which the tangents to the curves C_1 and C_2 at P make with the x – axis , respectively as shown in figure (8.5) .

Then the slope of the tangent to the curve C_1 is

$$\left(\frac{dy}{dx}\right)_1 = \tan\psi_1$$

and the slope of the tangent to the curve C_2 is

$$\left(\frac{dy}{dx}\right)_2 = \tan\psi_2 .$$

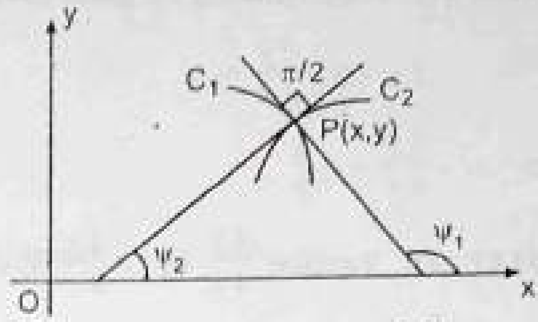


Figure (8.5)

Since C_1 and C_2 are orthogonal to each other, therefore the product of the slopes of their tangents is -1.

Thus $\tan\psi_1 \tan\psi_2 = -1$

or $\tan\psi_1 = -\dfrac{1}{\tan\psi_2}$

$$\left(\frac{dy}{dx}\right)_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_2}$$

where $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$ are the slopes of the tangents to the curves C_1 and C_2 at P respectively.

8.7 RULE FOR FINDING THE EQUATION OF ORTHOGONAL TRAJECTORIES OF A FAMILY OF CARTESIAN CURVES

Let the equation of the given family of curves be

$$f(x, y, C) = 0 \tag{1}$$

STEP (1): Differentiate equation (1) w.r.t. x and eliminate the parameter C between equation (1) and the resulting equation. This gives the differential equation of the family of curves (1).

Let it be $F\left(x, y, \dfrac{dy}{dx}\right) = 0$ (2)

STEP (2): Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in equation (2) to get the differential equation of the family of orthogonal trajectories. This is because at the point of intersection the product of slopes must be -1. Let it be

$$F\left(x, y, -\frac{dx}{dy}\right) = 0 \tag{3}$$

STEP 3: Integrate equation (3) to get the equation of the required orthogonal trajectories of the given family of curves.

EXAMPLE (7): **Find the orthogonal trajectories of the family of curves $x^2 + y^2 = C^2$.**

SOLUTION: The equation of the family of curves is

$$x^2 + y^2 = C^2 \tag{1}$$

which consists of circles with centers at the origin and radii C.

STEP (1): Differentiating the given equation implicitly w.r.t. x , we get

$$2x + 2y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{x}{y} \qquad (2)$$

STEP (2): Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (2) , the differential equation of the orthogonal trajectories of family of curves (1) is given by

$$-\frac{dx}{dy} = -\frac{x}{y}$$

Separating the variables , we get

$$\frac{dy}{y} = \frac{dx}{x}$$

STEP (3): Integrating , we get

$$\ln y = \ln x + \ln k$$

or $$y = kx$$

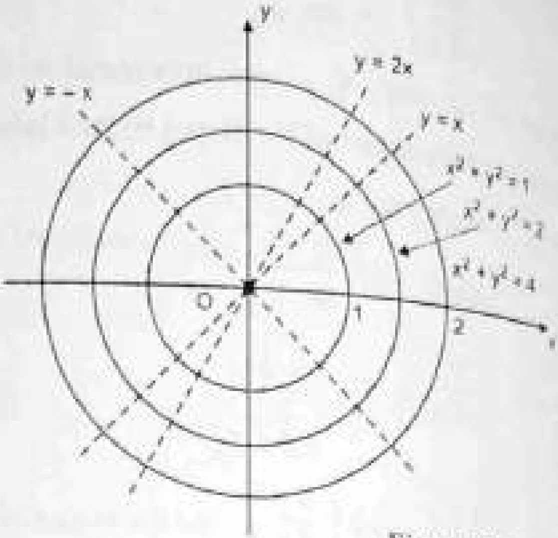


Figure (8.6)

which are straight lines through the origin and represent the orthogonal trajectories .

Figure (8.6) shows some members of the family of circles in solid lines and some members of the family of straight lines through the origin in dashed lines . Observe that straight line intersect each circle at right angles .

8.8 SELF – ORTHOGONAL FAMILY OF CURVES

A family of curves is said to be **self – orthogonal** in the sense that a member of the orthogonal trajectory is also a member of the original family .

EXAMPLE (8): **Prove that the family of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self – orthogonal , where a is a parameter .**

SOLUTION: The equation of the family of confocal and coaxial parabolas is

$$y^2 = 4a(x + a) \qquad (1)$$

STEP (1): Differentiating equation (1) w.r.t. x , we get

$$2y\frac{dy}{dx} = 4a$$

or $$a = \frac{1}{2}y\frac{dy}{dx} \qquad (2)$$

Eliminating the parameter a from equations (1) and (2) , we get

$$y^2 = 4\left(\frac{1}{2}y\frac{dy}{dx}\right)\left(x + \frac{1}{2}y\frac{dy}{dx}\right)$$

$$= 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

or $$y^2 = 2xyp + y^2p^2 \qquad \left(\text{where } p = \frac{dy}{dx}\right) \tag{3}$$

which is the differential equation of the family of curves (1).

STEP (2): Replacing p by $-\frac{1}{p}$, the differential equation of the orthogonal trajectories is

$$y^2 = 2xy\left(-\frac{1}{p}\right) + y^2\left(-\frac{1}{p}\right)^2$$

$$= -\frac{2xy}{p} + \frac{y^2}{p^2}$$

or $$y^2p^2 + 2xyp = y^2 \tag{4}$$

which is the same as equation (3).

Since the differential equations of the family of parabolas (1) and that of the orthogonal trajectories are same, the family of parabolas (1) is self – orthogonal.

8.9 ORTHOGONAL TRAJECTORIES IN POLAR COORDINATES

In finding the orthogonal trajectories of a given family of curves, it is sometimes convenient to use polar coordinates.

Let C_1 be a member of the family of given polar curves. Let C_2 be a member of the family of orthogonal trajectories intersecting C_1 at the point $P(r, \theta)$.

Let ϕ_1 and ϕ_2 be the angles which the radius vector OP makes with the tangents to the curves C_1 and C_2 at P respectively, as shown in figure (8.7).

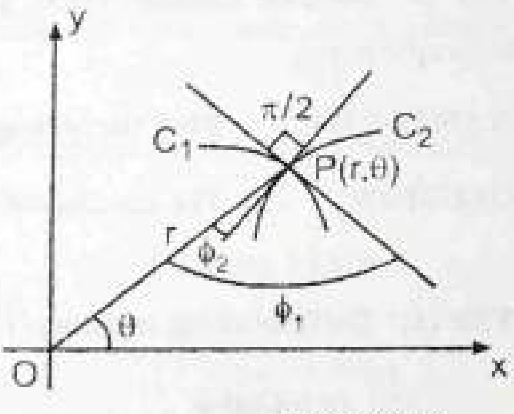


Figure (8.7)

We know that the angle between the curves C_1 and C_2 at P is defined as the angle between their tangents at the point P. Since C_1 and C_2 are orthogonal to each other, therefore the angle between their tangents is $\frac{\pi}{2}$. Now from the figure,

$$\phi_1 - \phi_2 = \frac{\pi}{2} \text{ so that } \phi_1 = \frac{\pi}{2} + \phi_2$$

Thus $$\tan\phi_1 = \tan\left(\frac{\pi}{2} + \phi_2\right) = -\cot\phi_2 = -\frac{1}{\tan\phi_2}$$

or $$\tan\phi_1 \tan\phi_2 = -1 \tag{1}$$

We know that $\tan\phi_1 = \left(r\frac{d\theta}{dr}\right)_1$ and $\tan\phi_2 = \left(r\frac{d\theta}{dr}\right)_2$, therefore equation (1) can be written as

$$\left(r\frac{d\theta}{dr}\right)_1 \left(r\frac{d\theta}{dr}\right)_2 = -1$$

from which, we get

$$\left(\frac{dr}{d\theta}\right)_1 = -r^2\left(\frac{d\theta}{dr}\right)_2 \qquad (2)$$

8.10 RULE FOR FINDING THE EQUATION OF ORTHOGONAL TRAJECTORIES OF A FAMILY OF POLAR CURVES

Let the equation of the given family of polar curves be

$$f(r, \theta, C) = 0 \qquad (1)$$

STEP (1): Differentiate equation (1) w.r.t. θ and eliminate the parameter C between equation (1) and the resulting equation. This gives the differential equation of the family of curves (1).

Let it be $F\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ (2)

STEP (2): Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in equation (2) to get the differential equation of the family of orthogonal trajectories. Let it be

$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0 \qquad (3)$$

STEP 3: Integrate equation (3) to get the equation of the required orthogonal trajectories of the given family of curves.

EXAMPLE (9): **Find the orthogonal trajectories of the family of cardioids $r = a(1 + \cos\theta)$.**

SOLUTION: The equation of the family of cardioids is

$$r = a(1 + \cos\theta) \qquad (1)$$

STEP (1): Differentiating equation (1) w.r.t. θ, we get

$$\frac{dr}{d\theta} = -a\sin\theta \qquad (2)$$

Eliminating the parameter a between equations (1) and (2), we get

$$r = -\frac{1}{\sin\theta}\frac{dr}{d\theta}(1 + \cos\theta)$$

or
$$\frac{dr}{d\theta} = -\frac{r\sin\theta}{1+\cos\theta} = \frac{-2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

or
$$\frac{dr}{d\theta} = -r\tan\frac{\theta}{2} \qquad (3)$$

which is the differential equation of the family of curves (1).

STEP (2): Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, the differential equation of the family of orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = -r\tan\frac{\theta}{2}$$

or $$\frac{dr}{r} = \cot\frac{\theta}{2}\,d\theta \tag{4}$$

STEP (3): Integrating equation (4) w.r.t. θ, we get

$$\ln r = 2\ln\sin\frac{\theta}{2} + \ln C_1$$

or $$\ln r = \ln\sin^2\frac{\theta}{2} + \ln C_1$$

or $$r = C_1\sin^2\frac{\theta}{2} = C_1\frac{(1-\cos\theta)}{2}$$

or $$r = C(1-\cos\theta) \quad \left(\text{where } C = \frac{1}{2}C_1\right)$$

which is the required equation .

8.11 OBLIQUE (OR ISOGONAL) TRAJECTORIES

A curve which cuts every member of a given family of curves at a constant angle $\alpha\ (\neq 90^\circ)$ is called an **oblique** trajectory or **isogonal** trajectory (or **α – trajectory**) of the family .

Let C_1 be a member of the family of given curves . Let C_2 be a member of the family of oblique trajectories intersecting C_1 at an angle α at the point $P(x, y)$. Let ψ_1 and ψ_2 be the angles of inclination which the tangents to the curves C_1 and C_2 at P make with the x – axis , respectively as shown in figure (8.8) .

We know that the angle α between C_1 and C_2 at P is defined as the angle between their tangents at the point P . Then from the figure ,

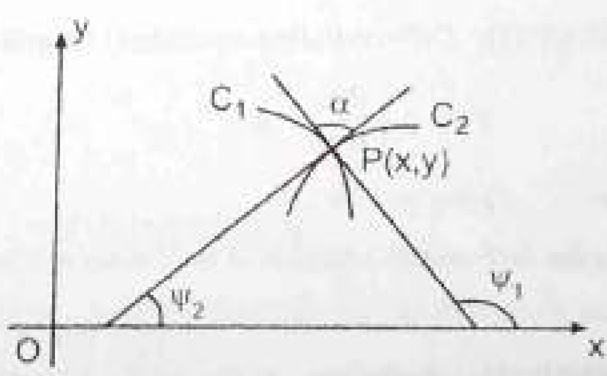


Figure (8.8)

$$\alpha = \psi_1 - \psi_2$$

so that $$\tan\alpha = \tan(\psi_1 - \psi_2) = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1\tan\psi_2}$$

or $$\tan\psi_1 - \tan\psi_2 = (1 + \tan\psi_1\tan\psi_2)\tan\alpha$$

or $$\tan\psi_1 - \tan\psi_1\tan\psi_2\tan\alpha = \tan\psi_2 + \tan\alpha$$

or $$\tan\psi_1(1 - \tan\psi_2\tan\alpha) = \tan\psi_2 + \tan\alpha$$

or $$\tan\psi_1 = \frac{\tan\psi_2 + \tan\alpha}{1 - \tan\psi_2\tan\alpha}$$

or $$\left(\frac{dy}{dx}\right)_1 = \frac{\left(\frac{dy}{dx}\right)_2 + \tan\alpha}{1 - \left(\frac{dy}{dx}\right)_2\tan\alpha}$$

where $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$ are the slopes of the tangents to the curves C_1 and C_2 at P respectively.

8.12 RULE FOR FINDING THE EQUATION OF OBLIQUE TRAJECTORIES OF A FAMILY OF CARTESIAN CURVES

Let the equation of the given family of curves be

$$f(x, y, C) = 0 \qquad (1)$$

STEP (1): Differentiate equation (1) w.r.t. x and eliminate the parameter C between equation (1) and the resulting equation. This gives the differential equation of the family of curves (1).

Let it be $F\left(x, y, \frac{dy}{dx}\right) = 0 \qquad (2)$

STEP (2): Replace $\frac{dy}{dx}$ by $\dfrac{\frac{dy}{dx} + \tan\alpha}{1 - \frac{dy}{dx}\cdot\tan\alpha}$ in equation (2) to get the differential equation of the family of orthogonal trajectories. Let it be

$$F\left(x, y, \frac{\frac{dy}{dx} + \tan\alpha}{1 - \frac{dy}{dx}\cdot\tan\alpha}\right) = 0 \qquad (3)$$

STEP 3: Integrate equation (3) to get the equation of the required orthogonal trajectories.

EXAMPLE (10): **Find the 45^0 trajectories of the family of concentric circles $x^2 + y^2 = C^2$.**

SOLUTION: The equation of family of concentric circles is

$$x^2 + y^2 = C^2 \qquad (1)$$

STEP (1): Differentiating equation (1) implicitly w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

or $\quad x + yp = 0 \qquad (2)$

is the differential equation of the family of curves (1).

STEP (2): Replacing p by $\dfrac{\frac{dy}{dx} + \tan 45^0}{1 - \frac{dy}{dx}\tan 45^0} = \dfrac{p+1}{1-p}$ in equation (2), the differential equation of 45^0 trajectories of the family of curves (1) is

$$x + y\frac{p+1}{1-p} = 0$$

or $\quad x(1-p)+y(p+1)=0$

or $\quad p(x-y)=x+y$

or $\quad p=\dfrac{x+y}{x-y}$

or $\quad \dfrac{dy}{dx}=\dfrac{x+y}{x-y} \qquad (3)$

which is a first order homogeneous equation .

STEP (3): Let $y = Vx$ in equation (3) so that $\dfrac{dy}{dx} = V + x\dfrac{dV}{dx}$

Thus $\quad V + x\dfrac{dV}{dx} = \dfrac{1+V}{1-V}$

or $\quad x\dfrac{dV}{dx} = \dfrac{1+V}{1-V} - V = \dfrac{1+V^2}{1-V}$

Separating the variables , we get

$$\frac{1-V}{1+V^2}\,dV = \frac{dx}{x}$$

or $\quad \left(\dfrac{1}{1+V^2} - \dfrac{1}{2}\,\dfrac{2V}{1+V^2}\right)dV = \dfrac{dx}{x}$

Integrating , we get

$$\tan^{-1}V - \frac{1}{2}\ln(1+V^2) = \ln x + \ln C$$

or $\quad \tan^{-1}V = \ln(Cx) + \ln\left(\sqrt{1+V^2}\right)$

or $\quad \tan^{-1}V = \ln\left(Cx\sqrt{1+V^2}\right)$

or $\quad \tan^{-1}\left(\dfrac{y}{x}\right) = \ln\left(Cx\sqrt{1+\dfrac{y^2}{x^2}}\right)$

or $\quad \tan^{-1}\left(\dfrac{y}{x}\right) = \ln\left(C\sqrt{x^2+y^2}\right)$

which is the required equation of 45^0 trajectories .

8.13 OBLIQUE TRAJECTORIES IN POLAR COORDINATES

Let C_1 be a member of the family of given polar curves . Let C_2 be a member of the family of oblique trajectories intersecting C_1 at an angle α at any point $P(r,\theta)$. Let ϕ_1 and ϕ_2 be the angles which the radius vector OP makes with the tangents to the curves C_1 and C_2 at P respectively , as shown in figure (8.9) . Then the angle α between C_1 and C_2 at P is defined as the angle between their tangents at the point P . Then from the figure

$$\alpha = \phi_1 - \phi_2$$

so that $\tan\alpha = \tan(\phi_1 - \phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$

or $\tan\phi_1 - \tan\phi_2 = (1 + \tan\phi_1 \tan\phi_2)\tan\alpha$

or $\tan\phi_1 (1 - \tan\phi_2 \tan\alpha) = \tan\phi_2 + \tan\alpha$

or $$\tan\phi_1 = \frac{\tan\phi_2 + \tan\alpha}{1 - \tan\phi_2 \tan\alpha} \qquad (1)$$

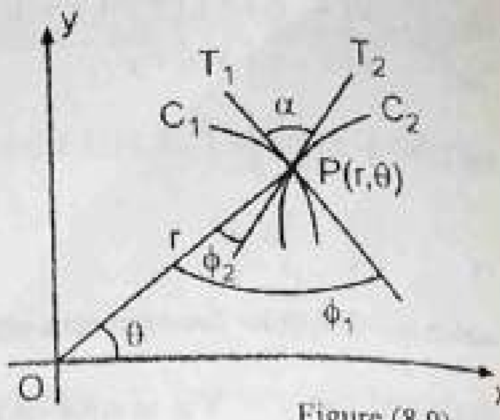


Figure (8.9)

But $\tan\phi_1 = \left(r\dfrac{d\theta}{dr}\right)_1$ and $\tan\phi_2 = \left(r\dfrac{d\theta}{dr}\right)_2$, therefore equation (1) becomes

$$\left(r\frac{d\theta}{dr}\right)_1 = \frac{\left(r\dfrac{d\theta}{dr}\right)_2 + \tan\alpha}{1 - \left(r\dfrac{d\theta}{dr}\right)_2 \tan\alpha} \qquad (2)$$

8.14 RULE FOR FINDING THE EQUATION OF OBLIQUE TRAJECTORIES OF A FAMILY OF POLAR CURVES

Let the equation of the given family of polar curves be

$$f(r, \theta, C) = 0 \qquad (1)$$

STEP (1): Differentiate equation (1) w.r.t. θ and eliminate the parameter C between equation (1) and the resulting equation. This gives the differential equation of the family of curves (1).

Let it be $F\left(r, \theta, \dfrac{dr}{d\theta}\right) = 0$ (2)

STEP (2): Replace $r\dfrac{d\theta}{dr}$ by $\dfrac{\left(r\dfrac{d\theta}{dr}\right) + \tan\alpha}{1 - \left(r\dfrac{d\theta}{dr}\right)\tan\alpha}$ in equation (2) to get the differential equation of the family of oblique trajectories. Let it be

$$F\left(r, \theta, \frac{\left(r\dfrac{d\theta}{dr}\right) + \tan\alpha}{1 - \left(r\dfrac{d\theta}{dr}\right)\tan\alpha}\right) = 0 \qquad (3)$$

STEP 3: Integrate equation (3) to get the equation of the required oblique trajectories of the given family of curves.

EXAMPLE (11): **Find the 45° trajectories of the family of curves $r = a\sin\theta$.**

SOLUTION: The equation of family of curves is

$$r = a\sin\theta \tag{1}$$

STEP (1): Differentiating equation (1) w.r.t. θ, we get

$$\frac{dr}{d\theta} = a\cos\theta \quad \text{so that} \quad \frac{d\theta}{dr} = \frac{1}{a\cos\theta}$$

and $$r\frac{d\theta}{dr} = \frac{1}{a\cos\theta}a\sin\theta = \tan\theta \tag{2}$$

is the differential equation of the family of curves (1).

STEP (2): Replacing $r\dfrac{d\theta}{dr}$ by $\dfrac{r\dfrac{d\theta}{dr}+\tan\dfrac{\pi}{4}}{1-\left(r\dfrac{d\theta}{dr}\right)\tan\dfrac{\pi}{4}} = \dfrac{r\dfrac{d\theta}{dr}+1}{1-r\dfrac{d\theta}{dr}}$ in equation (2), the differential equation of $\frac{\pi}{4}$ trajectories of the family of curves (1) is

$$\frac{r\dfrac{d\theta}{dr}+1}{1-r\dfrac{d\theta}{dr}} = \tan\theta$$

or $$r\frac{d\theta}{dr}+1 = \left(1-r\frac{d\theta}{dr}\right)\tan\theta$$

$$r\frac{d\theta}{dr}(1+\tan\theta) = \tan\theta - 1$$

Separating the variables, we get

$$\frac{dr}{r} = \frac{1+\tan\theta}{\tan\theta-1}d\theta = \frac{\cos\theta+\sin\theta}{\sin\theta-\cos\theta}d\theta$$

Integrating, we get

$$\ln r = \ln(\sin\theta-\cos\theta)+\ln C_1$$

$$r = C_1(\sin\theta-\cos\theta)$$

$$r = \sqrt{2}C_1\left(\frac{1}{\sqrt{2}}\sin\theta-\frac{1}{\sqrt{2}}\cos\theta\right)$$

$$= \sqrt{2}C_1\left(\cos\frac{\pi}{4}\sin\theta-\sin\frac{\pi}{4}\cos\theta\right)$$

$$= \sqrt{2}C_1\sin\left(\theta-\frac{\pi}{4}\right)$$

$$= C\sin\left(\theta-\frac{\pi}{4}\right) \quad \text{where} \quad C = \sqrt{2}C_1$$

8.15 PHYSICAL APPLICATIONS

The first order initial – value problem

$$\frac{dy}{dt} = Ky; \quad y(0) = y_0 \tag{1}$$

where K is a constant, occurs in many physical problems. The differential equation in (1) means that the time rate of change of a quantity $y(t)$ is proportional to the quantity present at that time t.

The solution of this problem can be obtained by separating the variables

i.e. $$\frac{dy}{y} = K\,dt$$

Integrating, we get

$$\ln y = Kt + A$$

or $$y = e^{Kt+A} = e^{Kt}e^{A}$$

or $$y(t) = Ce^{Kt} \quad (\text{where } C = e^{A}) \tag{2}$$

is the general solution of equation (1).

Using the initial condition that when $t = 0$, $y(0) = y_0$, we get from equation (1)

$$C = y_0$$

Thus equation (2) becomes

$$y(t) = y_0 e^{Kt} \tag{3}$$

which is the solution of the initial – value problem (1).

(a) GROWTH AND DECAY PROBLEMS

Exponential growth and exponential decay are quite important models in physics, biology, etc.

(i) POPULATION MODEL, EXPONENTIAL GROWTH OR DECAY

If relatively small populations (of humans, animals, bacteria, etc.) are left undisturbed, they often grow according to Malthus's law [named after the English social scientist (1766 – 1834)] which states that the time rate of change of a population (growth or decay) is proportional to the population present at any given time t.

Let $N(t)$ denote the number of a population that is either growing or decaying. The time rate of change is $\frac{dN}{dt}$. According to the law governing the process of growth or decay, $\frac{dN}{dt}$ is proportional to N. Thus $\frac{dN}{dt} \propto N$.

or $$\frac{dN}{dt} = KN$$

where K is the constant of proportionality.

If the initial population is N_0, then the initial condition is i.e. when $t = 0$, $N(0) = N_0$. This is the same initial – value problem as in equation (1).

As in equation (3), the solution of this problem is

$$N(t) = N_0 e^{Kt} \tag{4}$$

Equation (4) is called the law of exponential growth or exponential decay.

The exponential function e^{Kt} increases as t increases for $K > 0$, and decreases as t decreases. Thus in problems describing growth such as population, K will have a positive value, whereas in problems involving decay such as radioactive disintegration, K will yield a negative value.

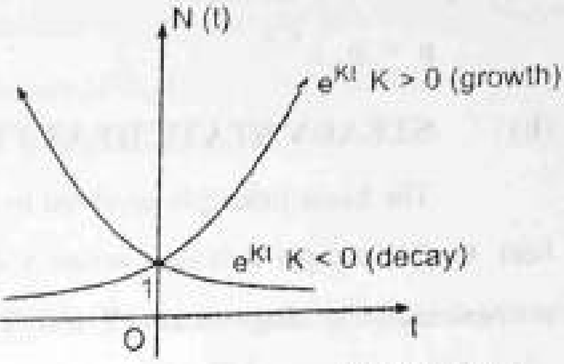


Figure (8.10)

(ii) RADIOACTIVITY, EXPONENTIAL DECAY

Experiments show that a radioactive substance such as radium decays at a rate proportional to the amount present. Let $y(t)$ be the amount of substance present at time t. The rate of change is $\frac{dy}{dt}$. According to the physical law governing the process of radiation, $\frac{dy}{dt}$ is proportional to y. Thus

$$\frac{dy}{dt} = Ky$$

where K is the constant of proportionality. This equation is a first order differential equation. As in equation (2) the solution of this equation is

$$y(t) = Ce^{Kt} \tag{5}$$

If the initial amount of substance is y_0 then the initial condition is i.e. when $t = 0$, $y(0) = y_0$. As in equation (3), the solution of this problem is

$$y(t) = y_0 e^{Kt} \tag{6}$$

(iii) ATMOSPHERIC PRESSURE, EXPONENTIAL DECAY

Experiments show that the rate of change of the atmospheric pressure p with altitude h is proportional to the pressure. Let $p(h)$ denote the pressure at a height h. The rate of change is $\frac{dp}{dh}$. According to the physical law governing the pressure, $\frac{dp}{dh}$ is proportional to p. Thus

$$\frac{dp}{dh} = Kp$$

where K is a constant of proportionality. This equation is a first order differential equation.

As in equation (2) the solution of this equation is

$$P(h) = C e^{Kh} \tag{7}$$

Assuming that the pressure at the sea level is p_0 i.e. when $h = 0$, $p = p_0$, we get from equation (7)

$$p_0 = C$$

Thus equation (7) becomes

$$p = p_0 e^{Kh}$$

(b) STEADY STATE HEAT FLOW PROBLEM

The basic principle involved in the problems of heat conduction in steady state is that quantity of heat Q flowing per unit time across a slab of cross – sectional area A and thickness Δx, whose faces are maintained at temperatures T and $T + \Delta T$, is given by

$$Q = -KA\frac{dT}{dx} \tag{8}$$

where K is the coefficient of thermal conductivity.

(c) NEWTON'S LAW OF COOLING

Newton's law of cooling states that the time rate of change of temperature of the body is proportional to the temperature difference between the body and its surrounding medium.

Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change of the temperature of the body is $\frac{dT}{dt}$, and Newton's law of cooling can be formulated as

$$\frac{dT}{dt} = -K(T - T_m)$$

or

$$\frac{dT}{dt} + KT = KT_m \tag{9}$$

where we have denoted the constant of proportionality by $-K$ in order to make $\frac{dT}{dt}$ negative, since the temperature of the body decreases with time. Note that $T > T_m$ in a cooling process. This law was verified experimentally by Newton.

Equation (9) is a first order linear differential equation. To solve it, the integrating factor is given by

$$\text{I.F.} = e^{\int K\,dt} = e^{Kt}$$

The solution of equation (9) is thus

$$T(t)(\text{I.F.}) = \int (\text{I.F.})\,KT_m\,dt + C$$

or

$$e^{Kt}T(t) = \int e^{Kt}KT_m\,dt + C = T_m e^{Kt} + C$$

or $T(t) = Ce^{-Kt} + T_m$ (10)

If the initial temperature of the body is T_0 i.e. when $t = 0$, $T(0) = T_0$, we get from equation (10)

$$T_0 = C + T_m \quad \text{or} \quad C = T_0 - T_m$$

Thus from equation (5), the temperature of the body at any time t is

$$T(t) = (T_0 - T_m)e^{-Kt} + T_m \qquad (11)$$

NOTE: (i) In this formula, it is assumed T_m remains constant and also we assume that the heat flows rapidly enough that the temperature of the body is the same at all points of the body at a given time.

(ii) Newton's law of cooling is equally applicable to heating. In this case, it takes the form

$$\frac{dT}{dt} = K(T_m - T) \qquad (12)$$

EXAMPLE (12): **A body whose temperature is initially 200 °C is immersed in a liquid whose temperature T_m is constantly 100 °C. If the temperature of the body is 150 °C at $t = 1$ minute, what is the temperature at $t = 2$ minutes?**

SOLUTION: We know that the differential equation of the Newton's law of cooling is

$$\frac{dT}{dt} + KT = -KT_m$$

whose solution is given by

$$T(t) = (T_0 - T_m)e^{-Kt} + T_m \qquad (1)$$

where T_0 is the initial temperature of the body.

In the present case, $T_0 = 200\,^{\circ}C$, $T_m = 100\,^{\circ}C$.

Thus equation (1) becomes

$$T(t) = (200 - 100)e^{-Kt} + 100$$

$$= 100e^{-Kt} + 100 \qquad (2)$$

Since at $t = 1$, $T = 150\,^{\circ}C$, we get from equation (2).

$$150 = 100e^{-K} + 100$$

or $$100e^{-K} = 50$$

or $$e^{-K} = \frac{1}{2}$$

Substituting in equation (2), we get

$$T(t) = 100(e^{-K})^t + 100$$

$$= 100\left(\frac{1}{2}\right)^t + 100 \qquad (3)$$

Thus at $t = 2$ minutes, we get

$$T(t) = 100\left(\frac{1}{2}\right)^2 + 100$$

$$= 25 + 100 = 125\,^{\circ}C$$

8.16 APPLICATIONS TO MECHANICS

Let a body of mass m start moving from a fixed point O, along the x – axis, under the action of several forces whose resultant is F. If P is the position of the body at any time t such that $OP = x$, then

(i) the velocity of the body at any time t is given by

$$v = \frac{dx}{dt} \text{ and}$$

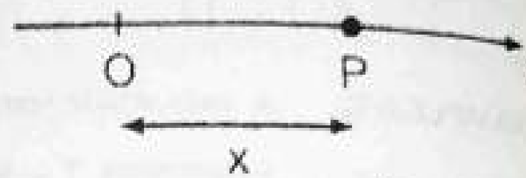


Figure (8.11)

(ii) the acceleration of the body at any time t is

$$a = \frac{dv}{dt}$$

$$= \frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

Also, Newton's second law of motion states that the time rate of change of linear momentum is equal to the total force acting on the body,

i.e. $$F = \frac{d}{dt}(mv)$$

For a constant mass m, this can be written as

$$F = m\frac{dv}{dt} = ma \qquad (1)$$

Equation (1) can also be written in the form

$$F = ma = m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} \qquad (2)$$

8.17 VERTICAL MOTION UNDER GRAVITY

It is found experimentally that when a particle moves in a straight line under gravity, near the surface of the earth, its acceleration is constant provided the air resistance is negligible. This acceleration is denoted by g and is called the **acceleration due to gravity**. The approximate value of g in SI system is $9.8\ m/s^2$.

In solving problems we may take the upward or downward direction as positive whichever is more convenient. When upward direction is taken positive, acceleration is $-g$, and when downward direction is taken positive, acceleration is g.

(a) VERTICALLY FALLING BODY PROBLEM

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and for convenience, choose the downward direction as the positive direction.

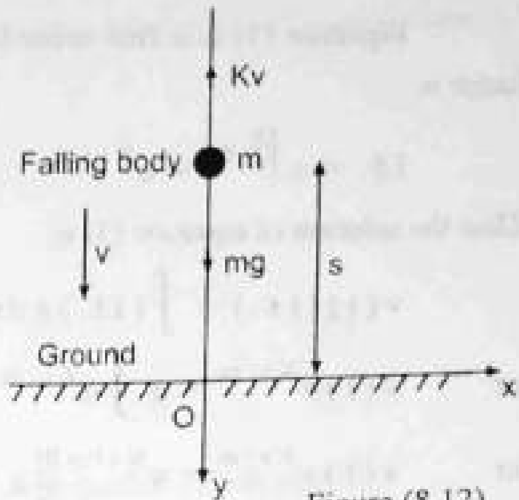


Figure (8.12)

Newton's second law of motion states that the time rate of change of the momentum of the body is equal to the total force acting on the body. For constant mass m, it can be written as

$$m\frac{dv}{dt} = F \tag{1}$$

where v is the velocity of the body, and F the total force on the body, both at time t.

In the present case, there are two forces acting on the body:

(i) the force due to gravity given by the weight W of the body, which equals mg and

(ii) the force due to air resistance given by $-Kv$, where $K \geq 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; i.e. it acts in the upward (or negative) direction.

The net force F acting on the body is, therefore

$$F = mg - Kv \tag{2}$$

From equations (1) and (2), we get

$$m\frac{dv}{dt} = mg - Kv$$

or $$\frac{dv}{dt} + \frac{K}{m}v = g \tag{3}$$

as the differential equation of motion of the body.

Note that equation (3) is valid only if the given conditions are satisfied. This equation is not valid if, for example, air resistance is not proportional to velocity but to the velocity squared, or if the upward direction is taken to be the positive direction.

If the upward direction is taken as the positive direction, then equation (3) becomes

$$m\frac{dv}{dt} = -mg + Kv$$

or $$\frac{dv}{dt} - \frac{K}{m}v = -g \tag{4}$$

SOLUTION OF EQUATION (3)

Equation (3) is a first order linear differential equation. To solve this equation the integrating factor is

$$\text{I.F.} = e^{\int \frac{K}{m} dt} = e^{Kt/m}$$

Thus the solution of equation (3) is

$$v(t)(\text{I.F.}) = \int (\text{I.F.})\, g\, dt + C_1$$

or $$v(t)e^{Kt/m} = g\int e^{Kt/m}\, dt + C_1$$

or $$v(t)e^{Kt/m} = e^{Kt/m}\frac{m}{K}g + C_1$$

or $$v(t) = C_1 e^{-Kt/m} + \frac{m}{K}g \qquad (5)$$

is the velocity at any time t.

If the initial velocity is v_0 i.e. when $t = 0$, $v = v_0$, we get from equation (5)

$$v_0 = C_1 + \frac{m}{K}g$$

or $$C_1 = v_0 - \frac{m}{K}g$$

Thus from equation (5), the velocity at any time t is given by

$$v(t) = \left(v_0 - \frac{m}{K}g\right)e^{-Kt/m} + \frac{m}{K}g \qquad (6)$$

or $$\frac{ds}{dt} = \left(v_0 - \frac{m}{K}g\right)e^{-Kt/m} + \frac{m}{K}g$$

Integrating again, we get

$$s = \left(v_0 - \frac{m}{K}g\right)\left(-\frac{m}{K}\right)e^{-Kt/m} + \frac{m}{K}gt + C_2$$

$$= \left(\frac{m^2}{K^2}g - \frac{m}{K}v_0\right)e^{-Kt/m} + \frac{m}{K}gt + C_2 \qquad (7)$$

If the initial position of the body is s_0 i.e. when $t = 0$, $s = s_0$, we get from equation (7)

$$s_0 = \left(\frac{m^2}{K^2}g - \frac{m}{K}v_0\right) + C_2$$

or $$C_2 = s_0 - \left(\frac{m^2}{K^2}g - \frac{m}{K}v_0\right)$$

Thus from equation (7), we get

$$s = \left(\frac{m^2}{K^2}g - \frac{m}{K}v_0\right)e^{-Kt/m} + \frac{m}{K}gt + s_0 - \left(\frac{m^2}{K^2}g - \frac{m}{K}v_0\right)$$

$$= s_0 + \left(\frac{m^2}{K^2}g - \frac{m}{K}v_0\right)\left(e^{-Kt/m} - 1\right) + \frac{m}{K}gt \qquad (8)$$

ZERO INITIAL VELOCITY

If the initial velocity is zero, then $v_0 = 0$, and from equation (6) the velocity at any time t is given by

$$v(t) = \frac{mg}{K}\left(1 - e^{-\frac{K}{m}t}\right) \tag{9}$$

The position of body at any time t is given from equation (8) as

$$s = s_0 + g\left(e^{-\frac{K}{m}t} - 1\right) \tag{10}$$

PARTICULAR CASE

If air resistance is negligible or non existent, then $K = 0$ and equation (3) simplifies to

$$\frac{dv}{dt} = g \tag{11}$$

Integrating, we get

$$v = gt + C_1 \tag{12}$$

If the initial velocity is v_0 i.e. when $t = 0$, $v = v_0$, we get from equation (12)

$$v_0 = C_1$$

Thus equation (12) becomes

$$v = gt + v_0$$

or $$\frac{ds}{dt} = gt + v_0$$

Integrating again, we get

$$s = \frac{1}{2}gt^2 + v_0 t + C_2 \tag{13}$$

If the initial position of the body is s_0 i.e. when $t = 0$, $s = s_0$, we get from equation (13)

$$s_0 = C_2$$

Thus equation (13) takes the form

$$s = \frac{1}{2}gt^2 + v_0 t + s_0 \tag{14}$$

If the initial velocity is zero, then $v_0 = 0$ and equation (14) reduces to

$$s = \frac{1}{2}gt^2 + s_0 \tag{15}$$

(b) VERTICALLY RISING BODY PROBLEM

Consider a vertically rising body of mass m that is being influenced only by gravity and an air resistance proportional to the velocity of the body. Assume that both gravity and mass remain constant and choose the upward direction as the positive direction.

Newton's second law of motion for a constant mass m takes the form

$$m\frac{dv}{dt} = F \quad (1)$$

where v is the velocity of the body, and F the total force on the body.

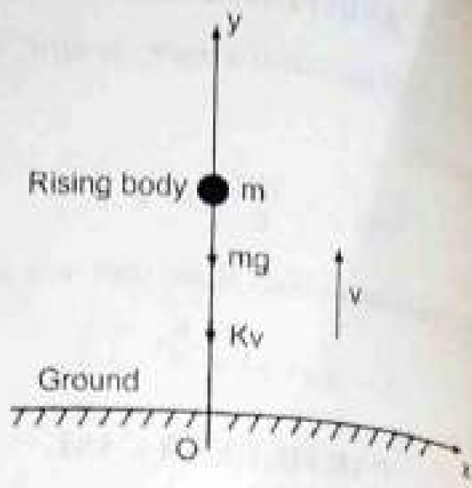


Figure (8.13)

In the present case, there are two forces acting on the body:

(i) the force due to gravity given by the weight W of the body, which equals mg and

(ii) the force due to air resistance given by Kv, where K is a constant of proportionality.

Since both these forces act in the downward (or negative) direction therefore, net force acting on the body is therefore

$$F = -mg - Kv \quad (2)$$

From equations (1) and (2), we get

$$m\frac{dv}{dt} = -mg - Kv$$

or $$\frac{dv}{dt} + \frac{K}{m}v = -g \quad (3)$$

as the differential equation of motion. This equation is a first order linear differential equation. To solve this equation the integrating factor is

$$\text{I.F.} = e^{\int \frac{K}{m} dt} = e^{Kt/m}$$

Thus the solution of equation (3) is

$$v(t)(\text{I.F.}) = \int (\text{I.F.})(-g)\,dt + C$$

or $$v(t)e^{Kt/m} = \int e^{Kt/m}(-g)\,dt + C$$

$$= -\frac{mg}{K}e^{Kt/m} + C$$

or $$v(t) = Ce^{-Kt/m} - \frac{m}{K}g \quad (4)$$

is the velocity at any time t.

If the initial velocity is v_0 i.e. when $t = 0$, $v = v_0$, we get from equation (4)

$$v_0 = C - \frac{m}{K}g$$

or $$C = v_0 + \frac{mg}{K}$$

Thus from equation (4), the velocity at any time t is given by

$$v(t) = \left(v_0 + \frac{m}{K}g\right)e^{-Kt/m} - \frac{m}{K}g \tag{5}$$

TIME TO REACH THE BODY AT MAXIM HEIGHT

The body reaches it maximum height when $v = 0$. Thus we require t when $v = 0$.

Substituting $v = 0$ in equation (5), we get

$$0 = \left(v_0 + \frac{m}{K}g\right)e^{-Kt/m} - \frac{m}{K}g$$

or $$e^{-Kt/m} = \frac{\frac{m}{K}g}{v_0 + \frac{m}{K}g} = \frac{1}{1 + \frac{v_0 K}{m g}}$$

or $$-\frac{K}{m}t = \ln\left(\frac{1}{1 + \frac{v_0 K}{m g}}\right)$$

or $$t = \frac{m}{K}\ln\left(1 + \frac{v_0 K}{m g}\right) \tag{6}$$

8.18 SIMPLE HARMONIC MOTION

A particle is said to execute simple harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.

Let O be the fixed point in the line A′OA and let P be the position of the particle at any time t, OP being equal to x, and v the velocity of the particle in the direction OP at this instant, then the acceleration of the particle is $\frac{d^2x}{dt^2}$ or $v\frac{dv}{dx}$.

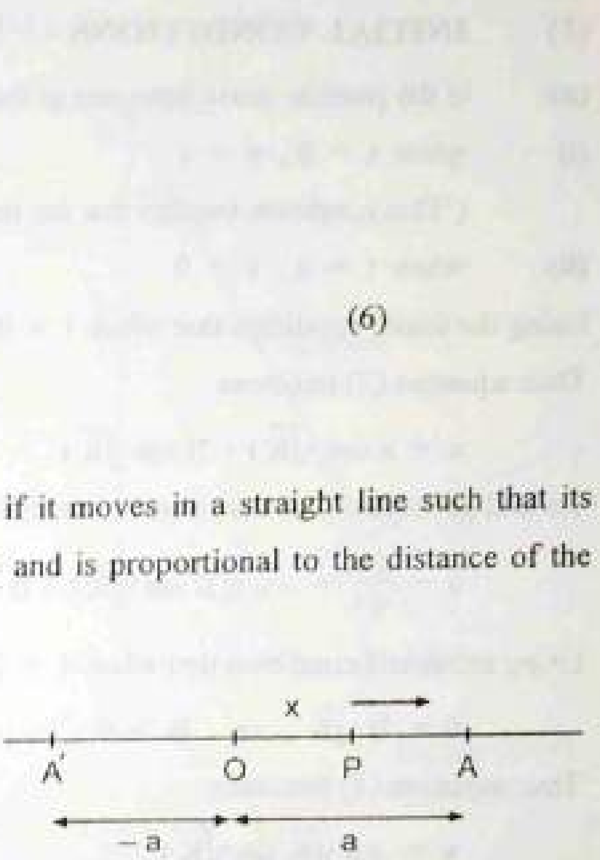


Figure (8.14)

For simple harmonic motion this acceleration is proportional to x and is directed towards O. That is, the acceleration is $-Kx$, where K is a constant of proportionality and negative sign indicates that the acceleration is directed against the direction in which x is increasing. Thus by Newton's second law, the motion is described by

$$\frac{d^2x}{dt^2} = -Kx$$

or $$\frac{d^2x}{dt^2} + Kx = 0 \tag{1}$$

which is a second order linear differential equation.

GENERAL SOLUTION OF THE EQUATION (1)

Equation (1) can be written in operator notation as

$$(D^2 + K)x = 0$$

The auxiliary equation is $D^2 + K = 0$ having characteristic roots as $D = \pm i\sqrt{K}$.

Therefore the general solution of equation (1) is

$$x = A\cos\sqrt{K}\,t + B\sin\sqrt{K}\,t \tag{2}$$

where A and B are arbitrary constants which can be determined by making use of the specified boundary conditions.

(1) INITIAL CONDITIONS

(a) If the particle starts from rest at the point A, then the initial conditions become :

(i) when $t = 0$, $x = a$

(This condition implies that the time is measured from the instant when the particle is at A) and

(ii) when $t = 0$, $v = 0$

Using the initial condition that when $t = 0$, $x = a$, we get from equation (2), $a = A$

Thus equation (2) becomes

$$x = a\cos\sqrt{K}\,t + B\sin\sqrt{K}\,t \tag{3}$$

To find the velocity at any time t, we differentiate equation (3) w.r.t. t. Thus

$$v = \frac{dx}{dt} = -a\sqrt{K}\sin\sqrt{K}\,t + B\sqrt{K}\cos\sqrt{K}\,t \tag{4}$$

Using the initial condition that when $t = 0$, $v = 0$, we get from equation (4)

$$0 = B\sqrt{K} \quad \text{or} \quad B = 0$$

Thus equation (4) becomes

$$v = -a\sqrt{K}\sin\sqrt{K}\,t \tag{5}$$

which is the velocity at any time t. The position of the particle at any time t from equation (3) is given by

$$x = a\cos\sqrt{K}\,t \tag{6}$$

(b) If the time is measured from the instant when the particle is at O, i.e.

when $t = 0$, $x = 0$, we get from equation (3)

$$0 = A$$

Thus equation (2) becomes

$$x = B\sin\sqrt{K}\,t \tag{7}$$

To find the velocity, we differentiate equation (7)

$$\frac{dx}{dt} = B\sqrt{K}\cos\sqrt{K}\,t$$

or

$$v = B\sqrt{K}\cos\sqrt{K}\,t \tag{8}$$

If the initial velocity is v_0 i.e. when $t = 0$, $v = v_0$, we get from equation (8)

$$v_0 = B\sqrt{K} \quad \text{or} \quad B = \frac{v_0}{\sqrt{K}}$$

Thus from equation (8), we get the velocity at any time t as

$$v(t) = \frac{v_0}{\sqrt{K}}\sqrt{K}\cos\sqrt{K}\,t = v_0\cos\sqrt{K}\,t \tag{9}$$

8.19 SIMPLE PENDULUM

If a heavy mass is attached to one end of a light inelastic string, the other and of which is fixed and oscillates under gravity in a vertical plane, the system is called a **simple pendulum**.

Let OA be an inelastic string of length ℓ whose one end is fixed at O and the other end is attached to a bob of mass m. Let θ be the angle which the string makes with the vertical line at any time t, when the bob is at the point P such that $OP = s$.

It is assumed that no resisting or other external forces are present.

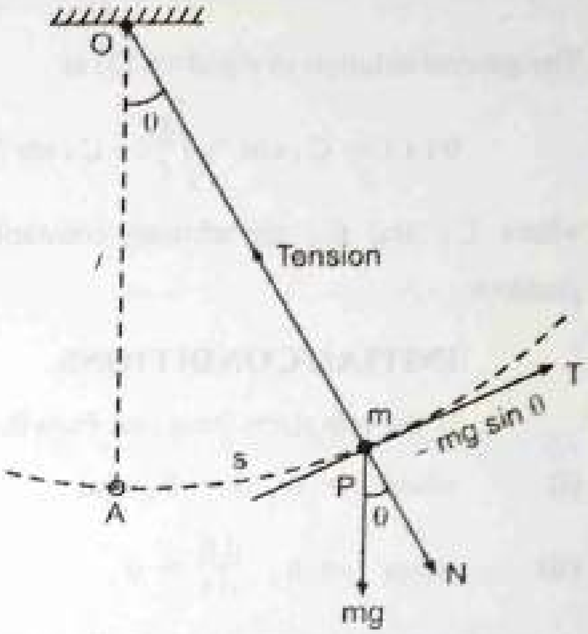


Figure (8.15)

The forces acting on the bob are :

(i) its weight mg acting vertically downward, and

(ii) the tension T in the string.

Resolve the weight mg of the bob into two components $mg\cos\theta$ along the normal PN and

$mg\cos\left(\theta + \frac{\pi}{2}\right) = -mg\sin\theta$ along the tangent PT.

The component $mg\cos\theta$ is balanced by the tension in the string.

By Newton's second law, the equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

or
$$\frac{d^2s}{dt^2} = g\sin\theta \tag{1}$$

where $\frac{d^2s}{dt^2}$ is the tangential acceleration.

Since $s = \ell\theta$, equation (1) becomes

$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta$$

or
$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0 \tag{2}$$

Since $\ell = \frac{s}{\theta}$, equation (2) is a second order non – linear differential equation whose solution cannot be obtained in elementary functions.

To get the solution in elementary functions, we make the further simplifying assumption.

We know that for small angle θ, $\sin\theta$ is very nearly equal to θ, where θ is in radians.

Thus equation (2) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \qquad (3)$$

which is a second order linear differential equation. This equation is of the same type as the equation for simple harmonic motion, with x replaced by θ and $K = \frac{g}{\ell}$.

The general solution of equation (3) is

$$\theta(t) = C_1 \cos\sqrt{\frac{g}{\ell}}\,t + C_2 \sin\sqrt{\frac{g}{\ell}}\,t \qquad (4)$$

where C_1 and C_2 are arbitrary constants whose values are determined using the initial conditions of the problem.

INITIAL CONDITIONS

If the bob starts from rest from the position where $\theta = \theta_0$ then the initial conditions become:

(i) when $t = 0$, $\theta = \theta_0$ and

(ii) when $t = 0$, $\frac{d\theta}{dt} = 0$.

Using these initial conditions, we find from equation (4) $C_1 = \theta_0$ and $C_2 = 0$.

Thus equation (4) reduces to

$$\theta(t) = \theta_0 \cos\sqrt{\frac{g}{\ell}}\,t \qquad (5)$$

8.20 MOTION OF A SIMPLE PENDULUM WITH RESISTANCE

Suppose that a simple pendulum of length ℓ [Figure (8.15)] oscillates through a small angle θ in a medium for which the resistance is proportional to the velocity.

The equation of motion along the tangent PT in this case is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta - Kv \quad (K = \text{constant of proportionality})$$

or $$m\frac{d^2s}{dt^2} = -mg\sin\theta - K\frac{ds}{dt}$$

or $$\frac{d^2s}{dt^2} + \frac{K}{m}\frac{ds}{dt} + g\sin\theta = 0 \qquad (1)$$

Since $s = \ell\theta$, equation (1) becomes

$$\ell\frac{d^2\theta}{dt^2}+\frac{K\ell}{m}\frac{d\theta}{dt}+g\sin\theta = 0$$

or
$$\frac{d^2\theta}{dt^2}+\frac{K}{m}\frac{d\theta}{dt}+\frac{g}{\ell}\sin\theta = 0 \qquad (2)$$

which is a second order differential equation.

For a small angle θ, $\sin\theta$ is very nearly equal to θ, therefore equation (2) takes the form

$$\frac{d^2\theta}{dt^2}+\frac{K}{m}\frac{d\theta}{dt}+\frac{g}{\ell}\theta = 0 \qquad (3)$$

In operator notation, this equation can be written as

$$\left(D^2+\frac{K}{m}D+\frac{g}{\ell}\right)\theta = 0$$

Writing $\frac{K}{m} = 2\lambda$ and $\frac{g}{\ell} = W^2$, this takes the form

$$(D^2+2\lambda D+W)\theta = 0$$

The auxiliary equation is $D^2+2\lambda D+W = 0$.

whose characteristic roots are $D = -\lambda \pm i\sqrt{W^2-\lambda^2}$.

The oscillatory motion of the bob is possible only when $K < W$. The solution of the differential equation is therefore

$$\theta = e^{-\lambda t}\left(C_1\cos\sqrt{W^2-\lambda^2}\,t+C_2\sin\sqrt{W^2-\lambda^2}\,t\right) \qquad (4)$$

which gives the vibratory motion.

Using the initial conditions that when

$$t = 0,\ \theta = \theta_0 \text{ and when } t = 0,\ \frac{d\theta}{dt} = 0,$$

we get from equation (4) that $C_1 = \theta_0$ and $C_2 = 0$.

Thus equation (4) reduces to

$$\theta = e^{-\lambda t}\left(\theta_0\cos\sqrt{W^2-K^2}\,t\right)$$

$$= e^{-\frac{Kt}{2m}}\left(\theta_0\cos\sqrt{\frac{g^2}{\ell^2}-\frac{K^2}{4m^2}}\,t\right) \qquad (5)$$

8.21 HANGING CABLE

Let a cable, like a telephone line, be hung from two points at the same level. Assume that the cable is flexible so that due to the load of the cable it takes the shape as shown in figure (8.16). Let A be the lowest point of the cable, and choose x – axis and y – axis so that y – axis passes through the point A.

Consider a portion of the hanging cable between the point A and an arbitrary point $P(x, y)$ on the cable so that $AP = s$. Let the weight of the cable per unit length be w, therefore the weight of the portion $AP = s$ of the cable is ws.

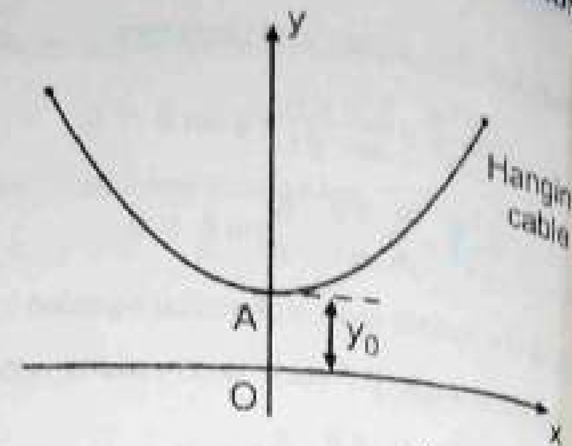


Figure (8.16)

The forces acting on the cable in equilibrium position are :

(1) H = horizontal tension pulling on the cable at A.

(2) T = tangential tension pulling on the cable at P.

(3) $W = ws$ = weight of the cable from A to P.

Then the equilibrium of the cable requires that the horizontal and vertical components of T balance H and W respectively. Thus

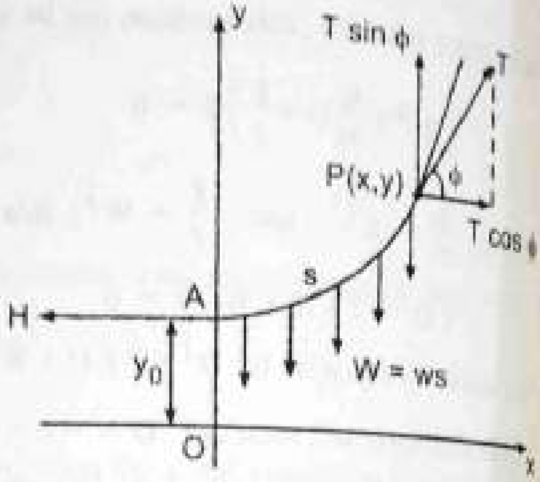


Figure (8.17)

$$T\cos\phi = H \tag{1}$$

and $$T\sin\phi = W = ws \tag{2}$$

Dividing equation (2) by equation (1), we get

$$\frac{T\sin\phi}{T\cos\phi} = \tan\phi = \frac{W}{H}$$

Since $\tan\phi = \dfrac{dy}{dx}$, therefore

$$\frac{dy}{dx} = \frac{ws}{H} \tag{3}$$

Differentiating equation (3), we get

$$\frac{d^2y}{dx^2} = \frac{w}{H}\frac{ds}{dx} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{4}$$

which is a second order differential equation.

The solution of this equation will involve two arbitrary constants, which will be determined using two boundary conditions.

The boundary conditions are

$$y = y_0 \text{ when } x = 0 \tag{5}$$

and $$\frac{dy}{dx} = 0 \text{ when } x = 0 \tag{6}$$

Equations (4), (5), and (6) constitute a second order boundary value problem.

To solve equation (4), let $\dfrac{dy}{dx} = P$ then $\dfrac{d^2y}{dx^2} = \dfrac{dP}{dx}$ so that equation (4) becomes

$$\frac{dP}{dx} = \frac{w}{H}\sqrt{1+P^2}$$

Separating the variables , we get

$$\frac{dP}{\sqrt{1+P^2}} = \frac{w}{H}\,dx$$

Integrating both sides we get

$$\int \frac{dP}{\sqrt{1+P^2}} = \int \frac{w}{H}\,dx + C_1$$

or $$\sinh^{-1} P = \frac{w}{H}x + C_1 \qquad (7)$$

Using the second boundary condition i.e. $P = \frac{dy}{dx} = 0$, when $x = 0$, we find from equation (7)

$$C_1 = \sinh^{-1} 0 = 0$$

Thus equation (7) becomes

$$\sinh^{-1} P = \frac{w}{H}x$$

or $$P = \sinh\left(\frac{w}{H}x\right)$$

or $$\frac{dy}{dx} = \sinh\left(\frac{w}{H}x\right)$$

Integrating again , we get

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) + C_2 \qquad (8)$$

Using the first boundary condition i.e. $y = y_0$ when $x = 0$, we find from equation (8)

$$y_0 = \frac{H}{w}\cosh 0 + C_2$$

or $$C_2 = y_0 - \frac{H}{w}$$

Thus equation (8) becomes

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) + y_0 - \frac{H}{w}$$

If we choose $y_0 = \frac{H}{w}$, then

$$y = \frac{H}{w}\cosh\left(\frac{w}{H}x\right) \qquad (9)$$

or $$y(x) = a\cosh\left(\frac{x}{a}\right) \quad \text{with} \quad a = \frac{H}{w}$$

Such a curve is sometimes called a **Catenary** (derived from the Latin word **Catena** meaning chain) . Many transmission lines , telegraph cables and cables of suspension bridges hang in the form of Catenaries .

TENSION IN THE CABLE AT P (x , y)

We know that $\frac{dy}{dx} = \tan\phi$

Since T acts along the tangent , and

$$T = \frac{H}{\cos\phi} = H\sec\phi$$

Also $y = a\cosh\left(\frac{x}{a}\right)$ with $a = \frac{H}{w}$

therefore $\frac{dy}{dx} = a\left(\frac{1}{a}\right)\sinh\left(\frac{x}{a}\right) = \sinh\left(\frac{x}{a}\right)$

and $\sec\phi = \sqrt{\sec^2\phi} = \sqrt{1+\tan^2\phi}$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\sinh^2\frac{x}{a}} = \sqrt{\cosh^2\frac{x}{a}} = \cosh\left(\frac{x}{a}\right)$$

Therefore $T = H\sec\phi = H\cosh\left(\frac{x}{a}\right)$ (10)

where $a = \frac{H}{w}$ or $H = wa$ (11)

From equations (10) and (11) , we get

$$T = wa\cosh\left(\frac{x}{a}\right) = wy$$

8.22 MASS SPRING SYSTEM

MODELING : FREE OSCILLATIONS

Linear homogeneous differential equations with constant coefficients have important engineering applications . In this section we shall consider such an application , which is fundamental . It is taken from mechanics , but we shall see later that it has a complete analog in electric circuits .

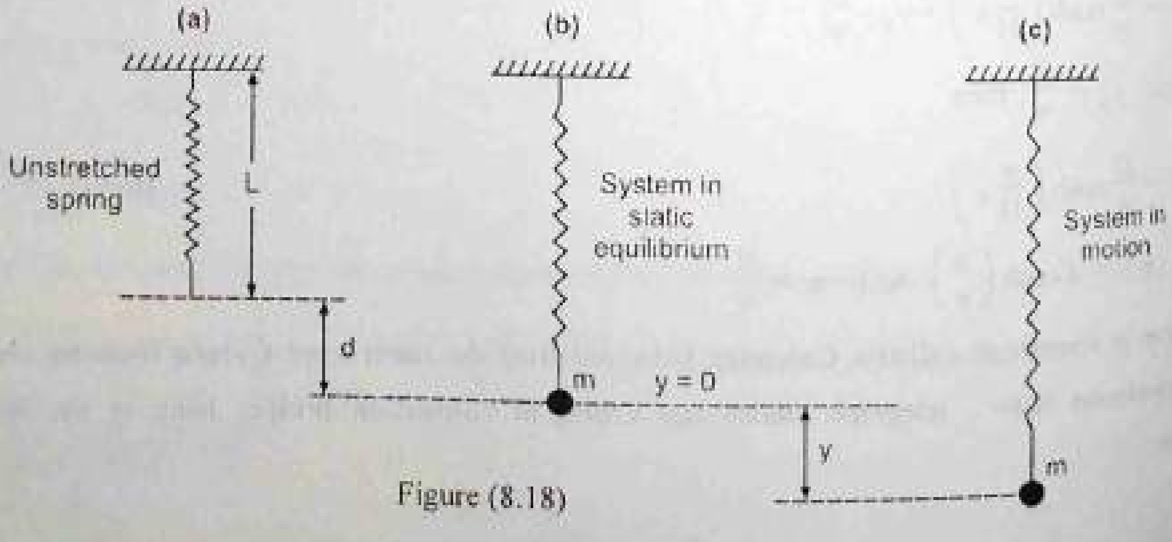


Figure (8.18)

We take an ordinary spring which resists compression as well as extension and suspend it vertically from a fixed support as shown in figure (8.18). At the lower end of the spring we attach a body of mass m. We assume m to be so large that we may neglect the mass of the spring. If we pull the body down a certain distance y and then release it, it undergoes a motion. We assume that the body moves strictly vertically.

We want to determine the motion of this mechanical system. For this purpose we consider the forces acting on the body during the motion. We may choose the **downward direction** as the positive direction and regard forces which act downward as positive and upward forces as negative.

The forces acting on the body are :

(i) its weight acting vertically downward

$$F_1 = mg \qquad (1)$$

where m is the mass of the body and g is the acceleration due to gravity.

(ii) the **spring force** exerted by the spring if the spring is stretched.

Experiments show that this force is proportional to the distance stretched, say

$F = K s$ (Hooke's Law)

where s is the distance stretched. The constant of proportionality K is called the **spring modulus**, and varies from one spring to another. If $s = 1$, then $F = K$. The larger the value of K, the stiffer the spring.

When the body is at rest (i.e. $y = 0$) we describe its position as the **static equilibrium position**. Clearly in this position the spring is stretched by an amount d such that the corresponding spring force is $-Kd$ (minus because the spring tends to pull the body upwards). Also, in this position the resultant of this spring force and the gravitational force in equation (1) is zero (i.e. in static equilibrium all the forces balance). Hence

$$-Kd + mg = 0$$

or

$$Kd = mg \qquad (2)$$

Let $y = y(t)$ denote the displacement of the body from the static equilibrium position, with the positive direction downwards [figure (c)].

From Hooke's law it follows that the total spring force corresponding to a displacement y is

$$F_2 = -Kd - Ky \qquad (3)$$

i.e. the resultant of the spring force $-Kd$ when the body is in static equilibrium position and the additional spring force $-Ky$ caused by the displacement. Note that the sign of the last term in equation (3) is properly chosen, because when y is positive, $-Ky$ is negative and, according to our assumption, represents an upward force, while for negative y the force $-Ky$ represents a downward force.

Adding the forces due to gravity and the spring given by equations (1) and (3), we have a total force

$$F_1 + F_2 = mg - Kd - Ky,$$

and because of equation (2) this becomes

$$F_1 + F_2 = -K y \tag{4}$$

Now consider the following two cases :

(1) UNDAMPED SYSTEM (FREE OSCILLATIONS)

Here we imagine that the damping effects (such as air resistance) are negligible . In this case equation (4) represents the resultant of all the forces acting on the body . Then by Newton's second law of motion

Mass x Acceleration = Force

where **force** means the resultant of the forces acting on the body at any instant . In our case , the acceleration is $\frac{d^2y}{dt^2}$ and that the resultant is given by equation (4) . Thus

$$m\frac{d^2y}{dt^2} = -K y$$

or $$\frac{d^2y}{dt^2} + \frac{K}{m} y = 0 \tag{5}$$

which is a second order linear differential equation with constant coefficients .

The general solution is

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{6}$$

where $\omega_0 = \sqrt{\frac{K}{m}}$ is called the natural frequency of the spring system and is a function of the stiffness of the spring and mass of the body . The constants A and B must be determined from additional data of the problem (such as initial position and initial velocity of the body) . The corresponding motion is called a **harmonic oscillation** . Equation (6) can be written in a more convenient form by first writing it as

$$y(t) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega_0 t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega_0 t\right) \tag{7}$$

Letting $\frac{A}{\sqrt{A^2+B^2}} = \cos\delta$, $\frac{B}{\sqrt{A^2+B^2}} = \sin\delta$, and $\sqrt{A^2+B^2} = d$

equation (7) takes the form

$$\begin{aligned} y(t) &= d(\cos\delta\cos\omega_0 t + \sin\delta\sin\omega_0 t) \\ &= d\cos(\omega_0 t - \delta) \end{aligned} \tag{8}$$

where d is called the amplitude of oscillations and $\delta = \tan^{-1}\left(\frac{B}{A}\right)$.

The amplitude of oscillations in equation (8) does not decay with time , and so they are said to be **undamped** oscillations . The quantity δ (an angle in radians) in equation (8) is called the **phase angle** for the oscillation , and by convention this angle is taken to lie in the interval $-\pi \le \delta \le \pi$.

Since the period of the trigonometric functions in equation (6) is $\frac{2\pi}{\omega_0}$, the body executes $\frac{\omega_0}{2\pi}$ cycles per second. The quantity $\frac{\omega_0}{2\pi}$ is called the **frequency** of the oscillation and is measured in cycles per second. A new name of cycles / sec is hertz (Hz).

(2) DAMPED SYSTEM

If the mass m is connected to a dashpot, as shown in figure (8.19), then a new force comes into play, tending to damp out (or retard) the motion. Experiment shows that the damping force is proportional to the velocity $\frac{dy}{dt}$ of the body and acts in the upward direction.

Thus the damping force is of the form

$$F_3 = -C\frac{dy}{dt}$$

for some constant C, called the **damping constant**.

Figure (8.19)

Let us show that the damping constant C is positive. If $\frac{dy}{dt}$ is positive, the body moves downward (in the positive y – direction) and $-C\frac{dy}{dt}$ must be an upward force i.e. by agreement $-C\frac{dy}{dt} < 0$ which implies $C > 0$. For negative $\frac{dy}{dt}$ the body moves upward and $-C\frac{dy}{dt}$ must represent a downward force i.e. $-C\frac{dy}{dt} > 0$ which implies $C > 0$.

The resultant of the forces acting on the body is now

$$F_1 + F_2 + F_3 = -Ky - C\frac{dy}{dt}$$

Hence, by Newton's second law

$$m\frac{d^2y}{dt^2} = -Ky - C\frac{dy}{dt}$$

or $$m\frac{d^2y}{dt^2} + C\frac{dy}{dt} + Ky = 0$$

or $$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = 0 \qquad (8)$$

Thus we see that the motion of the damped mechanical system is governed by the second order linear differential equation with constant coefficients.

In operator notation, equation (8) becomes

$$\left(D^2 + \frac{C}{m}D + \frac{K}{m}\right)y = 0$$

The auxiliary equation is $D^2 + \frac{C}{m} D + \frac{K}{m} = 0$ having characteristic roots as

$$D = \frac{-\frac{C}{m} \pm \sqrt{\frac{C^2}{m^2} - \frac{4K}{m}}}{2} = -\frac{C}{2m} \pm \frac{1}{2m}\sqrt{C^2 - 4mK}$$

Using the abbreviated notations

$$\alpha = \frac{C}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{C^2 - 4mK}$$

we can write $D_1 = -\alpha + \beta$ and $D_2 = -\alpha - \beta$

The form of the solutions of equation (8) will depend on the damping, and we now have the following three cases :

CASE (1): $C^2 > 4mK$ Distinct real roots D_1, D_2 **(overdamping)**.

CASE (2): $C^2 = 4mK$ A real repeated root **(critical damping)**.

CASE (3): $C^2 < 4mK$ Complex conjugate roots **(underdamping)**.

Let us discuss these three cases separately.

CASE (1): $C^2 > 4mK$ (overdamping)

When the damping constant C is so large that $C^2 > 4mK$, then D_1 and D_2 are distinct real roots and the general solution of equation (8) is

$$y(t) = C_1 e^{-(\alpha-\beta)t} + C_2 e^{-(\alpha+\beta)t} \tag{9}$$

and we see that in this case the body does not oscillate. For $t > 0$ both exponents in equation (9) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - \frac{K}{m} < \alpha^2$. Hence both terms in equation (9) approach zero, as $t \to \infty$. Thus after a sufficiently long time the mass will be at rest in the static equilibrium position ($y = 0$). That is, the motion dies out with time. This is physically obvious since the damping takes the energy from the system and there is no external force that keeps the motion going.

CASE (2): $C_2 = 4mK$ (critical damping)

In this case $\beta = 0$ and $D_1 = D_2 = -\alpha$ and the general solution of equation (8) is

$$y(t) = (C_1 + C_2 t)e^{-\alpha t} \tag{10}$$

As in case (1), $y(t)$ approaches zero as $t \to \infty$, and the motion dies out with time.

CASE (3): $C^2 < 4mK$ (underdamping)

If the damping constant C is so small that $C^2 < 4mK$, then β is pure imaginary, say

$$\beta = i\omega^* \quad \text{where} \quad \omega^* = \frac{1}{2m}\sqrt{4mK - C^2} \tag{11}$$

The roots of the auxiliary equation are now complex conjugate, i.e.

$$D_1 = -\alpha + i\omega^* \quad \text{and} \quad D_2 = -\alpha - i\omega^* \tag{12}$$

and the general solution of equation (8) is

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) \tag{13}$$

Equation (13) can be written in a more convenient form by writing it as

$$y(t) = e^{-\alpha t}\sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2+B^2}} \cos \omega^* t + \frac{B}{\sqrt{A^2+B^2}} \sin \omega^* t\right) \tag{14}$$

Letting $\frac{A}{\sqrt{A^2+B^2}} = \cos \delta$, $\frac{B}{\sqrt{A^2+B^2}} = \sin \delta$ and $d = \sqrt{A^2+B^2}$, equation (14) takes the form

$$\begin{aligned} y(t) &= d\, e^{-\alpha t}(\cos \delta \cos \omega^* t + \sin \delta \sin \omega^* t) \\ &= d\, e^{-\alpha t} \cos(\omega^* t - \delta) \end{aligned} \tag{15}$$

where $\delta = \tan^{-1}\left(\frac{B}{A}\right)$

This solution represents damped oscillations. Since $\cos(\omega^* t - \delta)$ varies between -1 and 1, the curve of the solution lies between the curves $y = d\, e^{-\alpha t}$ and $y = -d\, e^{-\alpha t}$ as shown in figure (8.20) touching these curves when $\omega^* t - \delta$ is an integer multiple of π. The frequency is $\frac{\omega^*}{2\pi}$ cycles per second. From equation (11) we see that smaller $C\ (>0)$ is, the larger is ω^* and more rapid the oscillations become. As $C \to 0$, ω^* approaches the value $\omega_0 = \sqrt{\frac{K}{m}}$ corresponding to harmonic oscillation of the undamped system.

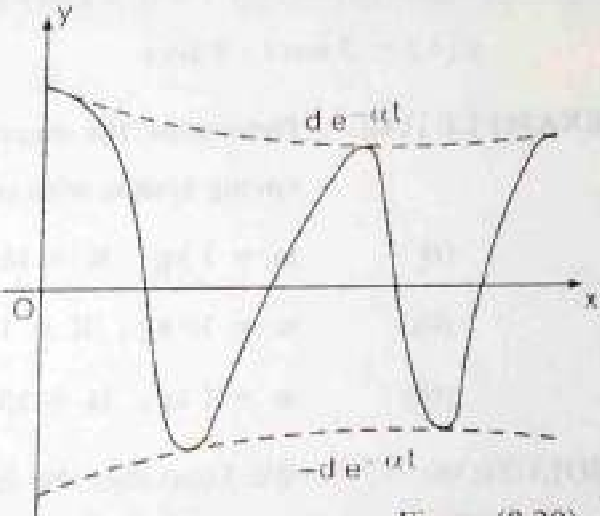


Figure (8.20)

EXAMPLE (13): Determine the equation of motion (i.e. displacement) of a mass – spring system in which a mass of 10 kg is attached to a spring with spring constant 10 N / m. The mass is initially displaced a distance 3 m from the equilibrium position and released with initial velocity 4 m / s subject to no damping or external forces.

SOLUTION: We know that the differential equation governing the motion of a mass – spring system in the case of undamped free oscillations is

$$\frac{d^2 y}{dt^2} + \omega_0^2 y = 0 \quad \left(\omega_0 = \sqrt{\frac{K}{m}}\right) \tag{1}$$

In the present case, $m = 10\,kg$, $k = 10\,N/m$, $y(0) = 3\,m$ and $y'(0) = 4\,m/s$.

Since $\omega_0 = \sqrt{\frac{10}{10}} = 1$, therefore equation (1) takes the form

$$\frac{d^2 y}{d t^2} + y = 0$$

The general solution of this equation is

$$y(t) = C_1 \cos t + C_2 \sin t \tag{2}$$

Using the first initial condition $y(0) = 3$, we get from equation (2)

$$C_1 = 3$$

Differentiating equation (2), we get

$$y'(t) = -C_1 \sin t + C_2 \cos t \tag{3}$$

Using the second initial condition $y'(0) = 4$, we get from equation (3)

$$C_2 = 4$$

Substituting the values of C_1 and C_2 in equation (2), we get

$$y(t) = 3 \cos t + 4 \sin t$$

EXAMPLE (14): **Determine the equation of motion (i.e. displacement) of a damped mass - spring system with each of the following data :**

(i) $m = 1\ kg,\ K = 16\ N/m,\ C = 10\ kg/s,\ y(0) = 4\ m,\ y'(0) = 0$

(ii) $m = 10\ kg,\ K = 1000\ N/m,\ C = 200\ kg/s,\ y(0) = 1\ m,\ y'(0) = 0$

(iii) $m = 1\ kg,\ K = 25\ N/m,\ C = 8\ kg/s,\ y(0) = 0,\ y'(0) = 3\ m/s$

SOLUTION: We know that the differential equation governing the motion of a mass – spring system in the case of damped free oscillations is

$$\frac{d^2 y}{d t^2} + \frac{C}{m}\frac{d y}{d t} + \frac{K}{m} y = 0$$

which in operator notation takes the form

$$\left(D^2 + \frac{C}{m} D + \frac{K}{m}\right) y = 0 \tag{1}$$

(i) In this case, $m = 1\ kg,\ K = 16\ N/m,\ C = 10\ kg/s,\ y(0) = 4\ m,\ y'(0) = 0$.
Thus equation (1) becomes

$$(D^2 + 10 D + 16) y = 0 \tag{2}$$

The auxiliary equation is $D^2 + 10 D + 16 = 0$ or $(D+2)(D+8) = 0$ having characteristic roots $D = -2, -8$.

Thus the general solution of equation (2) is

$$y(t) = C_1 e^{-2t} + C_2 e^{-8t} \tag{3}$$

Using the first initial condition $y(0) = 4$, we get from equation (3)

$$C_1 + C_2 = 4 \tag{4}$$

Differentiating equation (3), we get

$$y'(t) = -2C_1e^{-2t} - 8C_2e^{-8t} \tag{5}$$

Using the second initial condition $y'(0) = 0$, we get from equation (5)

$$0 = -2C_1 - 8C_2$$

or $$C_1 + 4C_2 = 0 \tag{6}$$

From equations (4) and (6), $C_1 = \frac{16}{3}$, $C_2 = -\frac{4}{3}$.

Substituting the values of C_1 and C_2, we get from equation (3)

$$y(t) = \frac{16}{3}e^{-2t} - \frac{4}{3}e^{-8t}$$

(ii) In this case, $m = 10$ kg, $K = 1000$ N/m, $C = 200$ kg/s, $y(0) = 1$ m, $y'(0) = 0$.

Thus equation (1) becomes

$$(D^2 + 20D + 100)y = 0 \tag{7}$$

The auxiliary equation is

$$D^2 + 20D + 100 = 0$$

or $$(D+10)^2 = 0$$

having characteristic roots $D = -10, -10$.

Thus the general solution of equation (7) is

$$y(t) = (C_1 + C_2t)e^{-10t} \tag{8}$$

Using the first initial condition $y(0) = 1$ we get from equation (8)

$$C_1 = 1$$

Differentiating equation (8), we get

$$y(t) = (C_1 + C_2t)(-10e^{-10t}) + C_2e^{-10t} \tag{9}$$

Using the second initial condition $y'(0) = 0$, we get from equation (9)

$$0 = -10C_1 + C_2 \quad \text{or} \quad C_2 = 10C_1 = 10(1) = 10$$

Substituting the values of C_1 and C_2 in equation (8), we get

$$y(t) = (1 + 10t)e^{-10t}$$

(iii) In this case, $m = 1$ kg, $K = 25$ N/m, $C = 8$ kg/s, $y(0) = 0$, $y'(0) = 3$ m/s

Thus equation (1) becomes

$$(D^2 + 8D + 25)y = 0 \tag{10}$$

The auxiliary equation is $D^2 + 8D + 25 = 0$ having characteristic roots $D = -4 \pm 3i$.

The general solution of equation (10) is

$$y(t) = e^{-4t}(C_1 \cos 3t + C_2 \sin 3t) \tag{11}$$

Using the first initial condition $y(0) = 0$, we get from equation (11)

$$C_1 = 0$$

Differentiating equation (11), we get

$$y'(t) = e^{-4t}(-3C_1 \sin 3t + 3C_2 \cos 3t) - 4e^{-4t}(C_1 \cos 3t + C_2 \sin 3t) \tag{12}$$

Using the second initial condition $y'(0) = 3$, we get from equation (12)

$$3 = 3C_2 - 4C_1 \quad \text{or} \quad 3 = 3C_2 - 0 \quad \text{or} \quad C_2 = 1$$

Substituting the values of C_1 and C_2 in equation (1), we get

$$y(t) = e^{-4t} \sin 3t$$

8.23 MODELING : FORCED OSCILLATIONS

So far, we have considered the free oscillations of a mass – spring system which are governed by the second order linear homogeneous differential equation

$$m\frac{d^2y}{dt^2} + C\frac{dy}{dt} + Ky = 0 \tag{1}$$

where m is the mass of the body, C the damping constant, and K the spring modulus.

We now extend the analysis by allowing an external variable force $F(t)$ to act on the system. Remember that equation (1) was obtained by considering the forces acting on the body and using Newton's second law. From this it is clear that the differential equation corresponding to the present situation is obtained from equation (1) by adding the force $F(t)$. This gives

$$m\frac{d^2y}{dt^2} + C\frac{dy}{dt} + Ky = F(t)$$

or
$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{1}{m}F(t) \tag{2}$$

Here $F(t)$ is called the **input** or **driving force**, and a corresponding solution is called an **output** or a **response** of the system to the driving force. The resulting motion is called a **forced motion**, in contrast to the **free motion** when $F(t) = 0$.

We shall consider the commonly encountered case of a periodic driving force, say

$$F(t) = F_0 \cos \omega t$$

where F_0 and ω are positive constants and ω is called the input frequency. We then have from equation (2)

$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{F_0}{m}\cos \omega t \tag{3}$$

SOLUTION OF EQUATION (3)

The general solution of equation (2) is of the form

$$y(t) = y_c(t) + y_p(t) \tag{4}$$

where $y_c(t)$ is the complementary function (i.e. the general solution of the corresponding homogeneous equation) and $y_p(t)$ is the particular integral . Since the complementary function $y_c(t)$ has been obtained in all cases in considering free oscillations . Thus we must determine the particular integral . The method of undetermined coefficients is convenient for this choice of the driving force , so we take

$$y_p(t) = A\cos\omega t + B\sin\omega t \tag{5}$$

Then $\quad y_p'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$

and $\quad y_p''(t) = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$

Substituting the values of these derivatives in equation (3) , we get

$$-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t + \frac{C}{m}(-\omega A\sin\omega t + \omega B\cos\omega t)$$

$$+\frac{K}{m}(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t$$

or $\quad \left(-\omega^2 A + \frac{C\omega B}{m} + \frac{KA}{m}\right)\cos\omega t + \left(-\omega^2 B - \frac{C\omega A}{m} + \frac{KB}{m}\right)\sin\omega t = \frac{F_0}{m}\cos\omega t$

Equating the coefficients of cosine and sine terms on both sides , we get

$$-\omega^2 A + \frac{C\omega B}{m} + \frac{KA}{m} = \frac{F_0}{m}$$

$$-\omega^2 B + \frac{C\omega A}{m} + \frac{KB}{m} = 0$$

or $\quad (K - m\omega^2)A + C\omega B = F_0$

$$-C\omega A + (K - m\omega^2)B = 0$$

This is a system of two linear equations in two unknowns A and B . The solution can be obtained using Cramer's rule .

Thus $\quad A = \dfrac{F_0(K - m\omega^2)}{(K - m\omega^2)^2 + \omega^2 C^2}, \quad B = \dfrac{F_0\omega C}{(K - m\omega^2)^2 + \omega^2 C^2}$

provided the denominator is not zero . (This is true if $C \neq 0$ or $\omega_0 \neq \omega$).

If we set $\sqrt{\dfrac{K}{m}} = \omega_0 \; (> 0)$, then

$$A = \frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 C^2}, \quad B = \frac{F_0\omega C}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 C^2}$$

Thus the particular integral in equation (5) becomes

$$y_p(t) = \frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2C^2}\cos\omega t + \frac{F_0\omega C}{m^2(\omega_0^2-\omega^2)^2+\omega^2C^2}\sin\omega t \qquad (6)$$

and the general solution in equation (4) is given by

$$y(t) = y_c(t) + \frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2C^2}\cos\omega t + \frac{F_0\omega C}{m^2(\omega_0^2-\omega^2)+\omega^2C^2}\sin\omega t \qquad (7)$$

Now there are two cases $C = 0$ (no damping) and $C > 0$ (damping). These cases will correspond to two different types of solutions .

CASE (1): UNDAMPED FORCED OSCILLATIONS

If there is no damping , then $C = 0$. We first assume that $\omega^2 \neq \omega_0^2$ i.e. the input frequency is not equal to the natural frequency . This is called the non – resonance case .

The general solution in equation (7) reduces to

$$\begin{aligned} y(t) &= y_c(t) + \frac{F_0\cos\omega t}{m(\omega_0^2-\omega^2)} \\ &= d\cos(\omega_0 t-\delta) + \frac{F_0\cos\omega t}{m(\omega_0^2-\omega^2)} \qquad (8) \end{aligned}$$

Equation (8) shows that the motion of the undamped forced system consists of two oscillatory motions , one due to free oscillations and the other due to forced oscillations . The frequencies of the free and forced oscillations are $\frac{\omega_0}{2\pi}$ and $\frac{\omega}{2\pi}$ respectively , whereas the respective amplitudes are d and $\frac{F_0}{m(\omega_0^2-\omega^2)}$.

It is also observed that if the natural frequency ω_0 is very large , the amplitude of the forced oscillations will be very small .

Next we assume that $\omega = \omega_0$. In this case , equation (3) becomes

$$\frac{d^2y}{dt^2} + \omega_0^2 y = \frac{F_0}{m}\cos\omega_0 t \quad \left(\omega_0^2 = \frac{K}{m}\right) \qquad (9)$$

As already obtained , the complementary function is

$$y_c(t) = d\cos(\omega_0 t-\delta)$$

The particular integral is given by

$$\begin{aligned} y_p(t) &= \frac{1}{D^2+\omega_0^2}\frac{F_0}{m}\cos\omega_0 t \quad (\text{case of failure}) \\ &= \frac{F_0}{m}t\frac{1}{2D}\cos\omega_0 t = \frac{F_0}{2m\omega_0}t\sin\omega_0 t \end{aligned}$$

Thus the general solution of equation (9) is

$$y(t) = d\cos(\omega_0 t-\delta) + \frac{F_0}{2m\omega_0}t\sin\omega_0 t \qquad (10)$$

We see that due to the presence of the factor t in the particular integral, the solution (10) becomes larger and larger as $t \to \infty$. In practice, this means that systems with zero or little damping may undergo large vibrations which can destroy the system. This phenomena in which the input frequency is equal to the natural frequency is known as **resonance**, and is of basic importance in the study of vibrating systems. When designing a mechanical system, the phenomena of resonance is avoided, so that the system does not collapse.

CASE (2): DAMPED FORCED OSCILLATIONS

If there is damping, then $C > 0$. As in the case of free oscillations, we have three possibilities:

(i) **$C^2 > 4mK$ (overdamping)**

In this case, the general solution (7) becomes

$$y(t) = C_1 e^{-(\alpha-\beta)t} + C_2 e^{-(\alpha+\beta)t} + \frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2C^2}\cos\omega t + \frac{F_0\omega C}{m^2(\omega_0^2-\omega^2)+\omega^2C^2}\sin\omega t \qquad (11)$$

where $\alpha = \frac{C}{2m}$, $\beta = \frac{1}{2m}\sqrt{C^2-4mK}$

The exponential terms on the R.H.S. of equation (11) i.e. the complementary function $y_c(t)$ is called the **transient solution** and the last two terms on the R.H.S. of equation (11) is called the **steady – state solution**.

(ii) **$C^2 = 4mK$ (Critical Damping)**

In this case, the general solution (7) becomes

$$y(t) = (C_1 + C_2 t)e^{-\alpha t} + \frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)+\omega^2C^2}\cos\omega t + \frac{F_0\omega C}{m^2(\omega_0^2-\omega^2)+\omega^2C^2}\sin\omega t \qquad (12)$$

Again, as $t \to \infty$, $y(t) \to y_p(t)$ the steady – state solution.

(iii) **$C^2 < 4mk$ (underdamping)**

In this case, the general solution (7) takes the form

$$y(t) = d\,e^{-\alpha t}\cos(\omega^* t - \delta) + \frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)+\omega^2C^2}\cos\omega t + \frac{F_0\omega C}{m^2(\omega_0^2-\omega^2)+\omega^2C^2}\sin\omega t \qquad (13)$$

where $\omega^* = \frac{1}{2m}\sqrt{4mK-C^2}$

Here the first term on the R.H.S. of equation (13) (i.e. the complementary function) approach zero as $t \to \infty$, and $y(t) \to y_p(t)$ the steady – state solution. In all cases of damped forced oscillations, the motion approaches the steady – state motion with increasing time.

EXAMPLE (15): **Determine the equation of motion (i.e. displacement) of an undamped forced mass – spring system with the following data :**

$$m = 20\text{ kg},\ K = 1000\text{ N/m},\ F = \sin 10t,\ y(0) = 1\text{ m},\ y'(0) = 0.$$

SOLUTION: The differential equation governing the motion of a mass – spring system in the case of undamped forced oscillations is

$$\frac{d^2y}{dt^2} + \frac{K}{m}y = \frac{1}{m}F(t) \tag{1}$$

In the present case, $m = 20$ kg, $K = 1000$ N / m, $F = \sin 10t$, $y(0) = 1$, $y'(0) = 0$

Thus equation (1) becomes

$$\frac{d^2y}{dt^2} + 50\,y = \frac{1}{20}\sin 10t$$

In operator notation, this equation takes the form

$$(D^2 + 50)\,y = \frac{1}{20}\sin 10t \tag{2}$$

The auxiliary equation is $D^2 + 50 = 0$ having characteristic roots $D = \pm\sqrt{50}\,i$.

The complementary function is

$$y_c(t) = C_1\cos\sqrt{50}\,t + C_2\sin\sqrt{50}\,t$$

The particular integral is given by

$$y_p(t) = \frac{1}{20}\,\frac{1}{D^2+50}\sin 10t$$

$$= \frac{1}{20}\left(\frac{1}{-100+50}\right)\sin 10t$$

$$= -\frac{1}{1000}\sin 10t$$

The solution of equation (2) is

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1\cos\sqrt{50}\,t + C_2\sin\sqrt{50}\,t - \frac{1}{1000}\sin 10t \tag{3}$$

Using the first initial condition $y(0) = 1$, we get from equation (3)

$$C_1 = 1$$

Differentiating equation (3), we get

$$y'(t) = -\sqrt{50}\,C_1\sin\sqrt{50}\,t + \sqrt{50}\,C_2\cos\sqrt{50}\,t - \frac{1}{100}\cos 10t \tag{4}$$

Using the second initial condition $y'(0) = 0$, we get from equation (4)

$$0 = \sqrt{50}\,C_2 - \frac{1}{100} \quad \text{or} \quad C_2 = \frac{1}{100\sqrt{50}}$$

Substituting the values of C_1 and C_2 in equation (3), we get

$$y(t) = \cos\sqrt{50}\,t + \frac{1}{100\sqrt{50}}\sin\sqrt{50}\,t - \frac{1}{1000}\sin 10\,t$$

$$= \frac{1}{1000}\left(1000\cos\sqrt{50}\,t + \sqrt{2}\sin\sqrt{50}\,t - \sin 10\,t\right)$$

which is the required displacement of the undamped forced mass – spring system.

EXAMPLE (16): **Determine the equation of motion (i.e. displacement) of a damped forced mass – spring system with each of the following data :**

(i) $m = 1\text{ kg},\ K = 16\text{ N/m},\ C = 10\text{ kg/s},\ F = 4\sin 4t\text{ N},$

$y(0) = 4\text{ m},\ y'(0) = 0$

(ii) $m = 10\text{ kg},\ K = 1000\text{ N/m},\ C = 200\text{ kg/s},\ F = \sin 10t\text{ N},$

$y(0) = 1\text{ m},\ y'(0) = 0$

(iii) $m = 1\text{ kg},\ K = 25\text{ N/m},\ C = 8\text{ kg/s},\ F = \sin 3t\text{ N}$

$y(0) = 0,\ y'(0) = 3\text{ m/s}$

SOLUTION: The differential equation governing the motion of a mass – spring system in the case of damped forced oscillations is

$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{1}{m}F(t) \qquad (1)$$

(i) In this case, $m = 1\text{ kg},\ K = 16\text{ N/m},\ C = 10\text{ kg/s},\ F(t) = 4\sin 4t,$

$y(0) = 1,\ y'(0) = 0.$

Thus equation (1) becomes

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 16y = 4\sin 4t$$

In operator notation, this equation takes the form

$$(D^2 + 10D + 16)y = 4\sin 4t \qquad (2)$$

The auxiliary equation is $D^2 + 10D + 16 = 0$ having characteristic roots $D = -2, -8$.

The complementary function is

$$y_c(t) = C_1e^{-2t} + C_2e^{-8t}$$

The particular integral is given by

$$y_p(t) = \frac{1}{D^2 + 10D + 16}4\sin 4t$$

$$= 4\left(\frac{1}{-16 + 10D + 16}\right)\sin 4t$$

$$= \frac{2}{5}\frac{1}{D}\sin 4t = -\frac{1}{10}\cos 4t$$

Thus the general solution of equation (2) is

$$y(t) = y_c(t)+y_p(t)$$

$$= C_1 e^{-2t}+C_2 e^{-8t}-\frac{1}{10}\cos 4t \tag{3}$$

Using the first initial condition $y(0) = 4$, we get from equation (3)

$$4 = C_1+C_2-\frac{1}{10} \quad \text{or} \quad C_1+C_2 = \frac{41}{10} \tag{4}$$

Differentiating equation (3), we get

$$y'(t) = -2C_1 e^{-2t}-8C_2 e^{-8t}+\frac{2}{5}\sin 4t \tag{5}$$

Using the second initial condition $y'(0) = 0$, we get from equation (4)

$$0 = -2C_1-8C_2$$

or $$C_1+4C_2 = 0 \tag{6}$$

From equations (4) and (6), $C_1 = \frac{82}{15}$, $C_2 = -\frac{41}{30}$.

Substituting the values of C_1 and C_2 in equation (3), we get

$$y(t) = \frac{82}{15}e^{-2t}-\frac{41}{30}e^{-8t}-\frac{1}{10}\cos 4t$$

which the required displacement of this mass – spring system .

(ii) In this case, $m = 10\text{ kg}$, $K = 1000\text{ N/m}$, $C = 200\text{ kg/s}$, $F = \sin 10t\text{ N}$, $y(0) = 1\text{ m}$, $y'(0) = 0$.

Thus equation (1) becomes

$$(D^2+20D+100)y = \frac{1}{10}\sin 10t \tag{7}$$

The auxiliary equation is $D^2+20D+100 = 0$ having characteristic roots $D = -10, -10$.

The complementary function is

$$y_c(t) = (C_1+C_2 t)e^{-10t}$$

The particular integral is given by

$$y_p(t) = \frac{1}{(D2+20D+100)}\frac{1}{10}\sin 10t$$

$$= \frac{1}{10}\left(\frac{1}{-100+20D+100}\right)\sin 10t$$

$$= \frac{1}{200}\frac{1}{D}\sin 10t = -\frac{1}{2000}\cos 10t$$

$$y(t) = (C_1+C_2 t)e^{-10t}-\frac{1}{2000}\cos 10t \tag{8}$$

Using the first initial condition $y(0) = 1$, we get from equation (8)

$$1 = C_1 - \frac{1}{2000} \quad \text{or} \quad C_1 = 1 + \frac{1}{2000} = \frac{2001}{2000}$$

Differentiating equation (8), we get

$$y'(t) = (C_1 + C_2 t)(-10 e^{-10t}) + C_2 e^{-10t} + \frac{1}{200} \sin 10 t \tag{9}$$

Using the second initial condition $y'(0) = 0$, we get from equation (9)

$$0 = -10 C_1 + C_2$$

or $$C_2 = 10 C_1 = 10\left(\frac{2001}{2000}\right) = \frac{2001}{200}$$

Substituting the values of C_1 and C_2 in equation (8), we get

$$y(t) = \left(\frac{2001}{2000} + \frac{2001}{200} t\right) e^{-10t} - \frac{1}{2000} \cos 4 t$$

$$= \frac{2001}{2000}(1 + 10 t) e^{-10t} - \frac{1}{2000} \cos 4 t$$

(iii) In this case, $m = 1$ kg, $K = 25$ N/m, $C = 8$ kg/s, $F(t) = \sin 3t$ N, $y(0) = 0$, $y'(0) = 3$ m/s.

Thus equation (1) becomes

$$\frac{d^2 y}{dt^2} + 8 \frac{dy}{dt} + 25 y = \sin 3 t$$

In operator notation, this equation takes the form

$$(D^2 + 8 D + 25) y = \sin 3 t \tag{10}$$

The auxiliary equation is $D^2 + 8 D + 25 = 0$ having characteristic roots $D = 4 \pm 3 i$.

The complementary function is

$$y_c(t) = e^{-4t}(C_1 \cos 3 t + C_2 \sin 3 t) \tag{11}$$

The particular integral is given by

$$y_p(t) = \frac{1}{D^2 + 8 D + 25} \sin 3 t$$

$$= \frac{1}{-9 + 8 D + 25} \sin 3 t = \frac{1}{8 D + 16} \sin 3 t$$

$$= \frac{1}{8}\left(\frac{D - 2}{D^2 - 4}\right) \sin 3 t = \frac{1}{8} \frac{D - 2}{-9 - 4} \sin 3 t$$

$$= -\frac{1}{104}(D - 2) \sin 3 t$$

$$= -\frac{1}{104}(3 \cos 3 t - 2 \sin 3 t)$$

The general solution of equation (7) is

$$y(t) = y_c(t) + y_p(t)$$

$$= e^{-4t}(C_1 \cos 3t + C_2 \sin 3t) - \frac{1}{104}(3\cos 3t - 2\sin 3t) \qquad (12)$$

Using the first initial condition $y(0) = 0$, we get from equation (12)

$$0 = C_1 - \frac{3}{104} \quad \text{or} \quad C_1 = \frac{3}{104}$$

Differentiating equation (12), we get

$$y'(t) = e^{-4t}(-3C_1 \sin 3t + 3C_2 \cos 3t)$$

$$-4e^{-4t}(C_1 \cos 3t + C_2 \sin 3t) - \frac{1}{104}(-9\sin 3t - 6\cos 3t) \qquad (13)$$

Using the second initial condition $y'(0) = 3$, we get from equation (13)

$$3 = 3C_2 - 4C_1 + \frac{6}{104}$$

or $$3C_2 = 3 + 4C_1 - \frac{6}{104} = 3 + \frac{12}{104} - \frac{6}{104} = \frac{318}{104}$$

or $$C_2 = \frac{106}{104}$$

Substituting the values of C_1 and C_2 in equation (12), we get

$$y(t) = e^{-4t}\left(\frac{3}{104}\cos 3t + \frac{106}{104}\sin 3t\right) - \frac{1}{104}(3\cos 3t - 2\sin 3t)$$

$$= \frac{1}{104}\left[e^{-4t}(3\cos 3t + 106\sin 3t) - 3\cos 3t + 2\sin 3t\right]$$

which is the required displacement of this mass – spring system.

8.24 COUPLED MASS – SPRING SYSTEM

Consider the coupled mass – spring system as shown in figure (8.21), in which the mass m_1 is attached to a fixed support by a spring with a spring constant K_1, and the mass m_2 is attached to m_1 by a spring with a spring constant K_2. The downward direction is taken as positive. Let $y_1(t)$ and $y_2(t)$ be the vertical displacements of masses m_1 and m_2, respectively, from their equilibrium positions ($y_1 = 0$ and $y_2 = 0$ correspond to the position of static equilibrium). We assume no damping and the masses of springs neglected.

When the system is in motion, the upper spring is subject to elongation (expansion) only, while the lower spring is subject to both elongation and compression. The upper spring is changed in length by the amount y_1 and the lower spring is changed in length by the amount $y_2 - y_1$. Then from Hooke's law, it follows that the springs exert forces equal to $-K_1 y_1$ and $K_2(y_2 - y_1)$ respectively, on m_1. Thus the net force exerted on m_1 is $-K_1 y_1 + K_2(y_2 - y_1)$. Applying Newton's second law to mass m_1, we have

$$m_1 \frac{d^2 y_1}{dt^2} = -K_1 y_1 + K_2(y_2 - y_1)$$

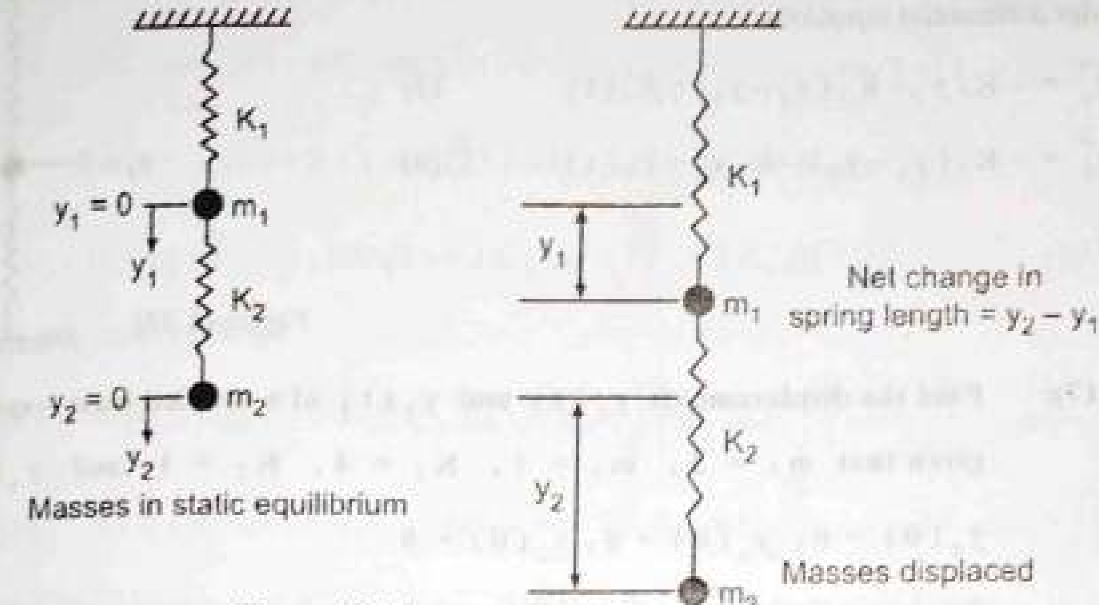


Figure (8.21)

Similarly, the net force exerted on m_2 due to elongation of the lower spring only is $-K_2(y_2 - y_1)$.

Then from Newton's second law, we have

and $$m_2 \frac{d^2 y_2}{dt^2} = -K_2(y_2 - y_1)$$

Thus the vertical motion of the coupled system is governed by the system of differential equations

$$m_1 y_1'' = -K_1 y_1 + K_2(y_2 - y_1) \tag{1}$$

$$m_2 y_2'' = -K_2(y_2 - y_1) \tag{2}$$

where the primes denote differentiation w.r.t. t.

COUPLED SYSTEM WITH EXTERNAL DRIVING FORCES

If there are external driving forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2 respectively. Then the motion is governed by the system.

$$m_1 y_1'' = -K_1 y_1 + K_2(y_2 - y_1) + F_1(t) \tag{3}$$

$$m_2 y_2'' = -K_2(y_2 - y_1) + F_2(t) \tag{4}$$

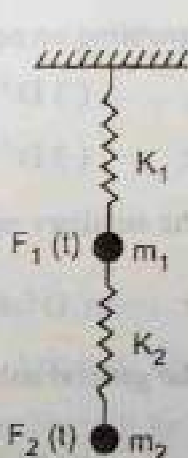


Figure (8.22)

TWO MASSES CONNECTED WITH THREE SPRINGS

Consider the coupled mass – spring system shown in figure (8.23). Proceeding as above, we see that the motion of this system is governed by the two second order differential equations

$$m_1 y_1'' = -K_1 y_1 + K_2(y_2 - y_1) + F_1(t) \qquad (5)$$

$$m_2 y_2'' = -K_2(y_1 - y_2) - K_3 y_2 + F_2(t) \qquad (6)$$

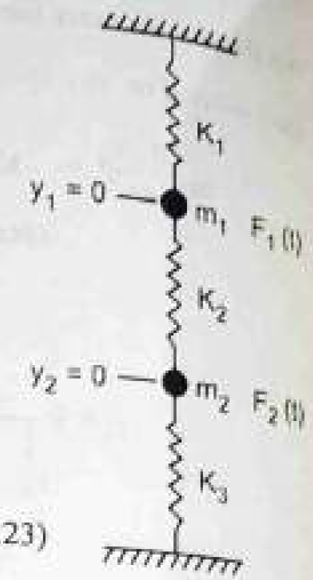


Figure (8.23)

EXAMPLE (17): **Find the displacements $y_1(t)$ and $y_2(t)$ of a coupled mass – spring system given that $m_1 = 3$, $m_2 = 1$, $K_1 = 4$, $K_2 = 1$ and $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = 0$.**

SOLUTION: We know that the system of differential equations governing the vibrations of a coupled system is

$$m_1 y_1'' = -(K_1 + K_2) y_1 + K_2 y_2$$

$$m_2 y_2'' = -K_2(y_2 - y_1)$$

In the present case, $m_1 = 3$, $m_2 = 1$, $K_1 = 4$, $K_2 = 1$. Thus the above system takes the form

$$3y_1'' = -5y_1 + y_2 \qquad (1)$$

$$y_2'' = -y_2 + y_1 \qquad (2)$$

In operator notation, this system takes the form

$$(3D^2 + 5)y_1 - y_2 = 0 \qquad (3)$$

$$-y_1 + (D^2 + 1)y_2 = 0 \qquad (4)$$

Operating on equation (3) by $D^2 + 1$ and adding equation (4), we get

$$(3D^2 + 5)(D^2 + 1)y_1 - y_1 = 0$$

or $$(3D^4 + 8D^2 + 4)y_1 = 1 \qquad (5)$$

The auxiliary equation is $3D^4 + 8D^2 + 4 = 0$

or $(D^2 + 2)(3D^2 + 2) = 0$ having characteristic roots as $D = \pm\sqrt{2}\,i$, $\pm\sqrt{\frac{2}{3}}\,i$.

The general solution of equation (5) is

$$y_1(t) = C_1 \cos\sqrt{2}\,t + \frac{1}{2}\sin\sqrt{2}\,t + C_3\cos\sqrt{\frac{2}{3}}\,t + C_4 \sin\sqrt{\frac{2}{3}}\,t \qquad (6)$$

From equation (3), we get

$$y_2 = (3D^2+5)y_1$$

$$= (3D^2+5)\left(C_1\cos\sqrt{2}\,t+\frac{1}{2}\sin\sqrt{2}\,t+C_3\cos\sqrt{\frac{2}{3}}\,t+C_4\sin\sqrt{\frac{2}{3}}\,t\right)$$

$$= -6C_1\cos\sqrt{2}\,t-6C_2\sin\sqrt{2}\,t-2C_3\cos\sqrt{\frac{2}{3}}\,t-2C_4\sin\sqrt{\frac{2}{3}}\,t$$

$$+5C_1\cos\sqrt{2}\,t+5C_2\sin\sqrt{2}\,t+5C_3\cos\sqrt{\frac{2}{3}}\,t+5C_4\sin\sqrt{\frac{2}{3}}\,t$$

$$= -C_1\cos\sqrt{2}\,t-C_2\sin\sqrt{2}\,t+3C_3\cos\sqrt{\frac{2}{3}}\,t+3C_4\sin\sqrt{\frac{2}{3}}\,t \qquad (7)$$

From equation (6) and (7), we get

$$y_1'(t) = -\sqrt{2}C_1\sin\sqrt{2}\,t+\sqrt{2}C_2\cos\sqrt{2}\,t-\sqrt{\frac{2}{3}}C_3\sin\sqrt{\frac{2}{3}}\,t+\sqrt{\frac{2}{3}}C_4\cos\sqrt{\frac{2}{3}}\,t \quad (8)$$

$$y_2'(t) = \sqrt{2}C_1\sin\sqrt{2}\,t-\sqrt{2}C_2\cos\sqrt{2}\,t-2\sqrt{3}C_3\sin\sqrt{\frac{2}{3}}\,t+2\sqrt{3}C_4\cos\sqrt{\frac{2}{3}}\,t \quad (9)$$

Using the initial conditions $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(t) = 0$, we get from equations (6), (7), (8) and (9)

$$C_1 + C_3 = 1$$

$$-C_1 + 3C_3 = 0$$

$$\sqrt{2}C_2 + \sqrt{\frac{2}{3}}C_4 = 0$$

$$-\sqrt{2}C_2 + 2\sqrt{3}C_4 = 0$$

Solving these equations, we get

$$C_1 = \frac{3}{4},\quad C_2 = 0,\quad C_3 = \frac{1}{4},\quad C_4 = 0$$

Substituting these values in equations (6) and (7), we get

$$y_1(t) = \frac{3}{4}\cos\sqrt{2}\,t+\frac{1}{4}\cos\sqrt{\frac{2}{3}}\,t$$

$$y_2(t) = -\frac{3}{4}\cos\sqrt{2}\,t+\frac{3}{4}\cos\sqrt{\frac{2}{3}}\,t$$

8.25 MODELING ELECTRIC CIRCUITS

We know that **modeling** means setting up mathematical models of physical or other systems. Here we shall model electric circuits. Their models will be linear differential equations. Although of particular interest to students of electrical engineering, computer engineering, etc., our discussion will be useful to **all** students because modeling skills can be acquired most successfully by considering practical problems from **various** fields.

BASIC ELEMENTS OF AN ELECTRIC CIRCUIT

The simplest electric circuit is a series circuit in which there is a source of electric energy supplying an **electromotive force** such as a battery or a generator , and a resistor which uses energy , for example an electric light bulb as shown in figure (8.24) .

If we close the switch , a current I will flow through the resistor , and this will cause a voltage drop , that is , the electric potential at the two ends of the resistor will be different . This potential difference or potential drop , or voltage drop can be measured by a voltmeter .

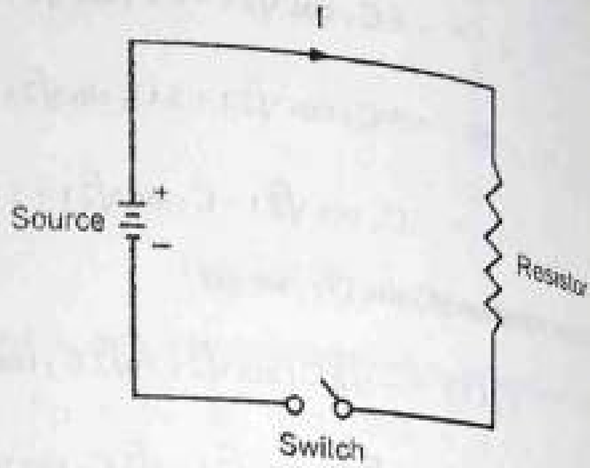


Figure (8.24)

Experiments show that the following law holds .

The voltage drop E_R across a resistor is proportional to the instantaneous current I , say

$$E_R = RI \qquad \text{(Ohm's law)} \tag{1}$$

where the constant of proportionality R is called the **resistance** of the resistor . The current I is measured in **amperes** , the resistance R in **ohms** , and the voltage E_R in **volts** . The other two important elements in more complicated circuits are **inductors** and **capacitors** .

INDUCTOR

An inductor opposes a change in current , having an inertia effect in electricity similar to that of mass in mechanics .

The voltage drop E_L across an inductor is proportional to the instantaneous time rate of change of the current I

i.e. $$E_L = L\frac{dI}{dt} \tag{2}$$

where the constant of proportionality L is called the **inductance** of the inductor and is measured in **henrys** , time t is measured in **seconds** .

CAPACITOR

A capacitor is an element of the circuit that stores energy . Experiments yield the following law .

The voltage drop E_C across a capacitor or **condenser** is proportional to the instantaneous electric charge Q on the capacitor

i.e. $$E_C = \frac{1}{C}Q \tag{3}$$

where C is called the **capacitance** and is measured in **farads** , the charge Q is measured in **coulombs** .

The current I is defined as the instantaneous time rate of change of the charge Q on the capacitor.

i.e. $I(t) = \dfrac{dQ}{dt}$

Thus equation (3) can be written as

$$E_C = \frac{1}{C}\int I(t)\,dt \tag{4}$$

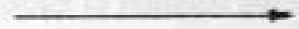
Note that the electromotive force (e.m.f.) provides a voltage of E i.e. a voltage drop of $-E$.

8.26 KIRCHHOFF'S VOLTAGE LAW (KVL)

This law is named after the German physicist Gustav Robert Kirchhoff (1824 – 1887). This states that the algebraic sum of all the voltage drops (potential drops) around any closed circuit is zero, or the voltage impressed on a closed circuit is equal to the sum of the voltage drops in the rest of the circuit.

8.27 NOTATIONS AND SYMBOLS

The following standard symbols and notations are used to describe electric circuits.

Electrical Quantity	Symbol	Unit	Diagrammatic Representation
Voltage or e.m.f.	V or E	Volt	E 
Current	I or i	Ampere	I 
Charge	Q or q	Coulomb	
Resistance	R	Ohm	R 
Inductance	L	Henry	L 
Capacitance	C	Farad	C. 
Key or switch			

Remember that the unit **volt** is named after the Italian physicist Alessandro Volta (1745 – 1827), the unit **ampere** is named after the French physicist Andre Marie Ampere (1775 – 1836), the unit **coulomb** is named after the French physicist and engineer Charles Augustin De Coulomb (1736 – 1806), the unit **ohm** is named after the German physicist Georg Simon Ohm (1789 – 1854), the unit **henry** is named after the American physicist Joseph Henry (1797 – 1878), and the unit **faraday** is named after the English physicist Michael Faraday (1791 – 1867).

8.28 RL – CIRCUIT

An RL – circuit is a circuit that contains a resistance R, the inductance L and an electromotive force (e.m.f.) E (t) as shown in figure (8.25).

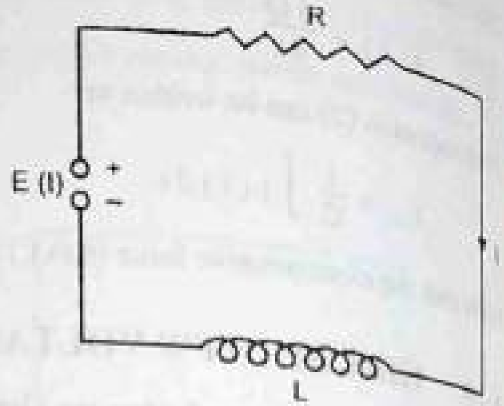


Figure (8.25)

We have to find the current I at any time t in the circuit assuming R and L to be constants.

We know that the voltage drop across the resistor is

$$E_R = R\,I \qquad (1)$$

Also, the voltage drop across the inductor is

$$E_L = L\frac{dI}{dt} \qquad (2)$$

By Kirchhoff's voltage law, the sum of two voltage drops must equal the electromotive force E (t).

Thus $\quad L\dfrac{dI}{dt} + R\,I = E(t)$

or $$\frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L} \qquad (3)$$

which is a first order linear differential equation. To solve this equation, we find the integrating factor

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

Thus the solution of equation (3) is given by

$$I(\text{I.F.}) = \int (\text{I.F.})\frac{E(t)}{L}dt + C$$

where C is a constant of integration.

or $$I\left(e^{\frac{Rt}{L}}\right) = \int e^{\frac{Rt}{L}} \cdot \frac{E(t)}{L}dt + C$$

or $$I(t) = e^{-\frac{Rt}{L}}\left[\int e^{\frac{Rt}{L}} \cdot \frac{E(t)}{L}dt + C\right] \qquad (4)$$

is the general solution of differential equation (3).

INITIAL CONDITION

Equation (4) involves an arbitrary constant C whose value will be determined using the initial condition. If the initial current in the circuit is of I_0 amperes, then the initial condition becomes

when $\quad t = 0,\ I(0) = I_0 \qquad (5)$

CASE (1): Constant Electromotive Force $E(t) = E_0$

In this case, the solution in equation (4) takes the form

$$I(t) = e^{-\frac{Rt}{L}}\left[\int e^{\frac{Rt}{L}} \cdot \frac{E_0}{L}dt + C\right]$$

$$= e^{-\frac{Rt}{L}}\left[e^{\frac{Rt}{L}}\cdot\frac{L}{R}\cdot\frac{E_0}{L}+C\right]$$

$$= e^{-\frac{Rt}{L}}\left[e^{\frac{Rt}{L}}\cdot\frac{E_0}{R}+C\right]$$

$$= \frac{E_0}{R}+Ce^{-\frac{Rt}{L}} \qquad (6)$$

Using the initial condition (5) i.e. when $t = 0$, $I(0) = I_0$, we get from equation (6)

$$I_0 = \frac{E_0}{R}+C \quad \text{or} \quad C = I_0-\frac{E_0}{R}$$

Substituting the value of C in equation (6), the current I at any time t is

$$I(t) = \frac{E_0}{R}+\left(I_0-\frac{E_0}{R}\right)e^{-\frac{Rt}{L}} \qquad (7)$$

The quantity $\left(I_0-\frac{E_0}{R}\right)e^{-\frac{Rt}{L}}$ in equation (7) is called the **transient current**, since this quantity goes to zero as $t \to \infty$. The quantity $\frac{E_0}{R}$ in equation (7) is called the **steady – state current**. As $t \to \infty$, the current I approach the constant value $\frac{E_0}{R}$ of the steady – state current.

CASE (2): **Periodic Electromotive Force $E(t) = E_0 \sin \omega t$**

In this case, the solution in equation (4) takes the form

$$I(t) = e^{-\frac{Rt}{L}}\left[\int e^{\frac{Rt}{L}}\frac{E_0}{L}\sin\omega t\;dt+C\right]$$

$$= e^{-\frac{Rt}{L}}\left[\frac{E_0}{L}\int e^{\frac{Rt}{L}}\sin\omega t\;dt+C\right] \qquad (8)$$

Using the formula $\int e^{ax}\sin bx\;dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$, equation (8) becomes

$$I(t) = Ce^{-\frac{Rt}{L}}+\frac{E_0}{L}e^{-\frac{Rt}{L}}\left[\frac{e^{\frac{Rt}{L}}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)}{\frac{R^2}{L^2}+\omega^2}\right]$$

$$= Ce^{-\frac{Rt}{L}}+\frac{E_0}{L^2}\cdot L^2\frac{(R\sin\omega t-\omega L\cos\omega t)}{R^2+\omega^2L^2}$$

$$= Ce^{-\frac{Rt}{L}}+E_0\left(\frac{R\sin\omega t-\omega L\cos\omega t}{R^2+\omega^2L^2}\right) \qquad (9)$$

Using the initial condition (5) i.e. when $t = 0$, $I(0) = I_0$, we get from equation (9)

$$I_0 = C-\frac{E_0\omega L}{R^2+\omega^2L^2} \quad \text{or} \quad C = I_0+\frac{E_0\omega L}{R^2+\omega^2L^2}$$

Substituting the value of C in equation (9), we get the current I at any time t as

$$I(t) = \left(I_0 + \frac{E_0 \omega L}{R^2+\omega^2 L^2}\right) e^{-\frac{Rt}{L}} + \frac{E_0}{R^2+\omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t) \qquad (10)$$

As mentioned in case (1), the exponential term on the R.H.S. of equation (10) is called the **transient current**, because this term approaches zero as $t \to \infty$. The steady – state is reached as $t \to \infty$. Let the steady – state current be denoted by I_s, then taking limits on both sides of equation (10) as $t \to \infty$, we get

$$I_s = \frac{E_0}{R^2+\omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t) \qquad (11)$$

Equation (11) may be written as

$$I_s = \frac{E_0}{\sqrt{R^2+\omega^2 L^2}}\left(\frac{R}{\sqrt{R^2+\omega^2 L^2}} \sin \omega t - \frac{\omega L}{\sqrt{R^2+\omega^2 L^2}} \cos \omega t\right) \qquad (12)$$

Letting $\frac{R}{\sqrt{R^2+\omega^2 L^2}} = \cos \delta$ and $\frac{\omega L}{\sqrt{R^2+\omega^2 L^2}} = \sin \delta$

Equation (12) takes the form

$$I_s = \frac{E_0}{\sqrt{R^2+\omega^2 L^2}}(\sin \omega t \cos \delta - \cos \omega t \sin \delta)$$

$$= \frac{E_0}{\sqrt{R^2+\omega^2 L^2}} \sin(\omega t - \delta) \quad \text{where} \quad \delta = \tan^{-1}\left(\frac{\omega L}{R}\right) \qquad (12)$$

EXAMPLE (18): **An RL circuit contains a resistor of resistance 8 ohms, an inductor of inductance 0.5 henries, and a battery of E volts. At $t = 0$, the current is zero. Find the current at any time t, the transient current, and steady – state current if (i) $E = 64$ and (ii) $E = 64 \sin 8t$.**

SOLUTION: We know that the differential equation for the current I in an RL – circuit is

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E(t)}{L} \qquad (1)$$

If $E(t) = E_0$, the solution of equation (1) is given by

$$I(t) = \frac{E_0}{R} + \left(I_0 - \frac{E_0}{R}\right) e^{-\frac{Rt}{L}} \qquad (2)$$

If $E(t) = E_0 \sin \omega t$, the solution of equation (1) is given by

$$I(t) = \left(I_0 + \frac{E_0 \omega L}{R^2+\omega^2 L^2}\right) e^{-\frac{Rt}{L}} + E_0\left(\frac{R \sin \omega t - \omega L \cos \omega t}{R^2+\omega^2 L^2}\right) \qquad (3)$$

In the present case, $R = 8$ ohms, $L = 0.5$ henries, therefore the differential equation (1) becomes

$$\frac{dI}{dt} + 16 I = 2E(t) \qquad (4)$$

(i) If $E(t) = E_0 = 64$, then the solution of equation (4) is given by from equation (2) as

$$I(t) = \frac{64}{8} + \left(I_0 - \frac{64}{8}\right) e^{-16t} \tag{5}$$

Since the initial current $I_0 = 0$, therefore from equation (5) the current I at any time t is given by

$$I(t) = -8\,e^{-16t} + 8 \tag{6}$$

The quantity $-8\,e^{-16t}$ is called the **transient current**, since this quantity goes to zero as $t \to \infty$. The quantity 8 in equation (4) is called the **steady – state current**.

(ii) If $E = 64 \sin 8t$, then the solution of equation (4) is given by from equation (3) as

$$I(t) = \left[I_0 + \frac{64\,(8)\,(1/2)}{64+16}\right] e^{-16t} + 64\left(\frac{8 \sin 8t - 4 \cos 8t}{64+16}\right)$$

$$= \left(I_0 + \frac{16}{5}\right) e^{-16t} + \frac{16}{5}\,(2 \sin 8t - \cos 8t) \tag{7}$$

Since the initial current $I_0 = 0$, therefore from equation (7) the current I at any time t is given by

$$I(t) = \frac{16}{5} e^{-16t} + \frac{16}{5}\,(2 \sin 8t - \cos 8t) \tag{8}$$

The quantity $\frac{16}{5} e^{-16t}$ is called the **transient current**, since this quantity goes to zero as $t \to \infty$.

The quantity $\frac{16}{5}(2 \sin 8t - \cos 8t)$ is called the **steady – state current**.

8.29 RC – CIRCUIT

An RC – circuit is a circuit that contains a resistance R, the capacitance C, and an electromotive force (e.m.f.) $E(t)$ as shown in figure (8.26).

We have to find the current I at any time t in the circuit assuming that R and C to be constants.

We know that the voltage drop across the resistor is

$$E_R = R\,I \tag{1}$$

Also, the voltage drop across a capacitor is

$$E_C = \frac{1}{C} \int I(t)\,dt \tag{2}$$

By Kirchoff's voltage law, the sum of two voltage drops must equal the electromotive force $E(t)$. Thus

$$R\,I + \frac{1}{C} \int I(t)\,dt = E(t) \tag{3}$$

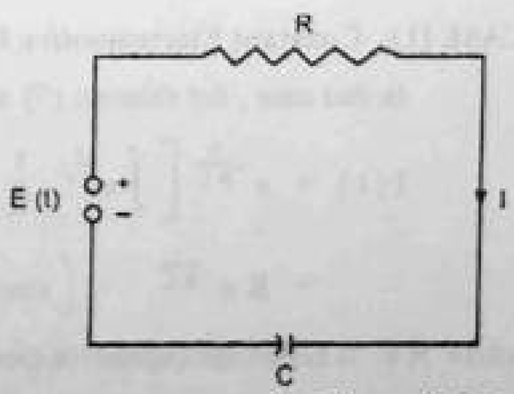


Figure (8.26)

To get rid of the integral we differentiate the equation w.r.t. t to get

$$R\frac{dI}{dt} + \frac{1}{C} I = \frac{dE}{dt}$$

or $$\frac{dI}{dt}+\frac{1}{RC}I=\frac{1}{R}\frac{dE}{dt} \tag{4}$$

which is a first order linear differential equation. To solve this equation, we find the integrating factor

$$\text{I.F.}=e^{\int\frac{1}{RC}dt}=e^{\frac{t}{RC}}$$

Thus the solution of equation (4) is given by

$$I(t)(\text{I.F.})=\int(\text{I.F.})\frac{1}{R}\frac{dE}{dt}dt+K$$

or $$I(t)\left(e^{\frac{t}{RC}}\right)=\int e^{\frac{t}{RC}}\cdot\frac{1}{R}\frac{dE}{dt}dt+K$$

or $$I(t)=e^{-\frac{t}{RC}}\left[\int e^{\frac{t}{RC}}\frac{1}{R}\frac{dE}{dt}dt+K\right] \tag{5}$$

is the general solution of differential equation (4).

INITIAL CONDITION

The initial condition for the current I is obtained from equation (3) by solving it for I and then setting $t=0$. Equation (3) can be written as

$$I(t)=\frac{1}{R}E(t)-\frac{1}{RC}Q(t)$$

At $t=0$, this becomes

$$I(0)=\frac{1}{R}E(0)-\frac{1}{RC}Q(0)$$

If the initial current is of I_0 amperes and the initial charge on the capacitor is of Q_0 coulombs, i.e. when $t=0$, $I(0)=I_0$ and $Q(0)=Q_0$, then the initial condition becomes

$$I_0=\frac{1}{R}E(0)-\frac{1}{RC}Q_0 \tag{6}$$

CASE (1): Constant Electromotive Force $E(t)=E_0$

In this case, the solution (5) takes the form

$$I(t)=e^{-\frac{t}{RC}}\left[\int e^{\frac{t}{RC}}\cdot\frac{1}{R}\cdot\frac{dE_0}{dt}dt+K\right]$$

$$=Ke^{-\frac{t}{RC}} \qquad \left(\text{since } \frac{dE_0}{dt}=0\right) \tag{7}$$

where RC is called the capacitive time constant of the circuit.

Using the initial condition (6) that when $t=0$, $I_0=\frac{1}{R}E(0)-\frac{1}{RC}Q_0$, we get from equation (7)

$$K=\frac{1}{R}E(0)-\frac{1}{RC}Q_0$$

Substituting the value of K in equation (7), the current I at any time t is given by

$$I(t) = \frac{1}{R}\left[E(0) - \frac{1}{RC}Q_0\right]e^{-\frac{t}{RC}} \tag{8}$$

CASE (2): **Periodic Electromotive Force $E(t) = E_0 \sin \omega t$**

In this case for $E(t) = E_0 \sin \omega t$, we have $\frac{dE}{dt} = \omega E_0 \cos \omega t$

Thus the solution in equation (5) becomes

$$I(t) = e^{-\frac{Rt}{L}}\left[\int e^{\frac{t}{RC}} \frac{1}{R} \omega E_0 \cos \omega t \; dt + K\right] \qquad (K \text{ is constant of integration})$$

$$= e^{-\frac{t}{RC}}\left[\frac{\omega E_0}{R} \int e^{\frac{t}{RC}} \cos \omega t \; dt + K\right] \tag{9}$$

Using the formula $\int e^{ax} \cos bx \; dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$, equation (9) becomes

$$I(t) = e^{-\frac{t}{RC}}\left[\frac{\omega E_0}{R} \frac{e^{\frac{t}{RC}}}{\left(\frac{1}{R^2C^2} + \omega^2\right)}\left(\frac{1}{RC}\cos \omega t + \omega \sin \omega t\right) + K\right]$$

$$= K e^{-\frac{t}{RC}} + \frac{\omega E_0}{R\left(\frac{1}{R^2C^2} + \omega^2\right)}\left(\frac{1}{RC}\cos \omega t + \omega \sin \omega t\right)$$

$$= K e^{-\frac{t}{RC}} + \frac{\omega E_0 C}{1 + \omega^2 R^2 C^2}(\cos \omega t + \omega RC \sin \omega t) \tag{10}$$

which is the required current at any time t.

Using the initial condition (6) that when $t = 0$, $I_0 = \frac{1}{R}E(0) - \frac{1}{RC}Q_0$, we get from equation (10)

$$\frac{1}{R}E(0) - \frac{1}{RC}Q_0 = K + \frac{\omega E_0 C}{1 + \omega^2 R^2 C^2}$$

or $$K = \frac{1}{R}E(0) - \frac{1}{RC}Q_0 - \frac{\omega E_0 C}{1 + \omega^2 R^2 C^2}$$

Substituting the value of K in equation (10), we get the current at any time t as

$$I(t) = \left(\frac{1}{R}E(0) - \frac{1}{RC}Q_0 - \frac{\omega E_0 C}{1 + \omega^2 R^2 C^2}\right)e^{-\frac{t}{RC}}$$

$$+ \frac{\omega E_0 C}{1 + \omega^2 R^2 C^2}(\cos \omega t + \omega RC \sin \omega t) \tag{11}$$

which may be written as

$$I(t) = \left(\frac{1}{R}E(0) - \frac{1}{RC}Q_0 - \frac{\omega E_0 C}{1 + \omega^2 R^2 C^2}\right)e^{-\frac{t}{RC}} + \frac{\omega E_0 C}{\sqrt{1 + \omega^2 R^2 C^2}}\sin(\omega t + \delta) \tag{12}$$

where $\delta = \tan^{-1}\left(\frac{1}{\omega RC}\right)$

DIFFERENTIAL EQUATION IN TERMS OF CHARGE Q

Equation (3) can be written in terms of charge Q on the capacitor. We know that

$I(t) = \frac{dQ}{dt}$ and $\int I(t)\,dt = Q$. Thus equation (3) takes the form

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

or $$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{1}{R}E(t) \tag{13}$$

which is a first order linear differential equation. To solve this equation, we find the integrating factor as

$$\text{I.F.} = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

Thus the general solution of equation (13) is

$$Q(t)(\text{I.F.}) = \int (\text{I.F.})\frac{1}{R}E(t)\,dt + K$$

$$Q(t)e^{\frac{t}{RC}} = \int e^{\frac{t}{RC}}\frac{1}{R}E(t)\,dt + K$$

or $$Q(t) = e^{-\frac{t}{RC}}\left[\int e^{\frac{t}{RC}}\frac{1}{R}E(t)\,dt + K\right] \tag{14}$$

The current I in the circuit can be obtained by differentiating equation (14) w.r.t. t.

INITIAL CONDITION

Equation (14) involves an arbitrary constant K which will be determined using the initial condition. If the initial charge on the capacitor is of Q_0 coulombs, then the initial condition becomes

when $t = 0,\ Q(0) = Q_0$ (15)

CASE (1): Constant Electromotive Force $E(t) = E_0$

In this case, the solution (14) takes the form

$$Q(t) = e^{-\frac{t}{RC}}\left[\int e^{\frac{t}{RC}}\frac{1}{R}E_0\,dt + K\right]$$

$$= e^{-\frac{t}{RC}}\left[e^{\frac{t}{RC}}RC\frac{1}{R}E_0 + K\right]$$

$$= CE_0 + Ke^{-\frac{t}{RC}} \tag{16}$$

Using the initial condition (15) i.e. when $t = 0,\ Q(0) = Q_0$, we get from equation (16)

$$Q_0 = CE_0 + K \quad \text{or} \quad K = Q_0 - CE_0$$

Substituting this value of K in equation (16), the charge Q at any time t is

$$Q(t) = CE_0 + (Q_0 - CE_0)e^{-\frac{t}{RC}} \tag{17}$$

The current I is obtained by differentiating equation (17) w.r.t. t.

CASE (2): Periodic Electromotive Force $E(t) = E_0 \sin \omega t$

In this case, the solution in equation (14) takes the form

$$Q(t) = e^{-\frac{t}{RC}}\left[\int e^{\frac{t}{RC}}\frac{1}{R}\,E_0 \sin \omega t + K\right]$$

$$= e^{-\frac{t}{RC}}\left[\frac{E_0}{R}\int e^{\frac{t}{RC}}\sin \omega t + K\right] \tag{18}$$

Using the formula $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$, equation (18) becomes

$$Q(t) = K e^{-\frac{t}{RC}} + \frac{E_0}{R}e^{-\frac{t}{RC}}\left[\frac{e^{\frac{t}{RC}}}{\frac{1}{R^2C^2}+\omega^2}\left(\frac{1}{RC}\sin \omega t - \omega \cos \omega t\right)\right]$$

$$= K e^{-\frac{t}{RC}} + \frac{E_0}{R}\,\frac{R^2C^2}{1+\omega^2R^2C^2}\left(\frac{\sin \omega t - \omega RC\cos \omega t}{RC}\right)$$

$$= K e^{-\frac{t}{RC}} + \frac{E_0 C}{1+\omega^2R^2C^2}(\sin \omega t - \omega RC\cos \omega t) \tag{19}$$

Using the initial condition (15), i.e. when $t = 0$, $Q(0) = Q_0$, we get from equation (19),

$$Q_0 = K - \frac{E_0\omega RC^2}{1+\omega^2R^2C^2} \quad \text{or} \quad K = Q_0 + \frac{E_0\omega RC^2}{1+\omega^2R^2C^2}$$

Substituting the value of K in equation (19), we get the charge at any time t as

$$Q(t) = \left(Q_0 + \frac{E_0\omega RC^2}{1+\omega^2R^2C^2}\right)e^{-\frac{t}{RC}} + \frac{E_0 C}{1+\omega^2R^2C^2}(\sin \omega t - \omega RC\cos \omega t) \tag{20}$$

The first term on the R.H.S. of equation (20) is the transient charge and the second term is the steady – state charge. The current I in the circuit can be obtained by differentiating equation (20) w.r.t. t.

