Q6 Determine which of the following linear transformations are one-to-one:

(i) $T: R^2 \longrightarrow R^3$ defined by $T(x_1,x_2) = (x_1+x_2\,,\ x_1-x_2\,,\ x_1+2x_2)$

Sol: Given linear transformation is

$$T(x_1,x_2) = (x_1+x_2\,,\ x_1-x_2\,,\ x_1+2x_2)$$

Let $x = (x_1,x_2)$

& $y = (y_1,y_2) \in R^2$

Then $T(x) = (x_1+x_2\,,\ x_1-x_2\,,\ x_1+2x_2)$

& $T(y) = (y_1+y_2\,,\ y_1-y_2\,,\ y_1+2y_2)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1+x_2, x_1-x_2, x_1+2x_2) = (y_1+y_2, y_1-y_2, y_1+2y_2)$

$$\Rightarrow \quad x_1+x_2 = y_1+y_2 \quad \text{——— (1)}$$

$$x_1-x_2 = y_1-y_2 \quad \text{——— (2)}$$

$$x_1+2x_2 = y_1+2y_2 \quad \text{——— (3)}$$

Adding (1) & (2)

$$2x_1 = 2y_1$$

$$\boxed{x_1 = y_1}$$

Put in (1)

$$x_1+x_2 = x_1+y_2$$

$$\Rightarrow \boxed{x_2 = y_2}$$

Hence $(x_1,x_2) = (y_1,y_2)$

or $x = y$

Hence $T(x) = T(y) \Rightarrow x = y$

Hence T is one-to-one

(ii) $T: R^3 \longrightarrow R^2$ defined by $T(x_1,x_2,x_3) = (x_1-x_2\,,\ x_3)$

Sol: Given linear transformation is

$$T(x_1,x_2,x_3) = (x_1-x_2\,,\ x_3)$$

Let $x = (x_1, x_2, x_3)$

& $y = (y_1, y_2, y_3) \in R^3$

Then $T(x) = (x_1 - x_2, x_3)$

& $T(y) = (y_1 - y_2, y_3)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1 - x_2, x_3) = (y_1 - y_2, y_3)$

$\Rightarrow \quad x_1 - x_2 = y_1 - y_2$ ——— (1)

$x_3 = y_3$ ——— (2)

From (1) we cannot conclude that

$x_1 = y_1$ & $x_2 = y_2$

Hence $T(x) = T(y) \not\Rightarrow x = y$

So T is not one-to-one.

(iii) $T : R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$

Sol. Given linear transformation is

$T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$

Let $x = (x_1, x_2)$

& $y = (y_1, y_2) \in R^2$

Then $T(x) = (x_1, x_1 + x_2, x_1 - x_2)$

& $T(y) = (y_1, y_1 + y_2, y_1 - y_2)$

Suppose $T(x) = T(y)$

$\Rightarrow (x_1, x_1 + x_2, x_1 - x_2) = (y_1, y_1 + y_2, y_1 - y_2)$

or $\quad x_1 = y_1$ ——— (1)

$x_1 + x_2 = y_1 + y_2$ ——— (2)

$x_1 - x_2 = y_1 - y_2$ ——— (3)

(1) $\Rightarrow \boxed{x_1 = y_1}$

Subt. (2) & (3)

$2x_2 = 2y_2 \quad \Rightarrow \boxed{x_2 = y_2}$

Hence $(x_1, x_2) = (y_1, y_2)$

or $x = y$

So $T(x) = T(y) \Rightarrow x = y$

Hence T is one-to-one.

Q7 Let C be the vector space of Complex number over the field of reals & $T : C \longrightarrow C$ be defined by $T(z) = \bar{z}$ where $\bar{z}$ denotes the Complex Conjugate of z. Show that T is linear.

Sol. Given transformation is

$$T(z) = \bar{z}$$

Let $z_1, z_2 \in C$ then we prove

(i) $T(z_1 + z_2) = T(z_1) + T(z_2)$

Now

$$\begin{aligned} T(z_1 + z_2) &= \overline{z_1 + z_2} \\ &= \bar{z}_1 + \bar{z}_2 \\ &= T(z_1) + T(z_2) \end{aligned}$$

(ii) Let $a \in R$ & $z_1 \in C$ then we prove

$T(az_1) = aT(z_1)$

Now

$$\begin{aligned} T(az_1) &= \overline{az_1} \\ &= a\bar{z}_1 \qquad \because a \in R \\ &= aT(z_1) \end{aligned}$$

Hence T is a linear transformation from C to C.

Q8 Let V be the vector space $P_n(x)$ of polynomials $p(x)$ with real coefficients & of degree not exceeding n together with the zero polynomial. Let $T : V \longrightarrow V$

be defined by $T(p(x)) = p(x