EXAMPLE (19): **An RC circuit has a resistor of resistance 20 ohms, a capacitor of capacitance 0.05 farad, and a battery of 60 volts. At $t = 0$, the charge on the capacitor is zero. Find the current in the circuit at any time t.**

SOLUTION: We know that the differential equation for the current I in an RC circuit is

$$\frac{dI}{dt} + \frac{1}{RC}I = \frac{1}{R}\frac{dE}{dt} \tag{1}$$

If $E(t) = E_0$, the solution of equation (1) is

$$I(t) = \frac{1}{R}\left[E(0) - \frac{1}{RC}Q_0\right]e^{-\frac{t}{RC}} \tag{2}$$

In the present case, $R = 20$ ohms, $C = 0.05$ farad, and $E(t) = E_0 = 60$, therefore equation (1) becomes

$$\frac{dI}{dt}+I=0 \tag{3}$$

Since $E(t)=60$, therefore $E(0)=60$, and solution (2) becomes

$$I(t)=\frac{1}{20}(60-Q_0)e^{-t} \tag{4}$$

But the initial charge $Q_0=0$ therefore from equation (4) the current I at any time t is given by

$$I(t)=3e^{-t} \tag{5}$$

ALTERNATIVE METHOD

We know that the differential equation for the charge Q in an RC – circuit is

$$\frac{dQ}{dt}+\frac{1}{RC}Q=\frac{1}{R}E(t) \tag{6}$$

If $E(t)=E_0$, the solution of equation (6) is

$$Q(t)=CE_0+(Q_0-CE_0)e^{-\frac{t}{RC}} \tag{7}$$

In the present case, $R=20$ ohms, $C=0.05$ farad, and $E(t)=60$, therefore equation (6) becomes

$$\frac{dQ}{dt}+I=3 \tag{8}$$

Since $E(t)=60$, therefore solution (7) becomes

$$\begin{aligned} Q(t) &= (0.05)60+\left[Q_0-(0.05)60\right]e^{-t} \\ &= 3+(Q_0-3)e^{-t} \end{aligned} \tag{9}$$

Since the initial charge $Q_0=0$, therefore from equation (7) the charge Q at any time t is

$$Q(t)=3(1-e^{-t}) \tag{10}$$

The current I is obtained by differentiating equation (10) w.r.t. t.

i.e. $\quad I=3e^{-t}$

EXAMPLE (20): **An RC circuit has a resistor of resistance 20 ohms, a capacitor of capacitance 0.05 farad, and an electromotive force of 100 sin 2 t volts. At $t=0$, the charge on the capacitor is zero. Find the current in the circuit at any time t.**

SOLUTION: We know that the differential equation for the current I in an RC circuit is

$$\frac{dI}{dt}+\frac{1}{RC}I=\frac{1}{R}\frac{dE}{dt} \tag{1}$$

If $E(t)=E_0\sin\omega t$, the solution of equation (1) is given by

$$I(t)=\left(\frac{1}{R}E(0)-\frac{1}{RC}Q_0-\frac{\omega E_0 C}{1+\omega^2R^2C^2}\right)e^{-\frac{t}{RC}} + \frac{\omega E_0 C}{1+\omega^2R^2C^2}(\cos\omega t+\omega RC\sin\omega t) \tag{2}$$

In the present case, $R = 20$ ohms, $C = 0.05$ farad, $E(t) = 100 \sin 2t$ therefore equation (1) becomes

$$\frac{dI}{dt} + I = 10 \cos 2t \tag{3}$$

Since $E(t) = 100 \sin 2t$, therefore $E_0 = 100$ and $E(0) = 0$. Then solution (2) becomes

$$I(t) = \left[\frac{1}{20}(0) - Q_0 - \frac{2(100)(0.05)}{1 + 4(400)(0.05)^2}\right] e^{-t}$$
$$+ \frac{2(100)(0.05)}{1 + 4(400)(0.05)^2}\left[\cos 2t + 2(20)(0.05)\sin 2t\right] \tag{4}$$

Since the initial charge $Q_0 = 0$, therefore from equation (4) the current I at any time t is given by

$$I(t) = -\frac{10}{5} e^{-t} + \frac{10}{5}(\cos 2t + 2 \sin 2t)$$
$$= -2e^{-t} + (2 \cos 2t + 4 \sin 2t) \tag{5}$$

ALTERNATIVE METHOD

We know that the differential equation for the charge Q in an RC – circuit is

$$\frac{dQ}{dt} + \frac{1}{RC} Q = \frac{1}{R} E(t) \tag{6}$$

If $E(t) = E_0 \sin \omega t$, the solution of equation (6) is

$$Q(t) = \left(Q_0 + \frac{E_0 \omega R C^2}{1 + \omega^2 R^2 C^2}\right) e^{-\frac{t}{RC}} + \frac{E_0 C}{1 + \omega^2 R^2 C^2}(\sin \omega t - \omega R C \cos \omega t) \tag{7}$$

In the present case, $R = 20$ ohms, $C = 0.05$ farad, $E(t) = 100 \sin 2t$, $E_0 = 100$, and $\omega = 2$ therefore equation (7) becomes

$$\frac{dQ}{dt} + Q = 5 \sin 2t \tag{8}$$

Since $E(t) = 100 \sin 2t$, therefore solution (7) becomes

$$Q(t) = \left[Q_0 + \frac{100(2)(20)(0.05)^2}{1 + 4(400)(0.05)^2}\right] e^{-t}$$
$$+ \frac{100(0.05)}{1 + 4(400)(0.05)^2}\left[\sin 2t - 2(20)(0.05)\cos 2t\right]$$
$$= \left(Q_0 + \frac{10}{5}\right) e^{-t} + \frac{5}{5}(\sin 2t - 2 \cos 2t)$$
$$= (Q_0 + 2) e^{-t} + (\sin 2t - 2 \cos 2t) \tag{9}$$

Since the initial charge $Q_0 = 0$, therefore from equation (9), the charge at any time t is

$$Q(t) = 2e^{-t} + (\sin 2t - 2 \cos 2t) \tag{10}$$

The current I is obtained by differentiating equation (10) w.r.t. t,

i.e. $I = \frac{dQ}{dt} = -2e^{-t} + (2 \cos 2t + 4 \sin 2t)$ which is the same as equation (5) above.

8.30 LC – CIRCUIT

An LC – circuit is a circuit that contains an inductor of inductance L, a capacitor of capacitance C and an electromotive force (e.m.f.) $E(t)$ as shown in figure (8.27). We have to find the current I at any time in the circuit assuming that L and C to be constants.

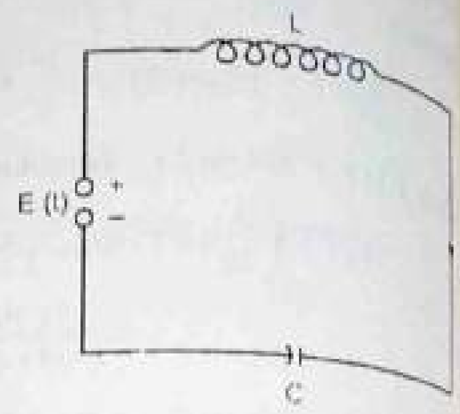


Figure (8.27)

We know that the voltage drop E_L across the inductor

$$E_L = L\frac{dI}{dt}$$

Also, the voltage drop across the capacitor is

$$E_C = \frac{1}{C}\int I(t)\,dt$$

By Kirchoff's voltage law, the sum of two voltage drops must equal the electromotive force $E(t)$. Thus

$$L\frac{dI}{dt}+\frac{1}{C}\int I(t)\,dt = E(t) \qquad (1)$$

To get rid of the integral we differentiate the equation w.r.t. t, we get

$$L\frac{d^2I}{dt^2}+\frac{1}{C}I = \frac{dE}{dt}$$

or
$$\frac{d^2I}{dt^2}+\frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt} \qquad (2)$$

which is a second order linear differential equation with constant coefficients.

INITIAL CONDITIONS

The general solution of equation (2) will involve two arbitrary constants, which will be determined using two initial conditions. If the initial current is of I_0 amperes, then the first initial condition is

when $\quad t = 0, \quad I(0) = I_0 \qquad (3)$

The second initial condition is obtained from equation (1) by solving it for $\frac{dI}{dt}$ and then setting $t = 0$.

Equation (1) can be written as

$$\frac{dI}{dt} = \frac{1}{L}E(t)-\frac{1}{LC}Q(t)$$

At $t = 0$, this becomes

$$I'(0) = \left(\frac{dI}{dt}\right)_{t=0} = \frac{1}{L}E(0)-\frac{1}{LC}Q(0)$$

If the initial charge on the capacitor is of Q_0 coulombs, then this initial condition becomes

$$I'(0) = \left(\frac{dI}{dt}\right)_{t=0} = \frac{1}{L}E(0)-\frac{1}{LC}Q_0 \qquad (4)$$

Equations (2), (3), and (4) constitute a second order initial – value problem, whose solution will give the current I in the circuit at any time t.

CASE (1): Zero Electromotive Force $E(t) = 0$

This is the case of LC – circuit without an electromotive force. In this case, equation (2) reduces to

$$\frac{d^2 I}{dt^2} + \omega_0^2 I = 0 \qquad \left(\frac{1}{LC} = \omega_0^2\right) \tag{5}$$

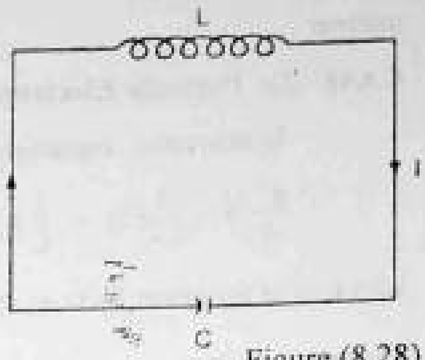


Figure (8.28)

which is similar to the differential equation of mass – spring system in the case of undamped free oscillations. The solution can be obtained and interpreted in the same manner.

CASE (2): Periodic Electromotive Force $E(t) = E_0 \sin \omega t$

In this case, equation (2) becomes

$$\frac{d^2 I}{dt^2} + \frac{1}{LC} I = \frac{1}{L} E_0 \omega \cos \omega t \tag{6}$$

which is similar to the differential equation of mass – spring system in the case of undamped forced oscillations and can similarly be solved and interpreted.

DIFFERENTIAL EQUATION IN TERMS OF CHARGE Q

Since $I = \dfrac{dQ}{dt}$ and $\int I(t)\, dt = Q$, therefore equation (1) can be written as

$$L\frac{d^2 Q}{dt^2} + \frac{1}{C} Q = E(t)$$

or

$$\frac{d^2 Q}{dt^2} + \frac{1}{LC} Q = \frac{1}{L} E(t) \tag{7}$$

which is a second order differential equation with constant coefficients in terms of Q.

INITIAL CONDITIONS

The general solution of equation (5) will involve two arbitrary constants, which will be determined using two initial conditions. If the initial charge on the capacitor is of Q_0 coulombs, then the initial conditions become

when $t = 0$, $Q(0) = Q_0$ (8)

and when $t = 0$, $I(0) = \left(\dfrac{dQ}{dt}\right)_{t=0} = 0$ (9)

Equations (7), (8), and (9) constitute a second order initial – value problem, whose solution will give the charge Q on the capacitor at any time t. The current I in the current at any time t is obtained by differentiating $Q(t)$ w.r.t. t.

CASE (1): Zero Electromotive Force $E(t) = 0$

In this case, equation (7) reduces to

$$\frac{d^2 Q}{dt^2} + \frac{1}{LC} Q = 0$$

which is of the same form as equation (5) above. The solution can be obtained and interpreted in the same manner.

CASE (2): Periodic Electromotive Force $E(t) = E_0 \sin \omega t$

In this case, equation (7) reduces to

$$\frac{d^2 Q}{dt^2} + \frac{1}{LC} Q = \frac{1}{L} E_0 \sin \omega t$$

which is of the same form as equation (6) above and can be solved and interpreted in the same manner.

EXAMPLE (21): **An LC – circuit contains an inductor of inductance 1 henry, a capacitor of capacitance 0.25 farad, and an electromotive force of 30 sin t volts. The initial current and the initial charge on the capacitor are zero. Find the current I flowing through the circuit at any time t.**

SOLUTION: We know that the differential equation for the current I in an LC circuit is

$$\frac{d^2 I}{dt^2} + \frac{1}{LC} I = \frac{1}{L}\frac{dE}{dt} \qquad (1)$$

In the present case, $L = 1$ henry, $C = 0.25$ farad, and $E(t) = 30 \sin t$ volts, therefore equation (1) becomes

$$\frac{d^2 I}{dt^2} + 4I = 30 \cos t \qquad (2)$$

In operator notation, this equation takes the form

$$(D^2 + 4) I = 30 \cos t \qquad \left(D = \frac{d}{dt}\right)$$

The auxiliary equation is $D^2 + 4 = 0$ having characteristic roots $D = \pm 2i$.

The complementary function is

$$I_c(t) = C_1 \cos 2t + C_2 \sin 2t$$

The particular integral is given by

$$I_p(t) = \frac{1}{D^2 + 4} 30 \cos t$$

$$= 30\left(\frac{1}{-1+4}\right) \cos t = 10 \cos t$$

The general solution of equation (2) is given by

$$I(t) = I_c(t) + I_p(t)$$

$$= C_1 \cos 2t + C_2 \sin 2t + 10 \cos t \qquad (3)$$

Using the first initial condition $I(0) = 0$, we get from equation (3)

$$0 = C_1 + 10 \quad \text{or} \quad C_1 = -10$$

Differentiating equation (3), we get

$$I'(t) = \frac{dI}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t - 10 \sin t \qquad (4)$$

Using the second initial condition

$$I'(0) = \left(\frac{dI}{dt}\right)_{t=0} = \frac{1}{L}E(0) - \frac{1}{LC}Q(0) = 0 - 4(0) = 0$$

we get from equation (4)

$$2C_2 = 0 \quad \text{or} \quad C_2 = 0$$

Substituting the values of C_1 and C_2 in equation (3), we get

$$\begin{aligned} I(t) &= -10 \cos 2t + 10 \cos t \\ &= 10(\cos t - \cos 2t) \end{aligned} \qquad (5)$$

ALTERNATIVE METHOD

We know that the differential equation for the current Q in an LC – circuit is

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = \frac{1}{L}E(t) \qquad (6)$$

In the present case, $L = 1$ henry, $C = 0.25$ farad, and $E(t) = 30 \sin t$ volts, therefore equation (6) becomes

$$\frac{d^2Q}{dt^2} + 4Q = 30 \sin t \qquad (7)$$

In operator notation, this equation takes the form

$$(D^2 + 4)Q = 30 \sin t$$

The auxiliary equation is $D^2 + 4 = 0$ having characteristic roots $D = \pm 2i$.

The complementary function is

$$Q_c(t) = C_1 \cos 2t + C_2 \sin 2t$$

The particular integral is given by

$$\begin{aligned} Q_p(t) &= \frac{1}{D^2+4} 30 \sin t \\ &= \left(\frac{1}{-1+4}\right) 30 \sin t = 10 \sin t \end{aligned}$$

The general solution of equation (7) is given by

$$\begin{aligned} Q(t) &= Q_c(t) + Q_p(t) \\ &= C_1 \cos 2t + C_2 \sin 2t + 10 \sin t \end{aligned} \qquad (8)$$

Using the first initial condition $Q(0) = 0$, we get from equation (8)

$$C_1 = 0$$

Differentiating equation (8), we get

$$Q'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t + 10 \cos t \qquad (9)$$

Using the second initial condition $Q'(0) = 0$, we get from equation (9)

$$0 = 2C_2 + 10 \quad \text{or} \quad C_2 = -5$$

Substituting the values of C_1 and C_2 in equation (8), we get

$$Q(t) = -5 \sin 2t + 10 \sin t \qquad (10)$$

The current $I(t)$ in the circuit is obtained by differentiating equation (10) w.r.t. t, i.e.

$$I(t) = -10 \cos 2t + 10 \cos t$$

which is the same as equation (5).

8.31 RLC – CIRCUIT

We now consider the flow of current through an RLC – circuit shown in figure (8.29). In it, an Ohm's resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C are connected in series to a source of electromotive force $E(t)$, where t is the time. The differential equation for the current $I(t)$ in RLC – circuit is obtained by considering the following three voltage drops :

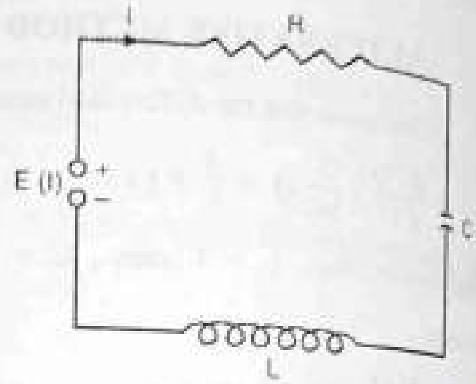


Figure (8.29)

The voltage drop across the resistor, by Ohm's law is

$$E_R = RI \qquad (1)$$

The voltage drop across the inductor is given by

$$E_L = L\frac{dI}{dt} \qquad (2)$$

The voltage drop across the capacitor is

$$E_C = \frac{1}{C}\int I(t)\,dt \qquad (3)$$

According to Kirchhoff's voltage law, the sum of three voltage drops must equal the electromotive force $E(t)$. Thus

$$RI + L\frac{dI}{dt} + \frac{1}{C}\int I(t)\,dt = E(t) \qquad (4)$$

To get rid of the integral in equation (4), we differentiate w.r.t. t, to get

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}I = \frac{dE}{dt}$$

or

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt} \qquad (5)$$

which is a second order non – homogeneous differential equation with constant coefficients.

INITIAL CONDITIONS

The general solution of equation (5) will involve two arbitrary constants , which will be determined using two initial conditions . If the initial current is I_0 and the initial charge is Q_0 , then the first initial condition is

when $t = 0$, $I(0) = I_0$ (6)

The second initial condition is obtained from equation (4) by solving it for $\frac{dI}{dt}$ and then setting $t = 0$.

Equation (4) can be written as

$$\frac{dI}{dt} = \frac{1}{L}E(t) - \frac{R}{L}I(t) - \frac{1}{LC}Q(t)$$

At $t = 0$, this becomes

$$\left(\frac{dI}{dt}\right)_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I_0 - \frac{1}{LC}Q_0 \tag{7}$$

Equations (5) , (6) , and (7) constitute an initial – value problem , whose solution will give the current I in the circuit at any time t .

CASE (1): Constant Electromotive Force $E(t) = E_0$

In this case , equation (5) reduces to

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0 \tag{8}$$

which is similar to the differential equation of mass – spring system in the case of damped free oscillations . The solution can be obtained and interpreted in the same manner .

CASE (2): Periodic Electromotive Force $E(t) = E_0 \sin \omega t$

In this case , equation (5) reduces to

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}E_0\,\omega \cos \omega t \tag{9}$$

which is similar to the differential equation of mass – spring system in the case of damped forced oscillations and can be solved and interpreted in the same manner .

DIFFERENTIAL EQUATION IN TERMS OF CHARGE Q

We know that $I = \frac{dQ}{dt}$. Hence $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$, and $\int I\,dt = Q$

Then equation (4) can be written as

$$R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2} + \frac{1}{C}Q = E(t)$$

or

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{1}{L}E(t) \tag{10}$$

which is a second order linear differential equation with constant coefficients .

INITIAL CONDITIONS

The solution of equation (10) will involve two arbitrary constants, which will be determined using two initial conditions. If the initial charge is Q_0, then the initial conditions are :

when $t = 0$, $Q(0) = Q_0$ (11)

when $t = 0$, $I(0) = \left(\frac{dQ}{dt}\right)_{t=0} = 0$ (12)

Equations (10), (11), and (12) constitute an initial – value problem, whose solution will give the charge Q on the capacitor at any time t. The current I at any time t is obtained by differentiating the charge $Q(t)$ w.r.t. t.

CASE (1): Constant Electromotive Force $E(t) = E_0$

In this case, equation (10) reduces to

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{1}{LC}Q = 0$$

which is of the same form as equation (8) above and can be solved and interpreted in the same manner.

CASE (2): Periodic Electromotive Force $E(t) = E_0 \sin \omega t$

In this case, equation (10) reduces to

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{1}{LC}Q = \frac{1}{L}E_0 \sin \omega t$$

which is of the same form as equation (9) above and can be solved and interpreted in the same manner.

EXAMPLE (22): **An RLC circuit has a resistance of 10 ohms, a capacitance of 10^{-2} farad, an inductance of 0.5 henry, and an applied voltage $E = 12$ volts. Assuming no initial current and no initial charge on the capacitor, find an expression for the current flowing through the circuit at any time t.**

SOLUTION: We know that the differential equation for the current I in an RLC circuit is

$$\frac{d^2I}{dt^2}+\frac{R}{L}\frac{dI}{dt}+\frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt} \quad (1)$$

In the present case, $R = 10$ ohms, $C = 10^{-2}$ farad, $L = 0.5$ henry and $E(t) = 12$ volts, therefore differential equation (1) becomes

$$\frac{d^2I}{dt^2}+20\frac{dI}{dt}+200\,I = 0 \quad (2)$$

In operator notation, this equation takes the form

$$(D^2+20D+200)I = 0 \quad \left(D = \frac{d}{dt}\right)$$

The auxiliary equation is $D^2+20D+200 = 0$ whose characteristic roots are $D = -10 \pm 10i$.

Thus the general solution of equation (2) is given by

$$I(t) = e^{-10t}(C_1 \cos 10t + C_2 \sin 10t) \tag{3}$$

Using the initial condition that when $t = 0$, $I(0) = 0$, we get from equation (3) $C_1 = 0$.

Thus equation (3) reduces to

$$I(t) = C_2 e^{-10t} \sin 10t \tag{4}$$

Differentiating equation (4) w.r.t. t, we get

$$\frac{dI}{dt} = C_2\left[e^{-10t}(10\cos 10t) - 10e^{-10t}\sin 10t\right] \tag{5}$$

The second initial condition takes the form

$$\left(\frac{dI}{dt}\right)_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I_0 - \frac{1}{LC}Q_0$$

$$= \frac{12}{1/2} - \left(\frac{10}{1/2}\right)(0) - \frac{1}{(1/2)(10^{-2})}(0) = 24$$

Using this initial condition, we get from equation (5)

$$24 = 10C_2 \quad \text{or} \quad C_2 = \frac{12}{5}$$

Thus the current I in the circuit at any time t from equation (4) is given by

$$I = \frac{12}{5}e^{-10t}\sin 10t$$

ALTERNATIVE METHOD

We know that the differential equation for the charge Q in an RLC circuit is

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{1}{L}E(t) \tag{6}$$

In the present case, $R = 10$ ohms, $C = 10^{-2}$ farad, $L = 0.5$ henry and $E(t) = 12$ volts, therefore equation (6) becomes

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 200Q = 24 \tag{7}$$

which in operator notation takes the form

$$(D^2 + 20D + 200)Q = 24 \qquad \left(D = \frac{d}{dt}\right)$$

The auxiliary equation is $D^2 + 20D + 200 = 0$ which has characteristic roots $D = -10 \pm 10i$.

The complementary function is

$$Q_c(t) = e^{-10t}(C_1 \cos 10t + C_2 \sin 10t)$$

The particular integral is given by

$$Q_p(t) = \frac{1}{D^2 + 20D + 200}(24) = \frac{24}{200} = \frac{3}{25}$$

Thus the general solution of equation (7) is

$$Q(t) = Q_c(t) + Q_p(t)$$

$$= e^{-10t}(C_1 \cos 10t + C_2 \sin 10t) + \frac{3}{25} \tag{8}$$

Using the first initial condition that when $t = 0$, $Q = 0$, we get from equation (8)

$$0 = C_1 + \frac{3}{25} \quad \text{or} \quad C_1 = -\frac{3}{25}$$

Thus equation (8) reduces to

$$Q(t) = e^{-10t}\left(-\frac{3}{25}\cos 10t + C_2 \sin 10t\right) + \frac{3}{25} \tag{9}$$

Differentiating equation (9), we get

$$\frac{dQ}{dt} = e^{-10t}\left(\frac{30}{25}\sin 10t + 10\,C_2 \cos 10t\right) - 10\,e^{-10t}\left(-\frac{3}{25}\cos 10t + C_2 \sin 10t\right) \tag{10}$$

Using the second initial condition that when $t = 0$, $\left(\frac{dQ}{dt}\right)_{t=0} = 0$, we get from equation (10)

$$0 = 10\,C_2 + \frac{30}{25} \quad \text{or} \quad C_2 = -\frac{3}{25}$$

Thus from equation (9), the charge Q at any time t is given by

$$Q(t) = e^{-10t}\left(-\frac{3}{25}\cos 10t - \frac{3}{25}\sin 10t\right) + \frac{3}{25}$$

$$= -e^{-10t}\left(\frac{3}{25}\cos 10t + \frac{3}{25}\sin 10t\right) + \frac{3}{25} \tag{11}$$

The current I at any time t is obtained by differentiating equation (11) w.r.t. t

i.e. $$I(t) = \frac{dQ}{dt} = -e^{-10t}\left(-\frac{30}{25}\sin 10t + \frac{30}{25}\cos 10t\right) + 10\,e^{-10t}\left(\frac{3}{25}\cos 10t + \frac{3}{25}\sin 10t\right)$$

$$= \frac{30}{25}e^{-10t}\sin 10t + \frac{30}{25}e^{-10t}\sin 10t$$

$$= \frac{12}{5}e^{-10t}\sin 10t \tag{12}$$

Note that although the current is completely transient, the charge on the capacitor is the sum of both transient and steady – state terms.

8.32 MODELING ELECTRICAL NETWORK

EXAMPLE (23): **Set up the mathematical model of the network shown in figure (8.30). Find the currents $I_1(t)$ and $I_2(t)$ assuming that $R = 10$ ohms, $L = 20$ henrys, $C = 0.05$ farad, $E(t) = 20$ volts and $I_1(0) = 0$, $I_2(0) = 2$ amperes.**

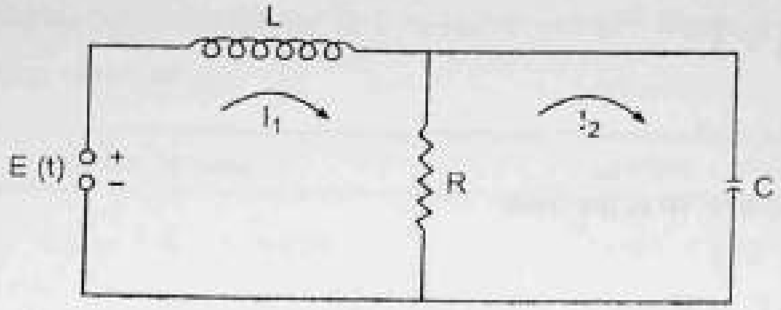


Figure (8.30)

SOLUTION: The mathematical model of the given network is obtained from the Kirchhoff's voltage law.

FOR THE LEFT LOOP

If I_1 is the current flowing in the left loop, then the voltage drop across an inductor L is

$$E_L = L\frac{dI_1}{dt}$$

The voltage drop across the resistor R is $E_R = R(I_1 - I_2)$

By Kirchoff's voltage law, the sum of two voltage drops must equal the electromotive force $E(t)$.

Thus $$L\frac{dI_1}{dt} + R(I_1 - I_2) = E(t) \qquad (1)$$

FOR THE RIGHT LOOP

If I_2 is the current flowing in the right loop, then the voltage drop across the capacitor C is

$$E_C = \frac{1}{C}Q = \frac{1}{C}\int I_2(t)\,dt$$

The voltage drop across the resistor R is $E_R = R(I_2 - I_1)$

By Kirchoff's voltage law, the sum of two voltage drops must equal zero. Thus

$$\frac{1}{C}\int I_2(t)\,dt + R(I_2 - I_1) = 0$$

Differentiating this equation w.r.t. t, we get

$$\frac{1}{C}I_2 + R\frac{dI_2}{dt} - R\frac{dI_1}{dt} = 0$$

or $$-R\frac{dI_1}{dt} + R\frac{dI_2}{dt} + \frac{1}{C}I_2 = 0 \qquad (2)$$

Note that this equation involves the derivatives of both the unknowns. Thus we see that the currents I_1 and I_2 in the network are governed by the system of differential equations (1) and (2).

In the present case, this system takes the form

$$20\frac{dI_1}{dt} + 10(I_1 - I_2) = 20$$

$$-10\frac{dI_1}{dt} + 10\frac{dI_2}{dt} + 20I_2 = 0$$

or $$2\frac{dI_1}{dt}+(I_1-I_2)=2 \tag{3}$$

$$-\frac{dI_1}{dt}+\frac{dI_2}{dt}+2I_2=0 \tag{4}$$

In operator notation, this system takes the form

$$(2D+1)I_1-I_2=2 \tag{5}$$

$$-DI_1+(D+2)I_2=0 \tag{6}$$

Operating on equation (5) by $(D+2)$ and adding equation (6), we get

$$(2D+1)(D+2)I_1-DI_1=(D+2)2$$

or $$(2D^2+5D+2-D)I_1=4$$

or $$(2D^2+4D+2)I_1=4$$

or $$(D+1)^2 I_1=2 \tag{7}$$

The auxiliary equation $(D+1)^2=0$ has characteristic roots as $D=-1,-1$.

The complementary function is $(I_1)_c=(C_1+C_2 t)e^{-t}$.

The particular integral is $(I_1)_p=2$.

The general solution of equation (7) is

$$I_1(t)=(C_1+C_2 t)e^{-t}+2 \tag{8}$$

From equation (5), we get

$$\begin{aligned} I_2(t) &= (2D+1)I_1-2 \\ &= (2D+1)\left[(C_1+C_2 t)e^{-t}+2\right]-2 \\ &= 2\left[-(C_1+C_2 t)e^{-t}+C_2 e^{-t}\right]+(C_1+C_2 t)e^{-t} \\ &= -(C_1+C_2 t)e^{-t}+2C_2 e^{-t} \end{aligned} \tag{9}$$

Using the initial conditions $I_1(0)=0$, $I_2(0)=2$, we get from equations (8) and (9)

$$C_1+2=0 \quad\text{and}\quad -C_1+2C_2=2.$$

Solving these equations, we get $C_1=-2$, $C_2=0$.

Thus the solution of this system is

$$I_1(t)=2-2e^{-t} \quad\text{and}\quad I_2(t)=2e^{-t}$$

8.33 COMPARISON OF RLC CIRCUITS AND FORCED DAMPED SPRING SYSTEMS

The equation of RLC circuit is

$$LI''+RI'+\frac{1}{C}I=E'(t)$$

and the equation for forced, damped spring motion is

$$my''+Cy'+Ky=F(t)$$

Observe that the differential equation for the RLC circuit is virtually identical to that of the mechanical system with the following observations .

Spring System	Electrical Circuit
$m\,y'' + C\,y' + K\,y = F(t)$	$L\,I'' + R\,I' + \frac{1}{C} I = E'$
displacement $y(t)$	current $I(t)$
driving force $F(t)$	derivative E' of electromotive force
mass m	Inductance L
damping constant C	Resistance R
Spring modulus K	reciprocal $\frac{1}{C}$ of capacitance

8.34 DEFLECTION OF BEAMS

Consider a horizontal beam of length L , which in its undeflected position lies along the positive direction of x – axis as shown in figure (8.31) . The beam is of constant (e.g. rectangular) cross – section and is made up of homogeneous material . If we apply a load to the beam in a vertical plane through its axis of symmetry (i.e. x – axis) the beam is bent or deflected . The deflection curve C of the beam , often called the **elastic curve** and shown dotted in the figure , is given by $y = f(x)$, where y is measured positive in the downward direction as shown in figure (8.32) .

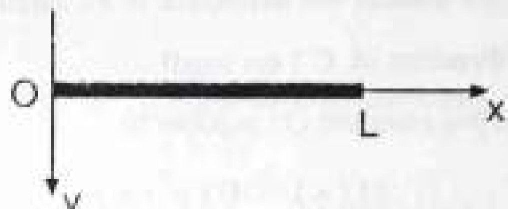


Figure (8.31)

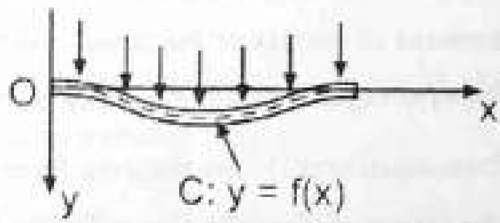


Figure (8.32)

The loads which cause the beam to bend may be of two types : They may be concentrated at one or more points along the beam , or they may be continuously distributed along the beam with a density function $W(x)$ known as the load per unit length . In either case , we have two important related quantities : one is the shear $V(x)$ at any point x of the beam which is defined to be the algebraic sum of all the vertical forces which act on the beam on the positive side of the point in question . The other is the bending moment $M(x)$ at x and is equal to the algebraic sum of the moments of all the forces on one side of x , the moments being taken as positive for forces in the positive y – direction and negative otherwise .

We shall consider the load per unit length and the shear to be positive if they act in the direction of the positive y – axis (direction in which loads usually act on a beam) .

It is shown in the theory of elasticity that the bending moment $M(x)$ is proportional to the curvature $K(x)$ of the deflection curve C

i.e. $$M(x) = EI\,K(x) \tag{1}$$

where EI is the constant of proportionality.

From calculus, we have

$$K(x) = \frac{y''}{\left[1+(y')^2\right]^{3/2}}$$

Thus equation (1) becomes

$$M(x) = EI\frac{y''}{\left[1+(y')^2\right]^{3/2}} \tag{2}$$

which is a second order nonlinear differential equation. Equation (2) is called the Bernoulli – Euler differential equation.

We assume the deflection to be small, so that $y(x)$ and its derivative $y'(x)$ (determining the tangent direction of C) are small.

Thus equation (2) reduces to

$$M(x) = EI\,y''(x) \tag{3}$$

where E is the **Young's modulus of elasticity** of the material from which beam is made, and I is the **moment of inertia** of the cross – section of the beam about its central axis. The quantity EI is called the **flexural rigidity** and is generally constant.

From equation (3), the shearing force is given by $EI\,y'''(x)$.

The theory of elasticity shows further that

$$M''(x) = W(x) \tag{4}$$

where $W(x)$ is the load per unit length.

From equations (3) and (4), we get

$$EI\,y^{(iv)}(x) = W(x) \tag{5}$$

which is a fourth order linear differential equation.

BOUNDARY CONDITIONS

The general solution of equation (2) or (4) will involve arbitrary constants which will be determined using the boundary conditions which depend upon how the beam is supported. The beam can be supported in various ways. The practically most important supports and corresponding boundary conditions are as follows :

(i) ENDS FIXED HORIZONTALLY

At the end fixed horizontally , the deflections is zero . Also , at this end the beam is horizontal , therefore the slope of the beam is zero . Thus the boundary conditions in this case :

$$y = y' = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L$$

i.e. $y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0$

Figure (8.33)

(ii) ENDS SIMPLY SUPPORTED

At the freely supported ends $x = 0$ and $x = L$, thus is no deflection of the beam so that $y = 0$.

Also there is no bending moment at this end , so that $\frac{d^2 y}{d x^2} = 0$.

Hence at freely supported ends $y = y'' = 0$ at $x = 0$ and $x = L$.

i.e. $y(0) = 0, \quad y''(0) = 0, \quad y(L) = 0, \quad y''(L) = 0$

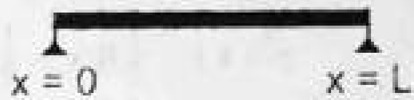


Figure (8.34)

(iii) DAMPED AT $x = 0$ AND FREE AT $x = L$

At the free and $x = L$ there is no bending moment or shear force , so that

$\frac{d^2 y}{d x^2} = 0$ and $\frac{d^3 y}{d x^3} = 0$ at this point .

Figure (8.35)

Thus the boundary conditions in this case are :

$$y(0) = y'(0) = 0, \quad y'''(L) = y'''(L) = 0$$

EXAMPLE (24): **A beam of length L is simply supported at both ends as shown in figure (8.36) .**

(i) Find the deflection if the beam has constant load per unit length $[W(x) = W_0]$

(ii) Determine the maximum deflection .

SOLUTION: (i) We know that

$$E I y^{(iv)}(x) = W(x)$$

or $y^{(iv)}(x) = \frac{W_0}{E I} \quad [\text{since} \;\; W(x) = W_0]$

$$= K \quad \left(K = \frac{W_0}{E I}\right)$$

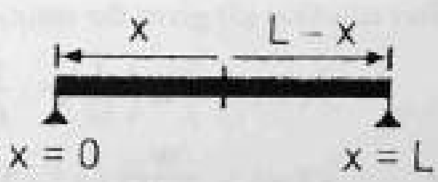


Figure (8.36)

Integrating twice , we get

$$y''(x) = \frac{1}{2} K x^2 + C_1 x + C_2 \qquad (1)$$

The boundary conditions in this case are :

$$y(0) = 0, \quad y''(0) = 0, \quad y(L) = 0, \quad y''(L) = 0$$

Using the boundary condition $y''(L) = 0$, we get from equation (1) $C_2 = 0$.

Thus equation (1) reduces to

$$y''(x) = \frac{1}{2}K x^2 + C_1 x \tag{2}$$

Using $y''(L) = 0$, we get

$$\frac{1}{2}K L^2 + C_1 L = 0$$

or $$C_1 = -\frac{1}{2}K L$$

Thus equation (2) reduces to

$$y''(x) = \frac{1}{2}K x^2 - \frac{1}{2}K L x$$

or $$y''(x) = \frac{1}{2}K(x^2 - L x) \tag{3}$$

Integrating equation (3) twice again, we get

$$y(x) = \frac{1}{2}K\left(\frac{1}{12}x^4 - \frac{1}{6}L x^3 + C_3 x + C_4\right) \tag{4}$$

Using $y(0) = 0$, we get $\frac{1}{2}K C_4 = 0$ or $C_4 = 0$.

Thus equation (4) becomes

$$y(x) = \frac{1}{2}K\left(\frac{1}{12}x^4 - \frac{1}{6}L x^3 + C_3 x\right) \tag{5}$$

Using $y(L) = 0$, we get

$$\frac{1}{2}K\left(\frac{1}{12}L^4 - \frac{1}{6}L^4 + C_3 L\right) = 0$$

or $$\frac{1}{12}L^3 - \frac{1}{6}L^3 + C_3 = 0$$

or $$C_3 = \frac{L^3}{12}$$

Thus equation (5) gives the solution as

$$y(x) = \frac{1}{2}K\left(\frac{1}{12}x^4 - \frac{1}{6}L x^3 + \frac{1}{12}L^3 x\right)$$

or $$y(x) = \frac{W_0}{24\,E\,I}(x^4 - 2 L x^3 + L^3 x)$$

(ii) The maximum deflection occurs at $x = \frac{L}{2}$. Thus

$$[y(x)]_{\max} = \frac{5\,W_0 L^4}{384\,E\,I}$$

8.35 SOLVED PROBLEMS

GEOMETRICAL APPLICATIONS

PROBLEM (1): **Find the equation of the curve for which the product of the subtangent at any point and the abscissa of that point is constant.**

SOLUTION: We know that the subtangent at any point $P(x, y)$ of the curve is $y\frac{dx}{dy}$.

Then according to the condition

$$x \cdot y\frac{dx}{dy} = C$$

Separating the variables, we get

or $$C\frac{dy}{y} = x\,dx$$

Integrating, we get

$$C \ln y = \frac{x^2}{2} + \ln A$$

or $$\ln y^C = \frac{x^2}{2} + \ln A$$

or $$y^C = A\,e^{x^2/2}$$

is the required equation of the curve.

PROBLEM (2): **Find the equation of the curve for which the angle between the radius vector and the tangent is half the vectorial angle.**

SOLUTION: Let ϕ be the angle between the radius vector and the tangent at $P(r, \theta)$, then

$$\tan\phi = r\frac{d\theta}{dr} \qquad (1)$$

But according to the condition

$$\phi = \frac{\theta}{2} \qquad (2)$$

From equations (1) and (2), we get

$$r\frac{d\theta}{dr} = \tan\frac{\theta}{2}$$

Separating the variables, we get

$$\frac{dr}{r} = \cot\frac{\theta}{2}\,d\theta$$

Integrating, we get

$$\ln r = 2\ln\sin\frac{\theta}{2} + \ln C$$

or $$\ln r = \ln\sin^2\frac{\theta}{2} + \ln C = \ln C\sin^2\frac{\theta}{2}$$

or $\quad r = C\sin^2\frac{\theta}{2} = C\left(\frac{1-\cos\theta}{2}\right)$

or $\quad r = a(1-\cos\theta) \quad \left(\text{where } a = \frac{C}{2}\right)$

which is a cardioid .

PROBLEM (3): **Find the equation of the curve for which the tangent at each point makes a constant angle α with the radius vector .**

SOLUTION: Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$. If ϕ is the angle between the radius vector and the tangent at P, then

$$\tan\phi = r\frac{d\theta}{dr}$$

According to the given condition

$$\phi = \alpha \text{ (constant)}$$

Thus $\quad \tan\alpha = r\frac{d\theta}{dr}$

Separating the variables , we get

$$\frac{dr}{r} = \cot\alpha \, d\theta$$

Integrating , we get

$$\ln r = \theta\cot\alpha + \ln C$$

or $\quad \ln r = \ln C e^{\theta\cot\alpha}$

or $\quad r = C e^{\theta\cot\alpha}$

is the required equation of the curve .

PROBLEM (4): **Find the equation of the curve for which the polar subnormal is proportional to the sine of the vectorial angle .**

SOLUTION: Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

Then the polar subnormal $= \frac{dr}{d\theta}$

According to the given condition

$$\frac{dr}{d\theta} \propto \sin\theta$$

or $\quad \frac{dr}{d\theta} = a\sin\theta$

where a is a constant of proportionality .

Separating the variables , we get

$$dr = a\sin\theta \, d\theta$$

Integrating , we get

$$r = -a\cos\theta + C$$

or $$r = C - a\cos\theta$$

which is the required equation of the curve .

In particular , if $C = a$, the curve is the cardioid .

$$r = a(1-\cos\theta)$$

PROBLEM (5): Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.

SOLUTION: The equation of the family of parabolas is

$$y^2 = 4ax \tag{1}$$

STEP (1): Differentiating equation (1) implicitly w.r.t. x , we get

$$2y\frac{dy}{dx} = 4a \tag{2}$$

Eliminating a from equations (1) and (2) , we get

$$y^2 = 2y\frac{dy}{dx}\cdot x$$

or $$\frac{dy}{dx} = \frac{y}{2x} \tag{3}$$

is the differential equation of the family of parabolas (1) .

STEP (2): Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, the differential equation of orthogonal trajectories is

$$-\frac{dx}{dy} = \frac{y}{2x}$$

or $$y\,dy = -2x\,dx \tag{4}$$

STEP (3): Integrating equation (4) , we get

$$\frac{1}{2}y^2 = -x^2 + C_1$$

or $$2x^2 + y^2 = 2C_1$$

or $$2x^2 + y^2 = C \qquad (\text{where } C = 2C_1)$$

is the required equation .

PROBLEM (6): Find the orthogonal trajectories of the family of parabolas $y = Cx^2$.

SOLUTION: The equation of the family of parabolas is

$$y = Cx^2 \tag{1}$$

STEP (1): Differentiating equation (1) w.r.t. x , we get

$$\frac{dy}{dx} = 2Cx \tag{2}$$

Eliminating C from equations (1) and (2), we get

$$\frac{dy}{dx} = 2 \cdot \frac{y}{x^2} \cdot x$$

or $$\frac{dy}{dx} = 2\,\frac{y}{x} \qquad (3)$$

STEP (2): Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (3), the differential equation of the orthogonal trajectories is

$$-\frac{dx}{dy} = 2\frac{y}{x}$$

or separating the variables, we get

or $$2\,y\,dy = -x\,dx \qquad (4)$$

STEP (3): Integrating equation (4), we get

$$y^2 = -\frac{1}{2}x^2 + k$$

or $$\frac{1}{2}x^2 + y^2 = k \qquad (5)$$

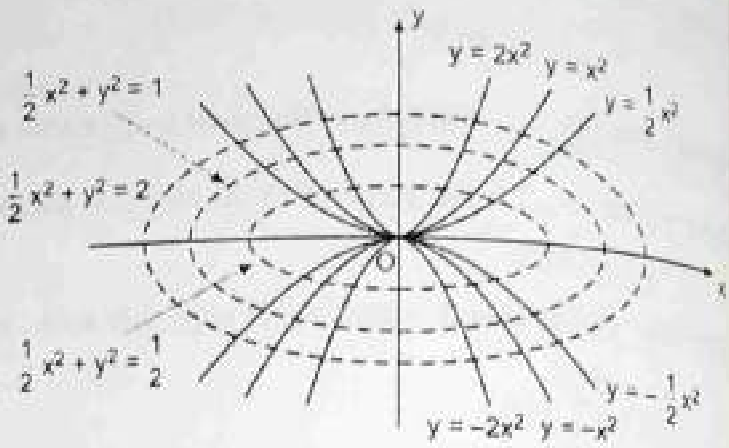


Figure (8.37)

is the equation of the family of orthogonal trajectories. These orthogonal trajectories are ellipses. Some members of the family of parabolas, along with some members of the family of ellipses are shown in figure (8.37). Note that each ellipse intersects each parabola at right angles.

PROBLEM (7): Find the orthogonal trajectories of the family of curves $x^2 + y^2 = Cx$.

SOLUTION: The equation of the family of given curves is

$$x^2 + y^2 = Cx \qquad (1)$$

which are circles with centres on the x – axis and radii $\frac{1}{2}C$.

STEP (1): Differentiating equation (1) implicitly w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = C \qquad (2)$$

Eliminating the parameter C from equations (1) and (2), we get

$$2x + 2y\frac{dy}{dx} = x + \frac{y^2}{x}$$

or $$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad (3)$$

STEP (2): Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (3), the differential equation for the family of orthogonal trajectories is

$$-\frac{dx}{dy} = \frac{y^2 - x^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \tag{4}$$

which is a homogeneous equation .

STEP (3): Let $y = Vx$ in equation (4) so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$.

Thus $$V + x\frac{dV}{dx} = \frac{2Vx^2}{x^2 - V^2x^2} = \frac{2V}{1 - V^2}$$

or $$x\frac{dV}{dx} = \frac{V + V^3}{1 - V^2}$$

or $$\frac{dx}{x} + \left[\frac{V^2 - 1}{V(1 + V^2)}\right] dV = 0$$

or $$\frac{dx}{x} + \left(-\frac{1}{V} + \frac{2V}{1 + V^2}\right) dV = 0$$

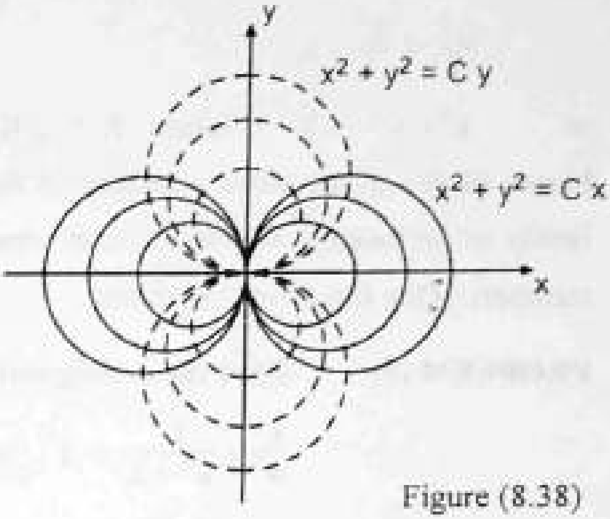


Figure (8.38)

Integrating , we get

$$\ln x - \ln V + \ln(1 + V^2) = \ln k$$

or $$\ln x + \ln(1 + V^2) = \ln V + \ln k$$

or $$x(1 + V^2) = kV \tag{5}$$

Let $V = \frac{y}{x}$ in equation (5) , we get

$$x\left(1 + \frac{y^2}{x^2}\right) = k\frac{y}{x}$$

or $$1 + \frac{y^2}{x^2} = k\frac{y}{x^2}$$

or $$x^2 + y^2 = ky \tag{6}$$

which is the equation of orthogonal trajectories .

PROBLEM (8): **Find the orthogonal trajectories of the family of rectangular hyperbolas $xy = C$.**

SOLUTION: The equation of the family of rectangular hyperbolas is

$$xy = C \tag{1}$$

STEP (1): Differentiating equation (1) implicitly w.r.t. x , we get

$$x\frac{dy}{dx} + y = 0$$

or $$\frac{dy}{dx} = -\frac{y}{x} \tag{2}$$

STEP (2): Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (2) , the differential equation of the family of orthogonal trajectories is

$$-\frac{dx}{dy} = -\frac{y}{x}$$

$$x\,dx = y\,dy \qquad (3)$$

STEP (3): Integrating both sides of equation (3), we get

$$\frac{x^2}{2} = \frac{y^2}{2} + K_1$$

or $\quad x^2 - y^2 = K \quad (\text{where } K = 2K_1)$

Figure (8.39) shows some members of the family of rectangular hyperbolas and some members of the family of hyperbolas.

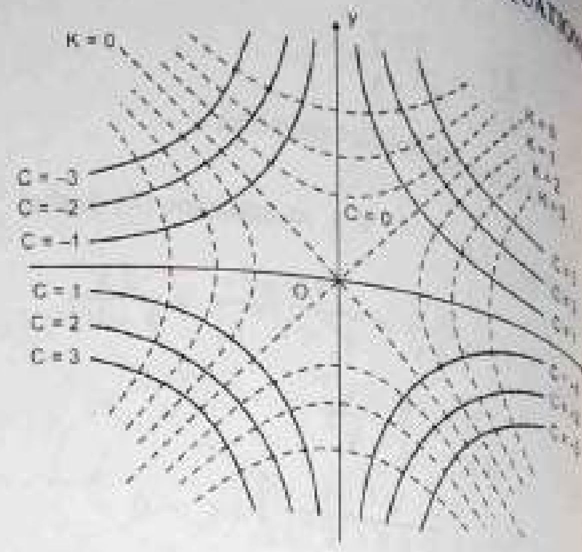


Figure (8.39)

PROBLEM (9): **Find the orthogonal trajectories of the family of curves**

$$\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1,$$ **where a is constant and λ is a parameter.**

SOLUTION: The equation of the family of curves is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1 \qquad (1)$$

STEP (1): Differentiating equation (1) implicitly w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{a^2+\lambda}\frac{dy}{dx} = 0$$

or $\quad \frac{x}{a^2} + \frac{y}{a^2+\lambda}p = 0 \quad \left(\text{where } p = \frac{dy}{dx}\right)$

or $\quad \frac{xy}{a^2} + \frac{y^2}{a^2+\lambda}p = 0$

or $\quad \frac{y^2}{a^2+\lambda} = -\frac{xy}{a^2 p} \qquad (2)$

From equations (1) and (2), we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2 p} = 1 \qquad (3)$$

which being free from the parameter λ, is the differential equation of the family of curves (1).

STEP (2): Replacing p by $-\frac{1}{p}$ in equation (3), the differential equation of the orthogonal trajectories is

$$\frac{x^2}{a^2} + \frac{xy}{a^2}p = 1$$

or $\quad p = \frac{a^2 - x^2}{xy}$

or $$\frac{dy}{dx} = \frac{a^2 - x^2}{xy}$$

Separating the variables, we get

$$y\,dy = \frac{a^2 - x^2}{x}\,dx$$

$$= \left(\frac{a^2}{x} - x\right)dx \qquad (4)$$

STEP (3): Integrating equation (4), we get

$$\frac{y^2}{2} = a^2 \ln x - \frac{x^2}{2} + C_1$$

or $$x^2 + y^2 = 2a^2 \ln x + 2C_1$$

or $$x^2 + y^2 = 2a^2 \ln x + C \quad (\text{where } C = 2C_1)$$

is the equation of the family of orthogonal trajectories.

SELF – ORTHOGONAL TRAJECTORIES

PROBLEM (10): **Prove that the family of confocal conics**

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \textbf{is self – orthogonal.}$$

SOLUTION: The equation of the family of confocal conics is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \qquad (1)$$

STEP (1): Differentiating equation (1) implicitly w.r.t. x, we get

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

or $$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}p = 0 \qquad \left(\text{where } p = \frac{dy}{dx}\right)$$

or $$(b^2+\lambda)x + (a^2+\lambda)yp = 0$$

or $$(x+yp)\lambda = -(a^2yp + b^2x)$$

or $$\lambda = -\frac{a^2yp + b^2x}{x+yp}$$

Thus $$a^2 + \lambda = a^2 - \frac{a^2yp + b^2x}{x+yp} = \frac{(a^2-b^2)x}{x+yp}$$

and $$b^2 + \lambda = b^2 - \frac{a^2yp + b^2x}{x+yp} = \frac{(b^2-a^2)yp}{x+yp}$$

Substituting in equation (1), we get

$$\frac{x^2(x+yp)}{(a^2-b^2)x} + \frac{y^2(x+yp)}{(b^2-a^2)yp} = 1$$

or $\quad (x+yp)\left(x-\frac{y}{p}\right)=a^2-b^2 \qquad (2)$

which being free from λ (the parameter) is the differential equation of the family of curves (1) .

STEP (2): Replacing p and $-\frac{1}{p}$, the differential equation of the orthogonal trajectories is

$$\left(x-\frac{y}{p}\right)(x+yp)=a^2-b^2 \qquad (3)$$

which is the same as equation (3) .

Since the differential equations of the family of curves (1) and that of orthogonal trajectories are the same , the family of curves (1) is self – orthogonal .

POLAR COORDINATES

PROBLEM (11): Find the orthogonal trajectories of the family of curves $r = 2a\cos\theta$.

SOLUTION: The equation of the family of curves is

$$r = 2a\cos\theta \qquad (1)$$

which are circles having centres on the line $\theta = 0°$ (i.e. x – axis) .

STEP (1): Differentiating equation (1) w.r.t. θ , we get

$$\frac{dr}{d\theta} = -2a\sin\theta \qquad (2)$$

Eliminating the parameter a from equations (1) and (2) , we get

$$\frac{dr}{d\theta} = -\frac{r}{\cos\theta}.\sin\theta$$

or $\quad \frac{dr}{d\theta} = -r\tan\theta \qquad (3)$

STEP (2): Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in equation (3) , the differential equation of the family of orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = -r\tan\theta$$

or $\quad \frac{dr}{r} = \cot\theta\, d\theta \qquad (4)$

STEP (3): Integrating both sides of equation (4) , we get

$$\ln r = \ln\sin\theta + \ln 2C$$

or $\quad r = 2C\sin\theta \qquad (5)$

The family of orthogonal trajectories is a family of circles having centres on the line

$\theta = \frac{\pi}{2}$ (i.e. y – axis) .

The family of orthogonal trajectories and the family of given curves is shown in figure (8.39) .

PROBLEM (12): **Find the equation of the family of orthogonal trajectories of a family of confocal and coaxial parabolas $r = \dfrac{2a}{1+\cos\theta}$.**

SOLUTION: The equation of the family of confocal and coaxial parabolas is

$$r = \frac{2a}{1+\cos\theta}$$

or $$r(1+\cos\theta) = 2a \qquad (1)$$

STEP (1): Differentiating equation (1) w.r.t. θ, we get

$$-r\sin\theta + (1+\cos\theta)\frac{dr}{d\theta} = 0$$

or $$\frac{dr}{d\theta} = \frac{r\sin\theta}{1+\cos\theta} = \frac{2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

or $$\frac{dr}{d\theta} = r\tan\frac{\theta}{2} \qquad (2)$$

which is the differential equation of the family of curves (1).

STEP (2): Replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in equation (2), the differential equation of the family of orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = r\tan\frac{\theta}{2}$$

or $$\frac{dr}{r} = -\cot\frac{\theta}{2}\,d\theta \qquad (3)$$

STEP (3): Integrating equation (3), we get

$$\ln r = -2\ln\sin\frac{\theta}{2} + \ln C$$

or $$\ln r = \ln\frac{C}{\sin^2\frac{\theta}{2}}$$

or $$r = \frac{C}{\sin^2\frac{\theta}{2}} = \frac{2C}{1-\sin\theta}$$

is the equation of the family of orthogonal trajectories.

PROBLEM (13): **Find the orthogonal trajectories of the family of logarithmic spirals $r = a^{\theta}$, where a is a parameter.**

SOLUTION: The equation of the family of curves is

$$r = a^{\theta} \qquad (1)$$

STEP (1): Taking logarithms of both sides , we get

$$\ln r = \theta \ln a \qquad (2)$$

Differentiating both sides w.r.t. θ , we get

$$\frac{1}{r}\frac{dr}{d\theta} = \ln a$$

or $$\frac{dr}{d\theta} = r \ln a = r\frac{\ln r}{\theta} \quad \left[\text{ using equation (2) }\right] \qquad (3)$$

which being free from the parameter a , is the differential equation of the family of curves (1) .

STEP (2): Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in equation (3) , the differential equation of the orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = r\frac{\ln r}{\theta}$$

or $$\frac{\ln r}{r} dr = -\theta\, d\theta \qquad (4)$$

STEP (3): Integrating equation (4) , we get

$$\int \ln r \cdot \frac{1}{r} dr = -\int \theta\, d\theta$$

or $$\frac{(\ln r)^2}{2} = -\frac{\theta^2}{2} + C_1$$

or $$(\ln r)^2 = -\theta^2 + C^2 \qquad \left(\text{where } C_1 = \frac{C^2}{2}\right)$$

or $$\ln r = \sqrt{C^2 - \theta^2}$$

or $$r = e^{\sqrt{C^2-\theta^2}}$$

which is the equation of the family of orthogonal trajectories .

PROBLEM (14): **Find the orthogonal trajectories of the family of curves $r^n = a^n \cos n\theta$ and hence find the orthogonal trajectories of the family of lemniscates $r^2 = a^2 \cos 2\theta$.**

SOLUTION: The equation of the family of curves is

$$r^n = a^n \cos n\theta \qquad (1)$$

STEP (1): Taking the logarithms of both sides of equation (1) , we get

$$n \ln r = n \ln a + \ln \cos n\theta$$

Differentiating both sides w.r.t. θ , we get

$$\frac{n}{r}\frac{dr}{d\theta} = -n \tan n\theta$$

or $$\frac{dr}{d\theta} = -r \tan n\theta \qquad (2)$$

is the differential equation of the family of curves (1).

STEP (2): Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in equation (2), the differential equation of the family of orthogonal trajectories is

$$-r^2 \frac{d\theta}{dr} = -r \tan n\theta$$

or $$\frac{dr}{r} = \cot n\theta \, d\theta \qquad (3)$$

STEP (3): Integrating both sides of equation (3), we get

$$\ln r = \frac{1}{n} \ln \sin n\theta + \ln C$$

or $$n \ln r = \ln \sin n\theta + n \ln C$$

or $$\ln r^n = \ln C^n \sin n\theta$$

or $$r^n = C^n \sin n\theta$$

is the equation of the family of orthogonal trajectories.

Put $n = 2$, the orthogonal trajectories of the family of curves

$$r^2 = a^2 \cos 2\theta \quad \text{is} \quad r^2 = C^2 \sin 2\theta .$$

PROBLEM (15): **Find the orthogonal trajectories of the family of curves $r^n \sin n\theta = a^n$.**

SOLUTION: The equation of the family of curves is

$$r^n \sin n\theta = a^n \qquad (1)$$

STEP (1): Taking logarithms of both sides, we get

$$n \ln r + \ln \sin n\theta = n \ln a$$

Differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} + \frac{n \cos n\theta}{\sin n\theta} = 0$$

or $$\frac{dr}{d\theta} = -r \cot n\theta \qquad (2)$$

which being free from the parameter λ, is the differential equation of the family of curves (1).

STEP (2): Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in equation (2), the differential equation of the family of orthogonal trajectories is

$$-r^2 \frac{d\theta}{dr} = -r \cot n\theta$$

$$\frac{dr}{r} = \tan n\theta \, d\theta \qquad (3)$$

STEP (3): Integrating equation (3) w.r.t. θ, we get

$$\ln r = \frac{1}{n}\ln \sec n\theta + \ln C$$

or $$n\ln r = \ln \sec n\theta + n\ln C$$

or $$\ln r^n = \ln(C^n \sec n\theta)$$

or $$r^n = C^n \sec n\theta$$

or $$r^n \cos n\theta = C^n$$

is the equation of the family of orthogonal trajectories.

PROBLEM (16): **Find the orthogonal trajectories of the family of curves**

$$\left(r + \frac{K^2}{r}\right)\cos\theta = \alpha, \quad \alpha \text{ being the parameter.}$$

SOLUTION: The equation of the family of curves is

$$\left(r + \frac{K^2}{r}\right)\cos\theta = \alpha \qquad (1)$$

STEP (1): Differentiating equation (1) w.r.t. θ, we get

$$\left(\frac{dr}{d\theta} - \frac{K^2}{r^2}\frac{dr}{d\theta}\right)\cos\theta - \left(r + \frac{K^2}{r}\right)\sin\theta = 0$$

or $$(r^2 - K^2)\cos\theta\frac{dr}{d\theta} = (r^3 + K^2 r)\sin\theta$$

or $$\frac{dr}{d\theta} = \frac{r^3 + K^2 r}{r^2 - K^2}\tan\theta \qquad (2)$$

is the differential equation of the family of curves (1).

STEP (2): Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in equation (2), the differential equation of the family of orthogonal trajectories is

$$-r^2\frac{d\theta}{dr} = \frac{r^3 + K^2 r}{r^2 - K^2}\tan\theta$$

or $$\frac{r^2 + K^2}{r^2 - K^2}\frac{dr}{r} = -\cot\theta\, d\theta \qquad (3)$$

STEP (3): Integrating both sides of equation (3), we get

$$\int \frac{r^2 + K^2}{r(r^2 - K^2)}\, dr = -\int \cot\theta\, d\theta$$

or $$\frac{1}{2}\int \frac{r^2 + K^2}{r^2(r^2 - K^2)}\, 2r\, dr = -\ln\sin\theta + \ln C$$

Let $r^2 = t$, then $2r\,dr = dt$. Thus

$$\frac{1}{2}\int \frac{t+K^2}{t(t-K^2)}\,dt = -\ln\sin\theta + \ln C$$

or $$\frac{1}{2}\int\left(\frac{2}{t-K^2}-\frac{1}{t}\right)dt = \ln\frac{C}{\sin\theta}$$

or $$\frac{1}{2}\left[2\ln(t-K^2)-\ln t\right] = \ln\frac{C}{\sin\theta}$$

or $$\ln(r^2-K^2)-\frac{1}{2}\ln r^2 = \ln\frac{C}{\sin\theta}$$

or $$\ln(r^2-K^2)-\ln r = \ln\frac{C}{\sin\theta}$$

$$\ln\left(\frac{r^2-K^2}{r}\right) = \ln\frac{C}{\sin\theta}$$

or $$\frac{r^2-K^2}{r} = \frac{C}{\sin\theta}$$

or $$\left(r-\frac{K^2}{r}\right)\sin\theta = C$$

is the equation of the family of orthogonal trajectories .

PROBLEM (17): Find the 45^0 trajectories of the family of curves $xy = C$.

SOLUTION: The equation of family of curves is

$$xy = C \tag{1}$$

STEP (1): Differentiating equation (1) implicitly w.r.t. x , we get

$$x\frac{dy}{dx}+y = 0$$

or $$xp+y = 0 \tag{2}$$

is the differential equation of the family of curves (1) .

STEP (2): Replacing p by $\dfrac{\frac{dy}{dx}+\tan 45^0}{1-\frac{dy}{dx}\tan 45^0} = \dfrac{p+1}{1-p}$ in equation (2) , the differential equation of 45^0 trajectories of the family of curves (1) is

$$x\left(\frac{p+1}{1-p}\right)+y = 0$$

or $$x(p+1)+y(1-p) = 0$$

or $$p(x-y) = -(x+y)$$

or $$p = \frac{y+x}{y-x}$$

or $$\frac{dy}{dx} = \frac{y+x}{y-x} \tag{3}$$

which is a first order homogeneous differential equation .

STEP (3): To solve equation (3) , let $y = Vx$ so that $\dfrac{dy}{dx} = V + x\dfrac{dV}{dx}$

Thus $\quad V + x\dfrac{dV}{dx} = \dfrac{Vx + x}{Vx - x} = \dfrac{V+1}{V-1}$

or $\quad x\dfrac{dV}{dx} = \dfrac{V+1}{V-1} - V = \dfrac{1+2V-V^2}{V-1}$

Separating the variables , we get

$$\frac{V-1}{1+2V-V^2}\,dV = \frac{dx}{x}$$

or $$\frac{1-V}{1+2V-V^2}\,dV = -\frac{dx}{x}$$

$$\frac{2-2V}{1+2V^2-V^2} = -\frac{2}{x}\,dx$$

or $$\ln(1+2V^2-V^2) = -2\ln x + \ln C$$

or $$\ln(1+2V-V^2) + 2\ln x = \ln C$$

or $$\ln(1+2V-V^2) + \ln x^2 = \ln C$$

or $$\ln\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right)x^2 = \ln C$$

or $$\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right)x^2 = C$$

or $$x^2 + 2xy - y^2 = C$$

is the required equation of 45^0 trajectories .

PROBLEM (18): Find the 45^0 trajectories of the family of curves $r^2 = a\sin 2\theta$.

SOLUTION: The equation of family of curves is

$$R^2 = a\sin 2\theta \tag{1}$$

STEP (1): Differentiating equation (1) w.r.t. θ , we get

$$2r\frac{dr}{d\theta} = 2a\cos 2\theta$$

or $$\frac{dr}{d\theta} = \frac{a\cos 2\theta}{r}$$

or $$r\frac{d\theta}{dr} = \frac{r^2}{a\cos 2\theta} = \frac{a\sin 2\theta}{a\cos 2\theta} = \tan 2\theta \tag{2}$$

STEP (2): Replacing $r\dfrac{d\theta}{dr} = \dfrac{r\dfrac{d\theta}{dr} + \tan\dfrac{\pi}{4}}{1-\left(r\dfrac{d\theta}{dr}\right)\tan\dfrac{\pi}{4}} = \dfrac{r\dfrac{d\theta}{dr}+1}{1-r\dfrac{d\theta}{dr}}$ in equation (2) , the differential equation of $\dfrac{\pi}{4}$ trajectories of the family of curves (1) is

$$\frac{r\frac{d\theta}{dr}+1}{1-r\frac{d\theta}{dr}} = \tan 2\theta$$

or

$$r\frac{d\theta}{dr}+1 = \left(1-r\frac{d\theta}{dr}\right)\tan 2\theta$$

$$r\frac{d\theta}{dr}(1+\tan 2\theta) = \tan 2\theta - 1$$

Separating the variables , we get

$$\frac{dr}{r} = \frac{1+\tan 2\theta}{\tan 2\theta - 1}d\theta = \frac{\cos 2\theta + \sin 2\theta}{\sin 2\theta - \cos 2\theta}$$

Integrating , we get

$$\ln r = \frac{1}{2}\ln(\sin 2\theta - \cos 2\theta) + \ln C_1$$

or

$$2\ln r = \ln(\sin 2\theta - \cos 2\theta) + 2\ln C_1$$

or

$$r^2 = C_1^2(\sin 2\theta - \cos 2\theta)$$

$$= \sqrt{2}\,C_1^2\left(\frac{1}{\sqrt{2}}\sin 2\theta - \sin\frac{1}{\sqrt{2}}\cos 2\theta\right)$$

$$= \sqrt{2}\,C_1^2\left(\cos\frac{\pi}{4}\sin 2\theta - \sin\frac{\pi}{4}\cos 2\theta\right)$$

$$= \sqrt{2}\,C_1^2\sin\left(2\theta - \frac{\pi}{4}\right)$$

or $\quad r^2 = C\sin\left(\theta - \frac{\pi}{4}\right) \quad$ where $\quad C = \sqrt{2}\,C_1^2$

PHYSICAL APPLICATIONS

PROBLEM (19): **Radium decays at a rate proportional to the instantaneous amount present at any time . If the half life of radium is T years , determine the amount present after t years .**

SOLUTION: Let $y(t)$ be the amount of radium present after t years . Then the time rate of change is $\frac{dy}{dt}$. According to the law

$$\frac{dy}{dt} \propto y \quad \text{or} \quad \frac{dy}{dt} = -Ky$$

If the initial amount is y_0 , then the initial condition becomes when $t = 0$, $y(0) = y_0$. The solution of this initial value problem is

$$y(t) = y_0 e^{-Kt} \qquad (1)$$

The half life is the time T when the amount present is half the original amount i.e. $\frac{y_0}{2}$. Then

$$\frac{y_0}{2} = y_0 e^{-KT} \quad \text{or} \quad e^{-KT} = \frac{1}{2} \quad \text{or} \quad e^{-K} = \left(\frac{1}{2}\right)^{1/T}$$

Thus equation (1) becomes

$$y(t) = y_0 (e^{-K})^t = y_0 \left(\frac{1}{2}\right)^{t/T}$$

ANOTHER FORM OF SOLUTION

$$e^{-KT} = \frac{1}{2} \quad \text{or} \quad -KT = \ln\left(\frac{1}{2}\right) = -\ln 2 \quad \text{or} \quad K = \frac{\ln 2}{T} \quad \text{and} \quad y(t) = y_0 e^{-t \ln 2/T}$$

NEWTON'S LAW OF COOLING

PROBLEM (20): **If an object cools from $80\,^{\circ}C$ to $60\,^{\circ}C$ in 20 minutes, find the temperature in 40 minutes if the surrounding temperature is $20\,^{\circ}C$.**

SOLUTION: We know that the differential equation of Newton's law of cooling is

$$\frac{dT}{dt} + KT = KT_m$$

whose solution is given by

$$T(t) = (T_0 - T_m) e^{-Kt} + T_m \qquad (1)$$

where T_0 is the initial temperature of the body and T_m the temperature of the surrounding medium.

In the present case, $T_0 = 80\,^{\circ}C$, $T_m = 20\,^{\circ}C$.

Thus equation (1) becomes

$$T(t) = (80 - 20) e^{-Kt} + 20$$

$$= 60\, e^{-Kt} + 20 \qquad (2)$$

Since after $t = 20$ minutes, $T = 60\,^{\circ}C$, we get from equation (2)

$$60 = 60\, e^{-20K} + 20$$

or $\quad 60\, e^{-20K} = 40$

or $\quad e^{-20K} = \frac{2}{3}$

or $\quad e^{-K} = \left(\frac{2}{3}\right)^{1/20}$

Substituting in equation (2), we get

$$T(t) = 60 (e^{-K})^t + 20$$

$$= 60 \left(\frac{2}{3}\right)^{t/20} + 20 \qquad (3)$$

When $t = 40$ minutes, we get

$$T(t) = 60\left(\frac{2}{3}\right)^{40/20} + 20$$

$$= 60\left(\frac{2}{3}\right)^{2} + 20 = 60\left(\frac{4}{9}\right) + 20$$

$$= \frac{240}{9} + 20 = 26.7 + 20 = 46.7\,^{\circ}C$$

PROBLEM (21): **Water at $100\,^{\circ}C$ cools to $80\,^{\circ}C$ in 10 minutes, in a room maintained at a temperature of $30\,^{\circ}C$. Find when the temperature of water will become $40\,^{\circ}C$.**

SOLUTION: We know that the differential equation of Newton's law of cooling is

$$\frac{dT}{dt} + KT = KT_m$$

whose solution is given by

$$T(t) = (T_0 - T_m)e^{-Kt} + T_m \qquad (1)$$

where T_0 is the initial temperature of the body and T_m the temperature of the surrounding medium.

In the present case, $T_0 = 100\,^{\circ}C$, $T_m = 30\,^{\circ}C$.

Thus equation (1) becomes

$$T(t) = (100 - 30)e^{-Kt} + 30$$

$$= 70e^{-Kt} + 30 \qquad (2)$$

Since after $t = 10$ minutes, $T = 80\,^{\circ}C$, we get from equation (2)

$$80 = 70e^{-10K} + 30$$

or $$70e^{-10K} = 50$$

or $$e^{-10K} = \frac{5}{7}$$

or $$e^{-K} = \left(\frac{5}{7}\right)^{1/10}$$

Substituting in equation (2), we get

$$T(t) = 70(e^{-K})^{t} + 30$$

$$= 70\left(\frac{5}{7}\right)^{t/10} + 30 \qquad (3)$$

when $T = 40\,^{\circ}C$, we get from equation (3)

$$40 = 70\left(\frac{5}{7}\right)^{t/10} + 30$$

or $\left(\frac{5}{7}\right)^{t/10} = \frac{1}{7}$

Taking logarithms on both sides , we get

$$\frac{t}{10}\ln\left(\frac{5}{7}\right) = \ln\left(\frac{1}{7}\right)$$

or $$t = 10\frac{\ln\left(\frac{1}{7}\right)}{\ln\left(\frac{5}{7}\right)} = 58 \text{ minutes}$$

or $t = 58$ minutes , the time that elapses before the temperature of water becomes $40\,^{\circ}C$.

APPLICATIONS TO MECHANICS

PROBLEM (22): **A particle free to move along a line , becomes subject to an acceleration $F_0 \cos p\,t$ along the line , where F_0 and p are constants . If initially , the particle is at rest at the origin , show that its displacement at any subsequent time is $\frac{F_0}{p}(1 - \cos p\,t)$.**

SOLUTION: If a is the acceleration at any time t , then

$$a = F_0 \cos p\,t$$

or $$\frac{dv}{dt} = F_0 \cos p\,t$$

Integrating , we get

$$v = \frac{F_0}{p}\sin p\,t + C_1 \qquad (1)$$

Using the first initial condition that when $t = 0$, $v = 0$, we get from equation (1)

$$C_1 = 0$$

Thus equation (1) becomes

$$v = \frac{F_0}{p}\sin p\,t$$

or $$\frac{ds}{dt} = \frac{F_0}{p}\sin p\,t \qquad (2)$$

Integrating again equation (2) , we get

$$s = -\frac{F_0}{p}\cos p\,t + C_2 \qquad (3)$$

Using the second initial condition that when $t = 0$, $s = 0$, we get from equation (3)

$$0 = -\frac{F_0}{p} + C_2 \quad \text{or} \quad C_2 = \frac{F_0}{p}$$

Thus equation (3) reduces to

$$s = -\frac{F_0}{p}\cos p t + \frac{F_0}{p}$$

$$= \frac{F_0}{p}(1 - \cos p t)$$

PROBLEM (23): **A particle initially at rest, moves from a fixed point in a straight line so that its acceleration at any time t is $\sin t + \frac{1}{(t+1)^2}$. Show that at time $t = \pi$ seconds from start, it is at a distance of $2\pi - \ln(\pi + 1)$ from the fixed point.**

SOLUTION: If a is the acceleration at any time t, then

$$a = \sin t + \frac{1}{(t+1)^2}$$

or $$\frac{dv}{dt} = \sin t + \frac{1}{(t+1)^2}$$

Integrating this equation, we get

$$v = -\cos t - \frac{1}{t+1} + C_1 \qquad (1)$$

Using the initial condition that when $t = 0$, $v = 0$, we get from equation (1)

$$0 = -1 - 1 + C_1 \quad \text{or} \quad C_1 = 2$$

Thus equation (1) becomes

$$v = -\cos t - \frac{1}{t+1} + 2$$

or $$\frac{ds}{dt} = -\cos t - \frac{1}{t+1} + 2$$

Integrating again, we get

$$s = -\sin t - \ln(t+1) + 2t + C_2 \qquad (2)$$

Using the initial condition, that when $t = 0$, $s = 0$, we get from equation (2) that

$$C_2 = 0$$

Thus equation (2) reduces to

$$s = -\sin t - \ln(t+1) + 2t \qquad (3)$$

When $t = \pi$ seconds, we get

$$s = -\sin\pi - \ln(\pi + 1) + 2\pi$$

or $$s = 2\pi - \ln(\pi + 1)$$

PROBLEM (24): **A particle moving in a straight line is subject to a resistance which produces retardation $K v^3$. Show that v and t are given in terms of s by equations**

$$v = \frac{u}{1 + K s u}, \quad t = \frac{1}{2}K s^2 + \frac{s}{u}$$ **where u is the initial velocity.**

SOLUTION: Given that retardation $= K v^3$

or $\quad a = -K v^3$

or $\quad v \dfrac{dv}{ds} = -K v^3$

Separating the variables, we get

$$\frac{dv}{v^2} = -K\,ds$$

Integrating, we get

$$-\frac{1}{v} = -K s + C_1 \qquad (1)$$

Using the initial condition that when $s = 0$, $v = u$, we get from equation (1)

$$-\frac{1}{u} = C_1$$

Thus equation (1) becomes

$$-\frac{1}{v} = -K s - \frac{1}{u}$$

or $\quad \dfrac{1}{v} = K s + \dfrac{1}{u}$

or $\quad v = \dfrac{u}{1 + K s u}$

or $\quad \dfrac{ds}{dt} = \dfrac{u}{1 + K s u}$

or $\quad (1 + K s u)\,ds = u\,dt$

Integrating, we get

$$s + \frac{1}{2} K u s^2 = u t + C_2 \qquad (2)$$

Using the initial condition that when $t = 0$, $s = 0$, we get from equation $C_2 = 0$.

Thus equation (2) becomes

$$s + \frac{1}{2} K u s^2 = u t$$

or $\quad t = \dfrac{1}{2} K s^2 + \dfrac{s}{u}$

PROBLEM (25): **The acceleration of a moving particle being proportional to the cube of the velocity and negative, find the distance moved in time t, the initial velocity being v_0 and the distance being measured from the position of the particle at time $t = 0$.**

SOLUTION: Let v be the velocity of the particle at any time t. Then

$$\frac{dv}{dt} = -K v^3 \qquad (1)$$

or $\quad \dfrac{dv}{v^3} = -K\,dt$

Integrating, we get

$$-\frac{1}{2v^2} = -Kt + C_1 \qquad (2)$$

Using the initial condition that when $t = 0$, $v = v_0$, we get from equation (2),

$$C_1 = -\frac{1}{2v_0^2}$$

Thus equation (2) becomes

$$-\frac{1}{2v^2} = -Kt - \frac{1}{2v_0^2}$$

or $\quad \dfrac{1}{2v^2} = Kt + \dfrac{1}{2v_0^2} = \dfrac{2Kv_0^2t+1}{2v_0^2}$

or $\quad v^2 = \dfrac{v_0^2}{2Kv_0^2t+1}$

or $\quad v = \dfrac{v_0}{\sqrt{2Kv_0^2t+1}}$

or $\quad \dfrac{ds}{dt} = \dfrac{v_0}{\sqrt{2Kv_0^2t+1}}$

Separating the variables, we get

$$ds = v_0\left(2Kv_0^2t+1\right)^{-1/2}dt$$

Integrating, we get

$$s = v_0\frac{1}{2Kv_0^2}(2)\sqrt{2Kv_0^2t+1} + C_2$$

$$= \frac{1}{Kv_0}\sqrt{2Kv_0^2t+1} + C_2 \qquad (3)$$

Using the condition that when $t = 0$, $s = 0$, we get from equation (3)

$$0 = \frac{1}{Kv_0} + C_2$$

or $\quad C_2 = -\dfrac{1}{Kv_0}$

Thus equation (3) becomes

$$s = \frac{1}{Kv_0}\sqrt{2Kv_0^2t+1} - \frac{1}{Kv_0}$$

$$= \frac{1}{Kv_0}\left(\sqrt{2Kv_0^2t+1} - 1\right)$$

which gives the distance moved by the particle in time t.

PROBLEM (26): **An object is thrown vertically upward from the ground with initial velocity 19.6 m / s. Neglecting air resistance, find**

(i) **the time taken to reach the maximum height**

(ii) **the maximum height reached**

SOLUTION: Let the object of mass m be located at distance s meters from the ground after time t seconds as shown in figure (8.40). Choose the upward direction as positive.

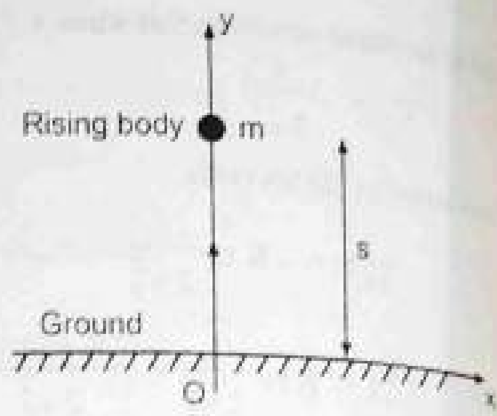


Figure (8.40)

By Newton's second law

$$m\frac{dv}{dt} = -mg$$

or $$\frac{dv}{dt} = -g = -9.8$$

Integrating w.r.t. t, we get

$$v = -9.8t + C_1 \qquad (1)$$

Using the initial condition that when $t = 0$, $v = 19.6$ m / s, we get from equation (1)

$$19.6 = C_1$$

Thus equation (1) becomes

$$v = -9.8t + 19.6 \qquad (2)$$

or $$\frac{ds}{dt} = -9.8t + 19.6$$

Integrating again, we get

$$s = -9.8\frac{t^2}{2} + 19.6t + C_2$$

$$= -4.9t^2 + 19.6t + C_2 \qquad (3)$$

Using the initial condition that when $t = 0$, $s = 0$, we get from equation (3)

$$0 = C_2$$

Thus equation (3) reduces to

$$s = -4.9t^2 + 19.6t \qquad (4)$$

(i) The maximum height will be reached when $v = 0$. Thus the time taken to reach at this height is obtained from equation (2) by taking $v = 0$. Thus $t = 2$.

(ii) The maximum height reached from equation (4) is

$$s = -(4.9)(4) + (19.6)(2)$$

$$= -19.6 + 39.2 = 19.6 \text{ m}$$

MASS – SPRING SYSTEMS

PROBLEM (27): **Find the displacement $y(t)$ of a damped forced mass – spring system when $m = 1$ kg, $K = 1$ N/m, $C = 3$ kg/s, $F(t) = 4\cos 3t$ N and $y(0) = 0$, $y'(0) = 6$.**

SOLUTION: We know that the differential equation governing the motion of a mass – spring system in the case of damped forced oscillations is

$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{1}{m}F(t) \qquad (1)$$

In the present case, $m = 1$, $K = 1$, $C = 3$, $F(t) = 4\cos 3t$, $y(0) = 0$, $y'(0) = 6$.

Thus equation (1) becomes

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 4\cos 3t \qquad (2)$$

In operator notation, this equation becomes

$$(D^2 + D + 3)y = 4\cos 3t$$

The auxiliary equation is $D^2 + D + 3 = 0$ having characteristic roots as $D = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$.

The complementary function is

$$y_c = e^{-t/2}\left(C_1 \cos\frac{\sqrt{11}}{2}t + C_2 \sin\frac{\sqrt{11}}{2}t\right)$$

The particular solution is

$$y_p = \frac{1}{D^2 + D + 3} 4\cos 3t$$

$$= 4\frac{1}{-9 + D + 3}\cos 3t$$

$$= 4\frac{1}{D - 6}\cos 3t = 4\frac{D + 6}{D^2 - 36}\cos 3t$$

$$= 4\left(\frac{1}{-45}\right)(D + 6)\cos 3t$$

$$= -\frac{4}{45}(-3\sin 3t + 6\cos 3t$$

$$= \frac{4}{15}(\sin 3t - 2\cos 3t)$$

Thus the general solution of equation (2) is

$$y(t) = e^{-t/2}\left(C_1 \cos\frac{\sqrt{11}}{2}t + C_2 \sin\frac{\sqrt{11}}{2}t\right) + \frac{4}{15}(\sin 3t - 2\cos 3t) \qquad (3)$$

Now $y'(t) = e^{-t/2}\left(-\frac{\sqrt{11}}{2}C_1\sin\frac{\sqrt{11}}{2}t+\frac{\sqrt{11}}{2}C_2\cos\frac{\sqrt{11}}{2}t\right)$

$$-\frac{1}{2}e^{-t/2}\left(C_1\cos\frac{\sqrt{11}}{2}t+C_2\sin\frac{\sqrt{11}}{2}t\right)+\frac{12}{15}(\cos 3t+2\sin 3t) \quad (4)$$

Using the initial conditions $y(0)=0$, $y'(0)=6$, we get from equations (3) and (4)

$$C_1-\frac{8}{15}=0 \quad \text{or} \quad C_1=\frac{8}{15}$$

and $$\frac{\sqrt{11}}{2}C_2-\frac{1}{2}C_1+\frac{12}{15}=6 \quad \text{or} \quad \frac{\sqrt{11}}{2}C_2=\frac{82}{15} \quad \text{or} \quad C_2=\frac{164}{15\sqrt{11}}$$

Substituting the values of C_1 and C_2 in equation (3), we get

$$y(t)=e^{-t/2}\left(\frac{8}{15}\cos\frac{\sqrt{11}}{2}t+\frac{164}{15\sqrt{11}}\sin\frac{\sqrt{11}}{2}t\right)+\frac{4}{15}(\sin 3t-2\cos 3t)$$

PROBLEM (28): **Find the displacements $y_1(t)$ and $y_2(t)$ of a coupled mass – spring system assuming that $m_1=6\,kg$, $m_2=2\,kg$, $K_1=4\,N/m$, $K_2=1\,N/m$ and $y_1(0)=2$, $y_2(0)=0$, $y_1'(0)=0$, $y_2'(0)=0$.**

SOLUTION: We know that the system of differential equations governing the vibrations of a coupled system is

$$m_1y_1''=-K_1y_1+K_2(y_2-y_1)$$
$$m_2y_2''=-K_2(y_2-y_1)$$

In the present case, $m_1=6$, $m_2=2$, $K_1=4$, $K_2=1$. Thus the above system takes the form

$$6y_1''+5y_1-y_2=0 \quad (1)$$
$$2y_2''+y_2-y_1=0 \quad (2)$$

In operator notation, this takes the form

$$(6D^2+5)y_1-y_2=0 \quad (3)$$
$$-y_1+(2D^2+1)y_2=0 \quad (4)$$

Operating on equation (3) by $(2D^2+1)$ and adding equation (4), we get

$$(6D^2+5)(2D^2+1)y_1-y_1=0$$
$$(12D^4+16D^2+5)y_1-y_1=0$$

or $$(12D^4+16D^2+4)y_1=0$$

or $$(3D^4+4D^2+1)y_1=0$$

The auxiliary equation is $3D^4+4D^2+1=0$ or $(D^2+1)(3D^2+1)=0$

having the characteristic roots as $\pm i, \pm\frac{i}{\sqrt{3}}$.

The complementary function is

$$y_1 = C_1 \cos t + C_2 \sin t + C_3 \cos \frac{1}{\sqrt{3}} t + C_4 \sin \frac{1}{\sqrt{3}} t \qquad (5)$$

From equation (3), we get

$$y_2 = (6D^2 + 5) y_1$$

$$= (6D^2 + 5)\left(C_1 \cos t + C_2 \sin t + C_3 \cos \frac{1}{\sqrt{3}} t + C_4 \sin \frac{1}{\sqrt{3}} t \right)$$

$$= -6 C_1 \cos t - 6 C_2 \sin t - 2 C_3 \cos \frac{1}{\sqrt{3}} t - 2 C_4 \sin \frac{1}{\sqrt{3}} t + 5 C_1 \cos t + 5 C_2 \sin t$$

$$+ 5 C_3 \cos \frac{1}{\sqrt{3}} t + 5 C_4 \sin \frac{1}{\sqrt{3}} t$$

$$= -C_1 \cos t - C_2 \sin t + 3 C_3 \cos \frac{1}{\sqrt{3}} t + 3 C_4 \sin \frac{4}{\sqrt{3}} \qquad (6)$$

From equations (5) and (6), we get

$$y_1'(t) = -C_1 \sin t + C_2 \cos t - \frac{1}{\sqrt{3}} C_3 \sin \frac{1}{\sqrt{3}} t + \frac{1}{\sqrt{3}} C_4 \cos \frac{1}{\sqrt{3}} t \qquad (7)$$

and $$y_2'(t) = C_1 \sin t - C_2 \cos t - \sqrt{3} C_3 \sin \frac{1}{\sqrt{3}} t + \sqrt{3} C_4 \cos \frac{1}{\sqrt{3}} t \qquad (8)$$

Using the initial conditions $y_1(0) = 2,\ y_2(0) = 0,\ y_1'(0) = 0,\ y_2'(0) = 0$, we get

$$C_1 + C_3 = 2, \quad -C_1 + 3 C_3 = 0, \quad C_2 + \frac{1}{\sqrt{3}} C_4 = 0, \quad -C_2 + \sqrt{3} C_4 = 0$$

Solving these equations, we get

$$C_1 = \frac{3}{2}, \quad C_2 = 0, \quad C_3 = \frac{1}{2}, \quad C_4 = 0$$

Substituting these values in equations (5) and (6), we get

$$y_1(t) = \frac{3}{2} \cos t + \frac{1}{2} \cos \frac{1}{\sqrt{3}} t$$

$$y_2(t) = -\frac{3}{2} \cos t + \frac{3}{2} \cos \frac{1}{\sqrt{3}} t$$

PROBLEM (29): **Find the displacements $y_1(t)$ and $y_2(t)$ of a coupled mass – spring system assuming that $m_1 = 4$ kg, $m_2 = 2$ kg, $K_1 = 2$ N/m, $K_2 = 2$ N/m, $K_3 = 1$N/m, $F_1(t) = 1$, $F_2(t) = 0$, $y_1(0) = 0$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = 0$.**

SOLUTION: We know that the system of differential equations governing the vibrations of the given system is

$$m_1 y_1'' = -K_1 y_1 + K_2 (y_2 - y_1) + F_1(t)$$

$$m_2 y_2'' = -K_2 (y_2 - y_1) - K_3 y_2 + F_2(t)$$

In the present case, this takes the form

$$4y_1'' + 4y_1 - 2y_2 = 1 \qquad (1)$$

$$2y_2'' + 3y_2 - 2y_1 = 0 \qquad (2)$$

In operator notation, this takes the form

$$(4D^2 + 4)y_1 - 2y_2 = 1 \qquad (3)$$

$$-2y_1 + (2D^2 + 3)y_2 = 0 \qquad (4)$$

Operating on equation (3) by $(2D^2 + 3)$ and multiplying equation (4) by 2 and adding, we get

$$(4D^2 + 4)(2D^2 + 3)y_1 - 4y_1 = (2D^2 + 3)1$$

or $$(8D^4 + 20D^2 + 12)y_1 - 4y_1 = 3$$

or $$(8D^4 + 20D^2 + 8)y_1 = 3$$

or $$(2D^4 + 5D^2 + 2)y_1 = 3 \qquad (5)$$

The auxiliary equation is $2D^4 + 5D^2 + 2 = 0$ or $(D^2 + 2)(2D^2 + 1)y_1 = 0$

having the characteristic roots as $\pm\sqrt{2}\,i, \pm\frac{i}{\sqrt{2}}$.

The complementary function is

$$(y_1)_c = C_1 \cos\sqrt{2}\,t + C_2 \sin\sqrt{2}\,t + C_3 \cos\frac{1}{\sqrt{2}}t + C_4 \sin\frac{1}{\sqrt{2}}t$$

The particular integral is $(y_1)_p = \frac{3}{8}$.

The general solution of equation (5) is

$$y_1(t) = C_1 \cos\sqrt{2}\,t + C_2 \sin\sqrt{2}\,t + C_3 \cos\frac{1}{\sqrt{2}}t + C_4 \sin\frac{1}{\sqrt{2}}t + \frac{3}{8} \qquad (6)$$

From equation (3), we get

$$2y_2 = (4D^2 + 4)y_1 - 1$$

$$= (4D^2 + 4)\left(C_1 \cos\sqrt{2}\,t + C_2 \sin\sqrt{2}\,t + C_3 \cos\frac{1}{\sqrt{2}}t + C_4 \sin\frac{1}{\sqrt{2}}t + \frac{3}{8}\right) - 1$$

$$= -8C_1 \cos\sqrt{2}\,t - 8C_2 \sin\sqrt{2}\,t - 2C_3 \cos\frac{1}{\sqrt{2}}t - 2C_4 \sin\frac{1}{\sqrt{2}}t + 4C_1 \cos\sqrt{2}\,t$$

$$+ 4C_2 \sin\sqrt{2}\,t + 4C_3 \cos\frac{1}{\sqrt{2}}t + 4C_4 \sin\frac{1}{\sqrt{2}}t + \frac{1}{2}$$

$$= -4C_1 \cos\sqrt{2}\,t - 4C_2 \sin\sqrt{2}\,t + 2C_3 \cos\frac{1}{\sqrt{2}}t + 2C_4 \sin\frac{1}{\sqrt{2}}t + \frac{1}{2}$$

or $$y_2(t) = -2C_1 \cos\sqrt{2}\,t - 2C_2 \sin\sqrt{2}\,t + C_3 \cos\frac{1}{\sqrt{2}}t + C_4 \sin\frac{1}{\sqrt{2}}t + \frac{1}{4} \qquad (7)$$

From equations (6) and (7), we get

$$y_1'(t) = -\sqrt{2}\,C_1 \sin\sqrt{2}\,t + \sqrt{2}\,C_2 \cos\sqrt{2}\,t - \frac{1}{\sqrt{2}}C_3 \sin\frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}}C_4 \cos\frac{1}{\sqrt{2}}t \qquad (8)$$

and $\quad y_2'(t) = 2\sqrt{2}\,C_1 \sin\sqrt{2}\,t - 2\sqrt{2}\,C_2 \cos\sqrt{2}\,t - \frac{1}{\sqrt{2}} C_3 \sin\frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}} C_4 \cos\frac{1}{\sqrt{2}}t \qquad (9)$

Using the initial conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = 0$, we get from equations (6), (7), (8), (9)

$$C_1 + C_3 = -\frac{3}{8}, \quad -2C_1 + C_3 = -\frac{1}{4}, \quad \sqrt{2}\,C_2 + \frac{1}{\sqrt{2}} C_4 = 0, \quad -2\sqrt{2}\,C_2 + \frac{1}{\sqrt{2}} C_4 = 0$$

Solving these equations, we get

$$C_1 = -\frac{1}{24}, \quad C_2 = 0, \quad C_3 = -\frac{1}{3}, \quad C_4 = 0$$

Substituting these values in equations (6) and (7), we get

$$y_1(t) = -\frac{1}{24}\cos\sqrt{2}\,t - \frac{1}{3}\cos\frac{1}{\sqrt{2}}t + \frac{3}{8}$$

$$y_2(t) = \frac{1}{12}\cos\sqrt{2}\,t - \frac{1}{3}\cos\frac{1}{\sqrt{2}}t + \frac{1}{4}$$

ELECTRIC CIRCUITS

PROBLEM (30): **An RL circuit contains a resistor of resistance 8 ohms, an inductor of inductance 0.5 henries, and a battery of E volts. At time $t = 0$ the current is zero. Find the current at any time $t > 0$ if**

(i) $E = 8\,t\,e^{-16t}$ (ii) $E = 32\,e^{-8t}$

SOLUTION: We know that the differential equation for the current I in an RL – circuit is

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E(t)}{L}$$

whose solution is given by

$$I(t) = e^{-\frac{Rt}{L}}\left[\int e^{\frac{Rt}{L}} \frac{E(t)}{L}\,dt + C\right]$$

In the present case, R = 8 ohms, L = 0.5 henries, therefore the differential equation becomes

$$\frac{dI}{dt} + 16I = 2E(t) \qquad (1)$$

whose solution is

$$I(t) = e^{-16t}\left[\int e^{16t}\,2E(t)\,dt + C\right] \qquad (2)$$

(i) When $E(t) = 8\,t\,e^{-16t}$, equation (1) takes the form

$$\frac{dI}{dt} + 16I = 16\,t\,e^{-16t}$$

whose solution from equation (2) is given by

$$I(t) = e^{-16t}\left[\int e^{16t}(16\,t\,e^{-16t})\,dt + C\right]$$

$$= e^{-16t}\left[\int 16t\,dt + C\right]$$

$$= e^{-16t}\left[8t^2 + C\right] \qquad (3)$$

Using the initial condition that when $t = 0$, $I = 0$, we get from equation (3)

$$0 = C$$

Thus from equation (3), the current I at any time t is given by

$$I(t) = 8t^2 e^{-16t} \qquad (4)$$

(ii) When $E(t) = 32e^{-8t}$, equation (1) takes the form

$$\frac{dI}{dt} + 16I = 64e^{-8t}$$

whose solution from equation (2) is given by

$$I(t) = e^{-16t}\left[\int e^{16t}(64e^{-8t})\,dt + C\right]$$

$$= e^{-16t}\left[\int 64e^{8t}\,dt + C\right]$$

$$= e^{-16t}\left[8e^{8t} + C\right]$$

$$= 8e^{-8t} + Ce^{-16t} \qquad (5)$$

Using the initial condition that when $t = 0$, $I = 0$, we get from equation (5)

$$0 = 8 + C \quad \text{or} \quad C = -8$$

Thus from equation (5), the current I at any time t is given by

$$I(t) = 8e^{-8t} - 8e^{-16t}$$

PROBLEM (31): **An RC circuit has an e.m.f. of $10 \sin t$ volts, a resistance of 100 ohms, a capacitance of 0.005 farad, and no initial charge on the capacitor. Find (i) the charge Q on the capacitor at any time t and (ii) the current I in the circuit at any time t.**

SOLUTION: (i) We know that the differential equation for the charge Q in the RC – circuit is

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{1}{R}E(t) \qquad (1)$$

In the present case, $R = 100$ ohms, $C = 0.005$ farad, $E = 10 \sin t$ volts, therefore equation (1) becomes

$$\frac{dQ}{dt} + 2Q = \frac{1}{10}\sin t \qquad (2)$$

which is a first order linear differential equation.

To solve this equation , the integrating factor is

$$\text{I.F.} = e^{\int 2\,dt} = e^{2t}$$

Thus the solution of equation (6) is

$$Q(\text{I.F.}) = \int \frac{1}{10}\sin t\,(\text{I.F.})\,dt + C$$

or

$$Q(e^{2t}) = \frac{1}{10}\int e^{2t}\sin t\,dt + C$$

$$= \frac{1}{10}\left[\frac{e^{2t}}{5}(2\sin t - \cos t)\right] + C$$

or

$$Q(t) = Ce^{-2t} + \frac{1}{50}(2\sin t - \cos t) \qquad (3)$$

Using the initial condition $I(t) = 0$, we get from equation (7), $C = \frac{1}{50}$.

Thus solution (7) becomes

$$Q(t) = \frac{1}{50}e^{-2t} + \frac{1}{50}(2\sin t - \cos t) \qquad (4)$$

(ii) The current I is obtained by differentiating equation (4) w.r.t. t

$$I(t) = \frac{dQ}{dt} = -\frac{1}{25}e^{-2t} + \frac{1}{50}(2\cos t - \sin t)$$

PROBLEM (32): **Find the current $I(t)$ in an LC – circuit when $L = 2$ henrys , $C = 0.005$ farad $E(t) = 210\sin 4t$ volts , and $I(0) = 0$, $I'(0) = 0$.**

SOLUTION: We know that the differential equation for the current $I(t)$ in an LC – circuit is

$$\frac{d^2I}{dt^2} + \frac{1}{LC}I = \frac{1}{L}\frac{dE}{dt} \qquad (1)$$

In the present case , $L = 2$, $C = 0.005$, $E = 210\sin 4t$, $I(0) = 0$, $I'(0) = 0$

Thus equation (1) becomes

$$\frac{d^2I}{dt^2} + 100I = 420\cos 4t$$

$$(D^2 + 100)I = 420\cos 4t \qquad (3)$$

The auxiliary equation is

$$D^2 + 100 = 0$$

$$D = \pm 10i$$

The complementary function is

$$I_c = C_1\cos 10t + C_2\sin 10t$$

The particular integral is

$$I_p = 420\frac{1}{D^2 + 100}\cos 4t$$

$$= 420 \frac{1}{-16+100} \cos 4t$$

$$= 420 \frac{1}{84} \cos 4t = 5 \cos 4t$$

Thus the general solution of equation (3), we get

$$I(t) = C_1 \cos 10t + C_2 \sin 10t + 5 \cos 4t \tag{4}$$

$$I'(t) = -10 C_1 \sin 10t + 10 C_2 \cos 10t - 20 \sin 4t \tag{5}$$

Using initial condition $I(0) = 0$ and $I'(0) = 0$, we get from equations (4) and (5)

$$C_1 + 5 = 0 \quad \text{or} \quad C_1 = -5$$

$$10 C_2 = 0 \quad \text{or} \quad C_2 = 0$$

Substituting the values of C_1 and C_2 in equation (4), we get the current $I(t)$ as

$$I(t) = -5 \cos 10t + 5 \cos 4t$$

PROBLEM (33): **Find the current $I(t)$ in the RLC – circuit when $R = 160$ ohms, $L = 20$ henrys, $C = 0.002$ farad, $E(t) = 37 \sin 10t$ volts, and $I(0) = 0$, $I'(0) = 1$.**

SOLUTION: We know that the differential equation for the current $I(t)$ in the RLC – circuit is

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC} I = \frac{1}{L}\frac{dE}{dt} \tag{1}$$

In the present case, $R = 160$, $L = 20$, $C = 0.002$, $E(t) = 37 \sin 10t$, $I(0) = 0$, $I'(0) = 0$.

Thus equation (1) becomes

$$\frac{d^2 I}{dt^2} + 8\frac{dI}{dt} + 25 I = \frac{37}{2} \cos 10t \tag{2}$$

In operator notation, this equation takes the form

$$(D^2 + 8D + 25) I = \frac{37}{2} \cos 10t \tag{3}$$

The auxiliary equation is $D^2 + 8D + 25 = 0$ having characteristic roots as $D = -4 \pm 3i$.

The complementary function is

$$I_C = e^{-4t} (C_1 \cos 3t + C_2 \sin 3t)$$

The particular integral is

$$I_p = \frac{37}{2} \frac{1}{D2 + 8D + 25} \cos 10t$$

$$= \frac{37}{2} \frac{1}{-100 + 8D + 25} \cos 10t$$

$$= \frac{37}{2} \frac{1}{8D - 75} \cos 10t = \frac{37}{2} \frac{8D + 75}{64 D^2 - 5625} \cos 10t$$

$$= \frac{37}{2}\,\frac{8\,D + 75}{-6400 - 5625}\cos 10\,t$$

$$= \frac{37}{2}\left(\frac{1}{-12025}\right)(8\,D + 75)\cos 10\,t$$

$$= -\frac{1}{650}(-80\sin 10\,t + 75\cos 10\,t)$$

$$= \frac{8}{65}\sin 10\,t - \frac{3}{26}\cos 10\,t$$

Thus the general solution of equation (3) is

$$I(t) = e^{-4t}(C_1\cos 3\,t + C_2\sin 3\,t) + \frac{8}{65}\sin 10\,t - \frac{3}{26}\cos 10\,t \qquad (4)$$

Now $\quad I'(t) = e^{-4t}(-3\,C_1\sin 3\,t + 3\,C_2\cos 3\,t) - 4\,e^{-4t}(C_1\cos 3\,t + C_2\sin 3\,t)$

$$+\frac{80}{65}\cos 10\,t + \frac{30}{26}\sin 10\,t \qquad (5)$$

Using the initial conditions $I(0) = 0$ and $I'(0) = 0$, we get from equations (4) and (5)

$$C_1 = \frac{3}{26},\quad C_2 = -\frac{10}{39}$$

Substituting these values in equation (4), we get

$$I(t) = e^{-4t}\left(\frac{3}{26}\cos 3\,t - \frac{10}{39}\sin 3\,t\right) + \frac{8}{65}\sin 10\,t - \frac{3}{26}\cos 10\,t$$

ELECTRICAL NETWORK

PROBLEM (34): **Set up the mathematical model of the network shown in figure (8.41). Find the currents $I_1(t)$ and $I_2(t)$ assuming that $L = 1$ henry, $R_1 = 4$ ohms, $R_2 = 6$ ohms, $C = 0.25$ farad, $E(t) = 12$ volts. Assume that $I_1(0) = 0$, $I_2(0) = 0$ when the switch is closed at $t = 0$.**

SOLUTION: The mathematical model of this network is obtained from Kirchhoff's voltage law.

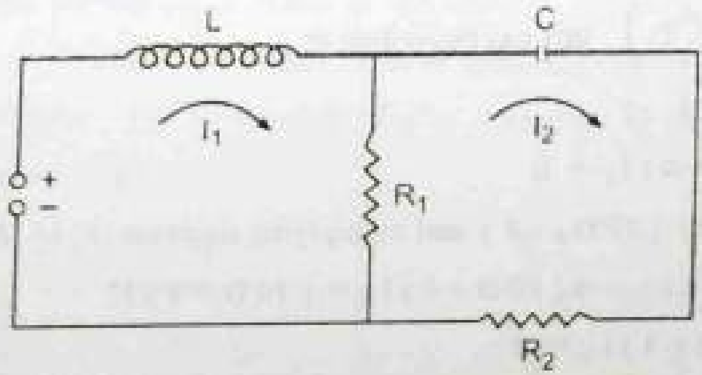


Figure (8.41)

FOR THE LEFT LOOP

If I_1 is the current flowing in the left loop, then the voltage drop across an inductor is

$$E_L = L\frac{d\,I_1}{d\,t}$$

The voltage drop across the resistor R_1 is $E_{R_1} = R_1(I_1 - I_2)$

By Kirchhoff's voltage law, the sum of two voltage drops must equal the electromotive force $E(t)$.

Thus $$L\frac{dI_1}{dt}+R_1(I_1-I_2) = E(t) \tag{1}$$

FOR THE RIGHT LOOP

If I_2 is the current flowing in the right loop, then the voltage drop across the capacitor is

$$E_C = \frac{1}{C}Q = \frac{1}{C}\int I_2(t)\,dt$$

The voltage drop across the resistor R_2 is $E_{R_2} = R_2 I_2$

The voltage drop across the resistor R_1 is $E_{R_1} = R_1(I_2-I_1)$

By Kirchhoff's voltage law, the sum of three voltage drops must equal zero. Thus

$$\frac{1}{C}\int I_2(t)\,dt + R_2 I_2 + R_1(I_2-I_1) = 0$$

Differentiating this equation w.r.t. t, we get

$$\frac{1}{C}I_2 + R_2\frac{dI_2}{dt} + R_1\left(\frac{dI_2}{dt}-\frac{dI_1}{dt}\right) = 0$$

Note that the second equation involves the derivatives of both the unknowns. Thus we see that the currents I_1 and I_2 in the network are governed by the system

$$L\frac{dI_1}{dt}+R_1(I_1-I_2) = E(t) \tag{1}$$

$$-R_1\frac{dI_1}{dt}+(R_1+R_2)\frac{dI_2}{dt}+\frac{1}{C}I_2 = 0 \tag{2}$$

In the present case, this system takes the form

$$\frac{dI_1}{dt}+4(I_1-I_2) = 12 \tag{3}$$

$$-4\frac{dI_1}{dt}+10\frac{dI_2}{dt}+4I_2 = 0 \tag{4}$$

In operator notation $\left(\frac{d}{dt} = D\right)$, this can be written as

or $$(D+4)I_1-4I_2 = 12 \tag{5}$$

$$-4DI_1+(10D+4)I_2 = 0 \tag{6}$$

Operating on equation (5) by $(10D_2+4)$ and multiplying equation (6) by 4

$$(D+4)(10D+4)I_1-4(10D+4)I_2 = (10D+4)12$$

$$16DI_1+4(10D+4)I_2 = 0$$

Adding, we get

$$(10D^2+44D+16-16D)I_1 = 48$$

or $$(10D^2+28D+16)I_1 = 48$$

or $$(5D^2+14D+8)I_1 = 24 \tag{7}$$

The auxiliary equation is $5D^2 + 14D + 8 = 0$.

or $(D+2)(5D+4) = 0$ having the characteristic roots as $D = -2, -\frac{4}{5}$.

The complementary function is $(I_1)_c = C_1 e^{-2t} + C_2 e^{-(4/5)t}$ and the particular integral is $(I_1)_p = 3$.

The general solution of equation (7) is

$$I_1(t) = C_1 e^{-2t} + C_2 e^{-(4/5)t} + 3 \tag{8}$$

Now from equation (5), we have

$$\begin{aligned} 4I_2 &= (D+4)\left(C_1 e^{-2t} + C_2 e^{-(4/5)t} + 3\right) - 12 \\ &= -2C_1 e^{-2t} - \frac{4}{5}C_2 e^{-(4/5)t} + 4C_1 e^{-2t} + 4C_2 e^{-(4/5)t} \\ &= 2C_1 e^{-2t} + \frac{16}{5}C_2 e^{-(4/5)t} \end{aligned}$$

or $$I_2(t) = \frac{1}{2}C_1 e^{-2t} + \frac{4}{5}C_2 e^{-(4/5)t} \tag{9}$$

Using the initial conditions that $I_1(0) = 0$ and $I_2(0) = 0$, we get from equations (8) and (9)

$$C_1 + C_2 = -3$$

and $$\frac{1}{2}C_1 + \frac{4}{5}C_2 = 0$$

Solving these equations, we get $C_1 = -8$, $C_2 = 5$.

Thus the solution of this system is

$$I_1(t) = -8e^{-2t} + 5e^{-(4/5)t} + 3$$

$$I_2(t) = -4e^{-2t} + 4e^{-(4/5)t}$$

PROBLEM (35): Set up the mathematical model of the network shown in figure (8.42). Find the currents $I_1(t)$ and $I_2(t)$ in the network assuming that $R_1 = 2$ ohms, $R_2 = 4$ ohms, $R_3 = 4$ ohms, $L_1 = 1$ henry, $L_2 = 2$ henrys, $E(t) = 195 \sin t$ volts, and $I_1(0) = I_2(0) = 0$.

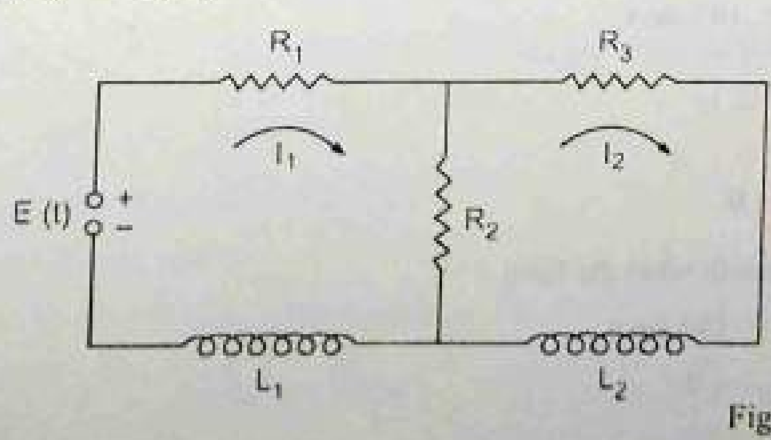


Figure (8.42)

SOLUTION: The mathematical model of the given network is obtained from the Kirchhoff's law.

FOR THE LEFT LOOP

If I_1 is the current flowing in the left loop, then the voltage drop across the resistor R_1 is

$$E_{R_1} = R_1 I_1$$

The voltage drop across the resistor R_2 is

$$E_{R_2} = R_2(I_1 - I_2)$$

The voltage across the inductor L_1 is

$$E_{L_1} = L_1 \frac{dI_1}{dt}$$

By Kirchhoff's voltage law, the sum of three voltage drops must equal the electromotive force $E(t)$.

Thus $$R_1 I_1 + R_2(I_1 - I_2) + L_1 \frac{dI_1}{dt} = E(t)$$

or $$L_1 \frac{dI_1}{dt} + R_1 I_1 + R_2(I_1 - I_2) = E(t) \tag{1}$$

FOR THE RIGHT LOOP

If I_2 is the current flowing in the right loop, then the voltage drop across the resistor R_3 is

$$E_{R_3} = R_3 I_2$$

The voltage drop across the inductor L_2 is

$$E_{L_2} = L_2 \frac{dI_2}{dt}$$

The voltage drop across the resistor R_2 is

$$E_{R_2} = R_2(I_2 - I_1)$$

By Kirchhoff's voltage law, the sum of three voltage drops must equal zero. Thus

$$R_3 I_2 + L_2 \frac{dI_2}{dt} + R_2(I_2 - I_1) = 0$$

or $$L_2 \frac{dI_2}{dt} + R_2(I_2 - I_1) + R_3 I_2 = 0 \tag{2}$$

In the present case, the system of differential equations (1) and (2) takes the form

$$\frac{dI_1}{dt} + 6I_1 - 4I_2 = 195 \sin t \tag{3}$$

$$2\frac{dI_2}{dt} + 8I_2 - 4I_1 = 0$$

or $$\frac{dI_2}{dt} + 4I_2 - 2I_1 = 0 \tag{4}$$

In operator notation*, this system takes the form

$$(D+6)I_1 - 4I_2 = 195 \sin t \tag{5}$$

$$-2I_1 + (D+4)I_2 = 0 \tag{6}$$

Operating on equation (5) by $(D+4)$ and multiplying equation (6) by 4 and adding , we get

$$(D+6)(D+4)I_1 - 8I_1 = (D+4)\,195\sin t$$

or
$$(D^2 + 10D + 24 - 8)I_1 = 195\cos t + 780\sin t$$

or
$$(D^2 + 10D + 16)I_1 = 195\cos t + 780\sin t \qquad (7)$$

The auxiliary equation is

$$D^2 + 10D + 16 = 0 \quad \text{or} \quad (D+2)(D+8) = 0$$

having the characteristic roots as $D = -2, -8$.

The complementary function is $(I_1)_c = C_1 e^{-2t} + C_2 e^{-8t}$ the particular integral is

$$(I_1)_p = \frac{195}{D^2 + 10D + 16}\cos t + \frac{780}{D^2 + 10D + 16}\sin t$$

$$= 195\frac{1}{10D+15}\cos t + 780\frac{1}{10D+15}\sin t$$

$$= 39\frac{1}{2D+3}\cos t + 156\frac{1}{2D+3}\sin t$$

$$= 39\frac{2D-3}{4D^2-9}\cos t + 156\frac{2D-3}{4D^2-9}\sin t$$

$$= -3(2D-3)\cos t - 12(2D-3)\sin t$$

$$= -3(-2\sin t - 3\cos t) - 12(2\cos t - 3\sin t)$$

$$= 6\sin t + 9\cos t - 24\cos t + 36\sin t$$

$$= -15\cos t + 42\sin t$$

Thus the general solution of equation (7) is

$$I_1(t) = C_1 e^{-2t} + C_2 e^{-8t} - 15\cos t + 42\sin t \qquad (8)$$

Now from equation (5) , we get

$$4I_2 = (D+6)(C_1 e^{-2t} + C_2 e^{-8t} - 15\cos t + 42\sin t) - 195\sin t$$

$$= -2C_1 e^{-2t} - 8C_2 e^{-8t} + 15\sin t + 42\cos t + 6C_1 e^{-2t}$$

$$+ 6C_2 e^{-8t} - 90\cos t + 252\sin t - 195\sin t$$

$$= 4C_1 e^{-2t} - 2C_2 e^{-8t} - 48\cos t + 72\sin t$$

or
$$I_2(t) = C_1 e^{-2t} - \frac{1}{2}C_2 e^{-8t} - 12\cos t + 18\sin t \qquad (9)$$

Using the initial conditions $I_1(0) = 0$ and $I_2(0) = 0$, we get from equations (8) and (9)

$$C_1 + C_2 = 15 \quad \text{and} \quad C_1 - \frac{1}{2}C_2 = 12$$

Solving these equations , we get $C_1 = 13$, $C_2 = 2$.

Thus the general solution of this system is

$$I_1(t) = 13e^{-2t} + 2e^{-8t} - 15\cos t + 42\sin t$$

$$I_2(t) = 13e^{-2t} - e^{-8t} - 12\cos t + 18\sin t$$

PROBLEM (36): **The deflection curve of a horizontal beam of length L fixed at one end $(x = 0)$ but otherwise unsupported, satisfies the equation $E\,I\,y'' = \frac{1}{2}W(L-x)^2$, where W is the uniform load per unit length of the beam. Find the equation of the deflection curve and the maximum deflection. This type of beam is called a cantilever beam.**

SOLUTION: The equation of the deflection curve is

$$E\,I\,y'' = \frac{1}{2}W(L-x)^2$$

Integrating, we get

$$E\,I\,y' = -\frac{1}{6}W(L-x)^3 + C_1 \qquad (1)$$

Using the boundary condition at the left end $(x = 0)$ i.e. when $x = 0$, $y'(0) = 0$. From equation (1), we get

$$0 = -\frac{1}{6}W L^3 + C_1 \quad \text{or} \quad C_1 = \frac{1}{6}W L^3$$

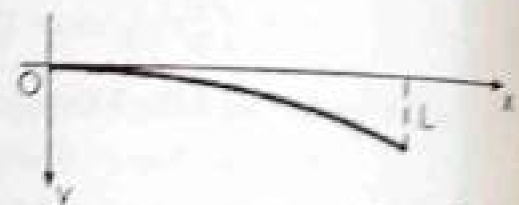


Figure (8.43)

Thus equation (1) becomes

$$E\,I\,y' = -\frac{1}{6}W(L-x)^3 + \frac{1}{6}W L^3$$

Integrating this equation again, we get

$$E\,I\,y = \frac{1}{24}W(L-x)^4 + \frac{1}{6}W L^3 x + C_2$$

Using the condition that when $x = 0$, $y = 0$, we get from equation (2)

$$0 = \frac{1}{24}W L^4 + C_2 \quad \text{or} \quad C_2 = -\frac{1}{24}W L^4$$

Thus equation (2) becomes

$$\begin{aligned} E\,I\,y &= \frac{1}{24}W(L-x)^4 + \frac{1}{6}W L^3 x^4 - \frac{1}{24}W L^4 \\ &= \frac{1}{24}W(L^4 - 4L^3x + 6L^2x^2 - 4Lx^3 + x^4) + \frac{1}{6}W L^3 x - \frac{1}{24}W L^4 \\ &= \frac{1}{24}W(6L^2x^2 - 4Lx^3 + x^4) \end{aligned}$$

or

$$y = \frac{W}{24\,E\,I}(6L^2x^2 - 4Lx^3 + x^4) \qquad (3)$$

The maximum deflection occurs at the right end $x = L$. Thus we get

$$y_{max} = \frac{W}{24\,E\,I}(6L^4 - 4L^4 + L^4) = \frac{1}{8}\,\frac{W L^4}{E\,I}$$

8.36 EXERCISE

GEOMETRICAL APPLICATIONS

PROBLEM (1): Find the equation of the curve for which the Cartesian subnormal is constant.

PROBLEM (2): Find the equation of the curve for which length of the subnormal is proportional to the square of the ordinate

PROBLEM (3): Find the equation of the curve for which the subtangent is n times the subnormal.

PROBLEM (4): Find the equation of the curve for which the length of the subnormal at any point is $\frac{a^2}{x^3}$.

PROBLEM (5): Find the equation of the curve for which the subtangent is proportional to the square of the abscissa.

PROBLEM (6): Find the equation of the curve whose slope at any point (x, y) is $3x + 4y$.

PROBLEM (7): Find the equation of the curve for which the normal at any point $P(x, y)$ passes through the origin.

PROBLEM (8): Find the equation of the curve for which the polar subtangent is proportional to the length of the radius vector.

PROBLEM (9): Find the equation of the curve for which polar subnormal is constant.

PROBLEM (10): Find the equation of the curve for which the angle between the radius vector and the tangent is

(i) equal to the vectorial angle (ii) twice the vectorial angle

PROBLEM (11): Find the orthogonal trajectories of each of the following families of curves:

(i) $y = Cx$ (ii) $x + 2y = C$

(iii) $ay^2 = x^3$ (iv) $x^{2/3} + y^{2/3} = a^{2/3}$

(v) $y = Ce^{-x}$ (vi) $y^3 + 3x^2y = C$

(vii) $y = x - 1 + Ce^{-x}$ (viii) $\frac{x^2}{a^2} + \frac{y^2}{b^2+1}$

PROBLEM (12): Find the orthogonal trajectories of each of the following families of curves:

(i) $r = a(1 - \sin\theta)$ (ii) $r^n \cos n\theta = a^n$

(iii) $r^n = a^n \sin n\theta$ (iv) $r = a + \sin 5\theta$

PROBLEM (13): Find the oblique trajectories of each of the following families of curves intersecting at the indicated angle.

(i) $y = Kx;\ \frac{\pi}{4}$ (ii) $y = Ke^x;\ \frac{\pi}{4}$

PROBLEM (14): Find the oblique trajectories of each of the following families of curves intersecting at the indicated angle .

(i) $r = a(1 - \cos\theta)$; $\alpha = \frac{\pi}{4}$ (ii) $r^2 = a\cos 2\theta$; $\alpha = \frac{\pi}{3}$

PHYSICAL APPLICATIONS

PROBLEM (15): The growth rate of a bacteria population is proportional to its size . Initially the population is 10,000 , while after 10 days its size is 25,000 . What is the population size after 20 days ?

PROBLEM (16): Suppose that the population P of bacteria in a culture at time t changes at a rate proportional to $P - P^3$. Solve for P at any time t , assuming that $P > 1$ for all t .

PROBLEM (17): In a culture of yeast the rate of growth is proportional to the population $p(t)$ present at time t and the population doubles in 1 day , how much can be expected after 1 week at the same rate of growth ?

PROBLEM (18): **Boyle – Mariotte Law for Ideal Gases:** Experiments show that for an ideal gas at constant temperature and low pressure p , the rate of change of the volume V of a gas with the pressure is proportional to $-\frac{V}{p}$. Find the volume of the gas at any time t .

PROBLEM (19): An object cools from $92\,^{o}C$ to $88\,^{o}C$ in one minute when the surrounding temperature is $24\,^{o}C$. How long a period must elapse , before the temperature of the object becomes $65\,^{o}C$.

MASS – SPRING SYSTEMS

PROBLEM (20): Find the displacement $y(t)$ of a mass – spring system subject to no damping or external forces for the following data :

$m = 10\,kg$, $K = 1000\,N/m$, $y(0) = 1\,m$, $y'(0) = 0$

PROBLEM (21): Find the displacement $y(t)$ of a mass – spring system subject to a damping but no external force for the following data :

$m = 10\,kg$, $K = 10\,N/m$, $C = 20\,kg/s$, $y(0) = 0$, $y'(0) = 1$

PROBLEM (22): Find the displacement $y(t)$ of a mass – spring system subject to a damping and an external force for the following data :

$m = 10\,kg$, $K = 10\,N/m$, $C = 20\,kg/s$, $F = \sin t$, $y(0) = 0$, $y'(0) = 1\,m/s$

COUPLED MASS – SPRING SYSTEMS

PROBLEM (23): Find the displacement $y_1(t)$ and $y_2(t)$ of the masses in a coupled mass – spring system with the following data :

(i) $m_1 = m_2 = 1\,kg,\ K_1 = 3\,N/m,\ K_2 = 2\,N/m,\ y_1(0) = 5,\ y_2(0) = 0,$
$y_1'(0) = 0,\ \ y_2'(0) = 0.$

(ii) $m_1 = 18\,kg,\ m_2 = 3\,kg,\ K_1 = 108\,N/m,\ K_2 = 18\,N/m,$
$y_1(0) = y_2(0) = 0,\ y_1'(0) = 1,\ \ y_2'(0) = 0.$

PROBLEM (24): Find the displacements $y_1(t)$ and $y_2(t)$ of the masses in a coupled mass – spring system with the following data :

$m_1 = m_2 = 1\,kg,\ K_1 = K_2 = K_3 = 3\,N/m,\ F_1(t) = F_2(t) = 0,$
$y_1(0) = 1,\ y_2(0) = 1,\ y_1'(0) = 3,\ y_2'(0) = -3.$

PROBLEM (25): Find the displacements $y_1(t)$ and $y_2(t)$ of the masses in a coupled mass – spring system with the following data :

$m_1 = m_2 = 2\,kg,\ K_1 = 6\,N/m,\ K_2 = 2\,N/m,\ K_3 = 3\,N/m,$
$F_1(t) = 2\,N,\ F_2(t) = 0,\ y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0.$

ELECTRIC CIRCUITS

PROBLEM (26): An RL circuit contains a resistor of resistance 10 ohms , an inductor of inductance 2 henrys , and an electromotive force $E(t)$ volts . At time $t = 0$, the current I is zero . Find the current I at any time t if

(i) $E(t) = 40$ (ii) $E(t) = 20\,e^{-3t}$

(iii) $E(t) = 50 \sin 5t$

PROBLEM (27): An RC circuit has an e.m.f. of 100 volts , a resistance of 5 ohms , a capacitance of 0.02 farad , and an initial charge on the capacitor of 5 coulombs . Find

(i) the charge Q on the capacitor at any time t and

(ii) the current in the circuit at any time t .

PROBLEM (28): Find the current $I(t)$ in the LC – circuit with the following data :

(i) $L = 10$ henrys , $C = 0.1$ farad , $E = 10t$ volts , $I(0) = I'(0) = 0$

(ii) $L = 2$ henrys , $C = 0.005$ farad , $E = 220 \sin 4t$ volts , $I(0) = 0 = I'(0) = 0$

PROBLEM (29): Find the current $I(t)$ in the RLC – circuit when $R = 16$ ohms, $L = 2$ henrys, $C = 0.02$ farad, $E(t) = 100 \sin 3t$ N, and $I(0) = 0$, $I'(0) = 0$.

ELECTRICAL NETWORKS

PROBLEM (30): Set up the mathematical model of the network shown in figure (8.44). Find the currents $I_1(t)$ and $I_2(t)$ in the network assuming that $R = 1$ ohms, $L = 1.25$ henrys, $C = 0.2$ farad, and $I_1(0) = 1$, $I_2(0) = 1$ ampere.

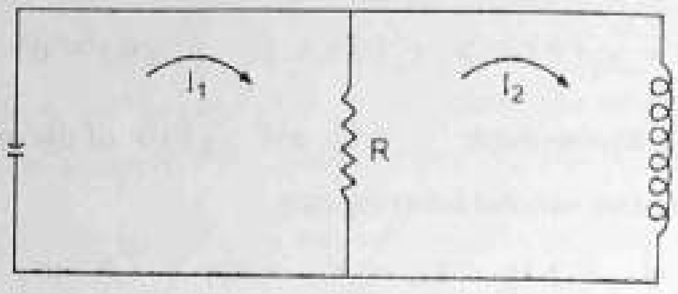


Figure (8.44)

Hint: The currents I_1 and I_2 in the network are governed by the system

$$R\left(\frac{dI_1}{dt} - \frac{dI_2}{dt}\right) + \frac{1}{C} I_1 = 0, \quad L\frac{dI_2}{dt} + R(I_2 - I_1) = 0$$

PROBLEM (31): Set up the mathematical model of the network shown in figure (8.45). Find the currents $I_1(t)$ and $I_2(t)$ in the network assuming that $R = 2.5$ ohms, $L = 1$ henry, $C = 0.04$ farad, $E(t) = 169 \sin t$ volts, $I_1(0) = I_2(0) = 0$.

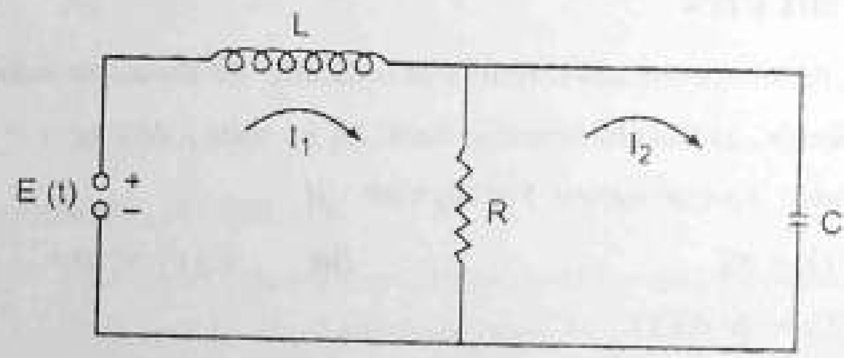


Figure (8.45)

Hint: The currents I_1 and I_2 in the network are governed by the system

$$R(I_1 - I_2) + L\frac{dI_1}{dt} = E(t), \quad R\left(\frac{dI_2}{dt} - \frac{dI_1}{dt}\right) + \frac{1}{C} I_2(t) = 0$$

PROBLEM (32): Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the electrical network shown in figure (8.45). Given that $R = 0.8$ ohms, $L = 1$ henry, $C = 0.25$ farad, $E(t) = \frac{4}{5}t + \frac{21}{25}$ volts, $I_1(0) = 1$, $I_2(0) = -3.8$.

PROBLEM (33): Set up the mathematical model of the network shown in figure (8.46). Find the currents $I_1(t)$ and $I_2(t)$ in the network assuming that $R_1 = 1$ ohm, $R_2 = 1.4$ ohm, $L_1 = 0.8$ henry, $L_2 = 1$ henry, $E(t) = 100$ volts and $I_1(0) = I_2(0) = 0$.

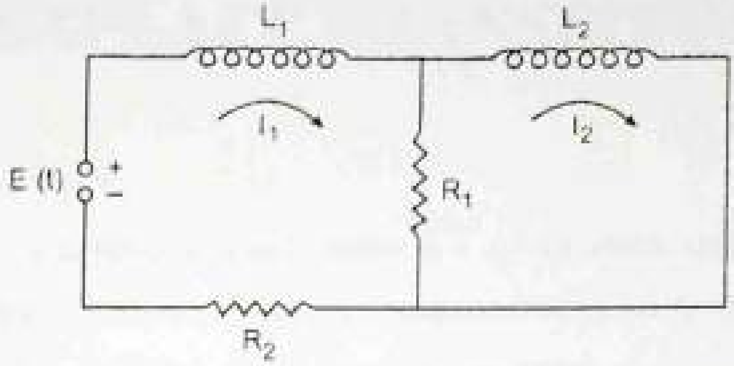


Figure (8.46)

Hint: The currents I_1 and I_2 in the network are governed by the system

$$L_1 \frac{dI_1}{dt} + R_1(I_1 - I_2) + R_2 I_1 = E(t), \quad L_2 \frac{dI_2}{dt} + R_1(I_2 - I_1) = 0$$

PROBLEM (34): Set up the mathematical model of the network shown in figure (8.47). Find the currents $I_1(t)$ and $I_2(t)$ in the network assuming that $R_1 = 5$ ohms, $R_2 = 10$ ohms, $C = 10^{-3}$ farad, $E(t) = 30$ volts and $I_1(0) = 0$, $I_2(0) = 3$.

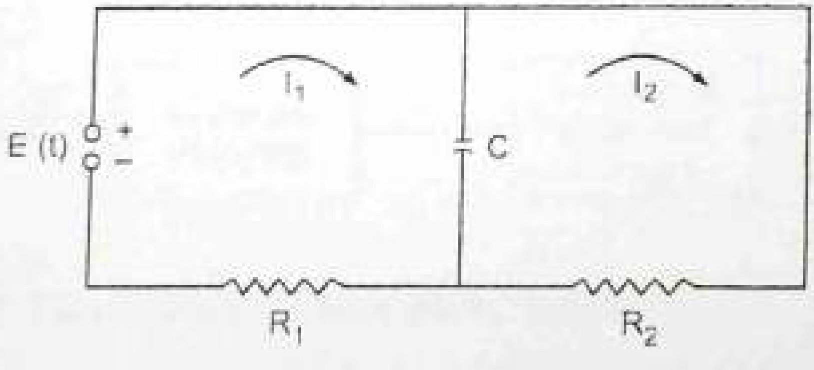


Figure (8.47)

Hint: The currents I_1 and I_2 in the network are governed by the system

$$\frac{1}{C}\left[I_1(t) - I_2(t)\right] + R_1 \frac{dI_1}{dt} = \frac{dE}{dt}, \quad R_2 \frac{dI_2}{dt} + \frac{1}{C}\left[I_2(t) - I_1(t)\right] = 0$$

PROBLEM (35): The deflection curve of a horizontal beam of length L fixed at both ends satisfies the equation $E\,I\,y'' = \frac{1}{2} W x^2 - \frac{1}{2} W L x - K$

where K is a constant and W is the uniform load per unit length of the beam. Find the deflection y of the beam and its maximum deflection.

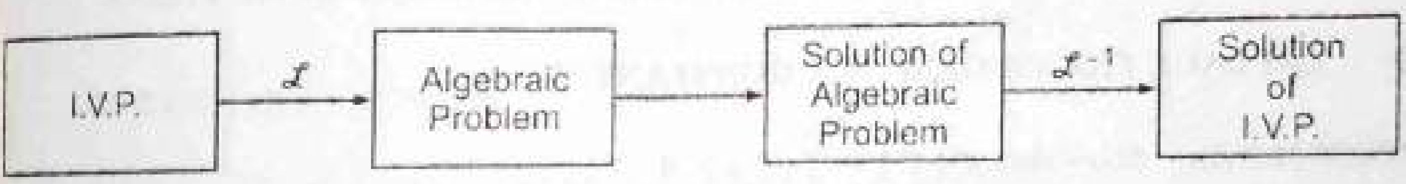
[Hint: Use the conditions (i) when $x = 0$ $y = 0$ (ii) when $x = 0$, $y' = 0$ (iii) when $x = L$, $y' = 0$]

CHAPTER 9

LAPLACE TRANSFORMS

9.1 INTRODUCTION

In mathematics , a **transform** usually refers to a device which changes one kind of a function or equation into another kind . That is , a transform reduces problems which we do not know how to solve , or which are difficult , into problems which are relatively easier to solve . The method of Laplace transforms (named after the French mathematician P.S. Laplace 1749–1827) is one of the efficient methods for solving linear differential equations and corresponding initial and boundary value problems . This method reduces the problem of solving a differential equation into an algebraic equation problem , whose solution (obtained by usual algebraic methods) is transformed back to obtain the solution of the given problem . The following flow chart illustrates the method of Laplace transforms for the solution of initial value problems (I.V.P) .



9.2 LAPLACE TRANSFORM OF A FUNCTION

If a real – valued function $f(t)$ defined for all $t \geq 0$ is multiplied by e^{-st} and integrated w.r.t. t from 0 to ∞ , a new function of the variable s say $\bar{f}(s)$ is obtained where s may be real or complex . The function $\bar{f}(s)$ is called the Laplace transform of the original function $f(t)$ and is also denoted by $\mathcal{L}\{f(t)\}$.

Thus $$\bar{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Clearly , the variable of integration t is a dummy variable and can be replaced by any other symbol . The process of obtaining the Laplace transform of a function is called the **Laplace transformation** .

9.3 THE LINEARITY THEOREM

THEOREM (9.1): **Prove that the Laplacian operator $\mathcal{L}$ is a linear operator i.e. if C_1 and C_2 are any constants and $f_1(t)$ and $f_2(t)$ any functions whose Laplace transforms exist, then**

$$\mathcal{L}\{C_1 f_1(t)+C_2 f_2(t)\} = C_1\,\mathcal{L}\{f_1(t)\}+C_2\,\mathcal{L}\{f_2(t)\}$$

$$= C_1 \bar{f}_1(s)+C_2 \bar{f}_2(s).$$

PROOF: By definition

$$\mathcal{L}\{C_1 f_1(t)+C_2 f_2(t)\} = \int_0^\infty e^{-st}[C_1 f_1(t)+C_2 f_2(t)]\,dt$$

$$= C_1\int_0^\infty e^{-st} f_1(t)\,dt + C_2\int_0^\infty e^{-st} f_2(t)\,dt$$

$$= C_1\,\mathcal{L}\{f_1(t)\}+C_2\,\mathcal{L}\{f_2(t)\}$$

and so $\mathcal{L}$ is a linear operator.

9.4 LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

(A) LAPLACE TRANSFORM OF A CONSTANT C

THEOREM (9.2): **Show that $\mathcal{L}\{C\} = \frac{C}{s}$, $s > 0$.**

PROOF: By definition

$$\mathcal{L}\{C\} = \int_0^\infty e^{-st}(C)\,dt = C\int_0^\infty e^{-st}\,dt = C\left|\frac{e^{-st}}{-s}\right|_0^\infty = \frac{C}{s},\quad s>0$$

In particular, $\mathcal{L}\{1\} = \frac{1}{s}$, $s>0$.

(B) LAPLACE TRANSFORM OF AN EXPONENTIAL FUNCTION

THEOREM (9.3): **Show that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$, a constant.**

PROOF: By definition,

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\,e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \quad s > a.$$

Note that the $\mathcal{L}\{f(t)\}$ may not exist for all values of s; but if it exists, then it will exist for suitably large values of s.

(C) LAPLACE TRANSFORMS OF TRIGONOMETRIC FUNCTIONS

THEOREM (9.4): Show that

(i) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \; s > 0$ (ii) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \; s > 0$

PROOF: We have

(i) $$\mathcal{L}\{\sin at\} = \mathcal{L}\left\{\frac{e^{iat}-e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i}\mathcal{L}\{e^{iat}\} - \frac{1}{2i}\mathcal{L}\{e^{-iat}\} \quad [\text{By linearity theorem (9.1)}]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{a}{s^2+a^2}, \quad s > 0$$

(ii) $$\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{e^{iat}+e^{-iat}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{iat}\} + \frac{1}{2}\mathcal{L}\{e^{-iat}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{s}{s^2+a^2}, \quad s > 0$$

ALTERNATE METHOD

$$\mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at + i\sin at\} \quad (\text{using Euler's formula})$$

$$= \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\} \tag{1}$$

Also $$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \tag{2}$$

From equations (1) and (2), we have on equating real and imaginary parts,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \text{ and } \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad s > 0$$

(D) LAPLACE TRANSFORMS OF HYPERBOLIC FUNCTIONS

THEOREM (9.5): Show that

(i) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \quad s > |a|$

(ii) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \quad s > |a|$

PROOF: (i) $\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2-a^2}, \quad s > a \text{ and } s > -a \text{ i.e. } s > |a|$$

(ii) $\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}, \quad s > |a|$$

(E) LAPLACE TRANSFORMS OF POLYNOMIAL FUNCTIONS

THEOREM (9.6): Show that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, \ldots$ **and** $s > 0$.

PROOF: We prove this result using the method of induction as follows ;

The result is true for $n = 0$, since $\mathcal{L}\{1\} = \frac{0!}{s} = \frac{1}{s}$. Assume that it is true for $n = k$.

i.e. $\mathcal{L}\{t^k\} = \frac{k!}{s^{n+1}}$ (1)

Now $\mathcal{L}\{t^{k+1}\} = \int_0^\infty e^{-st} t^{k+1}\, dt$

$$= \left|\frac{e^{-st}}{-s} t^{k+1}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}(k+1)t^k\, dt$$

$$= \left|-\frac{e^{-st}t^{k+1}}{s}\right|_0^\infty + \frac{k+1}{s}\int_0^\infty e^{-st} t^k\, dt$$

The first term on the R.H.S. is zero at $t = 0$ and for $t \to \infty$. Thus on using equation (1), we get

$$\mathcal{L}\{t^{k+1}\} = \frac{k+1}{s}\mathcal{L}\{t^k\} = \frac{k+1}{s}\,\frac{k!}{s^{k+1}} = \frac{k+1!}{s^{k+2}}$$

which shows that the result is true for $n = k+1$. Hence, by the principal of induction, it is true for all integral values of $n \geq 0$. In particular,

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t\} = \frac{1}{s^2}, \quad \mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad \mathcal{L}\{t^3\} = \frac{6}{s^4} \text{ and so on.}$$

GENERALIZATION

In general, the above formula takes the form :

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad n > -1, \quad s > 0. \qquad \text{(A)}$$

By definition $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$

Let $st = u$ therefore $dt = \frac{du}{s}$ and $0 \le u \le \infty$, then

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du = \frac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0$$

where $\Gamma(n+1) = \int_0^\infty e^{-u} u^n \, du$ is the Gamma function which converges if and only if $n > -1$.

One of the properties of Gamma function is $\Gamma(n+1) = n!$ for $n = 0, 1, 2, 3, \ldots$

so that the above formula in theorem (9.6) also follows from the general formula (A).

TABLE OF FORMULAS

	$f(t)$	$\mathcal{L}\{f(t)\} = \bar{f}(s)$
1	1	$\frac{1}{s}, \; s > 0$
2	t	$\frac{1}{s^2}, \; s > 0$
3	t^2	$\frac{2}{s^3}, \; s > 0$
4	$t^n, \; n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \; s > 0$
5	$t^n, \; n > -1$	$\frac{\Gamma(n+1)}{s^{n+1}}, \; s > 0$
6	e^{at}	$\frac{1}{(s-a)}, \; s > a$
7	$\sin at$	$\frac{a}{s^2 + a^2}, \; s > 0$
8	$\cos at$	$\frac{s}{s^2 + a^2}, \; s > 0$
9	$\sinh at$	$\frac{a}{s^2 - a^2}, \; s > \lvert a \rvert$
10	$\cosh at$	$\frac{s}{s^2 - a^2}, \; s > \lvert a \rvert$

EXAMPLE (1): **Find the following Laplace transforms :**

(i) $\mathcal{L}\{a\,e^{bt}\}$ (ii) $\mathcal{L}\{a+bt+ct^2\}$

(iii) $\mathcal{L}\{6\sin 2t - 3\cos 2t\}$ (iv) $\mathcal{L}\{3\cosh 5t - 4\sinh 5t\}$

SOLUTION: By the linearly theorem , we have

(i) $\mathcal{L}\{a\,e^{bt}\} = a\,\mathcal{L}\{e^{bt}\} = a\left(\frac{1}{s-b}\right) = \frac{a}{s-b}$

(ii)
$$\mathcal{L}\{a+bt+ct^2\} = \mathcal{L}\{a\} + b\,\mathcal{L}\{t\} + c\,\mathcal{L}\{t^2\}$$
$$= \frac{a}{s} + b\left(\frac{1}{s^2}\right) + c\left(\frac{2}{s^3}\right)$$
$$= \frac{a}{s} + \frac{b}{s^2} + \frac{2c}{s^3}$$

(iii)
$$\mathcal{L}\{6\sin 2t - 3\cos 2t\} = 6\,\mathcal{L}\{\sin 2t\} - 3\,\mathcal{L}\{\cos 2t\}$$
$$= 6\left(\frac{2}{s^2+4}\right) - 3\left(\frac{s}{s^2+4}\right)$$
$$= \frac{12}{s^2+4} - \frac{3s}{s^2+4}$$
$$= \frac{12-3s}{s^2+4}$$

(iv)
$$\mathcal{L}\{3\cosh 5t - 4\sinh 5t\} = 3\,\mathcal{L}\{\cosh 5t\} - 4\,\mathcal{L}\{\sinh 5t\}$$
$$= 3\left(\frac{s}{s^2-25}\right) - 4\left(\frac{5}{s^2-25}\right)$$
$$= \frac{3s}{s^2-25} - \frac{20}{s^2-25} = \frac{3s-20}{s^2-25}$$

9.5 EXISTENCE OF LAPLACE TRANSFORMS

We have seen that the Laplace transforms of standard functions existed under certain restrictions such as $s > 0$ or $s > a$ etc . In order to give the conditions that guarantee the existence of a Laplace transform , we require the following definitions .

PIECEWISE CONTINUOUS FUNCTIONS

A function $f(t)$ is called piecewise (or sectionally) continuous on an interval $a \le t \le b$, if the interval can be divided into a finite number of subintervals , in each of which $f(t)$ is continuous and has finite limits as t approaches either end point of the subinterval from the interior (i.e. finite left and right hand limits) .

An example of a function which is piecewise continuous is shown graphically in figure (9.1).

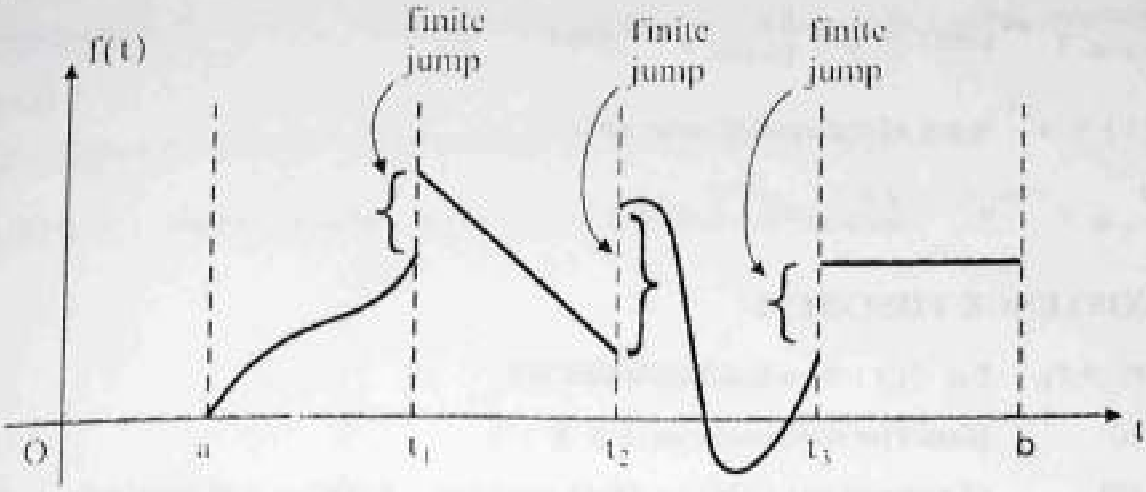


Figure (9.1)

The function has discontinuities at t_1, t_2 and t_3.

A specific example of a piecewise continuous function is :

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ t-1 & 1 \le t < 3 \\ 4-t & 3 \le t < 4 \\ -1 & 4 \le t < 5 \end{cases}$$

and is shown graphically in figure (9.2).

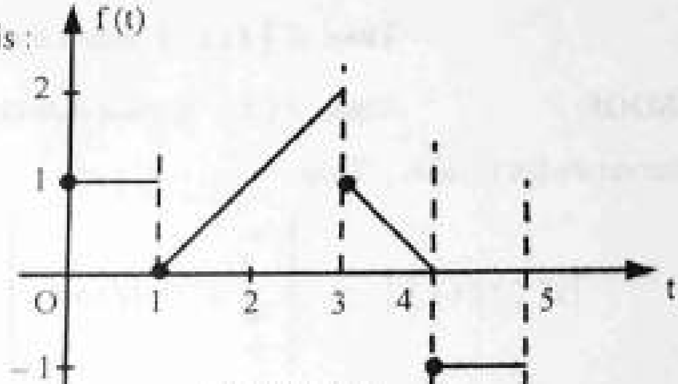


Figure (9.2)

FUNCTIONS OF EXPONENTIAL ORDER

A function $f(t)$ is said to be of exponential order a if $\underset{t \to \infty}{\text{Lt}}\ e^{-at} f(t)$ = a finite quantity.

In such case, there exists non – negative real constants a, M, and T such that

$$|e^{-at} f(t)| \le M \quad \text{or} \quad |f(t)| \le M e^{at} \quad \text{for all} \quad t \ge T.$$

Sometimes, this is written $f(t) = O(e^{at})$ as $t \to \infty$. For example,

(i) $f(t) = e^{at}$ is of exponential order since $\underset{t \to \infty}{\text{Lt}}\ e^{-at} e^{at} = \underset{t \to \infty}{\text{Lt}}\ 1 = 1$.

(ii) $f(t) = t^n$ $(n = 0, 1, 2, 3, \ldots..)$ is of exponential order since

$$\underset{t \to \infty}{\text{Lt}}\ e^{-at} t^n = \underset{t \to \infty}{\text{Lt}}\ \frac{t^n}{e^{at}} = \underset{t \to \infty}{\text{Lt}}\ \frac{n!}{e^{at}} = 0.$$

(iii) Any function which is bounded in absolute value for all $t \ge 0$ such as $\sin t$ and $\cos t$ are of exponential order since

$$\underset{t \to \infty}{\text{Lt}}\ e^{-at} \sin t = 0 = \underset{t \to \infty}{\text{Lt}}\ e^{-at} \cos t.$$

(iv) The functions $\sinh t$ and $\cosh t$ are of exponential order since

$$\underset{t \to \infty}{\mathrm{Lt}} \; e^{-at} \cosh t = \frac{1}{2} = \underset{t \to \infty}{\mathrm{Lt}} \; e^{-at} \sinh t$$

(v) $f(t) = e^{t^2}$ is not of exponential order since

$$\underset{t \to \infty}{\mathrm{Lt}} \; e^{-at} e^{t^2} = \underset{t \to \infty}{\mathrm{Lt}} \; e^{t^2 - at} = \infty$$

EXISTENCE THEOREM

THEOREM (9.7): Let $f(t)$ be a function which is :

(i) piecewise continuous on $t \geq 0$

(ii) of exponential order i.e. there exist non – negative real constants a, M, and T such that $|f(t)| \leq M e^{at}$ for all $t \geq T$.

Then $\mathcal{L}\{f(t)\}$ exists for all $s > a$.

PROOF: Since $f(t)$ is piecewise continuous, $e^{-st} f(t)$ is integrable over any finite interval on the t – axis. Then

$$|\mathcal{L}\{f(t)\}| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leq \int_0^\infty |e^{-st} f(t)|\, dt$$

$$\leq \int_0^\infty e^{-st} M e^{at}\, dt = M \int_0^\infty e^{-(s-a)t}\, dt = \frac{M}{s-a}$$

where the condition $s > a$ is needed for the existence of the last integral.

NOTE: It should be noted that the conditions stated in theorem (9.7) are only sufficient (and not necessary) i.e. if one or both the conditions are not satisfied, the Laplace transform of a function may or may not exist. For example,

(i) The function $f(t) = \dfrac{1}{\sqrt{t}}$ is not piecewise continuous on $t \geq 0$ because it is infinite at $t = 0$, but its Laplace transforms exists since

$$\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st} t^{-1/2}\, dt = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2}\, dx \quad (\text{with } st = x^2)$$

$$= \sqrt{\frac{\pi}{s}} \qquad \left(\text{since } \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \right)$$

(ii) The function $f(t) = 2te^{t^2}\cos e^{t^2}$ is not of exponential order but is Laplace transform exists .

(iii) The function $f(t) = e^{t^2}$ is not of exponential order and as such its Laplace transforms does not exist .

Note that if a Laplace transform of a given function exists , it is uniquely determined .

EXAMPLE (2): **Find the Laplace transform of the following function :**

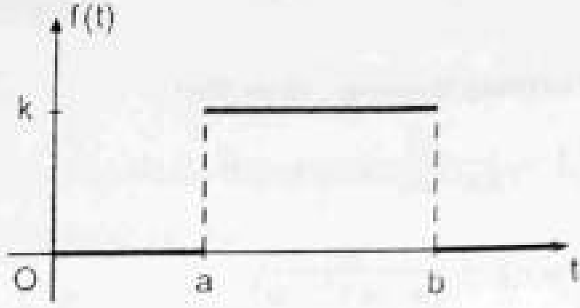


Figure (9.3)

SOLUTION: From the figure , we have

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < a \\ k & \text{if } a \le t \le b \\ 0 & \text{if } t > b \end{cases}$$

By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^a e^{-st} f(t)\,dt + \int_a^b e^{-st} f(t) + \int_b^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^b e^{-st}(k)\,dt + \int_b^\infty e^{-st}(0)\,dt$$

$$= k\left|\frac{e^{-st}}{-s}\right|_a^b = \frac{k}{s}\left(e^{-as} - e^{-bs}\right)$$

9.6 FIRST SHIFTING THEOREM FOR LAPLACE TRANSFORMS

THEOREM (9.8): **If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then $\mathcal{L}\{e^{at}f(t)\} = \bar{f}(s-a)$, $s > a$.**

PROOF: By definition ,

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

then $$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = \bar{f}(s-a)$$

and so the theorem is proved .

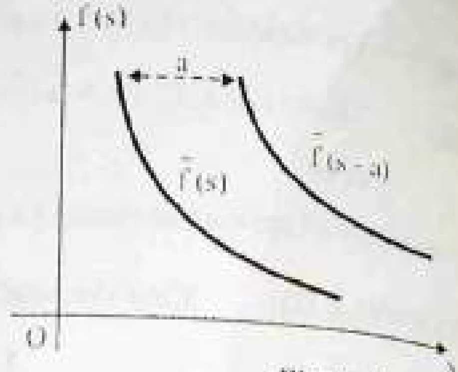


Figure (9.4)

EXAMPLE (3): **Using first shifting theorem , show that**

(i) $$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}},\ n = 0, 1, 2, \ldots\ldots$$

(ii) $$\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$$

(iii) $$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$$

SOLUTION: (i) Since $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\quad n = 0, 1, 2, \ldots\ldots$

Hence , by the first shifting theorem ,

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}},\quad n = 0, 1, 2, \ldots\ldots$$

(ii) Since $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$, therefore , by the first shifting theorem ,

$$\mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$$

(iii) Since $\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$, therefore , by the first shifting theorem ,

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$$

9.7 DIFFERENTIATION OF TRANSFORMS (MULTIPLICATION BY t^n)

THEOREM (9.9): If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s),\quad n = 1, 2, 3, \ldots\ldots$$

PROOF: We have $\bar{f}(s) = \int_0^\infty e^{-st}f(t)\,dt$

Then by Leibnitz's rule for differentiating under the integral sign

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= -\int_0^\infty e^{-st}\left[t f(t)\right] dt = -\mathcal{L}\{t f(t)\}$$

or $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$

which proves the theorem for $n = 1$. To establish the theorem, in general, we use mathematical induction. Assume the theorem is true for $n = k$.

i.e. $\mathcal{L}\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k}\bar{f}(s)$

or $\int_0^\infty e^{-st}\left[t^k f(t)\right] dt = (-1)^k \frac{d^k}{ds^k}\bar{f}(s)$

Then $\frac{d}{ds}\int_0^\infty e^{-st}\left[t^k f(t)\right] dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$

Using Leibnitz's rule, we get

$$-\int_0^\infty e^{-st}\left[t^{k+1} f(t)\right] dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$$

or $\int_0^\infty e^{-st}\left[t^{k+1} f(t)\right] dt = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\bar{f}(s)$

which shows that the theorem holds for $n = k + 1$. Thus the theorem is true for all positive integral values of n.

EXAMPLE (4): **Using the above theorem, find**

(i) $\mathcal{L}\{t \sin 2t\}$ **(ii)** $\mathcal{L}\{t^2 \sin 2t\}$

SOLUTION: Since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, hence by theorem (9.9), we get

(i) $\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$

(ii) $\mathcal{L}\{t^2 \sin 2t\} = \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) = \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right] = \frac{12s^2-16}{(s^2+4)^3}$

9.8 INTEGRAL OF TRANSFORMS (DIVISION BY t)

THEOREM (9.10): If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u)\,du$

provided $\underset{t \to 0+}{\text{Lt}} \left\{\frac{f(t)}{t}\right\}$ exists.

PROOF: **FIRST METHOD**

By definition, $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Therefore $\bar{f}(u) = \int_0^\infty e^{-ut} f(t)\,dt$

Integrating both sides w.r.t. u from s to ∞, we obtain

$$\int_s^\infty \bar{f}(u)\,du = \int_s^\infty \left[\int_0^\infty e^{-ut} f(t)\,dt\right] du$$

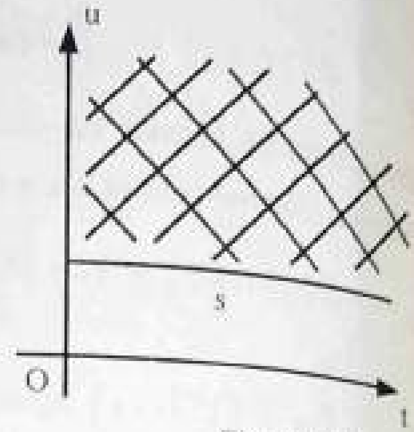


Figure (9.5)

The region of integration for the double integral is shown in figure (9.5). Interchanging the order of integration, we get

$$\int_s^\infty \bar{f}(u)\,du = \int_0^\infty \left[\int_s^\infty f(t)\, e^{-ut}\,du\right] dt$$

$$= \int_0^\infty f(t) \left|\frac{e^{-ut}}{-t}\right|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}\, e^{-st}\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

ALTERNATE METHOD

Let $G(t) = \frac{f(t)}{t}$, then $f(t) = t\,G(t)$. Taking the Laplace transform of both sides, we get

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\,G(t)\} = -\frac{d}{ds}\mathcal{L}\{G(t)\} \quad \text{or} \quad \bar{f}(s) = -\frac{d}{ds}\bar{G}(s)$$

Then integrating we get $\bar{G}(s) = -\int_\infty^s \bar{f}(u)\,du = \int_s^\infty \bar{f}(u)\,du$

i.e. $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u)\,du$

EXAMPLE (5): **Using theorem (9.10), find $\mathcal{L}\left\{\frac{\sin kt}{t}\right\}$.**

SOLUTION: Since $\underset{t \to 0+}{\text{Lt}} \left\{\frac{\sin kt}{t}\right\} = k$ exists, and

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \text{ hence } \mathcal{L}\left\{\frac{\sin kt}{t}\right\} = \int_s^\infty \frac{k}{u^2+k^2}\, du$$

$$= \left| \tan^{-1}\frac{u}{k} \right|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{k} = \cot^{-1}\frac{s}{k} \qquad \left(\text{since } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$$

9.9 UNIT STEP FUNCTION

The unit step function denoted by $u_a(t)$ is defined as

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \quad (a \geq 0) \end{cases}$$

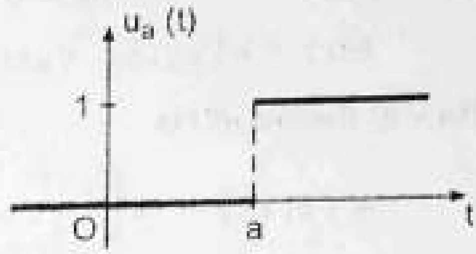


Figure (9.6)

and is shown graphically in figure (9.6).

Note that $u_a(t)$ has a jump discontinuity (of size 1) at $t = a$. In particular when $a = 0$,

$$u_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

The unit step function $u_a(t)$ is a basic building block of various functions, as we shall see, and it greatly increases the usefulness of Laplace transform methods.

NOTE: The unit step function is also called Heaviside's unit step function named after the English electrical engineer Oliver Heaviside (1850 – 1925).

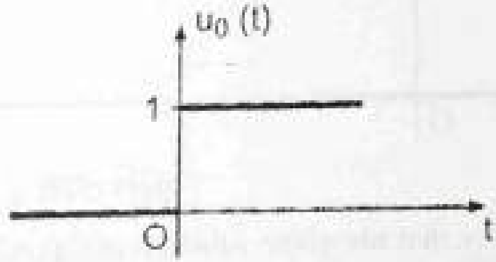


Figure (9.7)

THEOREM (9.11): Prove that $\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}$ **if** $s > 0$.

PROOF: We have $u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

so that $\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} u_a(t)\, dt$

$$= \int_0^a e^{-st} u_a(t)\, dt + \int_a^\infty e^{-st} u_a(t)\, dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left|\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s}, \quad s > 0.$$

EXAMPLE (6): **Represent the function $f(t)$ shown in figure (9.8), in terms of unit step functions and find its Laplace transforms :**

SOLUTION: From figure (9.8), we have

$$f(t) = k\left[u_a(t) - u_b(t)\right]$$

Hence by theorem (9.11)

$$\mathcal{L}\{f(t)\} = k\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$

$$= \frac{k}{s}\left(e^{-as} - e^{-bs}\right)$$

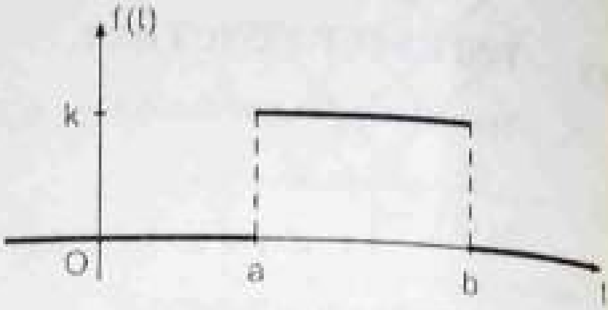


Figure (9.8)

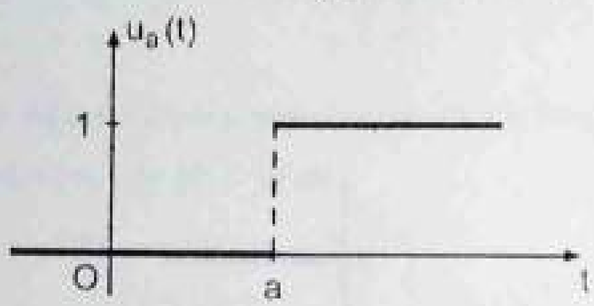


Figure (9.9)

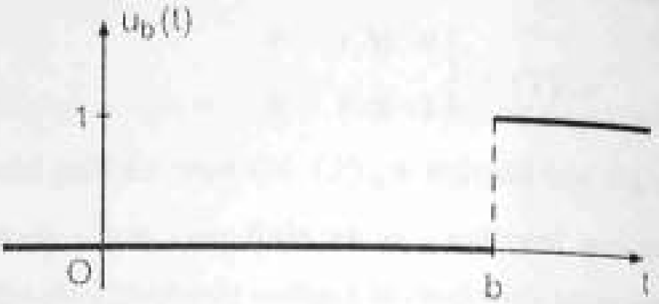


Figure (9.10)

Observe that the given function in figure (9.8) is obtained by subtracting the function in figure (9.10) from the function in figure (9.9).

EXAMPLE (7): **Represent the following functions in terms of unit step functions and find their Laplace transforms :**

(i)

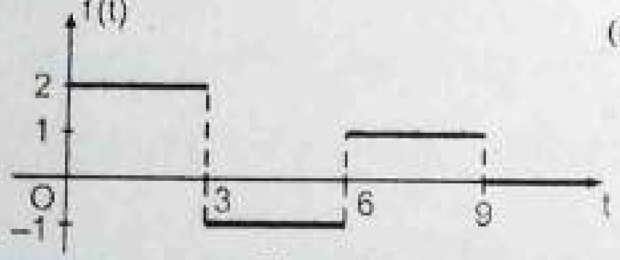


Figure (9.11)

(ii)

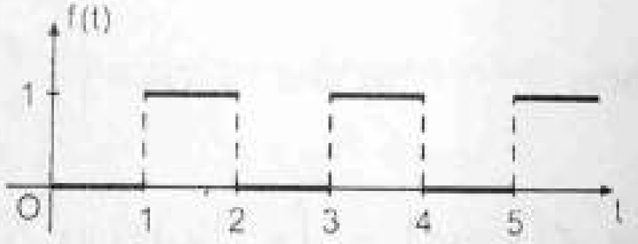


Figure (9.12)

SOLUTION: (i) From figure (9.11), we have

$$f(t) = 2\left[u_0(t) - u_3(t)\right] - 1\left[u_3(t) - u_6(t)\right] + 1\left[u_6(t) - u_9(t)\right]$$

$$= 2u_0(t) - 3u_3(t) + 2u_6(t) - u_9(t)$$

Hence, by theorem (9.11), we have

$$\mathcal{L}\{f(t)\} = 2\left(\frac{1}{s}\right)-3\left(\frac{e^{-3s}}{s}\right)+2\left(\frac{e^{-6s}}{s}\right)-\left(\frac{e^{-9s}}{s}\right)$$

$$= \frac{1}{s}(2-3e^{-3s}+2e^{-6s}-e^{-9s})$$

(ii) From figure (9.12), we have

$$f(t) = 1[u_1(t)-u_2(t)]+1[u_3(t)-u_4(t)]+u_5(t)+\ldots\ldots$$

$$= u_1(t)-u_2(t)+u_3(t)-u_4(t)+u_5(t)-+\ldots\ldots$$

Hence, by theorem (9.11), we have

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s}-\frac{e^{-2s}}{s}+\frac{e^{-3s}}{s}-\frac{e^{-4s}}{s}+\frac{e^{-5s}}{s}-+\ldots\ldots$$

$$= \frac{e^{-s}}{s}(1-e^{-s}+e^{-2s}-e^{-3s}+e^{-4s}-+\ldots\ldots)$$

$$= \frac{e^{-s}}{s}\left(\frac{1}{1+e^{-s}}\right) \qquad \left(\text{since the sum of a geometric series is } \frac{1}{1-r}\right)$$

9.10 SECOND SHIFTING THEOREM FOR LAPLACE TRANSFORMS

THEOREM (9.12): If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\{f(t-a)u_a(t)\} = e^{-as}\bar{f}(s)$, $a > 0$.

PROOF: Since $u_a(t) = \begin{cases} 0 & \text{if } 0 \le t < a \\ 1 & \text{if } t \ge a \end{cases}$

therefore $$\mathcal{L}\{f(t-a)u_a(t)\} = \int_0^\infty e^{-st}[f(t-a)u_a(t)]\,dt$$

$$= \int_0^a e^{-st}[f(t-a)u_a(t)]\,dt+\int_a^\infty e^{-st}[f(t-a)u_a(t)]\,dt$$

$$= \int_0^a e^{-st}(0)\,dt+\int_a^\infty e^{-st}f(t-a).1\,dt$$

Let $t-a = y$ therefore $dt = dy$ and $0 \le y \le \infty$

Hence $$\mathcal{L}\{f(t-a)u_a(t)\} = \int_0^\infty e^{-s(y+a)}f(y)\,dy$$

$$= e^{-as}\int_0^\infty e^{-sy}f(y)\,dy = e^{-as}\bar{f}(s)$$

EXAMPLE (8): **Find the Laplace transform of** $F(t) = \cos\left(t - \frac{2\pi}{3}\right) u_{2\pi/3}(t)$.

SOLUTION: Here $f(t) = \cos t$ and $a = \frac{2\pi}{3}$.

Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, hence, by theorem (9.12) with $a = \frac{2\pi}{3}$, we have

$$\mathcal{L}\{F(t)\} = \frac{s\,e^{-2\pi s/3}}{s^2+1}$$

EXAMPLE (9): **Find the Laplace transforms of the functions shown in figures below :**

(i)

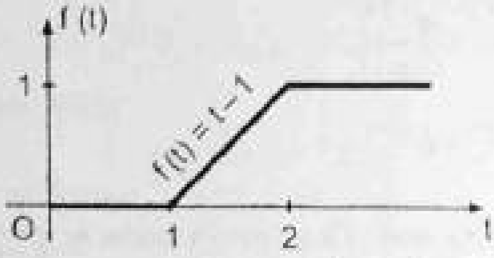


Figure (9.13)

(ii)

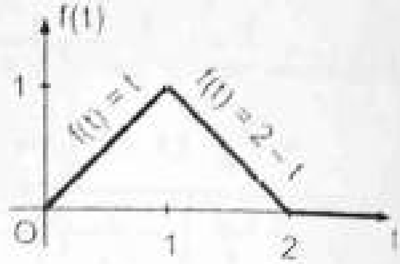


Figure (9.14)

SOLUTION: (i) From figure (9.13), we have

$$\begin{aligned} f(t) &= (t-1)\left[u_1(t) - u_2(t)\right] + 1\left[u_2(t)\right] \\ &= (t-1)u_1(t) - (t-1)u_2(t) + u_2(t) \\ &= (t-1)u_1(t) - (t-2)u_2(t) \end{aligned}$$

Hence, by theorem (9.12), we get

Therefore $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} = \frac{e^{-s} - e^{-2s}}{s^2}$

(ii) From figure (9.14), we have

$$\begin{aligned} f(t) &= t\left[u_0(t) - u_1(t)\right] + (2-t)\left[u_1(t) - u_2(t)\right] \\ &= t u_0(t) - 2(t-1)u_1(t) + (t-2)u_2(t) \end{aligned}$$

Hence, by theorem (9.12), we get

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$

9.11 DIRAC'S DELTA FUNCTION OR UNIT IMPULSE FUNCTION

We now consider a function which is very useful in many practical situations. This is the Dirac's delta function or unit impulse function named after the British theoretical physicist P.A.M. Dirac (1902 - 1984) in 1932. This function represents an extremely large force acting for a very short interval of time.

This situation occurs, for instance, when a tennis ball is hit, a system is given a blow by a hammer, a shock voltage applied to an electrical circuit, an aeroplane makes a **hard** landing and so on. To define this function we proceed as follows :

In mechanics, the impulse I of a force $f(t)$ over a time interval $a \le t \le a+\epsilon$ is defined as

$$I = \int_a^{a+\epsilon} f(t)\, dt$$

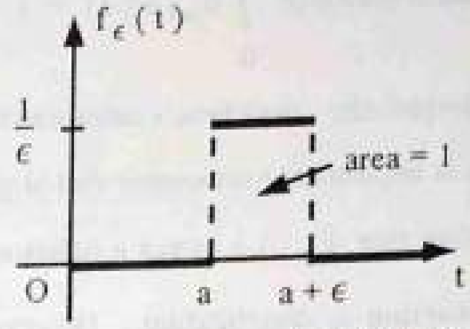


Figure (9.15)

Now consider the function

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{for } a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

where $\epsilon > 0$, and is shown graphically in figure (9.15).

Its impulse denoted by I_ϵ is given by

$$I_\epsilon = \int_a^{a+\epsilon} f_\epsilon(t)\, dt = \int_a^{a+\epsilon} \frac{1}{\epsilon}\, dt = 1$$

Clearly, I_ϵ represents the area of rectangle $\left(\text{i.e. } \frac{1}{\epsilon}.\epsilon = 1\right)$.

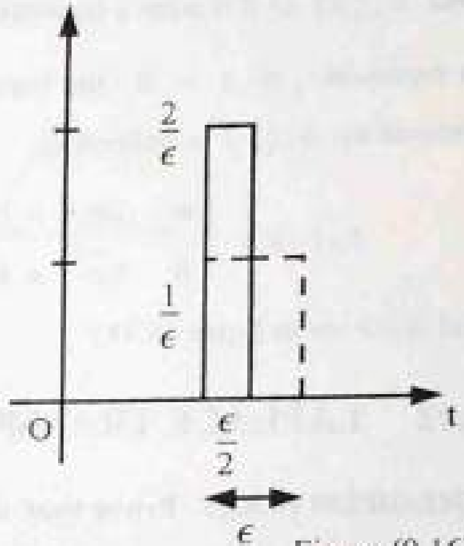


Figure (9.16)

We can write $f_\epsilon(t)$ in terms of unit step functions as

$$f_\epsilon(t) = \frac{1}{\epsilon}\left[u_a(t) - u_{a+\epsilon}(t)\right]$$

so that $$\mathcal{L}\{f_\epsilon(t)\} = \frac{1}{\epsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\right]$$

$$= \frac{e^{-as}(1-e^{-\epsilon s})}{\epsilon s} \qquad (1)$$

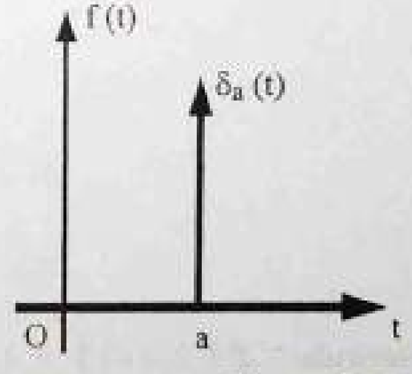


Figure (9.17)

If we reduce the width of the rectangle to $\frac{\epsilon}{2}$ and keep the area unchanged, the height of the rectangle will be $\frac{2}{\epsilon}$.

If we continue reducing the width of the rectangle retaining an area of unity, then as $\epsilon \to 0$, the height of the rectangle $\to \infty$. Thus we approach the final stage where the time interval is extremely small and the magnitude of the function extremely large. The function represented by this limiting stage (value) is known as the Dirac's delta function and is denoted by $\delta_a(t)$.

Thus $\delta_a(t)$ is defined as the limit of $f_\epsilon(t)$ as $\epsilon \to 0$

i.e. $$\delta_a(t) = \underset{\epsilon \to 0}{\mathrm{Lt}}\ f_\epsilon(t) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

and is such that $\int_0^\infty \delta_a(t)\, dt = 1$ (area of strip = 1)

Graphically , the Dirac's delta function is represented by a single verticle line with an arrow head at $t = a$.

It is important to remember that at all stages , including the limiting stage , the area of the strip is 1 .

Note that $\delta_a(t)$ is not a function in the ordinary sense as used in calculus , but a so called **generalized function or distribution** . However , because it is the limit of a piecewise continuous function , we may treat $\delta_a(t)$ as if it were a legitimate (valid) function .

In particular , if $a = 0$, the Dirac's delta function denoted by $\delta_0(t)$ is defined as

$$\delta_0(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

and is shown in figure (9.18) .

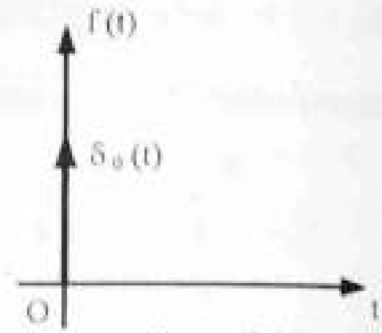


Figure (9.18)

9.12 LAPLACE TRANSFORM OF $\delta_a(t)$

THEOREM (9.13): Prove that $\mathcal{L}\{\delta_a(t)\} = e^{-as}$, $a \geq 0$, and $s > 0$.

PROOF:
$$\mathcal{L}\{\delta_a(t)\} = \mathcal{L}\left\{\underset{\epsilon \to 0}{\mathrm{Lt}}\ f_\epsilon(t)\right\}$$
$$= \underset{\epsilon \to 0}{\mathrm{Lt}}\ \mathcal{L}\{f_\epsilon(t)\}$$
$$= \underset{\epsilon \to 0}{\mathrm{Lt}}\ \frac{e^{-as}(1 - e^{-\epsilon s})}{\epsilon s} \quad [\text{ using equation (1) on page (17) }]$$
$$= \underset{\epsilon \to 0}{\mathrm{Lt}}\ e^{-as}\left(\frac{s e^{-\epsilon s}}{s}\right) = e^{-as} \quad [\text{ using L ' Hospital's rule }]$$

In particular , $\mathcal{L}\{\delta_0(t)\} = 1$.

9.13 PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic with period $P\ (>0)$, if

$f(t+P) = f(t)$ for all $t \geq 0$.

The least value of P is called the **least period** or simply the **period** of $f(t)$.

EXAMPLE (10): (i) The function $\sin t$ has periods $2\pi, 4\pi, 6\pi, \ldots\ldots$

since $\sin(t+2\pi) = \sin(t+4\pi) = \ldots\ldots = \sin t$. However, 2π is the least period or the period of $\sin t$. Similarly, the period of $\cos t$ is 2π, while the period of $\tan t$ is π, since $\cos(t+2\pi) = \cos t$ and $\tan(t+\pi) = \tan t$. Thus graphs of these functions are shown in figure (9.19).

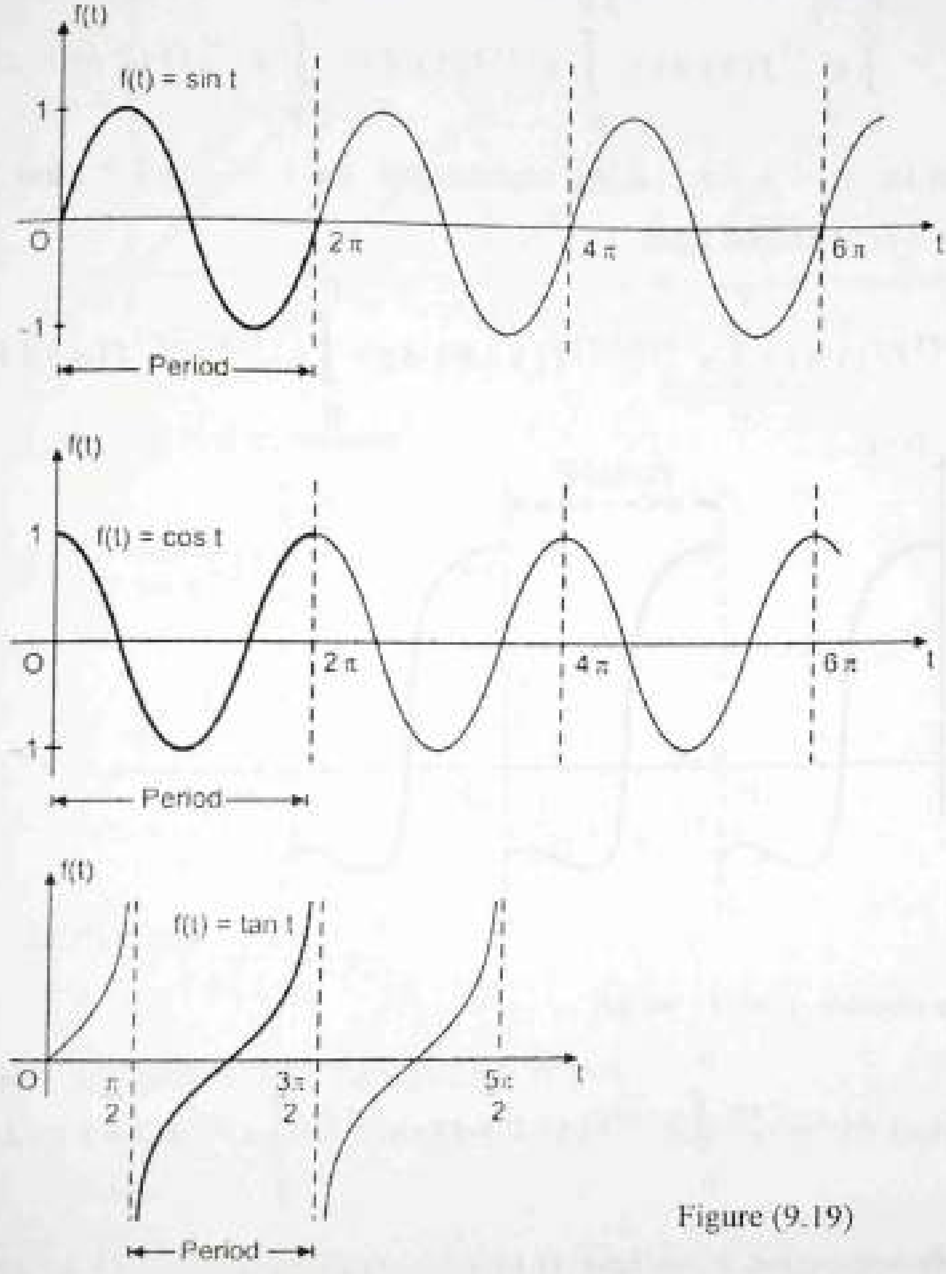


Figure (9.19)

(ii) The period of $\sin nt$ or $\cos nt$, where n is a positive integer, is $\dfrac{2\pi}{n}$,

since $\sin n\left(t+\dfrac{2\pi}{n}\right) = \sin nt$ and $\cos n\left(t+\dfrac{2\pi}{n}\right) = \cos nt$.

THEOREM (9.14): If $f(t)$ is periodic with period P and piecewise continuous on $t \geq 0$,

$$\textbf{then } \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sP}} \int_0^P e^{-st} f(t)\, dt, \quad s > 0.$$

PROOF: We have

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= \int_0^{P} e^{-st} f(t)\, dt + \int_P^{2P} e^{-st} f(t)\, dt + \int_{2P}^{3P} e^{-st} f(t)\, dt + \ldots\ldots\ldots$$

In the second integral let $t = y + P$, in the third integral let $t = y + 2P$, and, in general, let $t = y + nP$ in the $(n+1)$ st integral, then

$$\mathcal{L}\{f(t)\} = \int_0^{P} e^{-st} f(t)\, dt + \int_0^{P} e^{-s(y+P)} f(y+P)\, dy + \int_0^{P} e^{-s(y+2P)} f(y+2P)\, dy + \ldots\ldots$$

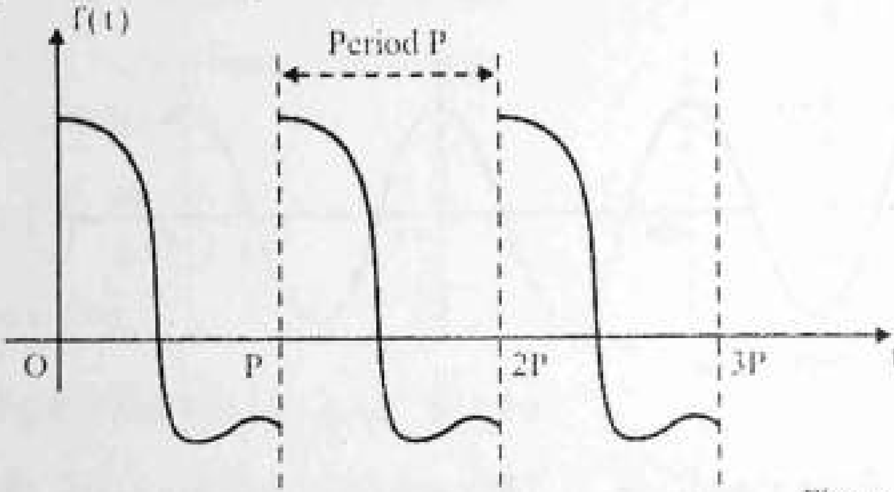


Figure (9.20)

Replacing the dummy variable y by t, we get

$$= \int_0^{P} e^{-st} f(t)\, dt + e^{-sP} \int_0^{P} e^{-st} f(t+P)\, dt + e^{-2sP} \int_0^{P} e^{-st} f(t+2P)\, dt + \ldots\ldots$$

Since $f(t)$ is periodic with period P, we have $f(t+P) = f(t+2P) = \ldots\ldots\ldots = f(t)$ and thus

$$\mathcal{L}\{f(t)\} = (1 + e^{-sP} + e^{-2sP} + \ldots\ldots\ldots) \int_0^{P} e^{-st} f(t)\, dt$$

The series within brackets is a G.P. series whose sum is $\frac{1}{1 - e^{-sP}}$. Hence

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sP}} \int_0^{P} e^{-st} f(t)\, dt, \quad s > 0.$$

EXAMPLE (11): **Find the Laplace transform of the following periodic function :**

$$f(t) = t, \qquad f(t+2\pi) = f(t)$$

SOLUTION: The graph of the given function is shown in figure (9.21) .

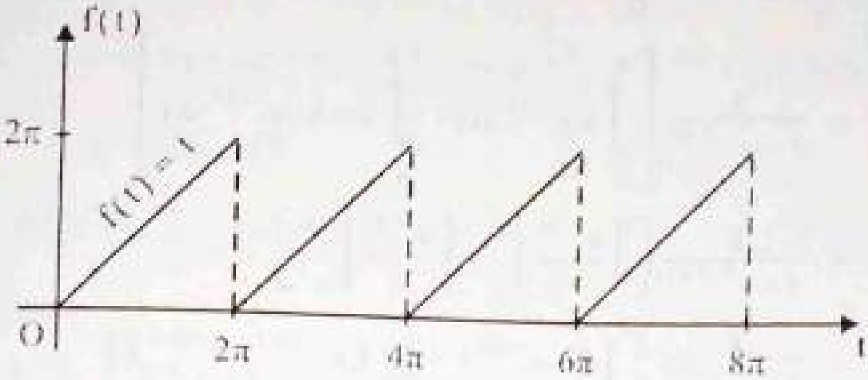


Figure (9.21)

By theorem (9.14) , since $P = 2\pi$, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}\, t\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{e^{-st} t}{s} - \frac{e^{-st}}{s^2} \right]_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{2\pi}{s} e^{-2\pi s} + \frac{1}{s^2}(1-e^{-2\pi s}) \right]$$

$$= \frac{1}{s^2} - \frac{2\pi e^{-2\pi s}}{s(1-e^{-2\pi s})}$$

EXAMPLE (12): **PERIODIC RECTANGULAR WAVE**

Find the Laplace transform of the rectangular wave shown in figure (9.22) below.

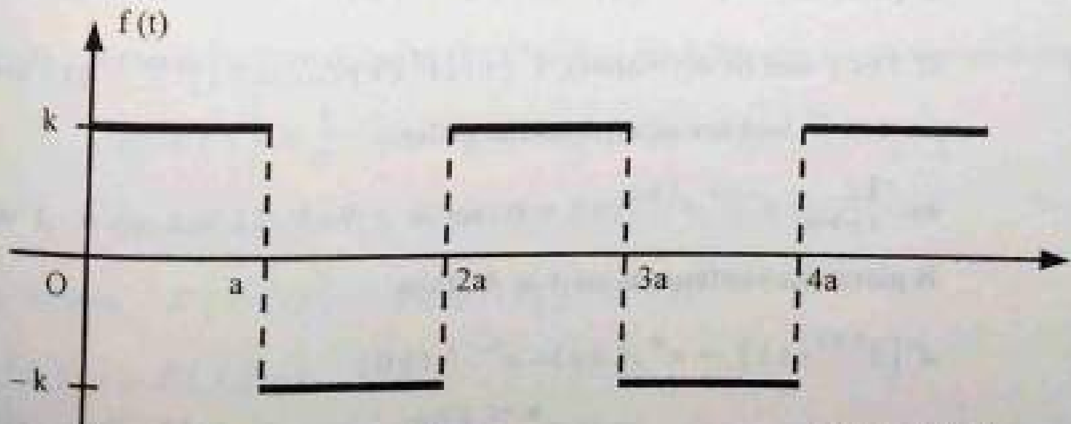


Figure (9.22)

SOLUTION: Since $P = 2a$, we obtain from theorem (9.14),

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} k e^{-st}\,dt + \int_a^{2a} (-k) e^{-st}\,dt \right]$$

$$= \frac{k}{1-e^{-2as}} \left[\left| \frac{e^{-st}}{-s} \right|_0^a + \left| \frac{e^{-st}}{s} \right|_a^{2a} \right]$$

$$= \frac{k}{1-e^{-2as}} \frac{1}{s} \left[(-e^{-as}+1) + (e^{-2as}-e^{-as}) \right]$$

$$= \frac{k}{s} \frac{1}{1-e^{-2as}} (1-e^{-as})^2 = \frac{k}{s} \left(\frac{1-e^{-as}}{1+e^{-as}} \right)$$

$$= \frac{k}{s} \frac{e^{-as/2}(e^{as/2}-e^{-as/2})}{e^{-as/2}(e^{as/2}+e^{-as/2})} = \frac{k}{s} \tanh \frac{as}{2}$$

9.14 LAPLACE TRANSFORMS OF DERIVATIVES

THEOREM (9.15): (i) If $f(t)$ is continuous on $t \geq 0$ and is of exponential order

i.e. $\underset{t \to \infty}{\mathrm{Lt}}\, e^{-st} f(t) = 0$, while $f'(t)$ is piecewise continuous on $t \geq 0$,

then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = s\bar{f}(s) - f(0)$

(ii) If $f(t)$ and $f'(t)$ are continuous on $t \geq 0$ and are of exponential order

i.e. $\underset{t \to \infty}{\mathrm{Lt}}\, e^{-st} f(t) = 0 = \underset{t \to \infty}{\mathrm{Lt}}\, e^{-st} f'(t)$, while $f''(t)$ is piecewise continuous on $t \geq 0$, then

$$\mathcal{L}\{f''(t)\} = s^2 \bar{f}(s) - s f(0) - f'(0).$$

(iii) If $f(t)$ and its derivatives $f'(t), f''(t), \ldots\ldots\ldots, f^{(n-1)}(t)$ are continuous on $t \geq 0$ and are of exponential order

i.e. $\underset{t \to \infty}{\mathrm{Lt}}\, e^{-st} f^{(k)}(t) = 0$, for $k = 0, 1, 2, \ldots\ldots\ldots\ldots, n-1$ while $f^{(n)}(t)$ is piecewise continuous on $t \geq 0$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots\ldots\ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

PROOF: (i) By definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \left|e^{-st} f(t)\right|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\} \quad \left(\text{since } \underset{t\to\infty}{\text{Lt}}\ e^{-st} f(t) = 0\right)$$

$$= s\,\bar{f}(s) - f(0)$$

(ii) By part (i) $\mathcal{L}\{u'(t)\} = s\mathcal{L}\{u(t)\} - u(0)$

Let $u(t) = f'(t)$, then

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left[s\,\bar{f}(s) - f(0)\right] - f'(0)$$

$$= s^2\,\bar{f}(s) - s\,f(0) - f'(0)$$

(iii) To prove the generalized result, we use the method of induction.

The result is true for $n = 1, 2$ as proved above.

Assume that the result is true for $n = k$.

i.e. $\mathcal{L}\{f^{(k)}(t)\} = s^k\,\bar{f}(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \ldots\ldots\ldots - f^{(k-1)}(0)$

Now by part (i)

$$\mathcal{L}\{f^{(k+1)}(t)\} = s\mathcal{L}\{f^{(k)}(t)\} - f^{(k)}(0)$$

$$= s\left[s^k\bar{f}(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \ldots\ldots\ldots - f^{(k-1)}(0)\right] - f^{(k)}(0)$$

$$= s^{k+1}\bar{f}(s) - s^k f(0) - s^{k-1} f'(0) - \ldots\ldots\ldots - s f^{(k-1)}(0) - f^{(k)}(0)$$

which shows that the result is true for $n = k+1$. Hence by the principle of induction, the result is true for all positive integral values of n.

EXAMPLE (13): Use the above theorem (9.15) to find the following Laplace transforms :

(i) $\mathcal{L}\{1\} = \frac{1}{s}$ (ii) $\mathcal{L}\{t\} = \frac{1}{s^2}$ (iii) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

SOLUTION: (i) Let $f(t) = 1$, therefore $f(0) = 1$ and $f'(t) = 0$,

hence from the formula $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$,

we have $\mathcal{L}\{0\} = s\,\mathcal{L}\{1\} - 1$

or $0 = s\,\mathcal{L}\{1\} - 1$ i.e. $\mathcal{L}\{1\} = \frac{1}{s}$

(ii) Let $f(t) = t$, therefore $f(0) = 0$ and $f'(t) = 1$

Hence from $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

we get $\mathcal{L}\{1\} = s\,\mathcal{L}\{t\} - 0$

or $\mathcal{L}\{t\} = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$

(iii) Let $f(t) = e^{at}$, therefore $f(0) = e^0 = 1$ and $f'(t) = a\,e^{at}$

Hence from the formula $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$

we get $\mathcal{L}\{a\,e^{at}\} = s\,\mathcal{L}\{e^{at}\} - 1$

or $a\,\mathcal{L}\{e^{at}\} = s\,\mathcal{L}\{e^{at}\} - 1$

or $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

EXAMPLE (14): Show that (i) $\mathcal{L}\{\sin^2\omega t\} = \frac{2\omega^2}{s(s^2+4\omega^2)}$ (ii) $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{(s^2+\omega^2)}$.

SOLUTION: (i) Let $f(t) = \sin^2\omega t$, therefore $f(0) = 0$

and $f'(t) = 2\omega\sin\omega t\cos\omega t = \omega\sin 2\omega t$.

Hence from the result $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$

we get $\mathcal{L}\{\omega\sin 2\omega t\} = s\,\mathcal{L}\{\sin^2\omega t\} - 0$

or $\mathcal{L}\{\sin^2\omega t\} = \frac{1}{s}\mathcal{L}\{\omega\sin 2\omega t\} = \frac{\omega}{s}\mathcal{L}\{\sin 2\omega t\}$

$$= \frac{\omega}{s}\left(\frac{2\omega}{s^2+4\omega^2}\right) = \frac{2\omega^2}{s(s^2+4\omega^2)}$$

(ii) Let $f(t) = \sin\omega t$, therefore $f(0) = 0$

and $f'(t) = \omega\cos\omega t$, $f'(0) = \omega$

$f''(t) = -\omega^2\sin\omega t$.

Hence from theorem (9.15) part (ii), we have

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

or $\mathcal{L}\{-\omega^2\sin\omega t\} = s^2\,\mathcal{L}\{\sin\omega t\} - 0 - \omega$

or $-\omega^2\,\mathcal{L}\{\sin\omega t\} = s^2\,\mathcal{L}\{\sin\omega t\} - \omega$

or $(s^2+\omega^2)\,\mathcal{L}\{\sin\omega t\} = \omega$

and so $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$

9.15 LAPLACE TRANSFORMS OF INTEGRALS

THEOREM (9.16): If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{\bar{f}(s)}{s}$.

PROOF: Let $u(t) = \int_0^t f(x)\,dx$ (1)

then by the property of definite integrals, if $F(u) = \int_a^u f(x)\,dx$ then $\frac{d}{du}F(u) = f(u)$.

we have from equation (1), $u'(t) = f(t)$ and $u(0) = 0$.

Taking the Laplace transform of both sides, we get $\mathcal{L}\{u'(t)\} = \mathcal{L}\{f(t)\}$

or $s\,\mathcal{L}\{u(t)\} - u(0) = \mathcal{L}\{f(t)\}$

or $\mathcal{L}\{u(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

or $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{\bar{f}(s)}{s}$

EXAMPLE (15): **Find the Laplace transforms of the following integrals :**

(i) $\int_0^t \cos 3x\,dx$ (ii) $\int_0^t (x^2 - x + e^{-x})\,dx$

SOLUTION: (i) Since $\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$

therefore $\mathcal{L}\left\{\int_0^t \cos 3x\,dx\right\} = \frac{1}{s}\left(\frac{s}{s^2+9}\right) = \frac{1}{s^2+9}$

(ii) Since $\mathcal{L}\{t^2 - t + e^{-t}\} = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{s+1} = \frac{s^3 - s^2 + s + 2}{s^3(s+1)}$

therefore, $\mathcal{L}\left\{\int_0^t (x^2 - x + e^{-x})\,dx\right\} = \frac{1}{s}\left(\frac{s^3 - s^2 + s + 2}{s^3(s+1)}\right) = \frac{s^3 - s^2 + s + 2}{s^4(s+1)}$

9.16 INVERSE LAPLACE TRANSFORMS

If the Laplace transform of a function $f(t)$ is $\bar{f}(s)$ i.e. $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $f(t)$ is called the inverse Laplace transform of $\bar{f}(s)$ and we write $f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$, where $\mathcal{L}^{-1}$ is the inverse Laplacian operator .

EXAMPLE (16): **Find the following inverse Laplace transforms :**

(i) $\mathcal{L}^{-1}\{t\}$ (ii) $\mathcal{L}^{-1}\{e^{-3t}\}$ (iii) $\mathcal{L}^{-1}\{\cos at\}$

SOLUTION: (i) Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, therefore $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$.

(ii) Since $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$, therefore $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$

(iii) Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, therefore $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

INVERSE LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

	$\bar{f}(s)$	$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^3}$	$\frac{t^2}{2}$
4	$\frac{1}{s^{n+1}}$, $n = 0, 1, 2, \ldots\ldots$	$\frac{t^n}{n!}$
5	$\frac{1}{s^{n+1}}$, $n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6	$\frac{1}{(s-a)}$	e^{at}
7	$\frac{1}{(s^2+a^2)}$	$\frac{\sin at}{a}$
8	$\frac{s}{(s^2+a^2)}$	$\cos at$
9	$\frac{1}{(s^2-a^2)}$	$\frac{\sinh at}{a}$
10	$\frac{s}{(s^2-a^2)}$	$\cosh at$

9.17 LINEARITY THEOREM FOR INVERSE LAPLACE TRANSFORMS

THEOREM (9.17): **Prove that $\mathcal{L}^{-1}$ is a linear operator, i.e. if C_1 and C_2 are any constants and $\bar{f}_1(s)$ and $\bar{f}_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then**

$$\mathcal{L}^{-1}\{C_1\bar{f}_1(s)+C_2\bar{f}_2(s)\} = C_1\mathcal{L}^{-1}\{\bar{f}_1(s)\}+C_2\mathcal{L}^{-1}\{\bar{f}_2(s)\}$$
$$= C_1 f_1(t)+C_2 f_2(t).$$

PROOF: By the Linearity theorem (9.1) on Laplace transforms, we have

$$\mathcal{L}\{C_1 f_1(t)+C_2 f_2(t)\} = C_1\mathcal{L}\{f_1(t)\}+C_2\mathcal{L}\{f_2(t)\}$$
$$= C_1\bar{f}_1(s)+C_2\bar{f}_2(s)$$

Therefore $\mathcal{L}^{-1}\{C_1\bar{f}_1(s)+C_2\bar{f}_2(s)\} = C_1 f_1(t)+C_2 f_2(t)$

$$= C_1\mathcal{L}^{-1}\{\bar{f}_1(s)\}+C_2\mathcal{L}^{-1}\{\bar{f}_2(s)\}$$

which show that $\mathcal{L}^{-1}$ is a linear operator.

EXAMPLE (17): Find $\mathcal{L}^{-1}\left\{\frac{4}{s-2}+\frac{3s}{s^2+16}-\frac{5}{s^2+4}\right\}$.

SOLUTION: By theorem (9.17), we have

$$\mathcal{L}^{-1}\left\{\frac{4}{s-2}+\frac{3s}{s^2+16}-\frac{5}{s^2+4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}-5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$
$$= 4e^{2t}+3\cos 4t-\frac{5}{2}\sin 2t$$

9.18 INVERSE VERSION OF THEOREM (9.8)

THEOREM (9.18): **If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$, then $\mathcal{L}^{-1}\{\bar{f}(s-a)\} = e^{at}f(t)$.**

PROOF: This follows at once from theorem (9.8) by shifting the operator on the other side.

EXAMPLE (18): Find (i) $\mathcal{L}^{-1}\left\{\frac{1}{(s+9)^2}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$ (iii) $\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2-4}\right\}$

SOLUTION: (i) Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$, hence by theorem (9.18), $\mathcal{L}^{-1}\left\{\frac{1}{(s+9)^2}\right\} = e^{-9t}t$

(ii) Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$, hence $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} = e^{-2t}\cos t$

(iii) Since $\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2-4}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} = \cosh 2t$, $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{\sinh 2t}{2}$, therefore

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2-4}\right\} = e^t \cosh 2t + e^t \frac{\sinh 2t}{2}$$

THEOREM (9.19): Prove that

(i) $$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}$$

(ii) $$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty \bar{f}(u)\,du\right\}$$

PROOF: From theorem (9.9), we have $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$

Also, from theorem (9.10), we have $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u)\,du$

From these equations, the required results follow by shifting the Laplacian operator sign on the right side

EXAMPLE (19): Find the following inverse Laplace transforms :

(i) $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\ln\left(\frac{s+5}{s-5}\right)\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)^2}\right\}$. Using theorem (9.19), we have

$$f(t) = t\mathcal{L}^{-1}\left\{\int_s^\infty \left\{\frac{4}{(u+1)^2}\right\}du\right\} = t\mathcal{L}^{-1}\left|-\frac{4}{u+1}\right|_s^\infty$$

$$= t\mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} = 4te^{-t}$$

(ii) $\mathcal{L}^{-1}\left\{\ln\left(\frac{s+5}{s-5}\right)\right\}$. Using theorem (9.19), we have

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\ln\left(\frac{s+5}{s-5}\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}[\ln(s+5)-\ln(s-5)]\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s+5}-\frac{1}{s-5}\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{10}{s^2-25}\right\} = \frac{2\sinh 5t}{t}$$

INVERSE VERSION OF THEOREM (9.11)

THEOREM (9.20): **Prove that** $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t).$

PROOF: This follows at once from theorem (9.11) by shifting the operator on the other side.

INVERSE VERSION OF THEOREM (9.12)

THEOREM (9.21): **If** $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$, **then** $\mathcal{L}^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)\,u_a(t).$

PROOF: This follows at once from theorem (9.12) by shifting the operator on the other side.

EXAMPLE (20): **Find** (i) $\mathcal{L}^{-1}\left\{\frac{s\,e^{-2s}}{s^2+16}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{(s-2)^4}\right\}$

SOLUTION: (i) Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$ and $a = 2$, hence by theorem (9.21)

$$\mathcal{L}^{-1}\left\{\frac{s\,e^{-2s}}{s^2+16}\right\} = \cos 4(t-2)\,u_2(t) = \begin{cases} 0 \;, & 0 \le t < 2 \\ \cos 4(t-2) \;, & t \ge 2 \end{cases}$$

(ii) Since $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = \frac{e^{2t}t^3}{3!}$, and $a = 6$, hence by theorem (9.21)

$$\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{(s-2)^4}\right\} = \frac{e^{2(t-6)}(t-6)^3}{3!}\,u_6(t)$$

INVERSE VERSION OF THEOREM (9.16)

THEOREM (9.22): **If** $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$, **then** $\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(x)\,dx.$

PROOF: By theorem (9.16) on Laplace transforms, we have

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{\bar{f}(s)}{s}. \text{ Therefore } \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(x)\,dx$$

EXAMPLE (21): **Find the following inverse Laplace transforms :**

(i) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\}$

SOLUTION: (i) Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}$, therefore, by theorem (9.22), we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \frac{1}{\omega}\int_0^t \sin \omega x\,dx = \frac{1}{\omega^2}(1-\cos \omega t)$$

(ii) $$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s \cdot s(s^2+\omega^2)}\right\}$$

$$= \frac{1}{\omega^2}\int_0^t (1-\cos\omega x)\,dx = \frac{1}{\omega^2}\left(t-\frac{\sin\omega t}{\omega}\right)$$

9.19 CONVOLUTION

Let $f(t)$ and $g(t)$ be two piecewise continuous functions. Then the convolution of $f(t)$ and $g(t)$ denoted by $f * g$ is defined as

$$f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$$

Note that since u is a dummy variable of integration, $f * g$ is a function of t.

EXAMPLE (22): Find the following convolution :

(i) $1 * 1$ **(ii)** $1 * \cos t$

(iii) $t * e^t$ **(iv)** $u_1(t) * t$

SOLUTION: (i) By definition $1 * 1 = \int_0^t (1)(1)\,du = \int_0^t du = |u|_0^t = t$

(ii) By definition $1 * \cos t = \int_0^t (1)\cos(t-u)\,du = |-\sin(t-u)|_0^t = \sin t$

(iii) By definition $t * e^t = \int_0^t u e^{t-u}\,du = e^t\int_0^t u e^{-u}\,du$

$$= e^t\left|-ue^{-u}-e^{-u}\right|_0^t = e^t(-te^{-t}-e^{-t}+1) = e^t-t-1$$

(iv) Since $u_1(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$, therefore

if $0 \le t < 1$, $u_1(t) * t = 0 * t = 0$

if $t \ge 1$, $u_1(t) * t = 1 * t = \int_1^t (1)(t-u)\,du = \left|tu-\frac{u^2}{2}\right|_1^t = \frac{(t-1)^2}{2}$

In this example, $f(t)$ is a discontinuous function at $t = 1$, but $f * g$ is continuous (with value 0 at $t = 1$). In general, the convolution of two continuous or piecewise continuous functions is continuous.

9.20 PROPERTIES OF CONVOLUTION

THEOREM (9.23): **Prove that $f(t)*g(t) = g(t)*f(t)$.**

PROOF: We have

$$f(t)*g(t) = \int_0^t f(u)g(t-u)\,du \tag{1}$$

Let $t-u = v$, then $du = -dv$, and $t \le v \le 0$. Thus equation (1) becomes

$$f(t)*g(t) = \int_t^0 f(t-v)g(v)(-dv) = \int_0^t f(t-v)g(v)\,dv$$

$$= \int_0^t g(v)f(t-v)\,dv = g(t)*f(t)$$

This shows that the convolution of $f(t)$ and $g(t)$ obeys the commutative law of algebra.

NOTE: The Convolution also obeys the following properties :

(i) $f*(g*h) = (f*g)*h$ (Associative Law)

(ii) $f*(g_1+g_2) = f*g_1 + f*g_2$ (Distributive Law)

(iii) $f*0 = 0*f = 0$

However, $1*g \neq g$ in general. For example, if $g(t) = t$, then $1*g = \int_0^t 1.(t-u)\,du = \frac{t^2}{2}$

9.21 CONVOLUTION THEOREM

THEOREM (9.24): **If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ and $\mathcal{L}^{-1}\{\bar{g}(s)\} = g(t)$, then**

$$\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t)*g(t).$$

PROOF: The required result follows if we can show

$$\mathcal{L}\{f(t)*g(t)\} = \bar{f}(s)\bar{g}(s)$$

By definition, $\mathcal{L}\{f(t)*g(t)\} = \int_0^\infty e^{-st}[f(t)*g(t)]\,dt$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)\,du\right]dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u) g(t-u)\, du\, dt \qquad (1)$$

since e^{-st} is constant w.r.t. u.

The region of integration for the double integral in equation (1) in the $t\,u$ - plane is shown shaded in figure (9.23). Interchanging the order of integration, we get

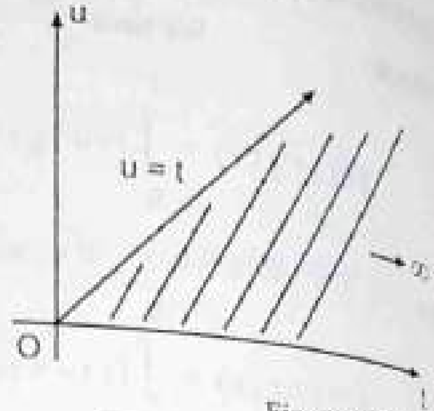


Figure (9.23)

$$\mathcal{L}\{f(t)*g(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\, dt\, du$$

$$= \int_0^\infty f(u) e^{-su} \left[\int_u^\infty e^{-s(t-u)} g(t-u)\, dt \right] du$$

Let $t - u = v$ in the inner integral, then it no longer depends on u, and we have

$$\mathcal{L}\{f(t)*g(t)\} = \int_0^\infty e^{-su} f(u) \left[\int_0^\infty e^{-sv} g(v)\, dv \right] du$$

$$= \left(\int_0^\infty e^{-su} f(u)\, du \right) \left(\int_0^\infty e^{-sv} g(v)\, dv \right) = \bar{f}(s)\bar{g}(s)$$

Hence $\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t)*g(t)$ as required

ALTERNATIVE METHOD

By definition,

$$\bar{f}(s)\bar{g}(s) = \left(\int_0^\infty e^{-su} f(u)\, du \right) \left(\int_0^\infty e^{-sv} g(v)\, dv \right)$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u) g(v)\, dv\, du$$

Let $u + v = t$, with u fixed, then $dv = dt$ and thus

$$\bar{f}(s)\bar{g}(s) = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\, dt\, du \qquad (2)$$

The region of integration for the double integral in equation (2) is shown in figure (9.23). Interchanging the order of integration, equation (2) becomes

$$\bar{f}(s)\bar{g}(s) = \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)\, g(t-u)\, du\right] dt$$

$$= \int_0^\infty e^{-st}\left[f(t) * g(t)\right] dt = \mathcal{L}\{f(t) * g(t)\}$$

and hence $\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t) * g(t)$.

EXAMPLE (23): **Using Convolution theorem, find**

(i) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s-a)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$

SOLUTION: (i) Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$

hence by the Convolution theorem (9.24), we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = t * e^{at}$$

$$= \int_0^t u\, e^{a(t-u)}\, du = e^{at}\int_0^t u\, e^{-au}\, du$$

$$= e^{at}\left|\frac{1}{a^2} - \frac{t\,e^{-at}}{a} - \frac{e^{-at}}{a^2}\right|_0^t = \frac{1}{a^2}(e^{at} - at - 1)$$

(ii) Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = t\,e^{-t}$

hence by the Convolution theorem (9.24), we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = t * t\,e^{-t} = t\,e^{-t} * t$$

$$= \int_0^t u\, e^{-u}(t-u)\, du = \int_0^t e^{-u}(ut - u^2)\, du$$

$$= \left|(ut-u^2)(-e^{-u})-(t-2u)e^{-u}+2e^{-u}\right|_0^t$$

$$= te^{-t}+2e^{-t}+t-2$$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$

Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$

hence , by the convolution theorem (9.24) , we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = e^{-t} * \sin t = \sin t * e^{-t}$$

$$= \int_0^t \sin u\, e^{-(t-u)}\, du = e^{-t}\int_0^t e^{u} \sin u\, du$$

$$= e^{-t}\left|\frac{e^{u}(\sin u-\cos u)}{2}\right|_0^t$$

$$= e^{-t}\left[\frac{e^{t}(\sin t-\cos t)}{2}+\frac{1}{2}\right]$$

$$= \frac{1}{2}(\sin t-\cos t+e^{-t})$$

9.22 LAPLACE INVERSE BY PARTIAL FRACTIONS

THEOREM (9.25): Show that

$$\mathcal{L}^{-1}\left\{\frac{1}{[(s-\alpha)^2+\beta^2]^2}\right\} = \frac{1}{2\beta^3}e^{\alpha t}(\sin\beta t-\beta t\cos\beta t)$$

$$\mathcal{L}^{-1}\left\{\frac{s-\alpha}{[(s-\alpha)^2+\beta^2]^2}\right\} = \frac{1}{2\beta}e^{\alpha t}t\sin\beta t.$$

PROOF: We can show with the help of convolutions that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\beta^2)^2}\right\} = \frac{1}{2\beta^3}(\sin\beta t-\beta t\cos\beta t)$$

and $$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\beta^2)^2}\right\} = \frac{1}{2\beta}t\sin\beta t$$

The required results then follow by the use of the first sifting theorem on inverse Laplace transforms.

EXAMPLE (24): Find the inverse Laplace transform of $\dfrac{s^2+2s}{(s^2+2s+2)^2}$.

SOLUTION: We have

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s}{(s^2+2s+2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s^2+2s+2)-2}{(s^2+2s+2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2} - \frac{2}{(s^2+2s+2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1} - \frac{2}{[(s+1)^2+1]^2}\right\}$$

$$= e^{-t}\sin t - 2\left(\frac{1}{2}\right)(\sin t - t\cos t)e^{-t}$$

$$= e^{-t}\, t\cos t$$

9.23 LAPLACE INVERSION FORMULA

THEOREM (9.26): **If $\mathcal{L}\{f(t)\} = \overline{f}(s)$, then prove that**

$$f(t) = \mathcal{L}^{-1}\{\overline{f}(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\,\overline{f}(s)\, ds$$

where all the singularities of $\overline{f}(s)$ lie to the left of the line $x = \gamma$ for some real constant γ.

(The Laplace inversion formula is also called complex inversion formula).

PROOF: Let AB be the line $x = \gamma$ such that $A = (\gamma, y)$ and $B = (\gamma, -y)$. Draw a semi-circular arc Γ of radius y with AB as diameter lying to the right of AB. Let C be a closed contour consisting of the line AB and the arc Γ i.e. $C = AB + \Gamma$.

Let $\overline{f}(z)$ be a complex-valued function.

Since all the singularities of $\overline{f}(z)$ lie to the left of the line $x = \gamma$, therefore $\overline{f}(z)$ is analytic within and on the contour C. Hence if s is any point within C, then by the Cauchy's integral formula

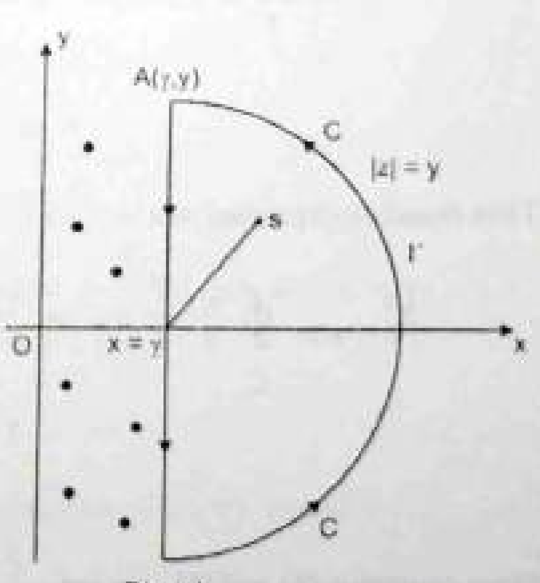


Figure (9.24)

$$\overline{f}(s) = \frac{1}{2\pi i}\oint_C \frac{\overline{f}(z)}{z-s}\, dz \qquad (1)$$

Regarding $\overline{f}(z)$ as the Laplace transform of $f(t)$, then

$$\overline{f}(z) = \int_0^{\infty} e^{-zt} f(t)\, dt \qquad (2)$$

From equations (1) and (2), we get

$$\bar{f}(s) = \frac{1}{2\pi i}\oint_C \left[\int_0^\infty e^{-zt} f(t)\, dt\right]\frac{1}{z-s}\, dz$$

Interchanging the order of integration, we get

$$\bar{f}(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left[\oint_C e^{-zt}\frac{1}{z-s}\, dz\right] dt \qquad (3)$$

Since y is the radius of the semi – circle, therefore the equation of the circular arc Γ is $|z| = y$.

Taking the limit of both sides of equation (3) as $|z| \to \infty$, we get

$$\underset{|z|\to\infty}{\mathrm{Lt}}\ \bar{f}(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left[\underset{|z|\to\infty}{\mathrm{Lt}}\ \oint_C \frac{e^{-zt}}{z-s}\, dz\right] dt \qquad (4)$$

Now $\underset{|z|\to\infty}{\mathrm{Lt}}\ \oint_C \frac{e^{-zt}}{z-s}\, dz = \underset{|z|\to\infty}{\mathrm{Lt}}\ \int_{AB} \frac{e^{-zt}}{z-s}\, dz + \underset{|z|\to\infty}{\mathrm{Lt}}\ \int_{\Gamma} \frac{e^{-zt}}{z-s}\, dz$

$$= \underset{y\to\infty}{\mathrm{Lt}}\ \int_{\gamma+iy}^{\gamma-iy} \frac{e^{-zt}}{z-s}\, dz + \underset{|z|\to\infty}{\mathrm{Lt}}\ \int_{\Gamma} \frac{e^{-zt}}{z-s}\, dz \qquad (5)$$

But $\underset{|z|\to\infty}{\mathrm{Lt}}\ \int_{\Gamma} \frac{e^{-zt}}{z-s}\, dz = \underset{|z|\to\infty}{\mathrm{Lt}}\ \int_{\Gamma} e^{zt}\frac{1}{z+s}\, dz$ (replacing $z = -z$)

$$= 0 \qquad \left(\text{since } \underset{|z|\to\infty}{\mathrm{Lt}}\ \int_{\Gamma} e^{zt} f(z)\, dz = 0\right)$$

Thus equation (5) reduces to

$$\underset{|z|\to\infty}{\mathrm{Lt}}\ \oint_C \frac{e^{-zt}}{z-s}\, dz = \underset{y\to\infty}{\mathrm{Lt}}\ \int_{\gamma+iy}^{\gamma-iy} \frac{e^{-zt}}{z-s}\, dz$$

$$= \int_{\gamma+i\infty}^{\gamma-i\infty} \frac{e^{-zt}}{z-s}\, dz = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-zt}}{s-z}\, dz \qquad (6)$$

From equations (4) and (6), we get

$$\bar{f}(s) = \frac{1}{2\pi i}\int_0^\infty f(t)\left[\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{-zt}}{s-z}\, dz\right] dt$$

Again interchanging the order of integration, we get

$$\bar{f}(s) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\left[\int_0^{\infty} e^{-zt} f(t)\,dt\right]\frac{1}{s-z}\,dz$$

$$= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(z)\frac{1}{s-z}\,dz$$

Taking the inverse Laplace transform of both sides, we get

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(z)\,\mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\}dz = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt}\bar{f}(z)\,dz$$

Replacing the dummy variable z by s, we get

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\bar{f}(s)\,ds$$

9.24 USE OF RESIDUE THEOREM IN FINDING INVERSE LAPLACE TRANSFORM

THEOREM (9.27): Suppose that the only singularities of $\bar{f}(s)$ are the poles all of which lie to the left of the line $x = \gamma$, for some real constant γ. Suppose further that

$$\underset{R\to\infty}{\mathrm{Lt}}\int_{\Gamma} e^{st}\bar{f}(s)\,ds = 0. \text{ Prove that}$$

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \sum \text{residues of } e^{st}\bar{f}(s) \text{ at all the poles of } \bar{f}(s)$$

PROOF: Take points A and B on the line $x = \gamma$ such that $A = (\gamma, y)$ and $B = (\gamma, -y)$ and form the circular arc Γ of radius R with centre at O. Thus $C = \Gamma + BA$

Now by Cauchy's residue theorem

$$\frac{1}{2\pi i}\oint_C e^{st}\bar{f}(s)\,ds = \sum \text{residues of } e^{st}\bar{f}(s) \text{ at all the poles of } \bar{f}(s) \text{ inside C.}$$

or

$$\frac{1}{2\pi i}\int_{BA} e^{st}\bar{f}(s)\,ds + \frac{1}{2\pi i}\int_{\Gamma} e^{st}\bar{f}(s)\,ds = \sum \text{residues of } e^{st}\bar{f}(s) \text{ at all the poles of } \bar{f}(s) \text{ inside C.}$$

Now as $R \to \infty$, $y \to \infty$. Therefore taking the limit of both sides as $R \to \infty$, we get

$$\underset{y\to\infty}{\mathrm{Lt}} \frac{1}{2\pi i}\int_{\gamma-iy}^{\gamma+iy} e^{st}\bar{f}(s)\,ds + \underset{R\to\infty}{\mathrm{Lt}} \frac{1}{2\pi i}\int_{\Gamma} e^{st}\bar{f}(s)\,ds$$

$= \sum$ residues of $e^{st}\bar{f}(s)$ at all the poles of $\bar{f}(s)$ to the left of the line $x = \gamma$ (1)

But by assumption $\underset{R\to\infty}{\mathrm{Lt}} \frac{1}{2\pi i}\int_{\Gamma} e^{st}\bar{f}(s)\,ds = 0$

Thus equation (1) becomes

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\bar{f}(s)\,ds$$

$= \sum$ residues of $e^{st}\bar{f}(s)$ at all the poles of $\bar{f}(s)$

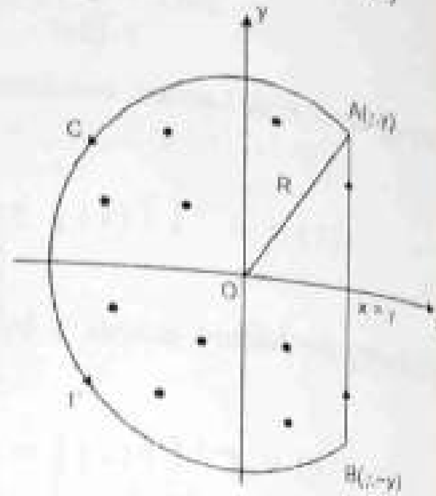


Figure (9.25)

or $f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \sum$ residues of $e^{st}\bar{f}(s)$ at all the poles of $\bar{f}(s)$

METHOD FOR CALCULATING INVERSE LAPLACE TRANSFORM

STEP (1): Find all the poles of $\bar{f}(s)$

STEP (2): Find the corresponding residues of $e^{st}\bar{f}(s)$ at these poles.

STEP (3): Use the formula $f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \sum$ residues of $e^{st}\bar{f}(s)$

EXAMPLE (25): **Find the following inverse Laplace transforms using Laplace inversion formula**

(i) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s+3)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

SOLUTION: (i) Here $\bar{f}(s) = \frac{s}{(s-2)(s+3)}$

The poles of $\bar{f}(s)$ are the simple poles $s = 2$, and $s = -3$.

The residues of $e^{st}\bar{f}(s) = \frac{e^{st}s}{(s-2)(s+3)}$ at these poles are :

$$\underset{s=2}{\mathrm{Res}}\, e^{st}\bar{f}(s) = \underset{s\to 2}{\mathrm{Lt}}\,(s-2)\frac{s\,e^{st}}{(s-2)(s+3)} = \underset{s\to 2}{\mathrm{Lt}}\,\frac{s\,e^{st}}{s+3} = \frac{2}{5}e^{2t}$$

$$\underset{s=-3}{\mathrm{Res}}\, e^{st}\bar{f}(s) = \underset{s\to -3}{\mathrm{Lt}}\,(s+3)\frac{s\,e^{st}}{(s-2)(s+3)} = \underset{s\to -3}{\mathrm{Lt}}\,\frac{s\,e^{st}}{s-2} = \frac{3}{5}e^{-3t}$$

Hence by the Laplace inversion formula , we get

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s+3)}\right\} = \sum \text{residues of } e^{st}\,\overline{f}(s) = \frac{2}{5}e^{2t} + \frac{3}{5}e^{-3t}$$

(ii) Here $\overline{f}(s) = \dfrac{1}{s^2+4}$

The poles of $\overline{f}(s)$ are the simple poles $s = \pm 2i$.

The residues of $e^{st}\,\overline{f}(s) = \dfrac{e^{st}}{s^2+4}$ at these poles are :

$$\operatorname*{Res}_{s=2i} e^{st}\,\overline{f}(s) = \operatorname*{Lt}_{s\to 2i} (s-2i)\frac{e^{st}}{(s+2i)(s-2i)} = \operatorname*{Lt}_{s\to 2i} \frac{e^{st}}{s+2i} = \frac{e^{2it}}{4i}$$

$$\operatorname*{Res}_{s=-2i} e^{st}\,\overline{f}(s) = \operatorname*{Lt}_{s\to -2i} (s+2i)\frac{e^{st}}{(s+2i)(s-2i)} = \operatorname*{Lt}_{s\to -2i} \frac{e^{st}}{s-2i} = -\frac{e^{-2it}}{4i}$$

Hence by the Laplace inversion formula , we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \sum \text{residues of } e^{st}\,\overline{f}(s)$$

$$= \frac{e^{2it}}{4i} - \frac{e^{-2it}}{4i} = \frac{1}{2}\left(\frac{e^{2it}-e^{-2it}}{2i}\right) = \frac{1}{2}\sin 2t$$

EXAMPLE (26): **Find the following inverse Laplace transforms using Laplace inversion formula**

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

SOLUTION: Here $\overline{f}(s) = \dfrac{1}{(s^2+4)^2}$

The poles of $\overline{f}(s)$ are $s = \pm 2i$ which are of order 2 each .

The residues of $e^{st}\,\overline{f}(s) = \dfrac{e^{st}}{(s^2+4)^2}$ at these poles are :

$$\operatorname*{Res}_{s=2i} e^{st}\,\overline{f}(s) = \operatorname*{Lt}_{s\to 2i} \frac{d}{ds}\left[\frac{(s-2i)^2 e^{st}}{(s-2i)^2(s+2i)^2}\right]$$

$$= \operatorname*{Lt}_{s\to 2i} \frac{d}{ds}\left[\frac{e^{st}}{(s+2i)^2}\right] = \operatorname*{Lt}_{s\to 2i}\left[\frac{(s+2i)te^{st}-2e^{st}}{(s+2i)^3}\right]$$

$$= \frac{4ite^{2it}-2e^{2it}}{(4i)^3} = -\frac{1}{16}te^{2it} + \frac{1}{32i}e^{2it}$$

$$\operatorname*{Res}_{s=-2i} e^{st}\,\overline{f}(s) = \operatorname*{Lt}_{s\to -2i} \frac{d}{ds}\left[\frac{(s+2i)^2 e^{st}}{(s-2i)^2(s+2i)^2}\right]$$

$$= \operatorname*{Lt}_{s\to -2i} \frac{d}{ds}\left[\frac{e^{st}}{(s-2i)^2}\right] = \operatorname*{Lt}_{s\to -2i}\left[\frac{(s-2i)te^{st}-2e^{st}}{(s-2i)^3}\right]$$

$$= \frac{-4ite^{-2it}-2e^{-2it}}{(-4i)^2} = -\frac{1}{16}te^{-2it}-\frac{1}{32i}e^{-2it}$$

Hence by the Laplace inversion formula , we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \sum \text{residues of } e^{st}\bar{f}(s)$$

$$= -\frac{1}{16}te^{2it}+\frac{1}{32i}e^{2it}-\frac{1}{16}te^{-2it}-\frac{1}{32i}e^{-2it}$$

$$= -\frac{1}{8}t\left(\frac{e^{2it}+e^{-2it}}{2}\right)+\frac{1}{16}\left(\frac{e^{2it}-e^{-2it}}{2i}\right)$$

$$= -\frac{1}{8}t\cos 2t+\frac{1}{16}\sin 2t$$

9.25 APPLICATION TO INITIAL – VALUE PROBLEMS (I.V.P)

An initial – value problem is a problem which involves a differential equation (ordinary or partial) along with the conditions specified at an initial time $t = 0$. Here we will restrict ourselves to the discussion of the problems involving ordinary differential equations . The following is the procedure to be followed for the solution of initial – value problems :

STEP (1): Take the Laplace transform of both sides of the given differential equation .

STEP (2): Use the initial conditions to obtain an algebraic equation and then solve this equation for the Laplace transform of the required solution .

STEP (3): Take the inverse Laplace transform to get the required solution .

We explain the method wit the help of following examples :

EXAMPLE (27): **Solve the initial – value problem :**

$$y''+y=0,\quad y(0)=1,\quad y'(0)=-1\quad [y=y(t)].$$

SOLUTION: **STEP (1):** Taking the Laplace transform of both sides of the differential equation , we get $[s^2\bar{y}(s)-sy(0)-y'(0)]+\bar{y}(s)=0$.

STEP (2): using the initial conditions , we get

$$s^2\bar{y}(s)-s+1+\bar{y}(s)=0$$

or $\quad (s^2+1)\bar{y}(s)=s-1$

or $\quad \bar{y}(s)=\frac{s-1}{s^2+1}=\frac{s}{s^2+1}-\frac{1}{s^2+1}$

STEP (3): Taking the inverse Laplace transform , we have

$$\mathcal{L}^{-1}\{\bar{y}(s)\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

or $\quad y(t)=\cos t-\sin t$

EXAMPLE (28): **Solve the initial – value problem :**

$$y'' - 9y = -8e^{t}; \quad y(0) = 0, \quad y'(0) = 10.$$

SOLUTION: **STEP (1):** Taking the Laplace transform of both sides of the differential equation ,

we get $\left[s^2\bar{y}(s) - s\,y(0) - y'(0)\right] - 9\bar{y}(s) = -8\left(\frac{1}{s-1}\right)$

STEP (2): Using the initial conditions , we get

$$s^2\bar{y}(s) - 10 - 9\bar{y}(s) = -\frac{8}{s-1}$$

or $$(s^2-9)\bar{y}(s) = 10 - \frac{8}{s-1} = \frac{10s-18}{s-1}$$

or $$\bar{y}(s) = \frac{10s-18}{(s-1)(s^2-9)} = \frac{10s-18}{(s-1)(s-3)(s+3)}$$

STEP (3): Taking the inverse Laplace transform we have

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{10s-18}{(s-1)(s-3)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s-3} - \frac{2}{s+3}\right\}$$

or $$y(t) = e^{t} + e^{3t} - 2e^{-3t}$$

EXAMPLE (29): **Solve the initial – value problem :**

$$y'' + y = 3\cos 2t; \quad y(0) = 0, \quad y'(0) = 0.$$

SOLUTION: **STEP (1):** Taking the Laplace transform of both sides of the differential equation ,

we get $\left[s^2\bar{y}(s) - s\,y(0) - y'(0)\right] + \bar{y}(s) = 3\left(\frac{s}{s^2+4}\right)$

STEP (2): Using the initial conditions , we have

$$s^2\bar{y}(s) + \bar{y}(s) = \frac{3s}{s^2+4}$$

or $$\bar{y}(s) = \frac{3s}{(s^2+1)(s^2+4)} = \frac{s}{s^2+1} - \frac{s}{s^2+4}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+4}\right\}$$

or $$y(t) = \cos t - \cos 2t$$

9.26 APPLICATION TO INTEGRAL EQUATIONS

An equation is called an **integral equation** if the unknown function occurs under the integral sign. Convolution also helps in solving such integral equations .

EXAMPLE (30): **Solve the integral equation** $y(t) = 1 + \int_0^t y(u)\sin(t-u)\,du$

SOLUTION: The given equation can be written as

$$y(t) = 1 + y(t) * \sin t$$

Taking the Laplace transform of both sides of this equation, we get

$$\bar{y}(s) = \frac{1}{s} + \mathcal{L}\{y(t) * \sin t\}$$

Using the convolution theorem, the above equation takes the form

$$\bar{y}(s) = \frac{1}{s} + \bar{y}(s) \cdot \frac{1}{s^2+1}$$

or $$\left(1 - \frac{1}{s^2+1}\right)\bar{y}(s) = \frac{1}{s}$$

or $$\frac{s^2}{s^2+1}\bar{y}(s) = \frac{1}{s} \quad \text{or} \quad \bar{y}(s) = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^3}\right\}$$

or $$y(t) = 1 + \frac{t^2}{2}$$

9.27 SYSTEMS OF DIFFERENTIAL EQUATIONS

The Laplace transformation may also be used for solving systems of differential equations. We explain the method with the following example.

EXAMPLE (31): **Solve the following system of differential equations :**

$$\left.\begin{aligned} y_1' &= 2y_1 - 3y_2 \\ y_2' &= y_2 - 2y_1 \end{aligned}\right\}; \quad y_1(0) = 8, \quad y_2(0) = 3.$$

SOLUTION: **STEP (1):** Taking the Laplace transforms of both the differential equations, we get

$$s\bar{y}_1(s) - y_1(0) = 2\bar{y}_1(s) - 3\bar{y}_2(s)$$

$$s\bar{y}_2(s) - y_2(0) = \bar{y}_2(s) - 2\bar{y}_1(s)$$

STEP (2): Using the initial conditions, we get

$$(s-2)\bar{y}_1(s) + 3\bar{y}_2(s) = 8$$

$$2\bar{y}_1(s) + (s-1)\bar{y}_2(s) = 3$$

Solving this system using Cramer's rule ,

$$\bar{y}_1(s) = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}}$$

$$= \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$\bar{y}_2(s) = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}}$$

$$= \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$y_1(t) = \mathcal{L}^{-1}\{\bar{y}_1(s)\} = 5e^{-t} + 3e^{4t}$$

$$y_2(t) = \mathcal{L}^{-1}\{\bar{y}_2(s)\} = 5e^{-t} - 2e^{4t}$$

EXAMPLE (32): **Solve the following system of differential equations using Laplace transforms :**

$$\left.\begin{aligned} y_1'' &= -5y_1 + 2y_2 \\ y_2'' &= 2y_1 - 2y_2 \end{aligned}\right\} \quad y_1(0) = 3,\ y_2(0) = 1,\ y_1'(0) = y_2'(0) = 0.$$

SOLUTION: **STEP (1):** Taking the Laplace transforms of both the differential equations , we get

$$s^2\bar{y}_1(s) - s\,y_1(0) - y_1'(0) = -5\bar{y}_1(s) + 2\bar{y}_2(s)$$

$$s^2\bar{y}_2(s) - s\,y_2(0) - y_2'(0) = 2\bar{y}_1(s) - 2\bar{y}_2(s)$$

STEP (2): Using the initial conditions , we get

$$(s^2+5)\bar{y}_1(s) - 2\bar{y}_2(s) = 3s$$

$$-2\bar{y}_1(s) + (s^2+2)\bar{y}_2(s) = s$$

Solving this system using Cramer's rule ,

$$\bar{y}_1(s) = \frac{\begin{vmatrix} 3s & -2 \\ s & s^2+2 \end{vmatrix}}{\begin{vmatrix} s^2+5 & -2 \\ -2 & s^2+2 \end{vmatrix}}$$

$$= \frac{3s^3+8s}{(s^2+1)(s^2+6)} = \frac{s}{s^2+1} + \frac{2s}{s^2+6}$$

$$\bar{y}_2(s) = \frac{\begin{vmatrix} s^2+5 & 3s \\ -2 & s \end{vmatrix}}{\begin{vmatrix} s^2+5 & -2 \\ -2 & s^2+2 \end{vmatrix}}$$

$$= \frac{s^3+11s}{(s^2+1)(s^2+6)} = \frac{2s}{s^2+1} - \frac{s}{s^2+6}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$y_1(t) = \mathcal{L}^{-1}\{\bar{y}_1(s)\} = \cos t + 2\cos\sqrt{6}\,t$$

$$y_2(t) = \mathcal{L}^{-1}\{\bar{y}_2(s)\} = 2\cos t - \cos\sqrt{6}\,t$$

EXAMPLE (33): **Solve the following initial – value problem using Laplace transforms :**

$$\left.\begin{aligned} 2y_1' - y_2' - y_3' &= 0 \\ y_1' + y_2' &= 4t+2 \\ y_2' + y_3 &= t^2+2 \end{aligned}\right\} \qquad y_1(0) = y_2(0) = y_3(0) = 0$$

SOLUTION: **STEP (1):** Taking the Laplace transforms of all the differential equations , we get

$$2s\bar{y}_1(s) - 2y_1(0) - s\bar{y}_2(s) - y_2(0) - s\bar{y}_3(s) + y_3(0) = 0$$

$$s\bar{y}_1(s) - y_1(0) + s\bar{y}_2(s) - y_2(0) = \frac{4}{s^2} + \frac{2}{s}$$

$$s\bar{y}_2(s) - y_2(0) + \bar{y}_3(s) = \frac{2}{s^3} + \frac{2}{s}$$

STEP (2): Using the initial conditions , we get

$$2s\bar{y}_1(s) - s\bar{y}_2(s) - s\bar{y}_3(s) = 0$$

$$s\bar{y}_1(s) + s\bar{y}_2(s) + 0 = \frac{4+2s}{s^2}$$

$$0 + s\bar{y}_2(s) + \bar{y}_3(s) = \frac{2+2s^2}{s^3}$$

Solving this system using Cramer's rule , we get

$$\bar{y}_1(s) = \frac{\begin{vmatrix} 0 & -s & -s \\ \frac{4+2s}{s^2} & s & 0 \\ \frac{2+2s^2}{s^3} & s & 1 \end{vmatrix}}{\begin{vmatrix} 2s & -s & -s \\ s & s & 0 \\ 0 & s & 1 \end{vmatrix}} = \frac{\frac{2(3-s)}{s}}{s^2(3-s)} = \frac{2}{s^3}$$

$$\bar{y}_2(s) = \frac{\begin{vmatrix} 2s & 0 & -s \\ s & \frac{4+2s}{s^2} & 0 \\ 0 & \frac{s+2s^2}{s^3} & 1 \end{vmatrix}}{\begin{vmatrix} 2s & -s & -s \\ s & s & 0 \\ 0 & s & 1 \end{vmatrix}} = \frac{\frac{2(s+1)(3-s)}{s}}{s^2(3-s)} = \frac{2(s+1)}{s^3} = \frac{2}{s^2} + \frac{2}{s^3}$$

$$\bar{y}_3(s) = \frac{\begin{vmatrix} 2s & -s & 0 \\ s & s & \frac{4+2s}{s^2} \\ 0 & s & \frac{2+2s^2}{s^3} \end{vmatrix}}{\begin{vmatrix} 2s & -s & -s \\ s & s & 0 \\ 0 & s & 1 \end{vmatrix}} = \frac{\frac{2(s-1)(s-3)}{s}}{s^2(3-s)} = \frac{2(1-s)}{s^3} = \frac{2}{s^3} - \frac{2}{s^2}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$y_1(t) = \mathcal{L}^{-1}\left\{\bar{y}_1(s)\right\} = t^2$$

$$y_2(t) = \mathcal{L}^{-1}\left\{\bar{y}_2(s)\right\} = 2t + t^2$$

$$y_3(t) = \mathcal{L}^{-1}\left\{\bar{y}_3(s)\right\} = t^2 - 2t$$

28 MASS – SPRING SYSTEM

UNDAMPED FORCED SYSTEM

Consider the forced oscillations of a mass – spring system with no damping . Assuming that the mass is initially at rest in the static equilibrium position , then we know from chapter (6) that the motion of this system is governed by the initial – value problem

$$\frac{d^2y}{dt^2} + \frac{K}{m}y = \frac{1}{m}F(t), \qquad y(0) = 0, \; y'(0) = 0 \tag{1}$$

Here m is the mass of the body , K the spring modulus , and $F(t)$ is the external driving force acting on the body . Setting $\frac{K}{m} = \omega_0^2$, equation (1) can be written as

$$\frac{d^2y}{dt^2} + \omega_0^2\, y = \frac{1}{m}F(t) \tag{2}$$

Taking the Laplace transform of both sides of equation (2) , we get

$$\left[s^2\bar{y}(s) - s\,y(0) - y'(0)\right] + \omega_0^2\,\bar{y}(s) = \frac{1}{m}\bar{F}(s)$$

Using the initial conditions $y(0) = 0$, $y'(0) = 0$, we get

$$(s^2+\omega_0^2)\bar{y}(s) = \frac{1}{m}\bar{F}(s)$$

or $$\bar{y}(s) = \frac{1}{m}\frac{\bar{F}(s)}{s^2+\omega_0^2}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{\bar{F}(s)}{s^2+\omega_0^2}\right\}$$

Since $\mathcal{L}^{-1}\{\bar{F}(s)\} = F(t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} = \frac{\sin\omega_0 t}{\omega_0}$, hence by the convolution theorem,

we get $$y(t) = \frac{1}{m}\left[F(t) * \frac{\sin\omega_0 t}{\omega_0}\right]$$

$$= \frac{1}{m\omega_0}\int_0^t F(u)\sin\omega_0(t-u)\,du \qquad (3)$$

PERIODIC DRIVING FORCE

In case of periodic driving force $F(t) = A\sin\omega t$, equation (3) takes the form

$$y(t) = \frac{B}{\omega_0}\int_0^t \sin\omega u\sin\omega_0(t-u)\,du \qquad \left(B = \frac{A}{m}\right) \qquad (4)$$

Using the formula $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, equation (4) becomes

$$y(t) = \frac{B}{2\omega_0}\int_0^t [\cos\{(\omega+\omega_0)u - \omega_0 t\} - \cos\{(\omega-\omega_0)u + \omega_0 t\}]\,du \qquad (5)$$

Now there are two cases to be considered.

CASE (1): If $\omega \neq \omega_0$, then after integration we get from equation (5)

$$y(t) = \frac{B}{2\omega_0}\left|\frac{\sin\{(\omega+\omega_0)u-\omega_0 t\}}{\omega+\omega_0} - \frac{\sin\{(\omega-\omega_0)u+\omega_0 t\}}{\omega-\omega_0}\right|_0^t$$

$$= \frac{B}{2\omega_0}\left[\frac{\sin\omega t + \sin\omega_0 t}{\omega+\omega_0} - \frac{\sin\omega t - \sin\omega_0 t}{\omega-\omega_0}\right]$$

$$= \frac{B}{2\omega_0}\left(\frac{2\omega\sin\omega_0 t - 2\omega_0\sin\omega t}{\omega^2-\omega_0^2}\right) = \frac{B}{\omega^2-\omega_0^2}\left(\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right)$$

This is the non – resonance case. With no damping, the resulting motion is simply a periodic oscillation.

CASE (2): If $\omega = \omega_0$, then from equation (5), we get

$$y(t) = \frac{B}{2\omega}\int_0^t [\cos(2\omega u - \omega t) - \cos\omega t]\,du$$

$$= \frac{B}{2\omega}\left|\frac{\sin(2\omega u - \omega t)}{2\omega} - u\cos\omega t\right|_0^t$$

$$= \frac{B}{2\omega}\left(\frac{\sin\omega t}{2\omega} + \frac{\sin\omega t}{2\omega} - t\cos\omega t\right)$$

$$= \frac{B}{2\omega}\left(\frac{\sin\omega t}{\omega} - t\cos\omega t\right)$$

$$= \frac{B}{2\omega^2}(\sin\omega t - \omega t\cos\omega t) \tag{6}$$

Note that as t increases, the second term increases and the solution increases in amplitude (i.e. it has large oscillations). This phenomenon is called **resonance**.

DAMPED FORCED SYSTEM

We now consider the forced oscillations of a damped mass – spring system. In this case, the motion of the system is governed by the initial – value problem

$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{1}{m}F(t), \qquad y(0) = 0, \quad y'(0) = 0 \tag{7}$$

Taking the Laplace transform of both sides of the differential equation, we get

$$[s^2\bar{y}(s) - s\,y(0) - y'(0)] + \frac{C}{m}[s\bar{y}(s) - y(0)] + \frac{K}{m}\bar{y}(s) = \frac{1}{m}\bar{F}(s)$$

Using the initial conditions $y(0) = 0$, $y'(0) = 0$, we get

$$\left(s^2 + \frac{C}{m}s + \frac{K}{m}\right)\bar{y}(s) = \frac{1}{m}\bar{F}(s)$$

or
$$\bar{y}(s) = \frac{\bar{F}(s)}{m\left(s^2 + \frac{C}{m}s + \frac{K}{m}\right)}$$

$$= \frac{\bar{F}(s)}{m\left[\left(s^2 + \frac{C}{m}s + \frac{C^2}{4m^2}\right) + \left(\frac{K}{m} - \frac{C^2}{4m^2}\right)\right]}$$

$$= \frac{\bar{F}(s)}{m\left[\left(s + \frac{C}{2m}\right)^2 + R\right]} \tag{8}$$

where $R = \frac{K}{m} - \frac{C^2}{4m^2}$. Now there are three cases to be considered.

CASE (1): When $R > 0$ (under damping). In this case, let $R = \omega^2$. From equation (8), we get

$$\bar{y}(s) = \frac{\bar{F}(s)}{m\left[\left(s+\frac{C}{2m}\right)^2+\omega^2\right]}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{\bar{F}(s)}{\left(s+\frac{C}{2m}\right)^2+\omega^2}\right\} \tag{9}$$

Since $\mathcal{L}^{-1}\{\bar{F}(s)\} = F(t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{C}{2m}\right)^2+\omega^2}\right\} = e^{-Ct/2m}\frac{\sin\omega t}{\omega}$,

hence by convolution theorem, equation (9) becomes

$$y(t) = \frac{1}{m}\left(F(t) * e^{-Ct/2m}\frac{\sin\omega t}{\omega}\right)$$

$$= \frac{1}{m\omega}\int_0^t F(u)\,e^{-C(t-u)/2m}\sin\omega(t-u)\,du \tag{10}$$

CASE (2): When $R = 0$ (critical damping). In this case, equation (8) reduces to

$$\bar{y}(s) = \frac{\bar{F}(s)}{m\left(s+\frac{C}{2m}\right)^2}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{\bar{F}(s)}{\left(s+\frac{C}{2m}\right)^2}\right\} \tag{11}$$

Since $\mathcal{L}^{-1}\{\bar{F}(s)\} = F(t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{C}{2m}\right)^2}\right\} = t\,e^{-Ct/2m}$, hence by convolution

theorem, equation (11) becomes

$$y(t) = \frac{1}{m}\left(F(t) * t\,e^{-Ct/2m}\right)$$

$$= \frac{1}{m}\int_0^t F(u)(t-u)\,e^{-C(t-u)/2m}\,du \tag{12}$$

CASE (3): When $R < 0$ (over damping). In this case, let $R = -\omega^2$. From equation (8), we get

$$\bar{y}(s) = \frac{\bar{F}(s)}{m\left[\left(s+\frac{C}{2m}\right)^2 - \omega^2\right]}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{\bar{F}(s)}{\left(s+\frac{C}{2m}\right)^2 - \omega^2}\right\} \qquad (13)$$

Since $\mathcal{L}^{-1}\{\bar{F}(s)\} = F(t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{C}{2m}\right)^2 - \omega^2}\right\} = e^{-Ct/2m}\frac{\sinh \omega t}{\omega}$,

hence by convolution theorem, equation (13) becomes

$$y(t) = \frac{1}{m}\left(F(t) * e^{-Ct/2m}\frac{\sinh \omega t}{\omega}\right)$$

$$= \frac{1}{m\omega}\int_0^t F(u)\, e^{-C(t-u)/2m} \sinh \omega (t-u)\, du \qquad (14)$$

EXAMPLE (34): **Determine the equation of motion (i.e. displacement) of a damped forced mass – spring system, given that $m = 1$ kg, $K = 16$ N / m, $C = 10$ kg / s, $F = 4 \sin 4t$ N, and $y(0) = 0$, $y'(0) = 0$.**

SOLUTION: We know that the differential equation governing the motion of a mass – spring system in the case of damped forced oscillations is

$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{1}{m}F(t) \qquad (1)$$

In the present case, $m = 1$, $K = 16$, $C = 10$, $F = 4 \sin 4t$, and $y(0) = 4$, $y'(0) = 0$.

Thus equation (1) becomes

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 16y = 4 \sin 4t \qquad (2)$$

STEP (1): Taking the Laplace transform of equation (2), we get

$$[s^2\bar{y}(s) - s\,y(0) - y'(0)] + 10[s\bar{y}(s) - y(0)] + 16\bar{y}(s) = \frac{16}{s^2+16}$$

STEP (2): Using the initial conditions, we get

$$(s^2 + 10s + 16)\bar{y}(s) = \frac{16}{s^2+16} \qquad (3)$$

STEP (3): Since $R = \frac{K}{m} - \frac{C^2}{4m^2} = 16 - 25 = -9 = -3^2$, therefore from equation (14) with $\omega = 3$ on page (515), we have

$$y(t) = \frac{1}{m\omega}\int_0^t F(u)\,e^{-C(t-u)/2m}\sinh\omega(t-u)\,du$$

$$= \frac{1}{3}\int_0^t 4\sin 4u\,e^{-5(t-u)}\sinh 3(t-u)\,du$$

$$= \frac{4}{3}e^{-5t}\int_0^t e^{5u}\sin 4u\sinh 3(t-u)\,du$$

$$= \frac{4}{3}e^{-5t}\int_0^t e^{5u}\left(\frac{e^{3(t-u)}-e^{-3(t-u)}}{2}\right)\sin 4u\,du$$

$$= \frac{2}{3}e^{-5t}\int_0^t \left(e^{3t}.e^{2u}-e^{-3t}.e^{8u}\right)\sin 4u\,du$$

$$= \frac{2}{3}e^{-2t}\int_0^t e^{2u}\sin 4u\,du - \frac{2}{3}e^{-8t}\int_0^t e^{8u}\sin 4u\,du \qquad (4)$$

Using the formula $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$, equation (4) becomes

$$y(t) = \frac{2}{3}e^{-2t}\left|\frac{e^{2u}(2\sin 4u - 4\cos 4u)}{20}\right|_0^t - \frac{2}{3}e^{-8t}\left|\frac{e^{8u}(8\sin 4u - 4\cos 4u)}{80}\right|_0^t$$

$$= \frac{2}{3}e^{-2t}\left[e^{2t}\left(\frac{2\sin 4t - 4\cos 4t}{20}\right)+\frac{1}{5}\right] - \frac{2}{3}e^{-8t}\left[e^{8t}\left(\frac{8\sin 4t - 4\cos 4t}{80}\right)+\frac{1}{20}\right]$$

$$= \frac{1}{15}\sin 4t - \frac{2}{15}\cos 4t + \frac{2}{15}e^{-2t} - \frac{1}{15}\sin 4t + \frac{1}{30}\cos 4t - \frac{1}{30}e^{-8t}$$

$$= \frac{2}{15}e^{-2t} - \frac{1}{30}e^{-8t} - \frac{1}{10}\cos 4t$$

ALTERNATIVE METHOD

From equation (3), we have

$$(s^2 + 10s + 16)\,\bar{y}(s) = \frac{16}{s^2+16}$$

or
$$\bar{y}(s) = \frac{16}{(s+2)(s+8)(s^2+16)} = \frac{2}{15(s+2)} - \frac{1}{30(s+8)} - \frac{s}{10(s^2+16)}$$

Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{15(s+2)} - \frac{1}{30(s+8)} - \frac{s}{10(s^2+16)}\right\}$$

or $$y(t) = \frac{2}{15}e^{-2t} - \frac{1}{30}e^{-8t} - \frac{1}{10}\cos 4t$$

9.29 COUPLED MASS – SPRING SYSTEM

EXAMPLE (35): **Find the displacements $y_1(t)$ and $y_2(t)$ of a coupled mass – spring system given that $m_1 = 3$, $m_2 = 1$, $K_1 = 4$, $K_2 = 1$ and $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = 0$.**

SOLUTION: We know that the system of differential equations governing the vibrations of a coupled system is

$$m_1 y_1'' = -(K_1 + K_2)y_1 + K_2 y_2$$

$$m_2 y_2'' = -K_2(y_2 - y_1)$$

In the present case , $m_1 = 3$, $m_2 = 1$, $K_1 = 4$, $K_2 = 1$. Thus the above system takes the form

$$3y_1'' = -5y_1 + y_2 \qquad (1)$$

$$y_2'' = -y_2 + y_1 \qquad (2)$$

Taking the Laplace transform of equations (1) and (2) , we get

$$3[s^2\bar{y}_1(s) - sy_1(0) - y_1'(0)] = -5\bar{y}_1(s) + \bar{y}_2(s)$$

$$s^2\bar{y}_2(s) - sy_2(0) - y_2'(0) = -\bar{y}_2(s) + \bar{y}_1(s)$$

Using the initial conditions , we get

$$(3s^2+5)\bar{y}_1(s) - \bar{y}_2(s) = 3s$$

$$-\bar{y}_1(s) + (s^2+1)\bar{y}_2(s) = 0$$

Using the Cramer's rule , we get

$$\bar{y}_1(s) = \frac{\begin{vmatrix} 3s & -1 \\ 0 & s^2+1 \end{vmatrix}}{\begin{vmatrix} 3s^2+5 & -1 \\ -1 & s^2+1 \end{vmatrix}} = \frac{3s(s^2+1)}{3s^4+8s^2+4} = \frac{3s^3+3s}{(3s^2+2)(s^2+2)}$$

$$= \frac{3}{4}\left(\frac{s}{3s^2+2}\right) + \frac{3}{4}\left(\frac{s}{s^2+2}\right) = \frac{1}{4}\left(\frac{s}{s^2+\frac{2}{3}}\right) + \frac{3}{4}\left(\frac{s}{s^2+2}\right)$$

and $\bar{y}_2(s) = \dfrac{\begin{vmatrix} 3s^2+5 & 3s \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 3s^2+5 & -1 \\ -1 & s^2+1 \end{vmatrix}} = \dfrac{3s}{3s^4+8s^2+4} = \dfrac{3s}{(3s^2+2)(s^2+2)}$

$$= \frac{9}{4}\left(\frac{s}{3s^2+2}\right) - \frac{3}{4}\left(\frac{s}{s^2+2}\right) = \frac{3}{4}\left(\frac{s}{s^2+\frac{2}{3}}\right) - \frac{3}{4}\left(\frac{s}{s^2+2}\right)$$

Taking the inverse Laplace transforms , we get

$$\mathcal{L}^{-1}\{\bar{y}_1(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{2}{3}}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$$

$$\mathcal{L}^{-1}\{\bar{y}_2(s)\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{2}{3}}\right\} - \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$$

or $y_1(t) = \frac{1}{4}\cos\sqrt{\frac{2}{3}}\,t + \frac{3}{4}\cos\sqrt{2}\,t$

$y_2(t) = \frac{3}{4}\cos\sqrt{\frac{2}{3}}\,t - \frac{3}{4}\cos\sqrt{2}\,t$ [compare with solved example (17) of chapter (8)]

9.30 ELECTRIC CIRCUIT

EXAMPLE (36): **An RLC circuit has a resistance of 10 ohms , a capacitance of 10^{-2} farad , an inductance of 0.5 henry , and an applied voltage E = 12 volts . Assuming no initial current and no initial charge on the capacitor , find an expression for the current flowing through the circuit at any time t .**

SOLUTION: We know that the differential equation for the charge Q in an RLC circuit is

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{1}{L}E(t) \tag{1}$$

In the present case , R = 10 ohms , C = 10^{-2} farad , L = 0.5 henry , and E (t) = 12 volts .
Thus equation (1) becomes

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 200\,Q = 24 \tag{2}$$

Taking the Laplace transform of both sides of equation (2) , we get

$$[s^2\bar{Q}(s) - sQ(0) - Q'(0)] + 20[s\bar{Q}(s) - Q(0)] + 200\,\bar{Q}(s) = \frac{24}{s}$$

Using the initial conditions $Q(0) = 0$ and $Q'(0) = 0$, we get

$$(s^2+20s+200)\,\bar{Q}(s) = \frac{24}{s}$$

or $$\bar{Q}(s) = \frac{24}{s(s^2+20s+200)}$$

$$= \frac{3}{25s} - \frac{3s}{25(s^2+20s+200)} - \frac{12}{5(s^2+20s+200)}$$

$$= \frac{3}{25}\left[\frac{1}{s} - \frac{s}{(s+10)^2+100}\right] - \frac{12}{5}\left[\frac{1}{(s+10)^2+100}\right]$$

$$= \frac{3}{25}\left[\frac{1}{s} - \frac{(s+10)-10}{(s+10)^2+100}\right] - \frac{12}{5}\left[\frac{1}{(s+10)^2+100}\right]$$

$$= \frac{3}{25}\left[\frac{1}{s} - \frac{s+10}{(s+10)^2+100}\right] - \frac{6}{5}\left[\frac{1}{(s+10)^2+100}\right]$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{Q}(s)\} = \frac{3}{25}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+10}{(s+10)^2+100}\right\} - \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+10)^2+100}\right\}$$

or $$Q(t) = \frac{3}{25}(1 - e^{-10t}\cos 10t) - \frac{6}{5}e^{-10t}\frac{\sin 10t}{10}$$

$$= \frac{3}{25}(1 - e^{-10t}\cos 10t) - \frac{3}{25}e^{-10t}\sin 10t$$

$$= \frac{3}{25} - \frac{3}{25}e^{-10t}(\cos 10t + \sin 10t)$$

9.31 ELECTRICAL NETWORK

EXAMPLE (37): **Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the electrical network shown in figure (9.26), assuming that $R = 10$ ohms, $L = 20$ henrys, $C = 0.05$ farad, $E(t) = 20$ volts and $I_1(0) = 0$, $I_2(0) = 2$ amperes.**

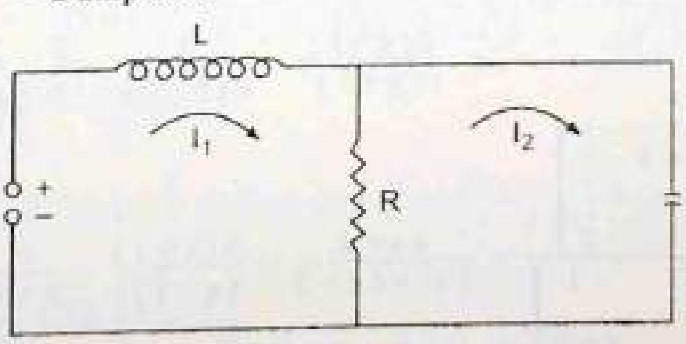


Figure (9.26)

SOLUTION: We know from chapter (5) that the system of differential equations governing the currents I_1 and I_2 in the given network is

$$L\frac{dI_1}{dt} + R(I_1 - I_2) = E(t) \tag{1}$$

$$-R\frac{dI_1}{dt} + R\frac{dI_2}{dt} + \frac{1}{C}I_2 = 0 \tag{2}$$

In the present case, this system takes the form

$$20\frac{dI_1}{dt} + 10(I_1 - I_2) = 20$$

$$-10\frac{dI_1}{dt} + 10\frac{dI_2}{dt} + 20I_2 = 0$$

or
$$2\frac{dI_1}{dt} + (I_1 - I_2) = 2 \tag{3}$$

$$-\frac{dI_1}{dt} + \frac{dI_2}{dt} + 2I_2 = 0 \tag{4}$$

Taking the Laplace transforms of both the differential equations (3) and (4), we get

$$2[s\bar{I}_1(s) - I_1(0)] + \bar{I}_1(s) - \bar{I}_2(s) = \frac{2}{s}$$

$$-[s\bar{I}_1(s) - I_1(0)] + [s\bar{I}_2(s) - I_2(0)] + 2\bar{I}_2(s) = 0$$

Using the initial conditions $I_1(0) = 0$, $I_2(0) = 2$, we get

$$(2s+1)\bar{I}_1(s) - \bar{I}_2(s) = \frac{2}{s}$$

$$-s\bar{I}_1(s) + (s+2)\bar{I}_2(s) = 2$$

Using the Cramer's rule, we get

$$\bar{I}_1(s) = \frac{\begin{vmatrix} \frac{2}{s} & -1 \\ 2 & s+2 \end{vmatrix}}{\begin{vmatrix} 2s+1 & -1 \\ -s & s+2 \end{vmatrix}} = \frac{\frac{4s+4}{s}}{2s^2+4s+2} = \frac{4s+4}{2(s)(s+1)^2}$$

$$= \frac{2(s+1)}{s(s+1)^2} = \frac{2}{s(s+1)} = \frac{2}{s} - \frac{2}{s+1}$$

$$\bar{I}_2(s) = \frac{\begin{vmatrix} 2s+1 & \frac{2}{s} \\ -s & 2 \end{vmatrix}}{\begin{vmatrix} 2s+1 & -1 \\ -s & s+2 \end{vmatrix}} = \frac{4s+4}{2s^2+4s+2} = \frac{2(s+1)}{(s+1)^2} = \frac{2}{s+1}$$

Taking the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\{\bar{I}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s+1}\right\}$$

$$\mathcal{L}^{-1}\{\bar{I}_2(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\}$$

or $\quad I_1(t) = 2 - 2e^{-t}$ and $I_2(t) = 2e^{-t}$

9.32 SOLVED PROBLEMS

LAPLACE TRANSFORMS

PROBLEM (1): **Find the following Laplace transforms, where a and b are constants.**

(i) $\mathcal{L}\{4+t^2-5e^{2t}+6\cos 3t\}$ (ii) $\mathcal{L}\{(at+b)^2\}$

(iii) $\mathcal{L}\{\sin(at+b)\}$ (iv) $\mathcal{L}\{\cos^2 at\}$

(v) $\mathcal{L}\{\sinh^2 at\}$ (vi) $\mathcal{L}\{(\sin t-\cos t)^2\}$

SOLUTION: By the linearity theorem (9.1), we have

(i) $$\mathcal{L}\{4+t^2-5e^{2t}+6\cos 3t\} = \mathcal{L}\{4\}+\mathcal{L}\{t^2\}-5\mathcal{L}\{e^{2t}\}+6\mathcal{L}\{\cos 3t\}$$
$$= \frac{4}{s}+\frac{2}{s^3}-5\left(\frac{1}{s-2}\right)+6\left(\frac{s}{s^2+9}\right)$$
$$= \frac{4}{s}+\frac{2}{s^3}-\frac{5}{s-2}+\frac{6s}{s^2+9}$$

(ii) $$\mathcal{L}\{(at+b)^2\} = \mathcal{L}\{a^2t^2+2abt+b^2\}$$
$$= a^2\mathcal{L}\{t^2\}+2ab\mathcal{L}\{t\}+b^2\mathcal{L}\{1\}$$
$$= a^2\left(\frac{2}{s^3}\right)+2ab\left(\frac{1}{s^2}\right)+b^2\left(\frac{1}{s}\right) = \frac{2a^2}{s^3}+\frac{2ab}{s^2}+\frac{b^2}{s}$$

(iii) $$\mathcal{L}\{\sin(at+b)\} = \mathcal{L}\{\sin at\cos b+\cos at\sin b\}$$
$$= \cos b\,\mathcal{L}\{\sin at\}+\sin b\,\mathcal{L}\{\cos at\}$$
$$= \cos b\left(\frac{a}{s^2+a^2}\right)+\sin b\left(\frac{s}{s^2+a^2}\right) = \frac{a\cos b+s\sin b}{s^2+a^2}$$

(iv) $$\mathcal{L}\{\cos^2 at\} = \mathcal{L}\left\{\frac{1+\cos 2at}{2}\right\}$$
$$= \frac{1}{2}\mathcal{L}\{1\}+\frac{1}{2}\mathcal{L}\{\cos 2at\} = \frac{1}{2s}+\frac{s}{2(s^2+4a^2)}$$

(v) $$\mathcal{L}\{\sinh^2 at\} = \mathcal{L}\left\{\frac{\cosh 2at-1}{2}\right\}$$
$$= \frac{1}{2}\mathcal{L}\{\cosh 2at\}-\frac{1}{2}\mathcal{L}\{1\} = \frac{1}{2}\left(\frac{s}{s^2-4a^2}\right)-\frac{1}{2s}$$

(vi) $$\mathcal{L}\{(\sin t-\cos t)^2\} = \mathcal{L}\{\sin^2 t+\cos^2 t-2\sin t\cos t\}$$
$$= \mathcal{L}\{1-\sin 2t\} = \frac{1}{s}-\frac{2}{s^2+4}$$

PROBLEM (2): **Find the Laplace transform of the function** $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 3 \\ 2 & \text{if } t \ge 3 \end{cases}$.

SOLUTION: By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st} f(t)\, dt + \int_3^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st}(1)\, dt + \int_3^\infty e^{-st}(2)\, dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_0^3 + 2 \left| \frac{e^{-st}}{-s} \right|_3^\infty$$

$$= \frac{1}{s} - \frac{e^{-3s}}{s} + \frac{2e^{-3s}}{s} = \frac{1}{s} + \frac{e^{-3s}}{s} = \frac{1}{s}(1 + e^{-3s})$$

PROBLEM (3): **Find the Laplace transform of the following function :**

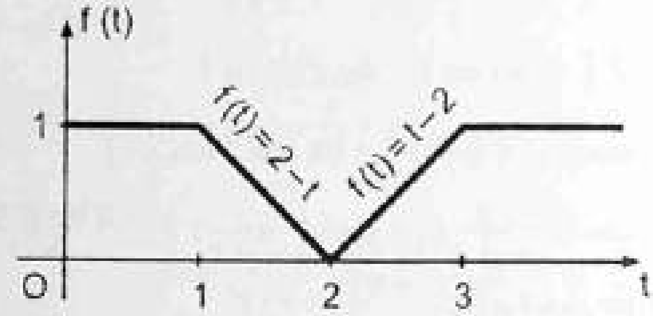


Figure (9.27)

SOLUTION: By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^2 e^{-st} f(t)\, dt + \int_2^3 e^{-st} f(t)\, dt + \int_3^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st}(1)\, dt + \int_1^2 e^{-st}(2-t)\, dt + \int_2^3 e^{-st}(t-2)\, dt + \int_3^\infty e^{-st}(1)\, dt$$

$$= \left| -\frac{e^{-st}}{s} \right|_0^1 + \left| -\frac{e^{-st}}{s}(2-t) + \frac{e^{-st}}{s^2} \right|_1^2 + \left| -\frac{e^{-st}}{s}(t-2) - \frac{e^{-st}}{s^2} \right|_2^3 + \left| -\frac{e^{-st}}{s} \right|_3^\infty$$

$$= \left(-\frac{e^{-s}}{s} + \frac{1}{s} \right) + \left(\frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right) + \left(-\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s^2} \right) + \frac{e^{-3s}}{s}$$

$$= \frac{1}{s} - \frac{(e^{-s} - 2e^{-2s} + e^{-3s})}{s^2}$$

PROBLEM (4): **Find the following Laplace transforms using the first shifting theorem (9.8) :**

(i) $\mathcal{L}\{4t^3 e^{-t}\}$ (ii) $\mathcal{L}\{e^{-2t}(2\cos 3t - \sin 3t)\}$

(iii) $\mathcal{L}\{e^{-t}\sin(\omega t + \theta)\}$ (iv) $\mathcal{L}\{e^{3t}(1 - t^2 + \sin t)\}$

(v) $\mathcal{L}\{e^{-t}\sin^2 t\}$ (vi) $\mathcal{L}\{e^{at}\sinh bt\}$

SOLUTION: (i) Since $\mathcal{L}\{4t^3\} = 4 \cdot \frac{3!}{s^4} = \frac{24}{s^4}$, hence by the first shifting theorem

$$\mathcal{L}\{4t^3 e^{-t}\} = \frac{24}{(s+1)^4}$$

(ii) Since $\mathcal{L}\{2\cos 3t - \sin 3t\} = \frac{2s}{s^2+9} - \frac{3}{s^2+9} = \frac{2s-3}{s^2+9}$, hence by the first shifting theorem

$$\mathcal{L}\{e^{-2t}(2\cos 3t - \sin 3t)\} = \frac{2(s+2)-3}{(s+2)^2+9} = \frac{2s+1}{s^2+4s+13}$$

(iii) Since $\mathcal{L}\{\sin(\omega t + \theta)\} = \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$, hence by the first shifting theorem

$$\mathcal{L}\{e^{-t}\sin(\omega t + \theta)\} = \frac{\omega\cos\theta + (s+1)\sin\theta}{(s+1)^2+\omega^2}$$

(iv) Since $\mathcal{L}\{1 - t^2 + \sin t\} = \frac{1}{s} - \frac{2}{s^3} + \frac{1}{s^2+1}$, hence by the first shifting theorem

$$\mathcal{L}\{e^{3t}(1 - t^2 + \sin t)\} = \frac{1}{s-3} - \frac{2}{(s-3)^3} + \frac{1}{(s-3)^2+1}$$

(v) Since $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$, hence by the first shifting theorem

$$\mathcal{L}\{e^{-t}\sin^2 t\} = \frac{1}{2(s+1)} - \frac{s+1}{2[(s+1)^2+4]}$$

$$= \frac{1}{2(s+1)} - \frac{s+1}{2(s^2+2s+5)} = \frac{2}{(s+1)(s^2+2s+5)}$$

(vi) Since $\mathcal{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$, $s > |b|$, hence by the first shifting theorem

$$\mathcal{L}\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}, \quad s > a + |b|$$

PROBLEM (5): **Represent the hyperbolic function in terms of exponential functions and applying the first shifting theorem (9.8), show that**

$$\mathcal{L}\{\cosh at \cos at\} = \frac{s^3}{s^4+4a^4}.$$

SOLUTION: We can write

$$\mathcal{L}\{\cosh at \cos at\} = \mathcal{L}\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\cos at\} + \frac{1}{2}\mathcal{L}\{e^{-at}\cos at\}$$

Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, hence by the first shifting theorem, we get

$$\mathcal{L}\{\cosh at \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}\right] + \frac{1}{2}\left[\frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s^2+2a^2)-2as} + \frac{s+a}{(s^2+2a^2)+2as}\right]$$

$$= \frac{1}{2}\left(\frac{2s^3}{s^4+4a^4}\right) = \frac{s^3}{s^4+4a^4}$$

PROBLEM (6): **Find the following Laplace transforms using theorem (9.9):**

(i) $\mathcal{L}\{t^2 e^t\}$ (ii) $\mathcal{L}\{t^2 \sinh 2t\}$

(iii) $\mathcal{L}\{t e^{-2t} \sin \omega t\}$ (iv) $\mathcal{L}\{t e^{-t} \cosh 2t\}$

SOLUTION: (i) Since $\mathcal{L}\{e^t\} = \frac{1}{s-1}$, therefore by theorem (9.9),

$$\mathcal{L}\{t^2 e^t\} = \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = \frac{d}{ds}\left[-\frac{1}{(s-1)^2}\right] = \frac{2}{(s-1)^3}$$

(ii) Since $\mathcal{L}\{\sinh 2t\} = \frac{2}{s^2-4}$, therefore by theorem (9.9),

$$\mathcal{L}\{t^2 \sinh 2t\} = \frac{d^2}{ds^2}\left(\frac{2}{s^2-4}\right) = \frac{d}{ds}\left[\frac{-4s}{(s^2-4)^2}\right] = \frac{12s^2+16}{(s^2-4)^3}$$

(iii) Since $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$, therefore by the first shifting theorem (9.8)

$$\mathcal{L}\{e^{-2t}\sin \omega t\} = \frac{\omega}{(s+2)^2+\omega^2}$$

Hence, by theorem (9.9), we get

$$\mathcal{L}\{t e^{-2t}\sin \omega t\} = -\frac{d}{ds}\left[\frac{\omega}{(s+2)^2+\omega^2}\right] = \frac{2\omega(s+2)}{[(s+2)^2+\omega^2]^2}$$

(iv) Since $\mathcal{L}\{\cosh 2t\} = \frac{s}{s^2-4}$, therefore by the first shifting theorem (9.8)

$$\mathcal{L}\{e^{-t}\cosh 2t\} = \frac{s+1}{(s+1)^2-4}$$

Hence, by theorem (9.9), we get

$$\mathcal{L}\{t\,e^{-t}\cosh 2t\} = -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2-4}\right] = \frac{s^2+2s+5}{(s^2+2s-3)^2}$$

PROBLEM (7): Find the following Laplace transforms using theorem (9.10) :

(i) $\mathcal{L}\left\{2\left(\frac{1-\cos at}{t}\right)\right\}$ (ii) $\mathcal{L}\left\{\frac{\sinh kt}{t}\right\}$

SOLUTION: (i) Since $\underset{t\to 0+}{\mathrm{Lt}}\left[2\left(\frac{1-\cos at}{t}\right)\right] = \underset{t\to 0+}{\mathrm{Lt}}\left(\frac{2a\sin at}{1}\right) = 0$ exists, and

$\mathcal{L}\{2(1-\cos at\} = \frac{2}{s} - \frac{2s}{s^2+a^2}$, hence by theorem (9.10), we have

$$\mathcal{L}\left\{2\left(\frac{1-\cos at}{t}\right)\right\} = \int_s^\infty \left(\frac{2}{u} - \frac{2u}{u^2+a^2}\right)du$$

$$= \left|2\ln u - \ln(u^2+a^2)\right|_s^\infty$$

$$= \left|\ln\frac{u^2}{u^2+a^2}\right|_s^\infty = \ln\left|\frac{1}{1+\frac{a^2}{u^2}}\right|_s^\infty$$

$$= \ln 1 - \ln\frac{1}{1+\frac{a^2}{s^2}} = 0 - \ln\left(\frac{s^2}{s^2+a^2}\right) = \ln\left(\frac{s^2+a^2}{s^2}\right)$$

(ii) Since $\underset{t\to 0+}{\mathrm{Lt}}\left[\frac{\sinh kt}{t}\right] = k$ exists, and

$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$, hence by theorem (9.10), we have

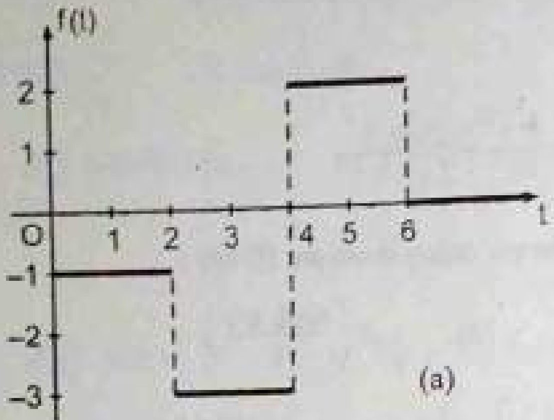
$$\mathcal{L}\left\{\frac{\sinh kt}{t}\right\} = \int_s^\infty \frac{k}{u^2-k^2}\,du = k\cdot\frac{1}{2k}\left|\ln\left(\frac{u-k}{u+k}\right)\right|_s^\infty$$

$$= \frac{1}{2}\left|\ln\left(\frac{1-\frac{k}{u}}{1+\frac{k}{u}}\right)\right|_s^\infty = \frac{1}{2}\left|\ln 1 - \ln\left(\frac{1-\frac{k}{s}}{1+\frac{k}{s}}\right)\right|$$

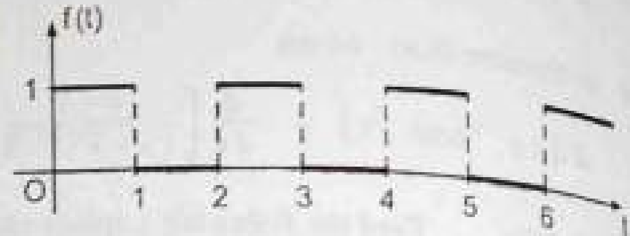
$$= \frac{1}{2}\left|0 - \ln\left(\frac{s-k}{s+k}\right)\right| = \frac{1}{2}\ln\left(\frac{s+k}{s-k}\right)$$

PROBLEM (8): Represent the following functions in terms of unit step functions and find their Laplace transform .

(i)

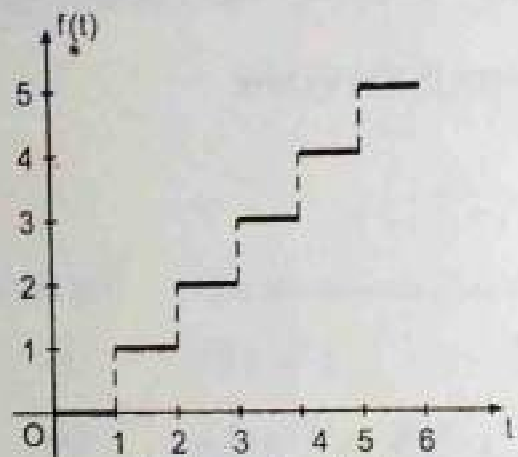


(a)



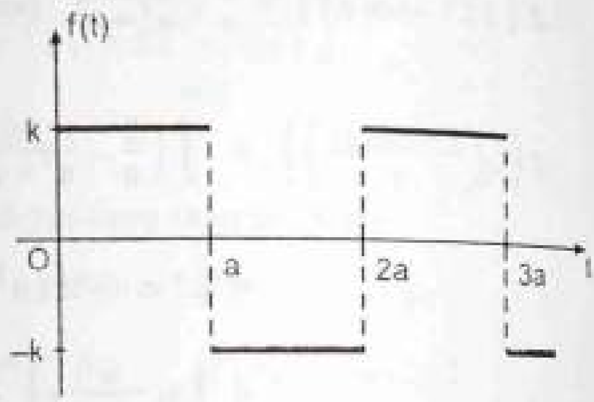
(b)

(iii)



(Staircase function)

(c)



(d)

Figure (9.28)

SOLUTION: (i) From figure [9.28 (a)] , we have

$$f(t) = -1\left[u_0(t)-u_2(t)\right]-3\left[u_2(t)-u_4(t)\right]+2\left[u_4(t)-u_6(t)\right]$$

$$= -u_0(t)-2u_2(t)+5u_4(t)-2u_6(t)$$

Hence by theorem (9.11) , we have

$$\mathcal{L}\{f(t)\} = -\frac{1}{s}-2\left(\frac{e^{-2s}}{s}\right)+5\left(\frac{e^{-4s}}{s}\right)-2\left(\frac{e^{-6s}}{s}\right)$$

$$= -\frac{1}{s}\left(1+2e^{-2s}-5e^{-4s}+2e^{-6s}\right)$$

(ii) From figure [9.28 (b)] , we have

$$f(t) = \left[u_0(t)-u_1(t)\right]+\left[u_2(t)-u_3(t)\right]+\left[u_4(t)-u_5(t)\right]+u_6(t)+\ldots\ldots$$

$$= u_0(t)-u_1(t)+u_2(t)-u_3(t)+u_4(t)-u_5(t)+\ldots\ldots$$

Hence by theorem (9.11) , we get

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s} + \ldots\ldots$$

$$= \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s} + \ldots\ldots\right)$$

$$= \frac{1}{s}\left(\frac{1}{1+e^{-s}}\right), \quad \text{since sum of G.P.} = \frac{1}{1-r}.$$

(iii) From figure [9.28 (c)] , we have

$$f(t) = 1[u_1(t) - u_2(t)] + 2[u_2(t) - u_3(t)] + 3[u_3(t) - u_4(t)] + 4[u_4(t) - u_5(t)] + \ldots\ldots$$

$$= u_1(t) + u_2(t) + u_3(t) + u_4(t) + \ldots\ldots$$

Hence , by theorem (9.11) , we get

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} + \ldots\ldots$$

$$= \frac{e^{-s}}{s}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\ldots\right)$$

$$= \frac{e^{-s}}{s(1-e^{-s})} \qquad \left(\text{since sum of a G.P.} = \frac{1}{1-r}\right)$$

(iv) From figure [9.28 (d)] , we have

$$f(t) = k[u_0(t) - u_a(t)] - k[u_a(t) - u_{2a}(t)] + k[u_{2a}(t) - u_{3a}(t)] - k\,u_{3a}(t) + \ldots\ldots$$

$$= k[u_0(t) - 2u_a(t) + 2u_{2a}(t) - 2u_{3a}(t) + \ldots\ldots]$$

Hence by theorem (9.11) , we get

$$\mathcal{L}\{f(t)\} = k\left[\frac{1}{s} - 2\frac{e^{-as}}{s} + 2\frac{e^{-2as}}{s} - 2\frac{e^{-3as}}{s} + \ldots\ldots\right]$$

$$= \frac{k}{s}\left[1 - 2e^{-as}\left(1 - e^{-as} + e^{-2as} - + \ldots\ldots\right)\right]$$

$$= \frac{k}{s}\left[1 - 2e^{-as}\left(\frac{1}{1+e^{-as}}\right)\right] \qquad \left(\text{since sum of a G.P.} = \frac{1}{1-r}\right)$$

$$= \frac{k}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

$$= \frac{k}{s}\,\frac{e^{-as/2}\left(e^{as/2} - e^{-as/2}\right)}{e^{-as/2}\left(e^{as/2} + e^{-as/2}\right)} = \frac{k}{s}\tanh\frac{as}{2}$$

PROBLEM (9): **Represent the following function in terms of unit step functions and find its Laplace transform :**

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 3 & \text{if } 1 < t < 3 \\ 5 & \text{if } \quad t > 3 \end{cases}$$

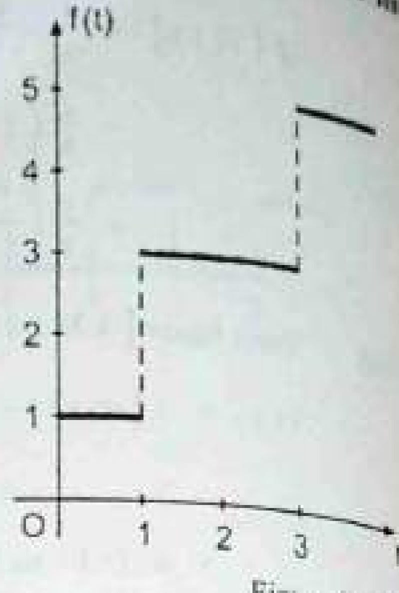


Figure (9.29)

SOLUTION: We can represent the function graphically as follows . From figure (9.29) , we have

$$f(t) = 1\left[u_0(t) - u_1(t)\right] + 3\left[u_1(t) - u_3(t)\right] + 5\,u_3(t)$$
$$= u_0(t) + 2\,u_1(t) + 2\,u_3(t)$$

Hence by theorem (9.11) , we get

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + 2\frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} = \frac{1}{s}\left(1 + 2e^{-s} + 2e^{-3s}\right)$$

PROBLEM (10): **Find the following Laplace transforms using the second shifting theorem (9.12) :**

(i) $\mathcal{L}\{(t-2)^3 u_2(t)\}$ (ii) $\mathcal{L}\{t\,u_1(t)\}$

(iii) $\mathcal{L}\{t^2 u_1(t)\}$ (iv) $\mathcal{L}\{e^{-2t} u_1(t)\}$

(v) $\mathcal{L}\{\sin \pi(t-3)\,u_3(t)\}$ (vi) $\mathcal{L}\{u_\pi(t)\sin t\}$

(vii) $\mathcal{L}\{\sinh a(t-b)\,u_b(t)\}$ (viii) $\mathcal{L}\{\cosh t\,u_1(t)\}$

SOLUTION: (i) $\mathcal{L}\{(t-2)^3 u_2(t)\}$

Here $a = 2$ and $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$. Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{(t-2)^3 u_2(t)\} = e^{-2s}\frac{3!}{s^4} = \frac{6e^{-2s}}{s^4}$$

(ii) $$\mathcal{L}\{t\,u_1(t)\} = \mathcal{L}\{(\overline{t-1}+1)\,u_1(t)\}$$
$$= \mathcal{L}\{(t-1)\,u_1(t)\} + \mathcal{L}\{u_1(t)\}$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{t\,u_1(t)\} = e^{-s}\left(\frac{1}{s^2}\right) + \frac{e^{-s}}{s} = \frac{e^{-s}}{s}\left(1 + \frac{1}{s}\right)$$

(iii) $$\mathcal{L}\{t^2 u_1(t)\} = \mathcal{L}\{(\overline{t-1}+1)^2 u_1(t)\}$$
$$= \mathcal{L}\{[(t-1)^2 + 2(t-1) + 1]\,u_1(t)\}$$
$$= \mathcal{L}\{(t-1)^2 u_1(t)\} + 2\mathcal{L}\{(t-1)\,u_1(t)\} + \mathcal{L}\{u_1(t)\}$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{t^2 u_1(t)\} = e^{-s}\left(\frac{2}{s^3}\right) + 2e^{-s}\left(\frac{1}{s^2}\right) + \frac{e^{-s}}{s}$$

$$= \frac{e^{-s}}{s}\left(1 + \frac{2}{s} + \frac{2}{s^2}\right)$$

(iv) $$\mathcal{L}\{e^{-2t}u_1(t)\} = \mathcal{L}\{e^{-2(\overline{t-1}+1)}u_1(t)\}$$

$$= e^{-2}\mathcal{L}\{e^{-2(t-1)}u_1(t)\}$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{e^{-2t}u_1(t)\} = e^{-2}.e^{-s}\left(\frac{1}{s+2}\right) - \frac{e^{-(s+2)}}{s+2}$$

(v) $$\mathcal{L}\{\sin\pi(t-3)u_3(t)\}$$

Here $a = 3$ and $\mathcal{L}\{\sin\pi t\} = \frac{\pi}{s^2+\pi^2}$.

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{\sin\pi(t-3)u_3(t)\} = e^{-3s}\left(\frac{\pi}{s^2+\pi^2}\right)$$

(vi) $$\mathcal{L}\{u_\pi(t)\sin t\} = -\mathcal{L}\{\sin(t-\pi)u_\pi(t)\} = -e^{-\pi s}\left(\frac{1}{s^2+1}\right)$$

(vii) $$\mathcal{L}\{\sinh a(t-b)u_b(t)\}$$

Here $a = b$ and $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{\sinh a(t-b)u_b(t)\} = e^{-bs}\left(\frac{a}{s^2-a^2}\right)$$

(viii) $$\mathcal{L}\{\cosh t\, u_1(t)\} = \mathcal{L}\{\cosh(\overline{t-1}+1)u_1(t)\}$$

$$= \mathcal{L}\{[\cosh(t-1)\cosh 1 + \sinh(t-1)\sinh 1]u_1(t)\}$$

$$= \cosh 1 . \mathcal{L}\{\cosh(t-1)u_1(t)\} + \sinh 1 . \mathcal{L}\{\sin(t-1)u_1(t)\}$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{\cosh t\, u_1(t)\} = \cosh 1 . e^{-s}\left(\frac{s}{s^2-1}\right) + \sinh 1\, e^{-s}\left(\frac{1}{s^2-1}\right)$$

$$= \frac{e^{-s}}{s^2-1}(s\cosh 1 + \sinh 1)$$

PROBLEM (11): **Find the Laplace transforms of the following functions using the second shifting theorem (9.12) :**

(i) $f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}$ (ii) $f(t) = \begin{cases} t & \text{if } 1 < t < 4 \\ 0 & \text{otherwise} \end{cases}$

(iii) $f(t) = \begin{cases} k \cos \omega t & \text{if } 0 \le t < \dfrac{\pi}{\omega} \\ 0 & \text{if } t \ge \dfrac{\pi}{\omega} \end{cases}$ (iv) $f(t) = \begin{cases} e^t & \text{if } 0 \le t < 2\pi \\ e^t + \cos t & \text{if } t \ge 2\pi \end{cases}$

SOLUTION: (i) $f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}$

$$= t^2\left[u_0(t) - u_3(t)\right]$$

$$= t^2 u_0(t) - t^2 u_3(t)$$

$$= t^2 u_0(t) - \left[(t-3)+3\right]^2 u_3(t)$$

$$= t^2 u_0(t) - (t-3)^2 u_3(t) - 6(t-3)u_3(t) - 9u_3(t)$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u_0(t) - (t-3)^2 u_3(t) - 6(t-3)u_3(t) - 9u_3(t)\}$$

$$= \frac{2}{s^3} - \frac{2}{s^3}e^{-3t} - \frac{6}{s^2}e^{-3t} - 9\frac{e^{-3t}}{s}$$

$$= \frac{2}{s^3} - \left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)e^{-3t}$$

(ii) $f(t) = \begin{cases} t & \text{if } 1 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$

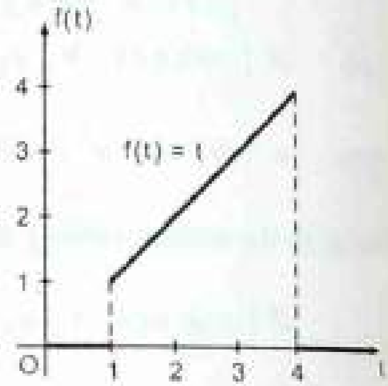


Figure (9.30)

$$= t\left[u_1(t) - u_4(t)\right]$$

$$= t u_1(t) - t u_4(t)$$

$$= \left[(t-1)u_1(t) + u_1(t)\right] - \left[(t-4)u_4(t) + 4u_4(t)\right]$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)u_1(t) + u_1(t)\} - \mathcal{L}\{(t-4)u_4(t) + 4u_4(t)\}$$

$$= \left[e^{-s}\left(\frac{1}{s^2}\right) + \frac{e^{-s}}{s}\right] - \left[e^{-4s}\left(\frac{1}{s^2}\right) + \frac{4e^{-4s}}{s}\right]$$

$$= (1+s)\frac{e^{-s}}{s^2} - (1+4s)\frac{e^{-4s}}{s^2}$$

(iii) $f(t) = \begin{cases} k\cos\omega t & \text{if } 0 \le t < \dfrac{\pi}{\omega} \\ 0 & \text{if } t \ge \dfrac{\pi}{\omega} \end{cases}$

$$= k\cos\omega t\left[u_0(t) - u_{\pi/\omega}(t)\right]$$

$$= k\cos\omega t\, u_0(t) - k\cos\omega t\, u_{\pi/\omega}(t)$$

$$= k\cos\omega t\, u_0(t) + k\cos\omega\left(t - \frac{\pi}{\omega}\right)u_{\pi/\omega}(t)$$

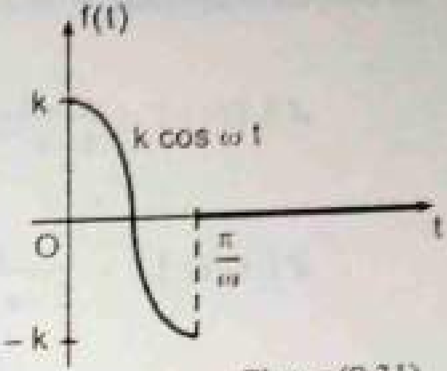


Figure (9.31)

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{f(t)\} = k\mathcal{L}\{\cos\omega t\, u_0(t)\} + k\mathcal{L}\left\{\cos\omega\left(t - \frac{\pi}{\omega}\right)u_{\pi/\omega}(t)\right\}$$

$$= k\frac{s}{s^2+\omega^2} + \frac{k e^{-\pi s/\omega} s}{s^2+\omega^2} = \frac{ks(1+e^{-\pi s/\omega})}{s^2+\omega^2}$$

(iv) $f(t) = \begin{cases} e^t & \text{if } 0 \le t < 2\pi \\ e^t + \cos t & \text{if } t \ge 2\pi \end{cases}$

$$= e^t\left[u_0(t) - u_{2\pi}(t)\right] + \left[e^t + \cos t\right]u_{2\pi}(t)$$

$$= e^t u_0(t) + \cos t\, u_{2\pi}(t) = e^t u_0(t) + \cos(t - 2\pi)u_{2\pi}(t)$$

Hence by the second shifting theorem (9.12)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^t u_0(t)\} + \mathcal{L}\{\cos(t-2\pi)u_{2\pi}(t)\} = \frac{1}{s-1} + \frac{s e^{-2\pi s}}{s^2+1}$$

PROBLEM (12): **Find the Laplace transforms of the following periodic functions :**

$$\mathbf{f(t) = 4\pi^2 - t^2, \qquad f(t+2\pi) = f(t).}$$

SOLUTION: (i) The graph of this function is shown in figure (9.32) :

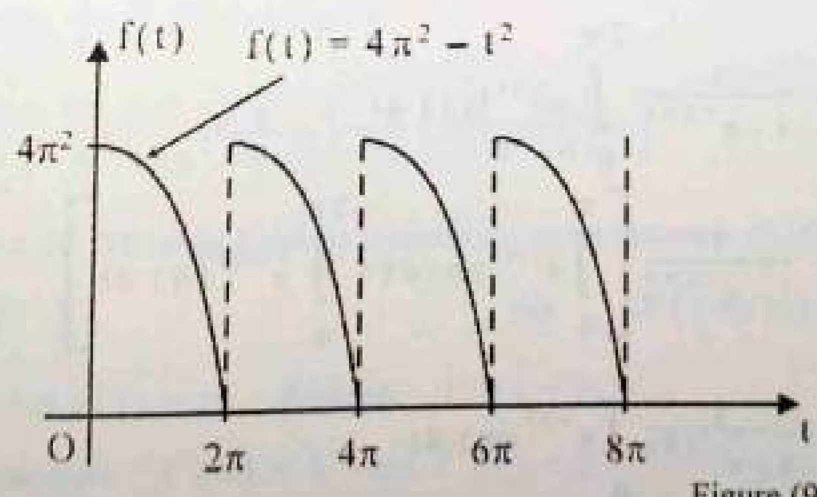


Figure (9.32)

By theorem (9.8), since $P = 2\pi$, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt, \quad s > 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} (4\pi^2 - t^2)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left| -\frac{e^{-st}(4\pi^2 - t^2)}{s} + \frac{2te^{-st}}{s^2} + \frac{2e^{-st}}{s^3} \right|_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{4\pi^2}{s} + \frac{4\pi e^{-2\pi s}}{s^2} + \frac{2e^{-2\pi s}}{s^3} - \frac{2}{s^3} \right]$$

$$= \frac{4\pi^2 s^2 - 2 + (4\pi s + 2)e^{-2\pi s}}{s^3(1-e^{-2\pi s})}$$

PROBLEM (13): **Find the Laplace transform of the following periodic function (9.14):**

$$f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ 0 & \pi < t < 2\pi \end{cases} \qquad f(t+2\pi) = f(t)$$

SOLUTION: The graph of the given function is shown in figure (9.33).

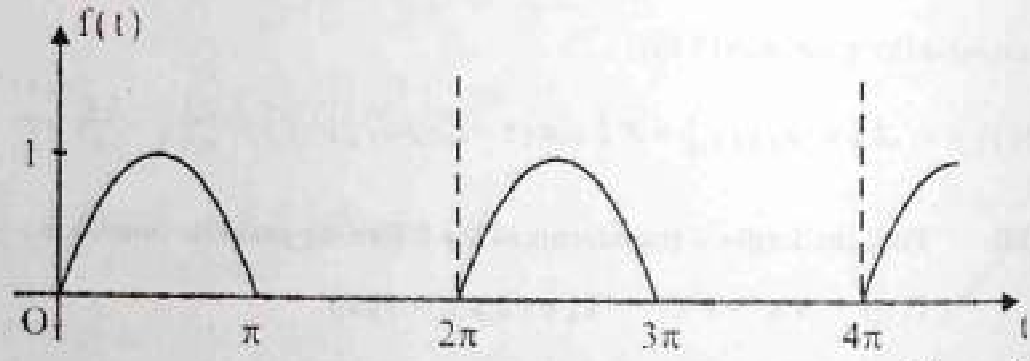


Figure (9.33)

By theorem (9.14), since $P = 2\pi$, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\, dt + \int_{\pi}^{2\pi} e^{-st}(0)\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt \qquad (1)$$

Using the formula $\int e^{ax}\sin bx \; dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$, equation (1) becomes

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\left|\frac{e^{-st}(-s\sin t-\cos t)}{s^2+1}\right|_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\frac{\left(e^{-\pi s}+1\right)}{s^2+1}$$

$$= \frac{1}{1-e^{-2\pi s}}\frac{\left(1+e^{-\pi s}\right)}{s^2+1} = \frac{1}{(s^2+1)\left(1-e^{-\pi s}\right)}$$

PROBLEM (14): Find the following Laplace transforms using theorem (9.15) :

(i) $\mathcal{L}\{t\,e^{at}\}$ (ii) $\mathcal{L}\{\cosh^2\omega t\}$

SOLUTION: (i) Let $f(t) = t\,e^{at}$ therefore $f(0) = 0$ and $f'(t) = e^{at} + a\,t\,e^{at}$

Hence using the formula $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

we get $\mathcal{L}\{e^{at} + a\,t\,e^{at}\} = s\mathcal{L}\{t\,e^{at}\} - 0$

or $\mathcal{L}\{e^{at}\} + a\mathcal{L}\{t\,e^{at}\} = s\mathcal{L}\{t\,e^{at}\}$

or $(s-a)\mathcal{L}\{t\,e^{at}\} = \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

or $\mathcal{L}\{t\,e^{at}\} = \dfrac{1}{(s-a)^2}$

(ii) Let $f(t) = \cosh^2\omega t$ therefore $f(0) = 1$

and $f'(t) = 2\omega\cosh\omega t\sinh\omega t = \omega\sinh 2\omega t$

Hence, by using the formula $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

we get $\mathcal{L}\{\omega\sinh 2\omega t\} = s\mathcal{L}\{\cosh^2\omega t\} - 1$

or $\mathcal{L}\{\cosh^2\omega t\} = \dfrac{1}{s} + \dfrac{\omega}{s}\mathcal{L}\{\sinh 2\omega t\}$

$$= \frac{1}{s} + \frac{\omega}{s}\left(\frac{2\omega}{s^2-4\omega^2}\right) = \frac{1}{s}\left(1+\frac{2\omega^2}{s^2-4\omega^2}\right)$$

PROBLEM (15): Find the following Laplace transforms using theorem (9.15) :

(i) $\mathcal{L}\{t^2\}$ (ii) $\mathcal{L}\{t\sin\omega t\}$

SOLUTION: (i) Let $f(t) = t^2$ therefore $f(0) = 0$

Also $f'(t) = 2t$ therefore $f'(0) = 0$ and $f''(t) = 2$

Hence , using the formula $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$

we get $\mathcal{L}\{2\} = s^2\mathcal{L}\{t^2\} - 0 - 0$

or $\mathcal{L}\{t^2\} = \frac{1}{s^2}\mathcal{L}\{2\} = \frac{1}{s^2}\left(\frac{2}{s}\right) = \frac{2}{s^3}$

(ii) Let $f(t) = t\sin\omega t$ therefore $f(0) = 0$

$f'(t) = \omega t\cos\omega t + \sin\omega t$ therefore $f'(0) = 0$

and $f''(t) = 2\omega\cos\omega t - \omega^2 t\sin\omega t$

Hence , using the formula $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$

we get $\mathcal{L}\{2\omega\cos\omega t - \omega^2 t\sin\omega t\} = s^2\mathcal{L}\{t\sin\omega t\} - 0 - 0$

or $2\omega\mathcal{L}\{\cos\omega t\} - \omega^2\mathcal{L}\{t\sin\omega t\} = s^2\mathcal{L}\{t\sin\omega t\}$

or $(s^2+\omega^2)\mathcal{L}\{t\sin\omega t\} = 2\omega\mathcal{L}\{\cos\omega t\} = 2\omega\left(\frac{s}{s^2+\omega^2}\right)$

or $\mathcal{L}\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$

PROBLEM (16): **Find the Laplace transforms of the following integrals using theorem (9.16):**

(i) $\int_0^t \cos^2 ax\,dx$ **(ii)** $\int_0^t \sin x\cos x\,dx$

SOLUTION: (i) Since $\mathcal{L}\{\cos^2 at\} = \mathcal{L}\left\{\frac{1+\cos 2at}{2}\right\}$

$$= \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cos 2at\}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+4a^2)}$$

therefore $\mathcal{L}\left\{\int_0^t \cos^2 ax\,dx\right\} = \frac{1}{s}\left(\frac{1}{2s} + \frac{s}{2(s^2+4a^2)}\right) = \frac{1}{2s^2} + \frac{1}{2(s^2+4a^2)}$

(ii) Since $\mathcal{L}\{\sin t\cos t\} = \frac{1}{2}\mathcal{L}\{\sin 2t\} = \frac{1}{2}\left(\frac{2}{s^2+4}\right) = \frac{1}{s^2+4}$

therefore $\mathcal{L}\left\{\int_0^t \sin x\cos x\,dx\right\} = \frac{1}{s}\left(\frac{1}{s^2+4}\right) = \frac{1}{s(s^2+4)}$

INVERSE LAPLACE TRANSFORMS

PROBLEM (17): **Find the following inverse Laplace transforms :**

(i) $\mathcal{L}^{-1}\left\{\frac{5}{s+3}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}$

(v) $\mathcal{L}^{-1}\left\{\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s^2-16}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{5}{s+3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = 5e^{-3t}$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{3!} = \frac{t^3}{6}$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$

(iv) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \cos t + \sin t$

(v) $\mathcal{L}^{-1}\left\{\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right\} = \cos\theta\,\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} + \sin\theta\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}$

$$= \sin\omega t\cos\theta + \cos\omega t\sin\theta = \sin(\omega t + \theta)$$

(vi) $\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s^2-16}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s^2-16}\right\} = 4\cosh 4t + \sinh 4t$

PROBLEM (18): **Find the following inverse Laplace transforms using theorem (9.18) :**

(i) $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^3}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{6}{s^2-4s-5}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+5}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{s+1-2\omega}{s^2+2s+\omega^2+1}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^3}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$, therefore by the first shifting theorem (9.18) for the inverse Laplace transforms,

we get $\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^3}\right\} = \frac{1}{2}e^{-t/2}t^2$

(ii) $$\mathcal{L}^{-1}\left\{\frac{6}{s^2-4s-5}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^2-9}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{3}{s^2-9}\right\} = 3\sinh 3t$, therefore by the first shifting theorem (9.18) for the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\left\{\frac{6}{s^2-4s-5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2-9}\right\} = 2e^{2t}\sinh 3t$$

(iii) $$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$, therefore by the first shifting theorem (9.18) for the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\} = e^{2t}\cos t$$

(iv) $$\mathcal{L}^{-1}\left\{\frac{s+1-2\omega}{s^2+2s+\omega^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)-2\omega}{(s+1)^2+\omega^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+\omega^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{\omega}{(s+1)^2+\omega^2}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t$, and $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t$, therefore by the first shifting theorem (9.18) for the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\left\{\frac{s+1-2\omega}{s^2+2s+\omega^2+1}\right\} = e^{-t}\cos\omega t - 2e^{-t}\sin\omega t = e^{-t}(\cos\omega t - 2\sin\omega t)$$

PROBLEM (19): **Find the following inverse Laplace transforms using theorem (9.19):**

(i) $\mathcal{L}^{-1}\left\{\frac{8s}{(s^2+4)^2}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{4s}{(s^2-4)^2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{(s-1)^2}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s}{\omega}\right)\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{8s}{(s^2+4)^2}\right\}$

Using theorem (9.19), we have

$$f(t) = t\mathcal{L}^{-1}\left\{\int_s^\infty \frac{8u}{(u^2+4)^2}\,du\right\} = t\mathcal{L}^{-1}\left\{\left| -\frac{4}{u^2+4}\right|_s^\infty\right\} = t\mathcal{L}^{-1}\left\{\frac{4}{s^2+4}\right\} = 2t\sin 2t$$

(ii) $\mathcal{L}^{-1}\left\{\frac{4s}{(s^2-4)^2}\right\}$

Using theorem (9.19), we have

$$f(t) = t\mathcal{L}^{-1}\left\{\int_s^\infty \frac{4u}{(u^2-4)^2}\,du\right\} = t\mathcal{L}^{-1}\left\{\left|-\frac{2}{u^2-4}\right|_s^\infty\right\} = t\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} = t\sinh 2t$$

(iii) $\mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{(s-1)^2}\right\}$

Using theorem (9.19), we have

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\ln\frac{s^2+1}{(s-1)^2}\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s^2+1)-\ln(s-1)^2\right]\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}-\frac{2}{s-1}\right\} = -\frac{1}{t}(2\cos t-2e^t)$$

(iv) $\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s}{\omega}\right)\right\}$

Using theorem (9.19), we have

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\cot^{-1}\left(\frac{s}{\omega}\right)\right\}$$

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{1}{\left(1+\frac{s^2}{\omega^2}\right)}\frac{1}{\omega}\right\} = \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \frac{\sin\omega t}{t}$$

PROBLEM (20): Find the following inverse Laplace transforms using second shifting theorem (9.21) :

(i) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^4}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{3(e^{-4s}-e^{-s})}{s}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{1-e^{-\pi s}}{s^2+4}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^4}\right\}$

Since $a = 1$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{6}$, hence by theorem (9.21), we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^4}\right\}=\frac{(t-1)^3}{6}u_1(t)=\begin{cases}0 & \text{if } 0\le t<1\\ \dfrac{(t-1)^3}{6} & \text{if } t\ge 1\end{cases}$$

(ii) $$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}$$

Since $a=\pi$ and $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\sin t$, hence by theorem (9.21), we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+2}\right\}=e^{-(t-\pi)}\sin(t-\pi)\,u_\pi(t)$$

$$=-e^{-(t-\pi)}\sin t\,u_\pi(t)=\begin{cases}0 & \text{if } 0\le t<\pi\\ -e^{-(t-\pi)}\sin t & \text{if } t\ge\pi\end{cases}$$

(iii) $$\mathcal{L}^{-1}\left\{\frac{3(e^{-4s}-e^{-s})}{s}\right\}=3\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\}-3\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\}$$

$$=3u_4(t)-3u_1(t)$$

$$=-3u_1(t)+3u_4(t)=\begin{cases}0 & \text{if } 0\le t<1\\ -3 & \text{if } 1<t<3\\ 0 & \text{if } t\ge 4\end{cases}$$

(iv) $$\mathcal{L}^{-1}\left\{\frac{1-e^{-\pi s}}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\}$$

$$=\frac{\sin 2t}{2}-\frac{1}{2}\sin 2(t-\pi)\,u_\pi(t)$$

$$=\frac{\sin 2t}{2}-\frac{1}{2}\sin 2t\,u_\pi(t)=\begin{cases}\dfrac{\sin 2t}{2} & \text{if } 0\le t<\pi\\ 0 & \text{if } t\ge\pi\end{cases}$$

PROBLEM (21): **Find the following inverse Laplace transforms using theorem (9.22):**

(i) $\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3-2s^2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{s-2}{s^2+4}\right)\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{1}{s^5-s^3}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$, hence by theorem (9.22), we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = \int_0^t e^{2x}\,dx = \left|\frac{e^{2x}}{2}\right|_0^t = \frac{e^{2t}}{2} - \frac{1}{2} = \frac{1}{2}(e^{2t}-1)$$

(ii) $$\mathcal{L}^{-1}\left\{\frac{1}{s^3-2s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s[s(s-2)]}\right\}$$

$$= \frac{1}{2}\int_0^t (e^{2x}-1)\,dx \qquad [\text{using part (i)}]$$

$$= \frac{1}{2}\left|\frac{e^{2x}}{2} - x\right|_0^t = \frac{1}{2}\left(\frac{e^{2t}}{2} - t - \frac{1}{2}\right)$$

(iii) $$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{s-2}{s^2+4}\right)\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+4}\right\} = \cos 2t - \sin 2t$, hence by theorem (9.22), we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{s-2}{s^2+4}\right)\right\} = \int_0^t (\cos 2x - \sin 2x)\,dx$$

$$= \left|\frac{\sin 2x}{2} + \frac{\cos 2x}{2}\right|_0^t = \frac{1}{2}(\sin 2t + \cos 2t - 1)$$

and $$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{s-2}{s^2+4}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\left[\frac{1}{s}\left(\frac{s-2}{s^2+4}\right)\right]\right\}$$

$$= \frac{1}{2}\int_0^t (\sin 2x + \cos 2x - 1)\,dx$$

$$= \frac{1}{2}\left|-\frac{\cos 2x}{2} + \frac{\sin 2x}{2} - x\right|_0^t = \frac{1}{2}\left(-\frac{\cos 2t}{2} + \frac{\sin 2t}{2} - t + \frac{1}{2}\right)$$

(iv) $$\mathcal{L}^{-1}\left\{\frac{1}{s^5-s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2-1)}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$, therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \int_0^t \sinh x\,dx = |\cosh x|_0^t = \cosh t - 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s[s(s^2-1)]}\right\}$$

$$= \int_0^t (\cosh x - 1)\, dx = \left| \sinh x - x \right|_0^t = \sinh t - t$$

and $$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s[s^2(s^2-1)]}\right\}$$

$$= \int_0^t (\sinh x - x)\, dx = \left| \cosh x - \frac{x^2}{2} \right|_0^t = \cosh t - \frac{t^2}{2} - 1$$

PROBLEM (22): **Find the following convolutions :**

(i) $e^{kt} * e^{-kt}$ (ii) $t^2 * e^{at}$

(iii) $u_\pi(t) * \cos t$ (iv) $e^{2t} * \cos 4t$

SOLUTION: (i) By definition

$$e^{kt} * e^{-kt} = \int_0^t e^{ku} e^{-k(t-u)}\, du = \int_0^t e^{-kt} e^{2ku}\, du$$

$$= e^{-kt} \left| \frac{e^{2ku}}{2k} \right|_0^t = e^{-kt}\left(\frac{e^{2kt}-1}{2k}\right)$$

$$= \frac{1}{2k}\left(e^{kt} - e^{-kt}\right) = \frac{1}{k} \sinh kt$$

(ii) By definition

$$t^2 * e^{at} = \int_0^t u^2 e^{a(t-u)}\, du = e^{at} \int_0^t u^2 e^{-au}\, du$$

$$= e^{at} \left| \frac{e^{-au} u^2}{-a} - \frac{2e^{-au} u}{a^2} - \frac{2e^{-au}}{a^3} \right|_0^t$$

$$= e^{at}\left(-\frac{e^{-at} t^2}{a} - \frac{2e^{-at} t}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right)$$

$$= -\frac{t^2}{a} - \frac{2t}{a^2} - \frac{2}{a^3} + \frac{2e^{at}}{a^3}$$

(iii) $u_\pi(t) * \cos t$

Since $u_\pi(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi \\ 1 & \text{if } t \ge \pi \end{cases}$, therefore if $0 \le t < \pi$, $u_\pi(t) * \cos t = 0 * \cos t = 0$ and

if $t \ge \pi$, $u_\pi(t) * \cos t = 1 * \cos t = \displaystyle\int_\pi^t (1)\cos(t-u)\, du$

$$= \left|-\sin(t-u)\right|_\pi^t = \sin(t-\pi) = -\sin t$$

Thus $u_\pi(t) * \cos t = -\sin t\, u_\pi(t)$

(iv) By definition

$$e^{2t} * \cos 4t = \cos 4t * e^{2t}$$

$$= \int_0^t \cos 4u\, e^{2(t-u)}\, du$$

$$= e^{2t}\int_0^t e^{-2u}\cos 4u\, du \qquad (1)$$

Using the formula $\displaystyle\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$ equation (1) becomes

$$e^{2t} * \cos 4t = e^{2t}\left|\frac{e^{-2u}(-2\cos 4u + 4\sin 4u)}{20}\right|_0^t$$

$$= e^{2t}\left[e^{-2u}\left(\frac{-2\cos 4t + 4\sin 4t}{20}\right) + \frac{1}{10}\right]$$

PROBLEM (23): **Find the following convolutions :**

(i) $\sin\omega t * \sin\omega t$ (ii) $\cos\omega t * \cos\omega t$ (iii) $\sin\omega t * \cos\omega t$

SOLUTION: (i) By definition

$$\sin\omega t * \sin\omega t = \int_0^t \sin\omega u \sin\omega(t-u)\, du$$

Using the formula $\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$, the above equation becomes

$$\sin\omega t * \sin\omega t = \frac{1}{2}\int_0^t \left[\cos\omega(2u-t) - \cos\omega t\right] du$$

$$= \frac{1}{2}\left|\frac{\sin\omega(2u-t)}{2\omega} - u\cos\omega t\right|_0^t = \frac{1}{2}\left(\frac{\sin\omega t}{\omega} - t\cos\omega t\right)$$

(ii) By definition

$$\cos \omega t * \cos \omega t = \int_0^t \cos \omega u \cos \omega (t-u)\, du$$

Using the formula $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$, the above equation becomes

$$\cos \omega t * \cos \omega t = \frac{1}{2}\int_0^t \left[\cos \omega t + \cos \omega (2u-t)\right] du$$

$$= \frac{1}{2}\left| u \cos \omega t + \frac{\sin \omega (2u-t)}{2\omega}\right|_0^t$$

$$= \frac{1}{2}\left(t \cos \omega t + \frac{\sin \omega t}{2\omega} + \frac{\sin \omega t}{2\omega}\right) = \frac{1}{2}\left(t \cos \omega t + \frac{\sin \omega t}{\omega}\right)$$

(iii) By definition

$$\sin \omega t * \cos \omega t = \int_0^t \sin \omega u \cos \omega (t-u)\, du$$

Using the formula $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$ the above equation becomes

$$\sin \omega t * \cos \omega t = \frac{1}{2}\int_0^t \left[\sin \omega t + \sin \omega (2u-t)\right] du$$

$$= \frac{1}{2}\left| u \sin \omega t - \frac{\cos \omega (2u-t)}{2\omega}\right|_0^t$$

$$= \frac{1}{2}\left[t \sin \omega t - \frac{\cos \omega t}{2\omega} + \frac{\cos \omega t}{2\omega}\right] = \frac{1}{2}\, t \sin \omega t$$

PROBLEM (24): **Find the following inverse Laplace transforms using convolution theorem (9.24)**

(i) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$, $a \neq b$ (ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega^2)^2}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$, $a \neq b$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s-b}\right\} = e^{bt}$, hence by convolution theorem, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = e^{at} * e^{bt}$$

$$= \int_0^t e^{au} . e^{b(t-u)} = e^{bt} \int_0^t e^{(a-b)u} \, du$$

$$= e^{bt} \left| \frac{e^{(a-b)u}}{a-b} \right|_0^t = \frac{e^{bt}}{a-b} \left[e^{(a-b)t} - 1 \right] = \frac{e^{at} - e^{bt}}{a-b}$$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}$ hence , by convolution theorem , we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = t * \frac{\sin \omega t}{\omega}$$

$$= \frac{1}{\omega} \int_0^t u \sin \omega (t-u) \, du$$

$$= \frac{1}{\omega} \left| \frac{u \cos \omega (t-u)}{\omega} + \frac{\sin \omega (t-u)}{\omega^2} \right|_0^t$$

$$= \frac{1}{\omega^3} \left| \omega u \cos \omega (t-u) + \sin \omega (t-u) \right|_0^t = \frac{1}{\omega^3} (\omega t - \sin \omega t)$$

(iii) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega^2)^2}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}$ and $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos \omega t$,

hence by the convolution theorem , we get

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega^2)^2}\right\} = \frac{\sin \omega t}{\omega} * \cos \omega t = \frac{1}{\omega} (\sin \omega t * \cos \omega t)$$

$$= \frac{1}{\omega}\left(\frac{1}{2} t \sin \omega t\right) \qquad \text{[Using problem (23) part (iii)]}$$

$$= \frac{1}{2\omega} t \sin \omega t$$

(iv) $\mathcal{L}^{-1}\left\{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\}$

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos \omega t$ and $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin \omega t$,

hence by convolution theorem , we get

$$\mathcal{L}^{-1}\left\{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+\omega^2)^2}\right\}-\mathcal{L}^{-1}\left\{\frac{\omega^2}{(s^2+\omega^2)^2}\right\}$$

$$= \cos\omega t * \cos\omega t - \sin\omega t * \sin\omega t \qquad (1)$$

But from problem (23) part (i)

$$\sin\omega t * \sin\omega t = \frac{1}{2}\left(\frac{\sin\omega t}{\omega} - t\cos\omega t\right)$$

and from problem (23) part (ii)

$$\cos\omega t * \cos\omega t = \frac{1}{2}\left(\frac{\sin\omega t}{\omega} + t\cos\omega t\right)$$

Hence from equation (1), we have

$$\mathcal{L}^{-1}\left\{\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right\} = \frac{1}{2}\left(\frac{\sin\omega t}{\omega} + t\cos\omega t\right) - \frac{1}{2}\left(\frac{\sin\omega t}{\omega} - t\cos\omega t\right)$$

$$= \frac{1}{2}t\cos\omega t + \frac{1}{2}t\cos\omega t = t\cos\omega t$$

PROBLEM (25): Find the following inverse Laplace transforms using partial fractions :

(i) $\mathcal{L}^{-1}\left\{\frac{s^2-6s+4}{s^3-3s^2+2s}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^3}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\frac{3s^2-2s-1}{(s-3)(s^2+1)}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{s^4+3(s+1)^3}{s^4(s+1)^3}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\frac{s^2-6s+4}{s^3-3s^2+2s}\right\}$

By resolving into partial fractions, we can write $\frac{s^2-6s+4}{s^3-3s^2+2s} = \frac{s^2-6s+4}{s(s-1)(s-2)} = \frac{2}{s}+\frac{1}{s-1}-\frac{2}{s-2}$

Thus $\mathcal{L}^{-1}\left\{\frac{s^2-6s+4}{s^3-3s^2+2s}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{1}{s-1}-\frac{2}{s-2}\right\} = 2+e^{t}-2e^{2t}$

(ii) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^3}\right\}$

By resolving into partial fractions, we can write $\frac{s}{(s-2)^3} = \frac{(s-2)+2}{(s-2)^3} = \frac{1}{(s-2)^2}+\frac{2}{(s-2)^3}$

Thus $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}+\frac{2}{(s-2)^3}\right\} = e^{2t}t+e^{2t}t^2 = e^{2t}(t+t^2)$

(iii) $\mathcal{L}^{-1}\left\{\frac{3s^2-2s-1}{(s-3)(s^2+1)}\right\}$

By resolving into partial fractions, we can write $\frac{3s^2-2s-1}{(s-3)(s^2+1)} = \frac{2}{s-3}+\frac{s+1}{s^2+1}$

Thus $\mathcal{L}^{-1}\left\{\dfrac{3s^2-2s-1}{(s-3)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s-3}+\dfrac{s+1}{s^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-3}+\frac{s}{s^2+1}+\frac{1}{s^2+1}\right\} = 2e^{3t}+\cos t+\sin t$$

(iv) $\mathcal{L}^{-1}\left\{\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3}\right\}$

By resolving into partial fractions, we can write $\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3} = \dfrac{3}{s^4}+\dfrac{1}{(s+1)^3}$

Thus $\mathcal{L}^{-1}\left\{\dfrac{s^4+3(s+1)^2}{s^4(s+1)^3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{3}{s^4}+\dfrac{1}{(s+1)^3}\right\} = 3\dfrac{t^3}{3!}+\dfrac{t^2}{2}e^{-t} = \dfrac{t^2}{2}(t+e^{-t})$

PROBLEM (26): **Find the following inverse Laplace transforms using partial fractions using theorem (9.25) :**

(i) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+13}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{s^2-6s+7}{(s^2-4s+5)^2}\right\}$

SOLUTION: (i) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+13}\right\}$

By resolving into partial fractions, we can write $\dfrac{s+1}{s^2+4s+13} = \dfrac{s+1}{(s^2+4s+4)+9} = \dfrac{(s+2)-1}{(s+2)^2+9}$

Thus $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\dfrac{(s+2)-1}{(s+2)^2+9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+9}-\frac{1}{(s+2)^2+9}\right\}$$

$$= e^{-2t}\cos 3t-\frac{1}{3}e^{-2t}\sin 3t = e^{-2t}\left(\cos 3t-\frac{1}{3}\sin 3t\right)$$

(ii) $\mathcal{L}^{-1}\left\{\dfrac{s^2-6s+7}{(s^2-4s+5)^2}\right\}$

By resolving into partial fractions, we can write $\dfrac{s^2-6s+7}{(s^2-4s+5)^2} = \dfrac{1}{s^2-4s+5}-\dfrac{2s-2}{(s^2-4s+5)^2}$

Thus $\mathcal{L}^{-1}\left\{\dfrac{s^2-6s+7}{(s^2-4s+5)^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2-4s+5}-\dfrac{2s-2}{(s^2-4s+5)^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+5}-\frac{2s-4}{(s^2-4s+5)^2}-\frac{2}{(s^2-4s+5)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}-\frac{2(s-2)}{[(s-2)^2+1]^2}-\frac{2}{[(s-2)^2+1]^2}\right\}$$

$$= e^{2t}\sin t-2e^{2t}\left(\frac{1}{2}\right)t\sin t-2\left(\frac{1}{2}\right)e^{2t}(\sin t-t\cos t)$$

$$= e^{2t}(\sin t-t\sin t-\sin t+t\cos t) = e^{2t}t(\cos t-\sin t)$$

PROBLEM (27): **Find the following inverse Laplace transform using Laplace inversion formula**

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\}$$

SOLUTION: Here $\bar{f}(s) = \dfrac{1}{(s+1)(s-2)^2}$

The poles of $\bar{f}(s)$ are $s = -1$ (simple pole), and $s = 2$ (double pole).

The residues of $e^{st}\bar{f}(s) = \dfrac{e^{st}}{(s+1)(s-2)^2}$ at these poles are :

$$\operatorname*{Res}_{s=-1} e^{st}\bar{f}(s) = \operatorname*{Lt}_{s\to -1}(s+1)\frac{e^{st}}{(s+1)(s-2)^2} = \operatorname*{Lt}_{s\to -1}\frac{e^{st}}{(s-2)^2} = \frac{e^{-t}}{9}$$

$$\operatorname*{Res}_{s=2} e^{st}\bar{f}(s) = \operatorname*{Lt}_{s\to 2}\frac{d}{ds}\left[\frac{(s-2)^2 e^{st}}{(s+1)(s-2)^2}\right]$$

$$= \operatorname*{Lt}_{s\to 2}\frac{d}{ds}\left(\frac{e^{st}}{s+1}\right) = \operatorname*{Lt}_{s\to 2}\left[\frac{(s+1)te^{st} - e^{st}}{(s+1)^2}\right]$$

$$= \frac{3te^{2t} - e^{2t}}{9} = \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$$

Hence by the Laplace inversion formula , we get

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s-2)^2}\right\} = \sum \text{residues of } e^{st}\bar{f}(s)$$

$$= \frac{1}{9}e^{-t} + \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$$

PROBLEM (28): **Find the following inverse Laplace transforms using Laplace inversion formula**

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2(s+1)^3}\right\}$$

SOLUTION: Here $\bar{f}(s) = \dfrac{1}{(s-1)^2(s+1)^3}$

The poles of $\bar{f}(s)$ are $s = 1$ and $s = -1$ which are of orders 2 and 3 respectively .

The residues of $e^{st}\bar{f}(s) = \dfrac{e^{st}}{(s-1)^2(s+1)^3}$ at these poles are :

$$\operatorname*{Res}_{s=1} e^{st}\bar{f}(s) = \operatorname*{Lt}_{s\to 1}\frac{d}{ds}\left[\frac{(s-1)^2 e^{st}}{(s-1)^2(s+1)^3}\right]$$

$$= \operatorname*{Lt}_{s\to 1}\frac{d}{ds}\left[\frac{e^{st}}{(s+1)^3}\right] = \operatorname*{Lt}_{s\to 1}\left[\frac{(s+1)te^{st} - 3e^{st}}{(s+1)^4}\right]$$

$$= \frac{2te^{t} - 3e^{t}}{16} = \frac{1}{16}(2t-3)e^{t}$$

$$\operatorname*{Res}_{s=-1} e^{st}\bar{f}(s) = \operatorname*{Lt}_{s\to -1} \frac{d^2}{ds^2}\left[(s+1)^3 \frac{e^{st}}{(s-1)^2(s+1)^3}\right]$$

$$= \operatorname*{Lt}_{s\to -1} \frac{d^2}{ds^2}\left(\frac{e^{st}}{(s-1)^2}\right) = \operatorname*{Lt}_{s\to -1} \frac{d}{ds}\left[\frac{te^{st}}{(s-1)^2} - \frac{2e^{st}}{(s-1)^3}\right]$$

$$= \operatorname*{Lt}_{s\to -1}\left[\frac{(s-1)t^2e^{st}-2te^{st}}{(s-1)^3} - \frac{(s-1)2te^{st}-6e^{st}}{(s-1)^4}\right]$$

$$= \frac{-2t^2e^{-t}-2te^{-t}}{-8} - \frac{-4te^{-t}-6e^{-t}}{16}$$

$$= \frac{1}{4}t^2e^{-t} + \frac{1}{4}te^{-t} + \frac{1}{4}te^{-t} + \frac{3}{8}e^{-t}$$

$$= \frac{1}{4}t^2e^{-t} + \frac{1}{2}te^{-t} + \frac{3}{8}e^{-t}$$

Hence by the Laplace inversion formula , we get

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2(s+1)^3}\right\} = \sum \text{residues of } e^{st}\bar{f}(s)$$

$$= \frac{1}{16}(2t-3)e^{t} + \left(\frac{1}{4}t^2 + \frac{1}{2}t + \frac{3}{8}\right)e^{-t}$$

INITIAL – VALUE PROBLEMS

PROBLEM (29): **Solve the initial – value problem :**

$$y'' - y = t; \quad y(0) = 1, \quad y'(0) = 1$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - \bar{y}(s) = \frac{1}{s^2}$$

STEP (2): Using the initial conditions , we get

$$(s^2-1)\bar{y}(s) - s - 1 = \frac{1}{s^2}$$

or $\quad \bar{y}(s) = \dfrac{s+1}{s^2-1} + \dfrac{1}{s^2(s^2-1)} = \dfrac{1}{s-1} + \dfrac{1}{s^2-1} - \dfrac{1}{s^2}$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

or $\quad y(t) = e^{t} + \sinh t - t$

PROBLEM (30): **Solve the initial – value problem :**

$$y'' + 4y' + 4y = 4\cos t + 3\sin t; \quad y(0) = 1, \quad y'(0) = 0$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$\left[s^2\bar{y}(s)-s\,y(0)-y'(0)\right]+4\left[s\bar{y}(s)-y(0)\right]+4\bar{y}(s)=4\left(\frac{s}{s^2+1}\right)+3\left(\frac{1}{s^2+1}\right)$$

STEP (2): Using the initial conditions , we get

$$(s^2+4s+4)\bar{y}(s)-s-4=\frac{4s}{s^2+1}+\frac{3}{s^2+1}$$

or
$$\bar{y}(s)=\frac{s+4}{(s+2)^2}+\frac{4s+3}{(s+2)^2(s^2+1)}$$

$$=\frac{(s+2)+2}{(s+2)^2}-\frac{1}{(s+2)^2}+\frac{1}{s^2+1}$$

$$=\frac{1}{s+2}+\frac{1}{(s+2)^2}+\frac{1}{s^2+1}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

or
$$y(t)=e^{-2t}+e^{-2t}t+\sin t$$

PROBLEM (31): **Solve the initial – value problem :**

$$y''+4y'+13y=145\cos 2t\,;\quad y(0)=9,\quad y'(0)=19$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$\left[s^2\bar{y}(s)-s\,y(0)-y'(0)\right]+4\left[s\bar{y}(s)-y(0)\right]+13\bar{y}(s)=\frac{145\,s}{s^2+4}$$

STEP (2): Using the initial conditions , we get

$$(s^2+4s+13)\bar{y}(s)-9s-19-36=\frac{145\,s}{s^2+4}$$

or
$$\bar{y}(s)=\frac{9s+55}{s^2+4s+13}+\frac{145\,s}{(s^2+4)(s^2+4s+13)}$$

$$=\frac{9s+55}{s^2+4s+13}+\frac{9s+16}{s^2+4}-\frac{9s+52}{s^2+4s+13}$$

$$=\frac{3}{s^2+4s+13}+\frac{9s+16}{s^2+4}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+9}\right\}+9\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}+8\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

or
$$y(t)=e^{-2t}\sin 3t+9\cos 2t+8\sin 2t$$

PROBLEM (32): **Solve the initial – value problem :**

$$y''' - 3y'' + 3y' - y = t^2 e^t; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2.$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$[s^3\bar{y}(s) - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2\bar{y}(s) - sy(0) - y'(0)]$$
$$+3[s\bar{y}(s) - y(0)] - \bar{y}(s) = \frac{2}{(s-1)^3}$$

STEP (2): Using the initial conditions , we get

$$s^3\bar{y}(s) - s^2 + 2 - 3s^2\bar{y}(s) + 3s + 3s\bar{y}(s) - 3 - \bar{y}(s) = \frac{2}{(s-1)^3}$$

or $$(s^3 - 3s^2 + 3s - 1)\bar{y}(s) = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

or $$(s-1)^3\bar{y}(s) = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

or $$\bar{y}(s) = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$
$$= \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\right\}$$

or $$y(t) = e^t - te^t - \frac{t^2 e^t}{2} + \frac{t^5 e^t}{60}$$

PROBLEM (33): **Solve the initial – value problem :**

$$y'' + 3y' + 2y = f(t); \quad y(0) = y'(0) = 0 \quad \text{where} \quad f(t) = \begin{cases} 4t & \text{if } 0 \le t < 1 \\ 8 & \text{if } \quad t \ge 1 \end{cases}$$

SOLUTION: We can write $f(t)$ as

$$f(t) = 4t + \begin{cases} 0 & \text{if } \quad 0 \le t < 1 \\ 8 - 4t & \text{if } \quad t \ge 1 \end{cases}$$
$$= 4t + (8 - 4t)u_1(t)$$
$$= 4t + 8u_1(t) - 4(t-1)u_1(t) - 4u_1(t)$$
$$= 4t + 4u_1(t) - 4(t-1)u_1(t)$$

Thus the differential equation becomes

$$y'' + 3y' + 2y = 4t + 4u_1(t) - 4(t-1)u_1(t)$$

STEP (1): Taking the Laplace transform of the differential equation , we get

$$\left[s^2\bar{y}(s)-s\,y(0)-y'(0)\right]+3\left[s\bar{y}(s)-y(0)\right]+2\bar{y}(s)=\frac{4}{s^2}+\frac{4e^{-s}}{s}-\frac{4e^{-s}}{s^2}$$

STEP (2): Using the initial conditions , we get

$$(s^2+3s+2)\bar{y}(s)=\frac{4}{s^2}+\frac{4(s-1)e^{-s}}{s^2}$$

or
$$\bar{y}(s)=\frac{4}{s^2(s+1)(s+2)}+\frac{4(s-1)e^{-s}}{s^2(s+1)(s+2)}$$
$$=\left(-\frac{3}{s}+\frac{2}{s^2}+\frac{4}{s+1}-\frac{1}{s+2}\right)+\left(\frac{5}{s}-\frac{2}{s^2}-\frac{8}{s+1}+\frac{3}{s+2}\right)e^{-s}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\}=\mathcal{L}^{-1}\left\{-\frac{3}{s}+\frac{2}{s^2}+\frac{4}{s+1}-\frac{1}{s+2}\right\}+\mathcal{L}^{-1}\left\{\frac{5e^{-s}}{s}-\frac{2e^{-s}}{s^2}-\frac{8e^{-s}}{s+1}+\frac{3e^{-s}}{s+2}\right\}$$

or
$$y(t)=-3+2t+4e^{-t}-e^{-2t}+\left[5-2(t-1)-8e^{-(t-1)}+3e^{-2(t-1)}\right]u_1(t)$$
$$=\begin{cases}-3+2t+4e^{-t}-e^{-2t} & \text{if } 0\le t<1\\ 4+4(1-2e)e^{-t}-(1-3e^2)e^{-2t} & \text{if } t\ge 1\end{cases}$$

PROBLEM (34): **Solve the initial – value problem :**

$$y''+4y=f(t);\quad y(0)=0,\quad y'(0)=3$$

$$\textbf{where } f(t)=\begin{cases}\sin 3t & \text{if } 0\le t<\pi\\ -\sin 3t & \text{if } t\ge\pi\end{cases}$$

SOLUTION: We can write $f(t)$ as

$$f(t)=3\sin t+\begin{cases}0 & \text{if } 0\le t<\pi\\ -6\sin t & \text{if } t\ge\pi\end{cases}$$
$$=3\sin t-6\sin t\,u_\pi(t)=3\sin t-6\sin(t-\pi)\,u_\pi(t)$$

Thus the differential equation becomes

$$y''+4y=3\sin t-6\sin(t-\pi)\,u_\pi(t)$$

STEP (1): Taking the Laplace transform of the differential equation , we get

$$\left[s^2\bar{y}(s)-s\,y(0)-y'(0)\right]+4\bar{y}(s)=\frac{3}{s^2+1}+\frac{6e^{-\pi s}}{s^2+1}$$

STEP (2): Using the initial conditions , we get

$$(s^2+4)\bar{y}(s)-3=\frac{3}{s^2+1}+\frac{6e^{-\pi s}}{s^2+1}$$

or $$\bar{y}(s) = \frac{3}{s^2+4} + \frac{3}{(s^2+1)(s^2+4)} + \frac{6e^{-\pi s}}{(s^2+1)(s^2+4)}$$

$$= \frac{3}{s^2+4} + \left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) + 2\left(\frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+4}\right)$$

STEP (3): Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+4}\right\}$$

or $$y(t) = \frac{3}{2}\sin 2t + \sin t - \frac{\sin 2t}{2} + 2\sin(t-\pi)u_\pi(t) - \sin 2(t-\pi)u_\pi(t)$$

$$= \sin 2t + \sin t - 2\sin t\, u_\pi(t) - \sin 2t\, u_\pi(t)$$

$$= \begin{cases} \sin 2t + \sin t & \text{if } 0 \le t < \pi \\ -\sin t & \text{if } t \ge \pi \end{cases}$$

PROBLEM (35): **Solve the following initial value problem :**

$$y'' + 2y' + y = \delta_1(t); \quad y(0) = 2, \quad y'(0) = 3$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] + \bar{y}(s) = = e^{-s}$$

STEP (2): Using the initial conditions , we get

$$s^2\bar{y}(s) - 2s - 3 + 2s\bar{y}(s) - 4 + \bar{y}(s) = = e^{-s}$$

or $$(s^2+2s+1)\bar{y}(s) = 2s + 7 + e^{-s}$$

or $$\bar{y}(s) = \frac{2s+7}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}$$

$$= \frac{2(s+1)+5}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}$$

$$= \frac{2}{(s+1)} + \frac{5}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1} + \frac{5}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}\right\} \quad (1)$$

Now $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$, therefore by the second shifting theorem of inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\} = (t-1)e^{-(t-1)}u_1(t)$$

Hence, from equation (1), we have

$$y(t) = 2e^{-t} + 5te^{-t} + (t-1)e^{-(t-1)}u_1(t)$$

$$= \begin{cases} 2e^{-t} + 5te^{-t} & \text{if } 0 \le t < 1 \\ 2e^{-t} + 5te^{-t} + (t-1)e^{-(t-1)} & \text{if } t \ge 1 \end{cases}$$

PROBLEM (36): $y'' - y = \sin t + \delta_{\pi/2}(t); \quad y(0) = 3, \quad y'(0) = -3.5.$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation, we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - \bar{y}(s) = \frac{1}{s^2+1} + e^{-\pi s/2}$$

STEP (2): Using the initial conditions, we get

$$(s^2-1)\bar{y}(s) - 3s + 3.5 = \frac{1}{s^2+1} + e^{-\pi s/2}$$

or
$$\bar{y}(s) = \frac{3s-3.5}{s^2-1} + \frac{1}{(s^2+1)(s^2-1)} + \frac{e^{-\pi s/2}}{s^2-1}$$

$$= \frac{6s-7}{2(s^2-1)} + \frac{1}{(s^2+1)(s^2-1)} + \frac{e^{-\pi s/2}}{s^2-1}$$

$$= \left[\frac{3}{s+1} - \frac{1}{2(s^2-1)}\right] + \frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right] + \frac{e^{-\pi s/2}}{s^2-1}$$

$$= \frac{3}{s+1} - \frac{1}{2(s^2+1)} + \frac{e^{-\pi s/2}}{s^2-1}$$

STEP (3): Taking the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} - \frac{1}{2(s^2+1)} + \frac{e^{-\pi s/2}}{s^2-1}\right\}$$

or
$$y(t) = 3e^{-t} - \frac{1}{2}\sin t + \sinh\left(t - \frac{\pi}{2}\right)u_{\pi/2}(t)$$

$$= \begin{cases} 3e^{-t} - \frac{1}{2}\sin t & \text{if } 0 \le t < \frac{\pi}{2} \\ 3e^{-t} - \frac{1}{2}\sin t + \sinh\left(t - \frac{\pi}{2}\right) & \text{if } t \ge \frac{\pi}{2} \end{cases}$$

PROBLEM (37) $y'' + 2y' - 3y = \delta_2(t) + \delta_3(t); \quad y(0) = 1, \quad y'(0) = 0$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation, we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] - 3\bar{y}(s) = e^{-2s} + e^{-3s}$$

STEP (2): Using the initial conditions, we get

$$(s^2+2s-3)\bar{y}(s) - s - 2 = e^{-2s} + e^{-3s}$$

or $\quad \bar{y}(s) = \dfrac{s+2}{(s-1)(s+3)} + \dfrac{e^{-2s}}{(s-1)(s+3)} + \dfrac{e^{-3s}}{(s-1)(s+3)}$

$$= \left[\frac{3}{4(s-1)} + \frac{1}{4(s+3)}\right] + \frac{1}{4}\left[\frac{e^{-2s}}{s-1} - \frac{e^{-2s}}{s+3}\right] + \frac{1}{4}\left[\frac{e^{-3s}}{s-1} - \frac{e^{-3s}}{s+3}\right]$$

STEP (3): Taking the inverse Laplace transforms , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \frac{3}{4}e^{t} + \frac{1}{4}e^{-3t} + \frac{1}{4}\left[e^{t-2}u_2(t) - e^{-3(t-2)}u_2(t)\right]$$
$$+ \frac{1}{4}\left[e^{(t-3)}u_3(t) - e^{-3(t-3)}u_3(t)\right]$$

or $$y(t) = \begin{cases} \frac{3}{4}e^{t} + \frac{1}{4}e^{-3t} & \text{if } 0 \le t < 2 \\ \left(\frac{3}{4} + \frac{1}{4}e^{-2}\right)e^{t} + \left(\frac{1}{4} - \frac{1}{4}e^{6}\right)e^{-3t} & \text{if } 2 < t < 3 \\ \left(\frac{3}{4} + \frac{1}{4}e^{-2} + \frac{1}{4}e^{-3}\right)e^{t} + \left(\frac{1}{4} - \frac{1}{4}e^{6} - \frac{1}{4}e^{9}\right)e^{-3t} & \text{if } t \ge 3 \end{cases}$$

PROBLEM (38): **Using convolution theorem , solve the following initial value problem :**

$$y'' + y = 2\cos t \; ; \quad y(0) = 2, \quad y'(0) = 0.$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + \bar{y}(s) = \frac{2s}{s^2+1}$$

STEP (2): Using the initial conditions , we get

$$(s^2+1)\bar{y}(s) - 2s = \frac{2s}{s^2+1}$$

or $$\bar{y}(s) = \frac{2s}{s^2+1} + \frac{2s}{(s^2+1)^2}$$

STEP (3): Taking the inverse Laplace transform ,

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

or $$y(t) = 2\cos t + 2(\sin t * \cos t)$$

$$= 2\cos t + 2\int_0^t \sin u \cos(t-u)\, du$$

$$= 2\cos t + \int_0^t [\sin t + \sin(2u - t)]\, du$$

$$= 2\cos t + \left| u \sin t - \frac{\cos(2u-t)}{2} \right|_0^t$$

$$= 2\cos t + t\sin t - \frac{\cos t}{2} + \frac{\cos t}{2} = 2\cos t + t\sin t$$

PROBLEM (39): **Solve the following initial – value problem using convolution theorem :**

$$y'' + y = \sin 3t; \quad y(0) = 0, \quad y'(0) = 0$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + \bar{y}(s) = \frac{3}{s^2+9}$$

STEP (2): Using the initial conditions , we get

$$(s^2+1)\bar{y}(s) = \frac{3}{s^2+9}$$

or $$\bar{y}(s) = \frac{3}{s^2+9}\,\frac{1}{s^2+1}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\,\frac{1}{s^2+1}\right\}$$

$$= \sin 3t * \sin t = \int_0^t \sin 3u \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos(4u-t) - \cos(2u+t)]\, du$$

$$= \frac{1}{2}\left| \frac{\sin(4u-t)}{4} - \frac{\sin(2u+t)}{2} \right|_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin 3t}{4} - \frac{\sin 3t}{2}\right) - \left(\frac{\sin(-t)}{4} - \frac{\sin t}{2}\right)\right]$$

$$= \frac{1}{2}\left(-\frac{1}{4}\sin 3t + \frac{3}{4}\sin t\right) = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t$$

PROBLEM (40): **Solve the following initial – value problem using convolution theorem :**

$$y'' + 25y = 5.2\,e^{-t}; \quad y(0) = 1.2, \quad y'(0) = -10.2$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + 25\bar{y}(s) = \frac{5.2}{s+1}$$

STEP (2): Using the initial conditions , we get

$$(s^2+25)\bar{y}(s)-1.2\,s+10.2 = \frac{5.2}{s+1}$$

or $$\bar{y}(s) = -\frac{10.2}{s^2+25}+\frac{1.2\,s}{s^2+25}+\frac{5.2}{(s+1)(s^2+25)}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = -10.2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\}+1.2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}+5.2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+25)}\right\} \quad (1)$$

Now $$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+25)}\right\} = e^{-t} * \frac{\sin 5t}{5} = \frac{1}{5}(\sin 5t * e^{-t})$$

$$= \frac{1}{5}\int_0^t \sin 5u\, e^{-(t-u)}\,du = \frac{e^{-t}}{5}\int_0^t e^{u}\sin 5u\,du$$

$$= \frac{e^{-t}}{5}\,\frac{1}{(26)}\left| e^{u}(\sin 5u-5\cos 5u)\right|_0^t$$

$$= \frac{e^{-t}}{130}(e^{t}\sin 5t-5e^{t}\cos 5t+5)$$

Hence , from equation (1) , we get

$$y(t) = -\frac{10.2}{5}\sin 5t+1.2\cos 5t+\frac{5.2}{130}(\sin 5t-5\cos 5t+5e^{-t})$$

$$= -2.04\sin 5t+1.2\cos 5t+0.04\sin 5t-0.2\cos 5t+0.2e^{-t}$$

$$= 0.2e^{-t}+\cos 5t-2\sin 5t$$

PROBLEM (41): **Solve the following initial – value problem using convolution theorem :**

$$y''+3y'+2y = 1-u_1(t); \quad y(0)=0, \quad y'(0)=1$$

SOLUTION: **STEP (1):** Taking the Laplace transform of the differential equation , we get

$$[s^2\bar{y}(s)-s\,y(0)-y'(0)]+3[s\bar{y}(s)-y(0)]+2\bar{y}(s) = \frac{1}{s}-\frac{e^{-s}}{s}$$

STEP (2): Using the initial conditions , we get

$$(s^2+3s+2)\bar{y}(s)-1 = \frac{1}{s}-\frac{e^{-s}}{s}$$

or $$(s+1)(s+2)\bar{y}(s) = 1+\frac{1}{s}-\frac{e^{-s}}{s}$$

or $$\bar{y}(s) = \frac{s+1}{s(s+1)(s+2)}-\frac{e^{-s}}{s(s+1)(s+2)}$$

$$= \frac{1}{s(s+2)}-\frac{e^{-s}}{s(s+1)(s+2)}$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)(s+2)}\right\} \tag{1}$$

Now $\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = 1 * e^{-2t} = e^{-2t} * 1 = \int_0^t e^{-2u}\,du = \frac{1}{2}(1-e^{-2t})$

and
$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\,\frac{1}{(s+1)(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\}$$

$$= u_1(t) * (e^{-t} - e^{-2t})$$

$$= \int_0^1 0\,du + \int_1^t \left[(1)e^{-(t-u)} - e^{-2(t-u)}\right] du$$

$$= \left| e^{-(t-u)} - \frac{1}{2}e^{-2(t-u)} \right|_1^t$$

$$= 1 - \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} \quad \text{for } t \geq 1$$

Hence , from equation (1) , we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \begin{cases} \frac{1}{2}(1-e^{-2t}) & \text{if } 0 \leq t < 1 \\ \frac{1}{2}(1-e^{-2t}) - \frac{1}{2} + e^{-(t-1)} - \frac{1}{2}e^{-2(t-1)}, & t \geq 1 \end{cases}$$

or
$$y(t) = \begin{cases} \frac{1}{2}(1-e^{-2t}) & \text{if } 0 \leq t < 1 \\ e^{-t+1} - \frac{1}{2}(1+e^{2})e^{-2t} & \text{if } t \geq 1 \end{cases}$$

APPLICATION TO INTEGRAL EQUATIONS

PROBLEM (42): **Solve the integral equations**

(i) $$y(t) = te^{t} - 2\int_0^t y(u)e^{(t-u)}\,du$$

(ii) $$y(t) = 1 - \sinh t + \int_0^t (1+u)\,y(t-u)\,du$$

SOLUTION: (i) The given equation can be written in the form

$$y(t) = t e^{t} - 2\,y(t) * e^{t}$$

STEP (1): Taking the Laplace transform of both sides of this equation, we get

$$\bar{y}(s) = \frac{1}{(s-1)^2} - 2\mathcal{L}\{y(t) * e^{t}\}$$

STEP (2): Using the convolution theorem, the above equation becomes

$$\bar{y}(s) = \frac{1}{(s-1)^2} - 2\,\bar{y}(s)\frac{1}{s-1}$$

or $$\left(1 + \frac{2}{s-1}\right)\bar{y}(s) = \frac{1}{(s-1)^2}$$

or $$\frac{s+1}{s-1}\bar{y}(s) = \frac{1}{(s-1)^2}$$

or $$\bar{y}(s) = \frac{1}{s^2-1}$$

STEP (3): Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}$$

or $$y(t) = \sinh t$$

(ii) The given equation can be written in the form

$$y(t) = 1 - \sinh t + (1+t) * y(t)$$

STEP (1): Taking the Laplace transform of both sides of this equation, we get

$$\bar{y}(s) = \frac{1}{s} - \frac{1}{s^2-1} + \mathcal{L}\{(1+t) * y(t)\}$$

STEP (2): Using the convolution theorem, this equation becomes

$$\bar{y}(s) = \frac{1}{s} - \frac{1}{s^2-1} + \left(\frac{1}{s} + \frac{1}{s^2}\right)\bar{y}(s)$$

$$= \frac{1}{s} - \frac{1}{s^2-1} + \left(\frac{s+1}{s^2}\right)\bar{y}(s)$$

or $$\left(1 - \frac{s+1}{s^2}\right)\bar{y}(s) = \frac{1}{s} - \frac{1}{s^2-1}$$

or $$\left(\frac{s^2-s-1}{s^2}\right)\bar{y}(s) = \frac{s^2-s-1}{s(s^2-1)}$$

or $$\bar{y}(s) = \frac{s}{s^2-1}$$

STEP (3): Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}$$

or $$y(t) = \cosh t$$

SYSTEMS OF DIFFERENTIAL EQUATIONS

PROBLEM (43): **Solve the following initial – value problems using Laplace transforms :**

$$\left.\begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned}\right\} \quad y_1(0) = 1, \quad y_2(0) = 0$$

SOLUTION: **STEP (1):** Taking the Laplace transforms of both the differential equations , we get

$$s\bar{y}_1(s) - y_1(0) = -\bar{y}_1(s) + \bar{y}_2(s)$$

$$s\bar{y}_2(s) - y_2(0) = -\bar{y}_1(s) - \bar{y}_2(s)$$

STEP (2): Using the initial conditions , we get

$$(s+1)\bar{y}_1(s) - \bar{y}_2(s) = 1$$

$$\bar{y}_1(s) + (s+1)\bar{y}_2(s) = 0$$

Solving this system using Cramer's rule ,

$$\bar{y}_1(s) = \frac{\begin{vmatrix} 1 & -1 \\ 0 & s+1 \end{vmatrix}}{\begin{vmatrix} s+1 & -1 \\ 1 & s+1 \end{vmatrix}} = \frac{s+1}{(s+1)^2+1}$$

$$\bar{y}_2(s) = \frac{\begin{vmatrix} s+1 & 1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} s+1 & -1 \\ 1 & s+1 \end{vmatrix}} = \frac{-1}{(s+1)^2+1}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$y_1(t) = \mathcal{L}^{-1}\{\bar{y}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = e^{-t}\cos t$$

$$y_2(t) = \mathcal{L}^{-1}\{\bar{y}_2(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{(s+1)^2+1}\right\} = -e^{-t}\sin t$$

PROBLEM (44): **Solve the following initial – value problems using Laplace transforms :**

$$\left.\begin{aligned} y_1'' + y_2 &= -5\cos 2t \\ y_2'' + y_1 &= 5\cos 2t \end{aligned}\right\} \quad \begin{aligned} y_1(0) &= 1, \quad y_1'(0) = 1 \\ y_2(0) &= -1, y_2'(0) = 1 \end{aligned}$$

SOLUTION: **STEP (1):** Taking the Laplace transforms of both the differential equations , we get

$$s^2\bar{y}_1(s) - s\,y_1(0) - y_1'(0) + \bar{y}_2(s) = -\frac{5s}{s^2+4}$$

$$s^2\bar{y}_2(s) - s\,y_2(0) - y_2'(0) + \bar{y}_1(s) = \frac{5s}{s^2+4}$$

STEP (2): Using the initial conditions , we get

$$s^2\bar{y}_1(s)+\bar{y}_2(s) = s+1-\frac{5s}{s^2+4} = \frac{s^3+s^2-s+4}{s^2+4}$$

$$\bar{y}_1(s)+s^2\bar{y}_2(s) = -s+1+\frac{5s}{s^2+4} = \frac{-s^3+s^2+s+4}{s^2+4}$$

Solving this system using Cramer's rule ,

$$\bar{y}_1(s) = \frac{\begin{vmatrix} \dfrac{s^3+s^2-s+4}{s^2+4} & 1 \\ \dfrac{-s^3+s^2+s+4}{s^2+4} & s^2 \end{vmatrix}}{\begin{vmatrix} s^2 & 1 \\ 1 & s^2 \end{vmatrix}} = \frac{(s-1)(s+1)(s^3+s^2+s+4)}{(s-1)(s+1)(s^2+1)(s^2+4)}$$

$$= \frac{s^3+s^2+s+4}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1}+\frac{s}{s^2+4}$$

$$\bar{y}_2(s) = \frac{\begin{vmatrix} s^2 & \dfrac{s^3+s^2-s+4}{s^2+4} \\ 1 & \dfrac{-s^3+s^2+s+4}{s^2+4} \end{vmatrix}}{\begin{vmatrix} s^2 & 1 \\ 1 & s^2 \end{vmatrix}} = \frac{(s-1)(s+1)(-s^3+s^2-s+4)}{(s-1)(s+1)(s^2+1)(s^2+4)}$$

$$= \frac{1}{s^2+1}-\frac{s}{s^2+4}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$y_1(t) = \mathcal{L}^{-1}\{\bar{y}_1(s)\} = \sin t + \cos 2t$$

$$y_2(t) = \mathcal{L}^{-1}\{\bar{y}_2(s)\} = \sin t - \cos 2t$$

PROBLEM (45): **Solve the following initial – value problem using Laplace transforms :**

$$\left.\begin{aligned} y_1'+y_2' &= 2\sinh t \\ y_2'+y_3' &= e^t \\ y_3'+y_1' &= 2e^t+e^{-t} \end{aligned}\right\} \quad y_1(0)=1, \quad y_2(0)=1, \quad y_3(0)=0$$

SOLUTION: **STEP (1):** Taking the Laplace transforms of all the differential equations , we get

$$s\bar{y}_1(s)-y_1(0)+s\bar{y}_2(s)-y_2(0) = \frac{2}{s^2-1}$$

$$s\bar{y}_2(s)-y_2(0)+s\bar{y}_3(s)-y_3(0) = \frac{1}{s-1}$$

$$s\bar{y}_3(s)-y_3(0)+s\bar{y}_1(s)-y_1(0) = \frac{2}{s-1}+\frac{1}{s+1}$$

STEP (2): Using the initial conditions , we get

$$s\bar{y}_1(s)+s\bar{y}_2(s)+0=1+1+\frac{2}{s^2-1}=\frac{2s^2}{s^2-1}$$

$$0+s\bar{y}_2(s)+s\bar{y}_3(s)=1+\frac{1}{s-1}=\frac{s}{s-1}$$

$$s\bar{y}_1(s)+0+s\bar{y}_3(s)=1+\frac{2}{s-1}+\frac{1}{s+1}=\frac{s^2+3s}{s^2-1}$$

Solving this system using Cramer's rule , we get

$$\bar{y}_1(s)=\frac{\begin{vmatrix}\frac{2s^2}{s^2-1} & s & 0\\ \frac{s}{s-1} & s & s\\ \frac{s^2+3s}{s^2-1} & 0 & s\end{vmatrix}}{\begin{vmatrix} s & s & 0\\ 0 & s & s\\ s & 0 & s\end{vmatrix}}=\frac{\frac{2s^3(s+1)}{s^2-1}}{2s^3}=\frac{1}{s-1}$$

$$\bar{y}_2(s)=\frac{\begin{vmatrix} s & \frac{2s^2}{s^2-1} & 0\\ 0 & \frac{s}{s-1} & s\\ s & \frac{s^2+3s}{s^2-1} & s\end{vmatrix}}{\begin{vmatrix} s & s & 0\\ 0 & s & s\\ s & 0 & s\end{vmatrix}}=\frac{\frac{2s^3(s-1)}{s^2-1}}{2s^3}=\frac{1}{s+1}$$

$$\bar{y}_3(s)=\frac{\begin{vmatrix} s & s & \frac{2s^2}{s^2-1}\\ 0 & s & \frac{s}{s-1}\\ s & 0 & \frac{s^2+3s}{s^2-1}\end{vmatrix}}{\begin{vmatrix} s & s & 0\\ 0 & s & s\\ s & 0 & s\end{vmatrix}}=\frac{\frac{4s^3}{s^2-1}}{2s^3}=\frac{2}{s^2-1}=\frac{1}{s-1}-\frac{1}{s+1}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$y_1(t)=\mathcal{L}^{-1}\{\bar{y}_1(s)\}=e^{t}$$

$$y_2(t)=\mathcal{L}^{-1}\{\bar{y}_2(s)\}=e^{-t}$$

$$y_3(t)=\mathcal{L}^{-1}\{\bar{y}_3(s)\}=e^{t}-e^{-t}$$

MASS – SPRING SYSTEM

PROBLEM (46): **Find the displacement $y(t)$ of a damped forced mass – spring system when $m = 1$ kg, $K = 1$ N/m, $C = 3$ kg/s, $F(t) = \sin 2t$ N and $y(0) = 0$, $y'(0) = 0$.**

SOLUTION: We know that the differential equation governing the motion of a mass – spring system in the case of damped forced oscillations is

$$\frac{d^2y}{dt^2} + \frac{C}{m}\frac{dy}{dt} + \frac{K}{m}y = \frac{1}{m}F(t) \tag{1}$$

In the present case, $m = 1$, $K = 1$, $C = 3$, $F(t) = \sin 2t$, $y(0) = 0$, $y'(0) = 0$.

Thus equation (1) becomes

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin 2t \tag{2}$$

STEP (1): Taking the Laplace transform of equation (2), we get

$$\left[s^2\bar{y}(s) - sy(0) - y'(0)\right] + 3\left[s\bar{y}(s) - y(0)\right] + \bar{y}(s) = \frac{2}{s^2+4}$$

STEP (2): Using the initial conditions, we get

$$(s^2+3s+1)\bar{y}(s) = \frac{2}{s^2+4} \tag{3}$$

or $$\bar{y}(s) = \frac{2}{(s^2+4)(s^2+3s+1)}$$

$$= -\frac{2s}{15(s^2+4)} - \frac{2}{15(s^2+4)} + \frac{2s}{15(s^2+3s+1)} + \frac{8}{15(s^2+3s+1)}$$

$$= -\frac{2}{15}\left[\frac{s}{s^2+4} + \frac{1}{s^2+4} - \frac{s}{s^2+3s+1} - \frac{4}{s^2+3s+1}\right]$$

$$= -\frac{2}{15}\left[\frac{s}{s^2+4} + \frac{1}{s^2+4} - \frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}} - \frac{\frac{5}{2}}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}}\right]$$

STEP (3): Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\} = -\frac{2}{15}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{s^2+4} - \frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}} - \frac{\frac{5}{2}}{\left(s+\frac{3}{2}\right)^2 - \frac{5}{4}}\right\}$$

or $$y(t) = -\frac{2}{15}\left(\cos 2t + \frac{\sin 2t}{2} - e^{-3t/2}\cosh\frac{\sqrt{5}}{2}t - \sqrt{5}\,e^{-3t/2}\sinh\frac{\sqrt{5}}{2}t\right)$$

COUPLED MASS – SPRING SYSTEM

PROBLEM (47): **Find the displacements $y_1(t)$ and $y_2(t)$ of a coupled mass – spring system assuming that $m_1 = 6\text{ kg}$, $m_2 = 2\text{ kg}$, $K_1 = 4\text{ N/m}$, $K_2 = 1\text{ N/m}$ and $y_1(0) = 2$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = 0$.**

SOLUTION: We know that the system of differential equations governing the vibrations of a coupled system is

$$m_1 y_1'' = -K_1 y_1 + K_2(y_2 - y_1)$$

$$m_2 y_2'' = -K_2(y_2 - y_1)$$

In the present case, $m_1 = 6$, $m_2 = 2$, $K_1 = 4$, $K_2 = 1$. Thus the above system takes the form

$$6y_1'' + 5y_1 - y_2 = 0 \qquad (1)$$

$$2y_2'' + y_2 - y_1 = 0 \qquad (2)$$

STEP (1): Taking the Laplace transforms of equations (1) and (2), we get

$$6\left[s^2\bar{y}_1(s) - s\,y_1(0) - y_1'(0)\right] + 5\bar{y}_1(s) - \bar{y}_2(s) = 0$$

$$2\left[s^2\bar{y}_2(s) - s\,y_2(0) - y_2'(0)\right] + \bar{y}_2(s) - \bar{y}_1(s) = 0$$

STEP (2): Using the initial conditions, we get

$$(6s^2 + 5)\bar{y}_1(s) - \bar{y}_2(s) = 12s$$

$$-\bar{y}_1(s) + (2s^2 + 1)\bar{y}_2(s) = 0$$

Using the Cramer's rule, we get

$$\bar{y}_1(s) = \frac{\begin{vmatrix} 12s & -1 \\ 0 & 2s^2+1 \end{vmatrix}}{\begin{vmatrix} 6s^2+5 & -1 \\ -1 & 2s^2+1 \end{vmatrix}} = \frac{24s^3 + 12s}{12s^4 + 16s^2 + 4} = \frac{6s^3 + 3s}{3s^4 + 4s^2 + 1}$$

$$= \frac{6s^3 + 3s}{(s^2+1)(3s^2+1)} = \frac{3s}{2(s^2+1)} + \frac{3s}{2(3s^2+1)}$$

and

$$\bar{y}_2(s) = \frac{\begin{vmatrix} 6s^2+5 & 12s \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 6s^2+5 & -1 \\ -1 & 2s^2+1 \end{vmatrix}} = \frac{12s}{12s^4 + 16s^2 + 4} = \frac{3s}{3s^4 + 4s^2 + 1}$$

$$= \frac{3s}{(s^2+1)(3s^2+1)} = -\frac{3s}{2(s^2+1)} + \frac{9s}{2(3s^2+1)}$$

STEP (3): Taking the inverse Laplace transforms , we get

$$\mathcal{L}^{-1}\{\bar{y}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{3s}{2(s^2+1)}+\frac{3s}{s(3s^2+1)}\right\}$$

$$\mathcal{L}^{-1}\{\bar{y}_2(s)\} = \mathcal{L}^{-1}\left\{\frac{-3s}{2(s^2+1)}+\frac{9s}{2(3s^2+1)}\right\}$$

or

$$y_1(t) = \frac{3}{2}\cos t + \frac{1}{2}\cos\frac{1}{\sqrt{3}}t$$

$$y_2(t) = -\frac{3}{2}\cos t + \frac{3}{2}\cos\frac{1}{\sqrt{3}}t$$

PROBLEM (48): **Find the displacements $y_1(t)$ and $y_2(t)$ of a coupled mass – spring system assuming that $m_1 = 4$ kg , $m_2 = 2$ kg , $K_1 = 2$ N / m , $K_2 = 2$ N / m , $K_3 = 1$N / m, $F_1(t) = 1$, $F_2(t) = 0$, $y_1(0) = 0$, $y_2(0) = 0$, $y_1'(0) = 0$, $y_2'(0) = 0$.**

SOLUTION: We know that the system of differential equations governing the vibrations of the given system is

$$m_1 y_1'' = -K_1 y_1 + K_2(y_2 - y_1) + F_1(t)$$

$$m_2 y_2'' = -K_2(y_2 - y_1) - K_3 y_2 + F_2(t)$$

In the present case , this takes the form

$$4y_1'' + 4y_1 - 2y_2 = 1 \qquad (1)$$

$$2y_2'' + 3y_2 - 2y_1 = 0 \qquad (2)$$

STEP (1): Taking the Laplace transforms of equations (1) and (2) , we get

$$4[s^2\bar{y}_1(s) - s y_1(0) - y_1'(0)] + 4\bar{y}_1(s) - 2\bar{y}_2(s) = \frac{1}{s}$$

$$2[s^2\bar{y}_2(s) - s y_2(0) - y_2'(0)] + 3\bar{y}_2(s) - 2\bar{y}_1(s) = 0$$

STEP (2): Using the initial conditions , we get

$$(4s^2+4)\bar{y}_1(s) - 2\bar{y}_2(s) = \frac{1}{s}$$

$$-2\bar{y}_1(s) + (2s^2+3)\bar{y}_2(s) = 0$$

Using the Cramer's rule , we get

$$\bar{y}_1(s) = \frac{\begin{vmatrix} \frac{1}{s} & -2 \\ 0 & 2s^2+3 \end{vmatrix}}{\begin{vmatrix} 4s^2+4 & -2 \\ -2 & 2s^2+3 \end{vmatrix}} = \frac{\frac{2s^2+3}{s}}{8s^4+20s^2+8}$$

$$= \frac{2s^2+3}{s(s^2+2)(8s^2+4)}$$

$$= \frac{3}{8s} - \frac{s}{24(s^2+2)} - \frac{8s}{3(8s^2+4)}$$

and $$\bar{y}_2(s) = \frac{\begin{vmatrix} 4s^2+4 & \frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 4s^2+4 & -2 \\ -2 & 2s^2+3 \end{vmatrix}} = \frac{2}{s(8s^4+20s^2+8)}$$

$$= \frac{2}{s(s^2+2)(8s^2+4)}$$

$$= \frac{1}{4s} + \frac{s}{12(s^2+2)} - \frac{8s}{3(8s^2+4)}$$

STEP (3): Taking the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\{\bar{y}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{8s} - \frac{s}{24(s^2+2)} - \frac{8s}{3(8s^2+4)}\right\}$$

$$\mathcal{L}^{-1}\{\bar{y}_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4s} + \frac{s}{12(s^2+2)} - \frac{8s}{3(8s^2+4)}\right\}$$

or $$y_1(t) = \frac{3}{8} - \frac{1}{24}\cos\sqrt{2}\,t - \frac{1}{3}\cos\sqrt{\frac{1}{2}}\,t$$

$$y_2(t) = \frac{1}{4} + \frac{1}{12}\cos\sqrt{2}\,t - \frac{1}{3}\cos\sqrt{\frac{1}{2}}\,t$$

ELECTRIC CIRCUITS

PROBLEM (49): **An RL circuit contains a resistor of resistance 8 ohms, an inductor of inductance 0.5 henries, and a battery of E volts. At $t = 0$, the current is zero. Find the current at any time t if (i) $E = 64$ and (ii) $E = 64\sin 8t$.**

SOLUTION: We know that the differential equation for the current I in an RL circuit is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L}$$

In the present case, $R = 8$ ohms, $L = 0.5$ henries, therefore the above differential equation becomes

$$\frac{dI}{dt} + 16I = 2E(t) \qquad (1)$$

(i) If $E(t) = 64$, then equation (1) takes the form

$$\frac{dI}{dt} + 16I = 128$$

STEP (1): Taking the Laplace transform of both sides of this equation, we get

$$[s\bar{I}(s) - I(0)] + 16\bar{I}(s) = \frac{128}{s}$$

STEP (2): Using the initial condition $I(0) = 0$, we get

$$(s+16)\bar{I}(s) = \frac{128}{s}$$

$$\bar{I}(s) = \frac{128}{s(s+16)} = 8\left(\frac{1}{s} - \frac{1}{s+16}\right)$$

STEP (3): Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{I}(s)\} = 8\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+16}\right\}$$

$$I(t) = 8(1 - e^{-16t})$$

(ii) If $E(t) = 64 \sin 8t$, then equation (1) takes the form

$$\frac{dI}{dt} + 16I = 128 \sin 8t$$

STEP (1): Taking the Laplace transform of this equation, we get

$$[s\bar{I}(s) - I(0)] + 16\bar{I}(s) = 128\left(\frac{8}{s^2+64}\right)$$

STEP (2): Using the initial condition $I(0) = 0$, we get

$$(s+16)\bar{I}(s) = \frac{1024}{s^2+64}$$

$$\bar{I}(s) = \frac{1024}{(s+16)(s^2+64)}$$

$$\bar{I}(s) = \frac{16}{5(s+16)} - \frac{16s}{5(s^2+64)} + \frac{256}{5(s^2+64)}$$

STEP (3): Taking the inverse Laplace transform of both sides, we get

$$\mathcal{L}^{-1}\{\bar{I}(s)\} = \mathcal{L}^{-1}\left\{\frac{16}{5(s+16)} - \frac{16s}{5(s^2+64)} + \frac{256}{5(s^2+64)}\right\}$$

$$I(t) = \frac{16}{5}e^{-16t} - \frac{16}{5}\cos 8t + \frac{32}{5}\sin 8t$$

$$= \frac{16}{5}(e^{-16t} - \cos 8t + 2\sin 8t)$$

Compare with example (18) of chapter (8).

PROBLEM (50): **An RC circuit has a resistor of resistance 20 ohms, a capacitor of capacitance 0.05 farad, and an electromotive force of 100 sin 2 t volts. The initial current and the initial charge on the capacitor are zero. Find the current in the circuit at any time t.**

SOLUTION: We know that the differential equation for the current I in an RC circuit is

$$\frac{dI}{dt} + \frac{1}{RC}I = \frac{1}{R}\frac{dE}{dt} \qquad (1)$$

In the present case, R = 20 ohms, C = 0.05 farad, and E (t) = 100 sin 2 t volts.

Therefore equation (1) becomes

$$\frac{dI}{dt}+I = 10\cos 2t$$

STEP (1): Taking the Laplace transform of both sides of this equation, we get

$$[s\bar{I}(s)-I(0)]+\bar{I}(s) = \frac{10s}{s^2+4}$$

STEP (2): Using the initial condition $I(0)=0$, we get

$$(s+1)\bar{I}(s) = \frac{10s}{s^2+4}$$

or $$\bar{I}(s) = \frac{10s}{(s+1)(s^2+4)} = -\frac{2}{s+1}+\frac{2s}{s^2+4}+\frac{8}{s^2+4}$$

STEP (3): Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{I}(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{s+1}+\frac{2s}{s^2+4}+\frac{8}{s^2+4}\right\}$$

or $$I(t) = -2e^{-t}+2\cos 2t+4\sin 2t$$

ALTERNATIVE METHOD

We know that the differential equation for the charge Q in an RC – circuit is

$$\frac{dQ}{dt}+\frac{1}{RC}Q = \frac{1}{R}E(t) \tag{2}$$

In the present case, R = 20 ohms, C = 0.05 farad, and E (t) = 10 sin 2 t. Thus equation (1) becomes $\frac{dQ}{dt}+Q = 5\sin 2t$.

Taking the Laplace transform of both sides of this equation, we get

$$[s\bar{Q}(s)-Q(0)]+\bar{Q}(s) = \frac{10}{s^2+4}$$

Using the initial condition $Q(0)=0$, we get

$$(s+1)\bar{Q}(s) = \frac{10}{s^2+4}$$

or $$\bar{Q}(s) = \frac{10}{(s+1)(s^2+4)} = \frac{2}{s+1}-\frac{2s}{s^2+4}+\frac{2}{s^2+4}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{Q}(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1}-\frac{2s}{s^2+4}+\frac{2}{s^2+4}\right\}$$

or $$Q(t) = 2e^{-t}-2\cos 2t+\sin 2t \tag{3}$$

The current I at any time t is given by differentiating (3) w.r.t. t

i.e. $$I(t) = \frac{dQ}{dt} = -2e^{-t}+4\sin 2t+2\cos 2t$$ [Compare with example (20) of chapter (8)]

PROBLEM (51): **An LC – circuit contains an inductor of inductance 1 henry, a capacitor of capacitance 0.25 farad, and an electromotive force of 30 sin t volts. The initial current and the initial charge on the capacitor are zero. Find the current I flowing through the circuit at any time t.**

SOLUTION: We know that the differential equation for the current I in an LC circuit is

$$\frac{d^2 I}{d t^2}+\frac{1}{LC} I = \frac{1}{L}\frac{dE}{dt} \tag{1}$$

In the present case, L = 1 henry, C = 0.25 farad, and E (t) = 30 sin t volts, therefore equation (1) becomes

$$\frac{d^2 I}{d t^2}+4 I = 30 \cos t \tag{2}$$

Taking the Laplace transform of both sides of equation (2), we get

$$\left[s^2 \bar{I}(s)-s I(0)-I'(0)\right]+4\bar{I}(s) = \frac{30 s}{s^2+1}$$

Using the initial conditions $I(0) = 0$, $I'(0) = 0$, we get

$$(s^2+4)\bar{I}(s) = \frac{30 s}{s^2+1}$$

or $$\bar{I}(s) = \frac{30 s}{(s^2+1)(s^2+4)} = \frac{10 s}{s^2+1} - \frac{10 s}{s^2+4}$$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{I}(s)\} = \mathcal{L}^{-1}\left\{\frac{10 s}{s^2+1} - \frac{10 s}{s^2+4}\right\}$$

or $$I(t) = 10 \cos t - 10 \cos 2t = 10(\cos t - \cos 2t)$$

ALTERNATIVE METHOD

We know that the differential equation for the current Q in an LC – circuit is

$$\frac{d^2 Q}{d t^2}+\frac{1}{LC} Q = \frac{1}{L} E(t) \tag{3}$$

In the present case, L = 1 henry, C = 0.25 farad, and E (t) = 30 sin t volts, therefore equation (3) becomes

$$\frac{d^2 Q}{d t^2}+4 Q = 30 \sin t \tag{4}$$

Taking the Laplace transform of both sides of equation (2), we get

$$\left[s^2 \bar{Q}(s)-s Q(0)-Q'(0)\right]+4\bar{Q}(s) = \frac{30}{s^2+1}$$

Using the initial conditions $Q(0) = 0$, $Q'(0) = 0$, we get

$$(s^2+4)\bar{Q}(s) = \frac{30}{s^2+1}$$

or $\quad \bar{Q}(s) = \dfrac{30}{(s^2+1)(s^2+4)} = \dfrac{10}{s^2+1} - \dfrac{10}{s^2+4}$

Taking the inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{\bar{Q}(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{s^2+1} - \frac{10}{s^2+4}\right\}$$

or $\quad Q(t) = 10 \sin t - 5 \sin 2t \qquad (5)$

The current I at any time t is obtained by differentiating equation (5) w.r.t. t.

Thus $\quad I(t) = \dfrac{dQ}{dt} = 10 \cos t - 10 \cos 2t = 10(\cos t - \cos 2t)$

Compare with example (21) of chapter (8).

PROBLEM (52): **Find the current $I(t)$ in the RLC – circuit when $R = 160$ ohms, $L = 20$ henrys, $C = 0.002$ farad, $E(t) = 37 \sin 10t$ volts, and $I(0) = 0$, $I'(0) = 0$.**

SOLUTION: We know that the differential equation for the current $I(t)$ in the RLC – circuit is

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC} I = \frac{1}{L}\frac{dE}{dt} \qquad (1)$$

In the present case, $R = 160$, $L = 20$, $C = 0.002$, $E(t) = 37 \sin 10t$, $I(0) = 0$, $I'(0) = 0$.

Thus equation (1) becomes

$$\frac{d^2 I}{dt^2} + 8\frac{dI}{dt} + 25\, I = \frac{37}{2}\cos 10t \qquad (2)$$

STEP (1): Taking the Laplace transform of equation (2), we get

$$[s^2 \bar{I}(s) - s I(0) - I'(0)] + 8[s\bar{I}(s) - I(0)] + 25\bar{I}(s) = \frac{37}{2}\left(\frac{s}{s^2+100}\right)$$

STEP (2): Using the initial condition, we get

$$(s^2 + 8s + 25)\bar{I}(s) = \frac{37}{2}\left(\frac{s}{s^2+100}\right)$$

or
$$\bar{I}(s) = \frac{37}{2}\left[\frac{s}{(s^2+100)(s^2+8s+25)}\right]$$

$$= \frac{37}{2}\left[\frac{-3s+32}{481(s^2+100)} + \frac{3s-8}{481(s^2+8s+25)}\right]$$

$$= -\frac{3}{26}\left(\frac{s}{s^2+100}\right) + \frac{32}{26}\left(\frac{1}{s^2+100}\right) + \frac{3}{26}\left(\frac{s}{s^2+18s+25}\right) - \frac{8}{26}\left(\frac{1}{s^2+8s+25}\right)$$

$$= -\frac{3}{26}\left(\frac{s}{s^2+100}\right) + \frac{32}{26}\left(\frac{1}{s^2+100}\right) + \frac{3}{26}\left[\frac{s+4}{(s+4)^2+9}\right] - \frac{20}{26}\left[\frac{1}{(s+4)^2+9}\right]$$

STEP (3): Taking the inverse Laplace transform , we get

$$\mathcal{L}^{-1}\{\bar{I}(s)\} = -\frac{3}{26}\mathcal{L}^{-1}\left\{\frac{s}{s^2+100}\right\} + \frac{32}{26}\mathcal{L}^{-1}\left\{\frac{1}{s^2+100}\right\}$$
$$+\frac{3}{26}\mathcal{L}^{-1}\left\{\frac{s+4}{(s+4)^2+9}\right\} - \frac{20}{26}\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+9}\right\}$$

or

$$I(t) = -\frac{3}{26}\cos 10t + \frac{8}{65}\sin 10t + \frac{3}{26}e^{-4t}\cos 3t - \frac{10}{39}e^{-4t}\sin 3t$$

Compare with problem (33) of chapter (8) .

ELECTRICAL NETWORKS

PROBLEM (53): **Find the currents $I_1(t)$ and $I_2(t)$ in the electrical network shown in figure (9.34) . Given that $L = 1$ henry , $R_1 = 4$ ohms , $R_2 = 6$ ohms , $C = 0.25$ farad, $E(t) = 12$ volts . Assume that $I_1(0) = 0$, $I_2(0) = 0$.**

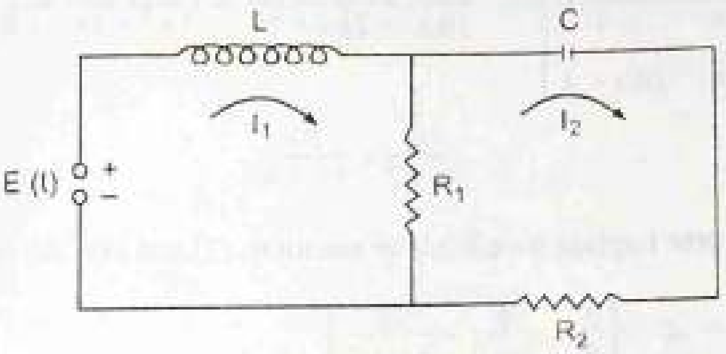


Figure (9.34)

SOLUTION: We know from chapter (8) that the system of differential equations governing the currents I_1 and I_2 in the given network is

$$L\frac{dI_1}{dt} + R_1(I_1 - I_2) = E(t)$$
$$-R_1\frac{dI_1}{dt} + (R_1+R_2)\frac{dI_2}{dt} + \frac{1}{C}I_2 = 0$$

In the present case , this system takes the form

$$\frac{dI_1}{dt} + 4(I_1 - I_2) = 12 \qquad (1)$$
$$-4\frac{dI_1}{dt} + 10\frac{dI_2}{dt} + 4I_2 = 0 \qquad (2)$$

STEP (1): Taking the Laplace transforms of both the differential equations , we get

$$[s\bar{I}_1(s) - I_1(0)] + 4[\bar{I}_1(s) - \bar{I}_2(s)] = \frac{12}{s}$$
$$-4[s\bar{I}_1(s) - I_1(0)] + 10[s\bar{I}_1(s) - I_2(0)] + 4\bar{I}_2(s) = 0$$

STEP (2): Using the initial conditions $I_1(0) = I_2(0) = 0$, we get

$$(s+4)\bar{I}_1(s) - 4\bar{I}_2(s) = \frac{12}{s}$$
$$-4s\bar{I}_1(s) + (10s+4)\bar{I}_2(s) = 0$$

Using the Cramer's rule , we get

$$\bar{I}_1(s) = \frac{\begin{vmatrix} \frac{12}{s} & -4 \\ 0 & 10s+4 \end{vmatrix}}{\begin{vmatrix} s+4 & -4 \\ -4s & 10s+4 \end{vmatrix}} = \frac{\frac{12}{s}(10s+4)}{10s^2+28s+16} = \frac{6(10s+4)}{s(5s^2+14s+8)}$$

$$= \frac{60s+24}{s(s+2)(5s+4)}$$

$$= \frac{3}{s} - \frac{8}{s+2} + \frac{25}{5s+4} \qquad (3)$$

$$\bar{I}_2(s) = \frac{\begin{vmatrix} s+4 & \frac{12}{s} \\ -4s & 0 \end{vmatrix}}{\begin{vmatrix} s+4 & -4 \\ -4s & 10s+4 \end{vmatrix}} = \frac{48}{10s^2+28s+16} = \frac{24}{5s^2+14s+8}$$

$$= -\frac{4}{s+2} + \frac{20}{5s+4} \qquad (4)$$

STEP (3): Taking the inverse Laplace transforms of equations (3) and (4) , we get

$$\mathcal{L}^{-1}\{\bar{I}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{8}{s+2} + \frac{25}{5s+4}\right\}$$

$$\mathcal{L}^{-1}\{\bar{I}_2(s)\} = \mathcal{L}^{-1}\left\{-\frac{4}{s+2} + \frac{20}{5s+4}\right\}$$

or $\quad I_1(t) = 3 - 8e^{-2t} + 5e^{-(4/5)t}$

$$I_2(t) = -4e^{-2t} + 4e^{-(4/5)t}$$

PROBLEM (54): **Find the currents $I_1(t)$ and $I_2(t)$ in the network shown in figure (9.35) assuming that $R_1 = 2$ ohms , $R_2 = 4$ ohms , $R_3 = 4$ ohms , $L_1 = 1$ henry , $L_2 = 2$ henrys , $E(t) = 195 \sin t$ volts , and $I_1(0) = I_2(0) = 0$.**

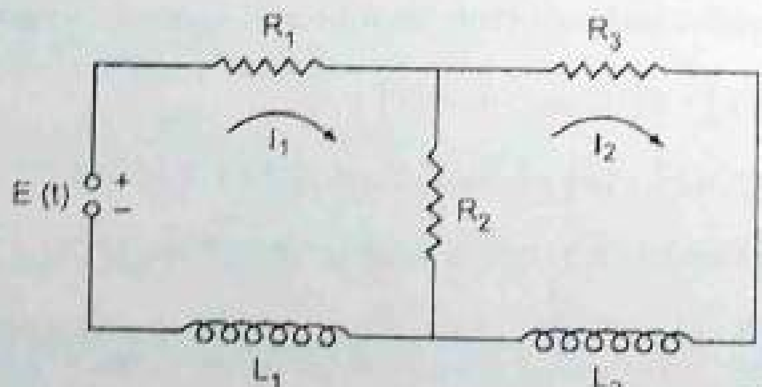


Figure (9.35)

SOLUTION: We know from chapter (5) that the system of differential equations governing the currents I_1 and I_2 in the given network is

$$\frac{dI_1}{dt}+6I_1-4I_2 = 195\sin t$$

$$2\frac{dI_2}{dt}+8I_2-4I_1 = 0$$

STEP (1): Taking the Laplace transforms of both the differential equations, we get

$$s\bar{I}_1(s)-I_1(0)+6\bar{I}_1(s)-4\bar{I}_2(s) = \frac{195}{s^2+1}$$

$$2s\bar{I}_2(s)-2I_2(0)+8\bar{I}_2(s)-4\bar{I}_1(s) = 0$$

STEP (2): Using the initial conditions, we get

$$(s+6)\bar{I}_1(s)-4\bar{I}_2(s) = \frac{195}{s^2+1}$$

$$-4\bar{I}_1(s)+(2s+8)\bar{I}_2(s) = 0$$

Using Cramer's rule, we get

$$\bar{I}_1(s) = \frac{\begin{vmatrix} \frac{195}{s^2+1} & -4 \\ 0 & 2s+8 \end{vmatrix}}{\begin{vmatrix} s+6 & -4 \\ -4 & 2s+8 \end{vmatrix}} = \frac{\frac{195(2s+8)}{s^2+1}}{2(s^2+10s+16)}$$

$$= \frac{195(s+4)}{(s+2)(s+8)(s^2+1)}$$

$$= \frac{13}{s+2}+\frac{2}{s+8}-\frac{15s}{s^2+1}+\frac{42}{s^2+1}$$

$$\bar{I}_2(s) = \frac{\begin{vmatrix} s+6 & \frac{195}{s^2+1} \\ -4 & 0 \end{vmatrix}}{\begin{vmatrix} s+6 & -4 \\ -4 & 2s+8 \end{vmatrix}} = \frac{\frac{780}{s^2+1}}{2(s^2+10s+16)}$$

$$= \frac{390}{(s+2)(s+8)(s^2+1)} = \frac{13}{s+2}-\frac{1}{s+8}-\frac{12s}{s^2+1}+\frac{18}{s^2+1}$$

STEP (3): Taking the inverse Laplace transforms, we get

$$\mathcal{L}^{-1}\{\bar{I}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{13}{s+2}+\frac{2}{s+8}-\frac{15s}{s^2+1}+\frac{42}{s^2+1}\right\}$$

$$\mathcal{L}^{-1}\{\bar{I}_2(s)\} = \mathcal{L}^{-1}\left\{\frac{13}{s+2}-\frac{1}{s+8}-\frac{12s}{s^2+1}+\frac{18}{s^2+1}\right\}$$

or

$$I_1(t) = 13e^{-2t}+2e^{-8t}-15\cos t+42\sin t$$

$$I_2(t) = 13e^{-2t}-e^{-8t}-12\cos t+18\sin t$$

9.33 EXERCISE

LAPLACE TRANSFORMS

PROBLEM (1): Find the Laplace transforms of the following functions, where a and b are constants.

(i) $\mathcal{L}\{e^{at+b}\}$ (ii) $\mathcal{L}\{\cos 2t - \sinh 4t - 4e^{-3t} + 3t^4\}$

(iii) $\mathcal{L}\{\cos(at+b)\}$ (iv) $\mathcal{L}\{\sinh(at+b)\}$

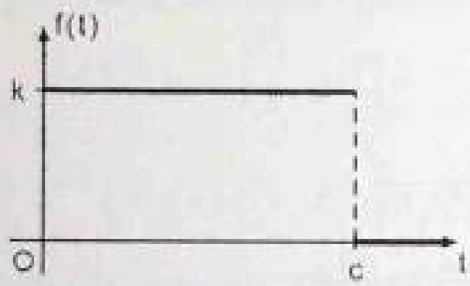
(v) $\mathcal{L}\{\sin^2 at\}$ (vi) $\mathcal{L}\{\cosh^2 3t\}$

PROBLEM (2): Find the Laplace transforms of the following functions :

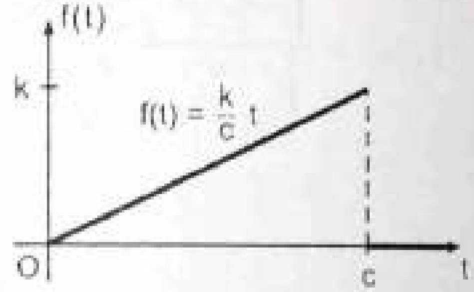
(i) $f(t) = \begin{cases} 3 & 0 \le t < 2 \\ -1 & 2 < t < 4 \\ 0 & t \ge 4 \end{cases}$ (ii) $f(t) = \begin{cases} 2t & \text{if } \quad 0 \le t < 4 \\ 1 & \quad t \ge 4 \end{cases}$

PROBLEM (3): Find the Laplace transforms of the following functions :

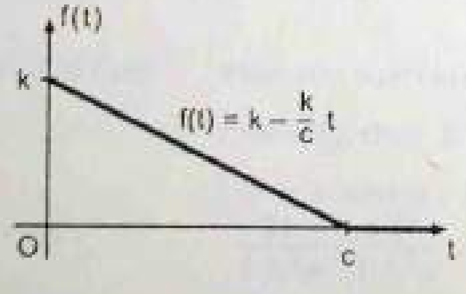
(i)



(ii)



(iii)



(iv)

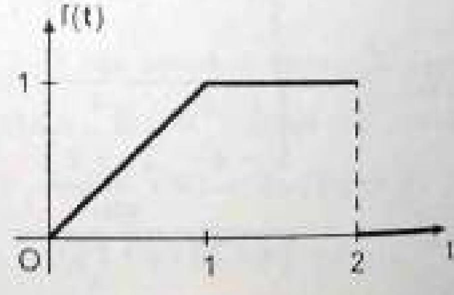


Figure (9.34)

PROBLEM (4): Find the following Laplace transforms using the first shifting theorem (9.8) :

(i) $\mathcal{L}\{t^2 e^{-2t}\}$ (ii) $\mathcal{L}\{2t^3 e^{-t/2}\}$

(iii) $\mathcal{L}\{e^{-t}\cos t\}$ (iv) $\mathcal{L}\{e^{-4t}\cosh 2t\}$

(v) $\mathcal{L}\{e^{4t}(t^3 - \cos t)\}$ (vi) $\mathcal{L}\{e^{t}\cos^2 2t\}$

(vii) $\mathcal{L}\{e^{-\alpha t}(A\cos\beta t + B\sin\beta t)\}$ (viii) $\mathcal{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

PROBLEM (5): Representing the hyperbolic functions in terms of exponential functions and applying the first shifting theorem (9.8), show that

(i) $\mathcal{L}\{\cosh at \sin at\} = \dfrac{a(s^2+2a^2)}{s^4+4a^4}$

(ii) $\mathcal{L}\{\sinh at \cos at\} = \dfrac{a(s^2-2a^2)}{s^4+4a^4}$

(iii) $\mathcal{L}\{\sinh at \sin at\} = \dfrac{2a^2 s}{s^4+4a^4}$

PROBLEM (6): Find the following Laplace transforms using the theorem (9.9):

(i) $\mathcal{L}\{4te^{2t}\}$ (ii) $\mathcal{L}\{2t\cos 2t\}$

(iii) $\mathcal{L}\{t\cosh t\}$ (iv) $\mathcal{L}\{t\sinh 3t\}$

(v) $\mathcal{L}\{t^2\cos\omega t\}$ (v) $\mathcal{L}\{te^{-t}\cos t\}$

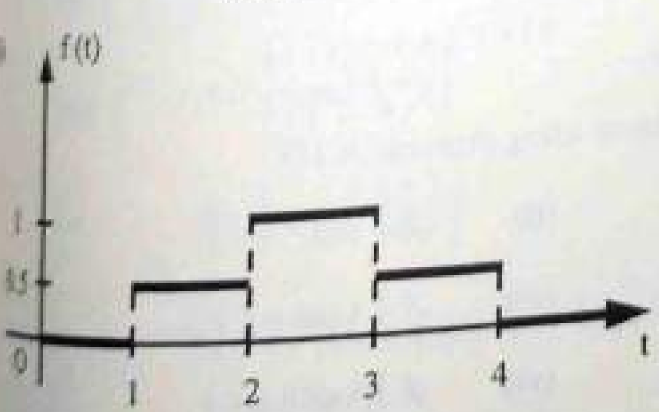
PROBLEM (7): Find the following Laplace transforms using theorem (9.10):

(i) $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\}$ (ii) $\mathcal{L}\left\{\dfrac{e^{-3t}-e^{-5t}}{t}\right\}$

(iii) $\mathcal{L}\left\{\dfrac{\cos 2t-\cos 3t}{t}\right\}$ (iv) $\mathcal{L}\left\{2\left(\dfrac{1-\cosh at}{t}\right)\right\}$

PROBLEM (8): Represent the following functions in terms of unit step functions and find their Laplace transforms:

(i)



(ii)

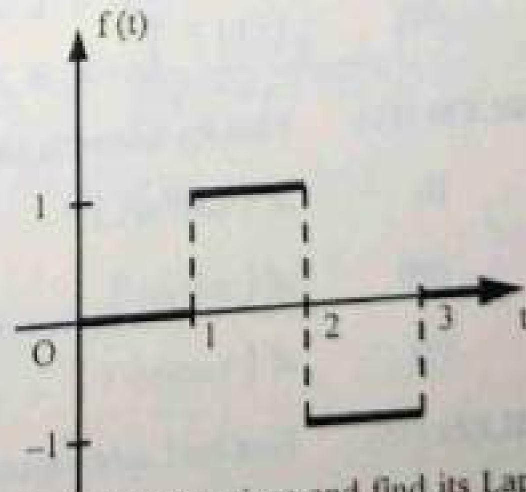


Figure (9.35)

PROBLEM (9): Represent the following function in terms of unit step functions and find its Laplace transform:

$$f(t) = \begin{cases} -2 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 < t < 10 \\ 2 & \text{if } t > 10 \end{cases}$$

PROBLEM (10): Find the following Laplace transforms the second shifting theorem (9.12):

(i) $\mathcal{L}\{(t-1)u_1(t)\}$ (ii) $\mathcal{L}\{(t-1)^2u_1(t)\}$

(iii) $\mathcal{L}\{e^t u_{1/2}(t)\}$ (iv) $\mathcal{L}\{\cos t\, u_\pi(t)\}$

(v) $\mathcal{L}\{\cos 4(t-3)u_3(t)\}$ (vi) $\mathcal{L}\{\sin t\, u_{\pi/2}(t)\}$

(vii) $\mathcal{L}\{\cosh a(t-b)u_b(t)\}$ (viii) $\mathcal{L}\{t^3 u_4(t)\}$

PROBLEM (11): Find the Laplace transforms of the following functions using the second shifting theorem (9.12):

(i) $f(t)=\begin{cases} t & \text{if } 0\le t<1 \\ 0 & \text{if } t>1 \end{cases}$ (ii) $f(t)=\begin{cases} e^t & \text{if } 0\le t<1 \\ 0 & \text{if } t>1 \end{cases}$

(iii) $f(t)=\begin{cases} \sin t & \text{if } 2\pi<t<4\pi \\ 0 & \text{otherwise} \end{cases}$ (iv) $f(t)=\begin{cases} \cos t & \text{if } 0\le t<\pi \\ \sin t & \text{if } t>\pi \end{cases}$

PROBLEM (12): Find the Laplace transforms of the following periodic functions using theorem (9.14)

(i) $f(t)=\pi-t \quad (0\le t<2\pi)$ (ii) $f(t)=t^2 \quad (0\le t<2\pi)$

(iii) $f(t)=e^t \quad (0\le t<2\pi)$

PROBLEM (13): Find the Laplace transforms of the following periodic functions using theorem (9.14)

(i) $f(t)=\begin{cases} t & \text{if } 0\le t<\pi \\ 0 & \text{if } \pi<t<2\pi \end{cases} \qquad f(t+2\pi)=f(t)$

(ii) $f(t)=\begin{cases} 0 & \text{if } 0\le t<\pi \\ t-\pi & \text{if } \pi<t<2\pi \end{cases} \qquad f(t+2\pi)=f(t)$

PROBLEM (14): Find the following Laplace transforms using theorem (9.15):

(i) $\mathcal{L}\{\cos^2\omega t\}$ (ii) $\mathcal{L}\{\sinh^2\omega t\}$

(iii) $\mathcal{L}\{\cos\omega t\}$ (iv) $\mathcal{L}\{t\cos\omega t\}$

(v) $\mathcal{L}\{t\cosh\omega t\}$ (vi) $\mathcal{L}\{t\sinh\omega t\}$

PROBLEM (15): Find the Laplace transforms of the following integrals using theorem (9.16):

(i) $\int_0^t (\sin x-\cos x)\,dx$ (ii) $\int_0^t (3e^{-x}+\sin 6x)\,dx$

(iii) $\int_0^t (x^2-3x+\cos 4x)\,dx$ (iv) $\int_0^t \sin^2 x\,dx$

INVERSE LAPLACE TRANSFORMS

PROBLEM (16): Find the following inverse Laplace transforms :

(i) $\mathcal{L}^{-1}\left\{\dfrac{2}{s-4}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}+\dfrac{1}{s^3}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+16}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s^2+4}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{s-9}{s^2-9}\right\}$

PROBLEM (17): Find the following inverse Laplace transforms using theorem (9.18) :

(i) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+\pi)^2}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{8}{\left(s-\frac{1}{2}\right)^3}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{10}{(s+\pi)^4}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-2s+2}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{s+4}{s^2+2s+10}\right\}$

(vii) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+3)^2+1}\right\}$ (viii) $\mathcal{L}^{-1}\left\{\dfrac{2s+2\alpha}{s^2+2\alpha s+\alpha^2-16}\right\}$

PROBLEM (18): Find the following inverse Laplace transforms using theorem (9.19) :

(i) $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-a)^3}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2+1)^2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{6s}{(s^2-9)^2}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{2s+6}{(s^2+6s+10)^2}\right\}$

(v) $\mathcal{L}^{-1}\left\{\ln\left(\dfrac{s+a}{s+b}\right)\right\}$ (vi) $\mathcal{L}^{-1}\left\{\cot^{-1}(s+1)\right\}$

PROBLEM (19): Find the following inverse Laplace transforms using theorem (9.21) :

(i) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^3}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}-e^{-4s}}{s-2}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{s\,e^{-\pi s}}{s^2+4}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2+\omega^2}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{s(1+e^{-\pi s})}{s^2+1}\right\}$

PROBLEM (20): Find the following inverse Laplace transforms using theorem (9.22) :

(i) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+s}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\left(\dfrac{s-1}{s+1}\right)\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{4}{s^3+4s}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{8}{s^4-4s^2}\right\}$

PROBLEM (21): Find the following convolutions :

(i) $1 * e^t$ (ii) $1 * \sin t$

(iii) $t * t^2$ (iv) $t^2 * t^2$

(v) $e^{kt} * e^{kt}$ (vi) $e^{at} * e^{bt}\ (a \neq b)$

(vii) $\sin t * \sin 2t$ (viii) $u_3(t) * e^{-2t}$

PROBLEM (22): Find the following inverse Laplace transforms using convolution theorem (9.24) :

(i) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s-1)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s-2)^2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^2}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s-1)}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+\omega^2)}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\}$

(vii) $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+\omega^2)^2}\right\}$ (viii) $\mathcal{L}^{-1}\left\{\dfrac{e^{-as}}{s(s-2)}\right\}$

PROBLEM (23): Find the following inverse Laplace transforms using partial fractions :

(i) $\mathcal{L}^{-1}\left\{\dfrac{3s}{s^2+2s-8}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{s^2+9s-9}{s^3-9s}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+1)^2}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{s^2+s-2}{(s+1)^3}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}\right\}$

PROBLEM (24): Find the following inverse Laplace transforms using Laplace inversion formula

(i) $\mathcal{L}^{-1}\left\{\dfrac{s+4}{s(s-2)(s+3)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+a^2)}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\dfrac{s^2-1}{(s^2+1)^2}\right\}$

(vii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$ (viii) $\mathcal{L}^{-1}\left\{\dfrac{3s-1}{s(s-1)^2(s+1)}\right\}$

INITIAL – VALUE PROBLEMS

Solve the following initial – value problems :

(25) $y' + 3y = 10 \sin t; \quad y(0) = 0$

(26) $y' - 5y = \frac{3}{2} e^{-4t}; \quad y(0) = 1$

(27) $y'' + 4y = 0; \quad y(0) = 2, \quad y'(0) = -8$

(28) $y'' + 2y' - 8y = 0; \quad y(0) = 0, \quad y'(0) = 6$

(29) $9y'' - 6y' + y = 0; \quad y(0) = 3, \quad y'(0) = 1$

(30) $y'' + 25y = t; \quad y(0) = 1, \quad y'(0) = \frac{1}{25}$

(31) $y'' + 4y = 1 - 2t; \quad y(0) = 0, \quad y'(0) = 0$

(32) $y'' - 4y = 8t^2 - 4; \quad y(0) = 5, \quad y'(0) = 10$

(33) $y''' + y' = e^t, \quad y(0) = y'(0) = y''(0) = 0$

(34) $y^{(iv)} - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

USE OF FIRST SHIFTING THEOREM

(35) $y'' + 2y' + 2y = 0; \quad y(0) = 0, \quad y'(0) = 1$

(36) $y'' - 4y' + 5y = 0; \quad y(0) = 1, \quad y'(0) = 2$

(37) $y'' + 2y' + y = 2 \cos t; \quad y(0) = 3, \quad y'(0) = 0$

(38) $y'' + 2y' + 5y = 9 \cosh 2t + 4 \sinh 2t; \quad y(0) = 1, \quad y'(0) = 2$

(39) $y'' + 2y' + y = e^{-t}; \quad y(0) = -1, \quad y'(0) = 1$

USE OF SECOND SHIFTING THEOREM

Solve the following initial – value problems :

(40) $y'' + 3y' + 2y = u_2(t); \quad y(0) = 0, \quad y'(0) = 0$

(41) $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 0$ where $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t-1 & \text{if } t > 1 \end{cases}$

(42) $y'' + y = f(t);\ y(0) = 0,\ y'(0) = 0,$ where $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } t > 1 \end{cases}$

(43) $y'' + y = f(t);\ y(0) = 0,\ y'(0) = 0$ where $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t > 1 \end{cases}$

(44) $y'' - 5y' + 6y = f(t);\ y(0) = 1,\ y'(0) = -2$ where $f(t) = \begin{cases} 4e^t & \text{if } 0 \le t < 2 \\ 0 & \text{if } t > 2 \end{cases}$

(45) $y'' + 9y = f(t);\ y(0) = 0,\ y'(0) = 4$ where $f(t) = \begin{cases} 8\sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

USE OF DIRAC DELTA FUNCTION

Solve the following initial – value problems :

(46) $y'' + 4y = \delta_\pi(t);\ y(0) = 2,\ y'(0) = 0$

(47) $y'' + 9y = \delta_1(t);\ y(0) = 0,\ y'(0) = 0$

(48) $y'' + y = \delta_\pi(t) - \delta_{2\pi}(t);\ y(0) = 0,\ y'(0) = 1$

(49) $y'' + 2y' + 2y = \delta_{2\pi}(t);\ y(0) = 1,\ y'(0) = -1$

(50) $y'' + 4y' + 4y = 1 + t + \delta_1(t);\ y(0) = 2,\ y'(0) = 2.25$

(51) $y'' + 2y' + 5y = 8e^t + \delta_1(t);\ y(0) = 2,\ y'(0) = 0$

USE OF CONVOLUTION THEOREM

Solve the following initial – value problems using convolution theorem :

(52) $y'' + y = t;\ y(0) = 0,\ y'(0) = 0$

(53) $y'' + y = \sin t;\ y(0) = 0,\ y'(0) = 0$

(54) $y'' + 3y' + 2y = e^{-t};\ y(0) = 0,\ y'(0) = 0$

(55) $y'' + 4y = u_1(t);\ y(0) = 0,\ y'(0) = 0$

(56) $y'' + y = -2\sin t;\ y(0) = 0,\ y'(0) = 1$

(57) $y'' + y = \frac{3}{2}\sin 2t;\ y(0) = 0,\ y'(0) = 0$

(58) $y'' + 4y = -6\cos t;\ y(0) = 1,\ y'(0) = 0$

APPLICATION TO INTEGRAL EQUATIONS

Solve the following integral equations :

(59) $y(t) = 1 + \int_0^t y(u)\,du$

(60) $y(t) = 1 - \int_0^t y(u)(t-u)\,du$

(61) $y(t) = 2t - 3 - \int_0^t e^u y(t-u)\, du$ (62) $y(t) = t + \int_0^t y(u) \sin(t-u)\, du$

(63) $y(t) = \sin 2t + \int_0^t y(u) \sin 2(t-u)\, du$ (64) $y(t) = \sin t + \int_0^t y(u) \sin(t-u)\, du$

SYSTEMS OF DIFFERENTIAL EQUATIONS

Solve the following systems of differential equations using Laplace transform :

(65) $\left.\begin{array}{l} y_1' = -y_2 \\ y_2' = y_1 \end{array}\right\}$ $y_1(0) = 1, \quad y_2(0) = 0$

(66) $\left.\begin{array}{l} y_1' + y_2 = 2\cos t \\ y_1 + y_2' = 0 \end{array}\right\}$ $y_1(0) = 0, \quad y_2(0) = 1$

(67) $\left.\begin{array}{l} y_1' = 2y_1 + 4y_2 \\ y_2' = y_1 + 2y_2 \end{array}\right\}$ $y_1(0) = -4, \quad y_2(0) = -4$

(68) $\left.\begin{array}{l} y_1' = 2y_1 - 4y_2 \\ y_2' = y_1 - 3y_2 \end{array}\right\}$ $y_1(0) = 3, \quad y_2(0) = 0$

(69) $\left.\begin{array}{l} y_1' = -2y_1 + 3y_2 \\ y_2' = 4y_1 - y_2 \end{array}\right\}$ $y_1(0) = 4, \quad y_2(0) = 3$

(70) $\left.\begin{array}{l} y_1'' = y_1 + 3y_2 \\ y_2'' = 4y_1 - 4e^t \end{array}\right\}$ $y_1(0) = 2, \quad y_1'(0) = 3, \quad y_2(0) = 1, \quad y_2'(0) = 2$

(71) $\left.\begin{array}{l} 2y_1'' + 3y_2' = 4 \\ 2y_2'' - 3y_1' = 0 \end{array}\right\}$ $y_1(0) = 0, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = 0$

(72) $\left.\begin{array}{l} y_1'' + y_2' = \cos t \\ y_2'' - y_1 = \sin t \end{array}\right\}$ $y_1(0) = -1, \quad y_1'(0) = -1, \quad y_2(0) = 1, \quad y_2'(0) = 0$

(73) $\left.\begin{array}{l} y_1'' + y_2 + y_1 = 0 \\ y_2' + y_1' = 0 \end{array}\right\}$ $y_1(0) = 0, \quad y_1'(0) = 0, \quad y_2(0) = 1$

(74) $\left.\begin{array}{l} y_1' + y_2 = \sin t \\ y_2' - y_3 = e^t \\ y_3' + y_1 + y_2 = 1 \end{array}\right\}$ $y_1(0) = 0, \quad y_2(0) = 1, \quad y_3(0) = 1$

(75) $\left.\begin{aligned} y_1' - y_2 &= 0 \\ y_1 + y_2' + y_3 &= 1 \\ y_1 - y_2 + y_3' &= 2\sin t \end{aligned}\right\}$ $y_1(0) = 1,\quad y_2(0) = 1,\quad y_3(0) = 1$

MASS – SPRING SYSTEMS

PROBLEM (76): Find the displacement $y(t)$ of a mass – spring system subject to no damping or external forces for the following data :

$m = 10\text{ kg},\quad K = 1000\text{ N/m},\quad y(0) = 1\text{ m},\quad y'(0) = 0$

PROBLEM (77): Find the displacement $y(t)$ of a mass – spring system subject to a damping but no external forces for the following data :

$m = 10\text{ kg},\quad K = 10\text{ N/m},\quad C = 20\text{ kg/s},\quad y(0) = 0,\quad y'(0) = 1$

PROBLEM (78): Find the displacement $y(t)$ of a mass – spring system subject to a damping and an external force for the following data :

$m = 10\text{ kg},\quad K = 10\text{ N/m},\quad C = 20\text{ kg/s},\quad F = \sin t,\quad y(0) = 0,$
$y'(0) = 1\text{ m/s}$

COUPLED MASS – SPRING SYSTEMS

PROBLEM (79): Find the displacement $y_1(t)$ and $y_2(t)$ of the masses in a coupled mass – spring system with the following data :

(i) $m_1 = m_2 = 1\text{ kg},\ K_1 = 3\text{ N/m},\ K_2 = 2\text{ N/m},\ y_1(0) = 5,\ y_2(0) = 0,$
$y_1'(0) = 0,\quad y_2'(0) = 0.$

(ii) $m_1 = 18\text{ kg},\ m_2 = 3\text{ kg},\ K_1 = 108\text{ N/m},\ K_2 = 18\text{ N/m},$
$y_1(0) = y_2(0) = 0,\ y_1'(0) = 1,\quad y_2'(0) = 0.$

PROBLEM (80): Find the displacements $y_1(t)$ and $y_2(t)$ of the masses in a coupled mass – spring system with the following data :

$m_1 = m_2 = 1\text{ kg},\ K_1 = K_2 = K_3 = 3\text{ N/m},\ F_1(t) = F_2(t) = 0,$
$y_1(0) = 1,\ y_2(0) = 1,\ y_1'(0) = 3,\ y_2'(0) = -3.$

PROBLEM (81): Find the displacements $y_1(t)$ and $y_2(t)$ of the masses in a coupled mass – spring system with the following data :

$m_1 = m_2 = 2\text{ kg},\ K_1 = 6\text{ N/m},\ K_2 = 2\text{ N/m},\ K_3 = 3\text{ N/m},$
$F_1(t) = 2\text{ N},\ F_2(t) = 0,\ y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0.$

ELECTRIC CIRCUITS

PROBLEM (82): An RL circuit contains a resistor of resistance 10 ohms, an inductor of inductance 2 henries, and an electromotive force $E(t)$ volts. At time $t = 0$, the current I is zero. Find the current I at any time t if

(i) $E(t) = 40$, (ii) $E(t) = 20e^{-3t}$, (iii) $E(t) = 50 \sin 5t$

PROBLEM (83): An RC circuit has an e.m.f. of 100 volts, a resistance of 5 ohms, a capacitance of 0.02 farad, and an initial charge on the capacitor of 5 coulombs. Find (i) the charge Q on the capacitor at any time t and (ii) the current in the circuit at any time t.

PROBLEM (84): Find the current $I(t)$ in the LC – circuit with the following data:

(i) $L = 10$ henrys, $C = 0.1$ farad, $E = 10t$ volts, $I(0) = I'(0) = 0$

(ii) $L = 2$ henrys, $C = 0.005$ farad, $E = 220 \sin 4t$ volts, $I(0) = 0 = I'(0) = 0$

PROBLEM (85): Find the current $I(t)$ in the RLC – circuit when $R = 16$ ohms, $L = 2$ henrys, $C = 0.02$ farad, $E(t) = 100 \sin 3t$ N, and $I(0) = 0$, $I'(0) = 0$.

ELECTRICAL NETWORKS

PROBLEM (86): Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the network shown in figure (9.36) assuming that $R = 1$ ohms, $L = 1.25$ henrys, $C = 0.2$ farad, and $I_1(0) = 1$, $I_2(0) = 1$ ampere.

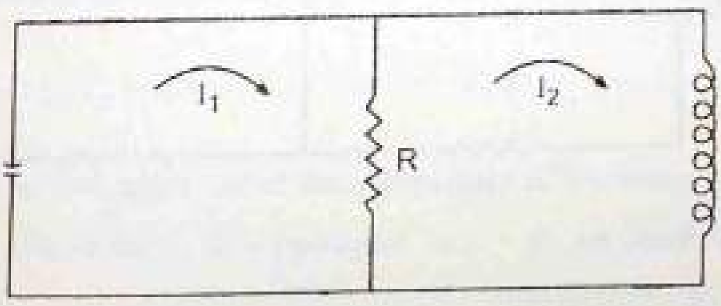


Figure (9.36)

PROBLEM (87): Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the network shown in figure (9.37) assuming that $R = 2.5$ ohms, $L = 1$ henry, $C = 0.04$ farad, $E(t) = 169 \sin t$ volts, and $I_1(0) = I_2(0) = 0$.

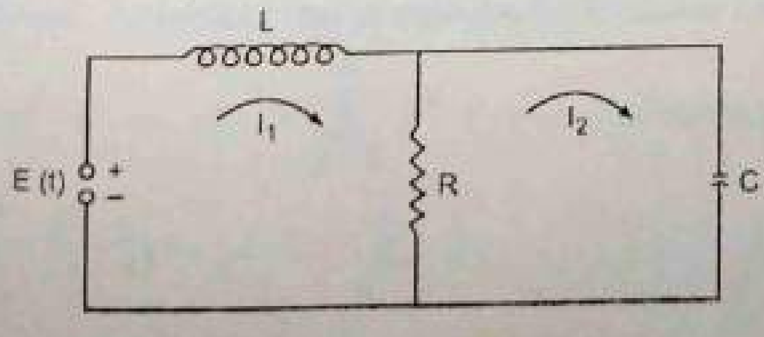


Figure (9.37)

PROBLEM (88): Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the electrical network shown in figure (9.37). Given that $R = 0.8$ ohms, $L = 1$ henry, $C = 0.25$ farad, $E(t) = \frac{4}{5}t + \frac{21}{25}$ volts, $I_1(0) = 1$, $I_2(0) = -3.8$.

PROBLEM (89): Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the network shown in figure (9.39) assuming that $R_1 = 1$ ohm, $R_2 = 1.4$ ohm, $L_1 = 0.8$ henry, $L_2 = 1$ henry, $E(t) = 100$ volts and $I_1(0) = I_2(0) = 0$.

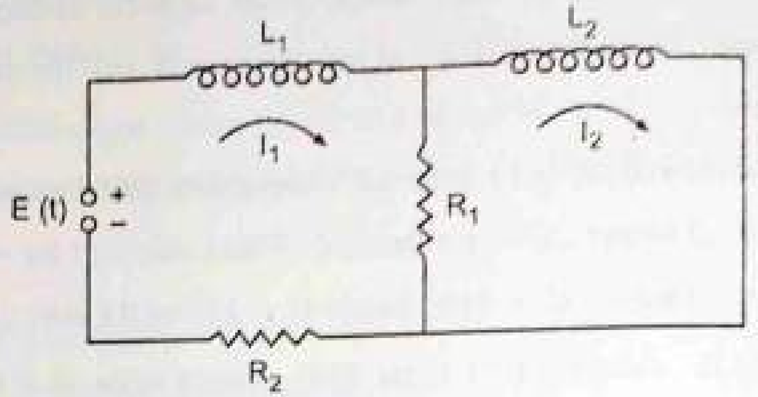


Figure (9.39)

PROBLEM (90): Using the method of Laplace transforms, find the currents $I_1(t)$ and $I_2(t)$ in the network shown in figure (9.40) assuming that $R_1 = 5$ ohms, $R_2 = 10$ ohms, $C = 10^{-3}$ farad, $E(t) = 30$ volts and $I_1(0) = 0$, $I_2(0) = 3$.

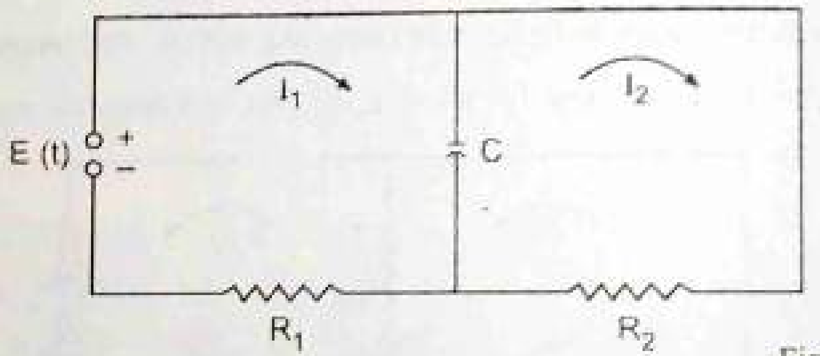


Figure (9.40)

CHAPTER 10

SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

10.1 INTRODUCTION

In chapter (4), we discussed linear differential equations with variable coefficients, whose solutions were obtained in closed form i.e. finite combination of familiar functions known from calculus. In many cases, the solutions of such equations cannot be obtained in closed form. Many of the most important differential equations in applied mathematics, for example, Bessel's equation, Legendre's equation, and hypergeometric equation, to be discussed in next chapters, are of this type. Since these and other equations and their solutions play an important role in applied mathematics, therefore in this chapter, we shall consider two standard methods of solutions — the power series method which yields solutions in the form of power series, and an extension of it, called the Frobenius method named after the German mathematician G. Frobenius (1849 – 1917). The method of Frobenius has great practical importance in applied mathematics.

10.2 POWER SERIES

We know from calculus that a power series in powers of $(x - x_0)$ is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots\dots\dots$$

where $a_0, a_1, a_2, \dots\dots$ are constants called the **coefficients** of the series. x_0 is a constant called the center of the series, and x is a variable. If in particular, $x_0 = 0$, we obtain a power series in powers of x

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots$$

Note that polynomials are also power series, since they have this form. We assume that all variables and constants are real. Examples of power series are the Maclaurin's series :

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots\dots\dots = \sum_{k=0}^{\infty} x^k \quad (|x| < 1, \text{Geometric series})$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\dots\dots = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\dots\dots = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\dots\dots = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$

$$\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\dots\dots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

10.3 ORDINARY AND SINGULAR POINTS OF A DIFFERENTIAL EQUATION

Consider the second order linear homogeneous differential equation with variable coefficients

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \qquad a_0(x) \neq 0 \tag{1}$$

Dividing the equation by $a_0(x)$, we can write it in the standard form (normal form or canonical form) as

$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$

where $P(x) = \frac{a_1(x)}{a_0(x)}$ and $Q(x) = \frac{a_2(x)}{a_0(x)}$.

ORDINARY POINT

Let $x = x_0$ be a point in some open interval I. A point x_0 is called an ordinary point of equation (2) if $a_0(x_0) \neq 0$ and $P(x)$ and $Q(x)$ are analytic at x_0. An ordinary point is also called a regular point of the differential equation.

SINGULAR POINT

A point x_0 is called a singular point of equation (2) if it is not an ordinary point. Thus x_0 is a singular point if $a_0(x_0) = 0$ or if either $P(x)$ or $Q(x)$ fails to be analytic at x_0.

EXAMPLE (1): **Find the ordinary and singular points of the differential equation**

$$x^3(x-2)^2y'' + 5(x+2)(x-2)y' + 3x^2y = 0$$

SOLUTION: Comparing it with equation (1), we have

$$a_0(x) = x^3(x-2)^2, \quad a_1(x) = 5(x+2)(x-2), \quad a_2(x) = 3x^2$$

Now $a_0(x) = x^3(x-2)^2 = 0$ implies $x = 0$ and $x = 2$

which are the singular points. Also, both

$$P(x) = \frac{a_1(x)}{a_0(x)} = \frac{5(x+2)}{x^3(x-2)} \text{ and } Q(x) = \frac{a_2(x)}{a_0(x)} = \frac{3}{x(x-2)^2}$$

are not analytic at $x = 0$ and $x = 2$.

All other real numbers except 0 and 2 are the ordinary (or regular) points of this equation.

10.4 REGULAR AND IRREGULAR SINGULAR POINTS

Writing equation (2) as

$$y'' + \frac{(x-x_0)P(x)}{x-x_0}y' + \frac{(x-x_0)^2Q(x)}{(x-x_0)^2}y = 0$$

or

$$y'' + \frac{P_1(x)}{x-x_0}y' + \frac{Q_1(x)}{(x-x_0)^2}y = 0 \qquad (3)$$

where $P_1(x) = (x-x_0)P(x) = \dfrac{(x-x_0)a_1(x)}{a_0(x)}$

and $Q_1(x) = (x-x_0)^2Q(x) = \dfrac{(x-x_0)^2a_2(x)}{a_0(x)}$

REGULAR SINGULAR POINT

A singular point x_0 of equation (2) is said to be regular singular point if the functions $P_1(x)$ and $Q_1(x)$ in equation (3) are analytic at x_0. In this case, $P_1(x)$ and $Q_1(x)$ have Taylor series expansions about the point $x = x_0$.

IRREGULAR SINGULAR POINT

A singular point x_0 that is not regular is said to be an irregular singular point. In other words, if either $P_1(x)$ or $Q_1(x)$ or both $P_1(x)$ and $Q_1(x)$ are not analytic at x_0, then x_0 is an irregular singular point.

EXAMPLE (2): **Classify the singular points of the differential equation**

$$x^3(x-2)^2y'' + 5(x+2)(x-2)y' + 3x^2y = 0$$

SOLUTION: As seen in example (1), $x = 0$ and $x = 2$ are the singular points of this equation. In this example, $a_0(x) = x^3(x-2)^2$, $a_1(x) = 5(x+2)(x-2)$ and $a_2(x) = 3x^2$.

First consider the point $x_0 = 0$.

Now $P_1(x) = \dfrac{(x-x_0)a_1(x)}{a_0(x)} = \dfrac{5x(x+2)(x-2)}{x^3(x-2)^2} = \dfrac{5(x+2)}{x^2(x-2)}$

is not defined at $x = 0$, hence it is not analytic there.

But $Q_1(x) = \dfrac{(x-x_0)^2a_2(x)}{a_0(x)} = \dfrac{x^2 \cdot 3x^2}{x^3(x-2)^2} = \dfrac{3x}{(x-2)^2}$ is analytic at $x = 0$.

Thus $x = 0$ is an irregular singular point of the differential equation.

Next, consider the point $x = 2$.

Now $P_1(x) = \dfrac{(x-x_0)a_1(x)}{a_0(x)} = \dfrac{5(x+2)(x-2)^2}{x^3(x-2)^2} = \dfrac{5(x+2)}{x^3}$ is analytic at $x = 2$.

Also, $Q_1(x) = \dfrac{(x-x_0)^2a_2(x)}{a_0(x)} = \dfrac{(x-2)^2 \cdot 3x^2}{x^3(x-2)^2} = \dfrac{3}{x}$ is analytic at $x = 2$.

Thus $x = 2$ is a regular singular point of the differential equation.

10.4 POWER SERIES METHOD

The power series method is the standard basic method for solving linear differential equations with variable coefficients. It gives solutions in the form of power series about an ordinary point $x = x_0$. We begin by describing the method.

Consider the second order linear homogeneous differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \qquad (1)$$

STEP (1): Assume a solution in the form of power series in powers of $(x - x_0)$ with unknown coefficients,

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots\ldots \qquad (2)$$

Differentiate equation (2) termwise to get

$$y'(x) = \sum_{k=1}^{\infty} k\, a_k (x - x_0)^{k-1}$$

$$= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \ldots\ldots \qquad (3)$$

and

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k (x - x_0)^{k-2}$$

$$= 2a_2 + 6a_3(x - x_0) + 12a_4(x - x_0)^2 + \ldots\ldots. \qquad (4)$$

Note that the series for y' begins at $k = 1$ and that for y'' at $k = 2$.

STEP (2): Substitute y, y', and y'' in equation (1) and collect like powers of $(x - x_0)$. Then equate the coefficients of each occurring power of $(x - x_0)$ to zero, starting with the constant term, the terms containing $(x - x_0)$, the terms containing $(x - x_0)^2$, etc. This gives relations from which we can determine the unknown coefficients of equation (2).

STEP (3): Substitute these coefficients in equation (1) to get the series solution of differential equation (1).

We illustrate the method for some simple differential equations that can also be solved by elementary methods.

EXAMPLE (3): **Find the power series solution of the differential equation $y' - y = 0$ about an ordinary point $x = 0$.**

SOLUTION: We know that the general solution of this equation is $y = Ce^x$.

STEP (1): Let $y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots \qquad (1)$

be the power series solution of the given differential equation .

Then $$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots\ldots \qquad (2)$$

STEP (2): Substituting y and y' from equations (1) and (2) into the given different equation

$$(a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots) - (a_0 + a_1 x + a_2 x^2 + \ldots\ldots) = 0$$

Then collecting like powers of x , we get

$$(a - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \ldots\ldots = 0$$

Equating the coefficient of each power of x to zero , we have

$$a_1 - a_0 = 0 , \quad 2a_2 - a_1 = 0 , \quad 3a_3 - a_2 = 0 , \ldots\ldots$$

Solving these equations , we may express $a_1 , a_2 , \ldots\ldots$ in terms of a_0 , which remains arbitrary .

$$a_1 = a_0 , \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!} , \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!} , \ldots\ldots$$

STEP (3): With these coefficients , the solution in equation (1) becomes

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \ldots\ldots$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right) = a_0 e^x$$

EXAMPLE (4): **Solve the differential equation $y'' + y = 0$ using the power series method about an ordinary point $x = 0$.**

SOLUTION: We know that the general solution of this equation is $y(x) = A\cos x + B\sin x$

STEP (1): Let the power series solution be

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots \qquad (1)$$

Then $$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots\ldots\ldots$$

and $$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots\ldots\ldots$$

STEP (2): Substituting $y(x)$ and $y''(x)$ in the given differential equation , we get

$$(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots\ldots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots) = 0$$

or $$(2a_2 + a_0) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + (20a_5 + a_3)x^3 + \ldots\ldots = 0$$

Equating the coefficients of various powers of x to zero , we get

$$2a_2 + a_0 = 0 , \quad 6a_3 + a_1 = 0 , \quad 12a_4 + a_2 = 0 , \quad 20a_5 + a_3 = 0 , \ldots\ldots\ldots$$

or $$a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!} , \quad a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!} , \quad a_4 = -\frac{a_2}{12} = \frac{a_0}{24} = \frac{a_0}{4!}$$

$$a_5 = -\frac{a_3}{20} = \frac{a_1}{120} = \frac{a_1}{5!} , \ldots\ldots\ldots \quad (a_0 \text{ and } a_1 \text{ are arbitrary})$$

Substituting the values of $a_2, a_3, a_4, \ldots\ldots\ldots$, in equation (1), we get the required solution as

$$y(x) = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \ldots\ldots\ldots \tag{2}$$

Rearranging the terms in solution (2), we get

$$y(x) = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

10.5 METHOD OF FROBENIUS

This method, like the power series method, is a method for solving linear differential equations with variable coefficients. The power series method gives the series solution of a differential equation about an ordinary (or regular) point $x = x_0$. This method no longer works if the solution is required about a regular singular point $x = x_0$. In this case, the method of Frobenius is used. Thus the Frobenius method applies to more general equations for which the power series method fails.

Consider the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{1}$$

Let $x = x_0$ be a regular singular point of equation (1) i.e. $a_0(x_0) = 0$. Then there may be no solution as a power series about $x = x_0$. In this case, we proceed as follows:

Since x_0 is a regular singular point, we can write equation (1) as

$$y'' + \frac{P_1(x)}{x - x_0}y' + \frac{Q_1(x)}{(x - x_0)^2}y = 0$$

where $P_1(x) = (x - x_0)\dfrac{a_1(x)}{a_0(x)}$ and $Q_1(x) = (x - x_0)^2\dfrac{a_2(x)}{a_0(x)}$

or $$(x - x_0)^2 y'' + (x - x_0)P_1(x)y' + Q_1(x)y = 0 \tag{2}$$

where $P_1(x)$ and $Q_1(x)$ are analytic at $x = x_0$.

STEP (1): Let the series solution of equation (2) be

$$y(x) = \sum_{k=-\infty}^{\infty} a_k (x - x_0)^{k+\beta} \tag{3}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots$. This series is called a Frobenius series. A Frobenius series may not be a power series, since β may be negative or noninteger.

Differentiate equation (3) to get

$$y'(x) = \sum_{k=-\infty}^{\infty} (k+\beta)a_k(x - x_0)^{k-\beta-1}$$

and $$y''(x) = \sum_{k=-\infty}^{\infty} (k+\beta)(k+\beta-1)a_k(x - x_0)^{k-\beta-2}$$

Since $P_1(x)$ and $Q_1(x)$ are analytic at $x = x_0$, we first represent these functions by power series in powers of $(x - x_0)$

i.e. $P_1(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots\dots\dots$

and $Q_1(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \dots\dots\dots$

Often $P_1(x)$ and $Q_1(x)$ are polynomials, therefore nothing needs to be done in that case.

STEP (2): Substituting for $P_1(x)$, $Q_1(x)$, y, y', and y'' in equation (2), we get

$$(x-x_0)^2 \sum_{k=-\infty}^{\infty} (k+\beta)(k+\beta-1)a_k(x-x_0)^{k-\beta-2}$$

$$+[b_0+b_1(x-x_0)+b_2(x-x_0)^2+\dots\dots](x-x_0) \sum_{k=-\infty}^{\infty} (k+\beta)a_k(x-x_0)^{k+\beta-1}$$

$$+[c_0+c_1(x-x_0)+c_2(x-x_0)^2+\dots\dots] \sum_{k=-\infty}^{\infty} a_k(x-x_0)^{k+\beta} = 0$$

or $$\sum_{k=-\infty}^{\infty} (k+\beta)(k+\beta-1)a_k(x-x_0)^{k+\beta}$$

$$+[b_0+b_1(x-x_0)+b_2(x-x_0)^2+\dots\dots] \sum_{k=-\infty}^{\infty} (k+\beta)a_k(x-x_0)^{k+\beta}$$

$$+[c_0+c_1(x-x_0)+c_2(x-x_0)^2+\dots\dots] \sum_{k=-\infty}^{\infty} a_k(x-x_0)^{k+\beta} = 0 \qquad (4)$$

Now collect the coefficients of various powers of $(x - x_0)$. Since the equation is an identity, the coefficients of various powers of $(x - x_0)$ must vanish. The coefficient of the lowest degree term $(x-x_0)^\beta$ is obtained by substituting $k = 0$ in equation (4)

i.e. $[\beta(\beta-1)a_0 + b_0\beta a_0 + c_0 a_0](x - x_0)^\beta = 0$

Since $(x-x_0)^\beta \neq 0$, therefore

$$\beta(\beta-1)a_0 + b_0\beta a_0 + c_0 a_0 = 0$$

or $$[\beta(\beta-1) + b_0\beta + c_0]a_0 = 0$$

Since $a_0 \neq 0$ by assumption, we get

$$\beta(\beta-1) + b_0\beta + c_0 = 0 \qquad (5)$$

This equation is called the **indicial equation** of the differential equation. This quadratic equation has two roots $\beta = \beta_1$ and $\beta = \beta_2$. These roots are called the **indicial roots**. One of these roots gives a solution in the form of series (3). Let this solution be $y_1(x)$. The form of the second independent solution $y_2(x)$ depends on the value of the second indicial root. Equating the coefficients of the remaining powers of $(x - x_0)$ to zero, we get a recurrence relation relating the coefficients a_k. Substituting the indicial roots β_1 and β_2, the values of the coefficients a_k are obtained.

STEP (3): The general solution of the differential equation can now be written as

$$y(x) = A y_1(x) + B y_2(x) \tag{6}$$

where A and B are arbitrary constants.

Depending on the nature of the indicial roots β_1 and β_2, there are the following three cases:

CASE (1): Unequal Indicial Roots not Differing by an Integer

Let the two indicial roots be β_1 and β_2. Substituting $\beta = \beta_1$ and $\beta = \beta_2$ in the recurrence relation for the coefficients, we obtain two sets of values of the coefficients.

Let these sets of coefficients be $a_0, a_1, a_2, \ldots\ldots$ and $a_0^*, a_1^*, a_2^*, \ldots\ldots$ respectively.

Note that $a_1, a_2, \ldots\ldots$ are expressed in terms of a_0 and $a_1^*, a_2^*, \ldots\ldots$ are expressed in terms of a_0^*.

Then the two solutions are

$$y_1(x) = (x - x_0)^{\beta_1}\left[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots\ldots\right]$$

and $$y_2(x) = (x - x_0)^{\beta_2}\left[a_0^* + a_1^*(x - x_0) + a_2^*(x - x_0)^2 + \ldots\ldots\right]$$

Since $\beta_1 - \beta_2$ is not an integer, $\dfrac{y_1}{y_2}$ is never a constant. Thus $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the given equation.

CASE (2): Equal Indicial Roots

In this case $\beta_1 = \beta_2 = \beta$. The first solution is the Frobenius solution

$$y_1(x) = (x - x_0)^{\beta}\left[a_0 + a_1(x - x_0) + a_2(x - x_0) + \ldots\ldots\right]$$

and the second linearly independent solution is obtained by substituting the value of β in $\dfrac{\partial y}{\partial \beta}$

i.e. $$y_2(x) = \left(\frac{\partial y}{\partial \beta}\right)_{\beta = 0}$$. It will be of the form

$$y_2(x) = y_1(x)\ln x + (x - x_0)^{\beta}\left[a_1^*(x - x_0) + a_2^*(x - x_0)^2 + \ldots\ldots\right]$$

The second solution will always consist of the product of the first solution $y_1(x)$ (or a numerical multiple of it) and $\ln x$, added to another series.

CASE (3): Unequal Indicial Roots Differing by an Integer Making a Coefficient of y Infinite

In this case, the indicial equation has two unequal roots β_1 and β_2 (say $\beta_1 > \beta_2$) differing by an integer, and if some of the coefficients of $y(x)$ become infinite when $\beta = \beta_2$, we modify the form of y by replacing a_0 by $c_0(\beta - \beta_2)$. We then get two linearly independent solutions by putting

$\beta = \beta_2$ in the modified form of y and $\frac{\partial y}{\partial \beta}$. The result of putting $\beta = \beta_1$ in y simply gives a numerical multiple of that obtained by putting $\beta = \beta_2$.

CASE (4): Unequal Indicial Roots Differing by an Integer Making a Coefficient of y Indeterminate

In this case, the indicial equation has two unequal roots β_1 and β_2 (say $\beta_1 > \beta_2$) differing by an integer, and if one of the coefficients of y becomes indeterminate when $\beta = \beta_2$, the complete solution is given by putting $\beta = \beta_2$ in y, which then contains two arbitrary constants. The result of putting $\beta = \beta_1$ in y simply gives a numerical multiple of one of the series obtained in the first solution.

CASE (1)

EXAMPLE (5): **Solve the differential equation $4xy'' + 2y' + y = 0$ by the method of Frobenius about the regular singular point $x = 0$.**

SOLUTION: Let the series solution be

$$y(x) = x^{\beta}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \qquad (1)$$

where β and a_k are constants such that $a_k = 0$ for $k = -1, -2, -3, \dots\dots\dots$

By differentiation, omitting the summation index k, we have

$$y'(x) = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y''(x) = \sum (k-\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

Substituting y, y', and y'' into the given differential equation, we get

$$4x \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2} + 2\sum (k+\beta) a_k x^{k+\beta-1} + \sum a_k x^{k+\beta} = 0$$

or $$\sum 4(k+\beta)(k+\beta-1) a_k x^{k+\beta-1} + \sum 2(k+\beta) a_k x^{k+\beta-1} + \sum a_k x^{k+\beta} = 0$$

Shifting index in the third term to write this equation as

$$\sum 4(k+\beta)(k+\beta-1) a_k x^{k+\beta-1} + \sum 2(k+\beta) a_k x^{k+\beta-1} + \sum a_{k-1} x^{k+\beta-1} = 0$$

or $$\sum [\{4(k+\beta)(k+\beta-1) + 2(k+\beta)\} a_k + a_{k-1}] x^{k+\beta-1} = 0 \qquad (2)$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by putting $k = 0$ in equation (2)

i.e. $$[\{4\beta(\beta-1) + 2\beta\} a_0 + a_{-1}] x^{\beta-1} = 0$$

Since the coefficient of $x^{\beta-1}$ must be zero, therefore

$$[4\beta(\beta-1) + 2\beta] a_0 + a_{-1} = 0$$

But $a_{-1} = 0$, we get the indicial equation

$$[4\beta(\beta-1)+2\beta]a_0 = 0$$

Now $a_0 \neq 0$, therefore $4\beta(\beta-1)+2\beta = 0$

or $\quad 4\beta^2 - 2\beta = 0 \quad$ or $\quad \beta = 0 \quad$ and $\quad \beta = \frac{1}{2}$ are the indicial roots which are distinct and not differing by an integer. Equating to zero the coefficient of $x^{k+\beta-1}$, we get from equation (2), the recurrence relation

$$[4(k+\beta)(k+\beta-1)+2(k+\beta)]a_k + a_{k-1} = 0$$

or $\quad 2(k+\beta)(2k+2\beta-1)a_k + a_{k-1} = 0$

or $$a_k = -\frac{a_{k-1}}{2(k+\beta)(2k+2\beta-1)} \tag{3}$$

Let $k = 1$ in equation (3), then $a_1 = -\dfrac{a_0}{2(\beta+1)(2\beta+1)}$

Let $k = 2$ in equation (3), $a_2 = -\dfrac{a_1}{2(\beta+2)(2\beta+3)} = \dfrac{a_0}{4(\beta+1)(\beta+2)(2\beta+1)(2\beta+3)}$

When $\beta = 0$, we get $\quad a_1 = -\dfrac{a_0}{2} = -\dfrac{a_0}{2!}, \quad a_2 = \dfrac{a_0}{4.1.2.1.3} = \dfrac{a_0}{24} = \dfrac{a_0}{4!}$

Thus from equation (1), one Frobenius solution corresponding to $\beta = 0$ is

$$y_1(x) = a_0\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \ldots\ldots\ldots\right]$$

When $\beta = \frac{1}{2}$, we get $\quad a_1 = -\dfrac{a_0}{2\left(\frac{3}{2}\right)(2)} = -\dfrac{a_0}{6} = -\dfrac{a_0}{3!}$

$$a_2 = \frac{a_0}{4\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)(2)(4)} = \frac{a_0}{120} = \frac{a_0}{5!}$$

The second Frobenius solution corresponding to $\beta = \frac{1}{2}$ is

$$y_2(x) = a_0 x^{1/2}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} - \ldots\ldots\ldots\right]$$

The general solution is given by

$$\begin{aligned} y(x) &= A_1 y_1 + A_2 y_2 \\ &= A_1 a_0\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \ldots\ldots\ldots\right] + A_2 a_0 x^{1/2}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} + \ldots\ldots\ldots\right] \\ &= A\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \ldots\ldots\ldots\right] + B x^{1/2}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} - \ldots\ldots\ldots\right] \end{aligned}$$

where $A = A_1 a_0$ and $B = A_2 a_0$.

CASE (2)

EXAMPLE (6): **Solve the differential equation $x y'' + y' + x y = 0$ using the method of Frobenius about the regular singular point $x = 0$.**

SOLUTION: **STEP (1):** Let the series solution be

$$y(x) = x^{\beta}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \qquad (1)$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \dots\dots\dots$

By differentiation, omitting the summation index k, we get

$$y'(x) = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y''(x) = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$x \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2} + \sum (k+\beta) a_k x^{k+\beta-1} + x \sum a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} + \sum (k+\beta) a_k x^{k+\beta-1} + \sum a_k x^{k+\beta+1} = 0$$

Shifting index in the third term to write it as

$$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} + \sum (k+\beta) a_k x^{k+\beta-1} + \sum a_{k-2} x^{k+\beta-1} = 0$$

or $$\sum [(k+\beta)(k+\beta-1) a_k + (k+\beta) a_k + a_{k-2}] x^{k+\beta-1} = 0 \qquad (2)$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by substituting $k = 0$ in equation (2)

i.e. $$[\beta(\beta-1) a_0 + \beta a_0 + a_{-2}] x^{\beta-1} = 0$$

Since the coefficient of $x^{\beta-1}$ must be zero, therefore $\beta(\beta-1) a_0 + \beta a_0 + a_{-2} = 0$.

But $a_{-2} = 0$, therefore, we get the indicial equation

$$\beta(\beta-1) a_0 + \beta a_0 = 0 \quad \text{or} \quad \beta^2 a_0 = 0$$

Since $a_0 \neq 0$, therefore $\beta^2 = 0$ or $\beta = 0, \ 0$ are the indicial roots which are equal.

Equating to zero the coefficient of $x^{k+\beta-1}$, we get from equation (2)

$$(k+\beta)(k+\beta-1) a_k + (k+\beta) a_k + a_{k-2} = 0$$

or $$(k+\beta)^2 a_k = -a_{k-2}$$

or $$a_k = -\frac{a_{k-2}}{(k+\beta)^2} \qquad (3)$$

Let $k = 1$, then $a_1 = -\dfrac{a_{-1}}{(\beta+1)^2} = 0$ (since $a_{-1} = 0$)

Let $k = 2$, then $a_2 = -\dfrac{a_0}{(\beta+2)^2}$

Let $k = 3$, then $a_3 = -\dfrac{a_1}{(\beta+3)^2} = 0$ (since $a_1 = 0$)

Let $k = 4$, then $a_4 = -\dfrac{a_2}{(\beta+4)^2} = \dfrac{a_0}{(\beta+2)^2(\beta+4)^2}$

Let $k = 5$, then $a_5 = -\dfrac{a_3}{(\beta+5)^2} = 0$ (since $a_3 = 0$)

Let $k = 6$, then $a_6 = -\dfrac{a_4}{(\beta+6)^2} = -\dfrac{a_0}{(\beta+2)^2(\beta+4)^2(\beta+6)^2}$

STEP (3): Substituting these coefficients in equation (1), we get

$$y(x) = a_0 x^\beta\left[1 - \frac{x^2}{(\beta+2)^2} + \frac{x^4}{(\beta+2)^2(\beta+4)^2} - \frac{x^6}{(\beta+2)^2(\beta+4)^2(\beta+6)^2} + \ldots\ldots\right] \tag{4}$$

Putting $\beta = 0$, we get one solution as

$$y_1(x) = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots\right]$$

To get the second linearly independent solution, we differentiate equation (4) partially w.r.t. β and put $\beta = 0$.

Now $$\frac{\partial y}{\partial \beta} = a_0 x^\beta \ln x\left[1 - \frac{x^2}{(\beta+2)^2} + \frac{x^4}{(\beta+2)^2(\beta+4)^2} - \frac{x^6}{(\beta+2)^2(\beta+4)^2(\beta+6)^2} + \ldots\ldots\right]$$
$$+ a_0 x^\beta\left[\frac{2}{(\beta+2)^3}x^2 + \frac{-4\beta+12}{(\beta+2)^3(\beta+4)^3} + \ldots\ldots\right]$$

Then $$y_2(x) = \left(\frac{\partial y}{\partial \beta}\right)_{\beta=0} = a_0 \ln x\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\right] + a_0\left[\frac{1}{2^2}x^2 - \frac{3}{128}x^4 + \ldots\right]$$
$$= y_1 \ln x + a_0\left(\frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}\left(1 + \frac{1}{2}\right)x^4 + \ldots\ldots\right)$$

The general solution is given by

$$y(x) = A\,y_1(x) + B\,y_2(x)$$

CASE (3)

EXAMPLE (7): **Solve the differential equation $(x - x^2)y'' - 3xy' - y = 0$ using the method of Frobenius about the regular singular point $x = 0$.**

SOLUTION: **STEP (1):** Let the series solution be

$$y(x) = x^\beta(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \tag{1}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots\ldots$

By differentiation, omitting the summation index k, we get

$$y'(x) = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y''(x) = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$(x-x^2)\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2} - 3x\sum (k+\beta) a_k x^{k+\beta-1} - \sum a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} - \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta}$$
$$- \sum 3(k+\beta) a_k x^{k+\beta} - \sum a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} - \sum [(k+\beta)(k+\beta-1) + 3(k+\beta) + 1] a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} - \sum [(k+\beta)^2 + 2(k+\beta) + 1] a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} - \sum (k+\beta+1)^2 a_k x^{k+\beta} = 0$$

Shifting index in the second summation, we get

$$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} - \sum (k+\beta)^2 a_{k-1} x^{k+\beta-1} = 0 \qquad (2)$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by substituting $k = 0$ in equation (2)

i.e. $[\beta(\beta-1) a_0 - \beta^2 a_{-1}] x^{\beta-1} = 0$

Since the coefficient of $x^{\beta-1}$ must be zero, therefore $\beta(\beta-1) a_0 - \beta^2 a_{-1} = 0$.

But $a_{-1} = 0$, therefore $\beta(\beta-1) a_0 = 0$

Since $a_0 \neq 0$, therefore $\beta(\beta-1) = 0$ or $\beta = 1, \beta = 0$.

That is, $\beta_1 = 1, \beta_2 = 0$ are the indicial roots which are unequal and differ by an integer.

Equating to zero the coefficient of $x^{k+\beta-1}$, we get from equation (2) the recurrence relation

$$(k+\beta)(k+\beta-1) a_k - (k+\beta)^2 a_{k-1} = 0$$

or $$a_k = \frac{k+\beta}{k+\beta-1} a_{k-1}$$

For $k = 1$, $a_1 = \dfrac{\beta+1}{\beta} a_0$

For $k = 2$, $a_2 = \dfrac{\beta+2}{\beta+1} a_1 = \dfrac{\beta+2}{\beta} a_0$

For $k = 3$, $a_3 = \dfrac{\beta+3}{\beta+2} a_2 = \dfrac{\beta+3}{\beta} a_0$

STEP (3): Substituting the values of these coefficients in equation (1), we get

$$y(x) = a_0 x^{\beta}\left(1+\frac{\beta+1}{\beta}x+\frac{\beta+2}{\beta}x^2+\frac{\beta+3}{\beta}x^3+\dots\dots\right) \tag{3}$$

Let $\beta = 0$ in equation (3), we see that the coefficients of $x, x^2, x^3, \dots\dots$ become infinite due to the presence of the factor β in the denominators. To overcome this difficulty, we replace a_0 by $c_0(\beta-\beta_2) = c_0\beta$ in equation (3), we get

$$y(x) = c_0 x^{\beta}\left[\beta+(\beta+1)x+(\beta+2)x^2+(\beta+3)x^3+\dots\dots\right] \tag{4}$$

Now let $\beta = 0$ in this equation, we get one solution as

$$y_1(x) = c_0(1+x+x^2+x^3+\dots\dots)$$

To get the second linearly independent solution, we differentiate partially equation (4) w.r.t. β and put $\beta = 0$.

Now $\dfrac{\partial y}{\partial\beta} = c_0 x^{\beta}\ln x\left[\beta+(\beta+1)x+(\beta+2)x^2+(\beta+3)x^3+\dots\dots\right]$

$$+c_0 x^{\beta}\left[1+x+x^2+x^3+\dots\dots\right]$$

Then $y_2(x) = \left(\dfrac{\partial y}{\partial\beta}\right)_{\beta=0} = c_0\ln x\left[x+2x^2+3x^3+\dots\dots\right]+c_0(1+x+x^2+x^3+\dots\dots)$

$$= y_1\ln x+c_0(1+x+x^2+x^3+\dots\dots)$$

Thus the general solution is given by

$$y(x) = A\,y_1(x)+B\,y_2(x)$$

CASE (4)

EXAMPLE (8): **Solve the differential equation $(1-x^2)y''+2xy'+y = 0$ using the method of Frobenius about the point $x = 0$.**

SOLUTION: Here $x = 0$ is an ordinary point.

STEP (1): Let the series solution be

$$y(x) = x^{\beta}(a_0+a_1x+a_2x^2+a_3x^3+\dots\dots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1,-2,-3,\dots\dots$

Then by differentiation, omitting the summation index k, we get

$$y'(x) = \sum(k+\beta)a_k x^{k+\beta-1} \quad\text{and}\quad y''(x) = \sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$(1-x^2)\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2}+2x\sum(k+\beta)a_k x^{k+\beta-1}+\sum a_k x^{k+\beta}=0$$

or
$$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2}-\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta}$$
$$+\sum 2(k+\beta)a_k x^{k+\beta}+\sum a_k x^{k+\beta}=0$$

Shifting indices in the second, third, and last summations, we get

$$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2}-\sum(k+\beta-2)(k+\beta-3)a_{k-2}x^{k+\beta-2}$$
$$+\sum 2(k+\beta-2)a_{k-2}x^{k+\beta-2}+\Sigma a_{k-2}x^{k+\beta-2}=0$$

or
$$\sum[(k+\beta)(k+\beta-1)a_k$$
$$-\sum\{(k+\beta-2)(k+\beta-3)-2(k+\beta-2)-1\}a_{k-2}]x^{k+\beta-2}=0 \quad (2)$$

The coefficient of the lowest degree term $x^{\beta-2}$ is obtained by substituting $k = 0$ in equation (2)

i.e. $$[\beta(\beta-1)a_0-\{(\beta-2)(\beta-3)-2(\beta-2)-1\}a_{-2}]x^{\beta-2}=0$$

Since the coefficient of $x^{\beta-2}$ must be zero, therefore

$$\beta(\beta-1)a_0-[(\beta-2)(\beta-3)-2(\beta-2)-1]a_{-2}=0$$

But $a_{-2}=0$, therefore $\beta(\beta-1)a_0=0$

Since $a_0 \neq 0$, we get the indicial equation $\beta(\beta-1)=0$ or $\beta=1$ and $\beta=0$. That is, $\beta_1=1$ and $\beta_2=0$ are the indicial roots which differ by an integer.

Equating to zero the coefficient of $x^{k+\beta-2}$, we get from equation (2) the recurrence relation

$$(k+\beta)(k+\beta-1)a_k=[(k+\beta-2)(k+\beta-3)-2(k+\beta-2)-1]a_{k-2}$$

or
$$a_k=\frac{(k+\beta-2)(k+\beta-3)-2(k+\beta-2)-1}{(k+\beta)(k+\beta-1)}a_{k-2}$$

For $k=1$, then $a_1=\dfrac{(\beta-1)(\beta-2)-2(\beta-1)-1}{(\beta+1)\beta}a_{-1}$

For $k=2$, then $a_2=\dfrac{\beta(\beta-1)-2\beta-1}{(\beta+2)(\beta+1)}a_0$

For $k=3$, then $a_3=\dfrac{(\beta+1)\beta-2(\beta+1)-1}{(\beta+3)(\beta+2)}a_1$

For $k=4$, then $a_4=\dfrac{(\beta+2)(\beta+1)-2(\beta+2)-1}{(\beta+4)(\beta+3)}a_2$

For $k=5$, then $a_5=\dfrac{(\beta+3)(\beta+2)-2(\beta+3)-1}{(\beta+5)(\beta+4)}a_3$

and so on.

When $\beta = \beta_2 = 0$, the coefficient a_1 becomes indeterminate of the form $\left(\frac{0}{0}\right)$, since $a_{-1} = 0$ by assumption. In this case, the identity $a_1(0) = 0$ is satisfied for every value of a_1. Therefore, we can take a_1 as arbitrary constant. Thus

$$a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{3}{6}a_1 = -\frac{a_1}{2}, \quad a_4 = -\frac{3}{12}a_2 = -\frac{1}{4}a_2 = \frac{1}{8}a_0, \quad a_5 = -\frac{1}{20}a_3 = \frac{1}{40}a_1, \ldots\ldots$$

Substituting these coefficients in equation (1), we get

$$y(x) = a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{2}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{40}x^5 + \ldots\ldots\ldots$$

$$= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \ldots\ldots\ldots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{40}x^5 + \ldots\ldots\ldots\right)$$

Since it contains two arbitrary constants, therefore it may be taken as the general solution.

If we substitute $\beta = 1$, we get the solution as

$$y(x) = a_0 x\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \ldots\ldots\ldots\right)$$

That is, a constant multiple of the second series in the first solution. Hence, we reject this solution.

10. SOLVED PROBLEMS

PROBLEM (1): **Show that $x = 0$ is an ordinary point and $x = 1$ is a regular singular point of the differential equation $(x^2-1)y''+xy'-y=0$.**

SOLUTION: Comparing the given equation with the general form

$$a_0(x)y''+a_1(x)y'+a_2(x)y=0$$

we have $a_0(x)=x^2-1, \quad a_1(x)=x, \quad a_2(x)=-1$

Since $a_0(0)=-1\neq 0$ and $P(x)=\dfrac{a_1(x)}{a_0(x)}=\dfrac{x}{x^2-1}$ and $Q(x)=\dfrac{a_2(x)}{a_0(x)}=\dfrac{-1}{x^2-1}$

are analytic at $x=0$, therefore $x=0$ is an ordinary point of the differential equation.

Now $a_0(1)=0$, therefore $x=1$ is a singular point.

Also, $P_1(x)=\dfrac{(x-1)x}{x^2-1}=\dfrac{x}{x+1}$ and $Q_1(x)=\dfrac{(x-1)^2(-1)}{x^2-1}=-\dfrac{x-1}{x+1}$

are both analytic at $x=1$, thus $x=1$ is a regular singular point of the given differential equation.

PROBLEM (2): **Show that $x = 0$ is an irregular singular point and $x = -1$ is a regular singular point of the differential equation $x^2(x+1)^2+(x^2-1)y'+2y=0$.**

SOLUTION: Comparing the given equation with

$a_0(x)y''+a_1(x)y'+a_2y=0$, we have

$a_0(x)=x^2(x+1)^2, \quad a_1(x)=x^2-1, \quad a_2(x)=2$

Consider the point $x=0$.

Now $a_0(0)=0$ and $P_1(x)=\dfrac{(x-x_0)a_1(x)}{a_0(x)}=\dfrac{x(x^2-1)}{x^2(x+1)^2}=\dfrac{x-1}{x(x+1)}$

and $Q_1(x)=\dfrac{(x-x_0)^2a_2(x)}{a_0(x)}=\dfrac{x^2(2)}{x^2(x+1)^2}=\dfrac{2}{(x+1)^2}$

Since $P_1(x)$ is not analytic at $x=0$ and $Q_1(x)$ is analytic at $x=0$, therefore $x=0$ is an irregular singular point of the differential equation.

Next, consider the point $x=-1$.

Now $a_0(-1)=0$ and $P_1(x)=\dfrac{(x-x_0)a_1(x)}{a_0(x)}=\dfrac{(x+1)(x^2-1)}{x^2(x+1)^2}=\dfrac{x-1}{x^2}$

and $Q_1(x)=\dfrac{(x-x_0)^2a_2(x)}{a_0(x)}=\dfrac{(x+1)^2(2)}{x^2(x+1)^2}=\dfrac{2}{x^2}$ are both analytic at $x=-1$, therefore

$x=-1$ is a regular singular point of the given differential equation.

PROBLEM (3): **Find the power series solution of the differential equation $y'=2xy$ about an ordinary point $x=0$.**

SOLUTION: We know that the general solution of this equation using the method of separation of variables is $y = Ce^{x^2}$.

STEP (1): Let the power series solution be

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots\ldots \qquad (1)$$

then $$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots\ldots\ldots \qquad (2)$$

STEP (2): Substituting y and y' into the given different equation

$$\begin{aligned} a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \ldots\ldots \\ = 2x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots) \\ = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + \ldots\ldots \end{aligned}$$

Then collecting like powers of x, we get

$$\begin{aligned} a_1 + (2a_2 - 2a_0)x + (3a_3 - 2a_1)x^2 + (4a_4 - 2a_2)x^3 \\ + (5a_5 - 2a_3)x^4 + (6a_6 - 2a_4)x^5 + \ldots\ldots = 0 \end{aligned}$$

Equating the coefficient of each power of x to zero, we get

$$\begin{aligned} a_1 = 0, \quad 2a_2 - 2a_0 = 0, \quad 3a_3 - 2a_1 = 0, \quad 4a_4 - 2a_2 = 0, \\ 5a_5 - 2a_3 = 0, \quad 6a_6 - 2a_4 = 0, \ldots\ldots \end{aligned}$$

From these equations we have

$$a_1 = 0, \quad a_3 = 0, \quad a_5 = 0, \ldots\ldots$$

and for the coefficients with even subscripts

$$a_2 = a_0, \quad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \quad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \ldots\ldots \qquad (a_0 \text{ remains arbitrary})$$

STEP (3): Substituting these coefficients into equation (1), we get

$$\begin{aligned} y &= a_0 + a_0 x^2 + \frac{a_0}{2!}x^4 + \frac{a_0}{3!}x^6 + \ldots\ldots \\ &= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\ldots\right) = a_0 e^{x^2} \end{aligned}$$

PROBLEM (4): **Find the power series solution of the differential equation $(1 - x^2)y'' - 2xy' + 2y = 0$, about the point $x = 0$.**

SOLUTION: Here $x = 0$ is a regular point of the given differential equation.

STEP (1): Let the power series solution be

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots\ldots\ldots \qquad (1)$$

Then $y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots\dots\dots$

and $y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots\dots\dots$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$(1-x^2)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots\dots\dots)$$
$$-2x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots\dots\dots)$$
$$+2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots\dots\dots) = 0$$

or

$$(2a_2 + 2a_0) + (6a_3 - 2a_1 + 2a_1)x + (12a_4 - 2a_2 - 4a_2 + 2a_2)x^2$$
$$+ (20a_5 - 6a_3 - 6a_3 + 2a_3)x^3 + (30a_6 - 12a_4 - 8a_4 + 2a_4)x^4 + \dots\dots = 0$$

Equating the coefficients of like powers of x on both sides, we get

$2a_2 + 2a_0 = 0$

$6a_3 - 2a_1 + 2a_1 = 0$

$12a_4 - 2a_2 - 4a_2 + 2a_2 = 0$

$20a_5 - 6a_3 - 6a_3 + 2a_3 = 0$

$30a_6 - 12a_4 - 8a_4 + 2a_4 = 0$ and so on.

From these equations, we get

$$a_2 = -a_0,\ a_3 = 0,\ a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0,\ a_5 = \frac{1}{2}a_3 = 0,\ a_6 = \frac{3}{5}a_4 = -\frac{1}{5}a_0, \dots\dots\dots$$

STEP (3): Substituting the values of $a_2, a_3, a_4, a_5, \dots\dots\dots$ in equation (1), we get

$$y(x) = a_0 + a_1x - a_0x^2 + 0 - \frac{1}{3}a_0x^4 + 0 - \frac{1}{5}a_0x^6 + 0 + \dots\dots\dots$$
$$= a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 + \dots\dots\dots\right) + a_1x$$

PROBLEM (5): Find the power series solution in powers of $x-1$ of the differential equation

$$y'' - y = 0.$$

SOLUTION: Here $x_0 = 1$.

STEP (1): Let the power series solution be

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + \dots\dots\dots \quad (1)$$

Then $y'(x) = a_1 + 2a_2(x-1) + 3a_3(x-1)^2 + 4a_4(x-1)^3 + 5a_5(x-1)^4 + \dots\dots\dots$

and $y''(x) = 2a_2 + 6a_3(x-1) + 12a_4(x-1)^2 + 20a_5(x-1)^3 + \dots\dots\dots$

STEP (2): Substituting y, and y'' in the given differential equation, we get

$$[2a_2 + 6a_3(x-1) + 12a_4(x-1)^2 + 20a_5(x-1)^3 + \dots\dots\dots]$$
$$-[a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + \dots\dots\dots] = 0$$

or $\quad (2a_2-a_0)+(6a_3-a_1)(x-1)+(12a_4-a_2)(x-1)^2+(20a_5-a_3)(x-1)^3+\ldots\ldots = 0$

Equating the coefficients of like powers of $(x-1)$ on both sides, we get

$$2a_2-a_0=0,\quad 6a_3-a_1=0,\quad 12a_4-a_2=0,\quad 20a_5-a_3=0,\ldots\ldots$$

From these equations, we get

$$a_2=\frac{a_0}{2},\quad a_3=\frac{a_1}{6}=\frac{a_1}{3!},\quad a_4=\frac{a_2}{12}=\frac{a_0}{24}=\frac{a_0}{4!},\quad a_5=\frac{a_3}{20}=\frac{1}{20}\frac{a_1}{3!}=\frac{a_1}{5!},\ldots\ldots$$

STEP (3): Substituting the values of $a_2, a_3, a_4, a_5, \ldots\ldots$ in equation (1), we get

$$y(x) = a_0+a_1(x-1)+\frac{a_0}{2}(x-1)^2+\frac{a_1}{3!}(x-1)^3+\frac{a_0}{4!}(x-1)^4+\frac{a_1}{5!}(x-1)^5+\ldots\ldots$$

$$= a_0\left[1+\frac{(x-1)^2}{2!}+\frac{(x-1)^4}{4!}+\ldots\ldots\right]+a_1\left[(x-1)+\frac{(x-1)^3}{3!}+\frac{(x-1)^5}{5!}+\ldots\ldots\right]$$

METHOD OF FROBENIUS

CASE (1)

PROBLEM (6): **Solve the following differential equation using the method of Frobenius:**

$$(2x+x^3)y''-y'-6xy=0$$

SOLUTION: Let the Frobenius series solution be

$$y(x)=x^{\beta}(a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots)=\sum_{k=-\infty}^{\infty}a_kx^{k+\beta} \qquad (1)$$

where β and a_k are constants such that $a_k=0$ for $k=-1,-2,-3,\ldots\ldots$

By differentiation, omitting the summation index k, we have

$$y'=\sum(k+\beta)a_ka^{k+\beta-1} \quad \text{and} \quad y''=\sum(k+\beta)(k+\beta-1)a_ka^{k+\beta-2}$$

Substituting y, y', and y'' into the given differential equation, we get

$$(2x+x^3)\sum(k+\beta)(k+\beta-1)a_kx^{k+\beta-2}-\sum(k+\beta)a_kx^{k+\beta-1}-6x\sum a_kx^{k+\beta}=0$$

or $$\sum 2(k+\beta)(k+\beta-1)a_kx^{k+\beta-1}+\sum(k+\beta)(k+\beta-1)a_kx^{k+\beta+1}$$
$$-\sum(k+\beta)a_kx^{k+\beta-1}-\sum 6a_kx^{k+\beta+1}=0$$

Shifting indices in the second and last summations, we get

or $$\sum 2(k+\beta)(k+\beta-1)a_kx^{k+\beta-1}+\sum(k+\beta-2)(k+\beta-3)a_{k-2}x^{k+\beta-1}$$
$$-\sum(k+\beta)a_kx^{k+\beta-1}-\sum 6a_{k-2}x^{k+\beta-1}=0$$

or $$\sum\left[\{2(k+\beta)(k+\beta-1)-(k+\beta)\}a_k+\{(k+\beta-2)(k+\beta-3)-6\}a_{k-2}\right]x^{k+\beta-1}=0 \qquad (2)$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by putting $k = 0$ in this equation

i.e. $$\left[\{2\beta(\beta-1)-\beta\}a_0+\{(\beta-2)(\beta-3)-6\}a_{-2}\right]x^{\beta-1}=0$$

Since the coefficient of $x^{\beta-1}$ must be zero, therefore

$$[2\beta(\beta-1)-\beta]a_0+[(\beta-2)(\beta-3)-6]a_{-2}=0$$

But $a_{-2}=0$, we get the indicial equation as

$$[2\beta(\beta-1)-\beta]a_0=0$$

Now $a_0\neq 0$, therefore the indicial equation is

$$2\beta(\beta-1)-\beta=0 \quad \text{or} \quad 2\beta^2-3\beta=0$$

having the roots as $\beta=0,\ \beta=\frac{3}{2}$, which are unequal and differ not by an integer.

Equating to zero the coefficient of $x^{k+\beta-1}$, we get the recurrence relation from equation (2) as

$$[2(k+\beta)(k+\beta-1)-(k+\beta)]a_k+[(k+\beta-2)(k+\beta-3)-6]a_{k-2}=0$$

or $$(k+\beta)(2k+2\beta-1)a_k+[(k+\beta-2)(k+\beta-3)-6]a_{k-2}=0$$

or $$a_k=-\frac{(k+\beta-2)(k+\beta-3)-6}{(k+\beta)(2k+2\beta-3)}a_{k-2} \qquad (3)$$

Let $k=1$, then $a_1=-\frac{(\beta-1)(\beta-2)-6}{(\beta+1)(2\beta-1)}a_{-1}=0$ (since $a_{-1}=0$)

Let $k=2$ in equation (3), then

$$a_2=-\frac{\beta(\beta-1)-6}{(\beta+2)(2\beta+1)}a_0=-\frac{(\beta+2)(\beta-3)}{(\beta+2)(2\beta+1)}a_0=-\frac{\beta-3}{2\beta+1}a_0$$

Let $k=3$ in equation (3), then

$$a_3=-\frac{(\beta+1)\beta-6}{(\beta+3)(2\beta+3)}a_1=0 \qquad (\text{since } a_1=0)$$

Let $k=4$ in equation (3), we get

$$a_4=-\frac{(\beta+2)(\beta+1)-6}{(\beta+4)(2\beta+5)}a_2$$

$$=-\frac{(\beta-1)(\beta+4)}{(\beta+4)(2\beta+5)}\cdot\left[-\frac{\beta-3}{2\beta+1}a_0\right]$$

$$=\frac{(\beta-1)(\beta-3)}{(2\beta+1)(2\beta+5)}a_0$$

Let $k=5$ in equation (3), we get

$$a_5=-\frac{(\beta+3)(\beta+2)-6}{(\beta+5)(2\beta+7)}a_3=0 \qquad (\text{since } a_3=0)$$

Let $k = 6$ in equation (3), we get

$$a_6 = -\frac{(\beta+4)(\beta+3)-6}{(\beta+6)(2\beta+9)}a_4$$

$$= -\frac{(\beta+1)(\beta+6)}{(\beta+6)(2\beta+9)}\cdot\frac{(\beta-1)(\beta-3)}{(2\beta+1)(2\beta+5)}a_0$$

$$= -\frac{(\beta+1)(\beta-1)(\beta-3)}{(2\beta+1)(2\beta+5)(2\beta+9)}a_0$$

When $\beta = 0$, we get $a_2 = 3a_0$, $a_4 = \frac{3}{5}a_0$, $a_6 = -\frac{3}{1.5.9}a_0 = -\frac{1}{15}a_0$

Thus from equation (1), one Frobenius solution corresponding to $\beta = 0$, is

$$y_1 = a_0 + 3a_0x^2 + \frac{3}{5}a_0x^4 - \frac{1}{15}a_0x^6 + \ldots\ldots\ldots$$

$$= a_0\left[1 + 3x^2 + \frac{3}{5}x^4 - \frac{1}{15}x^6 + \ldots\ldots\ldots\right]$$

When $\beta = \frac{3}{2}$, we get $a_2 = \frac{3}{8}a_0$, $a_4 = -\frac{3}{128}a_0$, $a_6 = \frac{5}{1024}a_0, \ldots\ldots\ldots$

The second Frobenius solution corresponding to $\beta = \frac{3}{2}$ is given by

$$y_2 = x^{3/2}\left[a_0 + \frac{3}{8}a_0x^2 - \frac{3}{128}a_0x^4 + \frac{5}{1024}a_0x^6 + \ldots\ldots\ldots\right]$$

$$= a_0x^{3/2}\left[1 + \frac{3}{8}x^2 - \frac{3}{128}x^4 + \frac{1}{1024}x^6 + \ldots\ldots\ldots\right]$$

The general solution is given by $y(x) = Ay_1 + By_2$

CASE (2)

PROBLEM (7): **Solve the differential equation $(x - x^2)y'' + (1-5x)y' - 4y = 0$ using the method of Frobenius about the regular singular point.**

SOLUTION: **STEP (1):** Let the series solution be

$$y(x) = x^{\beta}(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \qquad (1)$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots\ldots$

By differentiation, omitting the summation index k, we get

$$y'(x) = \sum (k+\beta)a_kx^{k+\beta-1} \quad \text{and} \quad y''(x) = \sum (k+\beta)(k+\beta-1)a_kx^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$(x-x^2)\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2}$$
$$+(1-5x)\sum(k+\beta)a_k x^{k+\beta-1}-4\sum a_k x^{k+\beta}=0$$

or
$$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-1}-\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta}$$
$$+\sum(k+\beta)a_k x^{k+\beta-1}-\sum 5(k+\beta)a_k x^{k+\beta}-\sum 4a_k x^{k+\beta}=0$$

or
$$\sum[(k+\beta)(k+\beta-1)+(k+\beta)]a_k x^{k+\beta-1}$$
$$-\sum[(k+\beta)(k+\beta-1)+5(k+\beta)+4]a_k x^{k+\beta}=0$$

or
$$\sum(k+\beta)^2 a_k x^{k+\beta-1}-\sum[(k+\beta)^2+4(k+\beta)+4]a_k x^{k+\beta}=0$$

or
$$\sum(k+\beta)^2 a_k x^{k+\beta-1}-\sum[(k+\beta)+2]^2 a_k x^{k+\beta}=0$$

Shifting index in the second summation, we get

$$\sum(k+\beta)^2 a_k x^{k+\beta-1}-\sum[(k+\beta)+1]^2 a_{k-1} x^{k+\beta-1}=0$$

or
$$\sum[(k+\beta)^2 a_k-\{(k+\beta)+1\}^2 a_{k-1}]x^{k+\beta-1}=0 \qquad (2)$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by substituting $k=0$ in equation (2)

i.e. $[\beta^2 a_0-(\beta+1)^2 a_{-1}]x^{\beta-1}=0$

Since the coefficient of $x^{\beta-1}$ must be zero, therefore $\beta^2 a_0-(\beta+1)^2 a_{-1}=0$.

But $a_{-1}=0$, therefore $\beta^2 a_0=0$

Since $a_0 \neq 0$, therefore $\beta^2=0$ or $\beta=0, 0$. The roots of the indicial equation are equal.

Equating the coefficients of $x^{k+\beta-1}$ on both sides of equation (2), we get

$$(k+\beta)^2 a_k-(k+\beta+1)^2 a_{k-1}=0$$

or
$$a_k=\frac{(k+\beta+1)^2 a_{k-1}}{(k+\beta)^2}$$

For $k=1$, $a_1=\dfrac{(\beta+2)^2 a_0}{(\beta+1)^2}$

For $k=2$, $a_2=\dfrac{(\beta+3)^2 a_1}{(\beta+2)^2}=\dfrac{(\beta+3)^2}{(\beta+2)^2}\cdot\dfrac{(\beta+2)^2}{(\beta+1)^2}a_0=\dfrac{(\beta+3)^2}{(\beta+1)^2}a_0$

For $k=3$, $a_3=\dfrac{(\beta+4)^2 a_2}{(\beta+3)^2}=\dfrac{(\beta+4)^2}{(\beta+3)^2}\cdot\dfrac{(\beta+3)^2}{(\beta+1)^2}a_0=\dfrac{(\beta+4)^2}{(\beta+1)^2}a_0$

and so on.

STEP (3): Substituting these coefficients in equation (1), we get

$$y(x) = x^{\beta} a_0 \left[1 + \frac{(\beta+2)^2}{(\beta+1)^2} x + \frac{(\beta+3)^2}{(\beta+1)^2} x^2 + \frac{(\beta+4)^2}{(\beta+1)^2} x^4 + \ldots\ldots\ldots \right] \quad (3)$$

Putting $\beta = 0$ in equation (3), we have one solution as

$$y_1(x) = a_0 (1 + 2^2 x + 3^2 x^2 + 4^2 x^4 + \ldots\ldots\ldots) \quad (4)$$

To get the second linearly independent solution, we differentiate equation (3) partially w.r.t. β, and put $\beta = 0$.

Now $$\frac{\partial y}{\partial \beta} = a_0 x^{\beta} \ln x \left[1 + \frac{(\beta+2)^2}{(\beta+1)^2} x + \frac{(\beta+3)^2}{(\beta+1)^2} x^2 + \frac{(\beta+4)^2}{(\beta+1)^2} x^3 + \ldots\ldots\ldots \right]$$

$$+ x^{\beta} a_0 \left[\left\{ \frac{2(\beta+2)}{(\beta+1)^2} - \frac{2(\beta+2)^2}{(\beta+1)^3} \right\} x + \left\{ \frac{2(\beta+3)}{(\beta+1)^2} - \frac{2(\beta+3)^2}{(\beta+1)^3} \right\} x^2 + \ldots\ldots\ldots \right]$$

Putting $\beta = 0$, we get

$$y_2(x) = \left(\frac{\partial y}{\partial \beta} \right)_{\beta=0}$$

$$= a_0 \ln x \left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots\ldots\ldots \right]$$

$$+ a_0 \left[\{ 2.2 - 2(2)^2 \} x + \{ 2.(3) - 2(3)^2 \} x^2 + \ldots\ldots\ldots \right]$$

$$= a_0 \ln x \left[1 + 2^2 x + 3^2 x^2 + 4^2 x^2 + \ldots\ldots\ldots \right]$$

$$- 2 a_0 \left[1.2x + 2.3x^2 + 3.4x^3 + \ldots\ldots\ldots \right]$$

or $$y_2(x) = a_0 \left[y_1 \ln x - 2 (1.2x + 2.3x^2 + 3.4x^3 - \ldots\ldots\ldots) \right]$$

The required solution is $y(x) = A y_1 + B y_2$

CASE (3)

PROBLEM (8): **Solve the differential equation $x^2 y'' + x y' + (x^2 - 4) y = 0$ using the method of Frobenius about the regular singular point $x = 0$.**

SOLUTION: **STEP (1):** Let the series solution be

$$y(x) = x^{\beta} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots\ldots$

By differentiation, omitting the summation index k, we get

$$y'(x) = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y''(x) = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$x^2 \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

$$+ x \sum (k+\beta) a_k x^{k+\beta-1} + (x^2-4) \sum a_k x^{k+\beta} = 0$$

or
$$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta} + \sum (k+\beta) a_k x^{k+\beta}$$

$$+ \sum a_k x^{k+\beta+2} - \sum 4 a_k x^{k+\beta} = 0$$

or
$$\sum [(k+\beta)(k+\beta-1) a_k + (k+\beta) a_k - 4 a_k] x^{k+\beta} + \sum a_k x^{k+\beta+2} = 0$$

Shifting index in the last term, we get

$$\sum [(k+\beta)(k+\beta-1) a_k + (k+\beta) a_k - 4 a_k] x^{k+\beta} + \sum a_{k-2} x^{k+\beta} = 0$$

or
$$\sum [(k+\beta)(k+\beta-1) a_k + (k+\beta) a_k - 4 a_k + a_{k-2}] x^{k+\beta} = 0$$

or
$$\sum [(k+\beta)^2 a_k - 4 a_k + a_{k-2}] x^{k+\beta} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{β} is obtained by substituting $k = 0$ in equation (2)

i.e. $[\beta^2 a_0 - 4 a_0 + a_{-2}] x^{\beta} = 0$

Since the coefficient of x^{β} must be zero, therefore $\beta^2 a_0 - 4 a_0 + a_{-2} = 0$.

But $a_{-2} = 0$, therefore $(\beta^2 - 4) a_0 = 0$

Since $a \neq 0$, therefore $\beta^2 - 4 = 0$ or $\beta = 2, -2$. That is, $\beta_1 = 2$, $\beta_2 = -2$ are the indicial roots which are unequal and differ by an integer.

Equating to zero the coefficient of $x^{k+\beta}$, we get from equation (2)

$$[(k+\beta)^2 - 4] a_k + a_{k-2} = 0$$

or
$$a_k = -\frac{a_{k-2}}{(k+\beta)^2 - 4}$$

Let $k = 1$, then $a_1 = -\dfrac{a_{-1}}{(\beta+1)^2 - 4} = 0$ (since $a_{-1} = 0$)

Let $k = 2$, then $a_2 = -\dfrac{a_0}{(\beta+2)^2 - 4} = -\dfrac{a_0}{\beta(\beta+4)}$

Let $k = 3$, then $a_3 = -\dfrac{a_1}{(\beta+3)^2 - 4} = 0$ (since $a_1 = 0$)

Let $k = 4$, then $a_4 = -\dfrac{a_2}{(\beta+4)^2 - 4} = -\dfrac{a_2}{(\beta+2)(\beta+6)} = \dfrac{a_0}{\beta(\beta+2)(\beta+4)(\beta+6)}$

Let $k = 5$, then $a_5 = -\dfrac{a_3}{(\beta+5)^2-4} = 0$ (since $a_3 = 0$)

Let $k = 6$, then $a_6 = -\dfrac{a_4}{(\beta+6)^2-4} = -\dfrac{a_4}{(\beta+4)(\beta+8)} = -\dfrac{a_0}{\beta(\beta+2)(\beta+4)^2(\beta+6)(\beta+8)}$

Substituting the values of these coefficients in equation (1), we get

$$y(x) = a_0 x^{\beta}\left[1 - \frac{x^2}{\beta(\beta+4)} + \frac{x^4}{\beta(\beta+2)(\beta+4)(\beta+6)} - \frac{x^6}{\beta(\beta+2)(\beta+4)^2(\beta+6)(\beta+8)} + \dots\right] \tag{4}$$

Now if we put $\beta = -2$, the coefficients of $x^4, x^6, \dots$ become infinite due to the factor $(\beta+2)$ in the denominators. To overcome this difficulty, we replace a_0 by $c_0(\beta+2)$ in equation (4). Then equation (4) becomes

$$y(x) = c_0 x^{\beta}\left[(\beta+2) - \frac{(\beta+2)x^2}{\beta(\beta+4)} + \frac{x^4}{\beta(\beta+4)(\beta+6)} - \frac{x^6}{\beta(\beta+4)^2(\beta+6)(\beta+8)} + \dots\right] \tag{5}$$

Now let $\beta = -2$ in equation (5), we get

$$y_1(x) = c_0 x^{-2}\left[-\frac{x^4}{16} + \frac{x^6}{192} - \dots\dots\right] = c_0 x^{-2}\left[-\frac{x^4}{2^2.4} + \frac{x^6}{2^3.4.6} - \dots\dots\right]$$

To get the second linearly independent solution, we differentiate equation (5) partially w.r.t. β and put $\beta = -2$.

Now $$\frac{\partial y}{\partial \beta} = c_0 x^{\beta} \ln x\left[(\beta+2) - \frac{(\beta+2)x^2}{\beta(\beta+4)} + \frac{x^4}{\beta(\beta+4)(\beta+6)} - \dots\dots\right]$$
$$+ c_0 x^{\beta}\left[1 + \left\{\frac{1}{2\beta^2} + \frac{1}{2(\beta+4)^2}\right\}x^2 + \dots\dots\right]$$

Then $$y_2(x) = \left(\frac{\partial y}{\partial \beta}\right)_{\beta=-2}$$
$$= c_0 x^{-2}\ln x\left[-\frac{x^4}{2^2.4} + \frac{x^6}{2^3.4.6} - \dots\dots\right] + c_0 x^{-2}\left[1 + \frac{1}{4}x^2 + \dots\dots\right]$$
$$= y_1 \ln x + c_0 x^{-2}\left[1 + \frac{1}{4}x^2 + \dots\dots\right]$$

The general solution is given by $y(x) = A\,y_1(x) + B\,y_2(x)$

CASE (4)

PROBLEM (9): **Solve the differential equation $x^2 y'' + 4xy' + (x^2+2)y = 0$ using the method of Frobenius about the regular singular point $x = 0$.**

SOLUTION: **STEP (1):** Let the series solution be

$$y(x) = x^{\beta}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots\ldots$

By differentiation, omitting the summation index k, we get

$$y'(x) = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y''(x) = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in the given differential equation, we get

$$x^2 \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

$$+ 4x \sum (k+\beta) a_k x^{k+\beta-1} + (x^2+2) \sum a_k x^{k+\beta} = 0$$

or
$$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta} + \sum 4(k+\beta) a_k x^{k+\beta}$$

$$+ \sum a_k x^{k+\beta+2} + \sum 2 a_k x^{k+\beta} = 0$$

or
$$\sum \left[(k+\beta)(k+\beta-1) + 4(k+\beta) + 2 \right] a_k x^{k+\beta} + \sum a_k x^{k+\beta+2} = 0$$

Shifting index in the last summation, we get

$$\sum \left[(k+\beta)^2 + 3(k+\beta) + 2 \right] a_k x^{k+\beta} + \sum a_{k-2} x^{k+\beta} = 0$$

or
$$\sum \left[(k+\beta+1)(k+\beta+2) a_k + a_{k-2} \right] x^{k+\beta} = 0 \tag{2}$$

The coefficient of the lowest degree term x^{β} is obtained by substituting $k = 0$ in equation (2)

i.e. $\quad \left[(\beta+1)(\beta+2) a_0 + a_{-2} \right] x^{\beta} = 0$

Since the coefficient of x^{β} must be zero, therefore

$$(\beta+1)(\beta+2) a_0 + a_{-2} = 0$$

But $a_{-2} = 0$, therefore $(\beta+1)(\beta+2) a_0 = 0$

Since $a_0 \neq 0$, therefore $(\beta+1)(\beta+2) = 0$ or $\beta = -1, -2$. That is, $\beta_1 = -1$ and $\beta_2 = -2$ are the indicial roots which are unequal and differ by an integer.

Equating to zero the coefficient of $x^{k+\beta}$, we get from equation (2) the recurrence relation

$$(k+\beta+1)(k+\beta+2) a_k + a_{k-2} = 0$$

or
$$a_k = -\frac{a_{k-2}}{(k+\beta+1)(k+\beta+2)}$$

For $k = 1$, $\quad a_1 = -\dfrac{a_{-1}}{(\beta+2)(\beta+3)}$

For $k = 2$, $\quad a_2 = -\dfrac{a_0}{(\beta+3)(\beta+4)}$

For $k = 3$, $a_3 = -\frac{a_1}{(\beta+4)(\beta+5)}$

For $k = 4$, $a_4 = -\frac{a_2}{(\beta+5)(\beta+6)} = \frac{a_0}{(\beta+3)(\beta+4)(\beta+5)(\beta+6)}$

For $k = 5$, $a_5 = -\frac{a_3}{(\beta+6)(\beta+7)} = \frac{a_1}{(\beta+4)(\beta+5)(\beta+6)(\beta+7)}$

and so on .

For $\beta = -2$, a_1 becomes indeterminate of the form $\left(\frac{0}{0}\right)$ since $a_{-1} = 0$ by assumption .

But in this case , the identity $a_1(0) = 0$ is satisfied for every value of a_1. Therefore , we can take a_1 as arbitrary constant .

Thus $a_2 = -\frac{a_0}{2}$, $a_3 = -\frac{a_1}{6}$, $a_4 = \frac{a_0}{24}$, $a_5 = \frac{a_1}{120}$

STEP (3): Substituting the values of these coefficients , the Frobenius series solution corresponding to $\beta = -2$ is

$$y(x) = x^{-2}\left[a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{24}x^4 + \frac{a_1}{120}x^5 + \cdots\cdots\right]$$

$$= x^{-2}\left[a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\cdots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\cdots\right)\right]$$

Since it contains two arbitrary constants , therefore it is the general solution .

10. EXERCISE

PROBLEM (1): Show that $x = 0$ is an ordinary point of the differential equations :

(i) $y'' - x y' + 2y = 0$ (ii) $(1 + x^2) y'' + x y' - x y = 0$

PROBLEM (2): Show that $x = 0$ is a regular singular point of the following differential equations :

(i) $2x^2 y'' - x y' + (x - 5) y = 0$ (ii) $x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

(iii) $2x^2 y'' + x y' - (x + 1) y = 0$ (iv) $2x^2 y'' + 7x(x + 1) y' - 3y = 0$

PROBLEM (3): Show that $x = 0$ is an ordinary point of the differential equation :

$(x^2 - 1) y'' + x y' - y = 0$

but $x = 1$ is a regular singular point .

PROBLEM (4): Show that $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point of the differential equation :

$x(x - 1)^3 y'' + 2(x - 1)^3 y' + 3y = 0$

PROBLEM (5): Find the power series solutions in powers of x of the following differential equations:

(i) $y' + 3y = 0$ (ii) $y'' - 4y = 0$

(iii) $(x^2 - 1) y'' + 4x y' + 2y = 0$ (iv) $(1 - x^2) y'' + 2x y' - y = 0$

(v) $y'' + x y' + x^2 y = 0$ (vi) $(1 + x^2) y'' + x y' - y = 0$

PROBLEM (6): Find the power series solutions of the following differential equations about the given point :

(i) $y' = 2y \quad (x = 1)$ (ii) $x y' - y = 0 \quad (x = 1)$

(iii) $y'' - x y' + 2y = 0 \quad (x = 1)$ (iv) $y'' + (x - 3) y' + y = 0 \quad (x = 2)$

METHOD OF FROBENIUS

CASE (1)

PROBLEM (7): Solve the following differential equations using the method of Frobenius about a regular singular point $x = 0$:

(i) $2x^2 y'' - x y' + (x^2 + 1) y = 0$

(ii) $2x(1 - x) y'' + (1 - x) y' + 3y = 0$

(iii) $2x(1 - x) y'' + (5 - 7x) y' - 3y = 0$

(iv) $9x(1 - x) y'' - 12 y' + 4y = 0$

CASE (2)

PROBLEM (8): Solve the following differential equations using the method of Frobenius about a regular singular point $x = 0$:

(i) $x y'' + (1 + x) y' + 2 y = 0$ (ii) $(x - x^2) y'' + (1 - x) y' - y = 0$

CASE (3)

PROBLEM (9): Solve the following differential equations using the method of Frobenius about a regular singular point $x = 0$:

(i) $x y'' - y = 0$ (ii) $x(1 - x) y'' - (1 + 3x) - y = 0$

CASE (4)

PROBLEM (10): Solve the following differential equations using the method of Frobenius about the point $x = 0$:

(i) $(1 - x^2) y'' - 2 x y' + 2 y = 0$ (ii) $(2 + x^2) y'' + x y' + (1 + x) y = 0$

CHAPTER 11

BESSEL FUNCTIONS

11.1 INTRODUCTION

We know that homogeneous linear differential equations with constant coefficients can be solved by algebraic methods, and the solutions are elementary functions known from calculus like sine, cosine, exponential, and logarithmic, etc. However, if the coefficients are not constants but depend on x, the situation is more complicated and the solutions may not be finite combination of familiar functions. In many problems of applied mathematics, mathematical physics, and engineering, we come across second order homogeneous linear differential equations with variable coefficients, whose solutions give rise to **special functions**. A very important differential equation of this type that arises very frequently, particularly in boundary value problems involving cylinderical symmetry, is the Bessel's differential equation. This equation occurs in the process of obtaining solutions of Laplace's equation in cylinderical polar coordinates by the method of separation of variables. This equation arises in different fields such as elasticity, fluid flow, heat conduction, electric fields, potential theory, propagation of waves, astronomy, and so on. Since this equation and its solutions play an important role in science and engineering, therefore in this chapter, we shall obtain the series solutions of this equation using the method of Frobenius.

11.2 BESSEL'S DIFFERENTIAL EQUATION

The second order homogeneous linear differential equation with variable coefficients

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (n \geq 0) \qquad (1)$$

is called the Bessel's differential equation named after the German mathematician F. W. Bessel (1784 – 1846). The parameter n is a given non-negative real number.

11.3 BESSEL FUNCTIONS

The solutions of equation (1) are called the Bessel functions of order n. Although, the differential equation is of order 2, but it is traditional to call it a differential equation of order n. In fact n refers to the order of the Bessel functions.

The general solution of equation (1) was obtained by Bessel in 1824, although the special cases of this equation had been studied earlier by the Swiss mathematicians Jakob Bernoulli (1654 – 1705) in 1703, Danial Bernoulli (1700 – 1782) in 1732, and L. Euler (1707 – 1783) in 1764.

11.4 GAMMA FUNCTION

We first define Gamma function and some of its properties which will be used in expressing the solutions of Bessel's differential equation .

The gamma function denoted by $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

which is convergent for $n > 0$ and it is a function of n .

Note that $\Gamma(n) = \infty$ if $n = 0$ or a negative integer .

PROPERTIES OF GAMMA FUNCTIONS

THEOREM (11.1): Prove that

(i) $\Gamma(n+1) = n\Gamma(n), \quad n > 0$

(ii) $\Gamma(n+1) = n!, \quad n = 1, 2, 3, \ldots\ldots$

PROOF: (i) We have

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

$$= \left| x^n(-e^{-x}) \right|_0^\infty - \int_0^\infty (-e^{-x})^n x^{n-1}\, dx$$

$$= \left| -\frac{x^n}{e^x} \right|_0^\infty + n \int_0^\infty x^{n-1} e^{-x}\, dx$$

The first term on the right is zero for the lower limit and indeterminate for the upper limit ; however repeated application of L'Hospital's rule shows that it is zero .

Hence $\Gamma(n+1) = n\Gamma(n)$, if $n > 0$.

(ii) $\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = -\frac{1}{e^x}\Big|_0^\infty = 1$

Thus $\Gamma(1) = 1$

Put $n = 1, 2, 3, \ldots\ldots$ in $\Gamma(n+1) = n\Gamma(n)$.

Then $\Gamma(2) = 1\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2.1 = 2!$

$\Gamma(4) = 3\Gamma(3) = 3.2! = 3!$

$\Gamma(5) = 4\Gamma(4) = 4.3! = 4!$

In general, $\Gamma(n+1) = n!$ if n is a positive integer.

For this reason Gamma function is sometimes called the generalized **Factorial** function.

THEOREM (11.2): Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

PROOF: We have

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\, dx$$

Let $n = \frac{1}{2}$, we get $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx$

$$\text{Therefore } \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \left\{2\int_0^\infty e^{-x^2}\, dx\right\}\left\{2\int_0^\infty e^{-y^2}\, dy\right\}$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, y \qquad (1)$$

Changing to polar coordinates $x = r\cos\theta$, and $y = r\sin\theta$, we have

$$dx\,dy = r\,dr\,d\theta, \quad 0 \le r \le \infty \text{ and } 0 \le \theta \le \frac{\pi}{2}$$

Therefore integral (1) becomes

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left|-\frac{1}{2}e^{-r^2}\right|_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2.\frac{\pi}{2} = \pi$$

and so $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

11.5 METHOD OF FROBENIUS

It is a method for obtaining solutions of linear differential equations with variable coefficients. In this method we assume a solution of the form

$$y = x^{\beta}(a_0 + a_1 x + a_2 x^2 + \dots\dots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \tag{1}$$

where β and a_k are constants such that $a_k = 0$ for $k < 0$. Substituting (1) into the differential equation leads to an equation for β, called an **indicial equation**, and equations for the constants $a_0, a_1, \dots\dots$ in the form of a recursion formula. By solving for β and other constants, a series solution can be obtained.

11.6 SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION

We shall solve the Bessel's differential equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (n \geq 0) \tag{1}$$

using the method of Frobenius.

GENERAL SOLUTION WHEN n IS NOT AN INTEGER

We first find the particular solutions of equation (1), when $n \neq 0, 1, 2, 3, \dots\dots$

Assuming a solution of the form

$$y = x^{\beta}(a_0 + a_1 x + a_2 x^2 + \dots\dots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \tag{2}$$

where $a_k = 0$ for $k = -1, -2, -3, \dots\dots$

By differentiation, omitting the summation index k, we have

$$y' = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y'' = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

Substituting the values of y, y', and y'' in equation (1), we get

$$x^2 \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2} + x \sum (k+\beta) a_k x^{k+\beta-1} + (x^2 - n^2) \sum a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta} + \sum (k+\beta) a_k x^{k+\beta} + \sum a_k x^{k+\beta+2} - \sum n^2 a_k x^{k+\beta} = 0$$

Shifting the index k by $k-2$ in the third summation, we get

$$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta} + \sum (k+\beta) a_k x^{k+\beta} + \sum a_{k-2} x^{k+\beta} - \sum n^2 a_k x^{k+\beta} = 0$$

or $$\sum \left[(k+\beta)(k+\beta-1) a_k + (k+\beta) a_k + a_{k-2} - n^2 a_k \right] x^{k+\beta} = 0$$

or $$\sum \left[\{ (k+\beta)^2 - n^2 \} a_k + a_{k-2} \right] x^{k+\beta} = 0 \tag{3}$$

The coefficient of the lowest degree term x^{β} is obtained by putting $k = 0$ in this equation

$$[(\beta^2 - n^2)a_0 + a_{-2}]x^{\beta} = 0$$

Since $x^{\beta} \neq 0$, therefore the coefficient of x^{β} must be zero

i.e. $\quad (\beta^2 - n^2)a_0 + a_{-2} = 0$

But $a_{-2} = 0$, we get the indicial equation $(\beta^2 - n^2)a_0 = 0$

Since $a_0 \neq 0$, therefore $\beta^2 = n^2$.

Now there are two cases, given by $\beta = n$ and $\beta = -n$. We shall consider first the case $\beta = n$ and obtain the second case by replacing n by $-n$.

CASE (1): $\beta = n$

In this case, we obtain the recurrence relation from equation (3)

$$k(2n+k)a_k + a_{k-2} = 0$$

or
$$a_k = -\frac{a_{k-2}}{k(2n+k)} \qquad (4)$$

Putting $k = 1, 2, 3, 4, \ldots\ldots$ successively in equation (4), we get

$$a_1 = -\frac{a_{-1}}{2n+1} = 0 \quad (\text{since } a_{-1} = 0)$$

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0}{2^2\,1!(n+1)}$$

$$a_3 = -\frac{a_1}{3(2n+3)} = 0 \quad (\text{since } a_1 = 0)$$

$$a_4 = -\frac{a_2}{4(2n+4)} = -\frac{a_2}{2^2\,2(n+2)} = \frac{a_0}{2^4\,2!(n+1)(n+2)}$$

$$a_5 = -\frac{a_3}{5(2n+5)} = 0 \quad (\text{since } a_3 = 0)$$

$$a_6 = -\frac{a_4}{6(2n+6)} = -\frac{a_4}{2^2\,3(n+3)} = -\frac{a_0}{2^6\,3!(n+1)(n+2)(n+3)}$$

and, in general, writing the even subscript k as $2m$, we get

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\ldots\ldots(n+m)}$$

and $\quad a_{2m+1} = 0$

Thus one solution of Bessel's equation is given by

$$y = y_1 = a_0 x^n + a_2 x^{n+2} + a_4 x^{n+4} + \ldots\ldots + a_{2m} x^{n+2m} + \ldots\ldots$$

$$= a_0\left[x^n - \frac{x^{n+2}}{2^2\,1!(n+1)} + \frac{x^{n+4}}{2^4\,2!(n+1)(n+2)} - \ldots + \frac{(-1)^m x^{n+2m}}{2^{2m} m!(n+1)(n+2)\ldots(n+m)} \ldots\right] \qquad (5)$$

Since a_0 is still arbitrary and we are looking for a particular solution, it is convenient to choose

$$a_0 = \frac{1}{2^n \Gamma(n+1)}$$

where $\Gamma(n+1)$ is the Gamma function. The reason for selecting the value of a_0 in terms of gamma function is that n is a non – negative real number. If n were a positive integer then a factorial function could have been used. The series (5) now becomes

$$y_1 = \left[\frac{x^n}{2^n \Gamma(n+1)} - \frac{x^{n+2}}{2^{n+2} 1!(n+1)\Gamma(n+1)} + \frac{x^{n+4}}{2^{n+4} 2!(n+2)(n+1)\Gamma(n+1)} - \dots\dots + \frac{(-1)^m x^{n+2m}}{2^{n+2m} m!(n+m)\dots\dots(n+2)(n+1)\Gamma(n+1)} \dots\dots\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \qquad (6)$$

This series converges for all positive x.

BESSEL FUNCTIONS OF FIRST KIND

The functions defined by the infinite series (6) are known as the **Bessel functions of the first kind of order n** and are denoted by $J_n(x)$.

Thus $$y_1 = J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \qquad (7)$$

In the special case when n is a positive integer or zero, then $\Gamma(n+m+1) = (n+m)!$, $\Gamma(1) = 1$ therefore, equation (7) becomes

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\,(n+m)!} \quad (n = 0, 1, 2, \dots\dots) \qquad (8)$$

CASE (2): $\beta = -n$

Replacing n by $-n$ in series (7) of case (1), we find that the function

$$y_2 = J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)} \qquad (9)$$

is a second particular solution of Bessel's equation (1). The function $J_{-n}(x)$, when substituted for y in Bessel's equation (1) will satisfy it, since $J_{-n}(x)$ differs from the solution $J_n(x)$ only in the sign of n, and n appears only squared in the Bessel's equation. Now if $n = 0$, both of the series (7) and (9) are identical. If $n = 1, 2, 3, \dots\dots$ series (9) fails to exist and series (7) is the only solution. For $n = -1, -2, -3, \dots\dots$ series (7) fails to exist and series (9) is the only solution.

Thus when $n \neq 0, 1, 2, \ldots\ldots$, we have two linearly independent solutions $y_1 = J_n(x)$ and $y_2 = J_{-n}(x)$ of Bessel's equation of order n and so for this case the general solution is

$$y = A J_n(x) + B J_{-n}(x) \qquad (n \neq 0, 1, 2, \ldots\ldots) \tag{10}$$

where A and B are arbitrary constants.

To obtain the general solution in the case when n is an integer, we first prove the following result.

THEOREM (11.3): Prove that

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{for } n = 0, 1, 2, 3, \ldots\ldots$$

PROOF: We know that

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)}$$

$$= \sum_{m=0}^{n-1} \frac{(-1)^m \left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)} + \sum_{m=n}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)}$$

Now since $\Gamma(-n+m+1)$ is infinite for $m = 0, 1, 2, \ldots\ldots, n-1$, the first sum on the right is zero.

Letting $m = n+k$ in the second sum, it becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^{n+k} \left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,\Gamma(k+1)} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{\Gamma(n+k+1)\,k!} \qquad \left[\text{using theorem (11.1)}\right]$$

$$= (-1)^n J_n(x)$$

GENERAL SOLUTION WHEN n IS AN INTEGER

We know that $J_{-n}(x) = (-1)^n J_n(x) \qquad \text{for } n = 0, 1, 2, 3, \ldots\ldots$

That is, when n is an integer, the function $J_{-n}(x)$ is a constant multiple of $J_n(x)$.

Thus $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and so

$$y = A J_n(x) + B J_{-n}(x)$$
$$= A J_n(x) + B(-1)^n J_n(x)$$
$$= \left[A + B(-1)^n\right] J_n(x)$$
$$= C J_n(x)$$

where C is an arbitrary constant. Hence y cannot be the general solution of Bessel's equation since it contains only one arbitrary constant.

To obtain the general solution in this case, the function $J_n(x)$ and $J_{-n}(x)$ which are called Bessel functions of the first kind are not sufficient and therefore Bessel functions of the second kind are introduced.

BESSEL'S FUNCTIONS OF SECOND KIND

We know that when $n \neq 0, 1, 2, \ldots\ldots$ the general solution of Bessel's equation is

$$y = A J_n(x) + B J_{-n}(x)$$

Taking the arbitrary constants A and B as

$$A = C_1 + \frac{C_2 \cos n\pi}{\sin n\pi}, \quad B = -\frac{C_2}{\sin n\pi}$$

the general solution becomes

$$y = \left(C_1 + \frac{C_2 \cos n\pi}{\sin n\pi}\right) J_n(x) - \frac{C_2}{\sin n\pi} J_{-n}(x)$$

$$= C_1 J_n(x) + C_2 \left(\frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}\right)$$

The functions defined by the expression

$$\frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}, \quad n \neq 0, 1, 2, 3, \ldots\ldots$$

are called the **Bessel's functions of the second kind of order n** and are denoted by $Y_n(x)$.

i.e. $$Y_n(x) = \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}, \quad n \neq 0, 1, 2, 3, \ldots\ldots$$

Thus the general solution of Bessel's equation can be written as

$$y = C_1 J_n(x) + C_2 Y_n(x), \quad n \neq 0, 1, 2, 3, \ldots\ldots \tag{11}$$

where C_1 and C_2 are arbitrary constants.

Now the Bessel functions $Y_n(x) = \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}$ becomes an **indeterminate** of the form $\frac{0}{0}$ for the case when n is an integer. This is because for an integer n, $\cos n\pi = (-1)^n$, $\sin n\pi = 0$ and $J_{-n}(x) = (-1)^n J_n(x)$. This **indeterminate form** can be evaluated by using L' Hospital's rule

i.e. $$Y_n(x) = \underset{p \to n}{\mathrm{Lt}} \frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi} \tag{12}$$

Thus we define the Bessel's functions of the second kind of order n as

$$Y_n(x) = \begin{cases} \dfrac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}, & n \neq 0, 1, 2, 3, \ldots\ldots \\ \underset{p \to n}{\mathrm{Lt}} \dfrac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi}, & n = 0, 1, 2, 3, \ldots\ldots \end{cases} \tag{13}$$

This is also called the Neumann function named after the German mathematician C. Neumann (1832 – 1925). Formula (13) can be used to find a series expansion for $Y_n(x)$.

GENERAL SOLUTION OF BESSEL'S EQUATION

The general solution of Bessel's differential equation for all values of n can then be written as

$$y = C_1 J_n(x) + C_2 Y_n(x) \qquad (\pi \geq 0) \tag{14}$$

where C_1 and C_2 are arbitrary constants.

THEOREM (11.4): Prove that $Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, 3, \ldots\ldots\ldots$

PROOF: We know that

$$Y_n(x) = \underset{p \to n}{\text{Lt}} \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}, \quad n = 0, 1, 2, 3, \ldots\ldots\ldots$$

Using L'Hospital's rule, we get

$$Y_n(x) = \left[\frac{-\pi \sin p\pi\, J_p + \cos p\pi \frac{\partial J_p}{\partial p} - \frac{\partial J_{-p}}{\partial p}}{\pi \cos p\pi}\right]_{p=n}$$

$$= \frac{-\pi \sin n\pi\, J_n + \cos n\pi \frac{\partial J_n}{\partial n} - \frac{\partial J_{-n}}{\partial n}}{\pi \cos n\pi}$$

$$= \frac{0 + (-1)^n \frac{\partial J_n}{\partial n} - \frac{\partial J_{-n}}{\partial n}}{\pi(-1)^n}$$

$$= \frac{1}{\pi}\left[\frac{\partial J_n}{\partial n} - (-1)^{-n}\frac{\partial J_{-n}}{\partial n}\right] \tag{1}$$

Replacing n by $-n$ in equation (1), we get

$$Y_{-n}(x) = \frac{1}{\pi}\left[\frac{\partial J_{-n}}{\partial(-n)} - (-1)^n \frac{\partial J_n}{\partial(-n)}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\partial J_{-n}}{\partial n} + (-1)^n \frac{\partial J_n}{\partial n}\right]$$

$$= \frac{1}{\pi}(-1)^n\left[\frac{\partial J_n}{\partial n} - (-1)^{-n}\frac{\partial J_{-n}}{\partial n}\right]$$

$$= (-1)^n Y_n(x) \qquad [\text{using equation (1)}]$$

11.17 SERIES EXPANSION FOR $Y_n(x)$

We first discuss the case for $n = 0$. In this case we must evaluate

$$Y_0(x) = \underset{p \to 0}{\text{Lt}} \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}$$

Using L' Hospital's rule (differentiating numerator and denominator w.r.t p), we find

$$Y_0(x) = \underset{p \to 0}{\mathrm{Lt}} \left[\frac{(-\pi \sin p\pi) J_p + \cos p\pi \dfrac{\partial J_p}{\partial p} - \dfrac{\partial J_{-p}}{\partial p}}{\pi \cos p\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\partial J_p}{\partial p} - \frac{\partial J_{-p}}{\partial p} \right]_{p=0} \tag{14}$$

where the notation indicates that we are to take the partial derivatives of $J_p(x)$ and $J_{-p}(x)$ w.r.t p and put $p = 0$

Since $-\dfrac{\partial J_{-p}}{\partial p} = \dfrac{\partial J_{-p}}{\partial(-p)} = \dfrac{\partial J_p}{\partial p}$, therefore equation (14) can be written as

$$Y_0(x) = \frac{2}{\pi} \left[\frac{\partial J_p}{\partial p} \right]_{p=0} \tag{15}$$

To obtain $\dfrac{\partial J_p}{\partial p}$ we differentiate the series

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{p+2m}}{m!\,\Gamma(p+m+1)} \quad \text{w.r.t p and obtain}$$

$$\frac{\partial J_p}{\partial p} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\partial}{\partial p} \left[\frac{\left(\frac{x}{2}\right)^{p+2m}}{\Gamma(p+m+1)} \right] \tag{16}$$

Now if we let $\dfrac{\left(\frac{x}{2}\right)^{p+2m}}{\Gamma(p+m+1)} = G$, then

$$\ln G = (p+2m) \ln\left(\frac{x}{2}\right) - \ln \Gamma(p+m+1)$$

so that differentiation w.r.t p gives

$$\frac{1}{G} \frac{\partial G}{\partial p} = \ln\left(\frac{x}{2}\right) - \frac{\Gamma'(p+m+1)}{\Gamma(p+m+1)}$$

Therefore

$$\frac{\partial G}{\partial p} = G\left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(p+m+1)}{\Gamma(p+m+1)} \right] \tag{17}$$

From equations (16) and (17)

$$\frac{\partial J_p}{\partial p} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{x}{2}\right)^{p+2m}}{\Gamma(p+m+1)} \left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(p+m+1)}{\Gamma(p+m+1)} \right]$$

Then for $p = 0$, we have

$$\left.\frac{\partial J_p}{\partial p}\right|_{p=0} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{x}{2}\right)^{2m}}{\Gamma(m+1)} \left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(m+1)}{\Gamma(m+1)}\right] \tag{18}$$

From equations (15) and (18), we have

$$Y_0(x) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{x}{2}\right)^{2m}}{\Gamma(m+1)} \left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(m+1)}{\Gamma(m+1)}\right]$$

From the theory of Gamma functions, we have

$$\frac{\Gamma'(m+1)}{\Gamma(m+1)} = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m} \quad \text{and} \quad \frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$$

where $\gamma = 0.5772$ is Euler's constant. This can be written as

$$Y_0(x) = \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,m!} \left[\ln\left(\frac{x}{2}\right) - \left(-\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m}\right)\right]$$

$$= \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,m!} \left[\ln\left(\frac{x}{2}\right) + \gamma\right]$$

$$- \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,m!} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m}\right)$$

$$= \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] \left(1 + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,m!}\right)$$

$$- \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,m!} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m}\right)$$

$$Y_0(x) = \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_0(x) - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m!\,m!} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m}\right)$$

$$= \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_0(x) + \frac{2}{\pi}\left[\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \ldots\ldots\ldots\right] \tag{19}$$

Since $Y_0(x)$ contains the term $\ln\left(\frac{x}{2}\right)$, we have $Y_0(x) \to -\infty$ as $x \to 0$.

For small $x > 0$, $Y_0(x)$ behaves like $\ln(x)$.

In general, when $n = 1, 2, 3, \ldots\ldots$ we can obtain the series expansion for $Y_n(x)$ as

$$Y_n(x) = \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma\right]J_n(x)-\frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!\left(\frac{x}{2}\right)^{2m-n}}{m!}$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\left[\phi(m)+\phi(n+m)\right]\frac{\left(\frac{x}{2}\right)^{2m+n}}{m!(n+m)!} \tag{20}$$

where γ is Euler's constant and $\phi(p) = 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{p}$, $\phi(0) = 0$

In particular, if $n = 1$, we get from equation (20)

$$Y_1(x) = \frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)-\frac{1}{x}-\frac{1}{4}x+\frac{5x^3}{2^6}-\frac{5x^5}{2^7.3^2}+\ldots\ldots\ldots\right] \tag{21}$$

11.8 BESSEL FUNCTIONS OF FIRST KIND OF ZEROTH AND FIRST ORDER

We know that the Bessel functions of first kind for integer values of n are given by

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{n+2m}}{m!(n+m)!} \quad (n = 0, 1, 2, \ldots\ldots) \tag{1}$$

For $n = 0$ in equation (1), we get

$$J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{2m}}{(m!)^2}$$

$$= 1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\ldots\ldots-\ldots\ldots+\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

$$= 1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\ldots\ldots \tag{2}$$

and for $n = 1$ in equation (1), we get

$$J_1(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{1+2m}}{m!(m+1)!}$$

$$= \frac{x}{2}-\frac{x^3}{2^3 2!}+\frac{x^5}{2^5 2!\,3!}-\frac{x^7}{2^7 3!\,4!}+\ldots\ldots+\frac{(-1)^m x^{2m+1}}{2^{2m+1}m!(m+1)!}+\ldots\ldots$$

$$= \frac{x}{2}-\frac{x^3}{2^2.4}+\frac{x^5}{2^2.4^2.6}-\frac{x^7}{2^2.4^2.6^2.8}+\ldots\ldots \tag{3}$$

which are called Bessel functions of zeroth and first orders, respectively and are shown graphically in figure (11.1).

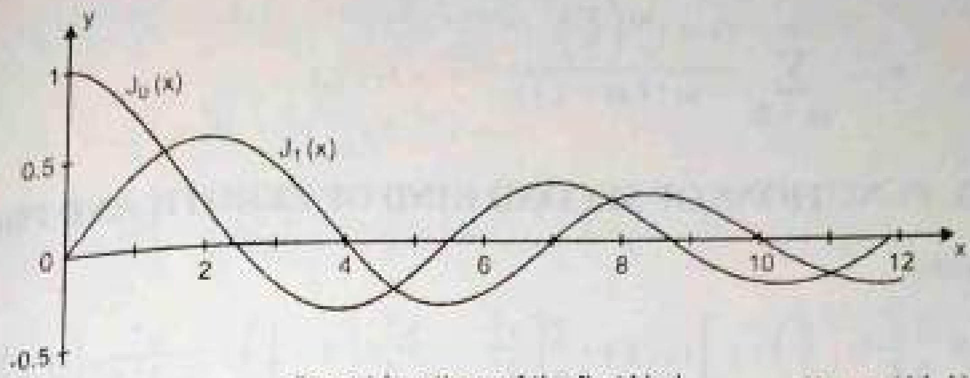


Figure (11.1)

From equation (2) and (3), we see that $J_0(x) = 1$ and $J_1(x) = 0$.

THEOREM (11.5): Prove that $\frac{d}{dx} J_0(x) = J_0'(x) = -J_1(x)$

PROOF: We know that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\,(n+m)!} \quad (n = 0, 1, 2, \ldots\ldots) \qquad (1)$$

Let $n = 0$ in equation (1)

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2}$$

Then
$$\frac{d}{dx} J_0(x) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2}$$

$$= \frac{d}{dx}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2}\right]$$

Replacing the summation index m by $m+1$, we get

$$\frac{d}{dx} J_0(x) = \frac{d}{dx}\left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \left(\frac{x}{2}\right)^{2m+2}}{[(m+1)!]^2}\right]$$

$$= \frac{d}{dx}\left[1 - \sum_{m=0}^{\infty} \frac{(-1)^{m} \left(\frac{x}{2}\right)^{2m+2}}{[(m+1)!]^2}\right]$$

$$= -\sum_{m=0}^{\infty} \frac{(-1)^m (2m+2) \left(\frac{x}{2}\right)^{2m+1} \left(\frac{1}{2}\right)}{(m+1)!\,(m+1)!}$$

$$= -\sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+1}}{m!(m+1)!} = -J_1(x)$$

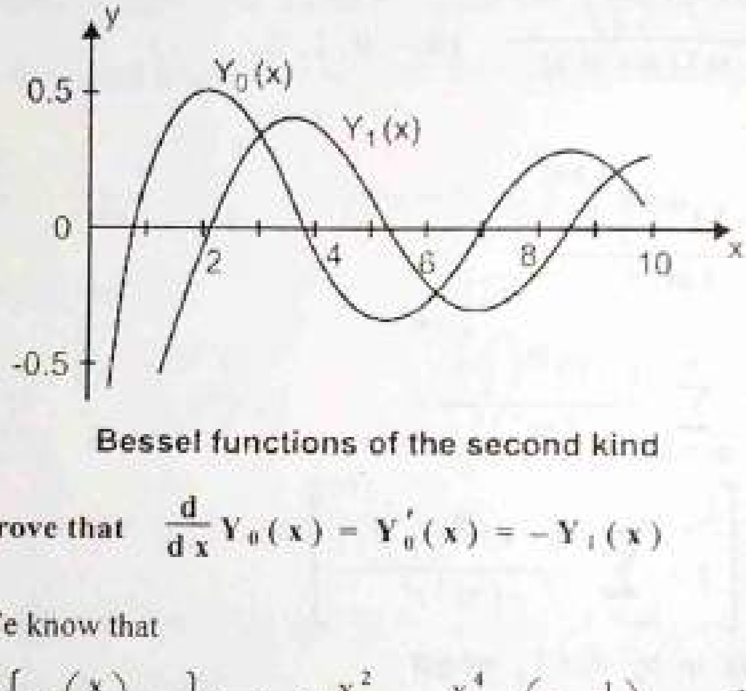
11.9 BESSEL FUNCTIONS OF SECOND KIND OF ZEROTH AND FIRST ORDER

We know that

$$Y_0(x) = \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma\right]J_0(x)+\frac{2}{\pi}\left[\frac{x^2}{2^2}-\frac{x^4}{2^2 4^2}\left(1+\frac{1}{2}\right)+\frac{x^6}{2^2 4^2 6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\ldots\right] \quad (5)$$

and $$Y_1(x) = \frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)-\frac{1}{x}-\frac{1}{4}x+\frac{5x^3}{2^6}-\frac{5x^5}{2^7.3^2}+\ldots\ldots\right] \quad (6)$$

which are called Bessel functions of second kind of zeroth and first order respectively are shown graphically in figure (11.2). We see that as $x \to 0+$, both $y_0(x)$ and $y_1(x) \to -\infty$.



Bessel functions of the second kind Figure (11.2)

THEOREM (11.6): Prove that $\frac{d}{dx}Y_0(x) = Y_0'(x) = -Y_1(x)$

PROOF: We know that

$$Y_0(x) = \frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_0(x)+\frac{x^2}{2^2}-\frac{x^4}{2^2.4^2}\left(1+\frac{1}{2}\right)+\frac{x^6}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\ldots\right] \quad (1)$$

and $$Y_1(x) = \frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)-\frac{1}{x}-\frac{1}{4}x+\frac{5x^3}{2^6}-\frac{5x^5}{2^7.3^2}+\ldots\ldots\right] \quad (2)$$

Differentiating equation (1) w.r.t. x, we get

$$Y_0'(x) = \frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_0'(x)+\frac{1}{x}J_0(x)+\frac{x}{2}-\frac{3x^3}{2^5}+\frac{11x^5}{2^8.3^2}-\ldots\ldots\right]$$

Since $J_0'(x) = -J_1(x)$ and

$$J_0(x) = 1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\cdots\cdots$$

$$= 1-\frac{x^2}{2^2}+\frac{x^4}{2^6}-\frac{x^6}{2^8.3^2}+\cdots\cdots$$

Therefore
$$Y_0'(x) = \frac{2}{\pi}\left[-\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)+\frac{1}{x}\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^6}-\frac{x^6}{2^8.3^2}+\cdots\cdots\right)\right.$$
$$\left.+\left(\frac{x}{2}-\frac{3x^3}{2^5}+\frac{11x^5}{2^8.3^2}-\cdots\cdots\right)\right]$$

$$= \frac{2}{\pi}\left[-\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)+\left(\frac{1}{x}-\frac{x}{2^2}+\frac{x^3}{2^6}-\frac{x^5}{2^8.3^2}+\cdots\cdots\right)\right.$$
$$\left.+\left(\frac{x}{2}-\frac{3x^3}{2^5}+\frac{11x^5}{2^8.3^2}-\cdots\cdots\right)\right]$$

$$= \frac{2}{\pi}\left[-\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)+\frac{1}{x}+\frac{1}{4}x-\frac{5x^3}{2^6}+\frac{5x^5}{2^7.3^2}-\cdots\cdots\right]$$

$$= -\frac{2}{\pi}\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}J_1(x)-\frac{1}{x}-\frac{1}{4}x+\frac{5x^3}{2^6}-\frac{5x^5}{2^7.3^2}+\cdots\cdots\right]$$

$$= -Y_1(x) \qquad [\text{ using equation (2) }]$$

11.10 ALTERNATIVE FORM OF THE GENERAL SOLUTION

THEOREM (11.7): Show that general solution of Bessel's differential equation can be written as

$$y(x) = A\,J_n(x)+B\,J_n(x)\int\frac{dx}{x\,J_n^2(x)} \qquad (\text{ for all } n \geq 0)$$

where A and B are arbitrary constants.

PROOF: We know that Bessel's differential equation is

$$x^2y''+xy'+(x^2-n^2)y = 0 \tag{1}$$

Let $y(x) = u(x)J_n(x) = u\,J_n$ (2)

be a solution of equation (1).

Then $y'(x) = u\,J_n'+u'\,J_n$

$$y''(x) = u\,J_n''+2u'\,J_n'+u''\,J_n$$

Substituting the values of y, y', and y'' in equation (1), we get

$$x^2\left(u\,J_n''+2u'\,J_n'+u''\,J_n\right)+x\left(u\,J_n'+u'\,J_n\right)+(x^2-n^2)u\,J_n = 0$$

or
$$u\left[x^2J_n''+x\,J_n'+(x^2-n^2)J_n\right]+x^2u''\,J_n+2x^2u'\,J_n'+x\,u'\,J_n = 0$$

The expression with square brackets is zero, since $J_n(x)$ is a solution of Bessel's equation (1).

Therefore $x^2 u'' J_n + 2x^2 u' J_n' + x u' J_n = 0$

Dividing throughout by $x^2 u' J_n$, we get

$$\frac{u''}{u'} + 2\frac{J_n'}{J_n} + \frac{1}{x} = 0$$

Integrating w.r.t. x, we get

$$\ln u' + 2 \ln J_n + \ln x = \ln B$$

or $\quad \ln\left(u' J_n^2 x\right) = \ln B$

or $\quad u' J_n^2 x = B \quad$ or $\quad u' = \dfrac{B}{x J_n^2}$

or $\quad u(x) = B \displaystyle\int \frac{dx}{x J_n^2(x)} + A$

Substituting the values of u in equation (2), we get

$$y(x) = \left[B \int \frac{dx}{x J_n^2(x)} + A\right] J_n(x)$$

$$= A J_n(x) + B J_n(x) \int \frac{dx}{x J_n^2(x)}$$

$$= A J_n(x) + B Y_n(x)$$

where $Y_n(x) = J_n(x) \displaystyle\int \frac{dx}{x J_n^2(x)}$ is another form of Bessel functions of the second kind.

11.11 GENERATING FUNCTION FOR $J_n(x)$

The function $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is called the generating function for the Bessel functions of the first kind for integer values of n. It is very useful in obtaining properties of these functions for integer values of n.

We now prove the following results.

THEOREM (11.8): Prove that $\quad e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \displaystyle\sum_{n=-\infty}^{\infty} J_n(x)\, t^n.$

PROOF: We have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}}\, e^{-\frac{x}{2t}} \tag{1}$$

Using the relations $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$, equation (1) becomes

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \left\{\sum_{r=0}^{\infty} \frac{\left(\frac{x t}{2}\right)^r}{r!}\right\}\left\{\sum_{k=0}^{\infty} \frac{\left(-\frac{x}{2t}\right)^k}{k!}\right\}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{r+k} t^{r-k}}{r!\,k!}$$

Let $r-k = n$ so that n varies from $-\infty$ to ∞. Then the sum becomes

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k} t^n}{(n+k)!\,k!}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\,(n+k)!}\right] t^n = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

11.12 RECURRENCE FORMULAS INVOLVING BESSEL FUNCTIONS OF FIRST KIND

THEOREM (11.9): Prove the recurrence formulas

(i) $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

(ii) $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \qquad \text{for all } n.$$

PROOF: We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \tag{1}$$

(i) $$\frac{d}{dx}\left\{x^n J_n(x)\right\} = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2n+2m}}{2^{n+2m}\, m!\,\Gamma(n+m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(n+m)\, x^{2n+2m-1}}{2^{n+2m}\, m!\,(n+m)\,\Gamma(n+m)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2n+2m-1}}{2^{n+2m-1}\, m!\,\Gamma(n+m)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{(n-1)+2m}}{2^{(n-1)+2m}\, m!\,\Gamma((n-1)+m+1)}$$

or $\quad \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(ii) $\quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{n+2m} m!\,\Gamma(n+m+1)}$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m)\, x^{2m-1}}{2^{n+2m} m!\,\Gamma(n+m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{n+2m-1} (m-1)!\,\Gamma(n+m+1)}$$

$$= x^{-n} \sum_{m=1}^{\infty} \frac{(-1)^m x^{n+2m-1}}{2^{n+2m-1} (m-1)!\,\Gamma(n+m+1)}$$

(since for $m = 0$, $(-1)! = \infty$)

Letting $m = k+1$, we get

$$= x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{n+2k+1}}{2^{n+2k+1} k!\,\Gamma(n+k+2)}$$

$$= (-1)\, x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k x^{(n+1)+2k}}{2^{(n+1)+2k} k!\,\Gamma((n+1)+k+1)}$$

or $\quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

THEOREM (11.10): Prove that

(i) $\quad x J_n'(x) + n J_n(x) = x J_{n-1}(x)$

(ii) $\quad x J_n'(x) - n J_n(x) = -x J_{n+1}(x)$

(iii) $\quad J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x)$

(iv) $\quad J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ **for all n**

PROOF: (i) We know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

or $\quad x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$

or $\quad x J_n'(x) + n J_n(x) = x J_{n-1}(x) \qquad (1)$

(ii) Also, we have

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

or $$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

or $$x J_n'(x) - n J_n(x) = -x J_{n+1}(x) \tag{2}$$

(iii) Adding equations (1) and (2), we get

$$2 x J_n'(x) = x\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

or $$J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x)$$

(iv) Subtracting equation (2) from equation (1), we get

$$2 n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

or $$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{4}$$

11.13 WRONSKIAN FORMULAS INVOLVING BESSEL FUNCTIONS

THEOREM (11.11): Prove that

(i) $$J_n(x) J_{-n}'(x) - J_n'(x) J_{-n}(x) = -\frac{2 \sin n\pi}{\pi x}$$

(ii) $$J_n(x) J_{-n+1}(x) + J_{-n}(x) J_{n-1}(x) = \frac{2 \sin n\pi}{\pi x}$$

PROOF: (i) Since $J_n(x)$, and $J_{-n}(x)$ abbreviated J_n, J_{-n} respectively, satisfy Bessel's equation, we have

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0 \tag{1}$$

and $$x^2 J_{-n}'' + x J_{-n}' + (x^2 - n^2) J_{-n} = 0 \tag{2}$$

Multiplying equation (1) by J_{-n}, equation (2) by J_n and then subtracting, we get

$$x^2\left[J_n'' J_{-n} - J_{-n}'' J_n\right] + x\left[J_n' J_{-n} - J_{-n}' J_n\right] = 0$$

or $$x\left[J_n'' J_{-n} - J_{-n}'' J_n\right] + \left[J_n' J_{-n} - J_{-n}' J_n\right] = 0$$

or $$\frac{d}{dx}\left[x\left(J_n' J_{-n} - J_{-n}' J_n\right)\right] = 0$$

Integrating, we find

$$x\left(J_n' J_{-n} - J_{-n}' J_n\right) = C$$

or $$J_n' J_{-n} - J_{-n}' J_n = \frac{C}{x} \tag{3}$$

We know that

$$J_n(x) = \frac{1}{2^n \Gamma(n+1)}\left(x^n - \frac{x^{n+2}}{2^2 (n+1)} + \dots\dots\right)$$

$$J_n'(x) = \frac{1}{2^n\Gamma(n+1)}\left(n x^{n-1} - \frac{(n+2)x^{n-1}}{4(n+1)} + \ldots\ldots\right)$$

$$J_{-n}(x) = \frac{1}{2^{-n}\Gamma(-n+1)}\left(x^{-n} - \frac{x^{2-n}}{2^2(1-n)} + \ldots\ldots\right)$$

$$J_{-n}'(x) = \frac{1}{2^{-n}\Gamma(-n+1)}\left(-n x^{-n-1} - \frac{(2-n)x^{1-n}}{4(1-n)} + \ldots\ldots\right)$$

Substituting in equation (3), we get

$$\frac{C}{x} = \frac{1}{2^n\Gamma(n+1)}\left[n x^{n-1} - \frac{(n+2)x^{n-1}}{4(n+1)} + \ldots\right] \cdot \frac{1}{2^{-n}\Gamma(-n+1)}\left[x^{-n} - \frac{x^{2-n}}{4(1-n)} + \ldots\right]$$
$$- \frac{1}{2^{-n}\Gamma(-n+1)}\left[-n x^{-n-1} - \frac{(2-n)x^{1-n}}{4(1-n)} + \ldots\right] \cdot \frac{1}{2^n\Gamma(n+1)}\left[x^n - \frac{x^{n+2}}{2^n(n+1)} + \ldots\right]$$

Comparing the coefficients of $\frac{1}{x}$ on both sides, we get

or $$C = \frac{n}{2^n\Gamma(n+1) \cdot 2^{-n}\Gamma(-n+1)} + \frac{n}{2^{-n}\Gamma(-n+1)2^n\Gamma(n+1)}$$

$$= \frac{2n}{\Gamma(n+1)\Gamma(-n+1)} = \frac{2n}{n\Gamma(n)\Gamma(1-n)}$$

$$= \frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2\sin n\pi}{\pi} \qquad \left[\text{since } \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

Substituting the value of C in equation (3), we get

$$\therefore J_n' J_{-n} - J_{-n}' J_n = \frac{2\sin n\pi}{\pi x}$$

or $$J_n(x)J_{-n}'(x) - J_n'(x)J_{-n}(x) = -\frac{2\sin n\pi}{\pi x} \tag{4}$$

(ii) From theorem (11.10), we have the recurrence relation

$$x J_n'(x) + n J_n(x) = x J_{n-1}(x)$$

or $$J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x) \tag{5}$$

Also, from the recurrence relation

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x)$$

or $$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

Replacing n by $-n$, we get

$$J_{-n}'(x) = -\frac{n}{x}J_{-n}(x) - J_{-n+1}(x) \tag{6}$$

Substituting equations (5) and (6) in equation (4), we get

$$J_n(x)\left[-\frac{n}{x}J_{-n}(x)-J_{-n+1}(x)\right]-\left[-\frac{n}{x}J_n(x)+J_{n-1}(x)\right]J_{-n}(x) = -\frac{2\sin n\pi}{\pi x}$$

or $$-J_{-n+1}(x)J_n(x)-J_{n-1}(x)J_{-n}(x) = -\frac{2\sin n\pi}{\pi x}$$

or $$J_n(x)J_{-n+1}(x)+J_{-n}(x)J_{n-1}(x) = \frac{2\sin n\pi}{\pi x}$$

11.14 BESSEL FUNCTIONS OF FIRST KIND WHEN n IS HALF AN ODD INTEGER

THEOREM (11.12): Prove that

(i) $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

(ii) $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

(iii) $$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$$

(iv) $$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x}+\sin x\right)$$

PROOF: We have

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \quad (n = 0, 1, 2, \ldots\ldots) \tag{1}$$

(i) Let $n = \frac{1}{2}$ in equation (1), we get

$$J_{1/2}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{(1/2)+2m}}{m!\,\Gamma\left(m+\frac{3}{2}\right)}$$

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\Gamma\left(\frac{3}{2}\right)}-\frac{\left(\frac{x}{2}\right)^{5/2}}{1!\,\Gamma\left(\frac{5}{2}\right)}+\frac{\left(\frac{x}{2}\right)^{9/2}}{2!\,\Gamma\left(\frac{7}{2}\right)}-\ldots\ldots$$

Using $\Gamma(n+1) = n\Gamma(n)$, and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get

$$J_{1/2}(x) = \frac{\left(\frac{x}{2}\right)^{1/2}}{\left(\frac{1}{2}\right)\sqrt{\pi}}-\frac{\left(\frac{x}{2}\right)^{5/2}}{1!\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}+\frac{\left(\frac{x}{2}\right)^{9/2}}{2!\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}-\ldots\ldots$$

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\left(\frac{1}{2}\right)\sqrt{\pi}}\left\{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\ldots\ldots\right\}$$

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\left(\frac{1}{2}\right)\sqrt{\pi}}\frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}}\sin x$$

(ii) Let $n = -\frac{1}{2}$ in equation (1), we get

$$J_{-1/2}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{x}{2}\right)^{-(1/2)+2m}}{m!\,\Gamma\left(m+\frac{1}{2}\right)}$$

$$= \frac{\left(\frac{x}{2}\right)^{-1/2}}{\Gamma\left(\frac{1}{2}\right)}-\frac{\left(\frac{x}{2}\right)^{3/2}}{1!\,\Gamma\left(\frac{3}{2}\right)}+\frac{\left(\frac{x}{2}\right)^{7/2}}{2!\,\Gamma\left(\frac{5}{2}\right)}-\ldots\ldots$$

$$= \frac{\left(\frac{x}{2}\right)^{-1/2}}{\sqrt{\pi}}-\frac{\left(\frac{x}{2}\right)^{3/2}}{1!\,\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}+\frac{\left(\frac{x}{2}\right)^{7/2}}{2!\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}-\ldots\ldots$$

$$= \frac{\left(\frac{x}{2}\right)^{-1/2}}{\sqrt{\pi}}\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\ldots\right\}$$

$$= \sqrt{\frac{2}{\pi x}}\cos x \qquad (3)$$

(iii) From theorem (11.10) Part (iv), we have

$$J_{n-1}(x)+J_{n+1}(x) = \frac{2n}{x}J_n(x) \qquad (2)$$

Substituting $n = \frac{1}{2}$ in equation (2), we get

$$J_{-1/2}(x)+J_{3/2}(x) = \frac{1}{x}J_{1/2}(x)$$

or $$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x)-J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$$

(iv) Substituting $n = -\frac{1}{2}$ in equation (2), we get

$$J_{-3/2}(x) + J_{1/2}(x) = -\frac{1}{x} J_{-1/2}(x)$$

or

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

11.15 INTEGRALS INVOLVING BESSEL FUNCTIONS OF FIRST KIND

THEOREM (11.13): Prove that

(i) $$\int x^n J_{n-1}(x)\, dx = x^n J_n(x) + C$$

(ii) $$\int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x) + C$$

PROOF: (i) We know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

Integrating both sides w.r.t. x, we get

$$\int x^n J_{n-1}(x)\, dx = x^n J_n(x) + C$$

(ii) We know that

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

Integrating both sides w.r.t. x, we get

$$\int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x) + C$$

INTEGRALS OF THE FORM $\int x^m J_n(x)$

In general, an integral of this form where m and n are integers such that $m + n \geq 0$ can be completely integrated if $m + n$ is odd. But if $m + n$ is even, the result depends on the residual integral $\int J_0(x)\, dx$. It is not possible to reduce $\int J_0(x)\, dx$ any further.

EXAMPLE (1): Prove that

(i) $$\int x^{-1} J_4(x)\, dx = -\frac{1}{x} J_3(x) - \frac{2}{x^2} J_2(x) + C$$

(ii) $$\int x^2 J_2(x)\, dx = -x^2 J_1(x) - 3x J_0(x) + 3\int J_0(x)\, dx$$

SOLUTION: (i) We know that

$$\int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x) + C \tag{1}$$

Let $n = 3$ in equation (1), we get

$$\int x^{-3} J_4(x)\, dx = -x^{-3} J_3(x) + C \tag{2}$$

Now
$$\begin{aligned}\int x^{-1} J_4(x)\, dx &= \int x^2 [x^{-3} J_4(x)]\, dx \\ &= x^2[-x^{-3} J_3(x)] + \int x^{-3} J_3(x) . 2x\, dx \\ &= -x^{-1} J_3(x) + 2\int x^{-2} J_3(x)\, dx \end{aligned} \tag{3}$$

Let $n = 2$ in equation (1), we get

$$\int x^{-2} J_3(x)\, dx = -x^{-2} J_2(x) \tag{4}$$

From equations (3) and (4), we get

$$\begin{aligned}\int x^{-1} J_4(x)\, dx &= -x^{-1} J_3(x) - 2x^{-2} J_2(x) + C \\ &= -\frac{1}{x} J_3(x) - \frac{2}{x^2} J_2(x) + C\end{aligned}$$

(ii)
$$\begin{aligned}\int x^2 J_2(x)\, dx &= \int x^3 [x^{-1} J_2(x)]\, dx \\ &= x^3[-x^{-1} J_1(x)] - \int [-x^{-1} J_1(x)]\, 3x^2\, dx \\ &= -x^2 J_1(x) + 3\int x J_1(x)\, dx\end{aligned}$$

Now
$$\begin{aligned}\int x J_1(x)\, dx &= -\int x J_0'(x)\, dx \\ &= -\left[x J_0(x) - \int J_0(x)\, dx\right] \\ &= -x J_0(x) + \int J_0(x)\, dx\end{aligned}$$

Then
$$\begin{aligned}\int x^2 J_2(x)\, dx &= -x^2 J_1(x) + 3\left\{-x J_0(x) + \int J_0(x)\, dx\right\} \\ &= -x^2 J_1(x) - 3x J_0(x) + 3\int J_0(x)\, dx\end{aligned}$$

11.16 RECURRENCE FORMULAS FOR BESSEL FUNCTIONS OF SECOND KIND

THEOREM (11.14): Prove that

(i) $x\,Y_n'(x) + n\,Y_n(x) = x\,Y_{n-1}(x)$

(ii) $x\,Y_n'(x) - n\,Y_n(x) = -x\,Y_{n+1}(x)$

(iii) $Y_{n-1}(x) - Y_{n+1}(x) = 2\,Y_n'(x)$

(iv) $Y_{n-1}(x) + Y_{n+1}(x) = \dfrac{2n}{x}\,Y_n(x)$

(v) $\dfrac{d}{dx}\left[x^n Y_n(x)\right] = x^n Y_{n-1}(x)$

(vi) $\dfrac{d}{dx}\left[x^{-n} Y_n(x)\right] = -x^{-n} Y_{n+1}(x)$

PROOF: We know from theorem (11.10) parts (i) and (ii) that

$$x\,J_n'(x) + n\,J_n(x) = x\,J_{n-1}(x) \tag{1}$$

and $$x\,J_n'(x) - n\,J_n(x) = -x\,J_{n+1}(x) \tag{2}$$

Replacing n by $-n$ in equation (2), we get

$$x\,J_{-n}'(x) + n\,J_{-n}(x) = -x\,J_{-n+1}(x) \tag{3}$$

Multiplying equation (1) by $\cos n\pi$ and subtracting from it equation (3), we get

$$x\left[\cos n\pi\,J_n'(x) - J_{-n}'(x)\right] + n\left[\cos n\pi\,J_n(x) - J_{-n}(x)\right] = x\left[\cos n\pi\,J_{n-1}(x) + J_{-(n-1)}(x)\right]$$

Dividing this relation by $\sin n\pi$, we get

$$x\left[\frac{\cos n\pi\,J_n'(x) - J_{-n}'(x)}{\sin n\pi}\right] + n\left[\frac{\cos n\pi\,J_n(x) - J_{-n}(x)}{\sin n\pi}\right] = x\left[\frac{\cos n\pi\,J_{n-1}(x) + J_{-(n-1)}(x)}{\sin n\pi}\right]$$

Since $\sin(n-1)\pi = -\sin n\pi$ and $\cos(n-1)\pi = -\cos n\pi$, we get

$$x\left[\frac{\cos n\pi\,J_n'(x) - J_{-n}'(x)}{\sin n\pi}\right] + n\left[\frac{\cos n\pi\,J_n(x) - J_{-n}(x)}{\sin n\pi}\right] = x\left[\frac{-\cos(n-1)\pi\,J_{n-1}(x) + J_{-(n-1)}(x)}{-\sin(n-1)\pi}\right]$$

Using the definition for $Y_n(x)$, we get

(i) $$x\,Y_n'(x) + n\,Y_n(x) = x\,Y_{n-1}(x) \tag{4}$$

Similarly we can obtain

(ii) $$x\,Y_n'(x) - n\,Y_n(x) = -x\,Y_{n+1}(x) \tag{5}$$

(iii) Adding equations (4) and (5), we find

$$2xY_n'(x) = xY_{n-1}(x) - xY_{n+1}(x)$$

or $$Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x)$$

(iv) Subtracting equation (5) from (4), we get

$$2nY_n(x) = xY_{n-1}(x) + xY_{n+1}(x)$$

or $$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x}Y_n(x)$$

(v) Multiplying equation (4) by x^{n-1}, we get

$$x^n Y_n'(x) + nx^{n-1}Y_n(x) = x^n Y_{n-1}(x)$$

or $$\frac{d}{dx}\left[x^n Y_n(x)\right] = x^n Y_{n-1}(x)$$

(vi) Multiplying equation (5) by x^{-n-1}, we get

$$x^{-n}Y_n'(x) - nx^{-n-1}Y_n(x) = -x^{-n}Y_{n+1}(x)$$

or $$\frac{d}{dx}\left[x^{-n}Y_n(x)\right] = -x^{-n}Y_{n+1}(x)$$

DEDUCTION

If we substitute $n = 0$ in equation (5) we find that

$$Y_0'(x) = -Y_1(x)$$

THEOREM (11.15): Prove that

(i) $$J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = \frac{2}{\pi x}$$

(ii) $$J_{n+1}(x)Y_n(x) - J_n(x)Y_{n+1}(x) = \frac{2}{\pi x}$$

PROOF: (i) We know that

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}$$

so that $$Y_n'(x) = \frac{J_n'(x)\cos n\pi - J_{-n}'(x)}{\sin n\pi}$$

Now $$J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = \frac{J_n(x)J_n'(x)\cos n\pi - J_n(x)J_{-n}'(x)}{\sin n\pi}$$

$$-\frac{J_n'(x)J_n(x)\cos n\pi - J_n'(x)J_{-n}(x)}{\sin n\pi}$$

$$= \frac{J_n'(x)\,J_{-n}(x) - J_{-n}'(x)\,J_n(x)}{\sin n\pi}$$

$$= \frac{2\sin n\pi}{\pi x \sin n\pi} = \frac{2}{\pi x} \qquad \left[\text{using theorem (11.11)}\right]$$

(ii) We know from the recurrence relations that

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x)$$

or $$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

Similarly, $Y_n'(x) = \frac{n}{x} Y_n(x) - Y_{n+1}(x)$

Substituting the values of $J_n'(x)$ and $Y_n'(x)$ in part (i), we get

$$J_n(x)\left[\frac{n}{x} Y_n(x) - Y_{n+1}(x)\right] - \left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right] Y_n(x) = \frac{2}{\pi x}$$

$$J_{n+1}(x)\,Y_n(x) - J_n(x)\,Y_{n+1}(x) = \frac{2}{\pi x}$$

11.17 BESSEL FUNCTIONS OF SECOND KIND WHEN n IS HALF AN ODD INTEGER

THEOREM (11.16): Prove that

(i) $$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x$$

(ii) $$Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

(iii) $$Y_{3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

(iv) $$Y_{-3/2}(x) = \frac{2}{\sqrt{\pi x}}\left(\cos x - \frac{\sin x}{x}\right)$$

PROOF: We know that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

and $$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right), \qquad J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

Now $$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} \qquad (1)$$

(i) Let $n = \frac{1}{2}$ in equation (1), we get

Now $$Y_{1/2}(x) = \frac{J_{1/2}(x)\cos\left(\frac{\pi}{2}\right) - J_{-1/2}(x)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0 - J_{-1/2}(x)}{1}$$

$$= -J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x$$

(ii) Let $n = -\frac{1}{2}$ in equation (1), we get

$$Y_{-1/2}(x) = \frac{J_{-1/2}(x)\cos\left(\frac{\pi}{2}\right) - J_{1/2}(x)}{\sin\left(-\frac{\pi}{2}\right)} = \frac{0 - J_{-1/2}(x)}{-1}$$

$$= J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

(iii) Let $n = \frac{3}{2}$ in equation (1), we get

$$Y_{3/2}(x) = \frac{J_{3/2}(x)\cos\left(\frac{3\pi}{2}\right) - J_{-3/2}(x)}{\sin\left(\frac{3\pi}{2}\right)}$$

$$= \frac{0 - J_{-3/2}(x)}{-1} = J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

(iv) Let $n = -\frac{3}{2}$ in equation (1), we get

$$Y_{-3/2}(x) = \frac{J_{-3/2}(x)\cos\left(-\frac{3\pi}{2}\right) - J_{3/2}(x)}{\sin\left(-\frac{3\pi}{2}\right)}$$

$$= \frac{0 - J_{3/2}(x)}{1} = -J_{3/2}(x)$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) = \frac{2}{\sqrt{\pi x}}\left(\cos x - \frac{\sin x}{x}\right)$$

11.18 INTEGRALS INVOLVING BESSEL FUNCTIONS OF SECOND KIND

THEOREM (11.17): Evaluate the integrals

(i) $$\int x^n Y_{n-1}(x)\,dx = x^n Y_n(x) + C$$

(ii) $$\int x^{-n} Y_{n+1}(x)\,dx = -x^{-n} Y_n(x) + C$$

PROOF: From theorem (11.14), we know that

(i) $$\frac{d}{dx}\left[x^n Y_n(x)\right] = x^n Y_{n-1}(x)$$

Therefore $$\int x^n Y_{n-1}(x)\,dx = x^n Y_n(x) + C$$

(ii) $$\frac{d}{dx}\left[x^{-n} Y_n(x)\right] = -x^{-n} Y_{n+1}(x)$$

Therefore $$\int x^{-n} Y_{n+1}(x)\,dx = -x^{-n} Y_n(x) + C$$

EXAMPLE (2): **Evaluate**

(i) $\int Y_3(x)\,dx$ (ii) $\int x^{-2} Y_2(x)\,dx$

SOLUTION: We have

(i) $$\int Y_3(x)\,dx = \int x^2\left[x^{-2} Y_3(x)\right] dx$$
$$= x^2\left[-x^{-2} Y_2(x)\right] - \int\left[-x^{-2} Y_2(x)\right] 2x\,dx$$
$$= -Y_2(x) + 2\int x^{-1} Y_2(x)\,dx$$

Since $\int x^{-1} Y_2(x)\,dx = -x^{-1} Y_1(x) + C$, therefore
$$\int Y_3(x)\,dx = -Y_2(x) - 2\left[\frac{Y_1(x)}{x}\right] + C$$

(ii) $$\int x^{-2} Y_2(x)\,dx = \int x^2 Y_2(x)\frac{1}{x^4}\,dx$$
$$= x^2 Y_2(x)\left(-\frac{1}{3x^3}\right) - \int\left(-\frac{1}{3x^3}\right) x^2 Y_1(x)\,dx$$
$$= -\frac{1}{3}x^{-1} Y_2(x) + \frac{1}{3}\int x^{-1} Y_1(x)\,dx$$

Now $$\int x^{-1} Y_1(x)\,dx = \int x Y_1(x)\frac{1}{x^2}\,dx$$
$$= x Y_1(x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right) x Y_0(x)\,dx$$
$$= -Y_1(x) + \int Y_0(x)\,dx$$

Therefore $$\int x^{-2} Y_2(x)\,dx = -\frac{1}{3}x^{-1} Y_2(x) + \frac{1}{3}\left(-Y_1(x) + \int Y_0(x)\,dx\right)$$
$$= -\frac{1}{3}x^{-1} Y_2(x) - \frac{1}{3}Y_1(x) + \frac{1}{3}\int Y_0(x)\,dx$$

11.19 EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

A number of second order differential equations with variable coefficients can be reduced to Bessel's equation by a suitable transformation , and hence can be solved in terms of Bessel functions .

EXAMPLE (3): **Transform the differential equation :**

$$x^2\frac{d^2 y}{d x^2}+x\frac{d y}{d x}+(\lambda^2 x^2-n^2)y=0 \qquad (\lambda = \text{constant})$$

to Bessel's equation and solve .

SOLUTION: Let $t=\lambda x$, then $\dfrac{dt}{dx}=\lambda$

and $\dfrac{dy}{dx}=\dfrac{dy}{dt}\dfrac{dt}{dx}=\lambda\dfrac{dy}{dt}$

$$\frac{d^2 y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=\frac{d}{dt}\left(\lambda\frac{dy}{dt}\right)\lambda=\lambda^2\frac{d^2 y}{dt^2}$$

Substituting the values of these derivatives in the given differential equation , we get

$$\frac{t^2}{\lambda^2}\left(\lambda^2\frac{d^2 y}{dt^2}\right)+\frac{t}{\lambda}\left(\lambda\frac{dy}{dt}\right)+(t^2-n^2)y=0$$

or $$t^2\frac{d^2 y}{dt^2}+t\frac{dy}{dt}+(t^2-n^2)y=0 \qquad (1)$$

which is the Bessel's differential equation whose solution is given by

$$y=A J_n(t)+B Y_n(t)$$

where A and B are constants .

The general solution of the given equation is therefore

$$y=A J_n(\lambda x)+B Y_n(\lambda x)$$

11.20 MISCELLANEOUS RESULTS

THEOREM (11.18): Prove that

(i) $\cos(x\sin\theta)=J_0(x)+2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta+\ldots\ldots\ldots$

(ii) $\sin(x\sin\theta)=2J_1(x)\sin\theta+2J_3(x)\sin 3\theta+2J_5(x)\sin 5\theta+\ldots\ldots\ldots$

(iii) $\cos(x\cos\theta)=J_0(x)-2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta-\ldots\ldots\ldots$

(iv) $\sin(x\cos\theta)=2J_1(x)\cos\theta-2J_3(x)\cos 3\theta+2J_5(x)\cos 5\theta-\ldots\ldots\ldots$

PROOF: We know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty} J_n(x)t^n$$

Put $t=e^{i\theta}$, we get

$$e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$

Since $\dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$, therefore

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)[\cos n\theta + i\sin n\theta]$$

$$\begin{aligned}
\cos(x\sin\theta) + i\sin(x\sin\theta) &= J_0(x) + J_{-1}(x)(\cos\theta - i\sin\theta) + J_1(x)(\cos\theta + i\sin\theta) \\
&\quad + J_{-2}(x)(\cos 2\theta - i\sin 2\theta) + J_2(x)(\cos 2\theta + i\sin 2\theta) + \ldots\ldots \\
&= \Big[J_0(x) + \{J_{-1}(x) + J_1(x)\}\cos\theta \\
&\quad + \{J_{-2}(x) + J_2(x)\}\cos 2\theta + \ldots\ldots\Big] \\
&\quad + i\Big[\{J_1(x) - J_{-1}(x)\}\sin\theta + \{J_2(x) - J_{-2}(x)\}\sin 2\theta + \ldots\ldots\Big]
\end{aligned}$$

Now by using $J_{-n}(x) = (-1)^n J_n(x),\quad n = 0, 1, 2, 3, \ldots\ldots$

$$= \Big[J_0(x) + 2J_2(x)\cos 2\theta + \ldots\ldots\Big] + i\Big[2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \ldots\ldots\Big]$$

Equating real and imaginary parts, we get

(i) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots\ldots$

(ii) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \ldots\ldots$

(iii) Put $\theta = \dfrac{\pi}{2} - \theta$ in part (i), we get the required results.

$\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots\ldots$

(iv) Put $\theta = \dfrac{\pi}{2} - \theta$ in part (ii), we get the required results.

$\sin(x\cos\theta) = 2J_1(x)\cos\theta - 2J_3(x)\cos 3\theta + 2J_5(x)\cos 5\theta + \ldots\ldots$

These series are known as Jacobi series.

DEDUCTION

Let $\theta = \dfrac{\pi}{2}$ in parts (i) and (ii) or $\theta = 0$ in parts (iii) and (iv), we get

$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \ldots\ldots$

$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \ldots\ldots$

11.21 INTEGRAL REPRESENTATION OF BESSEL FUNCTIONS OF FIRST KIND

THEOREM (11.19): Prove that $J_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta,\quad n = 0, 1, 2, \ldots\ldots$

PROOF: We know that

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \end{cases}, \qquad \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \end{cases}$$

Also , we know that

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots\ldots + 2J_n(x)\cos n\theta + \ldots\ldots \tag{1}$$

$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \ldots + 2J_n(x)\sin n\theta + \ldots\ldots \tag{2}$$

Multiply equations (1) and (2) by $\cos n\theta$ and $\sin n\theta$ respectively and integrate from 0 to π using the above resuls , we get when n is even or zero

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = 2J_n(x)\int_0^{\pi} \cos^2 n\theta \, d\theta = 2J_n(x)\frac{\pi}{2} = \pi J_n(x)$$

and $$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = 0 \qquad \text{(since all terms vanish on R.H.S.)}$$

Adding , we have

$$\pi J_n(x) = \int_0^{\pi} \left[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right] d\theta$$

or $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta, \quad n = 0, 2, 4, 6, \ldots\ldots \tag{1}$$

Similarly if n is odd , we get

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = 0 \qquad \text{(since all terms vaish on R.H.S.)}$$

and $$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = 2J_n(x)\int_0^{\pi} \sin^2 n\theta \, d\theta = 2J_n(x)\frac{\pi}{2} = \pi J_n(x)$$

and by adding

$$\pi J_n(x) = \int_0^{\pi} \left[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right] d\theta$$

or $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta, \quad n = 1, 3, 5, \ldots\ldots \tag{2}$$

Thus from equations (1) and (2), we have the required result as

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta, \quad n = 0, 1, 2, 3, 4, \ldots\ldots$$

DEDUCTION

Let $n = 0$ in equation (2), we get $\quad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta$

which is the integral representation of $J_0(x)$.

11.22 SPECIAL INTEGRALS INVOLVING BESSEL FUNCTIONS

THEOREM (11.20): Prove that if λ and μ are constants, then

(i) $$\int x J_n(\lambda x) J_n(\mu x)\, dx = \frac{x\left[\lambda J_n(\mu x) J_n'(\lambda x) - \mu J_n(\lambda x) J_n'(\mu x)\right]}{\mu^2 - \lambda^2}$$

if $\lambda \neq \mu$

(ii) $$\int x J_n^2(\lambda x)\, dx = \frac{x^2}{2} J_n'^2(\lambda x) + \frac{x^2}{2}\left(1 - \frac{n^2}{\lambda^2 x^2}\right) J_n^2(\lambda x) \quad \text{if } \lambda = \mu$$

PROOF: (i) We know that $y_1 = J_n(\lambda x)$ and $y_2 = J_n(\mu x)$ are the solutions of the Bessel's differential equation, therefore

$$x^2 y_1'' + x y_1' + (\lambda^2 x^2 - n^2) y_1 = 0 \tag{1}$$

and $$x^2 y_2'' + x y_2' + (\mu^2 x^2 - n^2) y_2 = 0 \tag{2}$$

Multiplying equation (1) by y_2, equation (2) by y_1 and subtracting, we find

$$x^2 [y_2 y_1'' - y_1 y_2''] + x[y_2 y_1' - y_1 y_2'] = (\mu^2 - \lambda^2) x^2 y_1 y_2$$

which on division by x can be written as

$$x \frac{d}{dx}[y_2 y_1' - y_1 y_2'] + [y_2 y_1' - y_1 y_2'] = (\mu^2 - \lambda^2) x y_1 y_2$$

or $$\frac{d}{dx}\left[x(y_2 y_1' - y_1 y_2')\right] = (\mu^2 - \lambda^2) x y_1 y_2$$

Then by integrating and omitting the constant of integration,

$$(\mu^2 - \lambda^2)\int x y_1 y_2\, dx = x(y_2 y_1' - y_1 y_2')$$

Using $y_1 = J_n(\lambda x)$, $y_2 = J_n(\mu x)$ and dividing by $(\mu^2 - \lambda^2) \neq 0$,

$$\int x J_n(\lambda x) J_n(\mu x)\, dx = \frac{x\left[\lambda J_n(\mu x) J_n'(\lambda x) - \mu J_n(\lambda x) J_n'(\mu x)\right]}{\mu^2 - \lambda^2} \tag{3}$$

(ii) If $\lambda = \mu$, then the R.H.S. of equation (3) becomes indeterminate of the form $\left(\frac{0}{0}\right)$. Thus

$$\int x J_n^2(\lambda x)\, dx = \underset{\mu \to \lambda}{\text{Lt}} \frac{x[\lambda J_n(\mu x) J_n'(\lambda x) - \mu J_n(\lambda x) J_n'(\mu x)}{\mu^2 - \lambda^2}$$

Using the L'Hospital rule, we get

$$\int x J_n^2(\lambda x)\, dx = \underset{\mu \to \lambda}{\text{Lt}} \frac{x[\lambda x J_n'(\mu x) J_n'(\lambda x) - J_n(\lambda x) J_n'(\mu x) - \mu x J_n(\lambda x) J_n''(\mu x)}{2\mu}$$

$$= \frac{1}{2\lambda}[\lambda x^2 J_n'^2(\lambda x) - x J_n(\lambda x) J_n'(\lambda x) - \lambda x^2 J_n(\lambda x) J_n''(\lambda x)]$$

$$= \frac{1}{2}\left[x^2 J_n'^2(\lambda x) - \frac{1}{\lambda} x J_n(\lambda x) J_n'(\lambda x) - x^2 J_n(\lambda x) J_n''(\lambda x)\right] \tag{4}$$

But since $y = J_n(\lambda x)$ is a solution of Bessel's equation

$$x^2 y'' + x y' + (\lambda^2 x^2 - n^2) y = 0$$

therefore $\lambda^2 x^2 J_n''(\lambda x) + \lambda x J_n'(\lambda x) + (\lambda^2 x^2 - n^2) J_n(\lambda x) = 0$

or $$J_n''(\lambda x) = -\frac{1}{\lambda x} J_n'(\lambda x) - \left(1 - \frac{n^2}{\lambda^2 x^2}\right) J_n(\lambda x) \tag{5}$$

From equations (4) and (5), we get

$$\int x J_n^2(\lambda x)\, dx = \frac{1}{2}\Bigg[x^2 J_n'^2(\lambda x) - \frac{1}{\lambda} x J_n(\lambda x) J_n'(\lambda x)$$

$$- x^2 J_n(\lambda x)\left\{-\frac{1}{\lambda x} J_n'(\lambda x) - \left(1 - \frac{n^2}{\lambda^2 x^2}\right) J_n(\lambda x)\right\}\Bigg]$$

$$= \frac{1}{2}\Bigg[x^2 J_n'^2(\lambda x) - \frac{1}{\lambda} x J_n(\lambda x) J_n'(\lambda x) + \frac{1}{\lambda} x J_n(\lambda x) J_n'(\lambda x)$$

$$+ x^2\left(1 - \frac{n^2}{\lambda^2 x^2}\right) J_n^2(\lambda x)\Bigg]$$

$$= \frac{1}{2} x^2 J_n'^2(\lambda x) + \frac{1}{2} x^2 \left(1 - \frac{n^2}{\lambda^2 x^2}\right) J_n^2(\lambda x) \tag{6}$$

DEDUCTION

$$\int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = \frac{\lambda J_n(\mu) J_n'(\lambda) - \mu J_n(\lambda) J_n'(\mu)}{\mu^2 - \lambda^2} \tag{7}$$

and $$\int_0^1 x J_n^2(\lambda x)\, dx = \frac{1}{2} J_n'^2(\lambda) + \frac{1}{2}\left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda) \tag{8}$$

11.23 ORTHOGONALITY OF BESSEL FUNCTIONS

THEOREM (11.21): Prove that if λ and μ are any two roots of $J_n(x) = 0$, then

$$\int_0^1 x\, J_n(\lambda x)\, J_n(\mu x)\, dx = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \frac{1}{2} J_{n+1}^2(\lambda) & \text{if } \lambda = \mu \end{cases}$$

i.e. $\sqrt{x}\, J_n(\lambda x)$ and $\sqrt{x}\, J_n(\mu x)$ are orthogonal over the interval $(0, 1)$.

PROOF: We know from the above theorem that if $\lambda \neq \mu$, then

$$\int_0^1 x\, J_n(\lambda x)\, J_n(\mu x)\, dx = \frac{\lambda J_n(\mu) J_n'(\lambda) - \mu J_n(\lambda) J_n'(\mu)}{\mu^2 - \lambda^2} \tag{1}$$

Since λ and μ are the roots of $J_n(x) = 0$, therefore $J_n(\lambda) = 0$ and $J_n(\mu) = 0$.

Thus from equation (1), we get

$$\int_0^1 x\, J_n(\lambda x)\, J_n(\mu x)\, dx = 0$$

Also, we know that if $\lambda = \mu$, then

$$\int_0^1 x\, J_n^2(\lambda x)\, dx = \frac{1}{2} J_n'^2(\lambda) + \frac{1}{2}\left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda) \tag{2}$$

Since $J_n(\lambda) = 0$, therefore equation (2) becomes

$$\int_0^1 x\, J_n^2(\lambda x)\, dx = \frac{1}{2} J_n'^2(\lambda) \tag{3}$$

Now from the recurrence relation

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x)$$

Let $x = \lambda$ in this relation, we get

$$\lambda J_n'(\lambda) - n J_n(\lambda) = -\lambda J_{n+1}(\lambda)$$

Since $J_n(\lambda) = 0$, therefore

$$J_n'(\lambda) = -J_{n+1}(\lambda) \tag{4}$$

From equations (3) and (4), we get

$$\int_0^1 x\, J_n^2(\lambda x)\, dx = \frac{1}{2} J_{n+1}^2(\lambda)$$

11.24 EXPANSION OF f (x) IN A SERIES OF BESSEL'S FUNCTIONS

THEOREM (11.22): Let $f(x)$ be defined over the interval $0 < x < 1$. Then prove that at every point of continuity of $f(x)$ in $0 < x < 1$, the Bessel series expansion of $f(x)$

is $f(x) = A_1 J_1(\lambda_1 x) + A_2 J_2(\lambda_2 x) + \ldots\ldots\ldots = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)$

where λ_p, $p = 1, 2, 3, \ldots\ldots$ are the positive roots of $J_n(x) = 0$,

and $A_p = \dfrac{2}{J_{n+1}^2(\lambda_p)} \displaystyle\int_0^1 x J_n(\lambda_p x)\, f(x)\, dx$

At a point of discontinuity of $f(x)$ in $0 < x < 1$, the left side be replaced by

$\frac{1}{2}\left[f(x+0) - f(x-0) \right]$

PROOF: Multiplying the series for $f(x)$ by $x J_n(\lambda_k x)$ and integrating term by term from 0 to 1, we get

$$\int_0^1 x J_n(\lambda_k x)\, f(x)\, dx = \sum_{p=1}^{\infty} A_p \int_0^1 x J_n(\lambda_k x) J_n(\lambda_p x)\, dx$$

$$= A_k \int_0^1 x J_n^2(\lambda_k x)\, dx \quad (\text{for } p = k)$$

since all other terms vanish using orthogonality condition. Therefore

$$\int_0^1 x J_n(\lambda_k x)\, f(x)\, dx = A_k \frac{1}{2} J_{n+1}^2(\lambda_k) \qquad [\text{using theorem } (11.21)]$$

or $$A_k = \frac{2}{J_{n+1}^2(\lambda_k)} \int_0^1 x J_n(\lambda_k x) f(x)\, dx \quad \text{or} \quad A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x) f(x)\, dx$$

PARTICULAR CASE

In particular, if $n = 0$, then

$f(x) = A_1 J_0(\lambda_1 x) + A_2 J_0(\lambda_2 x) + A_3 J_0(\lambda_3 x) + \ldots\ldots\ldots$

where $A_p = \dfrac{2}{J_1^2(\lambda_p)} \displaystyle\int_0^1 x f(x)\, J_0(\lambda_p x)\, dx$

EXAMPLE (4): **Expand $f(x) = 1$ in a Bessel series**

$$\sum_{p=1}^{\infty} A_p J_0(\lambda_p x) \text{ for } 0 < x < 1, \text{ if } \lambda_p,\ p = 1, 2, 3, \ldots\ldots \text{ are the}$$

positive roots of $J_0(x) = 0$.

SOLUTION: From theorem (11.22), we have

$$A_p = \frac{2}{J_1^2(\lambda_p)} \int_0^1 x J_0(\lambda_p x)\, dx$$

Let $\lambda_p x = v$, therefore $dx = \frac{1}{\lambda_p} dv$. Also $0 \le v \le \lambda_p$

$$\text{Therefore } A_p = \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \int_0^{\lambda_p} v J_0(v)\, dv$$

Using the formula $\int x J_0(x)\, dx = x J_1(x)$, we get

$$A_p = \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \left| v J_1(v) \right|_0^{\lambda_p} = \frac{2}{\lambda_p J_1(\lambda_p)}$$

Thus we have the required series

$$f(x) = 1 = \sum_{p=1}^{\infty} \frac{2}{\lambda_p J_1(\lambda_p)} J_0(\lambda_p x)$$

which can be written

$$\frac{J_0(\lambda_1 x)}{\lambda_1 J_1(\lambda_1)} + \frac{J_0(\lambda_2 x)}{\lambda_2 J_2(\lambda_2)} + \ldots\ldots\ldots\ldots = \frac{1}{2}$$

11.25 HANKEL FUNCTIONS

Hankel functions of first and second kinds of order n are defined respectively by

$$H_n^{(1)}(x) = J_n(x) + i\, Y_n(x)$$

$$H_n^{(2)}(x) = J_n(x) - i\, Y_n(x)$$

and are often called Bessel functions of the third kind of order n Hankel functions are named after the German mathematician H. Hankel (1839 – 1873). Hankel functions are particularly useful in the study of wave propagation.

In terms of these functions, the general solution of Bessel's equation can be written as

$y = C_1 H_n^{(1)}(x) + C_2 H_n^{(2)}(x)$, where C_1 and C_2 are arbitrary constants.

THEOREM (11.23): If n is not an integer, then show that

(i) $$H_n^{(1)}(x) = \frac{J_{-n}(x) - e^{-in\pi} J_n(x)}{i \sin n\pi}$$

(ii) $$H_n^{(2)}(x) = \frac{e^{in\pi} J_n(x) - J_{-n}(x)}{i \sin n\pi}$$

PROOF: We know that

(i)
$$\begin{aligned}
H_n^{(1)}(x) &= J_n(x) + i Y_n(x) \\
&= J_n(x) + i\left[\frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}\right] \\
&= \frac{J_n(x)\sin n\pi + i J_n(x)\cos n\pi - i J_{-n}(x)}{\sin n\pi} \\
&= i\left[\frac{J_n(x)(\cos n\pi - i\sin n\pi) - J_{-n}(x)}{\sin n\pi}\right] \\
&= i\left[\frac{J_n(x)e^{-in\pi} - J_{-n}(x)}{\sin n\pi}\right] \\
&= \frac{J_{-n}(x) - e^{-in\pi} J_n(x)}{i \sin n\pi}
\end{aligned}$$

(ii) Since $H_n^{(2)}(x) = J_n(x) - i Y_n(x)$,

we find on replacing i by $-i$ in the result of part (i)

$$\begin{aligned}
H_n^{(2)}(x) &= \frac{J_{-n}(x) - e^{in\pi} J_n(x)}{-i \sin n\pi} \\
&= \frac{e^{in\pi} J_n(x) - J_{-n}(x)}{i \sin n\pi}
\end{aligned}$$

11.26 MODIFIED BESSEL'S DIFFERENTIAL EQUATION

If we replace x by $i x$ in Bessel's equation, it becomes

$$x^2 y'' + x y' - (x^2 + n^2) y = 0 \qquad (1)$$

which is known as the modified Bessel's differential equation of order n. The solutions of equation (1) are called modified Bessel functions of order n.

GENERAL SOLUTION WHEN n IS NOT AN INTEGER

When n is not an integer, equation (1) has therefore the general solution as

$$y = A J_n(ix) + B J_{-n}(ix), \qquad n \neq 0, 1, 2, 3, \ldots\ldots\ldots \qquad (2)$$

where $J_n(ix)$ and $J_{-n}(ix)$ are the independent solutions of equation (1).

Now
$$J_n(ix) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{ix}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)}$$
$$= i^n \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \qquad \left[\text{since } (-1)^{2m} = 1\right]$$

Moreover, $J_n(ix)$ multiplied by any constant will also be a solution of the equation we are considering. Hence we multiply by i^{-n}, getting

$$i^{-n} J_n(ix) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \tag{3}$$

The functions defined by equation (3) are known as the **modified Bessel functions of the first kind of order** n and are denoted by $I_n(x)$.

Thus $I_n(x) = i^{-n} J_n(ix) = \left(e^{\frac{i\pi}{2}}\right)^{-n} J_n(ix)$
$$= e^{-\frac{n\pi i}{2}} J_n(ix) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \tag{4}$$

In the special case when n is a positive integer or zero, then $\Gamma(n+m+1) = (n+m)!$, $\Gamma(1) = 1$ therefore, equation (4) becomes

$$I_n(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,(n+m)!} \quad (n = 0, 1, 2, \ldots\ldots) \tag{5}$$

If n is not an integer, the function $I_{-n}(x)$ obtained from $I_n(x)$ by replacing n by $-n$ throughout is a second, independent solution of equation (1). Hence the general solution (2) of equation (1) can then be written as

$$y = A\,I_n(x) + B\,I_{-n}(x), \quad n \neq 0, 1, 2, 3, \ldots\ldots \tag{6}$$

To obtain the general solution in the case when n is an integer, we first prove the following result.

THEOREM (11.24): Prove that $I_{-n}(x) = I_n(x)$ for $n = 0, 1, 2, 3, \ldots\ldots$

PROOF: We have

$$I_{-n}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)}$$

$$= \sum_{m=0}^{n-1} \frac{\left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)} + \sum_{m=n}^{\infty} \frac{\left(\frac{x}{2}\right)^{-n+2m}}{m!\,\Gamma(-n+m+1)}$$

Now since $\Gamma(-n+m+1)$ is infinite for $m = 0, 1, 2, \ldots\ldots, n-1$, the first sum on the right is zero.
Letting $m = n+k$ in the second sum, it becomes

$$\sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{\Gamma(n+k+1)\,k!} = I_n(x)$$

GENERAL SOLUTION WHEN n IS AN INTEGER

We know that $I_{-n}(x) = I_n(x)$ for $n = 0, 1, 2, \ldots\ldots\ldots$

That is, when $n = 0, 1, 2, 3, \ldots\ldots$ the general solution $y = A\,I_n(x) + B\,I_{-n}(x)$ becomes

$$y = (A+B)\,I_n(x) = C\,I_n(x)$$

where C is an arbitrary constant. Thus when $n = 0, 1, 2, 3, \ldots\ldots$

$$A\,I_n(x) + B\,I_{-n}(x)$$

cannot be general solution of equation (1).

To obtain the general solution in this case, we therefore introduce modified Bessel's functions of the second kind of order n.

MODIFIED BESSEL FUNCTIONS OF SECOND KIND

We know that when $n \neq 0, 1, 2, 3, \ldots\ldots$, the general solution of modified Bessel's equation is $y = A\,I_n(x) + B\,I_{-n}(x)$

Taking the arbitrary constants A and B as

$$A = C_1 - \frac{C_2\pi}{2\sin n\pi}, \quad B = \frac{C_2\pi}{2\sin n\pi}$$

the general solution becomes

$$y = \left(C_1 - \frac{C_2\pi}{2\sin n\pi}\right) I_n(x) + \frac{C_2\pi}{2\sin n\pi} I_{-n}(x)$$

$$= C_1 I_n(x) + C_2\left[\frac{\pi}{2}\left(\frac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right)\right]$$

The function defined by the expression

$$\frac{\pi}{2}\left[\frac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right], \quad n \neq 0, 1, 2, 3, \ldots\ldots\ldots$$

are called the **modified Bessel function of the second kind of order n** and are denoted by $K_n(x)$.

Thus the general solution of modified Bessel's equation (1) can be written as

$$y = C_1 I_n(x) + C_2 K_n(x), \quad n \neq 0, 1, 2, 3, \ldots\ldots$$

where C_1 and C_2 are arbitrary constants.

Now the relation for modified Bessel functions

$$K_n(x) = \frac{\pi}{2}\left[\frac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right]$$

becomes an **indeterminate** of the form $\frac{0}{0}$ for the case when n is an integer. This is because for an integer n, $\sin n\pi = 0$ and $I_{-n}(x) = I_n(x)$. This **indeterminate form** can be evaluated by using L' Hospital's rule

i.e. $$K_n(x) = \underset{p \to n}{\text{Lt}}\ \frac{\pi}{2}\left[\frac{I_{-p}(x) - I_p(x)}{\sin p\pi}\right] \qquad (5)$$

Thus we define the Modified Bessel's function of the second kind of order n as

$$K_n(x) = \begin{cases} \dfrac{\pi}{2\sin n\pi}\left[I_{-n}(x) - I_n(x)\right] & , \quad n \neq 0, 1, 2, 3, \ldots\ldots \\ \underset{p \to n}{\text{Lt}}\ \dfrac{\pi}{2\sin p\pi}\left[I_{-p}(x) - I_p(x)\right] & , \quad n = 0, 1, 2, 3, \ldots\ldots \end{cases}$$

GENERAL SOLUTION OF MODIFIED BESSEL'S EQUATION

The general solution of the Modified Bessel's equation for all values of n can then be written as

$$y = C_1 I_n(x) + C_2 K_n(x) \qquad (n \geq 0)$$

where C_1 and C_2 are arbitrary constants.

THEOREM (11.25): Prove that $K_{-n}(x) = K_n(x), \quad n = 0, 1, 2, 3, \ldots\ldots\ldots$

PROOF: By definition

$$K_n(x) = \underset{p \to n}{\text{Lt}}\ \frac{\pi}{2}\left[\frac{I_{-p}(x) - I_p(x)}{\sin p\pi}\right], \quad n = 0, 1, 2, 3, \ldots\ldots\ldots$$

Using L'ospital's rule, we get

$$K_n(x) = \frac{\pi}{2}\left[\frac{\dfrac{\partial I_{-p}}{\partial p} - \dfrac{\partial I_p}{\partial p}}{\pi\cos p\pi}\right]_{p=n}$$

$$= \frac{1}{2}\left[\frac{\dfrac{\partial I_{-n}}{\partial n} - \dfrac{\partial I_n}{\partial n}}{\cos n\pi}\right]$$

$$= \frac{1}{2}(-1)^n\left[\frac{\partial I_{-n}}{\partial n} - \frac{\partial I_n}{\partial n}\right] \qquad (1)$$

Replacing n by $-n$ in equation (1), we get

$$K_{-n}(x) = \frac{1}{2}(-1)^{-n}\left[\frac{\partial I_n}{\partial(-n)} - \frac{\partial J_{-n}}{\partial(-n)}\right]$$

$$= \frac{1}{2}(-1)^{-n}\left[-\frac{\partial I_n}{\partial n} + \frac{\partial I_{-n}}{\partial n}\right]$$

$$= \frac{1}{2}(-1)^{-n}\left[\frac{\partial I_{-n}}{\partial n} - \frac{\partial I_n}{\partial n}\right]$$

$$= \frac{1}{2}(-1)^{n}\left[\frac{\partial I_{-n}}{\partial n} - \frac{\partial I_n}{\partial n}\right]$$

$$K_{-n}(x) = K_n(x) \qquad \left[\text{since } (-1)^{-n} = (-1)^n\right]$$

11.27 SERIES EXPANSION FOR $K_n(x)$

We first discuss the case for $n = 0$. In this case we must evaluate

$$K_0(x) = \underset{p \to n}{\mathrm{Lt}} \frac{\pi}{2}\left[\frac{I_{-p}(x) - I_p(x)}{\sin p\pi}\right]$$

Using L' Hospital's rule (differentiate numerator and denominator w.r.t. p), we find

$$K_0(x) = \underset{p \to 0}{\mathrm{Lt}} \frac{\pi}{2}\left[\frac{\dfrac{\partial I_{-p}}{\partial p} - \dfrac{\partial I_p}{\partial p}}{\pi\cos p\pi}\right]$$

$$= \frac{1}{2}\left[\frac{\partial I_{-p}}{\partial p} - \frac{\partial I_p}{\partial p}\right]_{p=0} \tag{6}$$

where the notation indicates that we are to take the partial derivatives of $I_{-p}(x)$ and $I_p(x)$ w.r.t. p and put $p = 0$

Since $\dfrac{\partial I_{-p}}{\partial p} = -\dfrac{\partial I_{-p}}{\partial(-p)} = -\dfrac{\partial I_p}{\partial p}$

equation (6) can be written as

$$K_0(x) = -\left[\frac{\partial I_p}{\partial p}\right]_{p=0} \tag{7}$$

To obtain $\dfrac{\partial I_p}{\partial p}$, we differentiate the series

$$I_p(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{p+2m}}{m!\,\Gamma(p+m+1)} \cdots\cdots$$

w.r.t. p and obtain

$$\frac{\partial I_p}{\partial p} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial}{\partial p} \frac{\left(\frac{x}{2}\right)^{p+2m}}{\Gamma(p+m+1)} \tag{8}$$

Now if we let $\dfrac{\left(\frac{x}{2}\right)^{p+2m}}{\Gamma(p+m+1)} = G$, then

$$\ln G = (p+2m)\ln\left(\frac{x}{2}\right) - \ln\Gamma(p+m+1)$$

so that differentiation w.r.t. p gives

$$\frac{1}{G}\frac{\partial G}{\partial p} = \ln\left(\frac{x}{2}\right) - \frac{\Gamma'(p+m+1)}{\Gamma(p+m+1)}$$

or

$$\frac{\partial G}{\partial p} = G\left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(p+m+1)}{\Gamma(p+m+1)}\right] \tag{9}$$

From equations (8) and (9)

$$\frac{\partial I_p}{\partial p} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\left(\frac{x}{2}\right)^{p+2m}}{\Gamma(p+m+1)}\left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(p+m+1)}{\Gamma(p+m+1)}\right]$$

Then for $p = 0$, we have

$$\left[\frac{\partial I_p}{\partial p}\right]_{p=0} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\left(\frac{x}{2}\right)^{2m}}{\Gamma(m+1)}\left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(m+1)}{\Gamma(m+1)}\right] \tag{10}$$

From equations (7) and (10)

$$K_0(x) = -\sum_{m=0}^{\infty} \frac{1}{m!} \frac{\left(\frac{x}{2}\right)^{2m}}{\Gamma(m+1)}\left[\ln\left(\frac{x}{2}\right) - \frac{\Gamma'(m+1)}{\Gamma(m+1)}\right]$$

and since $\dfrac{\Gamma'(m+1)}{\Gamma(m+1)} = -\gamma + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots\ldots + \dfrac{1}{m}$, and $\dfrac{\Gamma'(1)}{\Gamma(1)} = -\gamma$

where $\gamma = 0.5772\ldots\ldots$ is Euler's Constant. This can be written as

$$K_0(x) = -\left\{\ln\left(\frac{x}{2}\right) + \gamma\right\} I_0(x) + \sum_{m=1}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{m!\,m!}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m}\right)$$

$$= -\left[\ln\left(\frac{x}{2}\right) + \gamma\right] I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \ldots\ldots \tag{11}$$

Similarly, when $n = 1, 2, 3, \ldots\ldots$, we can obtain the series expansion for $K_n(x)$ as

$$K_n(x) = (-1)^{n+1}\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}I_n(x)+\frac{1}{2}\sum_{m=0}^{n-1}\frac{(-1)^m(n-m-1)!\left(\frac{x}{2}\right)^{2m-n}}{m!}$$

$$+\frac{(-1)^n}{2}\sum_{m=0}^{\infty}\left[\phi(m)+\phi(n+m)\right]\frac{\left(\frac{x}{2}\right)^{2m+n}}{m!(n+m)!} \qquad (12)$$

where γ is the Euler's constant and

$$\phi(p) = 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{p}, \quad \phi(0) = 0$$

In particular, if $n = 1$, we get from equation (12)

$$K_1(x) = \left[\ln\left(\frac{x}{2}\right)+\gamma\right]I_1(x)+\frac{1}{x}-\frac{1}{4}x-\frac{5x^3}{2^6}-\frac{5x^5}{2^7.3^2}-\ldots\ldots\ldots$$

11.28 MODIFIED BESSEL FUNCTIONS OF FIRST KIND OF ZEROTH AND FIRST ORDER

We know that the modified Bessel function of first kind of order n are given by

$$I_n(x) = \sum_{m=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \qquad (1)$$

In the special case when n is a positive integer or zero, then

$$\Gamma(n+m+1) = (n+m)! \text{ and } \Gamma(1) = 1$$

Therefore equation (1) becomes

$$I_n(x) = \sum_{m=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{n+2m}}{m!(n+m)!} \qquad (n = 0, 1, 2, \ldots\ldots\ldots) \qquad (2)$$

Put $n = 0$ in equation (2), we get

$$I_0(x) = \sum_{m=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2}$$

$$I_0(x) = 1+\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}+\frac{x^6}{2^6(3!)^2}+\ldots\ldots+\frac{x^{2m}}{2^{2m}(m!)^2}+\ldots\ldots$$

$$= 1+\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}+\ldots\ldots$$

(ii) Next put $n = 1$ in equation (2)

$$I_1(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{1+2m}}{m!(m+1)!}$$

$$= \frac{x}{2} + \frac{x^3}{2^3\,1!\,2!} + \frac{x^5}{2^5\,2!\,3!} + \cdots\cdots + \frac{x^{1+2m}}{2^{1+2m}\,m!(m+1)!} + \cdots\cdots$$

$$= \frac{x}{2} + \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \frac{x^7}{2^2.4^2.6^2.8} + \cdots\cdots$$

which are called modified Bessel functions of zeroth and first order and are shown graphically in figure (11.3).

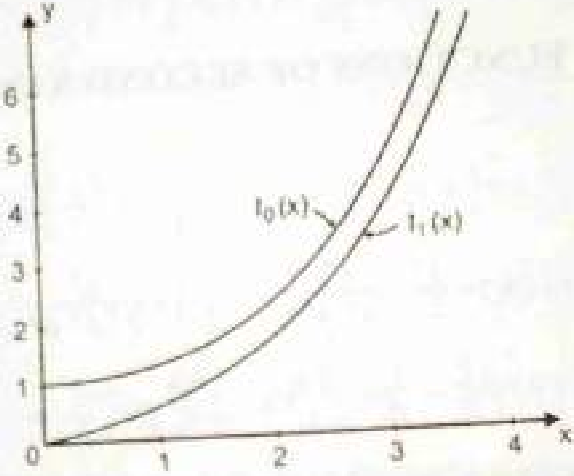


Figure (11.3)

THEOREM (11.26): Prove that $\frac{d}{dx} I_0(x) = I_0'(x) = I_1(x)$

PROOF: We know that

$$I_n(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!(n+m)!} \quad (n = 0, 1, 2, \ldots\ldots) \tag{1}$$

Let $n = 0$ in equation (1), we get

$$I_0(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2}$$

Then $$\frac{d}{dx} I_0(x) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2}$$

$$= \frac{d}{dx}\left[1 + \sum_{m=1}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2}\right]$$

Replacing the summation index m by m + 1, we get

$$\frac{d}{dx} I_0(x) = \frac{d}{dx}\left[1 + \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+2}}{[(m+1)!]^2}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(2m+2)\left(\frac{x}{2}\right)^{2m+1}\left(\frac{1}{2}\right)}{(m+1)!(m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+1}}{m!(m+1)!} = I_1(x)$$

11.29 MODIFIED BESSEL FUNCTIONS OF SECOND KIND OF ZEROTH AND FIRST ORDER

We know that

$$K_0(x) = -\left[\ln\left(\frac{x}{2}\right) + \gamma\right] I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \ldots\ldots \quad (1)$$

and $$K_1(x) = \left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_1(x) + \frac{1}{x} - \frac{1}{4}x - \frac{5x^5}{2^6} - \frac{5x^7}{2^7 . 3^2} - \ldots\ldots\ldots \quad (2)$$

which are called modified Bessel functions of second kind of zeroth and first order respectively and are shown graphically in figure (11.4).

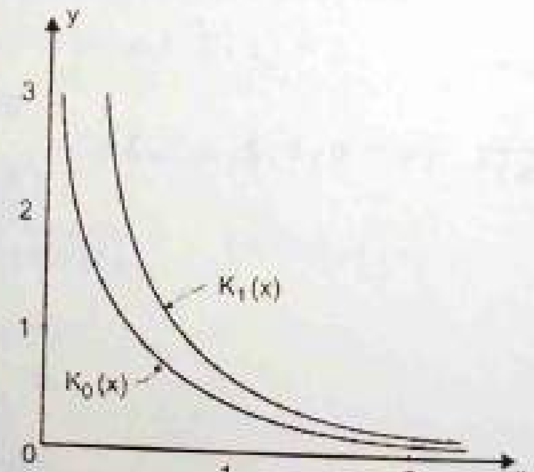


Figure (11.4)

THEOREM (11.27): Prove that $\frac{d}{dx} K_0(x) = K_0'(x) = -K_1(x)$

PROOF: We know that

$$K_0(x) = -\left[\ln\left(\frac{x}{2}\right) + \gamma\right] I_0(x) + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \ldots\ldots \quad (1)$$

and $$K_1(x) = \left[\ln\left(\frac{x}{2}\right)+\gamma\right]J_1(x)+\frac{1}{x}-\frac{1}{4}x-\frac{5x^3}{2^6}-\frac{5x^7}{2^7.3^2}-\ldots\ldots \quad (2)$$

Differentiating equation (1) w.r.t. x , we get

$$K_0'(x) = -\left[\ln\left(\frac{x}{2}\right)+\gamma\right]I_0'(x)-\frac{1}{x}I_0(x)+\frac{x}{2}+\frac{3x^3}{2^5}+\frac{11x^5}{2^8.3^2}+\ldots\ldots$$

Since $I_0'(x) = I_1(x)$ and

$$I_0(x) = 1+\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}+\ldots\ldots$$

$$= 1+\frac{x^2}{2^2}+\frac{x^4}{2^6}+\frac{x^6}{2^8.3^2}+\ldots\ldots$$

Therefore $$K_0'(x) = -\left[\ln\left(\frac{x}{2}\right)+\gamma\right]I_1(x)-\frac{1}{x}\left[1+\frac{x^2}{2^2}+\frac{x^4}{2^6}+\frac{x^6}{2^8.3^2}+\ldots\ldots\right]$$
$$+\left[\frac{x}{2}+\frac{3x^3}{2^5}+\frac{11x^5}{2^8.3^2}+\ldots\ldots\right]$$

$$= -\left[\ln\left(\frac{x}{2}\right)+\gamma\right]I_1(x)-\left[\frac{1}{x}+\frac{x}{2^2}+\frac{x^3}{2^6}+\frac{x^5}{2^8.3^2}+\ldots\ldots\right]$$
$$+\left[\frac{x}{2}+\frac{3x^3}{2^5}+\frac{11x^5}{2^8.3^2}+\ldots\ldots\right]$$

$$= -\left[\ln\left(\frac{x}{2}\right)+\gamma\right]I_1(x)-\frac{1}{x}+\frac{1}{4}x+\frac{5x^3}{2^6}+\frac{5x^5}{2^7.3^2}+\ldots\ldots$$

$$= -\left[\left\{\ln\left(\frac{x}{2}\right)+\gamma\right\}I_1(x)+\frac{1}{x}-\frac{1}{4}x-\frac{5x^3}{2^6}-\frac{5x^5}{2^7.3^2}-\ldots\ldots\right]$$

$$= -K_1(x) \qquad [\text{using equation (2)}]$$

11.30 ALTERNATIVE FORM OF THE GENERAL SOLUTION OF MODIFIED BESSEL'S EQUATION

THEOREM (11.28): Show that general solution of modified Bessel's differential equation for all values of n can be written as

$$y(x) = A\,I_n(x)+B\,I_n(x)\int\frac{dx}{x\,I_n^2(x)}$$

where A and B are arbitrary constants.

PROOF: We know that modified Bessel's differential equation is

$$x^2y''+xy'-(x^2+n^2)y = 0 \quad (1)$$

Let $$y(x) = u(x)I_n(x) = u\,I_n \quad (2)$$

be a solution of equation (1) when n is an integer.

Then $y'(x) = u I_n' + u' I_n$

$$y''(x) = u I_n'' + 2u' I_n' + u'' I_n$$

Substituting the values of y, y', and y'' in equation (1), we get

$$x^2(u I_n'' + 2u' I_n' + u'' I_n) + x(u I_n' + u' I_n) - (x^2 + n^2) u I_n = 0$$

or $$u[x^2 I_n'' + x I_n' + (x^2 - n^2) I_n] + x^2 u'' I_n + 2x^2 u' I_n' + x u' I_n = 0$$

The expression with square brackets is zero, since $I_n(x)$ is a solution of Bessel's equation (1).

Therefore $x^2 u'' I_n + 2x^2 u' I_n' + x u' I_n = 0$

Dividing throughout by $x^2 u' I_n$, we get

$$\frac{u''}{u'} + 2\frac{I_n'}{I_n} + \frac{1}{x} = 0$$

Integrating w.r.t. x, we get

$$\ln u' + 2 \ln I_n + \ln x = \ln B$$

or $$\ln(u' I_n^2 x) = \ln B$$

or $$u' I_n^2 x = B \quad \text{or} \quad u' = \frac{B}{x I_n^2}$$

or $$u(x) = B \int \frac{dx}{x I_n^2(x)} + A$$

Substituting the values of u in equation (2), we get

$$y(x) = \left[B \int \frac{dx}{x I_n^2(x)} + A\right] I_n(x)$$

$$= A I_n(x) + B I_n(x) \int \frac{dx}{x I_n^2(x)}$$

$$= A I_n(x) + B K_n(x)$$

where $K_n(x) = I_n(x) \int \frac{dx}{x I_n^2(x)}$ is another form of modified Bessel functions of the second kind.

11.31 GENERATING FUNCTION FOR $I_n(x)$

The function $e^{\frac{x}{2}\left(t + \frac{1}{t}\right)}$ is called the generating function for the modified Bessel functions of the first kind for integer values of n. It is very useful in obtaining properties of these functions for integer values of n.

THEOREM (11.29): Prove that $e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x)\, t^n$

PROOF: We have

$$e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} = e^{\frac{x t}{2}} \cdot e^{\frac{x}{2t}} \qquad (1)$$

Using the relation $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, equation (1) becomes

$$e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} = \left[\sum_{r=0}^{\infty} \frac{\left(\frac{x t}{2}\right)^r}{r!}\right]\left[\sum_{k=0}^{\infty} \frac{\left(\frac{x}{2t}\right)^k}{k!}\right]$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{r+k} t^{r-k}}{r!\,k!}$$

Let $r-k = n$, so that n varies from $-\infty$ to ∞, then the sum becomes

$$e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k} t^n}{(n+k)!\,k!}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,k!}\right] t^n$$

$$= \sum_{n=-\infty}^{\infty} I_n(x)\, t^n$$

11.32 RECURRENCE FORMULAS INVOLVING MODIFIED BESSEL FUNCTIONS OF FIRST KIND

THEOREM (11.30): Prove that

(i) $\frac{d}{dx}\left[x^n I_n(x)\right] = x^n I_{n-1}(x)$

(ii) $\frac{d}{dx}\left[x^{-n} I_n(x)\right] = x^{-n} I_{n+1}(x)$

PROOF: We have

$$I_n(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)}$$

(i) $$\frac{d}{dx}\left[x^n I_n(x)\right] = \frac{d}{dx}\sum_{m=0}^{\infty}\frac{x^{2n+2m}}{2^{n+2m}\,m!\,\Gamma(m+n+1)}$$

$$= \sum_{m=0}^{\infty}\frac{2(m+n)x^{2n+2m-1}}{2^{n+2m}\,m!\,\Gamma(m+n+1)}$$

$$= \sum_{m=0}^{\infty}\frac{x^{2n+2m-1}}{2^{n+2m-1}\,m!\,\Gamma(m+n)}$$

$$= x^n\sum_{m=0}^{\infty}\frac{x^{(n-1)+2m}}{2^{(n-1)+2m}\,m!\,\Gamma\left((n-1)+m+1\right)}$$

$$\frac{d}{dx}\left[x^n I_n(x)\right] = x^n I_{n-1}(x)$$

(ii) $$\frac{d}{dx}\left[x^{-n} I_n(x)\right] = \frac{d}{dx}\sum_{m=0}^{\infty}\frac{x^{2m}}{2^{n+2m}\,m!\,\Gamma(m+n+1)}$$

$$= \sum_{m=0}^{\infty}\frac{2m\,x^{2m-1}}{2^{n+2m}\,m!\,\Gamma(m+n+1)}$$

$$= \sum_{m=0}^{\infty}\frac{x^{2m-1}}{2^{n+2m-1}(m-1)!\,\Gamma(m+n+1)}$$

$$= x^{-n}\sum_{m=1}^{\infty}\frac{x^{n+2m-1}}{2^{n+2m-1}(m-1)!\,\Gamma(m+n+1)}$$

(since for $m = 0$, $(-1)! = \infty$)

Letting $m = k+1$, we get

$$= x^{-n}\sum_{k=0}^{\infty}\frac{x^{n+2k+1}}{2^{n+2k+1}\,k!\,\Gamma(n+k+2)}$$

$$= x^{-n}\sum_{k=0}^{\infty}\frac{x^{(n+1)+2k}}{2^{(n+1)+2k}\,k!\,\Gamma\left((n+1)+k+1\right)}$$

$$\frac{d}{dx}\left[x^{-n} I_n(x)\right] = x^{-n} I_{n+1}(x)$$

THEOREM (11.31): Prove that

(i) $\quad x I_n'(x) + n I_n(x) = x I_{n-1}(x)$

(ii) $\quad x I_n'(x) - n I_n(x) = x I_{n+1}(x)$

(iii) $\quad I_{n-1}(x) + I_{n+1}(x) = 2 I_n'(x)$

(iv) $\quad I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x) \quad$ for all n.

PROOF: (i) We know that

$$\frac{d}{dx}\left[x^n I_n(x)\right] = n I_{n-1}(x)$$

$$x^n I_n'(x) + n x^{n-1} I_n(x) = x^n I_{n-1}(x)$$

or $$x I_n'(x) + n I_n(x) = x I_{n-1}(x) \qquad (1)$$

i.e.

(ii) Also, we have

$$\frac{d}{dx}\left[x^{-n} I_n(x)\right] = x^{-n} I_{n+1}(x)$$

$$x^{-n} I_n'(x) - n x^{-n-1} I_n(x) = x^{-n} I_{n+1}(x)$$

or $$x I_n'(x) - n I_n(x) = x I_{n+1}(x) \qquad (2)$$

i.e.

(iii) Adding equations (1) and (2), then dividing by $2x$, we get

$$2x I_n'(x) = x\left[I_{n-1}(x) + I_{n+1}(x)\right]$$

or $$I_{n-1}(x) + I_{n+1}(x) = 2 I_n'(x)$$

(iv) Subtracting equation (2) from equation (1), we get

$$2n I_n(x) = x I_{n-1}(x) - x I_{n+1}(x)$$

or $$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x)$$

11.33 WRONSKIAN FORMULAS INVOLVING MODIFIED BESSEL FUNCTIONS OF FIRST KIND

THEOREM (11.32): Prove that

(i) $$I_n(x) I_{-n}'(x) - I_n'(x) I_{-n}(x) = -\frac{2 \sin n\pi}{\pi x}$$

(ii) $$I_{-n}(x) I_{n-1}(x) - I_n(x) I_{-n+1}(x) = \frac{2 \sin n\pi}{\pi x}$$

PROOF: (i) Since $I_n(x)$, and $I_{-n}(x)$ abbreviated I_n, I_{-n} respectively, satisfy modified Bessel's equation, we have

$$x^2 I_n'' + x I_n' - (x^2 + n^2) I_n = 0 \qquad (1)$$

and $$x^2 I_{-n}'' + x I_{-n}' - (x^2 + n^2) I_{-n} = 0 \qquad (2)$$

Multiplying equation (1) by I_{-n}, equation (2) by I_n and then subtracting, we get

$$x^2\left[I_n'' I_{-n} - I_{-n}'' I_n\right] + x\left[I_n' I_{-n} - I_{-n}' I_n\right] = 0$$

or $$x\left[I_n'' I_{-n} - I_{-n}'' I_n\right] + \left[I_n' I_{-n} - I_{-n}' I_n\right] = 0$$

or $$\frac{d}{dx}\left[x\left(I_n' I_{-n} - I_{-n}' I_n\right)\right] = 0$$

Integrating, we find

$$x\left(I_n' I_{-n} - I_{-n}' I_n\right) = C$$

or $$I_n' I_{-n} - I_{-n}' I_n = \frac{C}{x} \tag{3}$$

We know that

$$I_n(x) = \frac{1}{2^n \Gamma(n+1)}\left(x^n + \frac{x^{n+2}}{2^2(n+1)} + \ldots\ldots\right)$$

$$I_n'(x) = \frac{1}{2^n \Gamma(n+1)}\left(n x^{n-1} + \frac{(n+2)x^{n-1}}{4(n+1)} + \ldots\ldots\right)$$

$$I_{-n}(x) = \frac{1}{2^{-n}\Gamma(-n+1)}\left(x^{-n} + \frac{x^{2-n}}{2^2(1-n)} + \ldots\ldots\right)$$

$$I_{-n}'(x) = \frac{1}{2^{-n}\Gamma(-n+1)}\left(-n x^{-n-1} + \frac{(2-n)x^{1-n}}{4(1-n)} + \ldots\ldots\right)$$

Substituting in equation (3), we get

$$\frac{C}{x} = \frac{1}{2^n\Gamma(n+1)}\left[n x^{n-1} + \frac{(n+2)x^{n+1}}{4(n+1)} + \ldots\right] \cdot \frac{1}{2^{-n}\Gamma(-n+1)}\left[x^{-n} + \frac{x^{2-n}}{4(1-n)} + \ldots\right]$$
$$- \frac{1}{2^{-n}\Gamma(-n+1)}\left[-n x^{-n-1} + \frac{(2-n)x^{1-n}}{4(1-n)} + \ldots\right] \cdot \frac{1}{2^n\Gamma(n+1)}\left[x^n + \frac{x^{n+2}}{4(n+1)} + \ldots\right]$$

Equating the coefficients of $\frac{1}{x}$ on both sides, we get

$$C = \frac{n}{2^n\Gamma(n+1) . 2^{-n}\Gamma(-n+1)} + \frac{n}{2^n\Gamma(n+1)2^{-n}\Gamma(-n+1)}$$

$$= \frac{2n}{\Gamma(n+1)\Gamma(-n+1)} = \frac{2n}{n\Gamma(n)\Gamma(1-n)}$$

$$= \frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2\sin n\pi}{\pi} \qquad \left[\text{since } \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

or $$C = \frac{2\sin n\pi}{\pi}$$

Substituting the value of C in equation (3), we get

$$I_n' I_{-n} - I_{-n}' I_n = \frac{2\sin n\pi}{\pi x}$$

or $$I_n(x) I_{-n}'(x) - I_n'(x) I_{-n}(x) = -\frac{2\sin n\pi}{\pi x} \tag{4}$$

(ii) From theorem (11.31), we have the recurrence relation

$$x I_n'(x) + 2n I_n(x) = x I_{n-1}(x)$$

or $$I_n'(x) = -\frac{n}{x} I_n(x) + I_{n-1}(x) \tag{5}$$

Also, from the recurrence relation

$$x I_n'(x) - n I_n(x) = x I_{n+1}(x)$$

or

$$I_n'(x) = \frac{n}{x} I_n(x) + I_{n+1}(x)$$

Replacing n by − n, we get

$$I_{-n}'(x) = -\frac{n}{x} I_{-n}(x) + I_{-n+1}(x) \qquad (6)$$

Substituting equations (5) and (6) in equation (4), we get

$$\left[-\frac{n}{x} I_{-n}(x) + I_{-n+1}(x)\right] I_n(x) - \left[-\frac{n}{x} I_n(x) + I_{n-1}(x)\right] I_{-n}(x) = \frac{2 \sin n\pi}{\pi x}$$

or

$$I_{-n}(x) I_{n-1}(x) - I_n(x) I_{-n+1}(x) = \frac{2 \sin n\pi}{\pi x}$$

11.34 MODIFIED BESSEL FUNCTIONS OF FIRST KIND WHEN n IS HALF AN ODD INTEGER

THEOREM (11.33): Prove that

(i) $$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

(ii) $$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$$

(iii) $$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\cosh x - \frac{\sinh x}{x}\right)$$

(iv) $$I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\sinh x - \frac{\cosh x}{x}\right)$$

PROOF: We know that

$$I_n(x) = i^{-n} J_n(ix) \qquad (1)$$

(i) Let $n = \frac{1}{2}$ in equation (1), we get

$$I_{1/2}(x) = i^{-1/2} J_{1/2}(ix)$$

$$= i^{-1/2} \sqrt{\frac{2}{i\pi x}} \sin ix$$

$$= i^{-1/2} \sqrt{\frac{2}{i\pi x}}\, i \sinh x \qquad (\text{since } \sin ix = i \sinh x)$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

(ii) Let $n = -\frac{1}{2}$ in equation (1), we have

$$I_{-1/2}(x) = i^{1/2} J_{-1/2}(ix)$$

$$= i^{1/2}\sqrt{\frac{2}{i\pi x}}\cos i x \qquad (\text{since } \cos i x = \cosh x)$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

(iii) Let $n = \frac{3}{2}$ in equation (1), we get

$$I_{3/2}(x) = i^{-3/2} J_{3/2}(i x)$$

$$= i^{-3/2}\sqrt{\frac{2}{\pi i x}}\left(\frac{\sin i x}{i x} - \cos i x\right)$$

$$= \frac{1}{i^2}\sqrt{\frac{2}{\pi x}}\left(\frac{i \sinh x}{i x} - \cosh x\right)$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\sinh x}{x} - \cosh x\right) = \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right)$$

(iv) Let $n = -\frac{3}{2}$ in equation (1), we get

$$I_{-3/2}(x) = i^{3/2} J_{-3/2}(i x)$$

$$= i^{3/2}\left(-\sqrt{\frac{2}{\pi i x}}\right)\left(\frac{\cos i x}{i x} + \sin i x\right)$$

$$= -i\sqrt{\frac{2}{\pi x}}\left(\frac{\cosh x}{i x} + i \sinh x\right)$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cosh x}{x} - \sinh x\right) = \sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right)$$

11.35 RECURRENCE FORMULAS FOR MODIFIED BESSEL FUNCTIONS OF SECOND KIND

THEOREM (11.34): Prove that

(i) $x K_n'(x) + n K_n(x) = -x K_{n-1}(x)$

(ii) $x K_n'(x) - n K_n(x) = -x K_{n+1}(x)$

(iii) $K_{n-1}(x) + K_{n+1}(x) = -2 K_n'(x)$

(iv) $K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x)$

(v) $\frac{d}{dx}\{x^n K_n(x)\} = -x^n K_{n-1}(x)$

(vi) $\frac{d}{dx}[x^{-n} K_n(x)] = -x^{-n} K_{n+1}(x)$

PROOF: We know that from theorem (11.31) that

$$x I_n'(x) + n I_n(x) = x I_{n-1}(x) \quad (1)$$

and $$x I_n'(x) - n I_n(x) = x I_{n+1}(x) \quad (2)$$

Replacing n by $-n$ in the equation (2), we have

$$x I_{-n}'(x) + n I_{-n}(x) = x I_{-n+1}(x) \quad (3)$$

Subtracting equation (2) from equation (3), we have

$$x\left[I_{-n}'(x) - I_n'(x)\right] + n\left[I_{-n}(x) - I_n(x)\right] = x\left[I_{-n+1}(x) - I_{n-1}(x)\right]$$

Multiplying this relation by $\frac{\pi}{2\sin n\pi}$, we get

$$\frac{\pi x}{2\sin n\pi}\left[I_{-n}'(x) - I_n'(x)\right] + \frac{n\pi}{2\sin n\pi}\left[I_{-n}(x) - I_n(x)\right]$$

$$= \frac{\pi x}{2\sin n\pi}\left[I_{-(n-1)}(x) - I_{(n-1)}(x)\right]$$

Since $\sin(n-1)\pi = -\sin n\pi$ and using the definition for $k_n(x)$, we get

(i) $$x K_n'(x) + n K_n(x) = -x K_{n-1}(x) \quad (4)$$

Now replacing n by $-n$ in equation (4) using the fact that $K_{-n}(x) = K_n(x)$, we find

(ii) $$x K_n'(x) - n K_n(x) = -x K_{n+1}(x) \quad (5)$$

(iii) Adding equations (4) and (5), we find

$$2x K_n'(x) = -x\left[K_{n-1}(x) + K_{n+1}(x)\right]$$

or $$K_{n-1}(x) + K_{n+1}(x) = -2K_n'(x)$$

(iv) Subtracting equation (5) from equation (4), we have

$$2n K_n(x) = -x K_{n-1}(x) + x K_{n+1}(x)$$

or $$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x)$$

(v) Multiplying equation (4) by x^{n-1}, we get

$$x^n K_n'(x) + n x^{n-1} K_n(x) = -x^n K_{n-1}(x)$$

or $$\frac{d}{dx}\left[x^n K_n(x)\right] = -x^n K_{n-1}(x)$$

(vi) Multiplying equation (5) by x^{-n-1}, we get

$$x^{-n} K_n'(x) - n x^{-n-1} K_n(x) = -x^{-n} K_{n+1}(x)$$

or $$\frac{d}{dx}\left[x^{-n} K_n(x)\right] = -x^{-n} K_{n+1}(x)$$

DEDUCTION

If we substitute $n = 0$ in equation (6), we find that

$$K_0'(x) = -K_1(x)$$

11.36 WRONSKIAN FORMULAS INVOLVING MODIFIED BESSEL FUNCTIONS

THEOREM (11.35): Prove that

(i) $$I_n(x)K_n'(x) - I_n'(x)K_n(x) = -\frac{1}{x}, \text{ for all } x.$$

(ii) $$I_n(x)K_{n+1}(x) + I_{n+1}(x)K_n(x) = \frac{1}{x}$$

PROOF: (i) We know that

$$K_n(x) = \frac{\pi}{2\sin n\pi}\left[I_{-n}(x) - I_n(x)\right]$$

so that $$K_n'(x) = \frac{\pi}{2\sin n\pi}\left[I_{-n}'(x) - I_n'(x)\right]$$

Now $$I_n(x)K_n'(x) - K_n(x)I_n'(x) = \frac{\pi}{2\sin n\pi}\left[I_n(x)I_{-n}'(x) - I_n(x)I_n'(x)\right]$$
$$-\frac{\pi}{2\sin n\pi}\left[I_{-n}(x)I_n'(x) - I_n(x)I_n'(x)\right]$$
$$= \frac{\pi}{2\sin n\pi}\left[I_n(x)I_n'(x) - I_{-n}(x)I_n'(x)\right]$$
$$= \frac{\pi}{2\sin n\pi}\left(-\frac{2\sin n\pi}{\pi x}\right) = -\frac{1}{x} \quad \left[\text{using theorem (11.34)}\right]$$

(ii) We know from the recurrence relation that

$$x I_n'(x) - n I_n(x) = x I_{n+1}(x)$$

or $$I_n'(x) = \frac{n}{x} I_n(x) + I_{n+1}(x)$$

Also, we have

$$x K_n'(x) - n K_n(x) = -x K_{n+1}(x)$$

or $$K_n'(x) = \frac{n}{x} K_n(x) - K_{n+1}(x)$$

Substituting the values of $I_n'(x)$ and $K_n'(x)$ in part (i), we get

$$I_n(x)\left[\frac{n}{x}K_n(x) - K_{n+1}(x)\right] - \left[\frac{n}{x}I_n(x) + I_{n+1}(x)\right]K_n(x) = -\frac{1}{x}$$

or $$-I_n(x)K_{n+1}(x) - I_{n+1}(x)K_n(x) = -\frac{1}{x}$$

or $$I_n(x)K_{n+1}(x) + I_{n+1}(x)K_n(x) = \frac{1}{x}$$

11.37 MODIFIED BESSEL FUNCTIONS OF SECOND KIND WHEN n IS HALF AN ODD INTEGER

THEOREM (11.36): Prove that

(i) $$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}$$

(ii) $$K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x} = K_{1/2}(x)$$

(iii) $$K_{3/2}(x) = \sqrt{\frac{\pi}{2x}}\left(1+\frac{1}{x}\right)e^{-x}$$

(iv) $$K_{-3/2}(x) = \sqrt{\frac{\pi}{2x}}\left(1+\frac{1}{x}\right)e^{-x} = K_{3/2}(x)$$

PROOF: We know that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x \quad \text{and} \quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

Now $$K_n(x) = \frac{\pi}{2\sin n\pi}\left[I_{-n}(x) - I_n(x)\right], \qquad (n \neq 0, 1, 2, \ldots\ldots) \qquad (1)$$

(i) Let $n = \frac{1}{2}$ in equation (1), we get

$$K_{1/2}(x) = \frac{\pi}{2\sin\frac{\pi}{2}}\left[I_{-1/2}(x) - I_{1/2}(x)\right]$$

$$= \frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\cosh x - \sqrt{\frac{2}{\pi x}}\sinh x\right]$$

$$= \sqrt{\frac{\pi}{2x}}(\cosh x - \sinh x)$$

$$= \sqrt{\frac{\pi}{2x}}\left(\frac{e^{x}+e^{-x}}{2} - \frac{e^{x}-e^{-x}}{2}\right) = \sqrt{\frac{\pi}{2x}}\,e^{-x}$$

(ii) Let $n = -\frac{1}{2}$ in equation (1), we get

$$K_{-1/2}(x) = \frac{\pi}{2\sin\left(-\frac{\pi}{2}\right)}\left[I_{1/2}(x) - I_{-1/2}(x)\right]$$

$$= -\frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\sinh x - \sqrt{\frac{2}{\pi x}}\cosh x\right]$$

$$= -\frac{\pi}{2}\left(-\sqrt{\frac{2}{\pi x}}\right)(\cosh x - \sinh x)$$

$$= \sqrt{\frac{\pi}{2x}}(\cosh x - \sinh x) = \sqrt{\frac{\pi}{2x}}\,e^{-x} = K_{1/2}(x)$$

(iii) Let $n = \frac{3}{2}$ in equation (1), we get

$$K_{3/2}(x) = \frac{\pi}{2\sin\left(\frac{3\pi}{2}\right)}\left[I_{-3/2}(x) - I_{3/2}(x)\right]$$

$$= -\frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right) - \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right)\right]$$

$$= -\frac{\pi}{2}\sqrt{\frac{2}{\pi x}}\left[\sinh x - \frac{\cosh x}{x} - \cosh x + \frac{\sinh x}{x}\right]$$

$$= -\frac{\pi}{2x}\left[\sinh x - \cosh x + \frac{1}{x}(\sinh x - \cosh x)\right]$$

$$= -\sqrt{\frac{\pi}{2x}}\left[-e^{-x} + \frac{1}{x}(-e^{-x})\right] = \sqrt{\frac{\pi}{2x}}\left(1 + \frac{1}{x}\right)e^{-x}$$

(iv) Let $n = -\frac{3}{2}$ in equation (1), we get

$$K_{-3/2}(x) = \frac{\pi}{2\sin\left(-\frac{3\pi}{2}\right)}\left[I_{3/2}(x) - I_{-3/2}(x)\right]$$

$$= \frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right) - \sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right)\right]$$

$$= \frac{\pi}{2}\sqrt{\frac{2}{\pi x}}\left[\cosh x - \frac{\sinh x}{x} - \sinh x + \frac{\cosh x}{x}\right]$$

$$= \frac{\pi}{2x}\left[\cosh x - \sinh x + \frac{1}{x}(\cosh x - \sinh x)\right]$$

$$= \sqrt{\frac{\pi}{2x}}\left(e^{-x} + \frac{1}{x}e^{-x}\right) = \sqrt{\frac{\pi}{2x}}\left(1 + \frac{1}{x}\right)e^{-x} = K_{3/2}(x)$$

11.38 EQUATIONS REDUCIBLE TO MODIFIED BESSEL'S EQUATION

EXAMPLE (5): **Transform the differential equation**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (\lambda^2x^2 + n^2)y = 0$$

to modified Bessel's equation and solve.

SOLUTION: Let $t = \lambda x$, then $\frac{dt}{dx} = \lambda$ and $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \lambda\frac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\lambda\frac{dy}{dt}\right)\lambda = \lambda^2\frac{d^2y}{dt^2}$$

Substituting the values of these derivatives in the given differential equation, we get

$$\frac{t^2}{\lambda^2}\left(\lambda^2\frac{d^2y}{dt^2}\right)+\frac{t}{\lambda}\left(\lambda\frac{dy}{dt}\right)-(t^2+n^2)y=0$$

or

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-(t^2+n^2)y=0$$

which is modified Bessel's equation whose solution is given by

$$y=A\,I_n(t)+B\,K_n(t)$$

where A and B are arbitrary constants.

The general solution of the given differential equation is therefore

$$y(x)=A\,I_n(\lambda x)+B\,K_n(\lambda x)$$

11.39 MISCELLANEOUS RESULTS

THEOREM (11.37): Prove that

(i) $\cos(x\cosh\theta)=J_0(x)-2J_2(x)\cosh 2\theta+2J_4(x)\cosh 4\theta-\ldots\ldots$

(ii) $\sin(x\cosh\theta)=2J_1(x)\cosh\theta-2J_3(x)\cosh 3\theta+2J_5(x)\cosh 5\theta-\ldots\ldots$

(iii) $\cosh(x\cosh\theta)=I_0(x)+2I_2(x)\cosh 2\theta+2I_4(x)\cosh 4\theta+\ldots\ldots$

(iv) $\sinh(x\cosh\theta)=2I_1(x)\cosh\theta+2I_3(x)\cosh 3\theta+2I_5(x)\cosh 5\theta+\ldots$

PROOF: We know that

$$\cos(x\cos\theta)=J_0(x)-2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta-\ldots\ldots \quad (1)$$

$$\sin(x\cos\theta)=2J_1(x)\cos\theta-2J_3(x)\cos 3\theta+2J_5(x)\cos 5\theta-\ldots\ldots \quad (2)$$

(i) Replacing θ by $i\theta$ in equation (1), we get

$$\cos(x\cos i\theta)=J_0(x)-2J_2(x)\cos i(2\theta)+2J_4(x)\cos i(4\theta)-\ldots\ldots$$

or $$\cos(x\cosh\theta)=J_0(x)-2J_2(x)\cosh(2\theta)+2J_4(x)\cosh(4\theta)-\ldots\ldots \quad (3)$$

(ii) Similarly, replacing θ by $i\theta$ in equation (2), we get

$$\sin(x\cosh\theta)=2J_1(x)\cosh\theta-2J_3(x)\cosh 3\theta+2J_5(x)\cosh 5\theta-\ldots\ldots \quad (4)$$

(iii) Replacing x by ix in equation (3), we get

$$\cos(ix\cosh\theta)=J_0(ix)-2J_2(ix)\cosh 2\theta+2J_4(ix)\cosh 4\theta-\ldots\ldots$$

Since $J_n(ix)=i^n I_n(x)$, therefore

$$\cosh(x\cosh\theta)=I_0(x)+2I_2(x)\cosh 2\theta+2I_4(x)\cosh 4\theta+\ldots\ldots \quad (5)$$

(iv) Replacing x by ix in equation (4), we get

$$\sin(ix\cosh\theta)=2J_1(ix)\cosh\theta-2J_3(ix)\cosh 3\theta+2J_5(ix)\cosh 5\theta-\ldots\ldots$$

or $$i\sinh(x\cosh\theta)=2iI_1(x)\cosh\theta-2i^3I_3(x)\cosh 3\theta+2i^5J_5(x)\cosh 5\theta-\ldots\ldots$$

or $$\sinh(x\cosh\theta)=2I_1(x)\cosh\theta+2I_3(x)\cosh 3\theta+2I_5(x)\cosh 5\theta+\ldots\ldots \quad (6)$$

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DEDUCTIONS

Let $\theta = 0$ in equations (5) and (6), we get

$$\cosh x = I_0(x) + 2 I_2(x) + 2 I_4(x) + \ldots\ldots\ldots$$

and $$\sinh x = 2 I_1(x) + 2 I_3(x) + 2 I_5(x) + \ldots\ldots\ldots$$

THEOREM (11.38): Prove that $I_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cosh(x\cos\theta)\, d\theta = \dfrac{1}{\pi}\int_0^{\pi} \cosh(x\sin\theta)\, d\theta$

PROOF: We know that

$$\cos(x\cos\theta) = J_0(x) - 2 J_2(x)\cos 2\theta + 2 J_4(x)\cos 4\theta - \ldots\ldots\ldots \quad (1)$$

Replacing x by ix, we get

$$\cos(ix\cos\theta) = J_0(ix) - 2 J_2(ix)\cos 2\theta + 2 J_4(ix)\cos 4\theta - \ldots\ldots\ldots$$

Since $J_n(ix) = i^n I_n(x)$, therefore

$$\cosh(x\cos\theta) = I_0(x) + 2 I_2(x)\cos 2\theta + 2 I_4(x)\cos 4\theta - \ldots\ldots\ldots$$

Integrating both sides w.r.t. θ from $\theta = 0$ to π, we get

$$\int_0^{\pi} \cosh(x\cos\theta)\, d\theta = \int_0^{\pi} I_0(x)\, d\theta + 0 + 0 + \ldots\ldots\ldots$$

$$= I_0(x)\,|\,\theta\,|_0^{\pi} = \pi I_0(x)$$

or $$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\cos\theta)\, d\theta$$

Also, we know that

$$\cos(x\sin\theta) = J_0(x) + 2 J_2(x)\cos 2\theta + 2 J_4(x)\cos 4\theta + \ldots\ldots\ldots$$

Replacing x by ix, we get

$$\cos(ix\sin\theta) = J_0(ix) + 2 J_2(ix)\cos 2\theta + 2 J_4(ix)\cos 4\theta + \ldots\ldots\ldots$$

or $$\cosh(x\sin\theta) = I_0(x) - 2 I_2(x)\cos 2\theta + 2 I_4(x)\cos 4\theta + \ldots\ldots\ldots$$

Integrating both sides w.r.t. θ from $\theta = 0$ to π, we get

$$\int_0^{\pi} \cosh(x\sin\theta)\, d\theta = \int_0^{\pi} I_0(x)\, d\theta + 0 + 0 + \ldots\ldots\ldots$$

$$= I_0(x)\,|\,\theta\,|_0^{\pi} = \pi I_0(x)$$

or $$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\sin\theta)\, d\theta$$

11.40 SPECIAL INTEGRALS INVOLVING MODIFIED BESSEL FUNCTIONS

THEOREM (11.39): Prove that if λ and μ are constants, then

(i) $$\int x I_n(\lambda x) I_n(\mu x)\, dx = \frac{x\left[\lambda I_n(\mu x) I_n'(\lambda x) - \mu I_n(\lambda x) I_n'(\mu x)\right]}{\lambda^2 - \mu^2}$$ if $\lambda \neq \mu$

(ii) $$\int x I_n^2(\lambda x)\, dx = -\frac{1}{2}x^2 I_n'^2(\lambda x) + \frac{1}{2}x^2\left(1 + \frac{n^2}{\lambda^2 x^2}\right) I_n^2(\lambda x)$$ if $\lambda = \mu$

PROOF: (i) We know that $y_1 = I_n(\lambda x)$ and $y_2 = I_n(\mu x)$ are the solutions of modified Bessel's differential equation, therefore

$$x^2 y_1'' + x y_1' - (\lambda^2 x^2 + n^2) y_1 = 0 \qquad (1)$$

and $$x^2 y_2'' + x y_2' - (\mu^2 x^2 + n^2) y_2 = 0 \qquad (2)$$

Multiplying equation (1) by y_2, equation (2) by y_1 and subtracting, we find

$$x^2\left[y_2 y_1'' - y_1 y_2''\right] + x\left[y_2 y_1' - y_1 y_2'\right] = (\lambda^2 - \mu^2) x^2 y_1 y_2$$

which on division by x can be written as

$$x\frac{d}{dx}\left[y_2 y_1' - y_1 y_2'\right] + \left[y_2 y_1' - y_1 y_2'\right] = (\lambda^2 - \mu^2) x y_1 y_2$$

or $$\frac{d}{dx}\left[x\left(y_2 y_1' - y_1 y_2'\right)\right] = (\lambda^2 - \mu^2) x y_1 y_2$$

Then by integrating and omitting the constant of integration,

$$(\lambda^2 - \mu^2)\int x y_1 y_2\, dx = x\left(y_2 y_1' - y_1 y_2'\right)$$

Using $y_1 = I_n(\lambda x)$ and $y_2 = I_n(\mu x)$ and dividing by $(\lambda^2 - \mu^2) \neq 0$, we get

$$\int x I_n(\lambda x) I_n(\mu x)\, dx = \frac{x\left[\lambda I_n(\mu x) I_n'(\lambda x) - \mu I_n(\lambda x) I_n'(\mu x)\right]}{\lambda^2 - \mu^2} \qquad (3)$$

(ii) If $\lambda = \mu$, then the R.H.S. of equation (3) becomes indeterminate of the form $\left(\frac{0}{0}\right)$. Thus

$$\int x I_n^2(\lambda x)\, dx = \underset{\mu \to \lambda}{\mathrm{Lt}} \frac{x\left[\lambda I_n(\mu x) I_n'(\lambda x) - \mu I_n(\lambda x) I_n'(\mu x)\right)}{\lambda^2 - \mu^2}$$

Using the L'Hospital rule, we get

$$\int x I_n^2(\lambda x)\, dx = \underset{\mu \to \lambda}{\mathrm{Lt}} \frac{x\left[\lambda x I_n'(\mu x) I_n'(\lambda x) - I_n(\lambda x) I_n'(\mu x) - \mu x I_n(\lambda x) I_n''(\mu x)\right]}{-2\mu}$$

$$= -\frac{1}{2\lambda}\left[\lambda x^2 I_n'^2(\lambda x) - x I_n(\lambda x) I_n'(\lambda x) - \lambda x^2 I_n(\lambda x) I_n''(\lambda x)\right]$$

$$= -\frac{1}{2}\left[x^2 I_n'^2(\lambda x) - \frac{1}{\lambda} x I_n(\lambda x) I_n'(\lambda x) - x^2 I_n(\lambda x) I_n''(\lambda x)\right] \qquad (4)$$

But since $y = I_n(\lambda x)$ is a solution of Bessel's equation

$$x^2 y'' + x y' - (\lambda^2 x^2 + n^2) y = 0$$

therefore $\lambda^2 x^2 I_n''(\lambda x) + \lambda x I_n'(\lambda x) - (\lambda^2 x^2 + n^2) I_n(\lambda x) = 0$

or $$I_n''(\lambda x) = -\frac{1}{\lambda x} I_n'(\lambda x) + \left(1 + \frac{n^2}{\lambda^2 x^2}\right) I_n(\lambda x) \tag{5}$$

From equations (4) and (5), we get

$$\int x I_n^2(\lambda x)\, dx = -\frac{1}{2}\Big[x^2 I_n'^2(\lambda x) - \frac{1}{\lambda} x I_n(\lambda x) I_n'(\lambda x) - x^2 I_n(\lambda x)\left\{-\frac{1}{\lambda x} I_n'(\lambda x) + \left(1 + \frac{n^2}{\lambda^2 x^2}\right) I_n(\lambda x)\right\}\Big]$$

$$= -\frac{1}{2}\Big[x^2 I_n'^2(\lambda x) - \frac{1}{\lambda} x I_n(\lambda x) I_n'(\lambda x) + \frac{1}{\lambda} x I_n(\lambda x) I_n'(\lambda x) - x^2\left(1 + \frac{n^2}{\lambda^2 x^2}\right) I_n^2(\lambda x)\Big]$$

$$= -\frac{1}{2} x^2 I_n'^2(\lambda x) + \frac{1}{2} x^2 \left(1 + \frac{n^2}{\lambda^2 x^2}\right) I_n^2(\lambda x) \tag{6}$$

DEDUCTION

$$\int_0^1 x I_n(\lambda x) I_n(\mu x)\, dx = \frac{\lambda I_n(\mu) I_n'(\lambda) - \mu I_n(\lambda) I_n'(\mu)}{\lambda^2 - \mu^2} \tag{7}$$

and $$\int_0^1 x I_n^2(\lambda x)\, dx = -\frac{1}{2} I_n'^2(\lambda) + \frac{1}{2}\left(1 + \frac{n^2}{\lambda^2}\right) I_n^2(\lambda) \tag{8}$$

11.41 ORTHOGONALITY OF MODIFIED BESSEL FUNCTIONS

THEOREM (11.40): Prove that if λ and μ are any two roots of $J_n(x) = 0$, then

$$\int_0^1 x I_n(\lambda x) I_n(\mu x)\, dx = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ -\frac{1}{2} I_{n+1}^2(\lambda) & \text{if } \lambda = \mu \end{cases}$$

i.e. $\sqrt{x}\, I_n(\lambda x)$ and $\sqrt{x}\, I_n(\mu x)$ are orthogonal over the interval $(0, 1)$.

PROOF: We know from the above theorem that if $\lambda \neq \mu$, then

$$\int_0^1 x I_n(\lambda x) I_n(\mu x)\, dx = \frac{\lambda I_n(\mu) I_n'(\lambda) - \mu I_n(\lambda) I_n'(\mu)}{\lambda^2 - \mu^2} \tag{1}$$

Since λ and μ are the roots of $I_n(x) = 0$, therefore $I_n(\lambda) = 0$ and $I_n(\mu) = 0$.

Thus from equation (1), we get

$$\int_0^1 x\, I_n(\lambda x)\, I_n(\mu x)\, dx = 0$$

Also, we know that if $\lambda = \mu$, then

$$\int_0^1 x\, I_n^2(\lambda x)\, dx = -\frac{1}{2} I_n'^2(\lambda) + \frac{1}{2}\left(1 + \frac{n^2}{\lambda^2}\right) I_n^2(\lambda) \tag{2}$$

Since $I_n(\lambda) = 0$, therefore equation (2) becomes

$$\int_0^1 x\, I_n^2(\lambda x)\, dx = -\frac{1}{2} I_n'^2(\lambda) \tag{3}$$

Now from the recurrence relation

$$x I_n'(x) - n I_n(x) = x I_{n+1}(x)$$

Let $x = \lambda$ in this relation, we get

$$\lambda I_n'(\lambda) - n I_n(\lambda) = \lambda I_{n+1}(\lambda)$$

Since $I_n(\lambda) = 0$, therefore

$$I_n'(\lambda) = I_{n+1}(\lambda) \tag{4}$$

From equations (3) and (4), we get

$$\int_0^1 x\, I_n^2(\lambda x)\, dx = -\frac{1}{2} I_{n+1}^2(\lambda)$$

THEOREM (11.41): Prove that if λ and μ are constants, then

$$\int x\, I_n(\lambda x)\, J_n(\mu x)\, dx = \frac{x\left[\lambda J_n(\mu x) I_n'(\lambda x) - \mu I_n(\lambda x) J_n'(\mu x)\right]}{\lambda^2 + \mu^2}$$

PROOF: We know that $y_1 = I_n(\lambda x)$ and $y_2 = J_n(\mu x)$ are the solutions of the Bessel's differential equation, therefore

$$x^2 y_1'' + x y_1' - (\lambda^2 x^2 + n^2) y_1 = 0 \tag{1}$$

and

$$x^2 y_2'' + x y_2' + (\mu^2 x^2 - n^2) y_2 = 0 \tag{2}$$

Multiplying equation (1) by y_2, equation (2) by y_1 and subtracting, we find

$$x^2\left[y_2 y_1'' - y_1 y_2''\right] + x\left[y_2 y_1' - y_1 y_2'\right] = (\lambda^2 + \mu^2) x^2 y_1 y_2$$

which on division by x can be written as

$$x\frac{d}{dx}[y_2 y_1' - y_1 y_2'] + [y_2 y_1' - y_1 y_2] = (\lambda^2+\mu^2)x^2 y_1 y_2$$

or $$\frac{d}{dx}[x(y_2 y_1' - y_1 y_2')] = (\lambda^2+\mu^2)x y_1 y_2$$

Then by integrating and omitting the constant of integration ,

$$(\lambda^2+\mu^2)\int x y_1 y_2\, dx = x(y_2 y_1' - y_1 y_2')$$

Using $y_1 = I_n(\lambda x)$ and $y_2 = J_n(\mu x)$ and dividing by $(\lambda^2+\mu^2)$, we get

$$\int x I_n(\lambda x) I_n(\mu x)\, dx = \frac{x[\lambda J_n(\mu x) I_n'(\lambda x) - \mu I_n(\lambda x) J_n'(\mu x)]}{\lambda^2+\mu^2} \tag{3}$$

11.42 BER AND BEI FUNCTIONS

The functions $\mathrm{Ber}_n(x)$ and $\mathrm{Bei}_n(x)$ are the real and imaginary parts of $J_n(i^{3/2}x)$, where $i^{3/2} = e^{3\pi i/4}$ since $i = e^{\pi i/2}$

i.e. $$J_n(i^{3/2}x) = \mathrm{Ber}_n(x) + i\,\mathrm{Bei}_n(x)$$

Now $$J_n(i^{3/2}x) = \sum_{m=0}^{\infty} \frac{(-1)^m\left(\frac{i^{3/2}x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(i)^{2m}(i)^{\frac{3n}{2}+3m}\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \qquad \left[\text{since } (i)^{2m} = (-1)^m\right]$$

$$= \sum_{m=0}^{\infty} \frac{(i)^{\left(\frac{3n}{2}+m\right)}\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \qquad \left[\text{since } (i)^{4m} = (-1)^{2m} = 1\right]$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m} e^{\frac{(3n+2m)\pi i}{4}}}{m!\,\Gamma(n+m+1)} \qquad \left(\text{since } i = e^{\pi i/2}\right)$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)}\left[\cos\left(\frac{3n+2m}{4}\right)\pi + i\sin\left(\frac{3n+2m}{4}\right)\pi\right]$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)}\cos\left(\frac{3n+2m}{4}\right)\pi$$

$$+ i\sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)}\sin\left(\frac{3n+2m}{4}\right)\pi \tag{1}$$

But since $J_n(i^{3/2}x) = \text{Ber}_n(x) + i\,\text{Bei}_n(x)$ (2)

From equations (1) and (2) comparing real and imaginary parts, we find

$$\text{Ber}_n(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \cos\left(\frac{3n+2m}{4}\right)\pi$$

$$\text{Bei}_n(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2m}}{m!\,\Gamma(n+m+1)} \sin\left(\frac{3n+2m}{4}\right)\pi$$

THEOREM (11.42): Show that

(i) $$\textbf{Ber}_0(x) = 1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\ldots$$

(ii) $$\textbf{Bei}_0(x) = \frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \ldots\ldots$$

PROOF: We know that

$$J_0(i^{3/2}x) = \text{Ber}_0(x) + i\,\text{Bei}_0(x) \tag{1}$$

Now $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} + \ldots\ldots$$

$$J_0(i^{3/2}x) = 1 - \frac{(i^{3/2}x)^2}{2^2} + \frac{(i^{3/2}x)^4}{2^2.4^2} - \frac{(i^{3/2}x)^6}{2^2.4^2.6^2} + \frac{(i^{3/2}x)^8}{2^2.4^2.6^2.8^2} - \frac{(i^{3/2}x)^{10}}{2^2.4^2.6^2.8^2.10^2} + \ldots$$

$$= 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{2^2.4^2} - \frac{i^9x^6}{2^2.4^2.6^2} + \frac{i^{12}x^8}{2^2.4^2.6^2.8^2} - \frac{i^{15}x^{10}}{2^2.4^2.6^2.8^2.10^2} + \ldots\ldots$$

$$= 1 + \frac{i x^2}{2^2} - \frac{x^4}{2^2.4^2} - \frac{i x^6}{2^2.4^2.6^2} + \frac{x^8}{2^2.4^2.6^2.8^2} + \frac{i x^{10}}{2^2.4^2.6^2.8^2.10^2} + \ldots\ldots$$

$$= \left(1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\ldots\right)$$
$$+ i\left(\frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} + \ldots\ldots\right) \tag{2}$$

From equations (1) and (2) comparing real and imaginary parts, we find

$$\text{Ber}_0(x) = 1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\ldots$$

$$\text{Bei}_0(x) = \frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \ldots\ldots$$

NOTE: Sometimes the subscript 0 is omitted in $\text{Ber}_0(x)$ and $\text{Bei}_0(x)$.

THEOREM (11.43): Prove that

(i) $$\frac{d}{dx}\left(x\,\textbf{Ber}'\,x\right) = -x\,\textbf{Bei}\,x$$ (ii) $$\frac{d}{dx}\left(x\,\textbf{Bei}'\,x\right) = x\,\textbf{Ber}\,x$$

PROOF: (i) We know that

$$\operatorname{Ber} x = 1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\ldots\ldots$$

Differentiating w.r.t. x, we get

$$\operatorname{Ber}' x = -\frac{4x^3}{2^2.4^2} + \frac{8x^7}{2^2.4^2.6^2.8^2} - \ldots\ldots\ldots$$

and $$x \operatorname{Ber}' x = -\frac{x^4}{2^2.4} + \frac{x^8}{2^2.4^2.6^2.8} - \ldots\ldots\ldots$$

Thus $$\frac{d}{dx}\left(x \operatorname{Ber}' x\right) = -\frac{x^3}{2^2} + \frac{x^7}{2^2.4^2.6^2} - \ldots\ldots\ldots$$

$$= -x\left[\frac{x^2}{2^2} + \frac{x^6}{2^2.4^2.6^2} - \ldots\ldots\ldots\right] = -x \operatorname{Bei} x$$

(ii) We know that

$$\operatorname{Bei} x = \frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \ldots\ldots\ldots$$

Differentiating w.r.t. x, we get

$$\operatorname{Bei}' x = \frac{x}{2} - \frac{x^5}{2^2.4^2.6} + \frac{x^9}{2^2.4^2.6^2.8^2.10} - \ldots\ldots\ldots$$

and $$x \operatorname{Bei}' x = \frac{x^2}{2} - \frac{x^6}{2^2.4^2.6} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10} - \ldots\ldots\ldots$$

Thus $$\frac{d}{dx}\left(x \operatorname{Bei}' x\right) = x - \frac{x^5}{2^2.4^2} + \frac{x^9}{2^2.4^2.6^2.8^2} - \ldots\ldots\ldots$$

$$= x\left[1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\ldots\ldots\right] = x \operatorname{Ber} x$$

11.43 KER AND KEI FUNCTIONS

The functions $\operatorname{Ker}_n(x)$ and $\operatorname{Kei}_n(x)$ are the real and imaginary parts of

$$e^{-n\pi i/2} K_n(i^{1/2} x) = e^{-n\pi i/2} K_n(e^{\pi i/4} x), \quad \text{where } i^{1/2} = e^{\pi i/4}$$

i.e. $$e^{-n\pi i/2} K_n(i^{1/2} x) = \operatorname{Ker}_n(x) + i \operatorname{Kei}_n(x)$$

REMARK: These functions satisfy the equation

$$x^2 y'' + x y' - (i x^2 + n^2) y = 0$$

which arises in Electrical Engineering and other fields . The general solution of this equation is therefore given by

$$y = C_1 J_n(i^{3/2} x) + C_2 K_n(i^{1/2} x)$$

where C_1 and C_2 are arbitrary constants .

11.44 SOLVED PROBLEMS

PROBLEM (1): **Show that**

(i) $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

(ii) $J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$

(iii) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x)$

SOLUTION: From the recurrence relation

$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$, we have

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad (1)$$

(i) Let $n = 1$ in equation (1), we get

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \qquad (2)$$

(ii) Let $n = 2$ in equation (1), we get

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \qquad (3)$$

From equations (2) and (3), we get

$$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

(iii) Let $n = 3$ in equation (1), we get

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \qquad (4)$$

Substituting the values of $J_2(x)$ and $J_3(x)$ in equation (4), we get

$$J_4(x) = \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x)$$

PROBLEM (2): **Prove that**

(i) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

(ii) $J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right]$

(iii) $J_n'''(x) = \frac{1}{8}\left[J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\right]$

SOLUTION: (i) From the recurrence relation

$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$, we get

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \tag{1}$$

(ii) Differentiating equation (1) w.r.t. x, we get

$$J_n''(x) = \frac{1}{2}\left[J_{n-1}'(x) - J_{n+1}'(x)\right] \tag{2}$$

Replacing n by $n-1$ and $n+1$ in equation (1), we get

$$J_{n-1}'(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right]$$

and $$J_{n+1}'(x) = \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right]$$

Substituting these in equation (2), we get

$$J_n''(x) = \frac{1}{2}\left[\frac{1}{2}\{J_{n-2}(x) - J_n(x)\} - \frac{1}{2}\{J_n(x) - J_{n+2}(x)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}J_{n-2}(x) - J_n(x) + \frac{1}{2}J_{n+2}(x)\right]$$

or $$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right] \tag{3}$$

(iii) Differentiating equation (2) w.e.t. x, we get

$$J_n'''(x) = \frac{1}{2}\left[J_{n-1}''(x) - J_{n+1}''(x)\right] \tag{4}$$

Replacing n by $n-1$ and $n+1$ in equation (3), we get

$$J_{n-1}'' = \frac{1}{4}\left[J_{n-3}(x) - 2J_{n-1}(x) + J_{n+1}(x)\right]$$

and $$J_{n+1}'' = \frac{1}{4}\left[J_{n-1}(x) - 2J_{n+1}(x) + J_{n+3}(x)\right]$$

Substituting the values of these derivatives in equation (4), we get

$$J_n'''(x) = \frac{1}{2}\left[\frac{1}{4}\{J_{n-3}(x) - 2J_{n-1}(x) + J_{n+1}(x)\} - \frac{1}{4}\{J_{n-1}(x) - 2J_{n+1}(x) + J_{n+3}(x)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\{J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\}\right]$$

or $$J_n'''(x) = \frac{1}{8}\left[J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\right]$$

PROBLEM (3): **Prove that**

(i) $J_2(x) - J_0''(x) = -\dfrac{J_0'(x)}{x}$ (ii) $2J_0''(x) = J_2(x) - J_0(x)$

SOLUTION: (i) From the recurrence relation

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x) \qquad (1)$$

Let $n = 1$ in equation (1), we get

$$x J_1'(x) - J_1(x) = -x J_2(x) \qquad (2)$$

Since $J_0'(x) = -J_1(x)$, therefore $J_0''(x) = -J_1'(x)$.

Thus equation (2) becomes

$$-x J_0''(x) + J_0'(x) = -x J_2(x)$$

or

$$J_2(x) - J_0''(x) = -\frac{J_0'(x)}{x}$$

(ii) From problem (2), we have

$$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2 J_n(x) + J_{n+2}(x)\right] \qquad (3)$$

Let $n = 0$ in equation (3), we get

$$J_0''(x) = \frac{1}{4}\left[J_{-2}(x) - 2 J_0(x) + J_2(x)\right]$$

Since $J_{-2}(x) = (-1)^2 J_2(x) = J_2(x)$, therefore

$$J_0''(x) = \frac{1}{4}\left[J_2(x) - 2 J_0(x) + J_2(x)\right]$$

$$= \frac{1}{2}\left[J_2(x) - J_0(x)\right]$$

or

$$2 J_0''(x) = J_2(x) - J_0(x)$$

PROBLEM (4): **Prove that**

(i) $$\mathbf{J_1'(x) = J_0(x) - \frac{1}{x} J_1(x)}$$

(ii) $$\mathbf{J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)}$$

SOLUTION: We know from problem (2), that

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \qquad (1)$$

(i) Let $n = 1$ in equation (1), we get

$$J_1'(x) = \frac{1}{2}\left[J_0(x) - J_2(x)\right]$$

Since $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$, therefore

$$J_1'(x) = \frac{1}{2}\left[J_0(x) - \frac{2}{x} J_1(x) + J_0(x)\right]$$

$$= J_0(x) - \frac{1}{x} J_1(x)$$

(ii) Let $n = 2$ in equation (1), we get

$$J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$$

Since $J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$, therefore

$$\begin{aligned} J_2'(x) &= \frac{1}{2}\left[J_1(x) - \left\{\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right\}\right] \\ &= \frac{1}{2}\left[J_1(x) - \frac{8}{x^2} J_1(x) + J_1(x) + \frac{4}{x} J_0(x)\right] \\ &= \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x) \end{aligned}$$

PROBLEM (5): Prove that

$$x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x), \qquad n = 0, 1, 2, \ldots\ldots$$

SOLUTION: From the recurrence relation

$x J_n'(x) - n J_n(x) = -x J_{n+1}(x)$, we have

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \tag{1}$$

Differentiating both sides w.r.t. x, we get

$$x J_n''(x) + J_n'(x) = n J_n'(x) - x J_{n+1}'(x) - J_{n+1}(x)$$

or $$x^2 J_n''(x) = (n-1) x J_n'(x) - x^2 J_{n+1}'(x) - x J_{n+1}(x) \tag{2}$$

From the recurrence relation

$x J_n'(x) + n J_n(x) = x J_{n-1}(x)$, we have

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \tag{3}$$

Replacing n by $n+1$ in equation (3), we get

$$x J_{n+1}'(x) = -(n+1) J_{n+1}(x) + x J_n(x) \tag{4}$$

Substituting the values of $x J_n'(x)$ from equation (1) and $x J_{n+1}'(x)$ from equation (4) in equation (2), we get

$$\begin{aligned} x^2 J_n''(x) &= (n-1)[n J_n(x) - x J_{n+1}(x)] - x[-(n+1) J_{n+1}(x) + x J_n(x)] - x J_{n+1}(x) \\ &= (n^2 - n - x^2) J_n(x) + x J_{n+1}(x) \end{aligned}$$

PROBLEM (6): Show that $4 J_0'''(x) + 3 J_0'(x) + J_3(x) = 0$

SOLUTION: We know that

$$J_n'''(x) = \frac{1}{8}[J_{n-3}(x) - 3 J_{n-1}(x) + 3 J_{n+1}(x) - J_{n+3}(x)] \tag{1}$$

Let $n = 0$ in equation (1), we get

$$J_0'''(x) = \frac{1}{8}\left[J_{-3}(x) - 3J_{-1}(x) + 3J_1(x) - J_3(x)\right]$$

Since $J_{-n}(x) = (-1)^n J_n(x)$, therefore

$$J_0'''(x) = \frac{1}{8}\left[-J_3(x) + 3J_1(x) + 3J_1(x) - J_3(x)\right]$$

$$= \frac{1}{8}\left[6J_1(x) - 2J_3(x)\right] = \frac{3}{4}J_1(x) - \frac{1}{4}J_3(x)$$

or

$$4J_0'''(x) = 3J_1(x) - J_3(x)$$

or

$$4J_0'''(x) - 3J_1(x) + J_3(x) = 0$$

or

$$4J_0'''(x) + 3J_0'(x) + J_3(x) = 0 \qquad \left[\text{since } J_0'(x) = -J_1(x)\right]$$

PROBLEM (7): **Prove that**

$$\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

SOLUTION: We know that

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

or

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \qquad (1)$$

Also, from problem (2)

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

or

$$2J_n'(x) = \left[J_{n-1}(x) - J_{n+1}(x)\right] \qquad (2)$$

Multiplying equations (1) and (2), we get

$$2J_n(x)J_n'(x) = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

or

$$\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

PROBLEM (8): **Prove that**

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\frac{n}{x}J_n^2(x) - \left(\frac{n+1}{x}\right)J_{n+1}^2(x)\right]$$

SOLUTION: We have

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2J_n(x)J_n'(x) + 2J_{n+1}(x)J_{n+1}'(x) \qquad (1)$$

From the recurrence relation

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x), \text{ we have}$$

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \tag{2}$$

Also , from the recurrence relation

$$x J_n'(x) + n J_n(x) = x J_{n-1}(x), \text{ we have}$$

$$J_n'(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x)$$

Replacing n by $(n+1)$, we get

$$J_{n+1}'(x) = -\frac{n+1}{x} J_{n+1}(x) + J_n(x) \tag{3}$$

From equations (1) , (2) , and (3) , we get

$$\begin{aligned}\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] &= 2 J_n(x)\left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right] \\ &\quad + 2 J_{n+1}(x)\left[-\frac{n+1}{x} J_{n+1}(x) + J_n(x)\right] \\ &= \frac{2n}{x} J_n^2(x) - \frac{2(n+1)}{x} J_{n+1}^2(x) \\ &= 2\left[\frac{n}{x} J_n^2(x) - \left(\frac{n+1}{x}\right) J_{n+1}^2(x)\right]\end{aligned}$$

PROBLEM (9): **Prove that**

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

SOLUTION: We have

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = J_n(x) J_{n+1}(x) + x J_n'(x) J_{n+1}(x) + x J_n(x) J_{n+1}'(x) \tag{1}$$

From the recurrence relation

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x)$$

or $$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \tag{2}$$

Also , from the recurrence relation

$$x J_n'(x) + n J_n(x) = x J_{n-1}(x)$$

or $$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

Replacing n by $(n+1)$, we get

$$x J_{n+1}'(x) = -(n+1) J_{n+1}(x) + x J_n(x) \tag{3}$$

Substituting the values of $x J_n'(x)$ and $x J_{n+1}'(x)$ from equations (2) and (3) in equation (1), we get

$$\begin{aligned}\frac{d}{dx}\left[x J_n(x) + J_{n+1}(x)\right] &= J_n(x) J_{n+1}(x) + \left[n J_n(x) - x J_{n+1}(x)\right] J_{n+1}(x) \\ &\quad + \left[-(n+1) J_{n+1}(x) + x J_n(x)\right] J_n(x) \\ &= x\left[J_n^2(x) - J_{n+1}^2(x)\right]\end{aligned}$$

PROBLEM (10): **Prove that**

(i) $$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left[x^n J_n(x)\right] = x^{n-m} J_{n-m}(x)$$

(ii) $$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left[x^{-n} J_n(x)\right] = (-1)^m x^{-(n+m)} J_{n+m}(x)$$

SOLUTION: (i) We know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

or $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^{n-1} . x J_{n-1}(x)$$

or $$\left(\frac{1}{x}\frac{d}{dx}\right)\left[x^n J_n(x)\right] = x^{n-1} J_{n-1}(x)$$

Applying this rule rule , we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^2 \left[x^n J_n(x)\right] = x^{n-2} J_{n-2}(x)$$

Repeatedly applying this rule m times , we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left[x^n J_n(x)\right] = x^{n-m} J_{n-m}(x)$$

(ii) We know that

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

or $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = (-1) x^{-n-1} . x J_{n+1}(x)$$

or $$\left(\frac{1}{x}\frac{d}{dx}\right)\left[x^{-n} J_n(x)\right] = (-1) x^{-(n+1)} J_{n+1}(x)$$

Applying this rule again , we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^2 \left[x^{-n} J_n(x)\right] = (-1)^2 x^{-(n+2)} J_{n+2}(x)$$

Repeatedly applying this rule m times , we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \left[x^{-n} J_n(x)\right] = (-1)^m x^{-(n+m)} J_{n+m}(x)$$

PROBLEM (11): **Prove that**

$$J_0^2(x)+2\left[J_1^2(x)+J_2^2(x)+J_3^2+\dots\dots\right]=1$$

SOLUTION: We know from problem (8)

$$\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2\left[\frac{n}{x}J_n^2(x)-\frac{n+1}{x}J_{n+1}^2(x)\right] \qquad (1)$$

Let $n=0,1,2,3,\dots\dots$ successively in equation (1), we get

$$\frac{d}{dx}\left[J_0^2(x)+J_1^2(x)\right]=2\left[0-\frac{1}{x}J_1^2(x)\right]$$

$$\frac{d}{dx}\left[J_1^2(x)+J_2^2(x)\right]=2\left[\frac{1}{x}J_1^2(x)-\frac{2}{x}J_2^2(x)\right]$$

$$\frac{d}{dx}\left[J_2^2(x)+J_3^2(x)\right]=2\left[\frac{2}{x}J_2^2(x)-\frac{3}{x}J_3^2(x)\right]$$

Adding these column wise and noting that $J_n(x)\to 0$ as $n\to\infty$, we get

$$\frac{d}{dx}\left[J_0^2(x)+2\left\{J_1^2(x)+J_2^2(x)+J_3^2(x)+\dots\dots\right\}\right]=0$$

Integrating we get

$$J_0^2(x)+2\left[J_1^2(x)+J_2^2(x)+J_3^2(x)+\dots\dots\right]=C \qquad (2)$$

Let $x=0$ in equation (2) and noting that $J_0(0)=1$ and $J_n(0)=0$ for $n\geq 1$, we get

$$1+2(0+0+0\dots\dots)=C \quad \text{or} \quad C=1$$

Thus from equation (2), we get

$$J_0^2(x)+2\left[J_1^2(x)+J_2^2(x)+J_3^2(x)+\dots\dots\right]=1$$

PROBLEM (12): **Prove that**

$$x=2J_0(x)J_1(x)+6J_1(x)J_2(x)+\dots\dots$$
$$+2(2n+1)J_n(x)J_{n+1}+\dots\dots$$

SOLUTION: We know that

$$\frac{d}{dx}\left[xJ_n(x)J_{n+1}(x)\right]=x\left[J_n^2(x)-J_{n+1}^2(x)\right]$$

Replacing n by $0,1,2,3,\dots\dots$ successively in the above equation, we get

$$\frac{d}{dx}\left[xJ_0(x)J_1(x)\right]=x\left[J_0^2(x)-J_1^2(x)\right] \qquad (1)$$

$$\frac{d}{dx}\left[xJ_1(x)J_2(x)\right]=x\left[J_1^2(x)-J_2^2(x)\right] \qquad (2)$$

$$\frac{d}{dx}\left[x J_2(x) J_3(x)\right] = x\left[J_2^2(x) - J_3^2(x)\right] \qquad (3)$$

Multiplying equations (1), (2), (3), ... by 1, 3, 5, ... respectively and adding, we get

$$\frac{d}{dx}\left[x J_0(x) J_1(x) + 3 J_1(x) J_2(x) + 5 J_2(x) J_3(x) + \ldots\ldots\right]$$

$$= x\left[\{J_0^2(x) - J_1^2(x)\} + 3\{J_1^2(x) - J_2^2(x)\} + 5\{J_2^2(x) - J_3^2(x)\} + \ldots\ldots\right]$$

$$= x\left[J_0^2(x) + 2\{J_1^2(x) + J_2^2(x) + J_3^2(x) + \ldots\ldots\}\right]$$

$$= x(1) = x \qquad \left[\text{using problem (9)}\right]$$

Integrating both sides, we get

$$x\left[J_0(x) J_1(x) + 3 J_1(x) J_2(x) + 5 J_2(x) J_3(x) + \ldots\ldots\right] = \frac{x^2}{2} + C \qquad (4)$$

Let $x = 0$ in equation (4), we get $C = 0$

$$x\left[J_0(x) J_1(x) + 3 J_1(x) J_2(x) + 5 J_2(x) J_3(x) + \ldots\ldots\right] = \frac{x^2}{2}$$

or $$x = 2 J_0(x) J_1(x) + 6 J_1(x) J_2(x) + 10 J_2(x) J_3(x) + \ldots\ldots$$

PROBLEM (13): **Prove that**

(i) $$J_n(x) = \frac{2}{x}\left[(n+1) J_{n+1}(x) - (n+3) J_{n+3}(x) + (n+5) J_{n+5}(x) + \ldots\right]$$

(ii) $$J_{n-1}(x) = \frac{2}{x}\left[n J_n(x) - (n+2) J_{n+2}(x) + (n+4) J_{n+4} \ldots\ldots\right]$$

(iii) $$J_n'(x) = \frac{2}{x}\left[\frac{n}{2} J_n(x) - (n+2) J_{n+2}(x) + (n+4) J_{n+4}(x) + \ldots\ldots\right]$$

SOLUTION: From the recurrence relation

$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$, we have

$$2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right] \qquad (1)$$

Replacing n by $n+1$ in equation (1), we get

$$2(n+1) J_{n+1}(x) = x\left[J_n(x) + J_{n+2}(x)\right]$$

or $$\frac{1}{2} x J_n(x) = (n+1) J_{n+1}(x) - \frac{1}{2} x J_{n+2}(x) \qquad (2)$$

Replacing n by $n+2$ in equation (2)

$$\frac{1}{2} x J_{n+2}(x) = (n+3) J_{n+3}(x) - \frac{1}{2} x J_{n+4}(x) \qquad (3)$$

From equations (2) and (3), we get

$$\frac{1}{2}xJ_n(x)=(n+1)J_{n+1}(x)-(n+3)J_{n+3}(x)+\frac{1}{2}xJ_{n+4}(x) \qquad (4)$$

Replacing n by $n+4$ in equation (2), we get

$$\frac{1}{2}xJ_{n+4}(x)=(n+5)J_{n+5}(x)-\frac{1}{2}xJ_{n+6}(x) \qquad (5)$$

From equations (4) and (5), we get

$$\frac{1}{2}xJ_n(x)=(n+1)J_{n+1}(x)-(n+3)J_{n+3}(x)+(n+5)J_{n+5}(x)-\frac{1}{2}xJ_{n+6}(x)$$

Proceeding likewise and noting that $J_n(x)\to 0$ as $n\to\infty$, we get

$$\frac{1}{2}xJ_n(x)=(n+1)J_{n+1}(x)-(n+3)J_{n+3}(x)+(n+5)J_{n+5}(x)\ldots\ldots \qquad (6)$$

(ii) $$J_n(x)=\frac{2}{x}\left[(n+1)J_{n+1}(x)-(n+3)J_{n+3}(x)+(n+5)J_{n+5}(x)+\ldots\right]$$

Replacing n by $n-1$ in equation (6), we get

$$J_{n-1}(x)=\frac{2}{x}\left[nJ_n(x)-(n+2)J_{n+2}(x)+(n+4)J_{n+4}(x)+\ldots\ldots\right] \qquad (7)$$

(iii) From the recurrence relation

$$xJ_n'(x)+nJ_n(x)=xJ_{n-1}(x)$$

or $$J_n'(x)=-\frac{n}{x}J_n(x)+J_{n-1}(x)$$

Substituting the value of $J_{n-1}(x)$ from equation (6), we get

$$J_n'(x)=-\frac{n}{x}J_n(x)+\frac{2}{x}\left[nJ_n(x)-(n+2)J_{n+2}(x)+(n+4)J_{n+4}(x)-\ldots\ldots\right]$$

$$=\frac{2}{x}\left[\frac{n}{2}J_n(x)-(n+2)J_{n+2}(x)+(n+4)J_{n+4}(x)\ldots\ldots\right]$$

PROBLEM (14): **Use the generating function to prove**

(i) $$J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$$

(ii) $J_{n-1}-J_{n+1}=2J_n'(x)$ **for integer values of n.**

PROOF: We know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}J_n(x)t^n \qquad (1)$$

Differentiating both sides of the identity (1) w.r.t t, we get

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\frac{x}{2}\left(1+\frac{1}{t^2}\right)=\sum_{n=-\infty}^{\infty}nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1} \qquad \left[\text{using identity (1)}\right]$$

$$\sum_{n=-\infty}^{\infty} \frac{x}{2}\left(1+\frac{1}{t^2}\right) J_n(x)t^n = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}$$

which can be written as

$$\sum_{n=-\infty}^{\infty} \frac{x}{2}J_n(x)t^n + \sum_{n=-\infty}^{\infty} \frac{x}{2}J_n(x)t^{n-2} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}$$

or
$$\sum_{n=-\infty}^{\infty} \frac{x}{2}J_n(x)t^n + \sum_{n=-\infty}^{\infty} \frac{x}{2}J_{n+2}(x)t^n = \sum_{n=-\infty}^{\infty} (n+1)J_{n+1}(x)t^n$$

Equating the coefficients of t^n on both sides, we have

$$\frac{x}{2}J_n(x)+\frac{x}{2}J_{n+2}(x) = (n+1)J_{n+1}(x)$$

i.e.
$$J_n(x)+J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x)$$

Replacing n by $n-1$, we get the required result

$$J_{n-1}(x)+J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

(ii) Differentiating both sides of the identity (1) w.r.t x, we get

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\frac{1}{2}\left(t-\frac{1}{t}\right) = \sum_{n=-\infty}^{\infty} J_n'(x)t^n$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{2}J_n(x)t^n\left(t-\frac{1}{t}\right) = \sum_{n=-\infty}^{\infty} J_n'(x)t^n$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{2}J_n(x)t^{n+1} - \sum_{n=-\infty}^{\infty} \frac{1}{2}J_n(x)t^{n-1} = \sum_{n=-\infty}^{\infty} J_n'(x)t^n$$

or
$$\sum_{n=-\infty}^{\infty} \frac{1}{2}J_{n-1}(x)t^n - \sum_{n=-\infty}^{\infty} \frac{1}{2}J_{n+1}(x)t^n = \sum_{n=-\infty}^{\infty} J_n'(x)t^n$$

Equating the coefficient of t^n on both sides

$$\frac{1}{2}J_{n-1}(x)-\frac{1}{2}J_{n+1}(x) = J_n'(x)$$

or
$$J_{n-1}(x)-J_{n+1}(x) = 2J_n'(x)$$

PROBLEM (15): **Using recurrence relations , prove that**

(i) $$Y_{3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

(ii) $$Y_{-3/2}(x) = \frac{2}{\sqrt{\pi x}}\left(\cos x - \frac{\sin x}{x}\right)$$

SOLUTION: We have the recurrence relation

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x) \tag{1}$$

(i) Let $n = \frac{1}{2}$ in equation (1) , we get

$$Y_{-1/2}(x) + Y_{3/2}(x) = \frac{1}{x} Y_{1/2}(x)$$

or $$Y_{3/2}(x) = \frac{1}{x} Y_{1/2}(x) - Y_{-1/2}(x)$$

$$= \frac{1}{x}\left(-\sqrt{\frac{2}{\pi x}}\cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$$

(ii) Let $n = -\frac{1}{2}$ in equation (1) , we get

$$Y_{-3/2}(x) + Y_{1/2}(x) = -\frac{1}{x} Y_{-1/2}(x)$$

or $$Y_{-3/2}(x) = -Y_{1/2}(x) - \frac{1}{x} Y_{-1/2}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\cos x - \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x = \frac{2}{\sqrt{\pi x}}\left(\cos x - \frac{\sin x}{x}\right)$$

PROBLEM (16): **Using recurrence relations , prove that**

(i) $$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right)$$

(ii) $$I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\sinh x - \frac{\cosh x}{x}\right)$$

SOLUTION: We have the recurrence relation

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x), \text{ we have}$$

or $$I_{n+1}(x) = -\frac{2n}{x} I_n(x) + I_{n-1}(x) \tag{1}$$

(i) Let $n = \frac{1}{2}$ in the equation (1) , we get

$$I_{3/2}(x) = -\frac{1}{x} I_{1/2}(x) + I_{-1/2}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sinh x + \sqrt{\frac{2}{\pi x}}\cosh x = \sqrt{\frac{2}{\pi x}}\left(\cosh x - \frac{\sinh x}{x}\right)$$

(ii) Let $n = -\frac{1}{2}$ in equation (1) , we have

$$I_{1/2}(x) = \frac{1}{x} I_{-1/2}(x) + I_{-3/2}(x)$$

or
$$I_{-3/2}(x) = I_{1/2}(x) - \frac{1}{x} I_{-1/2}(x)$$

$$= \sqrt{\frac{2}{\pi x}} \sinh x - \frac{1}{x} \sqrt{\frac{2}{\pi x}} \cosh x = \sqrt{\frac{2}{\pi x}} \left(\sinh x - \frac{\cosh x}{x} \right)$$

PROBLEM (17): **Show that**

(i) $$x \sin x = 2 \left[2^2 J_2(x) - 4^2 J_4(x) + 6^2 J_6(x) \ldots\ldots\ldots \right]$$

(ii) $$x \cos x = 2 \left[1^2 J_1(x) - 3^2 J_3(x) + 5^2 J_5(x) \ldots\ldots\ldots \right]$$

SOLUTION: (i) We know that

$$\cos(x \sin\theta) = J_0(x) + 2 J_2(x) \cos 2\theta + 2 J_4(x) \cos 4\theta + 2 J_6(x) \cos 6\theta + \ldots\ldots\ldots \quad (1)$$

and
$$\sin(x \sin\theta) = 2 J_1(x) \sin\theta + 2 J_3(x) \sin 3\theta + 2 J_5(x) \sin 5\theta + 2 J_7(x) \sin 7\theta + \ldots\ldots\ldots \quad (2)$$

(i) Differentiating equation (1) w.r.t. θ , we get

$$\left[-\sin(x \sin\theta) \right] x \cos\theta = -4 J_2(x) \sin 2\theta - 8 J_4(x) \sin 4\theta - 12 J_6(x) \sin 6\theta \ldots\ldots\ldots \quad (3)$$

Again differentiating equation (3) w.r.t. θ , we get

$$\left[-\sin(x \sin\theta) \right] (-x \sin\theta) + \left[-\cos(x \sin\theta) \right] (x \cos\theta)^2$$
$$= -8 J_2(x) \cos 2\theta - 32 J_4(x) \cos 4\theta - 72 J_6(x) \cos 6\theta \ldots\ldots\ldots$$

or
$$\left[\sin(x \sin\theta) \right] (x \sin\theta) - \left[\cos(x \sin\theta) \right] (x \cos\theta)^2$$
$$= 2 \left[-2^2 J_2(x) \cos 2\theta - 4^2 J_4(x) \cos 4\theta - 6^2 J_6(x) \cos 6\theta \ldots\ldots\ldots \right] \quad (4)$$

Let $\theta = \frac{\pi}{2}$ in equation (4) , we get

$$x \sin x = 2 \left[2^2 J_2(x) - 4^2 J_4(x) + 6^2 J_6(x) \ldots\ldots\ldots \right]$$

(ii) Similarly , differentiating equation (2) w.r.t. θ and putting $\theta = \frac{\pi}{2}$, we get

$$x \cos x = 2 \left[1^2 J_1(x) - 3^2 J_3(x) + 5^2 J_5(x) - \ldots\ldots\ldots \right]$$

INTEGRALS INVOLVING BESSEL FUNCTIONS

PROBLEM (18): **Prove that**

(i) $$\int J_1(x)\,dx = -J_0(x) + C$$

(ii) $$\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx$$

(iii) $$\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x) + C$$

SOLUTION: (i) $\int J_1(x)\,dx = -\int J_0'(x)\,dx = -J_0(x) + C$

We know that

$$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) \tag{1}$$

Let $n = 1$ in equation (1), we get

$$\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x) \tag{2}$$

Let $n = 2$ in equation (1), we get

$$\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) \tag{3}$$

(ii) $$\int J_2(x)\,dx = \int x\left[x^{-1} J_2(x)\right] dx$$

$$= x\left[-x^{-1} J_1(x)\right] + \int x^{-1} J_1(x)\,dx + C \qquad \text{[using equation (2)]}$$

$$= -J_1(x) + \int x^{-1} J_1(x)\,dx + C \tag{4}$$

Now $$\int x^{-1} J_1(x)\,dx = \int x J_1(x) \frac{1}{x^2}\,dx$$

$$= x J_1(x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right) \cdot x J_0(x)\,dx$$

$$= -J_1(x) + \int J_0(x)\,dx$$

Thus equation (4) becomes

$$\int J_2(x)\,dx = -J_1(x) - J_1(x) + \int J_0(x)\,dx$$

$$= -2J_1(x) + \int J_0(x)\,dx$$

(iii)

$$\int J_3(x)\,dx = \int x^2\left[x^{-2}J_3(x)\right]dx$$

$$= x^2\left[-x^{-2}J_2(x)\right] + \int x^{-2}J_2(x)\,2x\,dx \quad \text{[using equation (3)]}$$

$$= -J_2(x) + 2\int x^{-1}J_2(x)\,dx$$

$$= -J_2(x) + 2\left[-x^{-1}J_1(x)\right] + C \quad \text{[using equation (2)]}$$

or

$$\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x) + C$$

PROBLEM (19): **Prove that** $\int x\,J_0^2(x)\,dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right] + C$

SOLUTION: We have

$$\int x J_0^2(x)\,dx = \int J_0^2(x).x\,dx$$

$$= J_0^2(x).\frac{x^2}{2} - \int \frac{x^2}{2}.2J_0(x)J_0'(x)\,dx$$

$$= \frac{1}{2}x^2 J_0^2(x) - \int x^2 J_0(x)\left[-J_1(x)\right]dx$$

$$= \frac{1}{2}x^2 J_0^2(x) + \int xJ_1(x)\left[xJ_0(x)\right]dx \qquad (1)$$

Since $\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x)$, therefore equation (1) becomes

$$\int xJ_0^2(x)\,dx = \frac{1}{2}x^2 J_0^2(x) + \int xJ_1(x)\frac{d}{dx}\left[xJ_1(x)\right]dx$$

$$= \frac{1}{2}x^2 J_0^2(x) + \frac{\left[xJ_1(x)\right]^2}{2} + C$$

$$= \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right] + C$$

PROBLEM (20): **Prove that** $\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C$

SOLUTION: We have

$$\int x^3 J_0(x)\,dx = \int x^2.xJ_0(x)\,dx$$

$$= x^2.xJ_1(x) - \int xJ_1(x).2x\,dx$$

$$= x^3 J_1(x) - 2\int x^2 J_1(x)\,dx$$

$$= x^3 J_1(x) - 2\left[x^2 J_2(x)\right] + C$$

$$= x^3 J_1(x) - 2x^2 J_2(x) + C \qquad (1)$$

ALTERNATIVE FORM

Since $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$, therefore equation (1) becomes

$$\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2\left[\frac{2}{x}J_1(x) - J_0(x)\right] + C$$
$$= x^3 J_1(x) - 4x J_1(x) + 2x^2 J_0(x) + C$$
$$= (x^3 - 4x)J_1(x) + 2x^2 J_0(x) + C$$

PROBLEM (21): **Show that** $\int x^4 J_1(x)\,dx = x^4 J_2(x) - 2x^3 J_3(x) + C$

$$= (4x^3 - 16x)J_1(x) + (8x^2 - x^4)J_0(x) + C$$

SOLUTION: Integration by parts gives

$$\int x^4 J_1(x)\,dx = \int x^2[x^2 J_1(x)\,dx]$$
$$= x^2[x^2 J_2(x)] - \int [x^2 J_2(x)][2x]\,dx$$
$$= x^4 J_2(x) - 2\int x^3 J_2(x)\,dx$$
$$= x^4 J_2(x) - 2x^3 J_3(x) + C \qquad (1)$$

ALTERNATIVE FORM

Since $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$

and $J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$

Thus equation (1) becomes

$$\int x^4 J_1(x)\,dx = x^4\left[\frac{2}{x}J_1(x) - J_0(x)\right] - 2x^3\left[\left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right]$$
$$= (4x^3 - 16x)J_1(x) + (8x^2 - x^4)J_0(x)$$

PROBLEM (22): **Show that**

$$\int \frac{J_3(x)}{x^3}\,dx = -\frac{1}{5x^2}J_3(x) - \frac{1}{15x}J_2(x) - \frac{1}{15}J_1(x) + \frac{1}{15}\int J_0(x)\,dx$$

SOLUTION: We have

$$\int \frac{J_3(x)}{x^3}\,dx = \int x^3 J_3(x)\frac{1}{x^6}\,dx$$
$$= x^3 J_3(x)\left(-\frac{1}{5x^5}\right) - \int\left(-\frac{1}{5x^5}\right)x^3 J_2(x)\,dx$$
$$= -\frac{1}{5x^2}J_3(x) + \frac{1}{5}\int \frac{J_2(x)}{x^2}\,dx$$

Now $\quad \displaystyle\int \frac{J_2(x)}{x^2}\,dx = \int x^2 J_2(x)\frac{1}{x^4}\,dx$

$$= x^2 J_2(x)\left(-\frac{1}{3x^3}\right) - \int\left(-\frac{1}{3x^3}\right)x^2 J_1(x)\,dx$$

$$= -\frac{1}{3x}J_2(x) + \frac{1}{3}\int \frac{J_1(x)}{x}\,dx$$

and $\quad \displaystyle\int \frac{J_1(x)}{x}\,dx = \int x J_1(x)\frac{1}{x^2}\,dx$

$$= x J_1(x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)x J_0(x)\,dx$$

$$= -J_1(x) + \int J_0(x)\,dx$$

Therefore $\quad \displaystyle\int \frac{J_3(x)}{x^3}\,dx = -\frac{1}{5x^2}J_3(x) + \frac{1}{5}\left[-\frac{1}{3x}J_2(x) + \frac{1}{3}\left(-J_1(x) + \int J_0(x)\,dx\right)\right]$

$$= -\frac{1}{5x^2}J_3(x) - \frac{1}{15x}J_2(x) - \frac{1}{15}J_1(x) + \frac{1}{15}\int J_0(x)\,dx$$

PROBLEM (23): **Evaluate**

(i) $\displaystyle\int_0^2 J_0'(x)\,dx \qquad$ (ii) $\displaystyle\int_1^2 x^{-5} J_6(x)\,dx$

SOLUTION: (i) Since $J_0'(x) = -J_1(x)$, therefore

$$\int_0^2 J_0'(x)\,dx = \left|-J_1(x)\right|_0^2 = -J_1(2) + J_1(0)$$

$$= -J_1(2) \qquad [\text{since } J_1(0) = 0]$$

(ii) $\displaystyle\int_1^2 x^{-5} J_6(x)\,dx$

Since $\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$, therefore

$$\int_1^2 x^{-5} J_6(x)\,dx = \left|-x^{-5} J_5(x)\right|_1^2 = J_5(1) - 2^{-5} J_5(2)$$

PROBLEM (24): **Evaluate** $\displaystyle\int_0^a x J_0(bx)\,dx$

where a and b are constants.

SOLUTION: Let $bx = t$, then $b\,dx = dt$ or $dx = \frac{dt}{b}$.

Also, when $x = 0$, $t = 0$, and when $x = a$, $t = ab$.

Thus $$\int_0^a x J_0(bx)\,dx = \int_0^{ab} \frac{t}{b} J_0(t)\frac{dt}{b} = \frac{1}{b^2}\int_0^{ab} tJ_0(t)\,dt$$

$$= \frac{1}{b^2}\left| tJ_1(t)\right|_0^{ab} = \frac{1}{b^2}\left[abJ_1(ab)\right] = \frac{a}{b}J_1(ab)$$

PROBLEM (25): **Prove that** $\int_a^b J_0(x)J_1(x)\,dx = \frac{1}{2}\left[J_0^2(a) - J_0^2(b)\right]$

SOLUTION: We have

$$\int_a^b J_0(x)J_1(x)\,dx = -\int_a^b J_0(x)J_0'(x)\,dx \qquad \left[\text{since } J_1(x) = -J_0'(x)\right]$$

$$= -\int_a^b \frac{1}{2}\frac{d}{dx}J_0^2(x)\,dx = -\frac{1}{2}\left|J_0^2(x)\right|_a^b = \frac{1}{2}\left[J_0^2(a) - J_0^2(b)\right]$$

PROBLEM (26): **Prove that**

(i) $$\int J_0(x)\cos x\,dx = xJ_0(x)\cos x + xJ_1(x)\sin x + C$$

(ii) $$\int J_1(x)\sin x\,dx = xJ_1(x)\sin x + (x\cos x - \sin x)J_0(x) + C$$

SOLUTION: (i) Integrating by parts taking 1 as the second function

$$\int J_0(x)\cos x \,.\, 1\,dx = xJ_0(x)\cos x - \int x\frac{d}{dx}\left[J_0(x)\cos x\right]dx$$

$$= xJ_0(x)\cos x - \int x\left[-J_0(x)\sin x + \cos x\, J_0'(x)\right]dx$$

$$= xJ_0(x)\cos x + \int x\left[J_0(x)\sin x + J_1(x)\cos x\right]dx$$

$$\left[\text{since } J_0'(x) = -J_1(x)\right]$$

$$= xJ_0(x)\cos x + \int xJ_0(x)\sin x\,dx + \int xJ_1(x)\cos x\,dx$$

$$= xJ_0(x)\cos x + \sin x \,.\, xJ_1(x) - \int xJ_1(x)\cos x\,dx + \int xJ_1(x)\cos x\,dx + C$$

$$= xJ_0(x)\cos x + xJ_1(x)\sin x + C$$

(ii) $$\int J_1(x)\sin x\,dx = -\int J_0'(x)\sin x\,dx \qquad [\text{since } J_0'(x) = -J_1(x)]$$

$$= -\left[J_0(x)\sin x - \int J_0(x)\cos x\,dx\right]$$

$$= -J_0(x)\sin x + \int J_0(x)\cos x\,dx$$

$$= -J_0(x)\sin x + [x J_0(x)\cos x + x J_1(x)\sin x] + C \quad [\text{using part (i)}]$$

$$= x J_1(x)\sin x + (x\cos x - \sin x) J_0(x) + C$$

PROBLEM (27): **Show that** $\displaystyle\int_0^{\pi/2} J_0(x\sin\theta)\cos\theta\sin\theta\,d\theta = \frac{J_1(x)}{x}$

SOLUTION: Let $x\sin\theta = t$, then $x\cos\theta\,d\theta = dt$.

Also, when $\theta = 0$, $t = 0$ and when $\theta = \frac{\pi}{2}$, $t = x$.

Thus $$\int_0^{\pi/2} J_0(x\sin\theta)\cos\theta\sin\theta\,d\theta = \int_0^x J_0(t)\cdot\frac{t}{x}\frac{dt}{x} = \frac{1}{x^2}\int_0^x t J_0(x)\,dt$$

$$= \frac{1}{x^2}\left|t J_1(t)\right|_0^x = \frac{1}{x^2}[x J_1(x) - 0] = \frac{J_1(x)}{x}$$

PROBLEM (28): **Show that** $\displaystyle J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta$

SOLUTION: We know that

$$\cos(x\cos\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots\ldots\ldots\ldots \qquad (1)$$

Integrating equation (1) w.r.t. θ from 0 to π, we get

$$\int_0^{\pi}\cos(x\cos\theta)\,d\theta = \int_0^{\pi} J_0(x)\,d\theta + 0 + 0 + \ldots\ldots\ldots\ldots$$

$$= J_0(x)\int_0^{\pi} d\theta = \pi J_0(x)$$

or $$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)\,d\theta$$

Similarly, we can prove $\displaystyle J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta$

PROBLEM (29): Prove that $\displaystyle\int_0^\infty e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$

SOLUTION: We know that $\displaystyle\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}, \quad a > 0$

Replacing b by $b\cos\theta$, we get

$$\int_0^\infty e^{-ax}\cos(bx\cos\theta)\,dx = \frac{a}{a^2+b^2\cos^2\theta}$$

Integrating under the integral sign w.r.t. θ from $\theta = 0$ to $\frac{\pi}{2}$, we get

$$\int_0^\infty e^{-ax}\left[\int_0^{\pi/2}\cos(bx\cos\theta)\,d\theta\right]dx = \int_0^{\pi/2}\frac{a}{a^2+b^2\cos^2\theta}\,d\theta \qquad (1)$$

Since $\displaystyle J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\cos\theta)\,d\theta$

therefore $\displaystyle J_0(bx) = \frac{2}{\pi}\int_0^{\pi/2}\cos(bx\cos\theta)\,d\theta$

or $$\frac{\pi}{2}J_0(bx) = \int_0^{\pi/2}\cos(bx\cos\theta)\,d\theta \qquad (2)$$

Also $$\int_0^{\pi/2}\frac{a}{a^2+b^2\cos^2\theta}\,d\theta = \frac{1}{\sqrt{a^2+b^2}}\left|\tan^{-1}\frac{a\tan x}{\sqrt{a^2+b^2}}\right|_0^{\pi/2} = \frac{1}{\sqrt{a^2+b^2}}\frac{\pi}{2} \qquad (3)$$

Thus from equations (1), (2), and (3), we get

$$\int_0^\infty e^{-ax}J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$$

DEDUCTIONS

(i) When $a \to 0$, we get if $b > 0$ $\displaystyle\int_0^\infty J_0(bx)\,dx = \frac{1}{b}$

(ii) In particular, if $b = 1$, we get $\displaystyle\int_0^\infty J_0(x)\,dx = 1$

PROBLEM (30): **Prove that**

(i) $$\int_0^{\pi/2} J_0(x\sin\theta)\sin\theta\, d\theta = \frac{\sin x}{x}$$ (ii) $$\int_0^{\pi/2} J_1(x\cos\theta)\, d\theta = \frac{1-\cos x}{x}$$

SOLUTION: (i) We know that $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2}$

Replacing x by $x\sin\theta$, we get

$$J_0(x\sin\theta) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x\sin\theta}{2}\right)^{2m}}{(m!)^2}$$

Then $$\int_0^{\pi/2} J_0(x\sin\theta)\sin\theta\, d\theta = \int_0^{\pi/2} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2} \sin^{2m+1}\theta\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2} \int_0^{\pi/2} \sin^{2m+1}\theta\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2} \left(\frac{2.4.6\ldots\ldots 2m}{1.3.5\ldots\ldots 2m+1}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2} \left(\frac{2.4.6\ldots\ldots 2m}{1.3.5\ldots\ldots 2m+1}\right)\left(\frac{2.4.6\ldots\ldots 2m}{2.4.6\ldots\ldots 2m}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2} \frac{(2^m.m!)(2^m.m!)}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}.(m!)^2} \frac{2^{2m}.(m!)^2}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!\,x} = \frac{\sin x}{x}$$

since $\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sin x$

(ii) We know that $J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{1+2m}}{m!(m+1)!}$

Replacing x by x cos θ, we get

$$J_1(x\cos\theta) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x\cos\theta}{2}\right)^{2m+1}}{m!(m+1)!}$$

Then $\int_0^{\pi/2} J_1(x\cos\theta)\, d\theta = \int_0^{\pi/2} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+1}}{m!(m+1)!} \cos^{2m+1}\theta\, d\theta$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+1}}{m!(m+1)!} \int_0^{\pi/2} \cos^{2m+1}\theta\, d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+1}}{m!(m+1)!} \left(\frac{2.4.6 \ldots\ldots\ldots 2m}{1.3.5 \ldots\ldots\ldots 2m+1}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+1}}{m!(m+1)!} \left(\frac{2.4.6 \ldots\ldots\ldots 2m}{1.3.5 \ldots\ldots\ldots 2m+1}\right) \left(\frac{2.4.6 \ldots\ldots\ldots 2m}{2.4.6 \ldots\ldots\ldots 2m}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{m!(m+1)!\, 2^{2m+1}} \frac{(2^m.m!)(2^m.m!)}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2(m+1)(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+2)!}$$

$$= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \ldots\ldots\ldots \tag{1}$$

Now $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots$

or $\frac{1-\cos x}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \ldots\ldots\ldots$ (2)

From equations (1) and (2), we get

$$\int_0^{\pi/2} J_1(x\cos\theta)\, d\theta = \frac{1-\cos x}{x}$$

PROBLEM (31): **Prove that**

(i) $$\int x^m J_0(x)\,dx = x^m J_1(x) + (m-1)x^{m-1} J_0(x) - (m-1)^2 \int x^{m-2} J_0(x)\,dx$$

(ii) $$\int x^{-m} J_0(x)\,dx = \frac{J_1(x)}{(m-1)^2 x^{m-2}} - \frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2}\int \frac{J_0(x)}{x^{m-2}}\,dx$$

(iii) $$\int x^m J_1(x)\,dx = -x^m J_0(x) + m\int x^{m-1} J_0(x)\,dx$$

(iv) $$\int x^{-m} J_1(x)\,dx = -\frac{J_1(x)}{m\,x^{m-1}} + \frac{1}{m}\int \frac{J_0(x)}{x^{m-1}}\,dx$$

SOLUTION: We have

(i) $$\begin{aligned}\int x^m J_0(x)\,dx &= \int x^{m-1}.\,xJ_0(x)\,dx \\ &= x^{m-1}.\,xJ_1(x) - \int xJ_1(x)(m-1)x^{m-2}\,dx \\ &= x^m J_1(x) - (m-1)\int x^{m-1} J_1(x)\,dx \\ &= x^m J_1(x) - (m-1) \\ &\qquad \left[x^{m-1}\{-J_0(x)\} - \int -J_0(x)(m-1)x^{m-2}\,dx\right] \\ &= x^m J_1(x) + (m-1)x^{m-1}J_0(x) - (m-1)^2\int x^{m-2}J_0(x)\,dx\end{aligned}$$

(ii) $$\begin{aligned}\int x^{-m} J_0(x)\,dx &= \frac{x^{-m+1}}{-m+1}J_0(x) - \int \frac{x^{-m+1}}{-m+1}J_0'\,dx \\ &= -\frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{m-1}\int x^{-m+1}J_1(x)\,dx \\ &= -\frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{m-1}\int x^{-m}.\,xJ_1(x)\,dx \\ &= -\frac{J_0(x)}{(m-1)x^{m-1}} - \frac{1}{m-1}\left[\frac{x^{-m+1}}{-m+1}xJ_1(x) - \int \frac{x^{-m+1}}{-m+1}xJ_0(x)\,dx\right] \\ &= -\frac{J_0(x)}{(m-1)x^{m-1}} + \frac{x^{-m+2}}{(m-1)^2}J_1(x) - \int \frac{x^{-m+2}}{(m-1)^2}J_0(x)\,dx \\ &= -\frac{J_0(x)}{(m-1)x^{m-1}} + \frac{J_1(x)}{(m-1)^2x^{m-2}} - \frac{1}{(m-1)^2}\int \frac{J_0(x)}{x^{m-2}}\,dx\end{aligned}$$

(iii) $$\begin{aligned}\int x^m J_1(x)\,dx &= x^m\{-J_0(x)\} + \int J_0(x)\,m\,x^{m-1}\,dx \\ &= -x^m J_0(x) + m\int x^{m-1}J_0(x)\,dx\end{aligned}$$

(iv) $\int x^{-m} J_1(x)\, dx = \int x^{-m-1} . x J_1(x)\, dx$

$$= \frac{x^{-m}}{-m} . x J_1(x) - \int \frac{x^{-m}}{-m} x J_0(x)\, dx$$

$$= -\frac{J_1(x)}{m\, x^{m-1}} + \frac{1}{m} \int \frac{J_0(x)}{x^{m-1}}\, dx$$

PROBLEM (32): Show that $J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta)\, d\theta$ satisfies Bessel's differential equation of order zero i.e. $y'' + \frac{1}{x} y' + y = 0$

SOLUTION: Let $y = J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta)\, d\theta$ (1)

then $y' = J_0'(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^{\pi} \cos(x \sin\theta)\, d\theta$

Using the rule for differentiating under the integral sign, we get

$$y' = J_0'(x) = \frac{1}{\pi} \int_0^{\pi} -\sin(x \sin\theta) . \sin\theta\, d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(x \sin\theta) . (-\sin\theta)\, d\theta \qquad (2)$$

Integrating by parts, we have

$$y' = \frac{1}{\pi} \left[\left| \sin(x \sin\theta) \cos\theta \right|_0^{\pi} - \int_0^{\pi} \cos\theta . \cos(x \sin\theta)\, x \cos\theta\, d\theta \right]$$

$$= -\frac{x}{\pi} \int_0^{\pi} \cos(x \sin\theta) \cos^2\theta\, d\theta$$

or $\frac{1}{x} y' = -\frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta) . \cos^2\theta\, d\theta$

Also, from equation (2), we have

$$y'' = J_0''(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^{\pi} \sin(x\sin\theta).(-\sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)(-\sin^2\theta)\, d\theta$$

$$= -\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)(\sin^2\theta)\, d\theta \qquad (3)$$

Substituting equations (1), (2), and (3) in the Bessel's differential equation, we get

$$y'' + \frac{1}{x}y' + y = -\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\sin^2\theta\, d\theta$$

$$-\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos^2\theta\, d\theta + \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\left[-(\sin^2\theta + \cos^2\theta) + 1\right] d\theta$$

$$= 0$$

Thus $J_0(x)$ is a solution of the given Bessel's differential equation.

EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

PROBLEM (33): **Transform the equation $4x^2y'' + 4xy' + (64x^2 - 9)y = 0$ to Bessel's equation using the transformation $4x = t$ and hence find the general solution.**

SOLUTION: Since $t = 4x$, therefore $\frac{dt}{dx} = 4$

Now $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 4\frac{dy}{dt}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$

$$= \frac{d}{dt}\left(4\frac{dy}{dt}\right)4 = 16\frac{d^2y}{dt^2}$$

Thus the given equation becomes

$$4\left(\frac{t^2}{16}\right)(16)\frac{d^2y}{dt^2} + 4\left(\frac{t}{4}\right)(4)\frac{dy}{dt} + \left[64\left(\frac{t^2}{16}\right) - 9\right]y = 0$$

or $$4t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+(4t^2-9)y=0$$

or $$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+\left(t^2-\frac{9}{4}\right)y=0$$

which is a Bessel's differential equation with $n=\frac{3}{2}$. Hence the general solution is given by

$$y(t)=AJ_{3/2}(t)+BJ_{-3/2}(t)$$

or $$y(x)=AJ_{3/2}(4x)+BJ_{-3/2}(4x)$$

PROBLEM (34): Transform the differential equation $xy''+y'+y=0$ to Bessel's equation using the transformation $t=2\sqrt{x}$ and hence find the general solution.

SOLUTION: Since $t=2\sqrt{x}$, therefore $\frac{dt}{dx}=\frac{1}{\sqrt{x}}=\frac{2}{t}$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{2}{t}\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}=\frac{d}{dt}\left(\frac{2}{t}\frac{dy}{dt}\right)\frac{2}{t}$$

$$=\left(\frac{2}{t}\frac{d^2y}{dt^2}-\frac{2}{t^2}\frac{dy}{dt}\right)\frac{2}{t}=\frac{4}{t^2}\frac{d^2y}{dt^2}-\frac{4}{t^3}\frac{dy}{dt}$$

Substituting into the given differential equation, we get

$$\frac{t^2}{4}\left(\frac{4}{t^2}\frac{d^2y}{dt^2}-\frac{4}{t^3}\frac{dy}{dt}\right)+\frac{2}{t}\frac{dy}{dt}+y=0$$

or $$\frac{d^2y}{dt^2}-\frac{1}{t}\frac{dy}{dt}+\frac{2}{t}\frac{dy}{dt}+y=0$$

or $$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}+y=0$$

or $$t^2\frac{dy}{dt}+t\frac{dy}{dt}+t^2y=0$$

which is a Bessel's differential equation of order zero. Hence its general solution is given by

$$y(t)=AJ_0(t)+BY_0(t)$$

or $$y(x)=AJ_0\left(2\sqrt{x}\right)+BY_0\left(2\sqrt{x}\right)$$

PROBLEM (35): Show that Bessel's differential equation

$$x^2y''+xy'+(x^2-n^2)y=0$$

can be reduced to the normal form $u''+\left(1-\dfrac{n^2-\frac{1}{4}}{x^2}\right)u=0$

using the substitution $y=x^{-1/2}u$. Discuss the case where $n=\pm\frac{1}{2}$

The normal form is one without the first derivative.

SOLUTION: Since $y = x^{-1/2}u$, therefore

$$y' = x^{-1/2}u' - \frac{1}{2}x^{-3/2}u$$

and
$$y'' = x^{-1/2}u'' - \frac{1}{2}x^{-3/2}u' - \frac{1}{2}x^{-3/2}u' + \frac{3}{4}x^{-5/2}u$$
$$= x^{-1/2}u'' - x^{-3/2}u' + \frac{3}{4}x^{-5/2}u$$

Substituting the values of y, y', and y'' in the Bessel's equation, we get

$$x^2\left(x^{-1/2}u'' - x^{-3/2}u' + \frac{3}{4}x^{-5/2}u'\right) + x\left(x^{-1/2}u' - \frac{1}{2}x^{-3/2}u\right) + (x^2 - n^2)x^{-1/2}u = 0$$

or
$$x^{3/2}u'' - x^{1/2}u' + \frac{3}{4}x^{-1/2}u + x^{1/2}u' - \frac{1}{2}x^{-1/2}u + x^{3/2}u - n^2x^{-1/2}u = 0$$

or
$$x^{3/2}u'' + \frac{1}{4}x^{-1/2}u + x^{3/2}u - n^2x^{-1/2}u = 0$$

or
$$u'' + \frac{1}{4}\frac{u}{x^2} + u - n^2\frac{u}{x^2} = 0$$

or
$$u'' + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right)u = 0 \qquad (1)$$

When $n = \pm\frac{1}{2}$, equation (1) reduces $u'' + u = 0$, whose solution is

$$u = A\cos x + B\sin x$$

or
$$x^{1/2}y = A\cos x + B\sin x$$

or
$$y = A\frac{\cos x}{\sqrt{x}} + B\frac{\sin x}{\sqrt{x}}$$ is the required solution.

PROBLEM (36): Reduce the differential equation

$$x y'' + (1 + 2n)y' + xy = 0$$

to Bessel's equation using the substitution $y = x^{-n}u$ where n is real. Hence find the general solution in terms of Bessel functions.

SOLUTION: Since $y = x^{-n}u$, therefore

$$y' = x^{-n}u' - nx^{-n-1}u$$

and
$$y'' = x^{-n}u'' - nx^{-n-1}u' - x^{-n-1}u' + n(n+1)x^{-n-2}u$$
$$= x^{-n}u'' - 2nx^{-n-1}u' + n(n+1)x^{-n-2}u$$

Substituting the values of y, y', and y'' into the given differential equation, we get

$$x\left[x^{-n}u'' - 2nx^{-n-1}u' + n(n+1)x^{-n-2}u\right] + (1+2n)\left[x^{-n}u' - nx^{-n-1}u\right] + x.x^{-n}u = 0$$

or
$$x^{-n+1}u'' - 2nx^{-n}u' + n(n+1)x^{-n-1}u + x^{-n}u' - nx^{-n-1}u$$
$$+ 2nx^{-n}u' - 2n^2x^{-n-1}u + x^{-n+1}u = 0$$

or $\quad x^{-n+1}u'' - n^2 x^{-n-1}u + x^{-n}u' + x^{-n+1}u = 0$

Dividing throughout by x^{-n-1}, we get

$$x^2 u'' - n^2 u + x u' + x u' + x^2 u = 0$$

or $\quad x^2 u'' + x u' + (x^2 - n^2) u = 0$

which is a Bessel's differential equation whose general solution can be written as

$$u = A J_n(x) + B J_{-n}(x)$$

or $\quad x^n y = A J_n(x) + B J_{-n}(x)$

or $\quad y = x^{-n}[A J_n(x) + B J_{-n}(x)] \quad$ if $n \neq 0, 1, 2, \ldots\ldots\ldots$

or $\quad y = x^{-n}[A J_n(x) + B Y_n(x)] \quad$ if $n = 0, 1, 2, \ldots\ldots\ldots$

PROBLEM (37): **(i) Reduce the general differential equation**

$$x^2 y'' + (1 - 2a) x y' + [b^2 c^2 x^{2c} + (a^2 - n^2 c^2)] y = 0$$

where a, b, c, and n are constants, to Bessel's equation using the substitutions $y = x^a u$, $b x^c = t$. Hence find its general solution.

(ii) Find the general solution of $x^2 y'' - 3 x y' + x^2 y = 0$.

SOLUTION: Since $y = x^a u$, therefore $y' = x^a u' + a x^{a-1} u$

and $\quad y'' = x^a u'' + a x^{a-1} u' + a x^{a-1} u' + a(a-1) x^{a-2} u$

$$= x^a u'' + 2a x^{a-1} u' + a(a-1) x^{a-2} u$$

Thus the given equation becomes

$$x^2[x^a u'' + 2a x^{a-1} u' + a(a-1) x^{a-2} u] + (1-2a) x [x^a u' + a x^{a-1} u] + [b^2 c^2 x^{2c} + (a^2 - n^2 c^2)] x^a u = 0$$

$$x^{a+2} u'' + 2a x^{a+1} u' + a(a-1) x^a u + x^{a+1} u' + a x^a u - 2a x^{a+1} u' - 2a^2 x^a u + [b^2 c^2 x^{2c} + (a^2 - n^2 c^2)] x^a u = 0$$

$$x^{a+2} u'' + x^{a+1} u' + (b^2 c^2 x^{2c} - n^2 c^2) x^a u = 0$$

or $\quad x^2 u'' + x u' + (b^2 c^2 x^{2c} - n^2 c^2) u = 0 \qquad (1)$

Next, let $b x^c = t$, then $\dfrac{dt}{dx} = b c x^{c-1} = c\dfrac{t}{x}$

$$\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \frac{du}{dt} c \frac{t}{x}$$

or $\quad x\dfrac{du}{dx} = c t \dfrac{du}{dt}$

and
$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx}$$
$$= \left[\frac{d}{dt}\left(c\frac{t}{x}\frac{du}{dt}\right)\right]\left(c\frac{t}{x}\right)$$
$$= c^2\left[\frac{t}{x}\frac{d^2u}{dt^2}+\frac{1}{x}\frac{du}{dt}-\frac{t}{x^2}\left(\frac{dx}{dt}\right)\frac{du}{dt}\right]\left(\frac{t}{x}\right)$$
$$= c^2\left[\frac{t^2}{x^2}\frac{d^2u}{dt^2}+\frac{t}{x^2}\frac{du}{dt}-\frac{t^2}{x^3}\left(\frac{1}{c}\frac{x}{t}\right)\frac{du}{dt}\right]$$
$$= \frac{c^2}{x^2}\left[t^2\frac{d^2u}{dt^2}+t\frac{du}{dt}-\frac{t}{c}\frac{du}{dt}\right]$$

or
$$x^2\frac{d^2u}{dx^2} = c^2t^2\frac{du^2}{dt^2}+c^2t\frac{du}{dt}-ct\frac{du}{dt}$$

Thus equation (1) becomes
$$c^2t^2\frac{d^2u}{dt^2}+c^2t\frac{du}{dt}-ct\frac{du}{dt}+ct\frac{du}{dt}+(c^2t^2-n^2c^2)u = 0$$

or
$$t^2\frac{d^2u}{dt^2}+t\frac{du}{dt}+(t^2-n^2)u = 0 \qquad (2)$$

which is the Bessel's differential equation of order n.

The solution of equation (2) is given by
$$u = AJ_n(t)+BJ_{-n}(t), \quad n \neq 0,1,2,\ldots\ldots\ldots$$

or
$$x^{-a}y = AJ_n(bx^c)+BJ_{-n}(bx^c)$$

or
$$y = x^a\left[AJ_n(bx^c)+BJ_{-n}(bx^c)\right] \qquad (3)$$

If $n = 0,1,2,\ldots\ldots\ldots$, the solution (3) may be written as
$$y = x^a\left[AJ_n(bx^c)+BY_n(bx^c)\right] \qquad (4)$$

(ii) $x^2y''-3xy'+x^2y = 0$

Comparing with the general form, we have
$$1-2a = -3, \quad b^2c^2 = 1, \quad 2c = 2, \quad a^2-n^2c^2 = 0$$

or $a = 2, \quad b = c = 1, \quad$ and $\quad n = 2$

Thus the solution (4) becomes
$$y = x^2\left[AJ_2(x)+BY_2(x)\right]$$

SERIES OF BESSEL FUNCTIONS

PROBLEM (38): **Expand $f(x) = x, \quad 0 < x < 1$ in a Bessel series**

$$\sum_{p=1}^{\infty} A_pJ_1(\lambda_px)$$ **where λ_p are the positive roots of $J_1(x) = 0$.**

SOLUTION: From theorem (11.22), we have

$$f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)$$

where $A_p^* = \dfrac{2}{J_{n+1}^2(\lambda_p)} \displaystyle\int_0^1 x J_n(\lambda_p x) f(x)\, dx$

When $n = 1$, we get

$$f(x) = \sum_{p=1}^{\infty} A_p J_1(\lambda_p x) \tag{1}$$

where $A_p = \dfrac{2}{J_2^2(\lambda_p)} \displaystyle\int_0^1 x^2 J_1(\lambda_p x)\, dx$

Let $\lambda_p x = v$, therefore $dx = \dfrac{1}{\lambda_p} dv$. Also $0 \le v \le \lambda_p$

$$\text{Therefore } A_p = \frac{2}{J_2^2(\lambda_p)} \int_0^{\lambda_p} \frac{v^2}{\lambda_p^2} J_1(v) \frac{1}{\lambda_p}\, dv$$

$$= \frac{2}{\lambda_p^3 J_2^2(\lambda_p)} \int_0^{\lambda_p} v^2 J_1(v)\, dv$$

$$= \frac{2}{\lambda_p^3 J_2^2(\lambda_p)} \left| v^2 J_2(v) \right|_0^{\lambda_p}$$

$$= \frac{2}{\lambda_p^3 J_2^2(\lambda_p)} \lambda_p^2 J_2(\lambda_p) = \frac{2}{\lambda_p J_2(\lambda_p)}$$

Thus we have the required Bessel series

$$f(x) = x = \sum_{p=1}^{\infty} \frac{2}{\lambda_p J_2(\lambda_p)} J_1(\lambda_p x)$$

PROBLEM (39): **Expand $f(x) = 1 - x^2$, $0 < x < 1$ in a Bessel series**

$$\sum_{p=1}^{\infty} A_p J_0(\lambda_p x)$$ **where λ_p are the positive roots of $J_1(x) = 0$.**

SOLUTION: From theorem (11.22), we have

$$f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)$$

where $A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x) f(x)\, dx$

If $n = 0$, we get

$$f(x) = \sum_{p=1}^{\infty} A_p J_0(\lambda_p x) \tag{1}$$

where $A_p = \frac{2}{J_1^2(\lambda_p)} \int_0^1 x J_0(\lambda_p x)(1-x^2)\, dx$

$$= \frac{2}{J_1^2(\lambda_p)} \int_0^1 [x J_0(\lambda_p x) - x^3 J_0(\lambda_p x)\, dx$$

Let $\lambda_p x = v$, therefore $dx = \frac{dv}{\lambda_p}$. Also $0 \le v \le \lambda_p$

$$\text{Thus}\quad A_p = \frac{2}{J_1^2(\lambda_p)} \int_0^{\lambda_p} \left[\frac{1}{\lambda_p} v J_0(v) - \frac{1}{\lambda_p^3} v^3 J_0(v)\right] \frac{1}{\lambda_p}\, dv \tag{2}$$

$$J_2(\lambda_p) = \frac{2}{\lambda_p} J_1(\lambda_p) - J_0(\lambda_p)$$

Since $\int v J_0(v)\, dv = v J_1(v)$

and $\int v^3 J_0(v) = v^3 J_1(v) - 2v^2 J_2(v)$

therefore equation (2) becomes

$$A_p = \frac{2}{\lambda_p J_1^2(\lambda_p)} \left| \frac{1}{\lambda_p} v J_1(v) - \frac{1}{\lambda_p^3} v^3 J_1(v) + \frac{2}{\lambda_p^3} v^2 J_2(v) \right|_0^{\lambda_p}$$

$$= \frac{2}{\lambda_p J_1^2(\lambda_p)} \left| J_1(\lambda_p) - J_1(\lambda_p) + \frac{2}{\lambda_p^3} \lambda_p^2 J_2(\lambda_p) \right|$$

$$= \frac{2}{\lambda_p J_1^2(\lambda_p)} \frac{2}{\lambda_p} J_2(\lambda_p) = \frac{4 J_2(\lambda_p)}{\lambda_p^2 J_1^2(\lambda_p)}$$

Thus from equation (1), we get

$$1 - x^2 = \sum_{p=1}^{\infty} \frac{4 J_2(\lambda_p) J_0(\lambda_p x)}{\lambda_p^2 J_1^2(\lambda_p)} \tag{1}$$

ALTERNATIVE FORM

Since $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$, therefore

$$J_2(\lambda p) = \frac{2}{\lambda_p}J_1(\lambda_p) - J_0(\lambda_p)$$

But $J_0(\lambda_p) = 0$ since λ_p are the roots $J_0(x) = 0$.

Thus $J_2(\lambda_p) = \frac{2}{\lambda_p}J_1(\lambda_p)$

Hence equation (1) takes the form

$$1 - x^2 = \sum_{p=1}^{\infty} \frac{4}{\lambda_p^2 J_1^2(\lambda_p)} \frac{2}{\lambda_p} J_1(\lambda_p) J_0(\lambda_p x)$$

or $$\frac{1-x^2}{8} = \sum_{p=1}^{\infty} \frac{J_0(\lambda_p x)}{\lambda_p^3 J_1(\lambda_p)}$$

PROBLEM (40): **Expand $f(x) = \ln x$, $0 < x < 1$ in a Bessel series**

$$\sum_{p=1}^{\infty} A_p J_0(\lambda_p x)$$ **where λ_p are the positive roots of $J_0(x) = 0$.**

SOLUTION: We have from theorem (11.22) that

$$f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)$$

where $$A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x) f(x)\, dx$$

If $n = 0$, the above series takes the form

$$\ln x = \sum_{p=1}^{\infty} A_p J_0(\lambda_p x) \tag{1}$$

where $$A_p = \frac{2}{J_1^2(\lambda_p)} \int_0^1 x J_0(\lambda_p x) \ln x\, dx$$

$$= \frac{2}{J_1^2(\lambda_p)} \int_0^1 \ln x \,.\, x J_0(\lambda_p x)\, dx$$

Let $\lambda_p x = v$, so that $dx = \frac{1}{\lambda_p} dv$. Also $0 \le v \le \lambda_p$

Thus
$$A_p = \frac{2}{J_1^2(\lambda_p)} \int_0^{\lambda_p} \ln\left(\frac{v}{\lambda_p}\right) \frac{1}{\lambda_p} v J_0(v) . \frac{1}{\lambda_p} dv$$

$$= \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \int_0^{\lambda_p} \ln\left(\frac{v}{\lambda_p}\right) v . J_0(v)\, dv$$

$$= \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \left[\left| \ln\left(\frac{v}{\lambda_p}\right) . v J_1(v) \right|_0^{\lambda_p} - \int_0^{\lambda_p} v J_1(v) . \left(\frac{\lambda_p}{v} . \frac{1}{\lambda_p}\right) dv \right]$$

$$= -\frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \int_0^{\lambda_p} J_1(v)\, dv = -\frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \int_0^{\lambda_p} -J_0'(v)\, dv$$

$$= \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} \left| J_0(v) \right|_0^{\lambda_p} = \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} [J_0(\lambda_p) - J_0(0)]$$

$$= \frac{2}{\lambda_p^2 J_1^2(\lambda_p)} (0 - 1) \qquad [\text{since } J_0(\lambda_p) = 0]$$

$$= -\frac{2}{\lambda_p^2 J_1^2(\lambda_p)}$$

This from equation (1), we get

$$\ln x = \sum_{p=1}^{\infty} \frac{-2}{\lambda_p^2 J_1^2(\lambda_p)} J_0(\lambda_p x)$$

or
$$-\frac{1}{2} \ln x = \sum_{p=1}^{\infty} \frac{J_0(\lambda_p x)}{\lambda_p^2 J_1^2(\lambda_p)}$$

WRONSKIAN OF HANKEL FUNCTIONS

PROBLEM (41): **Find the value of the Wronskian of the Hankel functions.**

SOLUTION: We know that

$$H_n^{(1)}(x) = J_n(x) + i Y_n(x)$$

and
$$H_n^{(2)}(x) = J_n(x) - i Y_n(x)$$

By definition, the Wronskian is given by

$$W = \begin{vmatrix} H_n^{(1)}(x) & H_n^{(2)}(x) \\ \frac{d}{dx} H_n^{(1)}(x) & \frac{d}{dx} H_n^{(2)}(x) \end{vmatrix}$$

$$= H_n^{(1)}(x)\frac{d}{dx}H_n^{(2)}(x) - H_n^{(2)}(x)\frac{d}{dx}H_n^{(1)}(x)$$

$$= [J_n(x)+iY_n(x)][J_n'(x)-iY_n'(x)] - [J_n(x)-iY_n(x)][J_n'(x)+iY_n'(x)]$$

$$= -iJ_n(x)Y_n'(x)+iY_n(x)J_n'(x)-iJ_n(x)Y_n'(x)+iY_n(x)J_n'(x)$$

$$= -2i[J_n(x)Y_n'(x)-Y_n(x)J_n'(x)]$$

$$= -2i\left(\frac{2}{\pi x}\right) \quad \left[\text{using theorem (11.15)}\right]$$

$$= -\frac{4i}{\pi x}$$

PROBLEM (42): **Prove that**

(i) $H_{-1/2}^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\,e^{ix}$ (ii) $H_{-1/2}^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\,e^{-ix}$

SOLUTION: We know that

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) \tag{1}$$

$$H_n^{(2)}(x) = J_n(x) - iY_n(x) \tag{2}$$

Let $n = -\frac{1}{2}$ in equations (1) and (2), we get

$$H_{-1/2}^{(1)}(x) = J_{-1/2}(x) + iY_{-1/2}(x) \tag{3}$$

and $$H_{-1/2}^{(2)}(x) = J_{-1/2}(x) - iY_{-1/2}(x) \tag{4}$$

Now $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ and $Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

Substituting these in equations (3) and (4), we get

$$H_{-1/2}^{(1)}(x) = \sqrt{\frac{2}{\pi x}}(\cos x + i\sin x) = \sqrt{\frac{2}{\pi x}}\,e^{i\theta}$$

and $$H_{-1/2}^{(2)}(x) = \sqrt{\frac{2}{\pi x}}(\cos x - i\sin x) = \sqrt{\frac{2}{\pi x}}\,e^{-i\theta}$$

MODIFIED BESSEL FUNCTIONS

PROBLEM (43): **Show that** $\frac{1}{4}xI_1(x) = I_2(x) + 2I_4(x) + 3I_6(x) + \ldots\ldots\ldots$

SOLUTION: We know that

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x}I_n(x)$$

or $$x[I_{n-1}(x) - I_{n+1}(x)] = 2nI_n(x) \tag{1}$$

Let $n = 2, 4, 6, \ldots\ldots$ in equation (1), we get

$$x\left[I_1(x) - I_3(x)\right] = 4\,I_2(x)$$

$$x\left[I_3(x) - I_5(x)\right] = 8\,I_4(x)$$

$$x\left[I_5(x) - I_7(x)\right] = 12\,I_6(x)$$

Adding, we get

$$x\,I_1(x) = 4\left[I_2(x) + 2\,I_4(x) + 3\,I_6(x) + \ldots\ldots\right]$$

or

$$\frac{1}{4}x\,I_1(x) = I_2(x) + 2\,I_4(x) + 3\,I_6(x) + \ldots\ldots$$

PROBLEM (44): **Prove that**

$$e^{x\cos\theta} = I_0(x) + 2\,I_1(x)\cos\theta + 2\,I_2(x)\cos 2\theta + 2\,I_3(x)\cos 3\theta + \ldots\ldots$$

SOLUTION: We know that

$$e^{\frac{x}{2}\left(t + \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x)\,t^n \qquad (1)$$

Let $x = e^{i\theta}$ in equation (1), we get

$$e^{\frac{x}{2}\left(e^{i\theta} + e^{-i\theta}\right)} = \sum_{n=-\infty}^{\infty} I_n(x)\,e^{in\theta}$$

or

$$e^{x\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(x)\left(\cos n\theta + i\sin n\theta\right)$$

$$= I_0(x) + I_1(x)(\cos\theta + i\sin\theta) + I_{-1}(x)(\cos\theta - i\sin\theta)$$
$$+ I_2(x)(\cos 2\theta + i\sin 2\theta) + I_{-2}(x)(\cos 2\theta - i\sin 2\theta)$$
$$+ I_3(x)(\cos 3\theta + i\sin 3\theta) + I_{-3}(x)(\cos 3\theta - i\sin 3\theta) + \ldots\ldots$$

Since $I_{-n}(x) = I_n(x)$, therefore

$$e^{x\cos\theta} = I_0(x) + 2\,I_1(x)\cos\theta + 2\,I_2(x)\cos 2\theta + 2\,I_3(x)\cos 3\theta + \ldots\ldots$$

PROBLEM (45): **Reduce the differential equation**

$$x^2 y'' + (1 + n)\,x\,y' - \lambda^2 x\,y = 0$$

to modified Bessel's equation using the substitution $y = x^{-n/2}u$ and find the general solution in terms of modified Bessel functions.

SOLUTION: Since $y = x^{-n/2}u$, therefore

$$y' = x^{-n/2}u' - \frac{n}{2}x^{-\frac{n}{2}-1}u$$

$$y'' = x^{-n/2}u'' - \frac{n}{2}x^{-\frac{n}{2}-1}u' - \frac{n}{2}\left[x^{-\frac{n}{2}-1}u' + \left(-\frac{n}{2}-1\right)x^{-\frac{n}{2}-2}u\right]$$

$$= x^{-n/2}u'' - n\,x^{-\frac{n}{2}-1}u' + \frac{n}{2}\left(\frac{n}{2}+1\right)x^{-\frac{n}{2}-2}u$$

Substituting the values of y, y', and y'' in the given differential equation, we get

$$x^2\left[x^{-n/2}u'' - n\,x^{-\frac{n}{2}-1}u' + \frac{n}{2}\left(\frac{n}{2}+1\right)x^{-\frac{n}{2}-2}u\right]$$
$$+(1+n)x\left[x^{-n/2}u' - \frac{n}{2}x^{-\frac{n}{2}-1}u\right] - \lambda^2 x \cdot x^{-n/2}u = 0$$

or $$x^2\left[u'' - n\,x^{-1}u' + \frac{n}{2}\left(\frac{n}{2}+1\right)x^{-2}u\right] + (1+n)x\left[u' - \frac{n}{2}x^{-1}u\right] - \lambda^2 x u = 0$$

or $$x^2u'' - n\,x\,u' + \frac{n}{2}\left(\frac{n}{2}+1\right)u + (1+n)x\,u' - \frac{n}{2}(1+n)u - \lambda^2 x u = 0$$

or $$x^2u'' + x\,u' + \frac{n^2}{4}u + \frac{n}{2}u - \frac{n}{2}u - \frac{n^2}{2}u - \lambda^2 x u = 0$$

or $$x^2u'' + x\,u' - \frac{n^2}{4}u - \lambda^2 x u = 0$$

or $$x^2u'' + x\,u' - \left(\lambda^2 x + \frac{n^2}{4}\right)u = 0 \qquad (1)$$

Now let $t = \sqrt{x}$, then $\dfrac{dt}{dx} = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2t}$

and $$\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \frac{1}{2t}\frac{du}{dt}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{2t}\frac{du}{dt}\right)\frac{1}{2t}$$

$$= \left(\frac{1}{2t}\frac{d^2u}{dt^2} - \frac{1}{2t^2}\frac{du}{dt}\right)\frac{1}{2t}$$

$$= \frac{1}{4t^2}\frac{d^2u}{dt^2} - \frac{1}{4t^3}\frac{du}{dt}$$

Thus equation (1) becomes

$$t^4\left(\frac{1}{4t^2}\frac{d^2u}{dt^2} - \frac{1}{4t^3}\frac{du}{dt}\right) + t^2\left(\frac{1}{2t}\frac{du}{dt}\right) - \left(\lambda^2t^2 + \frac{n^2}{4}\right)u = 0$$

$$\frac{1}{4}t^2\frac{d^2u}{dt^2}-\frac{1}{4}t\frac{du}{dt}+\frac{1}{2}t\frac{du}{dt}-\left(\lambda^2t^2+\frac{n^2}{4}\right)u = 0$$

or

$$t^2\frac{d^2u}{dt^2}+t\frac{du}{dt}-(4\lambda^2t^2+n^2)u = 0$$

or

which is the modified Bessel's differential equation whose solution is given by

$$u = A I_n(2\lambda t)+B I_{-n}(2\lambda t)$$

or

$$u = A I_n(2\lambda\sqrt{x})+B I_{-n}(2\lambda\sqrt{x})$$

or

$$y = x^{-n/2}[A I_n(2\lambda\sqrt{x})+B I_{-n}(2\lambda\sqrt{x})], \qquad (\text{n not an integer})$$

or

$$y = x^{-n/2}[A I_n(2\lambda\sqrt{x})+B K_n(2\lambda\sqrt{x})], \qquad (\text{n an integer})$$

PROBLEM (46): **Reduce the differential equation**

$$x^4y''+(e^{2/x}-n^2)y = 0$$

to Bessel's equation using the substitution $y = xu$ and then $t = e^{1/x}$. Hence find the general solution .

SOLUTION: Since $y = xu$, therefore

$$y' = xu'+u \text{ and } y'' = xu''+u'+u' = xu''+2u'$$

Substituting into the given differential equation , we get

$$x^4(xu''+2u')+(e^{2/x}-n^2)xu = 0$$

or

$$x^5u''+2x^4u'+(e^{2/x}-n^2)xu = 0 \qquad (1)$$

Now let $t = e^{1/x}$ or $\frac{1}{x} = \ln t$, then

$$u' = \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = e^{1/x}\left(-\frac{1}{x^2}\right)\frac{du}{dt} = -t\ln^2 t\frac{du}{dt}$$

and

$$u'' = \frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-t\ln^2 t\frac{du}{dt}\right)(-t\ln^2 t)$$

$$= \left[\frac{d}{dt}\left(t\ln^2 t\frac{du}{dt}\right)\right](t\ln^2 t)$$

$$= \left[t\ln^2 t\frac{d^2u}{dt^2}+(2t\ln t\frac{1}{t}+\ln^2 t)\frac{du}{dt}\right](t\ln^2 t)$$

$$= t^2\ln^4 t\frac{d^2u}{dt^2}+(2t\ln^3 t+t\ln^4 t)\frac{du}{dt}$$

Thus equation (I) becomes

$$\frac{1}{\ln^3 t}\left[t^2 \ln^4 t \frac{d^2 u}{d t^2}+(2 t \ln^3 t+t \ln^4 t) \frac{d u}{d t}\right]+\frac{2}{\ln^4 t}\left(-t \ln^2 t \frac{d u}{d t}\right)+(t^2-n^2) \frac{1}{\ln t} u=0$$

$$\frac{1}{\ln t} t^2 \frac{d^2 u}{d t^2}+\frac{2 t}{\ln^2 t} \frac{d u}{d t}+\frac{t}{\ln t} \frac{d u}{d t}-\frac{2 t}{\ln^2 t} \frac{d u}{d t}+(t^2-n^2) \frac{1}{\ln t} u=0$$

or $$t^2 \frac{d^2 u}{d t^2}+t \frac{d u}{d t}+(t^2-n^2) u=0$$

which is a Bessel's differential equation of order n, whose solution is given by

$$u=A J_n(t)+B J_{-n}(t)$$

or $$u=A J_n(e^{1/x})+B J_{-n}(e^{1/x})$$

or $$y=x\left[A J_n(e^{1/x})+B J_{-n}(e^{1/x})\right] \quad (n \text{ not an integer})$$

or $$y=x\left[A J_n(e^{1/x})+B Y_n(e^{1/x})\right] \quad (n \text{ an integer})$$

11.45 EXERCISE

PROBLEM (1): If $J_0(2) = a$ and $J_1(2) = b$, show that

(i) $J_2(2) = b - a$ (ii) $J_1'(2) = a - \frac{1}{2}b$ (iii) $J_2'(2) = a$

PROBLEM (2): Prove that

(i) $J_1''(x) = \frac{1}{x}J_2(x) - J_1(x) = \left(\frac{2}{x^2} - 1\right)J_1(x) - \frac{1}{x}J_0(x)$

(ii) $J_3'(x) = \left(\frac{12}{x^2} - 1\right)J_0(x) - \left(\frac{24}{x^3} - \frac{5}{x}\right)J_1(x)$

PROBLEM (3): Prove that

$\frac{d}{dx}[x J_0'(x)] = -x J_0(x)$ [Hint: $J_0'(x) = -J_1(x)$]

PROBLEM (4): Prove that

(i) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$

(ii) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right]$

PROBLEM (5): Prove that

(i) $Y_{5/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right]$

(ii) $Y_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$

PROBLEM (6): Show that $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$

PROBLEM (7): Prove that $J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x = \sqrt{\frac{2}{\pi x^3}}$

PROBLEM (8): Show that $[J_{3/2}(x)]^2 + [J_{-3/2}(x)]^2 = \frac{2}{\pi x}\left(\frac{1 + x^2}{x^2}\right)$

PROBLEM (9): Show that $J_{5/2}(x)\sin x + J_{-5/2}(x)\cos x = \sqrt{\frac{2}{\pi x}}\left(\frac{3}{x^2} - 1\right)$

PROBLEM (10): Show that

(i) $1 = J_0(x) + 2J_2(x) + 2J_4(x) + \ldots\ldots\ldots$

(ii) $x = 2[J_1(x) + 3J_3(x) + 5J_5(x) + \ldots\ldots]$

(iii) $x^2 = 2[4J_2(x) + 16J_4(x) + 36J_6(x) + \ldots\ldots]$

PROBLEM (11): Prove that

(i) $J_1(x) - J_3(x) + J_5(x) - J_7(x) + \dots\dots\dots = 0$

(ii) $\frac{1}{4} x J_1(x) = J_2(x) - 2J_4(x) + 3J_6(x) - \dots\dots\dots$

INTEGRALS INVOLVING BESSEL FUNCTIONS

PROBLEM (12): Prove that

(i) $\int x J_0(x)\, dx = x J_1(x) + C$

(ii) $\int x^2 J_0(x)\, dx = x^2 J_1(x) + x J_0(x) - \int J_0(x)\, dx + C$

(iii) $\int x^4 J_0(x)\, dx = (x^4 - 9x^2) J_1(x) + (3x^3 - 9x) J_0(x) + 9 \int J_0(x)\, dx + C$

$= (x^4 - 3x^2) J_1(x) - 3x^3 J_2(x) - 9x J_0(x) + 9 \int J_0(x)\, dx + C$

(iv) $\int x^5 J_0(x)\, dx = x^5 J_1(x) - 4x^4 J_2(x) + 8x^3 J_3(x) + C$

$= (x^5 - 16x^3 + 64x) J_1(x) + (4x^4 - 32x^2) J_0(x) + C$

(v) $\int x J_1(x)\, dx = -x J_0(x) + \int J_0(x)\, dx + C$

(vi) $\int x^2 J_1(x)\, dx = x^2 J_2(x) + C = 2x J_1(x) - x^2 J_0(x) + C$

(vii) $\int x^3 J_1(x)\, dx = x^3 J_2(x) + x^2 J_1(x) + 3x J_0(x) - 3 \int J_0(x)\, dx + C$

$= 3x^2 J_1(x) - (x^3 - 3x) J_0(x) - 3 \int J_0(x)\, dx + C$

PROBLEM (13): Prove that

(i) $\int x J_2(x)\, dx = -x J_1(x) - 2J_0(x) + C$

(ii) $\int x^3 J_2(x)\, dx = x^3 J_3(x) + C = (8x - x^3) J_1(x) - 4x^2 J_0(x) + C$

(iii) $\int x^4 J_2(x)\, dx = -(x^4 - 15x^2) J_1(x) - (5x^3 - 15x) J_0(x) - 15 \int J_0(x)\, dx + C$

(iv) $\int x J_3(x)\, dx = -x J_2(x) - 6J_1(x) + 3 \int J_0(x)\, dx + C$

$= -8J_1(x) + x J_0(x) + 3 \int J_0(x)\, dx + C$

(v) $\int x^2 J_3(x)\, dx = -x^2 J_2(x) - 4x J_1(x) - 8J_0(x) + C$

$= -6x^2 J_1(x) + (x^2 - 8) J_0(x) + C$

(vi) $$\int x^3 J_3(x)\,dx = -x^3 J_2(x) - 5x^2 J_1(x) - 15x J_0(x) + 15\int J_0(x)\,dx$$
$$= -7x^2 J_1(x) - (x^3 + 15x) J_0(x) + 15\int J_0(x)\,dx + C$$

(vii) $$\int x^4 J_3(x)\,dx = x^4 J_4(x) + C$$
$$= (48x - 8x^3) J_1(x) - (24x^2 - x^4) J_0(x) + C$$

PROBLEM (14): Show that

(i) $$\int x^{-2} J_0(x)\,dx = J_1(x) - \frac{J_0(x)}{x} - \int J_0(x)\,dx$$

(ii) $$\int x^{-4} J_0(x)\,dx = \left(\frac{1}{9x^2} - \frac{1}{9}\right) J_1(x) - \left(\frac{1}{3x^3} - \frac{1}{9x}\right) J_0(x) + \frac{1}{9}\int J_0(x)\,dx + C$$

(iii) $$\int x^{-1} J_1(x)\,dx = -J_1(x) + \int J_0(x)\,dx + C$$

(iv) $$\int x^{-3} J_1(x)\,dx = -\left(\frac{1}{3x^2} - \frac{1}{3}\right) J_1(x) - \frac{1}{3x} J_0(x) - \frac{1}{3}\int J_0(x)\,dx + C$$

(v) $$\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x) + C$$

(vi) $$\int x^{-2} J_2(x)\,dx = \frac{1}{3x} J_2(x) - \frac{1}{3} J_1(x) + \frac{1}{3}\int J_0(x)\,dx - C$$
$$= -\left(\frac{2}{3x^2} + \frac{1}{3}\right) J_1(x) + \frac{1}{3x} J_0(x) + \frac{1}{3}\int J_0(x)\,dx + C$$

(vii) $$\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) + C = -\frac{2}{x^3} J_1(x) - \frac{1}{x^2} J_0(x) + C$$

PROBLEM (15): Prove that

(i) $$\int J_0(x) J_1(x)\,dx = -\frac{1}{2} J_0^2(x) + C$$

(ii) $$\int x J_0(x) J_1(x)\,dx = -\frac{1}{2} x J_0^2(x) + \frac{1}{2}\int J_0^2(x)\,dx$$

(iii) $$\int x^2 J_0(x) J_1(x)\,dx = \frac{1}{2} x^2 J_1^2(x) + C$$

PROBLEM (16): Show that

(i) $$\int J_0(x)\sin x\,dx = x J_0(x)\sin x - x J_1(x)\cos x + C$$

(ii) $$\int J_1(x)\cos x\,dx = x J_1(x)\cos x - (x\sin x + \cos x) J_0(x) + C$$

PROBLEM (17): Prove that

(i) $$\int_0^{\pi/2} \sqrt{\pi x}\, J_{1/2}(2x)\, dx = 1$$ (ii) $$\int_0^{\infty} J_1(x)\, dx = 1$$

PROBLEM (18): If a is any root of $J_0(x) = 0$, show that

(i) $$\int_0^{a} J_1(x)\, dx = 1$$ (ii) $$\int_0^{1} J_1(ax)\, dx = \frac{1}{a}$$

PROBLEM (19): If $a\,(\neq 0)$ is any root of $J_1(x) = 0$, show that $\int_0^{1} x\, J_0(ax)\, dx = 0$

PROBLEM (20): $$\int_0^{x} J_0(x)\, dx = 2 \sum_{n=0}^{\infty} J_{2n+1}(x)$$

PROBLEM (21): Prove that

$$\int_0^{\pi/2} \sin(x \sin\theta)\, d\theta = \int_0^{\pi/2} \sin(x\cos\theta)\, d\theta = 2\sum_{n=0}^{\infty} \frac{J_{2n+1}(x)}{2n+1}$$

PROBLEM (22): Prove that $J_0(x) = \frac{2}{\pi}\int_0^{1} \frac{\cos xt}{\sqrt{1-t^2}}\, dt$

PROBLEM (23): Prove that $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\, d\theta$

PROBLEM (24): Prove that $J_0(x) = \frac{2}{\pi}\int_0^{\infty} \sin(x\cosh\theta)\, d\theta$

PROBLEM (25): Prove that $Y_0(x) = -\frac{2}{\pi}\int_0^{\infty} \cos(x\cosh u)\, du = -\frac{2}{\pi}\int_1^{\infty} \frac{\cos xt}{\sqrt{t^2-1}}\, dt$

PROBLEM (26): Prove that $I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\sin\theta}\, d\theta$

PROBLEM (27): Prove that

(i) $$\int_0^\infty \frac{J_n(bx)}{x}\,dx = \frac{1}{n}, \quad n = 1, 2, 3, \ldots\ldots$$

(ii) $$\int_0^\infty e^{-ax} J_0(b\sqrt{x})\,dx = \frac{1}{a} e^{-b^2/4a}$$

PROBLEM (28): Prove that

(i) $$J_n(x) = \frac{x^n}{2^n \sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(x\sin\theta)\cos^{2n}\theta\,d\theta, \quad n > -\frac{1}{2}$$

(ii) $$J_n(x) = \frac{x^n}{2^n \sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(x\cos\theta)\sin^{2n}\theta\,d\theta, \quad n > -\frac{1}{2}$$

PROBLEM (29): Prove that

$$J_n(x) = \frac{x^n}{2^{n-1}\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}} \int_0^1 (1-t^2)^{n-1/2} \cos xt\,dt$$

PROBLEM (30): Prove that

(i) $$\int_0^\infty \sin ax\, J_0(bx)\,dx = \begin{cases} 0 & , \quad b > a \\ \dfrac{1}{\sqrt{a^2+b^2}} & , \quad b < a \end{cases}$$

(ii) $$\int_0^\infty \cos ax\, J_0(bx)\,dx = \begin{cases} \dfrac{1}{\sqrt{b^2-a^2}} & , \quad b > a \\ 0 & , \quad b < a \end{cases}$$

PROBLEM (31): Find the general solutions of the following differential equations in terms of Bessel functions :

(i) $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ (ii) $x^2y'' + xy' + (x^2 - 9)y = 0$

PROBLEM (32): Transform each of the following differential equations to Bessel's equation using the indicated substitution and find the general solution in terms of Bessel functions :

(i) $x^2y'' + xy' + K^2x^2y = 0, \qquad (t = Kx)$

(ii) $y'' + e^{2x}y = 0 \quad (t = e^x)$

(iii) $xy'' + y' + y = 0 \quad (t = 2\sqrt{x})$

(iv) $y'' + \frac{1}{x}y' + 16x^6 y = 0 \quad (t = x^4)$

(v) $x^2y'' + xy' + n^2x^{2n}y = 0 \quad (t = x^n)$

(vi) $x^2y'' + 3xy' + (K^2x^2 + 1)y = 0 \quad (y = x^{-1}u)$

(vii) $x^2y'' + (1 + 2n)xy' + (x^2 + n^2)y = 0 \quad (y = x^{-n}u)$

(viii) $x^2y'' + \left(K^2x^2 + \frac{1}{4}\right)y = 0 \quad (y = x^{1/2}u)$

(ix) $x^2y'' + 2xy' + \left(K^2x^2 + \frac{1}{4}\right)y = 0 \quad (y = x^{-1/2}u)$

PROBLEM (33): Transform each of the following differential equations to Bessel's equation using the indicated substitution and find the general solution in terms of Bessel functions :

(i) $9x^2y'' + 9xy' + (81x^2 - 1)y = 0 \quad (t = 3x)$

(ii) $x^2y'' + xy' + (9x^2 - 4)y = 0 \quad (t = 3x)$

(iii) $x^2y'' + xy' + (4x^4 - 16)y = 0 \quad (t = x^2)$

(iv) $x^2y'' + xy' + \frac{1}{4}(x - 1)y = 0 \quad (t = \sqrt{x})$

(v) $x^2y'' + xy' + 4(x^2 - 1)y = 0 \quad (t = 2x)$

(vi) $x^2y'' + xy' + \left(3x^2 - \frac{1}{4}\right)y = 0 \quad (t = \sqrt{3}\,x)$

(vii) $xy'' - y' + xy = 0, \quad (y = xu)$

(viii) $xy'' + 5y' + xy = 0 \quad (y = x^{-2}u)$

(ix) $4x^2y'' + 2xy' + xy = 0 \quad \left(y = x^{1/4}u,\ \text{then}\ t = \sqrt{x}\right)$

(x) $x^2y'' - 2xy' + 4(x^4 - 1)y = 0 \quad (y = x^{3/2}u,\ \text{then}\ t = x^2)$

PROBLEM (34): Transform each of the following differential equations to Bessel's equation using the indicated substitution and find the general solution in terms of Bessel functions :

(i) $x^2y'' + xy' + 4(x^4 - n^2)y = 0 \quad (t = x^2)$

(ii) $y'' - (a^2e^{2xi} - n^2)y = 0 \quad (t = ae^{ix})$

(iii) $x^2y'' + xy' + 5\left(x^2 - \frac{n^2}{5}\right)y = 0 \quad (t = \sqrt{5}\,x)$

(iv) $x^6y'' + x^5y' + (x^2 - n^2)y = 0 \quad \left(t = \frac{1}{x}\right)$

PROBLEM (35): Find the general solutions of the following differential equations in terms of functions $x^a J_n(b x^c)$ and $x^a J_{-n}(b x^c)$ or $x^a Y_n(b x^c)$.

(i) $x^2 y'' + x y' + \left(4 x^4 - \frac{4}{9}\right) y = 0$

(ii) $x^2 y'' + 3 x y' + \left(16 x^4 - \frac{5}{4}\right) y = 0$

(iii) $x^2 y'' + 5 x y' + \left(81 x^6 + \frac{7}{4}\right) y = 0$

(iv) $x^2 y'' - x y' + (x^2 - 3) y = 0$

(v) $x^2 y'' - 7 x y' + (x^2 + 15) y = 0$

(vi) $x^2 y'' - 3 x y' + (4 x^4 - 60) y = 0$

PROBLEM (36): Expand $f(x) = x^2, \quad 0 < x < 1$ in a Bessel series

$$\sum_{p=1}^{\infty} A_p J_0(\lambda_p x), \quad \text{where } \lambda_p \text{ are the positive roots of } J_0(x) = 0$$

PROBLEM (37): Expand $f(x) = x - x^3, \quad 0 < x < 1$ in a Bessel series

$$\sum_{p=1}^{n} A_p J_1(\lambda_p x), \quad \text{where } \lambda_p \text{ are the positive roots of } J_1(x) = 0$$

PROBLEM (38): Expand $f(x) = x^2, \quad 0 < x < 1$ in a Bessel series

$$\sum_{p=1}^{n} A_p J_2(\lambda_p x), \quad \text{where } \lambda_p \text{ are the positive roots of } J_2(x) = 0$$

MODIFIED BESSEL FUNCTIONS

PROBLEM (39): Prove that

(i) $I_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3}{x^2} + 1 \right) \sinh x - \frac{3}{x} \cosh x \right]$

(ii) $I_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3}{x^2} + 1 \right) \cosh x - \frac{3}{x} \sinh x \right]$

PROBLEM (40): Prove that

(i) $\left(\frac{1}{x} \frac{d}{dx} \right)^m \left[x^{-n} I_n(x) \right] = x^{-(n+m)} I_{n+m}(x)$

(ii) $\left(\frac{1}{x} \frac{d}{dx} \right)^m \left[x^{n} I_n(x) \right] = x^{n-m} I_{n-m}(x)$

PROBLEM (41): Prove that

(i) $$\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{-n} K_n(x)] = (-1)^m x^{-(n+m)} K_{n+m}(x)$$

(ii) $$\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{n} K_n(x)] = (-1)^m x^{n-m} K_{n-m}(x)$$

PROBLEM (42): Show that

(i) $$\int x I_0(x)\, dx = x I_1(x) + C = x I_0'(x) + C$$

(ii) $$\int x I_1(x)\, dx = x I_0(x) - \int I_0(x)\, dx + C$$

(iii) $$\int x^2 I_0(x)\, dx = x^2 I_1(x) - x I_0(x) + \int I_0(x)\, dx + C$$

(iv) $$\int x^2 I_1(x)\, dx = x^2 I_2(x) + C = x^2 I_0(x) - 2x I_1(x) + C$$

(v) $$\int x^4 I_1(x)\, dx = (4x^3 - 16x) I_1(x) - (x^4 - 8x^2) I_0(x) + C$$

PROBLEM (43): Find the general solutions of the following differential equations in terms of modified Bessel functions :

(i) $$x^2 y'' + x y' - \lambda^2 x^2 y = 0$$

(ii) $$x^2 y'' + x y' - \left(3x^2 + \frac{1}{4}\right) y = 0$$

(iii) $$x^2 y'' + x y' - (i\lambda^2 x^2 + 1) y = 0$$

PROBLEM (44): Transform each of the following differential equations to modified Bessel's equation using the indicated substitution and find the general solution .

(i) $$x^6 y'' + x^5 y' - (x^2 + n^2) y = 0 \qquad \left(t = \frac{1}{x}\right)$$

(ii) $$y'' - \frac{1}{x} y = 0 \qquad (y = t u,\ \text{then}\ t = x^{1/2})$$

PROBLEM (45): Show that modified Bessel's equation $x^2 y'' + x y' - (x^2 + n^2) y = 0$ may be written in the normal form

$$u'' + \left(\frac{\frac{1}{4} - n^2}{x^2} - 1\right) u = 0$$

using the substituting $y = x^{-1/2} u$. Discuss the case when $n = \pm\frac{1}{2}$.

CHAPTER 12

LEGENDRE FUNCTIONS

12.1 INTRODUCTION

In solving partial differential equations by the method of separation of variables, we are often led to ordinary differential equations with variable coefficients, which can not be solved in terms of familiar functions. In chapter (11), we have discussed the solutions of Bessel's differential equation. Another very important differential equation of this type that arises very frequently in many physical problems, particularly in boundary – value problems involving spherical symmetry, is the **Legendre's differential equation**. This equation occurs in the process of obtaining solutions of Laplace's equation in spherical polar coordinates and hence is of great importance in applied mathematics, mathematical physics, and engineering.

12.2 LEGENDRE'S DIFFERENTIAL EQUATION

The second order homogeneous linear differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (1)$$

is called the Legendre's differential equation named after the French mathematician A. M. Legendre (1752–1833). The parameter n in equation (1) is a given real number. The solutions of equation (1) are called **Legendre functions** of order n.

12.3 SOLUTION OF LEGENDRE'S DIFFERENTIAL EQUATION

We shall solve Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (n = 0, 1, 2, \ldots\ldots) \qquad (1)$$

for the important special case in which the parameter n is zero or a positive integer. To obtain the general solution we use the method of Frobenius. Assuming a solution of the form

$$y = x^{\beta}(a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \qquad (2)$$

where $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots$

By differentiation, omitting the summation index k, we have

$$y' = \sum (k+\beta) a_k x^{k+\beta-1} \text{ and } y'' = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

Then substituting y, y', and y'' in equation (1), we get

$$(1-x^2)\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2} - 2x\sum(k+\beta)a_k x^{k+\beta-1} + n(n+1)\sum a_k x^{k+\beta} = 0$$

or $$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2} - \sum(k+\beta)(k+\beta-1)a_k x^{k+\beta} - \sum 2(k+\beta)a_k x^{k+\beta} + \sum n(n+1)a_k x^{k+\beta} = 0$$

or $$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2} - \sum\left[(k+\beta)(k+\beta-1)+2(k+\beta)-n(n+1)\right]a_k x^{k+\beta} = 0$$

or $$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2} - \sum\left[(k+\beta)^2+(k+\beta)-n(n+1)\right]a_k x^{k+\beta} = 0$$

Shifting the index k by $k-2$ in the second summation, we get

$$\sum(k+\beta)(k+\beta-1)a_k x^{k+\beta-2} - \sum\left[(k+\beta-2)^2+(k+\beta-2)-n(n+1)\right]a_{k-2} x^{k+\beta-2} = 0$$

or $$\sum\left[(k+\beta)(k+\beta-1)a_k - \left\{(k+\beta-2)^2+(k+\beta-2)-n(n+1)\right\}a_{k-2}\right]x^{k+\beta-2} = 0 \tag{3}$$

The coefficient of the lowest degree term $x^{\beta-2}$ is obtained by putting $k = 0$ in this equation

i.e. $$\left[\beta(\beta-1)a_0 - \left\{(\beta-2)^2+(\beta-2)-n(n+1)\right\}a_{-2}\right]x^{\beta-2} = 0$$

Since $x^{\beta-2} \neq 0$, the coefficient of $x^{\beta-2}$ must be zero, therefore

$$\beta(\beta-1)a_0 - \left\{(\beta-2)^2+(\beta-2)-n(n+1)\right\}a_{-2} = 0$$

But $a_{-2} = 0$, we get the indicial equation as

$$\beta(\beta-1)a_0 = 0$$

Now $a_0 \neq 0$, therefore the indicial equation is

$$\beta(\beta-1) = 0 \quad \text{or} \quad \beta = 0, \quad \beta = 1$$

are the indicial roots which are unequal and differ by an integer.

Now there are two cases, given by $\beta = 0$ and $\beta = 1$.

CASE (1): When $\beta = 0$

In this case, we obtain the recurrence relation from equation (3)

$$k(k-1)a_k - \left[(k-2)^2+(k-2)-n(n+1)\right]a_{k-2} = 0 \tag{4}$$

or $$a_k = \frac{(k-2)^2+(k-2)-n(n+1)}{k(k-1)}a_{k-2}$$

We see that when $k = 0, 1$, the coefficients a_0 and a_1 are indeterminate of the form $\left(\frac{0}{0}\right)$, since a_{-2} and a_{-1} are zero. Hence we can take the coefficients a_0 and a_1 as arbitrary constants.

If $k = 2$, then $a_2 = -\frac{n(n+1)}{2!} a_0$

If $k = 3$ $a_3 = -\frac{n(n+1)-2}{6} a_1 = -\frac{(n-1)(n+2)}{3!} a_1$

If $k = 4$ $a_4 = -\frac{n(n+1)-6}{12} a_2 = -\frac{(n-2)(n+3)}{12} a_2 = \frac{n(n-2)(n+1)(n+3)}{4!} a_0$

If $k = 5$ $a_5 = -\frac{n(n+1)-12}{20} a_3 = -\frac{(n-3)(n+4)}{20} a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1$

---

and so we obtain from equation (2)

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots\dots \\
&= a_0\left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 - \dots\dots\right] \\
&\quad + a_1\left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 - \dots\dots\right] \qquad (5) \\
&= a_0 y_1 + a_1 y_2
\end{aligned}
$$

Since the first series y_1 contains even powers of x only, while the second series y_2 contains odd powers of x only, their ratio y_1 / y_2 is not a constant, so that y_1 and y_2 are not proportional and are thus linearly independent solutions. It can be proved that these series converge for $-1 < x < 1$.

Since equation (5) a solution with two arbitrary constants a_0 and a_1, this is the general solution of Legendre's equation (1) and we need not consider case (2) i.e. $\beta = 1$. The solution obtained by $\beta = 1$ can be shown to be the second series of equation (5). This case illustrates the principle that when the indicial roots differ by an integer, the solutions obtained by using the two roots are linearly dependent.

If n is zero or an even positive integer, the coefficient of all power of x beyond x^n in the first series of equation (5) are zero, so that the first series terminates and becomes a polynomial of degree n in x. Likewise, if n is an odd positive integer, the second series terminates and reduces to a polynomial of degree n.

Thus for any integer $n \geq 0$, equation (5) has a polynomial solution. These polynomials multiplied by some constants are called **Legendre's polynomials** of order n or **Legendre's functions of the first kind** of order n and are denoted by $P_n(x)$, where $n = 0, 1, 2, 3 \dots\dots$

LEGENDRE POLYNOMIALS

To obtain the standard form for Legendre's polynomials of degree n, we go back to equation (4), which can be written as

$$k(k-1)a_k = [(k-1)(k-2) - n(n+1)]a_{k-2} \qquad (6)$$

We see that if $k = n+2$, then $a_{n+2} = 0$ and thus $a_{n+4} = 0,\ a_{n+6} = 0, \ldots\ldots$

Then letting $k = n, n-2, n-4, \ldots\ldots$, we find from equation (6)

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} a_n$$

....

....

Thus we have expressed all the non – vanishing coefficients in terms of the coefficient a_n of the highest power of x of the polynomial. The coefficient a_n is then arbitrary. This leads to the polynomial solutions

$$y = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \ldots\ldots\right]$$

The Legendre polynomials $P_n(x)$ are defined by choosing the arbitrary constant a_n as

$$a_n = 1, \text{ when } n = 0 \text{ and } a_n = \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{n!}, \qquad n = 1, 2, 3, \ldots\ldots$$

This choice is made in order that $P_n(1) = 1$, for every positive integral value of n i.e. all the polynomials will have the value 1 when $x = 1$. Hence the Lengendre Polynomials are defined by

$$P_n(x) = \frac{(2n-1)(2n-3)\ldots 3.1}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \ldots\right] \tag{7}$$

FIRST FEW LEGENDRE POLYNOMIALS

Note that $P_n(x)$ is a polynomial of degree n. The first few Legendre polynomials are given by

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

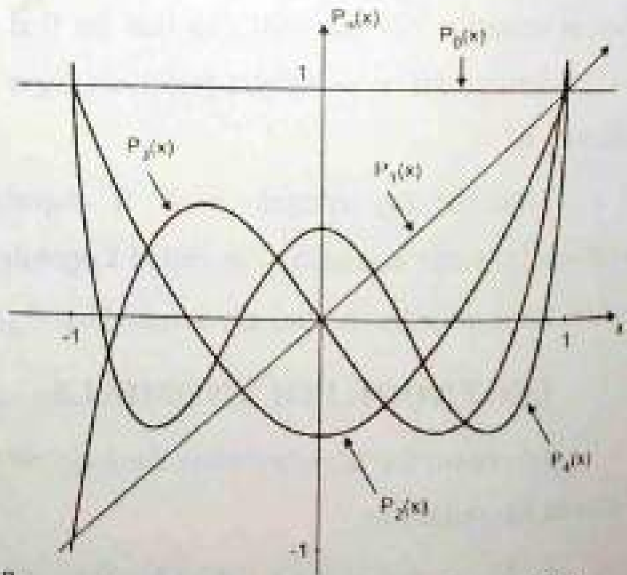

Figure (12.1)

Note that in all cases $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

12.4 RODRIGUE'S FORMULA

This formula named after the French mathematician O. Rodrigues (1794 – 1851) expresses the Legendre polynomials in a very concise form.

THEOREM (12.1): Derive Rodrigue's formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$, $n = 0, 1, 2, \ldots$

PROOF: We know that Legendre polynomials are given by

$$P_n(x) = \frac{(2n-1)(2n-3)\ldots 3.1}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

Now integrating this equation n time from 0 to x, we obtain

$$\int_0^x\int_0^x \cdots\cdots \int_0^x P_n(x)(dx)^n$$

$$= \frac{(2n-1)(2n-3)\ldots 3.1}{n!}\left[\frac{x^{2n}}{(n+1)\ldots 2n} - \frac{n(n-1)}{2(2n-1)}\frac{x^{2n-2}}{(n-1)n(n+1)\ldots(2n-2)}\right.$$
$$\left. + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\frac{x^{2n-4}}{(n-3)(n-2)(n-1)n(n+1)\ldots\ldots(2n-4)} - \ldots\ldots\right]$$

$$= \frac{(2n-1)(2n-3)\ldots 3.1}{n!}\left[\frac{x^{2n}}{2n\ldots\ldots(n+1)} - \frac{nx^{2n-2}}{2n(2n-1)(2n-2)\ldots\ldots(n+1)}\right.$$
$$\left. + \frac{n(n-1)}{2!}\frac{x^{2n-4}}{2n(2n-1)(2n-2)(2n-3)(2n-4)\ldots\ldots(n+1)} - \ldots\ldots\right]$$

$$= \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{2n!}\left[x^{2n} - nx^{2n-2} + \frac{n(n-1)}{2!}x^{2n-4} - \ldots\ldots\right]$$

$$= \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{2n(2n-1)(2n-2)\ldots\ldots 2.1}(x^2-1)^n$$

$$= \frac{1}{2n(2n-2)(2n-4)\ldots\ldots 2}(x^2-1)^n$$

$$= \frac{1}{2^n.n!}(x^2-1)^n$$

Differentiating this equation n times w.r.t. x, we get

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$$

12.5 GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

The function $\dfrac{1}{\sqrt{1-2xt+t^2}}$ is called the generating function for Legendre polynomials and is useful in obtaining their properties.

Replacing t by $-t$ in equation (1), we get

$$(1+2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)(-1)^n t^n \tag{3}$$

From equation (2) and (3), we get

or $$\sum_{n=0}^{\infty} P_n(-x)t^n = \sum_{n=0}^{\infty} (-1)^n P_n(x) t^n$$

Equating the coefficients of t^n on both sides, we get

$$P_n(-x) = (-1)^n P_n(x) \tag{4}$$

DEDUCTION

Replacing x by 1 in equation (4), we get result (ii), since

$$P_n(-1) = (-1)^n P_n(1) = (-1)^n \quad \left[\text{since } P_n(1) = 1\right]$$

Note that when n is odd, $(-1)^n = -1$ and equation (4) becomes

$$P_n(-x) = -P_n(x)$$

Thus $P_n(x)$ is an odd function of x when n is odd.

Similarly, $P_n(x)$ is an even function of x when x is even.

THEOREM (12.4): Prove that

(i) $P_n'(1) = \frac{1}{2}n(n+1)$ (ii) $P_n'(-1) = (-1)^{n+1}\frac{1}{2}n(n+1)$

(iii) $P_n'(-x) = (-1)^{n+1}P_n'(x)$

PROOF: Since $P_n(x)$ satisfies Legendre's differential equation, we get

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{1}$$

(i) Let $x = 1$ in equation (1), we get

$$-2P_n'(1) + n(n+1)P_n(1) = 0$$

or $$P_n'(1) = \frac{1}{2}n(n+1)P_n(1)$$

$$= \frac{1}{2}n(n+1) \quad \left[\text{since } P_n(1) = 1\right]$$

(ii) Let $x = -1$ in equation (1), we get

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$

$$P_n'(-1) = -\frac{1}{2}n(n+1)P_n(-1)$$

or
$$= \frac{1}{2}n(n+1)(-1)(-1)^n \qquad [\text{since } P_n(-1) = (-1)^n]$$

$$= (-1)^{n+1}\frac{1}{2}n(n+1)$$

(iii) We know that

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-1/2}$$

Differentiating w.r.t. x, we get

$$\sum_{n=0}^{\infty} P_n'(x)t^n = t(1-2xt+t^2)^{-3/2} \tag{1}$$

Replacing x by $-x$ in equation (1), we get

$$t(1+2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(-x)t^n \tag{2}$$

Replacing t by $-t$ in equation (1), we get

$$-t(1+2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(x)(-t)^n$$

or
$$t(1+2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} (-1)^{n+1}P_n'(x)t^n \tag{3}$$

From equations (2) and (3), we get

$$\sum_{n=0}^{\infty} P_n'(-x)t^n = \sum_{n=0}^{\infty} (-1)^{n+1}P_n'(x)t^n$$

Equating the coefficients of t^n on both sides, we get

$$P_n'(-x) = (-1)^{n+1}P_n'(x)$$

Note that if $x = 1$, we get result (ii).

THEOREM (12.5): Prove that

(i) $\mathbf{P_{2n+1}(0) = 0}$ (ii) $\mathbf{P_{2n}(0) = (-1)^n\dfrac{2n!}{2^{2n}(n!)^2}}$

(iii) $\mathbf{P_{2n}'(0) = 0}$ (iv) $\mathbf{P_{2n+1}'(0) = (-1)^n\dfrac{(2n+1)!}{2^{2n}(n!)^2}}$

Equating the coefficient of t^n on each side, we find

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2nxP_n(x)+(n-1)P_{n-1}(x)$$

or $\quad (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$

or $\quad P_{n+1}(x)=\dfrac{2n+1}{n+1}xP_n(x)-\dfrac{n}{n+1}P_{n-1}(x)$

NOTE: This formula is also called Bonnet's recurrence formula named after the French mathematician O. Bonnet (1819 – 1892).

THEOREM (12.7): Prove that

(i) $\quad xP_n'(x)-P_{n-1}'(x)=nP_n(x)$

(ii) $\quad P_{n+1}'(x)-xP_n'(x)=(n+1)P_n(x)$

(iii) $\quad P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$

(iv) $\quad (x^2-1)P_n'(x)=nxP_n(x)-nP_{n-1}(x)$

(v) $\quad (x^2-1)P_n'(x)=(n+1)[P_{n+1}(x)-xP_n(x)]$

PROOF: (i) From the generating function, we have

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \tag{1}$$

Differentiating both sides of the equation (1) w.r.t. x and t in turn, getting

$$\frac{t}{(1-2xt+t^2)^{3/2}}=\sum_{n=0}^{\infty}P_n'(x)t^n \tag{2}$$

$$\frac{x-t}{(1-2xt+t^2)^{3/2}}=\sum_{n=0}^{\infty}nP_n(x)t^{n-1} \tag{3}$$

From equations (2) and (3), equating the expressions for $(1-2xt+t^2)^{-3/2}$, we get

$$\sum_{n=0}^{\infty}P_n'(x)t^{n-1}=\sum_{n=0}^{\infty}\frac{n}{x-t}P_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty}(x-t)P_n'(x)t^{n-1}=\sum_{n=0}^{\infty}nP_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty}xP_n'(x)t^{n-1}-\sum_{n=0}^{\infty}P_n'(x)t^n=\sum_{n=0}^{\infty}nP_n(x)t^{n-1}$$

Equating the coefficients of t^{n-1} on each side, we get

$$xP_n'(x)-P_{n-1}'(x)=nP_n(x) \tag{4}$$

(ii) From theorem (12.6), we have

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

Differentiating both sides w.r.t. x, we have

$$P'_{n+1}(x) = \frac{2n+1}{n+1}\left[x P'_n(x) + P_n(x)\right] - \frac{n}{n+1} P'_{n-1}(x) \qquad (5)$$

Substituting the value of $P'_{n-1}(x)$ from equation (4) in equation (5), we get

$$P'_{n+1}(x) = \frac{2n+1}{n+1}\left[x P'_n(x) + P_n(x)\right] - \frac{n}{n+1}\left[x P'_n(x) - n P_n(x)\right]$$

$$= x P'_n(x) + (n+1) P_n(x)$$

or

$$P'_{n+1}(x) - x P'_n(x) = (n+1) P_n(x) \qquad (6)$$

(iii) Adding equations (4) and (6), we get

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$$

(iv) Multiplying equation (4) by x, we get

$$x^2 P'_n(x) - x P'_{n-1}(x) = n x P_n(x) \qquad (7)$$

Replacing n by $n-1$ in equation (6), we get

$$P'_n(x) - x P'_{n-1}(x) = n P_{n-1}(x) \qquad (8)$$

Subtracting equation (8) from equation (7), we have

$$(x^2-1) P'_n(x) = n x P_n(x) - n P_{n-1}(x) \qquad (9)$$

(v) From theorem (12.6), we have

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

$$= (n+1) x P_n(x) + n x P_n(x) - n P_{n-1}(x)$$

$$(n+1)\left[P_{n+1} - x P_n(x)\right] = n x P_n(x) - n P_{n-1}(x) \qquad (10)$$

From equations (9) and (10), we get

$$(x^2-1) P'_n(x) = (n+1)\left[P_{n+1}(x) - x P_n(x)\right]$$

12.8 BELTRAMI'S RESULT

THEOREM (12.8): Prove that

$$(2n+1)(x^2-1) P'_n(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

PROOF: From theorem (12.7) parts (iv) and (v), we have

Equating coefficients of t^{2n}, we have as required

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} \qquad \text{if } m = n.$$

12.10 EXPANSION OF f(x) IN A SERIES OF LEGENDRE PLOYNOMIALS

THEOREM (12.10): Let $f(x)$ be defined over the interval $-1 < x < 1$. Then prove that at every point of continuity of $f(x)$ in $-1 < x < 1$, the Legendre series expansion of $f(x)$ is

$$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \dots\dots\dots \sum_{k=0}^{\infty} A_kP_k(x)$$

where $A_k = \dfrac{2k+1}{2} \displaystyle\int_{-1}^{1} P_k(x)f(x)\,dx$

At a point of discontinuity of $f(x)$ in $-1 < x < 1$, the left side be replaced by $\frac{1}{2}\left[f(x+0) - f(x-0)\right]$.

PROOF: Multiplying the given series by $P_m(x)$ and integrating w.r.t. x from -1 to 1, we have on using theorem (12.8).

$$\int_{-1}^{1} P_m(x)f(x)\,dx = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_m(x)P_k(x)\,dx$$

$$= A_m \int_{-1}^{1} P_m^2(x)\,dx = \frac{2A_m}{2m+1}$$

or $$A_m = \frac{2m+1}{2} \int_{-1}^{1} P_m(x)f(x)\,dx$$

or $$A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x)f(x)\,dx$$

EXAMPLE (1): Expand the function $f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$ in a series of the form

$$\sum_{k=0}^{\infty} A_kP_k(x).$$

SOLUTION: We know that

$$A_k = \frac{2k+1}{2}\int_{-1}^{1} P_k(x) f(x)\, dx$$

$$= \frac{2k+1}{2}\int_{-1}^{0} P_k(x).0\, dx + \frac{2k+1}{2}\int_{0}^{1} P_k(x).1\, dx$$

$$= \frac{2k+1}{2}\int_{0}^{1} P_k(x)\, dx$$

Then $A_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 1.\,dx = \frac{1}{2}$

$$A_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x.\,dx = \frac{3}{4}$$

$$A_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \frac{5}{2}\int_0^1 \frac{3x^2-1}{2}\,dx = \frac{5}{4}\left| x^3 - x \right|_0^1 = 0$$

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{5x^3-3x}{2}\,dx = \frac{7}{4}\left| \frac{5}{4}x^4 - \frac{3}{2}x^2 \right|_0^1 = \frac{7}{4}\left(-\frac{1}{4}\right) = -\frac{7}{16}$$

$$A_4 = \frac{9}{2}\int_0^1 P_4(x)\,dx = \frac{9}{2}\int_0^1 \frac{35x^4-30x^2+3}{8}\,dx = \frac{9}{16}\left| 7x^5 - 10x^3 + 3x \right|_0^1 = 0$$

$$A_5 = \frac{11}{2}\int_0^1 P_5(x)\,dx = \frac{11}{2}\int_0^1 \frac{63x^5-70x^3+15x}{8}\,dx$$

$$= \frac{11}{16}\left| \frac{21}{2}x^6 - \frac{35}{2}x^4 + \frac{15}{2}x^2 \right|_0^1 = \frac{11}{16}\left(\frac{1}{2}\right) = \frac{11}{32}, \text{ etc.}$$

Thus $f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \ldots\ldots\ldots$ is the required series .

12.11 LEGENDRE POLYNOMIALS IN TERMS OF θ WHERE $x = \cos\theta$

If we let $x = \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ into the generating function , we get

$$\left[1 - t\left(e^{i\theta}+e^{-i\theta}\right)+t^2\right]^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos\theta)\, t^n \qquad (1)$$

THEOREM (12.12): Prove that

$$\int_{-1}^{1} P_n(x)\,dx = \begin{cases} 2 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n = 1, 2, 3, \ldots\ldots\ldots \end{cases}$$

PROOF: **CASE (1)**

If $n = 0$, then $P_n(x) = P_0(x) = 1$ and

$$\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} 1\,dx = |x|_{-1}^{1} = 2$$

CASE (2)

From Rodrigue's formula, we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

and
$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\left|\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right|_{-1}^{1}$$

$$= \frac{1}{2^n n!}\left|D^{n-1}(x^2-1)^n\right|_{-1}^{1} \qquad \left(\text{where } D = \frac{d}{dx}\right) \quad (1)$$

$$= \frac{1}{2^n n!}\left|D^{n-1}\left[(x+1)^n(x-1)^n\right]\right|_{-1}^{1}$$

Using the Leibnitz's rule for the nth derivative of a product

$$D^n(uv) = uD^n v + n(Du)(D^{n-1}v) + \frac{n(n-1)}{2!}(D^2u)(D^{n-2}v) + \ldots\ldots\ldots + vD^n u$$

we have

$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n n!}\Big|(x+1)^n D^{n-1}(x-1)^n + (n-1)\left[D(x+1)^n\right]D^{n-2}(x-1)^n$$

$$+\frac{(n-1)(n-2)}{2!}\left[D^2(x+1)^n\right]D^{n-3}(x-1)^n + \ldots\ldots\ldots + (x-1)^n D^{n-1}(x+1)^n\Big|_{-1}^{1}$$

Next using the formula $D^n(ax+b)^m = a^n\frac{m!}{(m-n)!}(ax+b)^{m-n}$, we get

$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n n!}\Big|(x+1)^n\frac{n!}{1!}(x-1)+(n-1)n(x+1)^{n-1}\frac{n!}{2!}(x-1)^2$$

$$+\frac{(n-1)(n-2)}{2!}n(n-1)(x+1)^{n-2}\frac{n!}{3!}(x-1)^3+\ldots\ldots\ldots+(x-1)^n\frac{n!}{1!}(x+1)\Big|_{-1}^{1}$$

$$= 0$$

since each term contains $x-1$ and $(x+1)$. This completes the proof.

THEOREM (12.13): Prove that

(i) $$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \quad \text{if } m < x.$$

(ii) $$\int_{-1}^{1} x^n P_n(x)\,dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

PROOF: Using Rodrigue's formula

$$\int_{-1}^{1} x^m P_n(x)\,dx = \int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} x^m D^n(x^2-1)^n\,dx \qquad \left(\text{where } D^n = \frac{d^n}{dx^n}\right)$$

$$= \frac{1}{2^n n!}\left[\left|x^m . D^{n-1}(x^2-1)^n\right|_{-1}^{1} - m\int_{-1}^{1} x^{m-1}D^{n-1}(x^2-1)^n\,dx\right]$$

The first term vanishes on using Leibnitz's rule. Therefore

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^m}{2^n n!}\int_{-1}^{1} x^{m-1}D^{n-1}(x^2-1)^n\,dx$$

$$= \frac{(-1)m}{2^n n!}\left[\left|x^{m-1} . D^{n-2}(x^2-1)^n\right|_{-1}^{1} - (m-1)\int_{-1}^{1} x^{m-2}D^{n-2}(x^2-1)^n\,dx\right]$$

The first term again vanishes on using Leibnitz's rule. Thus

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^m(m-1)}{2^n n!}\int_{-1}^{1} x^{m-2}D^{n-2}(x^2-1)^n\,dx$$

On integrating the R.H.S. (m − 2) times more and noting that $m < n$, we get

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{(-1)^m (m-1)(m-2)\ldots 2.1}{2^n n!}$$

$$\int_{-1}^{1} x^{m-2-(m-2)} D^{n-2-(m-2)} (x^2-1)^n \, dx$$

$$= \frac{(-1)^m m!}{2^n n!} \int_{-1}^{1} D^{n-m} (x^2-1)^n \, dx \qquad (1)$$

$$= \frac{(-1)^m m!}{2^n n!} \int_{-1}^{1} \frac{d}{dx}\left[\frac{d^{n-m-1}}{dx^{n-m-1}} (x^2-1)^n\right] dx$$

$$= \frac{(-1)^m m!}{2^n n!} \left| \frac{d^{n-m-1}}{dx^{n-m-1}} (x^2-1)^n \right|_{-1}^{1}$$

$$= \frac{(-1)^m m!}{2^n n!} \left| D^{n-m-1} (x+1)^n (x-1)^n \right|_{-1}^{1}$$

$$= 0 \qquad \text{(using Leibnitz's rule)}$$

(ii) Let $m = n$ in equation (1), we get

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{(-1)^n n!}{2^n n!} \int_{-1}^{1} (x^2-1)^n \, dx$$

$$= \frac{(-1)^n}{2^n} \int_{-1}^{1} (-1)^n (1-x^2)^n \, dx$$

$$= \frac{(-1)^{2n}}{2^n}\, 2 \int_{0}^{1} (1-x^2)^n \, dx$$

$$= \frac{1}{2^{n-1}} \int_{0}^{1} (1-x^2)^n \, dx \qquad (2)$$

Let $x = \sin\theta$, then $dx = \cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

Thus equation (2) becomes

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{1}{2^{n-1}} \int_{0}^{\pi/2} \cos^{2n+1}\theta \, d\theta \qquad (3)$$

But from calculus

$$\int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta = \frac{2.4.6 \ldots\ldots\ldots 2n}{1.3.5 \ldots\ldots\ldots 2n+1}, \quad n = 1, 2, 3, \ldots\ldots\ldots$$

$$= \frac{(2.4.6 \ldots\ldots\ldots 2n)(2.4.6 \ldots\ldots\ldots 2n)}{1.3.5 \ldots\ldots\ldots (2n+1)(2.4.6 \ldots\ldots\ldots 2n)}$$

$$= \frac{(2^n n!)(2^n n!)}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Thus equation (3) becomes

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{1}{2^{n-1}} \frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

12.13 LAPLACE'S DEFINITE INTEGRALS FOR $P_n(x)$

LAPLACE'S FIRST INTEGRAL FOR $P_n(x)$

THEOREM (12.14): Prove that

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x \pm \sqrt{x^2-1}\cos\phi \right]^n d\phi, \quad n = 1, 2, 3, \ldots\ldots\ldots$$

PROOF: From calculus, we have

$$\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad (\text{where } a^2 > b^2) \qquad (1)$$

Let $a = 1 - hx$ and $b = h\sqrt{x^2-1}$, then

$$a^2 - b^2 = (1-hx)^2 - h^2(x^2-1) = 1 - 2hx + h^2$$

Using the values of a, b, and $a^2 - b^2$ in equation (1), we get

$$\pi(1 - 2hx + h^2)^{-1/2} = \int_0^{\pi} \frac{d\phi}{1 - hx \pm h\sqrt{x^2-1}\cos\phi}$$

$$= \int_0^{\pi} \left[1 - hx \pm h\sqrt{x^2-1}\cos\phi \right]^{-1} d\phi$$

$$= \int_0^{\pi} \left[1 - h\left(x \pm \sqrt{x^2-1}\cos\phi \right) \right]^{-1} d\phi$$

or $$\pi \sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi} (1-ht)^{-1}\, d\phi \qquad \text{where } t = x \pm \sqrt{x^2-1}\cos\phi$$

or $$\pi \sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi} (1+ht+h^2t^2+\ldots\ldots+h^nt^n+\ldots\ldots)\, d\phi$$

Equating the coefficients of h^n on both sides, we get

$$\pi P_n(x) = \int_0^{\pi} t^n\, d\phi = \int_0^{\pi} \left(x \pm \sqrt{x^2-1}\cos\phi\right)^n d\phi$$

or $$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \left(x \pm \sqrt{x^2-1}\cos\phi\right)^n d\phi \tag{2}$$

DEDUCTION

Let $x = \cos\theta$. Then $\sqrt{x^2-1} = \sqrt{\cos^2\theta - 1} = \sqrt{-\sin^2\theta} = i\sin\theta$

With these values and positive sign, equation (2) gives

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi} (\cos\theta + i\sin\theta\cos\phi)^n\, d\phi$$

LAPLACE'S SECOND INTEGRAL FOR $P_n(x)$

THEOREM (12.15): Prove that

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}}, \quad n = 1, 2, 3, \ldots\ldots\ldots$$

PROOF: From calculus, we have

$$\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}}, \quad \text{where } a^2 > b^2 \tag{1}$$

Let $a = hx - 1$ and $b = h\sqrt{x^2-1}$, so that $a^2 - b^2 = 1 - 2xh + h^2$

Substituting these values in equation (1), we get

$$\pi(1-2xh+h^2)^{-1/2} = \int_0^{1} \left(hx - 1 \pm h\sqrt{x^2-1}\cos\phi)\right)^{-1} d\phi$$

or $$\frac{\pi}{h}\left(1-\frac{2x}{h}+\frac{1}{h^2}\right)^{-1/2} = \int_0^{\pi}\left[h\left(x\pm\sqrt{x^2-1}\cos\phi\right)-1\right]^{-1} d\phi$$

or $$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^{\pi}(ht-1)^{-1}\,d\phi \qquad \text{where } t = \left(x\pm\sqrt{x^2-1}\right)\cos\phi$$

or $$\pi\sum_{n=0}^{\infty}\frac{1}{h^{n+1}}P_n(x) = \int_0^{\pi}(ht)^{-1}\left(1-\frac{1}{ht}\right)^{-1} d\phi$$

$$= \int_0^{\pi}\frac{1}{ht}\left(1+\frac{1}{ht}+\frac{1}{h^2t^2}+\dots\dots\dots+\frac{1}{h^nt^n}+\dots\dots\dots\right)d\phi$$

$$= \int_0^{\pi}\left(\frac{1}{ht}+\frac{1}{h^2t^2}+\dots\dots\dots+\frac{1}{h^{n+1}t^{n+1}}+\dots\dots\dots\right)d\phi$$

Equating the coefficients of $\frac{1}{t^{n+1}}$ on both sides , we get

$$\pi P_n(x) = \int_0^{\pi}\frac{1}{t^{n+1}}\,d\phi$$

or $$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{\left(x\pm\sqrt{x^2-1}\cos\phi\right)^{n+1}}\,d\phi \qquad (2)$$

DEDUCTION

Replacing n by $-(n+1)$ in equation (2) , we get

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{\left(x\pm\sqrt{x^2-1}\cos\phi\right)^{-n}}\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(x\pm\sqrt{x^2-1}\cos\phi\right)^{n} d\phi$$

or $$P_{-(n+1)}(x) = P_n(x) \qquad \text{(Using Laplace's first integral)}$$

12.14 CHRISTOFFEL'S EXPANSION

THEOREM (12.16): Prove that

$$P_n'(x)=(2n-1)P_{n-1}(x)+(2n-5)P_{n-3}(x)+(2n-9)P_{n-5}(x)+\ldots$$

The last term of the series being $3P_1(x)$ or $P_0(x)$ according as n is even or odd.

PROOF: We have the recurrence relation

$$P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$$

or $$P_{n+1}'(x)=(2n+1)P_n(x)+P_{n-1}'(x)$$

Replacing n by $n-1$ in this relation, we get

$$P_n'(x)=(2n-1)P_{n-1}(x)+P_{n-2}'(x) \tag{1}$$

CASE (1): When n is Even

Replacing n by $n, n-2, n-4, \ldots\ldots\ldots, 4, 2$, successively in equation (1)

$$P_n'(x)=(2n-1)P_{n-1}(x)+P_{n-2}'(x)$$

$$P_{n-2}'(x)=(2n-5)P_{n-3}(x)+P_{n-4}'(x)$$

$$P_{n-4}'(x)=(2n-9)P_{n-5}(x)+P_{n-6}'(x)$$

$$P_4'(x)=7P_3(x)+P_2'(x)$$

$$P_2'(x)=3P_1(x)+P_0'(x)$$

Adding these and using the fact that $P_0(x)=1$ and $P_0'(x)=0$, we get

$$P_n'(x)=(2n-1)P_{n-1}(x)+(2n-5)P_{n-3}(x)+(2n-9)P_{n-5}(x)+\ldots\ldots\ldots+7P_3(x)+3P_1(x)$$

CASE (2): When n is Odd

Replacing n by $n, n-2, n-4, \ldots\ldots\ldots, 5, 3$, successively in equation (1), we get

$$P_n'(x)=(2n-1)P_{n-1}(x)+P_{n-2}'(x)$$

$$P_{n-2}'(x)=(2n-5)P_{n-3}(x)+P_{n-4}'(x)$$

$$P_{n-4}'(x)=(2n-9)P_{n-5}(x)+P_{n-6}'(x)$$

$$P_5'(x)=9P_4(x)+P_3'(x)$$

$$P_3'(x)=5P_2(x)+P_1'(x)$$

Adding these and using the fact that $P_1(x) = x$, $P_1'(x) = 1 = P_0(x)$, we get

$$P_n'(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \dots\dots + 5P_2(x) + P_0(x)$$

12.15 LEGENDRE FUNCTIONS OF THE SECOND KIND

The Legendre functions of the second kind are the series solutions of Legendre's equation which do not terminate. The general solution of Legendre's equation is

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \dots\dots\right] + a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \dots\dots\right] \quad (1)$$

From equation (1) if n is zero or even, the series which does not terminate is

$$a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \dots\dots\right]$$

while if n is odd, the series which does not terminate is

$$a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \dots\dots\right]$$

Since a_0 and a_1 are arbitrary constants we can take these as

$$a_1 = \frac{(-1)^{n/2}\, 2^n\left[\left(\frac{n}{2}\right)!\right]^2}{n!}$$

$$a_0 = \frac{(-1)^{(n+1)/2}\, 2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2}{n!}$$

Thus the Legendre functions of the second kind are given by the following according as n is even or odd respectively.

$$Q_n(x) = \frac{(-1)^{n/2}\, 2^n\left[\left(\frac{n}{2}\right)!\right]^2}{n!}\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \dots\dots\right] \quad (2)$$

$$n = 0, 2, 4, 6, \dots\dots$$

$$Q_n(x) = \frac{(-1)^{(n+1)/2}\, 2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2}{n!}\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \dots\right] \quad (3)$$

$$n = 1, 3, 5, \dots\dots$$

Hence the general solution of Legendre's equation in the case where $n = 0, 1, 2, 3$ ....

can be written as $y = A P_n(x) + B Q_n(x)$

where $P_n(x)$ are Legendre polynomials and $Q_n(x)$ are called the Legendre functions of the second kind of order n. The constant A and B are determined from the boundary conditions of the physical problem. The series (2) and (3) converge for $-1 < x < 1$.

FIRST FEW LEGENDRE FUNCTIONS OF THE SECOND KIND

(1) From equation (2) above, we have if $n = 0$

$$Q_0(x) = x + \frac{2}{3!}x^3 + \frac{1.3.2.4}{5!}x^5 + \frac{1.3.5.2.4.6}{7!}x^7 + \ldots\ldots$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots$$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

(2) From equation (3) above, we have if $n = 1$

$$Q_1(x) = -\left[1 - \frac{(1)(2)}{2!}x^2 + \frac{(1)(-1)(2)(4)}{4!}x^4 - \frac{(1)(-1)(-3)(2)(4)(6)}{6!}x^6 + \ldots\ldots\right]$$

$$= x\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right] - 1$$

$$= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

(3) The recurrence formulas for $Q_n(x)$ are identical with those of $P_n(x)$. Then

$$Q_{n+1}(x) = \frac{2n+1}{n+1}x\,Q_n(x) - \frac{n}{n+1}Q_{n-1}(x)$$

Putting $n = 1$, we have on using parts (1) and (2)

$$Q_2(x) = \frac{3}{2}x\,Q_1(x) - \frac{1}{2}Q_0(x)$$

$$= \frac{3}{2}x\left[\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right] - \frac{1}{4}\ln\left(\frac{1+x}{1-x}\right)$$

$$= \left(\frac{3x^2-1}{4}\right)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

(4) $$Q_3(x) = \frac{5}{3}x\,Q_2(x) - \frac{2}{3}Q_1(x)$$

$$= \frac{5}{3}x\left[\left(\frac{3x^2-1}{4}\right)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}\right] - \frac{2}{3}\left[\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right]$$

$$= \left(\frac{5x^3-3x}{4}\right)\ln\left(\frac{1+x}{1-x}\right) - \frac{5}{2}x^2 + \frac{2}{3}$$

12.16 RECURRENCE FORMULAS

The functions $Q_n(x)$ satisfy recurrence relation exactly similar to Legendre polynomials.

(1) $Q_{n+1}(x) = \frac{2n+1}{n+1} x Q_n(x) - \frac{n}{n+1} Q_{n-1}(x)$

(2) $x Q_n'(x) - Q_{n-1}'(x) = n Q_n(x)$

(3) $Q_{n+1}'(x) - x Q_n'(x) = (n+1) Q_n(x)$

(4) $Q_{n+1}'(x) - Q_{n-1}'(x) = (2n+1) Q_n(x)$

(5) $(x^2-1) Q_n'(x) = n x Q_n(x) - n Q_{n-1}(x)$

(6) $(x^2-1) Q_n'(x) = (n+1)[Q_{n+1}(x) - x Q_n(x)]$

12.17 ASSOCIATED LEGENDRE'S DIFFERENTIAL EQUATION

We know that Legendre's differential equation is

$$(1-x^2) y'' - 2xy' + n(n+1) y = 0$$

Since $y = P_n(x)$ are the solutions of equation (1), therefore

$$(1-x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0 \quad (1)$$

Differentiating equation (1) m times, we get

$$\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2 P_n}{dx^2}\right] + \frac{d^m}{dx^m}\left(-2x\frac{dP_n}{dx}\right) + n(n+1)\frac{d^m P_n}{dx^m} = 0 \quad (2)$$

The Leibnitz's rule for differentiation of the product is

$$D^n(uv) = uD^n v + n(Du)D^{n-1}v + \frac{n(n-1)}{2!}(D^2u)D^{n-2}v + \ldots\ldots + vD^n u$$

Using this rule, equation (2) becomes

$$(1-x^2)\frac{d^{m+2}P_n}{dx^{m+2}} + m\left[\frac{d}{dx}(1-x^2)\right]\frac{d^{m+1}P_n}{dx^{m+1}} + \frac{m(m-1)}{2!}\left[\frac{d^2}{dx^2}(1-x^2)\right]\frac{d^m P_n}{dx^m}$$
$$-2x\frac{d^{m+1}P_n}{dx^{m+1}} + m\left[\frac{d}{dx}(-2x)\right]\frac{d^m P_n}{dx^m} + n(n+1)\frac{d^m P_n}{dx^m} = 0$$

or
$$(1-x^2)\frac{d^{m+2}P_n}{dx^{m+2}} - 2mx\frac{d^{m+1}P_n}{dx^{m+1}} + \frac{m(m-1)}{2!}(-2)\frac{d^m P_n}{dx^m}$$
$$-2x\frac{d^{m+1}P_n}{dx^{m+1}} - 2m\frac{d^m P_n}{dx^m} + n(n+1)\frac{d^m P_n}{dx^m} = 0$$

or
$$(1-x^2)\frac{d^{m+2}P_n}{dx^{m+2}} - 2(m+1)x\frac{d^{m+1}P_n}{dx^{m+1}} + [n(n+1) - m(m-1) - 2m]\frac{d^m P_n}{dx^m} = 0$$

or
$$(1-x^2)\frac{d^{m+2}P_n}{dx^{m+2}} - 2(m+1)x\frac{d^{m+1}P_n}{dx^{m+1}} + [n(n+1) - m(m+1)]\frac{d^m P_n}{dx^m} = 0 \quad (3)$$

Let $u(x) = \frac{d^m P_n}{dx^m}$, then equation (3) becomes

$$(1-x^2)u'' - 2(m+1)xu' + [n(n+1) - m(m+1)]u = 0 \qquad (4)$$

Now let $u = (1-x^2)^{-m/2} w$, then

$$u' = (1-x^2)^{-m/2} w' + mx(1-x^2)^{(-m/2)-1} w$$

and $$u'' = (1-x^2)^{-m/2} w'' + 2mx(1-x^2)^{(-m/2)-1} w' + m(1-x^2)^{(-m/2)-1} w + m(m+2)x^2(1-x^2)^{(-m/2)-2} w$$

Substituting u, u', u'' in equation (4), we get

$$(1-x^2)[(1-x^2)^{-m/2} w'' + 2mx(1-x^2)^{(-m/2)-1} w' + m(1-x^2)^{(-m/2)-1} w + m(m+2)x^2(1-x^2)^{(-m/2)-2} w]$$
$$-2(m+1)x[(1-x^2)^{-m/2} w' + mx(1-x^2)^{(-m/2)-1} w]$$
$$+[n(n+1) - m(m+1)](1-x^2)^{-m/2} w = 0$$

Dividing both sides by $(1-x^2)^{-m/2}$, we get

$$(1-x^2)[w'' + 2mx(1-x^2)^{-1} w' + m(1-x^2)^{-1} w + m(m+2)x^2(1-x^2)^{-2} w]$$
$$-2(m+1)x[w' + mx(1-x^2)^{-1} w] + [n(n+1) - m(m+1)]w = 0$$

or $$(1-x^2)w'' + 2mxw' + mw + \frac{m(m+2)x^2 w}{1-x^2} - 2(m+1)xw'$$
$$- \frac{2m(m+1)x^2}{1-x^2} w + n(n+1)w - m(m+1)w = 0$$

or $$(1-x^2)w'' - 2xw' + n(n+1)w + \left[\frac{m(m+2)x^2}{1-x^2} - \frac{2m(m+1)x^2}{1-x^2} - m^2\right]w = 0$$

or $$(1-x^2)w'' - 2xw' + n(n+1)w - \frac{m^2}{1-x^2} w = 0$$

or $$(1-x^2)w'' - 2xw' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]w = 0 \qquad (5)$$

Equation (5) is called the associated Legendre's differential equation. If $m^2 = 0$, we get the Legendre's differential equation.

12.18 SOLUTION OF ASSOCIATED LEGENDRE'S DIFFERENTIAL EQUATION

Since $y = P_n(x)$ are the solutions of Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (1)$$

therefore $w = (1-x^2)^{m/2} u(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \qquad (2)$

are the solution of associated Legendre's differential equation

$$(1-x^2)w'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]w = 0 \qquad (3)$$

The functions defined by equation (2) are called the **associated Legendre functions of first kind of** degree n and order m and are denoted by $P_n^m(x)$.

Thus $$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \qquad (4)$$

The functions $P_n^m(x)$ are bounded at $x = \pm 1$.

From equation (4), we see that

$$P_n^0(x) = P_n(x) \quad \text{and} \quad P_n^m(x) = 0 \quad \text{if } m > n$$

Next $Q_n(x)$ are solution of Legendre's differential equation (1), therefore

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x) \qquad (5)$$

are the solution of associated Legendre's differential equation (3).

The functions defined by equation (5) are called the **associated Legendre functions of second kind of** degree n and order m and are denoted by $Q_n^m(x)$.

Thus $$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x)$$

The functions $Q_n^m(x)$ are unbounded at $x = \pm 1$.

GENERAL SOLUTION

Since $P_n^m(x)$ and $Q_n^m(x)$ are the independent solutions of associated Legendre's differential equation (3), therefore the general solution of this equation is

$$y(x) = A\,P_n^m(x) + B\,Q_n^m(x)$$

where A and B are arbitrary constants.

12.19 RODRIGUE'S FORMULA FOR ASSOCIATED LEGENDRE FUNCTION

We know that

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \qquad (1)$$

But the Rodrigue's formula for Legendre polynomials is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Thus equation (1) becomes

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}\left[\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\right]$$

$$= \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$$

which is the Rodrigue's formula for the associated Legendre function .

EXAMPLE (2): **Show that**

(i) $P_1^1(x) = (1-x^2)^{1/2}$ (ii) $P_2^1(x) = 3x(1-x^2)^{1/2}$

SOLUTION: We know that

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n \tag{1}$$

(i) Let $m = n = 1$ in equation (1), we get

$$P_1^1(x) = \frac{(1-x^2)^{1/2}}{2.1!}\frac{d^2}{dx^2}(x^2-1) = \frac{(1-x^2)^{1/2}}{2}2 = (1-x^2)^{1/2}$$

(ii) Let $m = 1$, $n = 2$ in equation (1), we get

$$P_2^1(x) = \frac{(1-x^2)^{1/2}}{2^2.2!}\frac{d^3}{dx^3}(x^2-1)^2$$

$$= \frac{(1-x^2)^{1/2}}{8}\frac{d^2}{dx^2}(4x^3-4x) = \frac{(1-x^2)^{1/2}}{8}\frac{d}{dx}(12x^2-4)$$

$$= \frac{(1-x^2)^{1/2}}{8}.24x = 3x(1-x^2)^{1/2}$$

12.20 GENERATING FUNCTION FOR ASSOCIATED LEGENDRE FUNCTION

THEOREM (12.17): Prove that

$$\frac{(2m)!(1-x^2)^{m/2}t^m}{2^m.m!(1-2tx+t^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x)t^n$$

PROOF: We know that the generating function for the Legendre polynomials is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Differentiating this equation m times w.r.t. x, we get

$$\frac{d^m}{dx^m}\left[\frac{1}{\sqrt{1-2xt+t^2}}\right] = \sum_{n=0}^{\infty}\frac{d^m}{dx^m}P_n(x)t^n \tag{1}$$

Now $$\frac{d^m}{dx^m}\left[\frac{1}{\sqrt{1-2xt+t^2}}\right] = \frac{d^{m-1}}{dx^{m-1}}\left[\frac{d}{dx}\frac{1}{\sqrt{1-2xt+t^2}}\right]$$

$$= \frac{d^{m-1}}{dx^{m-1}}\left[\left(-\frac{1}{2}\right)(1-2xt+t^2)^{-3/2}(-2t)\right]$$

$$= \frac{d^{m-1}}{dx^{m-1}}\left[\frac{(2.1)!\,t}{2.1(1-2xt+t^2)^{1+1/2}}\right]$$

$$= \frac{d^{m-2}}{dx^{m-2}}\left[\frac{d}{dx}\left\{\frac{(2.1)!\,t}{2.1(1-2xt+t^2)^{1+1/2}}\right\}\right]$$

$$= \frac{d^{m-2}}{dx^{m-2}}\left[\frac{(2.1)!\,t}{2.1}\left(-\frac{3}{2}\right)(1-2xt+t^2)^{-5/2}(-2t)\right]$$

$$= \frac{d^{m-2}}{dx^{m-2}}\left[\frac{12t^2}{2^2.1(1-2xt+t^2)^{5/2}}\right]$$

$$= \frac{d^{m-2}}{dx^{m-2}}\left[\frac{24t^2}{2^2.2.1(1-2xt+t^2)^{2+1/2}}\right]$$

$$= \frac{d^{m-2}}{dx^{m-2}}\left[\frac{(2.2)!\,t^2}{2^2.2.1(1-2xt+t^2)^{2+1/2}}\right]$$

$$= \frac{d^{m-3}}{dx^{m-3}}\left[\frac{(2.3)!\,t^3}{2^3.3.2.1(1-2xt+t^2)^{3+1/2}}\right]$$

and finally after differentiating $(m-3)$ times, we get

$$\frac{d^m}{dx^m}\left[\frac{1}{\sqrt{1-2xt+t^2}}\right] = \frac{(2m)!\,t^m}{2m.\,m!(1-2xt+t^2)^{m+1/2}}$$

Thus from equation (1), we get

$$\frac{(2m)!\,t^m}{2m.\,m!(1-2xt+t^2)^{m+1/2}} = \sum_{n=0}^{\infty}\frac{d^m}{dx^m}P_n(x)t^n$$

Multiplying both sides $(1-x^2)^{m/2}$, we get

$$\frac{(2m)!(1-x^2)^{m/2}t^m}{2^m m!(1-2xt+t^2)^{m+1/2}} = \sum_{n=0}^{\infty}\left[(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)t^n\right]t^n$$

$$= \sum_{n=0}^{\infty}P_n^m(x)t^n$$

$$= \sum_{n=0}^{m-1}P_n^m(x)t^n + \sum_{n=m}^{\infty}P_n^m(x)t^n$$

$$= 0 + \sum_{n=m}^{\infty}P_n^m(x)t^n \quad [\text{since } P_n^m(x) = 0 \text{ for } m > n]$$

$$= \sum_{n=m}^{\infty}P_n^m(x)t^n$$

12.21 RECURRENCE RELATIONS

THEOREM (12.18): Prove that

$$(n+1-m)P_{n+1}^{m}(x)-(2n+1)xP_{n}^{m}(x)+(n+m)P_{n-1}^{m}(x)=0$$

PROOF: We know from theorem (12.6) that

$$P_{n+1}(x)=\frac{2n+1}{n+1}xP_n(x)-\frac{n}{n+1}P_{n-1}(x)$$

or $$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x) \qquad (1)$$

Differentiating equation (1) m times

$$(2n+1)\frac{d^m}{dx^m}[xP_n(x)]=(n+1)\frac{d^m}{dx^m}P_{n+1}(x)+n\frac{d^m}{dx^m}P_{n-1}(x)$$

Using Leibnitz's rule on the L.H.S., we get

$$(2n+1)\left[x\frac{d^m}{dx^m}P_n(x)+m\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right]=(n+1)\frac{d^m}{dx^m}P_{n+1}(x)+n\frac{d^m}{dx^m}P_{n-1}(x)$$

Multiplying both sides by $(1-x^2)^{m/2}$, we get

$$(2n+1)\left[x(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)+m(1-x^2)^{m/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right]$$

$$=(n+1)(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n+1}(x)+n(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n-1}(x)$$

or $$(2n+1)\left[x(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)+m\sqrt{1-x^2}(1-x^2)^{(m-1)/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right]$$

$$=(n+1)(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n+1}(x)+n(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n-1}(x)$$

or $$(2n+1)xP_n^m(x)+(2n+1)m\sqrt{1-x^2}\,P_n^{m-1}(x)=(n+1)P_{n+1}^m(x)+nP_{n-1}^m(x) \qquad (2)$$

Also, we know from theorem (12.7) that

$$xP_n'(x)=nP_n(x)+P_{n-1}'(x)$$

Differentiating this equation $(m-1)$ times

$$\frac{d^{m-1}}{dx^{m-1}}[xP_n'(x)]=n\frac{d^{m-1}}{dx^{m-1}}P_n(x)+\frac{d^{m-1}}{dx^{m-1}}P_{n-1}'(x)$$

or $$x\frac{d^{m-1}}{dx^{m-1}}P_n'(x)+(m-1)\frac{d^{m-2}}{dx^{m-2}}P_n'(x)=n\frac{d^{m-1}}{dx^{m-1}}P_n(x)+\frac{d^{m-1}}{dx^{m-1}}P_{n-1}'(x)$$

Multiplying both sides by $(1-x^2)^{m/2}$, we get

$$x(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)+(m-1)(1-x^2)^{m/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x)$$

$$=n(1-x^2)^{m/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x)+(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n-1}(x)$$

or $$xP_n^m(x)+(m-1)\sqrt{1-x^2}\,P_n^{m-1}(x)=n\sqrt{1-x^2}\,P_n^{m-1}(x)+P_{n-1}^m(x)$$

or $$(n-m+1)\sqrt{1-x^2}\,P_n^{m-1}(x)=xP_n^m(x)-P_{n-1}^m(x)$$

or $$\sqrt{1-x^2}\,P_n^{m-1}(x)=\frac{1}{n-m+1}\left[xP_n^m(x)-P_{n-1}^m(x)\right] \qquad (3)$$

From equations (2) and (3), we get

$$(2n+1)xP_n^m(x)+\frac{(2n+1)m}{n-m+1}\left[xP_n^m(x)-P_{n-1}^m(x)\right]=(n+1)P_{n+1}^m(x)+nP_{n-1}^m(x)$$

or $$\left[(2n+1)+\frac{(2n+1)m}{n-m+1}\right]xP_n^m(x)=(n+1)P_{n+1}^m(x)+\left[n+\frac{(2n+1)m}{n-m+1}\right]P_{n-1}^m(x)$$

or $$\frac{(2n+1)(n+1)}{n-m+1}xP_n^m(x)=(n+1)P_{n+1}^m(x)+\frac{(n+m)(n+1)}{n-m+1}P_{n-1}^m(x)$$

or $$(n-m+1)P_{n+1}^m(x)-(2n+1)xP_n^m(x)+(n+m)P_{n-1}^m(x)=0$$

THEOREM (12.19): Prove that

$$\mathbf{P_n^{m+2}(x)-\frac{2(m+1)x}{\sqrt{1-x^2}}P_n^{m+1}(x)-(n-m)(n+m+1)P_n^m(x)=0}$$

SOLUTION: We know that the differential equation for u is

$$(1-x^2)u''-2(m+1)xu'+\left[n(n+1)-m(m+1)\right]u=0 \quad \text{where } u=\frac{d^mP_n}{dx^m}$$

or $$(1-x^2)\frac{d^{m+2}P_n}{dx^{m+2}}-2(m+1)x\frac{d^{m+1}P_n}{dx^{m+1}}+\left[n(n+1)-m(m+1)\right]\frac{d^mP_n}{dx^m}=0$$

Multiplying by $(1-x^2)^{m/2}$, we get

$$(1-x^2)^{(m+2)/2}\frac{d^{m+2}P_n}{dx^{m+2}}-2(m+1)x(1-x^2)^{m/2}\frac{d^{m+1}P_n}{dx^{m+1}}$$

$$+\left[n(n+1)-m(m+1)\right](1-x^2)^{m/2}\frac{d^mP_n}{dx^m}=0$$

$$(1-x^2)^{(m+2)/2}\frac{d^{m+2}P_n}{dx^{m+2}}-2(m+1)x(1-x^2)^{-1/2}(1-x^2)^{(m+1)/2}\frac{d^{m+1}P_n}{dx^{m+1}}$$

$$+\left[(n-m)(n+m+1)\right](1-x^2)^{m/2}\frac{d^mP_n}{dx^m}=0$$

or $$P_n^{m+2}(x)-\frac{2(m+1)x}{\sqrt{1-x^2}}P_n^{m+1}(x)+(n-m)(n+m+1)P_n^m(x)$$

12.22 ORTHOGONALITY OF ASSOCIATED LEGENDRE FUNCTIONS

THEOREM (12.20): Prove that

$$\int_{-1}^{1} P_n^m(x)\, P_\ell^m(x)\, dx = \begin{cases} 0 & \text{if } n \neq \ell \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{if } n = \ell \end{cases}$$

i.e. $P_n^m(x)$ and $P_\ell^m(x)$ are orthogonal in $-1 \le x \le 1$.

PROOF: Since $P_n^m(x)$ and $P_\ell^m(x)$ satisfy the associated Legendre's differential equation

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0, \text{ therefore}$$

$$(1-x^2)P_n^{m+2} - 2xP_n^{m+1} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]P_n^m = 0 \tag{1}$$

and
$$(1-x^2)P_\ell^{m+2} - 2xP_\ell^{m+1} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m = 0 \tag{2}$$

Multiplying equation (1) by P_ℓ^m, equation (2) by P_n^m and subtracting, we get

$$(1-x^2)\left(P_n^{m+2}P_\ell^m - P_\ell^{m+2}P_n^m\right) - 2x\left(P_n^{m+1}P_\ell^m - P_\ell^{m+1}P_n^m\right)$$
$$+\left[n(n+1) - \ell(\ell+1)\right]P_n^m P_\ell^m = 0$$

This can be written as

$$(1-x^2)\frac{d}{dx}\left(P_n^{m+1}P_\ell^m - P_\ell^{m+1}P_n^m\right) - 2x\left(P_n^{m+1}P_\ell^m - P_\ell^{m+1}P_n^m\right)$$
$$= \left[\ell(\ell+1) - n(n+1)\right]P_n^m P_\ell^m$$

or
$$\frac{d}{dx}\left[(1-x^2)\left(P_n^{m+1}P_\ell^m - P_\ell^{m+1}P_n^m\right)\right] = \left[\ell(\ell+1) - n(n+1)\right]P_n^m P_\ell^m$$

Integrating both sides w.r.t. x from -1 to 1, we get

$$\left[\ell(\ell+1) - n(n+1)\right]\int_{-1}^{1} P_n^m(x)P_\ell^m(x)\,dx = \left|(1-x^2)\left(P_n^{m+1}P_\ell^m - P_\ell^{m+1}P_n^m\right)\right|_{-1}^{1} = 0$$

Then since $n \neq \ell$

$$\int_{-1}^{1} P_n^m(x)P_\ell^m(x)\,dx = 0 \tag{3}$$

If $n = \ell$, then in this case, we have to prove

$$\int_{-1}^{1} \left[P_n^m(x)\right]^2 dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

Now $$\int_{-1}^{1} [P_n^m(x)]^2 \, dx = \int_{-1}^{1} (1-x^2)^m \frac{d^m P_n}{dx^m} \frac{d^m P_n}{dx^m} \, dx$$

Integrating by parts, we get

$$\int_{-1}^{1} [P_n^m(x)]^2 \, dx = \left| (1-x^2)^m \frac{d^m P_n}{dx^m} \frac{d^{m-1} P_n}{dx^{m-1}} \right|_{-1}^{1}$$

$$- \int_{-1}^{1} \frac{d^{m-1} P_n}{dx^{m-1}} \frac{d}{dx}\left[(1-x^2)^m \frac{d^m P_n}{dx^m} \right] dx$$

$$= - \int_{-1}^{1} \frac{d^{m-1} P_n}{dx^{m-1}} \frac{d}{dx}\left[(1-x^2)^m \frac{d^m P_n}{dx^m} \right] dx \qquad (4)$$

Now $$\frac{d}{dx}\left[(1-x^2)^m \frac{d^m P_n}{dx^m} \right] = (1-x^2)^m \frac{d^{m+1} P_n}{dx^{m+1}} - 2mx(1-x^2)^{m-1} \frac{d^m P_n}{dx^m} \qquad (5)$$

Next, recall that the differential equation satisfied by $u = \dfrac{d^m P_n}{dx^m}$ is

$$(1-x^2)u'' - 2(m+1)xu' + [n(n+1) - m(m+1)]u = 0$$

or $$(1-x^2)\frac{d^{m+2} P_n}{dx^{m+2}} - 2(m+1)\frac{d^{m+1} P_n}{dx^{m+1}} + [n(n-1) - m(m+1)] \frac{d^m P_n}{dx^m} = 0$$

Replacing m by $m-1$, we get

$$(1-x^2)\frac{d^{m+1} P_n}{dx^{m+1}} - 2mx\frac{d^m P_n}{dx^m} + [n(n+1) - m(m-1)] \frac{d^{m-1} P_n}{dx^{m-1}} = 0$$

Multiplying by $(1-x^2)^{m-1}$, we get

$$(1-x^2)^m \frac{d^{m+1} P_n}{dx^{m+1}} - 2mx(1-x^2)^{m-1}\frac{d^m P_n}{dx^m} + [n(n+1) - m(m-1)]$$
$$(1-x^2)^{m-1}\frac{d^{m-1} P_n}{dx^{m-1}} = 0$$

or $$(1-x^2)^m \frac{d^{m+1} P_n}{dx^{m+1}} - 2mx(1-x^2)^{m-1}\frac{d^m P_n}{dx^m}$$
$$= -[n(n+1) - m(m-1)] (1-x^2)^{m-1}\frac{d^{m-1} P_n}{dx^{m-1}}$$

Thus equation (5) becomes

$$\frac{d}{dx}\left[(1-x^2)^m \frac{d^m P_n}{dx^m} \right] = -[n(n+1) - m(m-1)] (1-x^2)^{m-1}\frac{d^{m-1} P_n}{dx^{m-1}}$$

$$= -\left[(n+m)(n-m+1)\right](1-x^2)^{m-1}\frac{d^{m-1}P_n}{dx^{m-1}}$$

and from equation (4) , we get

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = (n+m)(n-m+1)\int_{-1}^{1}(1-x^2)^{m-1}\frac{d^{m-1}P_n}{dx^{m-1}}\frac{d^{m-1}P_n}{dx^{m-1}}$$

$$= (n+m)(n-m+1)\int_{-1}^{1}\left[P_n^{m-1}(x)\right]^2 dx$$

Using this formula repeatedly m times to obtain

$$= (n+m)(n+m-1)(n-m+1)(n-m+2)\int_{-1}^{1}\left[P_n^{m-2}(x)\right]^2 dx$$

$$= \left[(n+m)(n+m-1)(n+m-2)\right]\left[(n-m+1)(n-m+2)(n-m+3)\right]\int_{-1}^{1}\left[P_n^{m-3}(x)\right]^2 dx$$

$$= \left[(n+m)(n+m-1)(n+m-2)\ldots(n+1)\right]$$

$$\left[(n-m+1)(n-m+2)(n-m+3)\ldots(n-1)n\right]\int_{-1}^{1}\left[P_n^0(x)\right]^2 dx \tag{6}$$

The binomial coefficients denoted by $\binom{n}{m}$ are defined as

$$\binom{n}{m} = \frac{n(n-1)(n-2)\ldots(n-m+1)}{m!} = \frac{n!}{m!(n-m)!}$$

This implies that

$$n(n-1)(n-2)\ldots(n-m+1) = \frac{n!}{(n-m)!} \tag{7}$$

From equations (6) and (7) , we get

$$\int_{-1}^{1}\left[P_n^m(x)\right]^2 dx = \left[(n+m)(n+m-1)(n+m-2)\ldots(n+1)\right]\frac{n!}{(n-m)!}\int_{-1}^{1}\left[P_n(x)\right]^2 dx$$

$$= \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}$$

12.23 EXPANSION OF $f(x)$ IN A SERIES OF ASSOCIATED LEGENDRE FUNCTIONS

THEOREM (12.21): Let $f(x)$ be defined over the interval $-1 < x < 1$. Then prove that at every point of continuity of $f(x)$ in $-1 < x < 1$, the expansion of $f(x)$ in associated Legendre functions is

$$f(x) = A_m P_m^m(x) + A_{m+1} P_{m+1}^m(x) + A_{m+2} P_{m+2}^m(x) + \ldots\ldots$$

$$= \sum_{n=m}^{\infty} A_n P_n^m(x)$$

$$\text{where } A_k = \frac{2k+1}{2} \frac{(k-m)!}{(k+m)!} \int_{-1}^{1} f(x)\, P_k^m(x)\, dx$$

At a point of discontinuity of $f(x)$ in $-1 < x < 1$, the left side be replaced by $\frac{1}{2}\left[f(x+0) - f(x-0) \right]$

PROOF: Multiplying the given series by $P_\ell^m(x)$ and integrating from -1 to 1, we have on using theorem (12.20).

$$\int_{-1}^{1} f(x) P_\ell^m(x)\, dx = \sum_{n=m}^{\infty} A_n \int_{-1}^{1} P_n^m(x)\, P_\ell^m(x)\, dx$$

$$= A_\ell \int_{-1}^{1} \left[P_\ell^m(x) \right]^2 dx$$

$$= A_\ell \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$$

or $$A_\ell = \frac{2\ell+1}{2} \frac{(\ell-m)!}{(\ell+m)!} \int_{-1}^{1} f(x)\, P_\ell^m(x)\, dx$$

or $$A_k = \frac{2k+1}{2} \frac{(k-m)!}{(k+m)!} \int_{-1}^{1} f(x)\, P_k^m(x)\, dx$$

12.24 SOLVED PROBLEMS

PROBLEM (1): **Using theorem (12.2), show that**

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1),$$

$$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

SOLUTION: We know that the generating function for Legendre polynomials is

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-1/2}$$

$$= \left[1-(2x-t)t\right]^{-1/2}$$

$$= 1+\frac{1}{2}(2x-t)t+\frac{1.3}{2.4}(2x-t)^2t^2+\frac{1.3.5}{2.4.6}(2x-t)^3t^3$$

$$+\frac{1.3.5.7}{2.4.6.8}(2x-t)^4t^4+\ldots\ldots$$

$$= 1+xt-\frac{1}{2}t^2+\frac{3}{2}x^2t^2-\frac{3}{2}xt^3+\frac{3}{8}t^4+\frac{5}{2}x^3t^3-\frac{15}{4}x^2t^4$$

$$+\frac{18}{5}xt^5-\frac{5}{16}t^6+\frac{35}{8}x^4t^4-\frac{35}{4}x^3t^5+\frac{105}{16}x^2t^6-\frac{35}{16}xt^7+\frac{35}{128}t^8+\ldots\ldots$$

or $$P_0(x)+P_1(x)t+P_2(x)t^2+P_3(x)t^3+P_4(x)t^4+\ldots\ldots$$

$$= 1+xt+\frac{1}{2}(3x^2-1)t^2+\frac{1}{2}(5x^3-3x)t^3+\frac{1}{8}(35x^4-30x^2+3)+\ldots\ldots$$

Comparing the coefficients of like powers on both sides, we get

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x), \quad P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

PROBLEM (2): **Given that $P_0(x) = 1$, $P_1(x) = x$, then using theorem (12.6) find**

(i) $P_2(x)$ **(ii)** $P_3(x)$ **(iii)** $P_4(x)$

SOLUTION: We have the recurrence formula of theorem (12.6)

$$P_{n+1}(x) = \frac{2n+1}{n+1} \times P_n(x) - \frac{n}{n+1} P_{n-1}(x) \qquad (1)$$

(i) Letting $n = 1$ in equation (1)

$$P_2(x) = \frac{3}{2} \times P_1(x) - \frac{1}{2}P_0(x) = \frac{3}{2}x^2-\frac{1}{2} = \frac{1}{2}(3x^2-1)$$

(ii) Put $n = 2$ in equation (1), we get

$$P_3(x) = \frac{5}{3} \times P_2(x) - \frac{2}{3}P_1(x) = \frac{5}{3}x\left(\frac{3x^2-1}{2}\right)-\frac{2}{3}x = \frac{1}{2}(5x^3-3x)$$

(iii) Similarly letting $n = 3$ in equation (1)

$$P_4(x) = \frac{7}{4}x\,P_3(x) - \frac{3}{4}P_2(x) = \frac{7}{4}x\left(\frac{5x^3 - 3x}{2}\right) - \frac{3}{4}\left(\frac{3x^2 - 1}{2}\right)$$

$$= \frac{1}{8}(35x^4 - 30x^2 + 3)$$

PROBLEM (3): **Given that $P_0(x) = 1$ and $P_1(x) = x$, then show that**

(i) $$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

(ii) $$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

(iii) $$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

(iv) $$x^5 = \frac{8}{63}P_5(x) + \frac{4}{9}P_3(x) + \frac{3}{7}P_1(x)$$

SOLUTION: We know that

(i) $$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

or $$3x^2 - 1 = 2P_2(x)$$

or $$3x^2 = 2P_2(x) + 1 = 2P_2(x) + P_0(x)$$

or $$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

(ii) $$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

or $$5x^3 - 3x = 2P_3(x)$$

or $$5x^3 = 2P_3(x) + 3x = 2P_3(x) + 3P_1(x)$$

or $$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

(iii) $$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

or $$35x^4 - 30x^2 + 3 = 8P_4(x)$$

or $$35x^4 = 8P_4(x) + 30x^2 - 3$$

$$= 8P_4(x) + 30\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - 3P_0(x)$$

$$= 8P_4(x) + 20P_2(x) + 7P_0(x)$$

or $$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

(iv) $$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

or $$63x^5 - 70x^3 + 15x = 8P_5(x)$$

or $\quad 63x^5 = 8P_5(x) + 70x^3 - 15x$

$$= 8P_5(x) + 70\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] - 15P_1(x)$$

$$= 8P_5(x) + 28P_3(x) + 27P_1(x)$$

$$x^5 = \frac{8}{63}P_5(x) + \frac{4}{9}P_3(x) + \frac{3}{7}P_1(x)$$

PROBLEM (4): **Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.**

SOLUTION: Using problem (3), we have

$$x^4 + 2x^3 + 2x^2 - x - 3 = \left[\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right] + 2\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right]$$

$$+ 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - P_1(x) - 3P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \left(\frac{4}{7} + \frac{4}{3}\right)P_2(x)$$

$$+ \left(\frac{6}{5} - 1\right)P_1(x) + \left(\frac{1}{5} + \frac{2}{3} - 3\right)P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{32}{15}P_0(x)$$

PROBLEM (5): **Express $P(x) = 3P_3(x) + 2P_2(x) + 4P_1(x) + 5P_0(x)$ as a polynomial in x.**

SOLUTION: We know that

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Thus $\quad f(x) = 3P_3(x) + 2P_2(x) + 4P_1(x) + 5P_0(x)$

$$= 3\left(\frac{1}{2}\right)(5x^3 - 3x) + 2\left(\frac{1}{2}\right)(3x^2 - 1) + 4x + 5$$

$$= \frac{3}{2}(5x^3 - 3x) + (3x^2 - 1) + 4x + 5$$

$$= \frac{15}{2}x^3 - \frac{9}{2}x + 3x^2 - 1 + 4x + 5$$

$$= \frac{15}{2}x^3 + 3x^2 - \frac{1}{2}x + 4 = \frac{1}{2}(15x^3 + 6x^2 - x + 8)$$

PROBLEM (6): **Prove that**

$$\frac{1 - t^2}{(1 - 2xt + t^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n$$

SOLUTION: We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad (1)$$

Differentiating both sides of equation (1) w.r.t. t, we get

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} \qquad (2)$$

Multiplying both sides of equation (2) with $2t$, we get

$$\frac{2t(x-t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} 2nP_n(x)t^n \qquad (3)$$

Adding equations (1) and (3)

$$\frac{1}{(1-2xt+t^2)^{1/2}} + \frac{2t(x-t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x)t^n + \sum_{n=0}^{\infty} 2nP_n(x)t^n$$

$$\frac{1-2xt+t^2+2xt-2t^2}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)P_n(x)t^n$$

$$\frac{1-t^2}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)P_n(x)t^n$$

PROBLEM (7): **Prove that**

$$\frac{1+t}{t\sqrt{1-2xt+t^2}} - \frac{1}{t} = \sum_{n=0}^{\infty} \left[P_n(x)+P_{n+1}(x)\right] t^n$$

SOLUTION: We have

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Now
$$\frac{1+t}{t\sqrt{1-2xt+t^2}} - \frac{1}{t} = \frac{1}{t}(1-2xt+t^2)^{-1/2} + (1-2xt+t^2)^{-1/2} - \frac{1}{t}$$

$$= \frac{1}{t}\sum_{n=0}^{\infty} P_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^n - \frac{1}{t}$$

$$= \frac{1}{t}\left[P_0(x) + \sum_{n=1}^{\infty} P_n(x)t^n\right] + \sum_{n=0}^{\infty} P_n(x)t^n - \frac{1}{t}$$

$$= \frac{1}{t} + \frac{1}{t}\sum_{n=1}^{\infty} P_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^n - \frac{1}{t} \quad \left[\text{since } P_0(x) = 1\right]$$

$$= \sum_{n=1}^{\infty} P_n(x)t^{n-1} + \sum_{n=0}^{\infty} P_n(x)t^n$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^n$$

$$= \sum_{n=0}^{\infty} [P_{n+1}(x)+P_n(x)]t^n \quad (\text{as required})$$

PROBLEM (8): **Prove that**

$$x P_n'(x) = n P_n(x) + (2n-3) P_{n-2}(x) + (2n-7) P_{n-4}(x) + \ldots\ldots\ldots$$

SOLUTION: We have from theorem (12.8) that

$$x P_n'(x) - P_{n-1}'(x) = n P_n(x) \tag{1}$$

Also, we have

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x) \tag{2}$$

Replacing n by $n-2,\ n-4,\ n-6, \ldots\ldots\ldots$ successively in equation (2), we get

$$\left.\begin{aligned} P_{n-1}'(x) - P_{n-3}'(x) &= (2n-3) P_{n-2}(x) \\ P_{n-3}'(x) - P_{n-5}'(x) &= (2n-7) P_{n-4}(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right] \tag{3}$$

Adding equation (1) and (3)

$$x P_n'(x) = n P_n(x) + (2n-3) P_{n-2}(x) + (2n-7) P_{n-4}(x) + \ldots\ldots\ldots$$

PROBLEM (9): **Prove that**

$$P_{n+1}'(x) + P_n'(x) = P_0(x) + 3 P_1(x) + 5 P_2(x) + \ldots\ldots + (2n+1) P_n(x)$$

SOLUTION: We have the recurrence formula

$$(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \tag{1}$$

Substituting $n = 1, 2, 3, \ldots\ldots\ldots, (n-1),\ n$ successively in equation (1), we get

$$3 P_1(x) = P_2'(x) - P_0'(x)$$

$$5 P_2(x) = P_3'(x) - P_1'(x)$$

$$7 P_3(x) = P_4'(x) - P_2'(x)$$

$$9 P_4(x) = P_5'(x) - P_3'(x)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(2n-3) P_{n-2}(x) = P_{n-1}'(x) - P_{n-3}'(x)$$

$$(2n-1)P_{n-1}(x) = P_n'(x) - P_{n-2}'(x)$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

Adding all these and noting that in the sum of the R.H.S. all the terms cancel except the first two of the second column and the last two of the first column , we get

$$3P_1(x)+5P_2(x)+7P_3(x)+\dots\dots\dots+(2n+1)P_n(x)$$

$$= -P_0'(x) - P_1'(x) + P_n'(x) + P_{n+1}'(x) \qquad (2)$$

Since $P_0(x) = 1$ and $P_1(x) = x$, we have $P_0'(x) = 0$ and $P_1'(x) = 1 = P_0(x)$.

Thus equation (2) becomes

$$3P_1(x)+5P_2(x)+7P_3(x)+\dots\dots\dots+(2n+1)P_n(x) = -P_0(x)+P_n'(x)+P_{n+1}'(x)$$

or $\quad P_0(x)+3P_1(x)+5P_2(x)+\dots\dots\dots+(2n+1)P_n(x) = P_n'(x)+P_{n+1}'(x)$

which may also be written in summation notation as

$$\sum_{n=0}^{\infty} (2n+1)P_n(x) = P_n'(x)+P_{n+1}'(x)$$

INTEGRALS INVOLVING LEGENDRE POLYNOMIALS

PROBLEM (10): **Prove that**

$$\int_{-1}^{1} (1-x^2)\, P_m'(x) P_n'(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2n(n+1)}{2n+1} & \text{if } m = n \end{cases}$$

SOLUTION: **CASE (1): When $m \neq n$**

Integrating by parts , we have

$$\int_{-1}^{1} (1-x^2)\, P_m'(x) P_n'(x)\, dx = \int_{-1}^{1} \left[(1-x^2)\, P_m'(x)\right] P_n'(x)\, dx$$

$$= \left|(1-x^2)P_m' \cdot P_n(x)\right|_{-1}^{1}$$

$$- \int_{-1}^{1} P_n(x)\left[(1-x^2)P_m''(x) - 2xP_m'(x)\right] dx$$

$$= -\int_{-1}^{1} \left[(1-x^2)P_m''(x) - 2xP_m'(x)\right] P_n(x)\, dx \qquad (1)$$

Since $P_m(x)$ satisfies the Legendre's equation, therefore

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0$$

or $$(1-x^2)P_m''(x) - 2xP_m'(x) = -m(m+1)P_m(x) \qquad (2)$$

From equations (1) and (2), we get

$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx = -\int_{-1}^{1} -m(m+1)P_m(x)P_n(x)\,dx$$

$$= m(m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx \qquad (3)$$

$$= 0$$

using orthogonality relation, since $m \neq n$.

CASE (2): When $m = n$

In this case, equation (3) takes the form

$$\int_{-1}^{1}(1-x^2)[P_n'(x)]^2\,dx = n(n+1)\int_{-1}^{1}P_n^2(x)\,dx$$

$$= n(n+1)\left(\frac{2}{2n+1}\right) \qquad \text{(using orthogonality relationship)}$$

$$= \frac{2n(n+1)}{2n+1}$$

PROBLEM (11): **Prove that** $\displaystyle\int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

SOLUTION: From theorem (12.6), we have the recurrence formula

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x)$$

or $$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$$

Multiplying both sides of this equation by $P_{n-1}(x)$, and integrating w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{n+1}{2n+1}\int_{-1}^{1}P_{n+1}(x)P_{n-1}(x)\,dx$$

$$+\frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2\,dx \qquad (1)$$

Using the orthogonality relationship

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$

we have $$\int_{-1}^{1} [P_{n-1}(x)]^2\,dx = \frac{2}{2(n-1)+1} = \frac{2}{2n-1}$$

Thus equation (1) becomes

$$\int_{-1}^{1} x P_n(x)P_{n-1}(x)\,dx = 0 + \frac{n}{2n+1}\,\frac{2}{2n-1} = \frac{2n}{4n^2-1}$$

PROBLEM (12): **Prove that**

$$\int_{-1}^{1} (x^2-1)P_{n+1}(x)P_n'(x)\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

SOLUTION: From the Beltrami's result in theorem (12.8), we have

$$(2n+1)(x^2-1)P_n'(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

or $$(x^2-1)P_n'(x) = \frac{n(n+1)}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x)\right] \qquad (1)$$

Multiplying both sides of equation (1) by $P_{n+1}(x)$ and integrating w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} (x^2-1)P_n'(x)P_{n+1}(x)\,dx = \frac{n(n+1)}{2n+1}\int_{-1}^{1} [P_{n+1}(x)]^2\,dx$$

$$- \frac{n(n+1)}{2n+1}\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)\,dx \qquad (2)$$

Using the orthogonality relationship

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}, \text{ we have}$$

$$\int_{-1}^{1} [P_{n+1}(x)]^2\,dx = \frac{2}{2(n+1)+1} = \frac{2}{2n+3}$$

Thus equation (2) becomes

$$\int_{-1}^{1} (x^2-1)P_n'(x)P_{n+1}(x)\,dx = \frac{n(n+1)}{2n+1}\left(\frac{2}{2n+3}\right) + 0 = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

PROBLEM (13): **Prove that**

$$\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

SOLUTION: From theorem (12.6), we have recurrence relation

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

or $$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \tag{1}$$

Replacing n by $n-1$ in equation (1), we get

$$(2n-1) x P_{n-1}(x) = n P_n(x) + (n-1) P_{n-2}(x) \tag{2}$$

Next, replacing n by $n+1$ in equation (1), we get

$$(2n+3) x P_{n+1}(x) = (n+2) P_{n+2}(x) + (n+1) P_n(x) \tag{3}$$

Multiplying equations (2) and (3) and integrating w.r.t. x from -1 to 1, we get

$$\begin{aligned}
(2n-1)(2n+3) \int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx
&= n(n+2) \int_{-1}^{1} P_n(x) P_{n+2}(x)\, dx + (n-1)(n+2) \int_{-1}^{1} P_{n-2}(x) P_{n+2}(x)\, dx \\
&\quad + n(n+1) \int_{-1}^{1} [P_n(x)]^2\, dx + n
\end{aligned} \tag{4}$$

Using the orthogonality relationship

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n-1} & \text{if } m = n \end{cases}$$

equation (4) becomes

$$(2n-1)(2n+3) \int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx = n(n+1) \frac{2}{2n+1}$$

or $$\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

DEDUCTION

Since $P_n(x)$ is a polynomial of degree n, so $x^2 P_{n-1}(x) P_{n+1}(x)$ is a polynomial of degree $2+(n-1)+(n+1)=2(n+1)$. Since $2(n+1)$ is even, we see that $x^2 P_{n-1} P_{n+1}$ is an even function of x and hence

$$\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx = 2\int_{0}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx$$

or

$$\int_{0}^{1} x^2 P_{n-1}(x) P_{n+1}\, dx = \frac{1}{2}\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}(x)\, dx$$

$$= \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

PROBLEM (14): Show that $\displaystyle\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}\, dx = \frac{2t^n}{2n+1}$.

SOLUTION: We have

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$= P_0(x) + P_1(x)t + P_2(x)t^2 + \dots\dots\dots + P_n(x)t^n + \dots\dots\dots$$

Multiplying both sides by $P_n(x)$ and integrating w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}\, dx = \int_{-1}^{1} P_0(x)P_n(x)\, dx + t\int_{-1}^{1} P_1(x)P_n(x)\, dx$$

$$+ t^2\int_{-1}^{1} P_2(x)P_n(x)\, dx + \dots\dots\dots + t^n\int_{-1}^{1} P_n(x)P_n(x)$$

$$+ t^{n+1}\int_{-1}^{1} P^{n+1}(x)P_n(x)\, dx + \dots\dots\dots$$

Using the orthogonality relation

$$\int_{-1}^{1} P_m(x)P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$

we get

$$\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}\,dx = t^n \int_{-1}^{1} P_n^2(x)\,dx$$

$$= t^n\left(\frac{2}{2n+1}\right) = \frac{2t^n}{2n+1}$$

PROBLEM (15): **Prove that** $\displaystyle\int_{-1}^{1} x\,P_n'(x)\,P_n(x)\,dx = \frac{2n}{2n+1}$.

SOLUTION: We know that

$$x\,P_n'(x) = n\,P_n(x) + (2n-3)\,P_{n-2}(x) + (2n-7)\,P_{n-4}(x) + \ldots\ldots\ldots$$

Multiplying both sides by $P_n(x)$ and integrating w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} x\,P_n'(x)\,P_n(x)\,dx = n\int_{-1}^{1} P_n^2(x)\,dx + (2n-3)\int_{-1}^{1} P_{n-2}(x)\,P_n(x)\,dx$$

$$+ (2n-7)\int_{-1}^{1} P_{n-4}(x)\,P_n(x)\,dx + \ldots\ldots\ldots$$

Using orthogonality relation $\displaystyle\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$

we get $\displaystyle\int_{-1}^{1} x\,P_n'(x)\,P_n(x)\,dx = n\cdot\frac{2}{2n+1} + 0 + 0 + \ldots\ldots\ldots = \frac{2n}{2n+1}$

PROBLEM (16): **Prove that** $\displaystyle\int_{-1}^{1} \left[P_n'(x)\right]^2 dx = n(n+1)$.

SOLUTION: From Christoffel's expansion, we have

$$P_n'(x) = (2n-1)\,P_{n-1}(x) + (2n-5)\,P_{n-3}(x) + (2n-9)\,P_{n-5}(x) + \ldots\ldots\ldots \quad (1)$$

The last term on the R.H.S. of equation (1) is $3P_1(x)$ or $P_0(x)$ according as n is even or odd.

Squaring both sides of equation (1), we get when n is even

$$\left[P_n'(x)\right]^2 = (2n-1)^2 P_{n-1}^2(x) + (2n-5)^2 P_{n-3}^2(x)$$

$$+ (2n-9)^2 P_{n-5}^2(x) + \ldots\ldots\ldots + (3)^2 P_1^2(x)$$

$$+ 2(2n-1)(2n-5)\,P_{n-3}(x)\,P_{n-5}(x) + \ldots\ldots\ldots \quad (2)$$

and when n is odd

$$[P_n'(x)]^2 = (2n-1)^2 P_{n-1}^2(x) + (2n-5)^2 P_{n-3}^2(x) + (2n-9)^2 P_{n-5}^2(x) + \dots\dots + P_0^2(x) + 2(2n-1)(2n-5) P_{n-3}(x) P_{n-5}(x) + \dots\dots \quad (3)$$

Now there are two cases :

CASE (1): Integrating equation (2) w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = (2n-1)^2 \int_{-1}^{1} P_{n-1}^2(x)\, dx + (2n-5)^2 \int_{-1}^{1} P_{n-3}^2(x)\, dx + \dots\dots + (3)^2 \int_{-1}^{1} P_1^2(x)\, dx + \dots\dots + 2(2n-1)(2n-5) \int_{-1}^{1} P_{n-3}(x) P_{n-5}(x)\, dx + \dots\dots \quad (4)$$

Using the orthogonality relationship ,

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$, we get from equation (4)

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = (2n-1)^2 \frac{2}{2(n-1)+1} + (2n-5)^2 \frac{2}{2(n-3)+1} + \dots\dots + (3)^2 \frac{2}{2.1+1} + 0 + 0 + \dots\dots$$

$$= 2[(2n-1) + (2n-5) + \dots\dots + 3] \quad (5)$$

Let m be the number of terms in A.P. on the R.H.S. of equation (5) , then

$$(2n-1) + (m-1)(-4) = 3 \quad \text{or} \quad m = \frac{n}{2}$$

But sum of A.P. $= \dfrac{\text{number of terms}}{2}$ (first term + last term)

$$= \frac{n}{4}[(2n-1)+3] = \frac{n(n+1)}{2}$$

Hence equation (5) reduces to

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = 2 . \frac{1}{2} n(n+1) = n(n+1)$$

CASE (2): Integrating equation (3) w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = (2n-1)^2 \int_{-1}^{1} P_{n-1}^2(x) \, dx + (2n-5)^2 \int_{-1}^{1} P_{n-3}^2(x) \, dx$$

$$+ \ldots\ldots\ldots + \int_{-1}^{1} P_1^2(x) \, dx + \ldots\ldots\ldots$$

$$+2(2n-1)(2n-5) \int_{-1}^{1} P_{n-3}(x) P_{n-5}(x) \, dx + \ldots\ldots\ldots \quad (6)$$

Using the orthogonality relationship, we get

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = (2n-1)^2 \frac{2}{2(n-1)+1} + (2n-5)^2 \frac{2}{2(n-5)+1}$$

$$+ \ldots\ldots\ldots + \frac{2}{2.0+1} + 0 + 0 + \ldots\ldots\ldots$$

$$= 2\left[(2n-1) + (2n-5) + \ldots\ldots\ldots + 1\right] \quad (7)$$

Let p be the number of terms in A.P. on the R.H.S. of equation (7), then

$$(2n-1) + (p-1)(-4) = 1 \quad \text{or} \quad p = \frac{1}{2}(n+1)$$

But sum of A.P. $= \dfrac{(n+1)/2}{2}[2n-1+1] = \dfrac{1}{2}n(n+1)$

Hence equation (7) reduces to

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = 2 . \frac{1}{2} n(n+1) = n(n+1)$$

Thus for all values of n, we have

$$\int_{-1}^{1} [P_n'(x)]^2 \, dx = n(n+1)$$

PROBLEM (17): **Prove that**

$$\int_{-1}^{1} f(x) P_n(x) \, dx = \frac{(-1)^n}{2^n 2!} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x) \, dx$$

SOLUTION: Using Rodrigue's formula, we have

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} f(x)D^n(x^2-1)^n\,dx \qquad \left(\text{where } D^n = \frac{d^n}{dx^n}\right)$$

$$= \frac{1}{2^n n!}\left[\left|f(x).D^{n-1}(x^2-1)^n\right|_{-1}^{1} - \int_{-1}^{1} f'(x)D^{n-1}(x^2-1)^n\,dx\right]$$

The first term vanishes since $\left|D^{n-1}(x^2-1)^n\right|_{-1}^{1} = 0$ on using **Leibnitz's theorem**. **Therefore**

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{-1}{2^n n!}\int_{-1}^{1} f'(x)D^{n-1}(x^2-1)^n\,dx$$

$$= \frac{-1}{2^n n!}\left[\left|f'(x).D^{n-2}(x^2-1)^n\right|_{-1}^{1} - \int_{-1}^{1} f''(x)D^{n-2}(x^2-1)^n\,dx\right]$$

The first term again vanishes on using Leibnitz's rule. Thus

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{(-1)^2}{2^n n!}\int_{-1}^{1} f''(x)D^{n-2}(x^2-1)^n\,dx$$

On integrating the R.H.S. $(n-2)$ times, we get

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)D^{n-n}(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\,dx$$

This completes the proof.

PROBLEM (18): Expand $f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$

SOLUTION: We have

$$A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) f(x)\, dx$$

$$= \frac{2k+1}{2} \int_{-1}^{0} P_k(x) . 0\, dx + \frac{2k+1}{2} \int_{0}^{1} P_k(x) . x\, dx$$

$$= \frac{2k+1}{2} \int_{0}^{1} P_k(x) . x\, dx$$

Then $A_0 = \frac{1}{2} \int_{0}^{1} P_0(x) . x\, dx = \frac{1}{2} \int_{0}^{1} 1 . x\, dx = \frac{1}{4}$

$$A_1 = \frac{3}{2} \int_{0}^{1} P_1(x) . x\, dx = \frac{3}{2} \int_{0}^{1} x . x\, dx = \frac{1}{2}$$

$$A_2 = \frac{5}{2} \int_{0}^{1} P_2(x) . x\, dx = \frac{5}{2} \int_{0}^{1} \frac{3x^2 - 1}{2} x\, dx$$

$$= \frac{5}{4} \int_{0}^{1} (3x^3 - x)\, dx = \frac{5}{4} \left| \frac{3}{4} x^4 - \frac{x^2}{2} \right|_0^1 = \frac{5}{4}\left(\frac{1}{4}\right) = \frac{5}{16}$$

$$A_3 = \frac{7}{2} \int_{0}^{1} P_3(x) . x\, dx = \frac{7}{2} \int_{0}^{1} \frac{5x^3 - 3x}{2} x\, dx$$

$$= \frac{7}{4} \int_{0}^{1} (5x^4 - 3x^2)\, dx = \frac{7}{4} \left| x^5 - x^3 \right|_0^1 = 0$$

$$A_4 = \frac{9}{2} \int_{0}^{1} P_4(x) . x\, dx = \frac{9}{2} \int_{0}^{1} \frac{35x^4 - 30x^2 + 3}{8} x\, dx$$

$$= \frac{9}{16} \int_{0}^{1} (35x^5 - 30x^3 + 3x)\, dx$$

$$= \frac{9}{16}\left|\frac{35}{6}x^6 - \frac{15}{2}x^4 + \frac{3}{2}x^2\right|_0^1$$

$$= \frac{9}{16}\left(\frac{35}{6} - \frac{15}{2} + \frac{3}{2}\right) = \frac{9}{16}\left(-\frac{1}{6}\right) = -\frac{3}{32}$$

Thus $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x)\ldots\ldots\ldots$

PROBLEM (19): **Expand the function $f(x) = |x|, -1 < x < 1$, in a series of the form**

$$\sum_{k=0}^{\infty} A_k P_k(x).$$

SOLUTION: By definition $|x| = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$, we have

$$A_k = \frac{2k+1}{2}\int_{-1}^{1} P_k(x) f(x)\, dx$$

$$= \frac{2k+1}{2}\int_{-1}^{0} P_k(x).(-x)\, dx + \frac{2k+1}{2}\int_0^1 P_k(x).x\, dx$$

Changing x to $-x$ in the first integral, we get

$$= \frac{2k+1}{2}\int_0^1 P_k(-x).x\, dx + \frac{2k+1}{2}\int_0^1 P_k(x).x\, dx$$

$$= \frac{2k+1}{2}\left[(-1)^k + 1\right]\int_0^1 P_k(x).x\, dx \qquad \left[\text{Since } P_k(-x) = (-1)^k P_k(x)\right]$$

Then $A_0 = \frac{1}{2}.2.\int_0^1 P_0(x).x\, dx = \int_0^1 x\, dx = \frac{1}{2}$

$$A_1 = \frac{3}{2}.0.\int_0^1 P_1(x).x\, dx = 0$$

$$A_2 = \frac{5}{2}.2.\int_0^1 P_2(x).x\, dx = 5\int_0^1 \frac{3x^2-1}{2}\, x\, dx$$

$$= \frac{5}{2}\int_0^1 (3x^3 - x)\, dx = \frac{5}{2}\left|\frac{3}{4}x^4 - \frac{1}{2}x^2\right|_0^1 = \frac{5}{2}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{5}{8}$$

$$A_3 = \frac{7}{2}.0.\int_0^1 P_3(x).x\,dx = 0$$

$$A_4 = \frac{9}{2}.2.\int_0^1 P_4(x).x\,dx = 9\int_0^1 \frac{35x^4-30x^2+3}{8}\,x\,dx$$

$$= \frac{9}{8}\int_0^1 (35x^5-30x^3+3x)\,dx$$

$$= \frac{9}{8}\left|\frac{35}{6}x^6-\frac{15}{2}x^4+\frac{3}{2}x^2\right|_0^1$$

$$= \frac{7}{8}\left(\frac{35}{6}-\frac{15}{2}+\frac{3}{2}\right) = \frac{9}{8}\left(-\frac{1}{6}\right) = -\frac{3}{16}\text{, etc.}$$

Thus $f(x) = \frac{1}{2}P_0(x)+\frac{5}{8}P_2(x)-\frac{3}{16}P_4(x)+\ldots\ldots\ldots$

PROBLEM (20): **Expand the function $f(x) = x^2$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.**

SOLUTION: **METHOD (1)**

We know that $x^2 = \sum_{k=0}^{\infty} A_k P_k(x)$

Then $A_k = \frac{2k+1}{2}\int_{-1}^{1} x^2 P_k(x)\,dx$

Putting $k = 0, 1, 2, 3 \ldots\ldots\ldots$, we find as before

$$A_0 = \frac{1}{3},\quad A_1 = 0,\quad A_2 = \frac{2}{3},\quad A_3 = 0,\quad A_4 = 0,\ldots\ldots\ldots$$

so that $x^2 = \frac{1}{3}P_0(x)+\frac{2}{3}P_2(x)$.

METHOD (2)

We must find A_k, $k = 0, 1, 2, 3, \ldots\ldots$ such that

$$x^2 = A_0P_0(x)+A_1P_1(x)+A_2P_2(x)+A_3P_3(x)+$$

$$= A_0.1+A_1.x+A_2\left(\frac{3x^2-1}{2}\right)+A_3\left(\frac{5x^3-3x}{2}\right)+$$

Since the left side is a polynomial of degree 2 , we must have

$$A_3 = 0, \quad A_4 = 0, \quad A_5 = 0, \ldots\ldots\ldots$$

Thus $x^2 = A_0 - \frac{A_2}{2} + A_1 x + \frac{3}{2} A_2 x^2$

from which $A_0 - \frac{A_2}{2} = 0, \quad A_1 = 0, \quad \frac{3}{2} A_2 = 1$

Thus $A_0 = \frac{1}{3}, \quad A_1 = 0, \quad A_2 = \frac{2}{3}$

i.e. $x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)$

PROBLEM (21): **Show that**

(i) $P_2^2(x) = 3(1-x^2)$ (ii) $P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$

(iii) $P_3^2(x) = 15(1-x^2)$ (iv) $P_3^3(x) = 15(1-x^2)^{3/2}$

SOLUTION: We know that

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n \qquad (1)$$

(i) Let $m = n = 2$ in equation (1) , we get

$$P_2^2(x) = \frac{(1-x^2)}{2^2 . 2!} \frac{d^4}{dx^4}(x^2-1)^2$$

$$= \frac{(1-x^2)}{8} \frac{d^3}{dx^3}(4x^3-4x)$$

$$= \frac{(1-x^2)}{8} \frac{d^2}{dx^2}(12x^2-4)$$

$$= \frac{(1-x^2)}{8} \frac{d}{dx}(24x) = \frac{1-x^2}{8} . 24 = 3(1-x^2)$$

(ii) Let $m = 1$, $n = 3$ in equation (1) , we get

$$P_3^1(x) = \frac{(1-x^2)^{1/2}}{2^3 . 3!} \frac{d^4}{dx^4}(x^2-1)^3$$

$$= \frac{(1-x^2)^{1/2}}{48}(360x^2-72)$$

$$= \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$$

(iii) Let $m = 2$, $n = 3$ in equation (1) , we get

$$P_3^2(x) = \frac{(1-x^2)^{1/2}}{2^3 . 3!} \frac{d^5}{dx^5}(x^2-1)^3$$

$$= \frac{(1-x^2)}{48}(720x) = 15x(1-x^2)$$

(iv) Let $m = n = 3$ in equation (1), we get

$$P_3^3(x) = \frac{(1-x^2)^{3/2}}{2^3 \cdot 3!} \frac{d^6}{dx^6}(x^2-1)^3$$

$$= \frac{(1-x^2)^{3/2}}{48} 720 = 15(1-x^2)^{3/2}$$

12.25 EXERCISE

PROBLEM (1): Express $2 - 3x + 4x^2$ in terms of Legendre polynomials.

PROBLEM (2): Express $x^4 - 3x^2 + x$ in terms of Legendre polynomials.

PROBLEM (3): Express $4P_3(x) + 6P_2(x) - 3P_1(x) - 2P_0(x)$ in terms of polynomial in x.

PROBLEM (4): Express $8P_4(x) + 2P_2(x) + P_0(x)$ in terms of polynomial in x.

PROBLEM (5): Evaluate using orthogonality relations :

(i) $$\int_{-1}^{1} P_1(x) P_2(x)\, dx$$ (ii) $$\int_{-1}^{1} [P_2(x)]^2\, dx$$

PROBLEM (6): Expand $f(x) = \begin{cases} 0 & -1 < x < 0 \\ 2x+1 & 0 < x < 1 \end{cases}$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

PROBLEM (7): Expand $f(x) = \begin{cases} -1 & \text{for } -1 \le x \le 0 \\ 1 & \text{for } 0 < x \le 1 \end{cases}$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

PROBLEM (8): If $f(x) = \sum_{k=0}^{\infty} A_k P_k(x)$, obtain the Parseval's identity

$$\int_{-1}^{1} [f(x)]^2\, dx = \sum_{k=0}^{\infty} \frac{A_k^2}{2k+1}$$

PROBLEM (9): Show that

(i) $$\int_{-1}^{1} x^2 P_3(x)\, dx = 0$$ (ii) $$\int_{-1}^{1} x^3 P_3(x)\, dx = \frac{4}{35}$$

(iii) $$\int_{-1}^{1} x^3 P_4(x)\, dx = 0$$ (iv) $$\int_{-1}^{1} x^6 P_4(x)\, dx = \frac{16}{231}$$

PROBLEM (10): Prove that

(i) $$\int_{-1}^{1} P_n(x)\,P'_{n+1}(x)\,dx = 2$$

(ii) $$\int_{-1}^{1} P'_{n-1}(x)\,P'_{n+1}(x)\,dx = n(n-1)$$

PROBLEM (11): Show that the general solution of the differential equation :

$$(1-x^2)y'' - 2xy' + 2ny = 0$$

is $y = C_1 x + C_2\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$

PROBLEM (12): Show that

$$P_4^1(x) = \frac{5}{7}(7x^3 - 3x)(1-x^2)^{1/2}$$

$$P_4^2(x) = \frac{15}{2}(7x^2 - 1)(1-x^2)$$

$$P_4^3(x) = 105\,x(1-x^2)^{3/2}$$

$$P_4^4(x) = 105(1-x^2)^2$$

CHAPTER 13

HYPERGEOMETRIC FUNCTIONS

13.1 INTRODUCTION

In the preceding two chapters, we have discussed the solutions of Bessel's and Legendre's differential equations. In this chapter, we shall discuss the solutions of another very important differential equation of applied mathematics and mathematical physics, known as hypergeometric equation. Furthermore, the solutions of confluent hypergeometric differential equation will also be discussed.

The major development of the theory of hypergeometric functions was carried out by Gauss. Some important results concerning the hypergeometric functions had been developed earlier by Euler and others, but it was Gauss who made the first symmetric study of hypergeometric functions.

13.2 HYPERGEOMETRIC DIFFERENTIAL EQUATION

The second order homogeneous linear differential equation

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

where a, b, c are constants, is known as Gauss's differential equation or hypergeometric differential equation named after the German mathematician C.F. Gauss (1777 – 1855). The solutions of this equation are called **hypergeometric functions**, which are more general than the other special functions.

13.3 THE POCHHAMMER SYMBOL

The Pochhammer symbol denoted by $(a)_n$ is defined as

$$(a)_n = a(a+1)(a+2)\ldots\ldots(a+n-1), \quad n = 1, 2, 3, \ldots\ldots \qquad (1)$$

with $(a)_0 = 1$ for $a \neq 0$.

DEDUCTION

(i) If $n = 1$ in equation (1), we see that

$$(1)_n = 1.2.3\ldots\ldots n = n!$$

Thus the Pochhammer symbol is the generalization of the ordinary factorial function.

(ii) $$(a)_{n+1} = a(a+1)(a+2)\ldots\ldots[a+(n+1)-1]$$
$$= a[(a+1)(a+2)\ldots\ldots\{(a+1)+n-1\}] = a(a+1)_n$$

(iii) $$(a+n)(a)_n = a(a+1)\ldots\ldots(a+n-1)(a+n)$$
$$= a(a+1)\ldots\ldots(a+n-1)[(a+(n+1)-1] = (a)_{n+1}$$

From (ii) and (iii), we have

$$(a+n)(a)_n = a(a+1)_n$$

RELATIONSHIP BETWEEN THE POCHHAMMER SYMBOL AND GAMMA FUNCTION

We can write

$$(a)_n = a(a+1)(a+2)\ldots\ldots(a+n-1)$$
$$= \frac{1.2.3\ldots\ldots(a-2)(a-1)a(a+1)(a+2)\ldots\ldots(a+n-1)}{1.2.3\ldots\ldots(a-2)(a-1)} = \frac{\Gamma(a+n)}{\Gamma(a)}$$

Thus $$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

13.4 SOLUTION OF HYPERGEOMETRIC DIFFERENTIAL EQUATION

We shall solve the hypergeometric differential equation

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \tag{1}$$

by the method of Frobenius.

STEP (1): Let the series solution be

$$y(x) = x^{\beta}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \tag{2}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots\ldots$

Then by differentiation, omitting the summation index k, we get

$$y' = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y'' = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

STEP (2): Substituting y, y', and y'' in equation (1), we get

$$x(1-x)\sum (k+\beta)(k+\beta-1)a_k x^{k+\beta-2}$$
$$+[c-(a+b+1)x]\sum (k+\beta)a_k x^{k+\beta-1} - ab\sum a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1)a_k x^{k+\beta-1} - \sum (k+\beta)(k+\beta-1)a_k x^{k+\beta}$$
$$+\sum c(k+\beta)a_k x^{k+\beta-1} - \sum (a+b+1)(k+\beta)a_k x^{k+\beta} - \sum ab\, a_k x^{k+\beta} = 0$$

or $$\sum [(k+\beta)(k+\beta-1)+c(k+\beta)]a_k x^{k+\beta-1}$$
$$-\sum [(k+\beta)(k+\beta-1)+(a+b+1)(k+\beta)+ab]a_k x^{k+\beta} = 0$$

or
$$\sum\left[(k+\beta)(k+\beta-1+c)\right]a_k x^{k+\beta-1}$$
$$-\sum\left[(k+\beta)^2+(a+b)(k+\beta)+ab\right]a_k x^{k+\beta}=0$$

Shifting index k by $k-1$ in the second summation, we get
$$\sum\left[(k+\beta)(k+\beta-1+c)\right]a_k x^{k+\beta-1}$$
$$-\sum\left[(k+\beta-1)^2+(a+b)(k+\beta-1)+ab\right]a_{k-1}x^{k+\beta-1}=0$$

or
$$\sum\left[(k+\beta)(k+\beta-1+c)a_k\right.$$
$$\left.-\left\{(k+\beta-1)^2+(a+b)(k+\beta-1)+ab\right\}a_{k-1}\right]x^{k+\beta-1}=0 \qquad (3)$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by substituting $k=0$ in equation (3)

i.e. $\left[\beta(\beta-1+c)a_0-\left\{(\beta-1)^2+(a+b)(\beta-1)+ab\right\}a_{-1}\right]x^{\beta-1}=0$

Since $x^{\beta-1}\neq 0$, the coefficient of $x^{\beta-1}$ must be zero, therefore
$$\beta(\beta-1+c)a_0-\left[(\beta-1)^2+(a+b)(\beta-1)+ab\right]a_{-1}=0.$$

But $a_{-1}=0$, we get indicial equation $\beta(\beta-1+c)a_0=0$

Since $a_0\neq 0$, therefore $\beta(\beta-1+c)=0$ or $\beta=0$, $\beta=1-c$ are the indicial roots which are unequal and differ not by an integer.

Equating to zero the coefficient of $x^{k+\beta-1}$, we get the recurrence relation from equation (3)

i.e. $$a_k=\frac{(k+\beta-1)^2+(a+b)(k+\beta-1)+ab}{(k+\beta)(k+\beta-1+c)}a_{k-1}$$
$$=\frac{(k+\beta-1+a)(k+\beta-1+b)}{(k+\beta)(k+\beta-1+c)}a_{k-1}$$

Let $k=1$, then $a_1=\dfrac{(\beta+a)(\beta+b)}{(\beta+1)(\beta+c)}a_0$

Let $k=2$, then $a_2=\dfrac{(\beta+1+a)(\beta+1+b)}{(\beta+2)(\beta+1+c)}a_1=\dfrac{(\beta+a)(\beta+b)(\beta+1+a)(\beta+1+b)}{(\beta+1)(\beta+2)(\beta+c)(\beta+1+c)}a_0$

Let $k=3$, then $a_3=\dfrac{(\beta+2+a)(\beta+2+b)}{(\beta+3)(\beta+2+c)}a_2$
$$=\frac{(\beta+a)(\beta+b)(\beta+1+a)(\beta+1+b)(\beta+2+a)(\beta+2+b)}{(\beta+1)(\beta+2)(\beta+3)(\beta+c)(\beta+1+c)(\beta+2+c)}a_0$$

and so on.

CASE (1): When $\beta=0$

In this case, the coefficients become
$$a_1=\frac{ab}{c}a_0,\quad a_2=\frac{ab(1+a)(1+b)}{1.2.c(1+c)}a_0,\quad a_3=\frac{ab(1+a)(1+b)(2+a)(2+b)}{1.2.3.c(1+c)(2+c)}a_0$$

Substituting the values of these coefficients in equation (2), we get one solution as

$$y_1(x) = a_0\left[1+\frac{ab}{1.c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots\ldots\right] \tag{4}$$

where $c \neq 0, -1, -2, \ldots\ldots\ldots$

If we take $a_0 = 1$ in equation (4), the series is called the **hypergeometric series**. Its sum $y_1(x)$ is denoted by $F(a,b,c;x)$ and is called the **hypergeometric function**.

Thus
$$F(a,b,c;x) = 1+\frac{ab}{1.c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots\ldots \tag{5}$$

In terms of Pochhammer symbol, equation (5) becomes

$$F(a,b,c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \tag{6}$$

CASE (2): When $\beta = 1-c$

In this case the coefficients are

$$a_1 = \frac{(1-c+a)(1-c+b)}{(2-c).1}a_0$$

$$a_2 = \frac{(1-c+a)(1-c+b)(2-c+a)(2-c+b)}{(2-c)(3-c)1.2}a_0$$

$$a_3 = \frac{(1-c+a)(1-c+b)(2-c+a)(2-c+b)(3-c+a)(3-c+b)}{(2-c)(3-c)(4-c)1.2.3}a_0$$

Substituting the values of coefficients in equation (2), we get

$$y_2(x) = a_0x^{1-c}\left[1+\frac{(1-c+a)(1-c+b)}{1.(2-c)}x+\frac{(1-c+a)(1-c+b)(2-c+a)(2-c+b)}{1.2(2-c)(3-c)}x^2 + \frac{(1-c+a)(1-c+b)(2-c+a)(2-c+b)(3-c+a)(3-c+b)}{1.2.3(2-c)(3-c)(4-c)}x^3\ldots\ldots\right] \tag{7}$$

where $c \neq 2, 3, 4, \ldots\ldots\ldots$

If we take $a_0 = 1$ in equation (7), the sum of series (7) is denoted by

$$x^{c-1}F(1-c+a, 1-c+b, 2-c; x)$$

which is a second independent solution of equation (1).

GENERAL SOLUTION

The general solution of equation (1) is therefore

$$y(x) = AF(a,b,c;x)+Bx^{1-c}F(1-c+a, 1-c+b, 2-c; x)$$

where A and B are arbitrary constants.

SPECIAL CASES

EXAMPLE: **Show that**

$$F(1,1,1;x) = F(1,b,b;x) = F(a,1,a;x) = \frac{1}{1-x}$$

SOLUTION: We know that

$$F(a,b,c;x) = 1+\frac{ab}{c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots \quad (1)$$

Let $a = b = c = 1$ in equation (1), we get

$$F(1,1,1;x) = 1+x+\frac{1.2.1.2}{1.2.1.2}x^2+\frac{1.2.3.1.2.3}{1.2.3.1.2.3}x^3+\ldots\ldots\ldots$$

$$= 1+x+x^2+x^3+\ldots\ldots\ldots = \frac{1}{1-x}$$

which is a geometric series. Since equation (1) reduces to a geometric series as a particular case, equation (1) is called hypergeometric series.

Next, let $a = 1$, $c = b$ in equation (1), we get

$$F(1,b,b;x) = 1+\frac{1.b}{b}x+\frac{1.2.b(b+1)}{1.2.b(b+1)}x^2+\frac{1.2.3.b(b+1)(b+2)}{1.2.3.b(b+1)(b+2)}x^3+\ldots$$

$$= 1+x+x^2+x^3+\ldots\ldots\ldots = \frac{1}{1-x}$$

Finally, let $b = 1$, $c = a$ in equation (1), we get

$$F(a,1,a;x) = 1+\frac{a.1}{a}x+\frac{a(a+1).1.2}{1.2.a(a+1)}x^2+\frac{a(a+1)(a+2)1.2.3}{1.2.3.a(a+1)(a+2)}x^3+\ldots$$

$$= 1+x+x^2+x^3+\ldots\ldots\ldots = \frac{1}{1-x}$$

13.5 SYMMETRIC PROPERTY OF HYPERGEOMETRIC FUNCTION

THEOREM (13.1): **Prove that**

$$F(a,b,c;x) = F(b,a,c;x)$$

i.e. hypergeometric function does not change if the parameters a and b are interchanged, keeping c fixed.

PROOF: We know that

$$F(a,b,c;x) = 1+\frac{ab}{1.c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

Interchanging a and b, we see that

$$F(a,b,c;x) = F(b,a,c;x)$$

13.6 DIFFERENTIATION OF HYPERGEOMETRIC FUNCTIONS

THEOREM (13.2): Prove that

(i) $$\frac{d}{dx}F(a,b,c;x) = \frac{ab}{c}F(a+1,b+1,c+1;x)$$

(ii) $$\frac{d^2}{dx^2}F(a,b,c;x) = \frac{a(a+1)b(b+1)}{c(c+1)}F(a+2,b+2,c+2;x)$$

PROOF: We know that

$$F(a,b,c;x) = 1+\frac{ab}{1.c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots \quad (1)$$

(i) Differentiating equation (1) w.r.t. x, we get

$$\frac{d}{dx}F(a,b,c;x) = \frac{ab}{c}+\frac{ab}{c}\frac{(a+1)(b+1)}{c+1}x+\frac{ab}{c}\frac{(a+1)(a+2)(b+1)(b+2)}{2(c+1)(c+2)}x^2+\ldots$$

$$= \frac{ab}{c}\left[1+\frac{(a+1)(b+1)}{c+1}x+\frac{(a+1)(a+2)(b+1)(b+2)}{1.2(c+1)(c+2)}x^2+\ldots\ldots\right] \quad (2)$$

$$= \frac{ab}{c}F(a+1,b+1,c+1,x)$$

(ii) Differentiating again equation (2) w.r.t. x, we get

$$\frac{d^2}{dx^2}F(a,b,c;x) = \frac{ab}{c}\left[\frac{(a+1)(b+1)}{c+1}+\frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)}x+\ldots\right]$$

$$= \frac{a(a+1)b(b+1)}{c(c+1)}\left[1+\frac{(a+2)(b+2)}{c+2}x\ldots\right]$$

$$= \frac{a(a+1)b(b+1)}{c(c+1)}F(a+2,b+2,c+2;x)$$

GENERALIZATION

In general, we have

$$\frac{d^n}{dx^n}F(a,b,c;x) = \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{c(c+1)\ldots(c+n-1)}F(a+n,b+n,c+n;x)$$

$$= \frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n,c+n;x)$$

13.7 INTEGRAL REPRESENTATION OF HYPERGEOMETRIC FUNCTION

THEOREM (13.3): Prove that

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt$$

...OOF: By definition

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{x^n}{n!} \quad (1)$$

$$= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} (a)_n \frac{\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}$$

...ltiplying and dividing by the R.H.S. $\Gamma(c-b)$, we get

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} (a)_n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} \frac{x^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} (a)_n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(b+n+c-b)} \frac{x^n}{n!} \quad (2)$$

...w the beta function denoted by $B(m,n)$ is ...

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt, \quad m>0, \quad n>0$$

...e relationship between the beta and gamma function is

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt \quad (3)$$

...om equations (2) and (3), we get

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} (a)_n \left[\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt\right] \frac{x^n}{n!}$$

...ere $b+n>0$, $c-b>0$ so that $c>b>0$.

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left[\sum_{n=0}^{\infty} (a)_n \frac{(xt)^n}{n!}\right] dt \quad (4)$$

...know by the binomial series that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}x^k + \dots\dots$$

Let $x = -xt$ and $n = -a$, we get

$$(1-xt)^{-a} = 1 - a(-xt) + \frac{-a(-a-1)}{2!}(-xt)^2 + \frac{-a(-a-1)(-a-2)}{3!}(-xt)^3$$
$$+ \ldots\ldots + \frac{-a(-a-1)(-a-2)\ldots\ldots(-a-k+1)}{k!}(-xt)^k + \ldots\ldots$$
$$= 1 + a(xt) + \frac{a(a+1)}{2!}(xt)^2 + \frac{a(a+1)(a+2)}{3!}(xt)^3$$
$$+ \ldots\ldots + (-1)^k \frac{a(a+1)(a+2)\ldots\ldots(a+k-1)}{k!}(-1)^k (xt)^k + \ldots\ldots$$
$$= \sum_{k=0}^{\infty} \frac{a(a+1)(a+2)\ldots+(a+k-1)}{k!}(xt)^k$$
$$= \sum_{k=0}^{\infty} (a)_k \frac{(xt)^k}{k!} = \sum_{n=0}^{\infty} (a)_n \frac{(xt)^k}{n!}$$

Thus equation (4) becomes

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt$$

13.8 GAUSS THEOREM

THEOREM (13.4): Prove that

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \qquad c > a+b$$

PROOF: We know that

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt$$

Let $x = 1$, then

$$F(a,b,c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}\, dt$$
$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\, dt$$
$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\Gamma(b)\Gamma(c-b-a)}{\Gamma(b+c-b-a)}$$
$$= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}$$

13.9 VANDERMONDE'S THEOREM

THEOREM (13.5): Prove that $F(-n, b, c; 1) = \dfrac{(c-b)_n}{(c)_n}$

PROOF: We know from Gauss theorem that

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \tag{1}$$

Let $a = -n$ in equation (1), we get

$$F(-n, b, c; 1) = \frac{\Gamma(c)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-b)}$$

$$= \frac{\Gamma(c)(c-b+n-1)(c-b+n-2)\ldots\ldots(c-b)\Gamma(c-b)}{(c+n-1)(c+n-2)\ldots\ldots(c+1)c.\Gamma(c)\Gamma(c-b)}$$

$$= \frac{(c-b+n-1)(c-b+n-2)\ldots\ldots(c-b)}{(c+n-1)(c+n-2)\ldots\ldots(c+1)c}$$

or $$F(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n}$$

13.10 KUMMER'S THEOREM

THEOREM (13.6): Prove that $F(a, b, b-a+1; -1) = \dfrac{\Gamma(b-a+1)\Gamma\left(\frac{b}{2}+1\right)}{\Gamma(b+1)\Gamma\left(\frac{b}{2}-a+1\right)}$

PROOF: We know that

$$F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt \tag{1}$$

Let $c = b-a+1$ and $x = -1$ in equation (1), we get

$$F(a, b, b-a+1; -1) = \frac{\Gamma(b-a+1)}{\Gamma(b)\Gamma(b-a+1-b)}\int_0^1 t^{b-1}(1-t)^{b-a+1-b-1}(1+t)^{-a}\,dt$$

$$= \frac{\Gamma(b-a+1)}{\Gamma(b)\Gamma(1-a)}\int_0^1 t^{b-1}(1-t)^{-a}(1+t)^{-a}\,dt$$

$$= \frac{\Gamma(b-a+1)}{\Gamma(b)\Gamma(1-a)}\int_0^1 t^{b-1}(1-t^2)^{-a}\,dt \tag{2}$$

Let $t^2 = u$, then $2t\,dt = du$ or $dt = \dfrac{du}{2t} = \dfrac{du}{2\sqrt{u}}$

Thus equation (2) becomes

$$F(a,b,b-a+1;-1)=\frac{\Gamma(b-a+1)}{\Gamma(b)\Gamma(1-a)}\int_0^1 u^{(b-1)/2}(1-u)^{-a}\frac{du}{2\sqrt{u}}$$

$$=\frac{\Gamma(b-a+1)}{2\Gamma(b)\Gamma(1-a)}\int_0^1 u^{(b-1)/2}(1-u)^{(1-a)-1}\,du$$

$$=\frac{\Gamma(b-a+1)}{2\Gamma(b)\Gamma(1-a)}\frac{\Gamma\left(\frac{b}{2}\right)\Gamma(1-a)}{\Gamma\left(\frac{b}{2}+1-a\right)}$$

$$=\frac{\Gamma(b-a+1)\Gamma\left(\frac{b}{2}\right)}{2\Gamma(b)\Gamma\left(\frac{b}{2}+1-a\right)}$$

Multiplying and dividing by b, we get

$$F(a,b,b-a+1;-1)=\frac{\Gamma(b-a+1)\,\frac{b}{2}\Gamma\left(\frac{b}{2}\right)}{b\Gamma(b)\Gamma\left(\frac{b}{2}+1-a\right)}$$

$$=\frac{\Gamma(b-a+1)\,\Gamma\left(\frac{b}{2}+1\right)}{\Gamma(b+1)\Gamma\left(\frac{b}{2}-a+1\right)}$$

13.11 RELATIONSHIP BETWEEN LEGENDRE POLYNOMIALS AND HYPERGEOMETRIC FUNCTIONS

THEOREM (13.7): Prove that $P_n(x)=F\left(-n,n+1,1;\frac{1-x}{2}\right)$

PROOF: By Rodrigue's formula, we have

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n$$

$$=\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n\frac{(1+x)^n}{2^n}\right]$$

$$=\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n\left(1-\frac{1-x}{2}\right)^n\right]$$

Using binomial theorem, we get

$$P_n(x) = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n\left\{1-n\left(\frac{1-x}{2}\right)+\frac{n(n-1)}{2!}\left(\frac{1-x}{2}\right)^2 \dots\right\}\right]$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n - \frac{n}{2}(1-x)^{n+1} + \frac{n(n-1)}{2!2^2}(1-x)^{n+2} - \dots\right] \quad (1)$$

Using the formula $\frac{d^n}{dx^n}(a-bx)^m = (-1)^n b^n \frac{m!}{(m-n)!}(a-bx)^{m-n}$ equation (1) becomes

$$P_n(x) = \frac{(-1)^n}{n!}\left[(-1)^n n! - (-1)^n \frac{n}{2}\frac{(n+1)!}{1!}(1-x) + (-1)^n \frac{n(n-1)}{2!2^2}\frac{(n+2)!}{2!}(1-x)^2 + \dots\right]$$

$$= \frac{(-1)^{2n}}{n!}\left[n! - \frac{n(n+1)}{2}n!(1-x) + \frac{n(n-1)}{2!2^2}\frac{(n+2)(n+1)}{2!}n!(1-x)^2 - \dots\right]$$

$$= 1 + \frac{(-n)(n+1)}{1.1!}\left(\frac{1-x}{2}\right) + \frac{(-n)(-n+1)(n+1)(n+2)}{2.1.2!}\left(\frac{1-x}{2}\right)^2 + \dots\dots$$

$$= F\left(-n, n+1, 1; \frac{1-x}{2}\right)$$

13.12 CONTIGUOUS RELATIONSHIP

THEOREM (13.8): Prove that

$$(a-b)F(a,b,c;x) = aF(a+1,b,c;x) - bF(a,b+1,c;x)$$

PROOF: We have

$$\text{R.H.S.} = aF(a+1,b,c;x) - bF(a,b+1,c;x)$$

$$= a\sum_{n=0}^{\infty}\frac{(a+1)_n(b)_n}{(c)_n}\frac{x^n}{n!} - b\sum_{n=0}^{\infty}\frac{(a)_n(b+1)_n}{(c)_n}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{a(a+1)_n(b)_n}{(c)_n}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(a)_n b(b+1)_n}{(c)_n}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(a+n)(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(a)_n(b+n)(b)_n}{(c)_n}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}[(a+n)-(b+n)]\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

$$= (a-b)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

$$= (a-b)F(a,b,c;x)$$

13.13 ALTERNATIVE FORMS OF HYPERGEOMETRIC FUNCTION

THEOREM (13.9): Prove that if $|x| < 1$ and $\left|\frac{x}{1-x}\right| < 1$, then

$$F(a,b,c;x) = (1-x)^{-a} F\left(a, c-b, c; \frac{x}{x-1}\right)$$

Also, deduce that

$$F\left(a,b,c;\frac{1}{2}\right) = 2^{a} F(a, c-b, c; -1)$$

PROOF: By the integral representation of hypergeometric function, we have

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt \qquad (1)$$

Let $1-t = u$ so that $dt = -du$ and $t = 1-u$, equation (1) becomes

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-u)^{b-1} u^{c-b-1}(1-x+xu)^{-a}\,du$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{c-b-1}(1-u)^{b-1}(1-x)^{-a}\left(1+\frac{xu}{1-x}\right)^{-a} du$$

$$= \frac{(1-x)^{-a}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{c-b-1}(1-u)^{b-1}\left(1-\frac{x}{x-1}u\right)^{-a} du \qquad (2)$$

Replacing b by $c-b$ and x by $\frac{x}{x-1}$ in equation (1), we get

$$F\left(a, c-b, c; \frac{x}{x-1}\right) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma[c-(c-b)]} \int_0^1 t^{c-b-1}(1-t)^{c-(c-b)-1}\left(1-\frac{xt}{x-1}\right)^{-a} dt$$

Replacing the dummy variable t by u, we get

$$F\left(a, c-b, c; \frac{x}{x-1}\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{c-b-1}(1-u)^{b-1}\left(1-\frac{x}{x-1}u\right)^{-a} du \qquad (3)$$

From equations (2) and (3), we get

$$F(a,b,c;x) = (1-x)^{-a} F\left(a, c-b, c; \frac{x}{x-1}\right) \qquad (4)$$

DEDUCTION

Let $x = \frac{1}{2}$ in equation (4), we get

$$F\left(a, b, c; \frac{1}{2}\right) = 2^{a} F(a, c-b, c; -1)$$

THEOREM (13.10): Prove that

$$F(a, b, c; x) = (1-x)^{c-a-b} F(c-a, c-b, c; x)$$

PROOF: We know that

$$F(a, b, c; x) = (1-x)^{-a} F\left(a, c-b, c; \frac{x}{x-1}\right) \qquad (1)$$

Using the symmetry property

$$F\left(a, c-b, c; \frac{x}{x-1}\right) = F\left(c-b, a, c; \frac{x}{x-1}\right)$$

$$= (1-x)^{-(c-b)} F\left(c-b, c-a, c; \frac{x}{x-1}\right) \quad [\text{ using equation (1) }]$$

Replacing x by $\frac{x}{x-1}$, we get

$$F(a, c-b, c; x) = \left(1 - \frac{x}{x-1}\right)^{-c+b} F(c-b, c-a, c; x)$$

$$= \left(\frac{-1}{x-1}\right)^{-c+b} F(c-b, c-a, c; x)$$

$$= \left(\frac{1}{1-x}\right)^{-c+b} F(c-b, c-a, c; x)$$

$$= (1-x)^{c-b} F(c-b, c-a, c; x) \qquad (2)$$

From equations (1) and (2), we get

$$F(a, b, c; x) = (1-x)^{-a}(1-x)^{c-b} F(c-b, c-a, c; x)$$

$$= (1-x)^{c-a-b} F(c-a, c-b, c; x)$$

13.14 CONFLUENT HYPERGEOMETRIC EQUATION

The second order homogeneous linear differential equation

$$x y'' + (c-x) y' - a y = 0$$

where a, and c are constants, is called the **confluent hypergeometric equation**. The solutions of this equation are called the **confluent hypergeometric functions**.

13.15 SOLUTION OF CONFLUENT HYPERGEOMETRIC EQUATION

We shall solve the confluent hypergeometric equation

$$x y'' + (c - x) y' - a y = 0 \tag{1}$$

by the method of Frobenius . Assuming a series solution of the form

$$y(x) = x^{\beta}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots) = \sum_{k=-\infty}^{\infty} a_k x^{k+\beta} \tag{2}$$

where β and a_k are constants such that $a_k = 0$, for $k = -1, -2, -3, \ldots\ldots\ldots$

By differentiation , omitting the summation index k, we get

$$y' = \sum (k+\beta) a_k x^{k+\beta-1} \quad \text{and} \quad y'' = \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2}$$

Substituting y, y', and y'' in equation (1) , we get

$$x \sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-2} + (c-x) \sum (k+\beta) a_k x^{k+\beta-1} - a \sum a_k x^{k+\beta} = 0$$

or $$\sum (k+\beta)(k+\beta-1) a_k x^{k+\beta-1} + \sum c(k+\beta) a_k x^{k+\beta-1}$$
$$- \sum (k+\beta) a_k x^{k+\beta} - \sum a . a_k x^{k+\beta} = 0$$

or $$\sum [(k+\beta)(k+\beta-1) + c(k+\beta)] a_k x^{k+\beta-1} - \sum [(k+\beta) + a] a_k x^{k+\beta} = 0$$

Shifting index k by $k-1$ in the second summation , we get

$$\sum [(k+\beta)(k+\beta-1) + c(k+\beta)] a_k x^{k+\beta-1} - \sum [(k+\beta-1) + a] a_{k-1} x^{k+\beta-1} = 0$$

or $$\sum [\{(k+\beta)(k+\beta-1) + c(k+\beta)\} a_k - \{(k+\beta-1) + a\} a_{k-1}] x^{k+\beta-1} = 0 \tag{3}$$

The coefficient of the lowest degree term $x^{\beta-1}$ is obtained by substituting $k = 0$ in equation (3)

i.e. $$[\{\beta(\beta-1) + c\beta\} a_0 - (\beta - 1 + a) a_{-1}] x^{\beta-1} = 0$$

Since $x^{\beta-1} \neq 0$, therefore the coefficient of $x^{\beta-1}$ must be zero . Thus

$$[\beta(\beta-1) + c\beta] a_0 - (\beta - 1 + a) a_{-1} = 0 .$$

But $a_{-1} = 0$, we get indicial equation $[\beta(\beta-1) + c\beta] a_0 = 0$

Since $a_0 \neq 0$, therefore $\beta(\beta-1) + c\beta = 0$ or $\beta(\beta - 1 + c) = 0$ or $\beta = 0$, $\beta = 1 - c$, are the indicial roots which are unequal and differ not by an integer .

Equating to zero the coefficient of $x^{k+\beta-1}$, we get the recurrence relation from equation (3)

i.e. $$a_k = \frac{(k+\beta-1)+a}{(k+\beta)(k+\beta-1+c)} a_{k-1}$$

Let $k = 1$, then $a_1 = \dfrac{\beta+a}{(\beta+1)(\beta+c)} a_0$

Let $k = 2$, then $a_2 = \dfrac{\beta+1+a}{(\beta+2)(\beta+1+c)} a_1 = \dfrac{(\beta+a)(\beta+1+a)}{(\beta+1)(\beta+2)(\beta+c)(\beta+1+c)} a_0$

Let $k = 3$, then $a_3 = \dfrac{\beta+2+a}{(\beta+3)(\beta+2+c)} a_2$

$$= \frac{(\beta+a)(\beta+1+a)(\beta+2+a)}{(\beta+1)(\beta+2)(\beta+3)(\beta+c)(\beta+1+c)(\beta+2+c)} a_0$$

and so on.

CASE (1): When $\beta = 0$

In this case, the coefficients become

$$a_1 = \frac{a}{c} a_0, \quad a_2 = \frac{a(a+1)}{1.2.c(c+1)} a_0$$

$$a_3 = \frac{a(a+1)(a+2)}{1.2.3.c(c+1)(c+2)} a_0$$

Substituting the values of these coefficients in equation (2), we get

$$y_1(x) = a_0\left[1 + \frac{a}{c}x + \frac{a(a+1)}{1.2.c(c+1)}x^2 + \frac{a(a+1)(a+2)}{1.2.3.c(c+1)(c+2)}x^3 + \ldots\ldots\right] \quad (4)$$

where $c \neq 0, -1, -2, \ldots\ldots\ldots$

If we take $a_0 = 1$ in equation (4), the series is called the **confluent hypergeometric series**. Its sum $y_1(x)$ is denoted by $F(a, c; x)$ and is called the **confluent hypergeometric function**.

Thus $$F(a,c;x) = 1 + \frac{a}{1.c}x + \frac{a(a+1)}{1.2.c(c+1)}x^2 + \frac{a(a+1)(a+2)}{1.2.3.c(c+1)(c+2)}x^3 + \ldots\ldots \quad (5)$$

In terms of Pochhammer's symbol, equation (5) can be written as

$$F(a,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!} \quad (6)$$

CASE (2): When $\beta = 1 - c$

In this case the coefficients are

$$a_1 = \frac{(1-c+a)}{(2-c).1} a_0, \quad a_2 = \frac{(1-c+a)(2-c+a)}{(2-c)(3-c)1.2} a_0$$

$$a_3 = \frac{(1-c+a)(2-c+a)(3-c+a)}{(2-c)(3-c)(4-c)1.2.3} a_0$$

Substituting the values of these coefficients in equation (2), we get

$$y_2(x) = a_0 x^{1-c}\left[1+\frac{(1-c+a)}{1.(2-c)}x+\frac{(1-c+a)(2-c+a)}{1.2(2-c)(3-c)}x^2 + \frac{(1-c+a)(2-c+a)(3-c+a)}{1.2.3(2-c)(3-c)(4-c)}x^3+\ldots\ldots\right] \quad (7)$$

where $c \neq 2, 3, 4, \ldots\ldots$

If we take $a_0 = 1$ in equation (7), the sum of series is denoted by $x^{1-c}F(1-c+a,\ 2-c;x)$ which is a second independent solution of equation (1).

GENERAL SOLUTION

The general solution of equation (1) is therefore

$$y(x) = AF(a, c; x) + Bx^{1-c}F(1-c+a,\ 2-c;\ x)$$

where A and B are arbitrary constants.

13.16 DIFFERENTIATION OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

THEOREM (13.11): Prove that

(i) $$\frac{d}{dx}F(a, c;\ x) = \frac{a}{c}F(a+1, c+1;\ x)$$

(ii) $$\frac{d^2}{dx^2}F(a, c;\ x) = \frac{a(a+1)}{c(c+1)}F(a+2, c+2;\ x)$$

PROOF: We know that

$$F(a, c;\ x) = 1+\frac{a}{1.c}x+\frac{a(a+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots \quad (1)$$

Differentiating equation (1) w.r.t. x, we get

$$\frac{d}{dx}F(a, c;\ x) = \frac{a}{c}+\frac{a(a+1)}{c(c+1)}x+\frac{a(a+1)(a+2)}{2c(c+1)(c+2)}x^2+\ldots\ldots$$

$$= \frac{a}{c}\left[1+\frac{a+1}{c+1}x+\frac{(a+1)(a+2)}{2(c+1)(c+2)}x^2+\ldots\ldots\right] \quad (2)$$

$$= \frac{a}{c}F(a+1, c+1;\ x)$$

(ii) Differentiating again equation (2) w.r.t. x, we get

$$\frac{d^2}{dx^2}F(a, c;\ x) = \frac{a}{c}\left[\frac{a+1}{c+1}+\frac{(a+1)(a+2)}{(c+1)(c+2)}x+\ldots\ldots\right]$$

$$= \frac{a(a+1)}{c(c+1)}\left[1+\frac{a+2}{c+2}x+\ldots\ldots\right]$$

$$= \frac{a(a+1)}{c(c+1)}F(a+2, c+2;\ x)$$

GENERALIZATION

In general, we have

$$\frac{d^n}{dx^n} F(a, c; x) = \frac{(a)_n}{(c)_n} F(a+n, c+n; x)$$

13.17 INTEGRAL REPRESENTATION OF CONFLUENT HYPERGEOMETRIC FUNCTION

THEOREM (13.12): Prove that

$$F(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt}\, dt, \quad c > a > 0$$

PROOF: By definition

$$F(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{x^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{x^n}{n!}$$

Multiplying and dividing the R.H.S. by $\Gamma(c-a)$, we get

$$F(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)} \frac{x^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c-a+a+n)} \frac{x^n}{n!}$$

Now by definition

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt, \quad m > 0, \quad n > 0$$

Thus equation (1) becomes

$$F(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} \int_0^1 t^{a+n-1}(1-t)^{c-a-1} \frac{x^n}{n!}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \left[\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right] dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt}\, dt$$

since $e^{xt} = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} = 1 + \frac{xt}{1!} + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + \ldots\ldots$

13.18 KUMMER'S RELATION

THEOREM (13.13): Prove that

$$F(a, c; x) = e^x F(c-a, c; -x)$$

PROOF: We know that

$$F(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt}\, dt \tag{1}$$

Replacing a by $c-a$ and x by $-x$, we get

$$F(c-a, c; -x) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma[c-(c-a)]} \int_0^1 t^{c-a-1}(1-t)^{c-(c-a)-1} e^{-xt}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{c-a-1}(1-t)^{a-1} e^{-xt}\, dt \tag{2}$$

Let $1-t = u$ so that $dt = -du$ and $t = 1-u$. Also, $1 \le u \le 0$.

Thus equation (2) becomes

$$F(c-a, c; -x) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} e^{-x(1-u)}\, du$$

$$= \frac{\Gamma(c)e^{-x}}{\Gamma(c-a)\Gamma(a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} e^{xu}\, du$$

Changing the dummy variable u by t, we get

$$F(c-a, c; -x) = e^{-x}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt}\, dt$$

$$= e^{-x} F(a, c; x)$$

or $\quad F(a, c; x) = e^x F(c-a, c; -x)$

13.19 OTHER DIFFERENTIAL EQUATIONS FOR SPECIAL FUNCTIONS

(1) HERMITE'S DIFFERENTIAL EQUATION

The differential equation

$$y'' - 2xy' + 2ny = 0$$

where n is real, is known as the Hermite's differential equation named after the French mathematician C. Hermite (1822 – 1901). If $n = 0, 1, 2, 3, \ldots\ldots$, then the solutions of this equation are called **Hermite polynomials**.

(2) LAGUERRE'S DIFFERENTIAL EQUATION

The differential equation

$$xy'' + (1-x)y' + ny = 0$$

where n is real, is called Laguerre's differential equation named after the French mathematician E. Laguerre (1834 – 1886). If $n = 0, 1, 2, 3, \ldots\ldots$, the solutions of this equation are called **Laguerre polynomials**.

(3) CHEBYSHEV'S DIFFERENTIAL EQUATION

The differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0, \qquad n = 0, 1, 2, \ldots\ldots$$

where n is a non – negative integer is called Chebyshev differential equation named after the Russian mathematician P. Chebyshev (1821 – 1894). If $n = 0, 1, 2, 3, \ldots\ldots$, the solutions of this equation are called **Chebyshev polynomials**.

It is interesting to note that the solutions of above mentioned differential equations play an important role in the development of special functions – an exciting branch of mathematics.

13.20 SOLVED PROBLEMS

PROBLEM (1): Show that $F(-n,1,1,-x) = (1+x)^n$

SOLUTION: We know that

$$F(a,b,c;x) = 1+\frac{a.b}{c}x+\frac{a(a+1)b(b+1)}{1.2c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\dots \quad (1)$$

Let $a=-n$, $b=c=1$, $x=-x$ in equation (1), we get

$$F(-n,1,1;-x) = 1+\frac{(-n).1}{1}(-x)+\frac{(-n)(-n+1).1.2}{1.2.1.2}(-x)^2$$

$$+\frac{(-n)(-n+1)(-n+2).1.2.3}{1.2.3.1.2.3}(-x)^3+\dots$$

$$= 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\dots$$

$$= (1+x)^n$$

PROBLEM (2): Show that $F(1,1,2;-x) = \dfrac{\ln(1+x)}{x}$

SOLUTION: We know that

$$F(a,b,c;x) = 1+\frac{a.b}{c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\dots \quad (1)$$

Let $a=b=1$, $c=2$, and $x=-x$ in equation (1), we get

$$F(1,1,2;-x) = 1+\frac{1.1}{2}(-x)+\frac{1.2.1.2}{1.2.2.3}(-x)^2+\frac{1.2.3.1.2.3}{1.2.3.2.3.4}(-x)^3+\dots\dots$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\dots \quad (2)$$

Also, the Taylor's series for $\ln(1+x)$ is

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\dots$$

or

$$\frac{\ln(1+x)}{x} = 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\dots \quad (3)$$

From equation (2) and (3), we get

$$F(1,1,2;-x) = \frac{\ln(1+x)}{x}$$

PROBLEM (3): Show that

$$F\left(\frac{1}{2},1,\frac{3}{2};x^2\right) = \frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right)$$

SOLUTION: We know that

$$F(a,b,c;x) = 1+\frac{a.b}{c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots \quad (1)$$

Let $a=\frac{1}{2}$, $b=1$, $c=\frac{3}{2}$, $x=x^2$ in equation (1), we get

$$F\left(\frac{1}{2},1,\frac{3}{2};x^2\right) = 1+\frac{\frac{1}{2}.1}{\frac{3}{2}}x^2+\frac{\frac{1}{2}\cdot\frac{3}{2}.1.2}{1.2.\frac{3}{2}\cdot\frac{5}{2}}(x^2)^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}.1.2.3}{1.2.3.\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}(x^2)^3+\ldots\ldots$$

$$= 1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\ldots\ldots \quad (4)$$

Now the Taylor series for $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ is

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\ldots$$

or

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) = 1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\ldots\ldots \quad (5)$$

From equation (4) and (5), we get

$$F\left(\frac{1}{2},1,\frac{3}{2};x^2\right) = \frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right)$$

PROBLEM (4): **Show that** $F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right) = \frac{\sin^{-1}x}{x}$

SOLUTION: We know that

$$F(a,b,c;x) = 1+\frac{a.b}{c}x+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3+\ldots \quad (1)$$

Let $a=b=\frac{1}{2}$, $c=\frac{3}{2}$, $x=x^2$ in equation (1), we get

$$F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right) = 1+\frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{3}{2}}x^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1.2.\frac{3}{2}\cdot\frac{5}{2}}(x^2)^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3.\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}(x^2)^3+\ldots\ldots$$

$$= 1+\frac{x^2}{2.3}+\frac{1.3}{2.4.5}x^4+\frac{1.3.5}{2.4.6.7}x^6+\ldots\ldots \quad (2)$$

Now the Taylor series for $\sin^{-1}x$ is

$$\sin^{-1}x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\ldots\ldots$$

or

$$\frac{\sin^{-1}x}{x} = 1+\frac{x^2}{2.3}+\frac{1.3}{2.4.5}x^4+\frac{1.3.5}{2.4.6.7}x^6+\ldots\ldots \quad (3)$$

From equation (1) and (3), we get

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2\right) = \frac{\sin^{-1} x}{x}$$

PROBLEM (5): **Show that** $F\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right) = \frac{\tan^{-1} x}{x}$

SOLUTION: We know that

$$F(a,b,c;x) = 1 + \frac{a.b}{c}x + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3 + \ldots \quad (1)$$

Let $a = \frac{1}{2}$, $b = 1$, $c = \frac{3}{2}$, $x = -x^2$ in equation (1), we get

$$F\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right) = 1 + \frac{\frac{1}{2}.1}{\frac{3}{2}}(-x^2) + \frac{\frac{1}{2}.\frac{3}{2}.1.2}{1.2.\frac{3}{2}.\frac{5}{2}}(-x^2)^2 + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.1.2.3}{1.2.3.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}(-x^2)^3 + \ldots\ldots$$

$$= 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \ldots\ldots\ldots\ldots \quad (4)$$

Now the Taylor series for $\tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots\ldots$$

or $$\frac{\tan^{-1} x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots\ldots\ldots\ldots \quad (5)$$

From equation (4) and (5), we get

$$F\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right) = \frac{\tan^{-1} x}{x}$$

PROBLEM (6): **Prove that**

$$\underset{n \to \infty}{\mathrm{Lt}}\, F\left(1, n; 1; \frac{x}{n}\right) = e^x$$

SOLUTION: We know that

$$F(a,b,c;x) = 1 + \frac{a.b}{c}x + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}x^3 + \ldots \quad (1)$$

Let $a = 1$, $b = n$, $c = 1$, $x = \frac{x}{n}$ in equation (1), we get

$$F\left(1, n; 1; \frac{x}{n}\right) = 1 + \frac{1.n}{1.1}\left(\frac{x}{n}\right) + \frac{1.2.(n)(n+1)}{1.2.1.2}\left(\frac{x}{n}\right)^2 + \frac{1.2.3(n)(n+1)(n+2)}{1.2.3.1.2.3}\left(\frac{x}{n}\right)^3 + \ldots\ldots\ldots$$

$$= 1+x+\frac{n(n+1)}{1.2}\left(\frac{x^2}{n^2}\right)+\frac{n(n+1)(n+2)}{1.2.3}\frac{x^3}{n^3}+\ldots\ldots$$

$$= 1+x+\left(\frac{n+1}{n}\right)\frac{x^2}{1.2}+\left(\frac{n^2+3n+2}{n^2}\right)\frac{x^3}{1.2.3}+\ldots\ldots$$

$$= 1+x+\left(1+\frac{1}{n}\right)\frac{x^2}{2!}+\left(1+\frac{3}{n}+\frac{2}{n^2}\right)\frac{x^3}{3!}+\ldots\ldots$$

$$\underset{n\to\infty}{\mathrm{Lt}} F\left(1,n;1;\frac{x}{n}\right) = \underset{n\to\infty}{\mathrm{Lt}}\left[1+x+\left(1+\frac{1}{n}\right)\frac{x^2}{2!}+\left(1+\frac{3}{n}+\frac{2}{n^2}\right)\frac{x^3}{3!}+\ldots\ldots\right]$$

$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots = e^x$$

13.21 EXERCISE

PROBLEM (1): Prove that $(1-x)^{-a} = F(a,b,b;x)$

PROBLEM (2): Prove that $F(1,1,2;x) = -\dfrac{\ln(1-x)}{x}$

PROBLEM (3): Prove that $F(a,b,c;x) = (1-x)^{-b} F\left(b, c-a, c; \dfrac{x}{x-1}\right)$

PROBLEM (4): Prove that

$$F(a,b+1,c+1;x) - F(a,b,c;x) = \frac{a(c-b)}{c(c+1)} x F(a+1,b+1,c+2;x)$$

PROBLEM (5): Show that $\left[\dfrac{d}{dx} F(a,b,c;x)\right]_{x=0} = \dfrac{ab}{c}$

PROBLEM (6): Prove that

$$F(a-1,b-1,c;x) - F(a,b-1,c;x) = \frac{1-b}{c} x F(a,b,c+1;x)$$

PROBLEM (7): Prove that

$$F(a,b,c;x) - F(a,b,c-1;x) = -\frac{ab}{c(c-1)} x F(a+1,b+1,c+1;x)$$

PROBLEM (8): Prove that $F\left(a, 1-a, c, \dfrac{1}{2}\right) = \dfrac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{c}{2}\right)\Gamma\left(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}\right)}$

PROBLEM (9): Prove that $F\left(a, b, \dfrac{a}{2}+\dfrac{b}{2}+\dfrac{1}{2}; \dfrac{1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}+\frac{b}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{b}{2}\right)}$

PROBLEM (10): Prove that

(i) $e^x = F(a,a;x)$

(ii) $e^x - 1 = x F(1,2;x)$

(iii) $\left(1+\dfrac{x}{a}\right)e^x = F(a+1,a;x)$

PROBLEM (11): Prove that

$$cF(a,c;x) = cF(c-1,c;x) + xF(a,c+1;x)$$

PROBLEM (12): Prove that

$$(a-c+1)F(a,c;x) = aF(a+1,c;x) - (c-1)F(a,c-1;x)$$

CHAPTER 14

STURM – LIOUVILLE SYSTEMS

14.1 INTRODUCTION

In the course of solving boundary value problems using the method of separation of variables, we often arrive at systems which are called the Sturm – Liouville systems or Sturm – Liouville problems [named after the Swiss mathematician J.C.F. Sturm (1803 – 1855) and French mathematician Joseph Liouville (1809–1982)]. These systems occur in various engineering applications, for example, in connection with vibrations of strings and membranes and in heat conduction.

14.2 STURM – LIOUVILLE DIFFERENTIAL EQUATION

A second order ordinary linear homogeneous differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0, \quad (a \le x \le b) \qquad (1)$$

where the functions $p(x), q(x)$ and $r(x)$ are continuous on the interval $a \le x \le b$ and λ is a real constant, is called a **Sturm – Liouville** equation or simply S – L equation.

If we define an operator $L \equiv \frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)$

then S – L equation (1) can be written as

$$L(y)+\lambda r(x)y=0 \qquad (2)$$

EXAMPLES OF STURM – LIOUVILLE DIFFERENTIAL EQUATION

(1) Consider the simplest second order differential equation $y''+\lambda y=0$

This can be written as $\frac{d}{dx}\left(\frac{dy}{dx}\right)+\lambda y=0$

Comparing it with equation (1), we have

$$p(x)=1, \quad q(x)=0, \quad r(x)=1$$

Thus the given differential equation is of the form of Sturm – Liouville equation (1).

(2) Consider Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

This can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\lambda y = 0, \quad \text{where } \lambda = n(n+1).$$

Here $p(x) = 1-x^2$, $q(x) = 0$, $r(x) = 1$

Thus Legendre's differential equation is of the form of Sturm – Liouville equation (1).

(3) Consider the Bessel's differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y = 0, \quad n \geq 0.$$

Replacing the independent variable x by Kx, where K is a constant, we get

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(K^2x^2-n^2)y = 0$$

Dividing throughout by x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\left(-\frac{n^2}{x}+K^2x\right)y = 0$$

or $$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\left(-\frac{n^2}{x}+\lambda x\right)y = 0 \quad (\lambda = K^2)$$

Here $p(x) = x$, $q(x) = -\frac{n^2}{x}$ and $r(x) = x$.

Thus Bessel's differential equation is of the form of Sturm – Liouville equation (1).

14.3 GENERAL SECOND ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION

Consider the general second order linear homogeneous differential equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+\left[Q(x)+\lambda R(x)\right]y = 0 \tag{1}$$

Now I.F. $= e^{\int P(x)\,dx}$ is an integrating factor of the first two terms, since

$$e^{\int P(x)\,dx}\left[\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}\right] = \frac{d}{dx}\left[e^{\int P(x)\,dx}\frac{dy}{dx}\right]$$

Multiplying equation (1) by the integrating factor, we obtain

$$e^{\int P(x)\,dx}\left[\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}\right]+e^{\int P(x)\,dx}\left[Q(x)+\lambda R(x)\right] = 0$$

or $$\frac{d}{dx}\left[e^{\int P(x)\,dx}\frac{dy}{dx}\right]+e^{\int P(x)\,dx}\left[Q(x)+\lambda R(x)\right]y = 0$$

or $$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y = 0$$

where $p(x) = e^{\int P(x)\,dx}$, $q(x) = Q(x)e^{\int P(x)\,dx}$ and $r(x) = R(x)e^{\int P(x)\,dx}$

Thus it is possible to reduce a general second order linear homogeneous differential equation into the Sturm – Liouville form.

EXAMPLES

(i) HERMITE'S DIFFERENTIAL EQUATION

Consider the Hermite's differential equation

$$y'' - 2xy' + 2ny = 0 \qquad (n \text{ real}) \qquad (2)$$

Comparing it with equation (1), we have $P(x) = -2x$. The integrating factor is

$$e^{\int P(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2}$$

Multiplying equation (2) with e^{-x^2}, we get

$$e^{-x^2} y'' - 2xe^{-x^2} y' + 2ne^{-x^2} y = 0$$

or $$\frac{d}{dx}\left[e^{-x^2}\frac{dy}{dx}\right] + \lambda e^{-x^2} y = 0 \qquad (\lambda = 2n)$$

Here $p(x) = e^{-x^2}$, $q(x) = 0$, $r(x) = e^{-x^2}$

Thus Hermite's differential equation is of form of Sturm – Liouville equation.

(ii) LAGUERRE'S DIFFERENTIAL EQUATION

Consider the Laguerre's differential equation

$$xy'' + (1-x)y' + ny = 0 \qquad (n \text{ real})$$

or $$y'' + \left(\frac{1}{x} - 1\right)y' + \frac{1}{x}ny = 0 \qquad (3)$$

Comparing it with equation (1), we have $P(x) = \frac{1}{x} - 1$. The integrating factor is

$$e^{\int P(x)\,dx} = e^{\int\left(\frac{1}{x}-1\right)dx} = e^{\ln x - x} = x.e^{-x}$$

Multiplying equation (3) with xe^{-x}, we get

$$xe^{-x}y'' + \left(\frac{1}{x} - 1\right)xe^{-x}y' + \frac{1}{x}(xe^{-x})ny = 0$$

or $$xe^{-x}y'' + (1-x)e^{-x}y' + e^{-x}ny = 0$$

or $$\frac{d}{dx}[xe^{-x}y'] + \lambda e^{-x}y = 0 \qquad (\text{where } \lambda = n)$$

Here $p(x) = xe^{-x}$, $q(x) = 0$, $r(x) = e^{-x}$

Thus Laguerre's differential equation is of the form of Sturm – Liouville equation.

(iii) SHEBYSHEV'S DIFFERENTIAL EQUATION

Consider the Shebyshev's differential equation

$$(1-x^2)y'' - xy' + n^2y = 0, \qquad n = 0, 1, 2, \ldots\ldots\ldots$$

$$y'' - \frac{x}{1-x^2} y' + \frac{1}{1-x^2} n^2 y = 0 \quad (4)$$

or

Comparing it with equation (1), we have $P(x) = -\frac{1}{1-x^2}$. The integrating factor is

$$e^{\int P(x)\,dx} = e^{\int -\frac{x}{1-x^2}\,dx} = e^{\frac{1}{2}\ln(1-x^2)} = e^{\ln\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Multiplying equation (4) with $\sqrt{1-x^2}$, we get

$$\sqrt{1-x^2}\, y'' - \frac{x}{\sqrt{1-x^2}} y' + \frac{1}{\sqrt{1-x^2}} n^2 y = 0$$

or
$$\frac{d}{dx}\left[\sqrt{1-x^2}\, y'\right] + \frac{1}{\sqrt{1-x^2}} \lambda y = 0 \qquad (\lambda = n^2)$$

Here $p(x) = \sqrt{1-x^2}$, $q(x) = 0$, $r(x) = \frac{1}{\sqrt{1-x^2}}$

Thus Shebyshev's differential equation is of the form of Sturm – Liouville equation.

14.4 REGULAR, SINGULAR AND PERIODIC S – L EQUATIONS

Consider the Sturm – Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right] y = 0 \quad (1)$$

defined over the interval $a \le x \le b$.

(i) If both $p(x)$ and $r(x)$ do not vanish at any point of the interval $a \le x \le b$, then equation (1) is called the **regular** Sturm – Liouville equation.

(ii) If $p(x)$ or $r(x)$ vanish at a or b (or at both a and b), then equation (1) is called the **singular** Sturm – Liouville equation.

(iii) If $p(x)$, $q(x)$, and $r(x)$ are periodic functions of period $b-a$ i.e. $p(x) = p(x+b-a)$ etc., then equation (1) is called the **periodic** Sturm – Liouville equation.

EXAMPLES

(i) Consider the equation $y'' + \lambda y = 0$ for $0 \le x \le 1$.

Here $p(x) = r(x) = 1 > 0$. Thus this equation is a regular Sturm – Liouville equation over the given interval. Furthermore, this equation can also be regarded as a periodic Sturm – Liouville equation, since $p(x) = 1$, $q(x) = 0$ and $r(x) = 1$ are periodic functions.

(ii) Consider the Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0, \quad \text{for } -1 \le x \le 1.$$

Here $p(x) = 1 - x^2$ and $r(x) = 1$.

Since $p(-1) = p(1) = 0$, therefore, this equation is a singular Sturm – Liouville equation over the given interval

(iii) Consider the Bessel's differential equation

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\left(-\frac{n^2}{x}+\lambda x\right)y=0, \quad \text{for } 0 \le x \le 1.$$

Here $p(x) = x$ and $r(x) = x$. These vanish at $x = 0$. Thus this equation is a singular Sturm – Liouville equation for any interval containing 0. However, it can be regarded as a regular Sturm – Liouville equation for any interval not containing zero.

14.5 TYPES OF STURM – LIOUVILLE SYSTEMS

There are three types of Sturm – Liouville systems or Sturm – Liouville boundary – value problems depending upon the boundary conditions.

(i) REGULAR STURM – LIOUVILLE SYSTEM

A system consisting of regular Sturm – Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0, \quad (a \le x \le b)$$

and the boundary conditions

$$\left.\begin{aligned}\alpha_1 y(a)+\alpha_2 y'(a)=0\\ \beta_1 y(b)+\beta_2 y'(b)=0\end{aligned}\right\} \qquad (A)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants such that α_1 and α_2 are not both zero and β_1, β_2 are not both zero, is called a regular Sturm – Liouville system. For example, consider the system

$$y''+\lambda y=0, \quad y(0)=0, \quad y(1)=0$$

Here the interval is $0 \le x \le 1$ and $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a = 0$, $b = 1$.

Furthermore, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$.

Since the Sturm – Liouville equation is regular and the boundary conditions are of the form (A), therefore the system is the regular Sturm – Liouville system.

(ii) PERIODIC STURM – LIOUVILLE SYSTEM

A system consisting of the periodic Sturm – Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0, \quad (a \le x \le b)$$

and the periodic boundary conditions

$$y(a)=y(b) \text{ and } y'(a)=y'(b) \qquad (B)$$

is called a periodic Sturm – Liouville system. For example, consider the system

$$y''+\lambda y=0, \quad y(-\pi)=y(\pi) \text{ and } y'(-\pi)=y'(\pi)$$

Here the interval is $-\pi \le x \le \pi$, whose length is 2π. Also, the boundary conditions are periodic with period 2π, since $y(-\pi) = y(-\pi+2\pi) = y(\pi)$ and $y'(-\pi) = y'(-\pi+2\pi) = y'(\pi)$. Since the Sturm – Liouville equation is periodic, and the boundary conditions are of the form (B), therefore, the given system is a periodic Sturm – Liouville system.

(iii) SINGULAR STURM – LIOUVILLE SYSTEM

A system consisting of the singular Sturm – Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0$$

satisfying one of the following three types of boundary conditions is called a singular Sturm – Liouville system.

(i) If $p(a) = 0$, there is no boundary condition at the point $x = a$ and the boundary condition at the point $x = b$ is

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

where β_1 and β_2 are not both zero. As a second condition, we require that the solution must be bounded at a.

(ii) If $p(b) = 0$, there is no boundary condition at $x = b$ and the boundary condition at $x = a$ is

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$

where α_1 and α_2 are not both zero. In this case, we require that the solution must be bounded at b.

(iii) If $p(a) = p(b) = 0$, we have no boundary conditions specified at a or b, but require that the solution must be bounded on $a \le x \le b$. For example, consider the system

$$\frac{d}{dx}\left[4x\frac{dy}{dx}\right] + \lambda x y = 0, \quad y(1) = 0 \quad \text{over } 0 < x \le 1.$$

Here $p(x) = 4x$, $q(x) = 0$, $r(x) = x$.

Since $p(0) = 0$ and $r(0) = 0$ and there is no boundary condition specified at the end point $x = 0$, therefore, this system is a singular Sturm – Liouville system. As a second condition, we may require that the solution must be bounded as $x \to 0+$.

As another example, consider the Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0 \quad \text{for } -1 \le x \le 1.$$

Here $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$.

Since $p(-1) = p(1) = 0$, therefore, there is no need for any boundary condition to be specified to obtain the solution. We simply require that the solution must be bounded over the given interval. Thus this is an example of the singular Sturm – Liouville system.

14.6 ORTHOGONAL AND ORTHONORMAL SETS OF FUNCTIONS WITH RESPECT TO A WEIGHT FUNCTION

Two real valued functions $f(x)$ and $g(x)$ are said to be orthogonal w.r.t. a weight function $w(x) \geq 0$ over the interval $a \leq x \leq b$ if

$$\int_a^b w(x) f(x) g(x)\, dx = 0 \tag{1}$$

A set of functions $\{\phi_k(x)\}$, $k = 1, 2, 3, \ldots\ldots$ is said to be orthogonal w.r.t. a weight function $w(x) \geq 0$ over the interval $a \leq x \leq b$ if for any two functions $\phi_m(x)$ and $\phi_n(x)$ in the set, the following relation holds.

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = \begin{cases} 0 & \text{for } m \neq n \\ r_n > 0 & \text{for } m = n \end{cases} \tag{2}$$

Here r_n is a positive finite constant.

A set of function $\{\phi_k(x)\}$, $k = 1, 2, 3, \ldots\ldots$ is said to be orthonormal w.r.t. a weight function $w(x) \geq 0$ on the interval $a \leq x \leq b$, if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases} \tag{3}$$

Equation (3) can be written as

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = \delta_{mn} \tag{4}$$

where δ_{mn}, called the Kronecker's symbol is defined as

$$\delta_{mn} = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases}$$

14.7 EIGENVALUES AND EIGENFUNCTIONS

We know that a non – trivial solution is the one which is not identically zero. Because of the presence of the parameter λ, the problem of S – L system is an eigenvalue problem. We look for the values of the parameter λ for which there exists a non – trivial solution of the system over the interval $a \leq x \leq b$. These values of λ are called the characteristic values or eigenvalues of the system, and the corresponding non – trivial solutions are called the characteristic functions or eigenfunctions of the system. Note that by definition the eigenfunction cannot be a zero function.

14.8 PROPERTIES OF REGULAR STURM - LIOUVILLE SYSTEM

ORTHOGONALITY OF EIGENFUNCTIONS

THEOREM (14.1): Let $y_m(x)$ and $y_n(x)$ be two eigenfunctions corresponding to two different eigenvalues λ_m and λ_n respectively, of a regular Sturm - Liouville system defined over the interval $a \le x \le b$. Prove that $y_m(x)$ and $y_n(x)$ are orthogonal over that interval w.r.t. the weight function $r(x)$.

PROOF: Since $y_m(x)$ and $y_n(x)$ are the solutions of the regular Sturm - Liouville system, then

$$\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right]+\left[q(x)+\lambda_m r(x)\right]y_m = 0 \tag{1}$$

$$\left.\begin{aligned} \alpha_1 y_m(a)+\alpha_2 y_m'(a) &= 0 \\ \beta_1 y_m(b)+\beta_2 y_m'(b) &= 0 \end{aligned}\right\} \tag{2}$$

and $$\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right]+\left[q(x)+\lambda_n r(x)\right]y_n = 0 \tag{3}$$

$$\left.\begin{aligned} \alpha_1 y_n(a)+\alpha_2 y_n'(a) &= 0 \\ \beta_1 y_n(b)+\beta_2 y_n'(b) &= 0 \end{aligned}\right\} \tag{4}$$

Multiplying equation (1) by y_n and equation (3) by y_m and subtracting, we get

$$y_n\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right]-y_m\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right]+(\lambda_m-\lambda_n)r(x)y_m y_n = 0$$

or $$p(x)\left[y_n y_m''-y_m y_n''\right]+p'(x)\left[y_n y_m'-y_m y_n'\right] = (\lambda_n-\lambda_m)r(x)y_m y_n$$

or $$p(x)\frac{d}{dx}\left[y_n y_m'-y_m y_n'\right]+p'(x)\left[y_n y_m'-y_m y_n'\right] = (\lambda_n-\lambda_m)r(x)y_m y_n$$

or $$\frac{d}{dx}\left[p(x)(y_n y_m'-y_m y_n')\right] = (\lambda_n-\lambda_m)r(x)y_m y_n$$

Integrating both sides w.r.t. x from a to b, we get

$$\left|p(x)(y_n y_m'-y_m y_n')\right|_a^b = \int_a^b(\lambda_n-\lambda_m)r(x)y_m y_n\,dx$$

$$p(b)\left[y_n(b)y_m'(b)-y_m(b)y_n'(b)\right]-p(a)\left[y_n(a)y_m'(a)-y_m(a)y_n'(a)\right]$$
$$= (\lambda_n-\lambda_m)\int_a^b r(x)y_m(x)y_n(x)\,dx$$

Substituting the values of $y_m'(a)$, $y_m'(b)$, $y_n'(a)$ and $y_n'(b)$ from equations (2) and (4), we get

$$p(b)\left[-\frac{\beta_1}{\beta_2}y_m(b)y_n(b)+\frac{\beta_1}{\beta_2}y_m(b)y_n(b)\right]$$
$$-p(a)\left[-\frac{\alpha_1}{\alpha_2}y_m(a)y_n(a)+\frac{\alpha_1}{\alpha_2}y_m(a)y_n(a)\right]$$
$$=(\lambda_n-\lambda_m)\int_a^b r(x)y_m(x)y_n(x)\,dx$$

or $$(\lambda_n-\lambda_m)\int_a^b r(x)y_m(x)y_n(x)\,dx=0$$

Since λ_m and λ_n are distinct, therefore $\lambda_n-\lambda_m\neq 0$.

Hence, $$\int_a^b r(x)y_m(x)y_n(x)\,dx=0$$

which shows that $y_m(x)$ and $y_n(x)$ are orthogonal w.r.t. the weight function $r(x)$ over the interval $a\le x\le b$. This complete the proof.

REALITY OF EIGENVALUES

THEOREM (14.2): If $r(x)>0$ over the interval $a\le x\le b$ (or $r(x)<0$ over the interval $a\le x\le b$), then prove that all the eigenvalues of a regular Sturm – Liouville system are real.

PROOF: Consider the regular Sturm – Liouville system

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \qquad (1)$$

$$\left.\begin{aligned}\alpha_1 y(a)+\alpha_2 y'(a)=0\\ \beta_1 y(b)+\beta_2 y'(b)=0\end{aligned}\right\} \qquad (2)$$

Let $\lambda=\alpha+i\beta$ be an eigenvalue of the system and let $y(x)=u(x)+iv(x)$ be a corresponding eigenfunction. Here, α, β, $u(x)$ and $v(x)$ are real.

Substituting into the system (1) and (2), we get

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}(u+iv)\right]+\left[q(x)+(\alpha+i\beta)r(x)\right](u+iv)=0$$
$$\alpha_1\left[u(a)+iv(a)\right]+\alpha_2\left[u'(a)+iv'(a)\right]=0$$
$$\beta_1\left[u(b)+iv(b)\right]+\beta_2\left[u'(b)+iv'(b)\right]=0$$

Separating into real and imaginary parts, we get

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+\left[q(x)+\alpha r(x)\right]u-\beta r(x)v=0 \tag{3}$$

$$\left.\begin{aligned}\alpha_1 u(a)+\alpha_2 u'(a)=0\\ \beta_1 u(b)+\beta_2 u'(b)=0\end{aligned}\right\} \tag{4}$$

and $$\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right]+\left[q(x)+\alpha r(x)\right]v+\beta r(x)u=0 \tag{5}$$

$$\left.\begin{aligned}\alpha_1 v(a)+\alpha_2 v'(a)=0\\ \beta_1 v(b)+\beta_2 v'(b)=0\end{aligned}\right\} \tag{6}$$

Multiplying equation (3) by v and equation (5) by u and subtracting, we get

$$v\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]-u\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right]-\beta(u^2+v^2)r(x)=0$$

or $$p(x)\left[vu''-uv''\right]+p'(x)\left[vu'-uv'\right]=\beta(u^2+v^2)r(x)$$

or $$p(x)\frac{d}{dx}\left[vu'-uv'\right]+p'(x)\left[vu'-uv'\right]=\beta(u^2+v^2)r(x)$$

$$\frac{d}{dx}\left[p(x)(vu'-uv')\right]=\beta(u^2+v^2)r(x)$$

Integrating this equation w.r.t. x from a to b, we get

$$\beta\int_a^b(u^2+v^2)r(x)\,dx=\left|p(x)(vu'-uv')\right|_a^b$$

$$=p(b)\left[v(b)u'(b)-u(b)v'(b)\right]-p(a)\left[v(a)u'(a)-u(a)v'(a)\right]$$

Because of the boundary conditions (4) and (6), the R.H.S. is zero, thus

$$\beta\int_a^b(u^2+v^2)r(x)\,dx=0 \tag{7}$$

Since y is an eigenfunction (i.e. non – trivial solution) of the system, $u^2+v^2\neq 0$. Furthermore, $r(x)>0$ (or $r(x)<0$) on the interval $a\le x\le b$, the integral on the left of equation (7) is not zero. Hence $\beta=0$ which means that $\lambda=\alpha$ is real. This completes the proof.

EXAMPLE (1): **Given the regular Sturm – Liouville system**

$$y''+\lambda y=0,\quad y(0)=0,\quad y(1)=0.$$

(i) **Find the eigenvalues and eigenfunctions of the system,**

(ii) **Show that the eigenfunctions are orthogonal over the interval $0\le x\le 1$.**

(iii) **Find the corresponding orthonormal set of eigenfunctions.**

SOLUTION: **CASE (1):** If $\lambda = 0$, then $y'' = 0$ with the general solution $y = Ax + B$. Since $y(0) = 0$ implies $B = 0$ and so $y = Ax$. Next, $y(0) = 1$ gives $A = 0$, so that we have only the trivial solution $y = 0$. Thus $\lambda = 0$ is not an eigenvalue of the problem.

Therefore, the given system is a regular Sturm – Liouville system.

CASE (2): If $\lambda < 0$, say $\lambda = -a^2$ with $a > 0$. Then $y'' - a^2 y = 0$ with the general solution as

$$y(x) = A e^{-ax} + B e^{ax}$$

Using the condition $y(0) = 0$ and $y(1) = 0$, we get

$$A + B = 0 \quad \text{and} \quad A e^{-a} + B e^{a} = 0$$

From these equations, we get $A = B = 0$

Thus we have only the trivial solution $y = 0$. Thus no number $\lambda < 0$ is an eigenvalue of the problem.

CASE (3): If $\lambda > 0$, the general solution of $y'' + \lambda y = 0$ is

$$y(x) = A \cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x$$

Using the boundary condition $y(0) = 0$, we get $A = 0$, and thus

$$y(x) = B \sin\sqrt{\lambda}\, x \tag{1}$$

Using $y(1) = 0$, we get $B \sin\sqrt{\lambda} = 0$

Since $B \neq 0$, because otherwise the solution will be trivial, therefore $\sin\sqrt{\lambda} = 0$

or $\sqrt{\lambda} = n\pi, \quad n = 1, 2, 3, \ldots\ldots$

or $\lambda = n^2 \pi^2 \tag{2}$

are the required eigenvalues.

We exclude the values $n = 0, -1, -2, \ldots\ldots\ldots$, since the corresponding solutions are trivial. For each value of n, we get a different eigenvalue and a different eigenfunction from equation (1) given by

$$y_n = B_n \sin n\pi x, \quad n = 1, 2, 3, \ldots\ldots \tag{3}$$

(ii) The eigenfunctions are orthogonal over $0 \le x \le 1$, since

$$\int_0^1 (B_m \sin m\pi x)(B_n \sin n\pi x)\, dx = B_m B_n \int_0^1 \sin m\pi x \sin n\pi x\, dx$$

$$= \frac{B_m B_n}{2} \int_0^1 [\cos(m-n)\pi x - \cos(m+n)\pi x]\, dx$$

$$= \frac{B_m B_n}{2} \left| \frac{\sin(m-n)\pi x}{(m-n)\pi} - \frac{\sin(m+n)\pi x}{(m+n)\pi} \right|_0^1$$

$$= 0, \quad m \neq n.$$

(iii) The eigenfunctions will be orthonormal if

$$\int_0^1 (B_n \sin n\pi x)^2 \, dx = 1$$

or $$B_n^2 \int_0^1 \left(\frac{1-\cos 2n\pi x}{2}\right) dx = 1$$

or $$B_n^2 = 2 \quad \text{or} \quad B_n = \sqrt{2}$$

Thus the set of eigenfunctions

$$y_n = \sqrt{2} \sin n\pi x, \quad n = 1, 2, 3, \ldots\ldots$$

is an orthonormal set.

14.9 PROPERTIES OF PERIODIC STURM – LIOUVILLE SYSTEM

ORTHOGONALITY OF EIGENFUNCTION

THEOREM (14.3): Let $y_m(x)$ and $y_n(x)$ be two eigenfnctions corresponding to two different eigenvalues λ_m and λ_n respectively, of a periodic Sturm – Liouville system defined over the interval $a \le x \le b$. Prove that $y_m(x)$ and $y_n(x)$ are orthogonal over that interval w.r.t. the weight function $r(x)$.

PROOF: Since $y_m(x)$ and $y_n(x)$ are the solutions of the periodic Sturm – Liouville system, then

$$\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] + \left[q(x) + \lambda_m r(x)\right] y_m = 0 \tag{1}$$

$$y_m(a) = y_m(b), \quad y_m'(a) = y_m'(b) \tag{2}$$

and $$\frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right] + \left[q(x) + \lambda_n r(x)\right] y_n = 0 \tag{3}$$

$$y_n(a) = y_n(b), \quad y_n'(a) = y_n'(b) \tag{4}$$

Multiplying equation (1) by y_n and equation (3) by y_m and subtracting, we get

$$y_n \frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] - y_m \frac{d}{dx}\left[p(x)\frac{dy_n}{dx}\right] + (\lambda_m - \lambda_n) r(x) y_m y_n = 0$$

or $$p(x)\left[y_n y_m'' - y_m y_n''\right] + p'(x)\left[y_n y_m' - y_m y_n'\right] = (\lambda_n - \lambda_m) r(x) y_m y_n$$

or $$\frac{d}{dx}\left[p(x)\left(y_n y_m' - y_n y_n'\right)\right] = (\lambda_n - \lambda_m) r(x) y_m y_n$$

Integrating both sides w.r.t. x from a to b, we get

$$\left| p(x)\left(y_n y_m' - y_m y_n'\right)\right|_a^b = \int_a^b (\lambda_n - \lambda_m) r(x) y_m y_n \, dx$$

or

$$p(b)\left[y_n(b) y_m'(b) - y_m(b) y_n'(b)\right] - p(a)\left[y_n(a) y_m'(a) - y_m(a) y_n'(a)\right]$$

$$= (\lambda_n - \lambda_m) \int_0^a r(x) y_m(x) y_n(x) \, dx \qquad (5)$$

Substituting the values of $y_m'(a)$, $y_m'(b)$, $y_n'(a)$, $y_n'(b)$ from equations (2) and (4), equation (5) becomes

$$p(b)\left[y_n(b) y_m'(a) - y_m(b) y_n'(a)\right] - p(a)\left[y_n(b) y_m'(a) - y_m(b) y_n'(a)\right]$$

$$= (\lambda_n - \lambda_m) \int_0^a r(x) y_m(x) y_n(x) \, dx$$

Since the system is periodic, $p(a) = p(b)$, therefore, L.H.S. of this equation is zero. Thus

$$(\lambda_n - \lambda_m) \int_a^b r(x) y_m(x) y_n(x) \, dx = 0$$

Since λ_m and λ_n are distinct, therefore $\lambda_n - \lambda_m \neq 0$.

Hence, $\displaystyle\int_a^b r(x) y_m(x) y_n(x) \, dx = 0$

which shows that $y_m(x)$ and $y_n(x)$ are orthogonal w.r.t. the weight function $r(x)$ over the interval $a \le x \le b$. This completes the proof.

REALITY OF EIGENVALUES

THEOREM (14.4): **If $r(x) > 0$ over the interval $a \le x \le b$ (or $r(x) < 0$ over the interval $a \le x \le b$), then prove that all the eigenvalues of a periodic Sturm – Liouville system are real.**

PROOF: Consider the periodic Sturm – Liouville system

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right] y = 0 \qquad (1)$$

$$y(a) = y(b), \quad y'(a) = y'(b) \qquad (2)$$

Let $\lambda = \alpha + i\beta$ be an eigenvalue of the system and let $y(x) = u(x) + iv(x)$ be a corresponding eigenfunction. Here α, β, $u(x)$ and $v(x)$ are real.

Substituting into equations (1) and (2), we get

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}(u+iv)\right] + [q(x) + (\alpha + i\beta)r(x)](u+iv) = 0$$

$$u(a) + iv(a) = u(b) + iv(b)$$

$$u'(a) + iv'(a) = u'(b) + iv'(b)$$

Separating into real and imaginary parts, we get

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + [q(x) + \alpha r(x)]u - \beta r(x)\, v = 0 \tag{3}$$

$$u(a) = u(b), \quad u'(a) = u'(b) \tag{4}$$

and
$$\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + [q(x) + \alpha r(x)]v + \beta r(x)\, u = 0 \tag{5}$$

$$v(a) = v(b), \quad v'(a) = v'(b) \tag{6}$$

Multiplying equation (3) by v and equation (5) by u and subtracting, we get

$$v\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] - u\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] - \beta(u^2+v^2)r(x) = 0$$

or
$$p(x)[vu'' - uv''] + p'(x)[vu' - uv'] = \beta(u^2+v^2)r(x)$$

or
$$p(x)\frac{d}{dx}[vu' - uv'] + p'(x)[vu' - uv'] = \beta(u^2+v^2)r(x)$$

or
$$\frac{d}{dx}[p(x)(vu' - uv')] = \beta(u^2+v^2)r(x)$$

Integrating this equation w.r.t. x from a to b, we get

$$\beta\int_a^b (u^2+v^2)r(x)\,dx = \left|p(x)(vu' - uv')\right|_a^b$$

$$= p(b)[v(b)u'(b) - u(b)v'(b)]$$
$$- p(a)[v(a)u'(a) - u(a)v'(a)]$$

Because of the boundary conditions (4) and (6)

$$\beta\int_a^b (u^2+v^2)r(x)\,dx = p(b)[v(b)u'(a) - u(b)v'(a)]$$
$$- p(a)[v(b)u'(a) - u(b)v'(a)]$$

Using the fact that the system is periodic, i.e. $p(a) = p(b)$, therefore R.H.S. is zero.

Thus $$\beta \int_a^b (u^2+v^2)\, r(x)\, dx = 0 \tag{7}$$

Since y is an eigenfunction (i.e. non – trivial solution) of the system, $u^2 + v^2 \neq 0$. Furthermore, $r(x) > 0$ (or $r(x) < 0$) on the interval $a \le x \le b$, the integral on the left of equation (7) is not zero. Hence $\beta = 0$ which means that $\lambda = \alpha$ is real. This completes the proof.

EXAMPLE (2): **Given the periodic Sturm – Liouville system**

$$y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

(i) **Find the eigenvalues and eigenfunctions of the system.**

(ii) **Find the corresponding set of normalized eigenfunctions.**

SOLUTION: (i) Since the Sturm – Liouville equation is periodic and the boundary conditions are of the type periodic, therefore this system is a periodic Sturm – Liouville system. It can be verified that the only solution corresponding to $\lambda < 0$ is the trivial solution. For the case $\lambda \ge 0$, the general solution of the equation $y'' + \lambda y = 0$ is

$$y(x) = A \cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x \tag{1}$$

Using the boundary condition $y(-\pi) = y(\pi)$, we get

$$A \cos\left(-\sqrt{\lambda}\,\pi\right) + B \sin\left(-\sqrt{\lambda}\,\pi\right) = A \cos\left(\sqrt{\lambda}\,\pi\right) + B \sin\left(\sqrt{\lambda}\,\pi\right)$$

or $$A \cos\sqrt{\lambda}\,\pi - B \sin\sqrt{\lambda}\,\pi = A \cos\sqrt{\lambda}\,\pi + B \sin\sqrt{\lambda}\,\pi$$

or $$2B \sin\sqrt{\lambda}\,\pi = 0 \tag{2}$$

From equation (1), we get

$$y'(x) = -\sqrt{\lambda}\, A \sin\sqrt{\lambda}\, x + \sqrt{\lambda}\, B \cos\sqrt{\lambda}\, x$$

Using the boundary condition $y'(-\pi) = y'(\pi)$, we get

$$-\sqrt{\lambda}\, A \sin\left(-\sqrt{\lambda}\,\pi\right) + \sqrt{\lambda}\, B \cos\left(-\sqrt{\lambda}\,\pi\right) = -\sqrt{\lambda}\, A \sin\sqrt{\lambda}\,\pi + \sqrt{\lambda}\, B \cos\sqrt{\lambda}\,\pi$$

or $$\sqrt{\lambda}\, A \sin\sqrt{\lambda}\,\pi + \sqrt{\lambda}\, B \cos\sqrt{\lambda}\,\pi = -\sqrt{\lambda}\, A \sin\sqrt{\lambda}\,\pi + \sqrt{\lambda}\, B \cos\sqrt{\lambda}\,\pi$$

or $$2\sqrt{\lambda}\, A \sin\sqrt{\lambda}\,\pi = 0 \tag{3}$$

From equations (2) and (3) if $\sin\sqrt{\lambda}\,\pi \neq 0$, then $A = B = 0$ and we have the trivial solution.

Thus $$\sin\sqrt{\lambda}\,\pi = 0 \quad \text{or} \quad \sqrt{\lambda}\,\pi = n\pi \tag{4}$$

or $$\lambda = n^2, \quad n = 0, 1, 2, 3, \ldots\ldots$$

are the required eigenvalues.

For each value of n, we get a different eigenvalue and a different eigenfunction given by equation (1) as

$$y_n(x) = A_n \cos nx + B_n \sin nx, \quad n = 0, 1, 2, 3, \ldots\ldots \tag{5}$$

where A_n and B_n are constants which cannot both be zero but are otherwise arbitrary.

Hence, the eigenfunctions are given by

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\ldots$$

(ii) By theorem (14.4), any two of these eigenfunctions belonging to different eigenvalues are orthogonal over the interval $-\pi \le x \le \pi$. Furthermore, $\cos nx$ and $\sin nx$ for the same n are also orthogonal since

$$\int_{-\pi}^{\pi} \cos nx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2nx \, dx = 0$$

Also $$\int_{-\pi}^{\pi} (1)^2 \, dx = 2\pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2 nx \, dx = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$$

Hence, the orthonormal set of eigenfunctions is given by

$$\frac{1}{\sqrt{2\pi}}, \; \frac{\cos x}{\sqrt{\pi}}, \; \frac{\sin x}{\sqrt{\pi}}, \; \frac{\cos 2x}{\sqrt{\pi}}, \; \frac{\sin 2x}{\sqrt{\pi}}, \ldots\ldots$$

14.10 EIGENVALUES AND EIGENFUNCTIONS OF SINGULAR S–L SYSTEM

(i) Consider the Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0 \quad \text{for} \quad -1 \le x \le 1$$

As already mentioned this equation is an example of a singular S – L differential equation.

Since $p(-1) = p(1) = 0$, so there are no boundary conditions at $x = -1$ and $x = 1$.

We want bounded solutions on this interval, so we choose $\lambda = n(n+1)$ with $n = 0, 1, 2, \ldots\ldots$.

These are the eigenvalues of this problem. The corresponding eigenfunctions are non – zero constant multiples of the Legendre polynomials $P_n(x)$.

(ii) Consider the Bessel's differential equation of order n

$$(xy')' + \left(-\frac{n^2}{x} + \lambda x\right)y = 0 \quad \text{with} \quad y(1) = 0 \quad \text{over the interval } 0 \le x \le 1.$$

Here $p(x) = x$, $r(x) = x$. Since $p(0) = 0$, so there is no boundary condition at $x = 0$.

The boundary condition at right end $x = 1$ is $y(1) = 0$. This is an example of type I singular Sturm – Liouville problem.

We know that $\lambda > 0$, the general solution of Bessel's equation is

$$y(x) = A J_n(\sqrt{\lambda} x) + B Y_n(\sqrt{\lambda} x) \tag{2}$$

(2) (i) $\dfrac{3-4e^{-2s}+e^{-4s}}{s}$ (ii) $\dfrac{2}{s^2}(1-e^{-4s})-\dfrac{7}{s}e^{-4s}$

(3) (i) $\dfrac{k}{s}(1-e^{-cs})$ (ii) $-\dfrac{ke^{-cs}}{s}-\dfrac{k(1-e^{-cs})}{cs^2}$

(iii) $\dfrac{k}{s}-\dfrac{k}{cs^2}(1-e^{-cs})$ (iv) $-\dfrac{e^{-2s}}{s}+\dfrac{1}{s^2}(1-e^{-s})$

(4) (i) $\dfrac{2}{(s+2)^3}$ (ii) $\dfrac{12}{\left(s+\frac{1}{2}\right)^4}$

(iii) $\dfrac{s+1}{s^2+2s+2}$ (iv) $\dfrac{s+4}{(s+4)^2-4}$

(v) $\dfrac{6}{(s-4)^4}-\dfrac{s-4}{(s-4)^2+1}$ (vi) $\dfrac{1}{2}\left[\dfrac{1}{s-1}+\dfrac{s-1}{(s-1)^2+16}\right]$

(vii) $\dfrac{A(s+\alpha)}{(s+\alpha)^2+\beta^2}+\dfrac{B\beta}{(s+\alpha)^2+\beta^2}$ (viii) $\dfrac{6}{(s+1)^2-4}-\dfrac{5s}{(s+1)^2-4}$

(6) (i) $\dfrac{4}{(s-2)^2}$ (ii) $\dfrac{2s^2-8}{(s^2+4)^2}$ (iii) $\dfrac{s^2+1}{(s^2-1)^2}$

(iv) $\dfrac{6s}{(s^2-9)^2}$ (v) $\dfrac{2s^3-6\omega^2 s}{(s^2+\omega^2)^3}$ (vi) $-\dfrac{s^2+2s}{(s^2+2s+2)^2}$

(7) (i) $\cot^{-1}(s)$ (ii) $\ln\left(\dfrac{s+5}{s+3}\right)$

(iii) $\dfrac{1}{2}\ln\left(\dfrac{s^2+9}{s^2+4}\right)$ (iv) $\ln\left(\dfrac{s^2-a^2}{s^2}\right)$

(8) (i) $\dfrac{1}{2s}(e^{-s}+e^{-2s}-e^{-3s}-e^{-4s})$ (ii) $\dfrac{1}{s}(e^{-s}-2e^{-2s}+e^{-3s})$

(9) $\dfrac{2}{s}(-1+e^{-s}+e^{-10s})$

(10) (i) $\dfrac{e^{-s}}{s^2}$ (ii) $\dfrac{2e^{-s}}{s^3}$

(iii) $\dfrac{e^{(1-s)/2}}{s-1}$ (iv) $-e^{-\pi s}\left(\dfrac{s}{s^2+1}\right)$

(v) $e^{-3s}\left(\dfrac{s}{s^2+16}\right)$ (vi) $e^{-\pi s/2}\left(\dfrac{s}{s^2+1}\right)$

(vii) $e^{-bs}\left(\dfrac{s}{s^2-a^2}\right)$ (viii) $e^{-4s}\left(\dfrac{6}{s^4}+\dfrac{24}{s^3}+\dfrac{48}{s^2}+\dfrac{64}{s}\right)$

(11) (i) $\dfrac{1}{s^2}-\dfrac{e^{-s}}{s^2}-\dfrac{e^{-s}}{s}$ (ii) $\dfrac{1}{s-1}(1-e^{1-s})$

(iii) $\dfrac{1}{s^2+1}(e^{-2\pi s}-e^{-4\pi s})$ (iv) $\dfrac{s+(s-1)e^{-\pi s}}{s^2+1}$

(12) (i) $\dfrac{\pi s-1+(\pi s+1)e^{-2\pi s}}{s^2(1-e^{-2\pi s})}$ (ii) $\dfrac{2(e^{2\pi s}-1-2\pi s-2\pi^2 s^2)}{s^3(e^{2\pi s}-1)}$

(iii) $\dfrac{e^{2(1-s)\pi}-1}{(1-s)(1-e^{-2\pi s})}$

(13) (i) $\dfrac{\frac{1}{s^2}(1-e^{-\pi s})-\frac{\pi}{s}e^{-\pi s}}{1-e^{-2\pi s}}$ (ii) $\dfrac{\frac{1}{s^2}(e^{-\pi s}-e^{-2\pi s})-\frac{\pi}{s}e^{-2\pi s}}{1-e^{-2\pi s}}$

(14) (i) $\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4\omega^2}\right)$ (ii) $\frac{1}{2}\left(\frac{s}{s^2-4\omega^2}-\frac{1}{s}\right)$ (iii) $\frac{s}{s^2+\omega^2}$

(iv) $\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ (v) $\frac{s^2+\omega^2}{(s^2-\omega^2)^2}$ (vi) $\frac{2\omega s}{(s^2-\omega^2)^2}$

(15) (i) $\frac{1-s}{s(s^2+1)}$ (ii) $\frac{3}{s+1}+\frac{6}{s^2+36}$

(iii) $\frac{6}{s^3}-\frac{3}{s^2}+\frac{1}{s^2+4}$ (iv) $\frac{1}{2s^2}-\frac{1}{2(s^2+4)}$

(16) (i) $2e^{4t}$ (ii) $t+\frac{t^4}{24}$ (iii) $1-e^{-t}$

(iv) $\frac{1}{2}\sin 4t$ (v) $2\cos 2t+\frac{1}{2}\sin 2t$ (vi) $\cosh 3t-3\sinh 3t$

(17) (i) $e^{-\pi t}t$ (ii) $4e^{t/2}t^2$ (iii) $\frac{5}{3}e^{-\pi t}t^3$

(iv) $\frac{1}{2}e^{-t}\sin 2t$ (v) $e^{-t}(\cos t+\sin t)$ (vi) $e^{-t}\cos 3t+e^{-t}\sin 3t$

(vii) $e^{-3t}(\cos t-3\sin t)$ (viii) $2e^{-\alpha t}\cosh 4t$

(18) (i) t^2e^{at} (ii) $t\sin t$ (iii) $t\sinh 3t$

(iv) $te^{-3t}\sin t$ (iv) $\frac{e^{-bt}-e^{-at}}{t}$ (vi) $\frac{1}{t}e^{-t}\sin t$

(19) (i) $\begin{cases} 0 & \text{if } 0 \le t < 3 \\ t-3 & \text{if } t > 3 \end{cases}$ (ii) $\begin{cases} 0 & \text{if } 0 \le t < 1 \\ \frac{(t-1)^2}{2} & \text{if } t > 1 \end{cases}$

(iii) $\begin{cases} 0 & \text{if } 0 \le t < 2 \\ e^{2(t-2)} & \text{if } 2 < t < 4 \\ e^{2(t-2)}-e^{2(t-4)} & \text{if } t > 4 \end{cases}$ (iv) $\begin{cases} 0 & \text{if } 0 \le t < \pi \\ \cos 2t & \text{if } t > \pi \end{cases}$

(v) $\frac{1}{\omega}\sin\omega(t-1)u_1(t)=\begin{cases} 0 & \text{if } 0 \le t < 1 \\ \frac{1}{\omega}\sin\omega(t-1) & \text{if } t > 1 \end{cases}$

(vi) $\cos t-\cos t\, u_\pi(t)=\begin{cases} \cos t & \text{if } 0 \le t < \pi \\ 0 & \text{if } t > \pi \end{cases}$

(20) (i) $1-e^{-t}$ (ii) $t+e^{-t}-1$ (iii) $2-t-2e^{-t}$

(iv) $1-\cos 2t$ (v) $-\cos t-\sin t+t+1$ (vi) $\sinh 2t-2t$

21) (i) e^t-1 (ii) $1-\cos t$ (iii) $\frac{t^4}{12}$

(iv) $\frac{t^5}{30}$ (v) te^{kt} (vi) $\frac{e^{at}-e^{bt}}{a-b}$

(vii) $\frac{2}{3}\sin t - \frac{1}{3}\sin 2t$

(viii) $\begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1-e^{-2(t-3)}}{2} & \text{if } t \ge 3 \end{cases}$

(22) (i) $e^{t} - 1$

(ii) $\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}$

(iii) te^{at}

(iv) $e^{t} - t - 1$

(v) $\frac{1}{\omega^2}(1 - \cos\omega t)$

(vi) $\frac{1}{2\omega^2}\left(\frac{\sin\omega t}{\omega} - t\cos\omega t\right)$

(vii) $\frac{1}{2}\left(\frac{\sin\omega t}{\omega} + t\cos\omega t\right)$

(viii) $\begin{cases} 0 \text{ if } 0 \le t < a \\ \frac{1}{2}\left[e^{2(t-a)} - 1\right] \text{ if } t > a \end{cases}$

(23) (i) $2e^{-4t} + e^{2t}$

(ii) $1 + 3\sinh 3t$

(iii) $(t+1)e^{-t}$

(iv) $e^{-t}(1 - t - t^2)$

(v) $e^{-t}(\cos t - \sin t)$

(vi) $e^{-t}(\cos 2t - 2t\sin t)$

(24) (i) $-\frac{2}{3} + \frac{3}{5}e^{2t} + \frac{1}{15}e^{-3t}$

(ii) $\cos at$

(iii) $\frac{1-\cos at}{a^2}$

(iv) te^{-t}

(v) $\frac{1}{2a}t\sin at$

(vi) $t\cos t$

(vii) $\frac{1}{2}t^2 + \cos t - 1$

(viii)

(25) $y(t) = e^{-3t} - \cos t + 3\sin t$

(26) $y(t) = \frac{7}{6}e^{5t} - \frac{1}{6}e^{-4t}$

(27) $y(t) = 2\cos 2t - 4\sin 2t$

(28) $y(t) = e^{2t} - e^{-4t}$

(29) $y(t) = 3e^{t/3}$

(30) $y(t) = \cos 5t + \frac{1}{25}t$

(31) $y(t) = \frac{1}{4}(1 - \cos 2t) - \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)$

(32) $y(t) = 5e^{2t} - 2t^2$

(33) $y(t) = -1 + \frac{1}{2}e^{x} + \frac{1}{2}\cos x - \frac{1}{2}\sin x$

(34) $y(t) = \frac{1}{4}e^{x} + \frac{1}{4}e^{-x} + \frac{1}{2}\cos x$

(35) $y(t) = e^{-t}\sin t$

(36) $y(t) = e^{2t}\cos t$

(37) $y(t) = 3e^{-t} + 2te^{-t} + \sin t$

(38) $y(t) = \cosh 2t + e^{-t}\sin 2t$

(39) $y(t) = -e^{-t} + \frac{1}{2}e^{-t}t^2$

(40) $y(t) = \begin{cases} 0 \text{ if } 0 \le t < 2 \\ \frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)} \text{ if } t > 2 \end{cases}$

(41) $y(t) = (t-1)u_1(t) - \sin(t-1)u_1(t) = \begin{cases} 0 \text{ if } 0 \le t < 1 \\ (t-1) - \sin(t-1) \text{ if } t > 1 \end{cases}$

(42) $y(t) = (1-\cos t) - [1-\cos(t-1)]u_1(t) = \begin{cases} 1-\cos t & \text{if } 0 \le t < 1 \\ -\cos t + \cos(t-1) & \text{if } t > 1 \end{cases}$

(43) $y(t) = t - \sin t - [1-\cos(t-1)]u_1(t) - [(t-1) - \sin(t-1)]u_1(t)$

$= \begin{cases} t - \sin t & \text{if } 0 \le t < 1 \\ \cos(t-1) + \sin(t-1) - \sin t & \text{if } t > 1 \end{cases}$

(44) $y(t) = (2e^t + e^{2t} - 2e^{3t}) - (2e^t - 4e^{2t-2} + 2e^{3t-4})u_2(t)$

$= \begin{cases} 2e^t + e^{2t} - 2e^{3t} & \text{if } 0 \le t < 2 \\ e^{2t} - 2e^{3t} + 4e^{2t-2} - 2e^{3t-4} & \text{if } t > 2 \end{cases}$

(45) $y(t) = \sin 3t + \sin t - \sin t\, u_\pi(t) + \frac{\sin 3t}{3} u_\pi(t) = \begin{cases} \sin 3t + \sin t & \text{if } 0 \le t < \pi \\ \frac{4}{3}\sin 3t & \text{if } t > \pi \end{cases}$

(46) $y(t) = \begin{cases} 2\cos t & \text{if } 0 \le t < \pi \\ 2\cos t - \frac{1}{2}\sin 2t & \text{if } t > \pi \end{cases}$

(47) $y(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ \frac{1}{3}\sin 3(t-1) & \text{if } t > 1 \end{cases}$

(48) $y(t) = \sin t - \sin t\, u_\pi(t) - \sin t\, u_{2\pi}(t) = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ -\sin t & \text{if } t > 2\pi \end{cases}$

(49) $y(t) = e^{-t}\cos t + e^{-(t-2\pi)}\sin t\, u_{2\pi}(t) = \begin{cases} e^{-t}\cos t & \text{if } 0 \le t < 2\pi \\ e^{-t}(\cos t + e^{2\pi}\sin t) & \text{if } t > 2\pi \end{cases}$

(50) $y(t) = \begin{cases} 2te^{-2t} + \frac{1}{4}t & \text{if } 0 \le t < 1 \\ \frac{1}{4}t + e^{-2t}[2t + (t-1)e^2] & \text{if } t > 1 \end{cases}$

(51) $y(t) = e^t + e^{-t}\cos 2t + \frac{1}{2}e^{-(t-1)}\sin 2(t-1)u_1(t)$

$= \begin{cases} e^t + e^{-t}\cos 2t & \text{if } 0 \le t < 1 \\ e^t + e^{-t}\cos 2t + \frac{1}{2}e^{-(t-1)}\sin 2(t-1) & \text{if } t > 1 \end{cases}$

(52) $y(t) = t - \sin t$

(53) $y(t) = \frac{1}{2}(\sin t - t\cos t)$

(54) $y(t) = te^{-t} - e^{-t} + e^{-2t}$

(55) $y(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ \frac{1}{4} - \frac{1}{4}\cos(2t-2) & \text{if } t > 1 \end{cases}$

(56) $y(t) = t\cos t$

(57) $y(t) = \sin t - \frac{1}{2}\sin 2t$

(58) $y(t) = 3\cos 2t - 2\cos t$

(59) $y(t) = e^{t}$

(60) $y(t) = \cos t$

(61) $y(t) = 5t - 3 - t^2$

(62) $y(t) = t + \frac{1}{6}t^3$

(63) $y(t) = \sqrt{2}\sin\sqrt{2}\,t$

(64) $y(t) = t$

(65) $y_1(t) = \cos t, \quad y_2(t) = \sin t$

(66) $y_1(t) = \sin t, \quad y_2(t) = \cos t$

(67) $y_1(t) = -6e^{4t} + 2, \quad y_2(t) = -3e^{4t} - 1$

(68) $y_1(t) = 4e^{t} - 2e^{-2t}, \quad y_2(t) = e^{t} - e^{-2t}$

(69) $y_1(t) = 3e^{2t} + e^{-5t}, \quad y_2(t) = 4e^{2t} - e^{-5t}$

(70) $y_1(t) = e^{t} + e^{2t}, \quad y_2(t) = e^{2t}$

(71) $y_1(t) = \frac{8}{9}\left(1 - \cos\frac{3}{2}t\right), \quad y_2(t) = \frac{4}{3}t - \frac{8}{9}\sin\frac{3}{2}t$

(72) $y_1(t) = -\cos t - \sin t, \quad y_2(t) = \cos t$

(73) $y_1(t) = -\frac{1}{2}t^2, \quad y_2(t) = 1 + \frac{1}{2}t^2$

(74) $y_1(t) = 1 - e^{t}, \quad y_2(t) = e^{t} + \sin t, \quad y_3(t) = \cos t$

(75) $y_1(t) = \cos t + \sin t, \quad y_2(t) = \cos t - \sin t, \quad y_3(t) = 1$

(76) $y(t) = \cos 10t$

(77) $y(t) = te^{-t}$

(78) $y(t) = \frac{1}{20}\left[(1 + 21t)e^{-t} - \cos t\right]$

(79) (i) $y_1(t) = \cos t + 4\cos\sqrt{6}\,t, \quad y_2(t) = 2\cos t - 2\cos\sqrt{6}\,t$

(ii) $y_1(t) = \frac{1}{5}\sin 2t + \frac{1}{5}\sin 3t, \quad y_2(t) = \frac{3}{5}\sin 2t - \frac{2}{5}\sin 3t$

(80) $y_1(t) = \cos\sqrt{3}\,t + \sin 3t, \quad y_2(t) = \cos\sqrt{3}\,t - \sin 3t$

(81) $y_1(t) = \frac{5}{18} - \frac{1}{10}\cos 2t - \frac{8}{45}\cos 3t, \quad y_2(t) = \frac{1}{9} - \frac{1}{5}\cos 2t + \frac{4}{45}\cos 3t$

(82) (i) $I(t) = 4(1 - e^{5t})$ (ii) $I(t) = 5(e^{-3t} - e^{-5t})$

(iii) $I(t) = \frac{5}{2}(\sin 5t - \cos 5t) + \frac{5}{2}e^{-5t}$

(83) (i) $Q(t) = 2 + 3e^{-10t}$ (ii) $I(t) = -30e^{-10t}$

(84) (i) $I(t) = 1 - \cos t$ (ii) $I(t) = \frac{110}{21}(\cos 4t - \cos 10t)$

(85) $I(t) = \frac{75}{52}(2\cos 3t + 3\sin 3t) - e^{-4t}\left(\frac{150}{52}\cos 3t + \frac{425}{52}\sin 3t\right)$

(86) $I_1(t) = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}$, $I_2(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$

(87) $I_1(t) = (19 + 32.5t)e^{-5t} - 19\cos t + 62.5\sin t$

$I_2(t) = (-6 - 32.5t)e^{-5t} + 6\cos t + 2.5\sin t$

(88) $I_1(t) = e^{-4t} + t$, $I_2(t) = -4e^{-4t} + \frac{1}{5}$

(89) $I_1(t) = 500\left(\frac{1}{7} - \frac{1}{12}e^{-t/2} - \frac{5}{84}e^{-7t/2}\right)$, $I_2(t) = 500\left(\frac{1}{7} - \frac{1}{6}e^{-t/2} + \frac{1}{42}e^{-7t/2}\right)$

(90) $I_1 = 2 - 2e^{-300t}$, $I_2 = 2 + e^{-300t}$

CHAPTER 10

(5) (i) $y(x) = a_0\left(1 + \frac{4}{1!}x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \ldots\ldots\right) = a_0 e^{4x}$

(ii) $y(x) = a_0\left(1 + 2x^2 + \frac{2}{3}x^4 + \ldots\ldots\right) + a_1\left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5 + \ldots\ldots\right)$

(iii) $y(x) = a_0(1 + x^2 + x^4 + \ldots\ldots) + a_1(x + x^3 + x^5 + \ldots\ldots)$

(iv) $y(x) = a_0\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\ldots\right) + a_1\left(x - \frac{x^3}{3!} - \frac{x^5}{5!} - \ldots\ldots\right)$

(v) $y(x) = a_0\left(1 - \frac{x^4}{12} + \frac{x^6}{90} + \ldots\ldots\right) + a_1\left(x - \frac{x^3}{6} - \frac{x^5}{40} - \ldots\ldots\right)$

(vi) $y(x) = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} \ldots\ldots\right) + a_1 x$

(6) (i) $y(x) = a_0\left[1 + 2(x-1) + 2(x-1)^2 + \frac{4}{3}(x-1)^3 + \ldots\ldots\right]$

(ii) $y(x) = a_0[1 + (x-1)]$

(iii) $y(x) = a_0\left[1 - (x-1)^2 - \frac{1}{3}(x-1)^3 - \ldots\ldots\right] + a_1\left[(x-1) + \frac{1}{2}(x-1)^2 - \ldots\ldots\right]$

(iv) $y(x) = a_0\left[1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 + \ldots\ldots\right]$

$+ a_1\left[(x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 + \ldots\ldots\right]$

(7) (i) $y(x) = A_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \ldots\ldots\right) + B_1 x^{1/2}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} - \ldots\ldots\right)$

where $A_1 = a_0 A$ and $B_1 = a_0 B$

(ii) $y(x) = a_0\left[1 - 3x + \frac{3}{1.3}x^2 + \frac{3}{1.3.5}x^3 + \ldots\ldots\right] + a_0 x^{1/2}(1 - x)$

(iii) $y(x) = A\left(1+\frac{3}{5}x+\frac{3}{7}x^2+\frac{3}{9}x^3+\ldots\ldots\ldots\right)+Bx^{-3/2}$

(iv) $y(x) = a_0\left[1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+\ldots\ldots\ldots\right]$

$+a_0x^{7/3}\left[1+\frac{8}{10}x+\frac{8.11}{10.13}x^2+\frac{8.11.14}{10.13.16}x^3+\ldots\ldots\ldots\right]$

(8) (i) $y_1(x) = \left(1-2x+\frac{3}{2!}x^2-\frac{4}{3!}x^3+\ldots\ldots\ldots\right)$

$y_2(x) = y_1\ln x+2\left(2-\frac{1}{2}\right)x-\frac{3}{2!}\left(2+\frac{1}{2}-\frac{1}{3}\right)x^2+\ldots\ldots\ldots$

(ii) $y_1(x) = 1+x+\frac{2}{4}x^2+\frac{2.5}{4.9}x^3+\ldots\ldots\ldots$

$y_2(x) = y_1\ln x+a_0\left(-2x-x^2-\frac{14}{27}x^3-\ldots\ldots\ldots\right)$

(9) (i) $y_1(x) = a_0\left(x+\frac{1}{2}x^2+\frac{1}{12}x^3+\frac{1}{144}x^4+\ldots\ldots\ldots\right)$

$y_2(x) = y_1\ln x+1-\frac{3}{4}x^2-\frac{7}{36}x^3-\frac{135}{1728}x^4-\ldots\ldots\ldots$

(ii) $y_1(x) = 1.2x^2+2.3x^3+3.4x^4+\ldots\ldots\ldots$

$y_2(x) = y_1\ln x+(-1+x+3x^2+5x^3+7x^4+\ldots\ldots\ldots)$

(10) (i) $y(x) = a_0\left(1-x^2-\frac{1}{3}x^4-\frac{1}{5}x^6-\ldots\ldots\ldots\right)+a_1x$

(ii) $y(x) = a_0\left(1-\frac{1}{4}x^2-\frac{1}{12}x^3+\frac{5}{96}x^4+\ldots\right)+a_1\left(x-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{24}x^5-\ldots\right)$

CHAPTER 11

(31) (i) $y(x) = AJ_{1/2}(x)+BJ_{-1/2}(x)$

(ii) $y(x) = AJ_3(x)+BY_3(x)$

(32) (i) $y(x) = AJ_0(Kx)+BY_0(Kx)$

(ii) $y(x) = AJ_0(e^x)+BY_0(e^x)$

(iii) $y(x) = AJ_0(2\sqrt{x})+BY_0(2\sqrt{x})$

(iv) $y(x) = AJ_0(x^4)+BY_0(x^4)$

(v) $y(x) = AJ_0(x^n)+BY_0(x^n)$

(vi) $y(x) = \frac{1}{x}\left[AJ_0(Kx)+BY_0(Kx)\right]$

(vii) $y(x) = x^{-n}\left[AJ_0(x)+BY_0(x)\right]$

(viii) $y(x) = x^{1/2}\left[AJ_0(Kx)+BY_0(Kx)\right]$

(ix) $y(x) = x^{-1/2}\left[AJ_0(Kx)+BY_0(Kx)\right]$

(33) (i) $y(x) = AJ_{1/3}(3x) + BJ_{-1/3}(3x)$

(ii) $y(x) = AJ_2(3x) + BY_2(3x)$

(iii) $y(x) = AJ_2(x^2) + BY_2(x^2)$

(iv) $y(x) = AJ_1(\sqrt{x}) + BY_1(\sqrt{x})$

(v) $y(x) = AJ_2(2x) + BY_2(2x)$

(vi) $y(x) = AJ_{1/2}(\sqrt{3}\,x) + BJ_{-1/2}(\sqrt{3}\,x)$

(vii) $y(x) = AxJ_1(x) + BxY_1(x)$

(viii) $y(x) = x^{-2}[AJ_2(x) + BY_2(x)]$

(ix) $y(x) = x^{1/4}[AJ_{1/2}(x^{1/2}) + BJ_{-1/2}(x^{1/2})]$

(x) $y(x) = x^{3/2}[AJ_{5/4}(x^2) + BJ_{-5/4}(x^2)]$

(34) (i) $y(x) = AJ_n(x^2) + BJ_{-n}(x^2)$, (n not an integer)

or $y(x) = AJ_n(x^2) + BY_n(x^2)$, (n an integer)

(ii) $y(x) = AJ_n(ae^{ix}) + BJ_{-n}(ae^{ix})$ (n not an integer)

$= AJ_n(ae^{ix}) + BY_n(ae^{ix})$ (n an integer)

(iii) $y(x) = AJ_n(\sqrt{5}\,x) + BJ_{-n}(\sqrt{5}\,x)$ (n non – integer)

or $y(x) = AJ_n(\sqrt{5}\,x) + BY_n(\sqrt{5}\,x)$ (n an integer)

(iv) $y(x) = AJ_n\left(\frac{1}{x}\right) + BJ_{-n}\left(\frac{1}{x}\right)$, (n not an integer)

or $y(x) = AJ_n\left(\frac{1}{x}\right) + BY_n\left(\frac{1}{x}\right)$, (n an integer)

(35) (i) $y(x) = AJ_{1/3}(x^2) + BJ_{-1/3}(x^2)$

(ii) $y(x) = \frac{1}{x}[AJ_{3/4}(2x^2) + BJ_{-3/4}(2x^2)]$

(iii) $y(x) = \frac{1}{x^2}[AJ_{1/2}(3x^3) + BJ_{-1/2}(3x^3)]$

(iv) $y(x) = x[AJ_2(x) + BY_2(x)]$

(v) $y(x) = x^4[AJ_1(x) + BY_1(x)]$

(vi) $y(x) = x^2[AJ_4(x^2) + BY_4(x^2)]$

(36) $$x^2 = \sum_{p=1}^{\infty} \frac{[2\lambda_p J_1(\lambda_p) - 4J_2(\lambda_p)]\, J_0(\lambda_p x)}{\lambda_p^2 J_1^2(\lambda_p)}$$

(37) $x - x^3 = \sum_{p=1}^{\infty} \frac{4 J_3(\lambda_p) J_1(\lambda_p x)}{\lambda_p^2 J_2^2(\lambda_p)}$

(38) $x^2 = \sum_{p=1}^{\infty} \frac{2 J_2(\lambda_p x)}{\lambda_p J_3(\lambda_p)}$

(43) (i) $y(x) = A I_0(\lambda x) + B K_0(\lambda x)$

(ii) $y(x) = A I_{1/2}(\sqrt{3}\, x) + B I_{-1/2}(\sqrt{3}\, x)$

(iii) $y(x) = A I_1(\lambda x i^{1/2}) + B K_1(\lambda x i^{1/2})$

(44) (i) $y(x) = A I_n\left(\frac{1}{x}\right) + B I_{-n}\left(\frac{1}{x}\right)$, (n not an integer)

or $y(x) = A I_n\left(\frac{1}{x}\right) + B K_n\left(\frac{1}{x}\right)$, (n an integer)

(ii) $y(x) = \sqrt{x}\,[A I_1(2\sqrt{x}) + B K_1(2\sqrt{x})]$

(45) $y(x) = A \frac{e^{x}}{\sqrt{x}} + B \frac{e^{-x}}{\sqrt{x}}$

CHAPTER 12

(1) $\frac{8}{3} P_2(x) - 3 P_1(x) + \frac{10}{3} P_0(x)$

(2) $\frac{8}{35} P_4(x) - \frac{10}{7} P_2(x) + P_1(x) - \frac{4}{5} P_0(x)$

(3) $10x^3 + 9x^2 - 9x - 5$ (4) $35x^4 - 27x^2 + 3$

(5) (i) 0 (ii) $\frac{2}{5}$

(6) $f(x) = P_0(x) + \frac{7}{4} P_1(x) + \frac{5}{8} P_2(x) - \frac{7}{16} P_3(x) + \ldots\ldots$

(7) $f(x) = \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) + \ldots\ldots$

CHAPTER 14

(1) $\lambda = \left[\frac{(2n+1)\pi}{2L}\right]^2$, $n = 0, 1, 2, \ldots\ldots, y_n(x) = B_n \sin \frac{(2n+1)\pi x}{2L}$

(2) $\lambda = \left[\frac{(2n+1)\pi}{2L}\right]^2$, $n = 0, 1, 2, \ldots\ldots, y_n(x) = A_n \cos \frac{(2n+1)\pi x}{2L}$

(3) $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \ldots\ldots, y_n(x) = A_n \cos \frac{n\pi x}{L}$

(4) $\lambda = \frac{n^2}{9}$, $n = 0, 1, 2, \ldots\ldots, y_n(x) = A_n \cos\frac{n}{3}x + B_n \sin\frac{n}{3}x$

where A_n and B_n are not both zero.

(5) An infinite number of eigenvalues are determined from the equation $\tan\sqrt{\lambda}\,\pi = -2\sqrt{\lambda}$. The eigenfunction corresponding to an eigenvalue λ_n is $y_n(x) = B_n \sin\sqrt{\lambda_n}\,x$.

(6) An infinite number of eigenvalues are determined from the equation $\tan\sqrt{\lambda} = \frac{1}{2\sqrt{\lambda}}$. The eigenfunction corresponding to an eigenvalue λ_n is

$$y_n = B_n\left(2\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x + \sin\sqrt{\lambda_n}\,x\right)$$

(7) $\lambda = \left[\frac{(2n+1)\pi}{2}\right]^2$, $n = 0, 1, 2, \ldots\ldots, y_n(x) = B_n \sin\left[\frac{(2n+1)\pi}{2}\ln x\right]$

(8) $\lambda = \left(\frac{n\pi}{2}\right)^2$, $n = 1, 2, 3, \ldots\ldots, y_n(x) = B_n \sin\left(\frac{n\pi}{2}\ln x\right)$

(9) $\lambda = (n\pi)^2$, $n = 1, 2, 3, \ldots\ldots, y_n(x) = B_n x \sin(n\pi \ln x)$

(10) $\lambda = \left(\frac{n\pi}{2}\right)^2$, $n = 1, 2, 3, \ldots\ldots, y_n(x) = B_n x^2 \sin\left(\frac{n\pi}{2}\ln x\right)$

(11) $\lambda = 1 + n^2$, $n = 1, 2, 3, \ldots\ldots, y_n(x) = B_n\left[e^{-x}\sin(nx)\right]$

(12) $\lambda = 1 + \frac{n^2\pi^2}{9}$, $n = 1, 2, 3, \ldots\ldots, y_n(x) = B_n\frac{1}{x}\left(\sin\frac{n\pi}{3}\ln x\right)$

(13) $G(x,\xi) = \begin{cases} -\ln\xi, & 0 < x < \xi \\ -\ln x, & \xi < x \le 1 \end{cases}$

(14) $G(x,\xi) = \begin{cases} \frac{1}{2}e^{-\xi}e^{x}, & -\infty < x < \xi \\ \frac{1}{2}e^{\xi}e^{-x}, & \xi < x < \infty \end{cases}$

(15) $G_m(x,\xi) = \begin{cases} \frac{1}{4}x^2 + \frac{1}{4}\xi^2 - \frac{1}{2}x\xi + \frac{1}{2}x - \frac{1}{2}\xi + \frac{1}{6}, & -1 \le x < \xi \\ \frac{1}{4}x^2 + \frac{1}{4}\xi^2 - \frac{1}{2}x\xi - \frac{1}{2}x + \frac{1}{2}\xi + \frac{1}{6}, & \xi < x \le 1 \end{cases}$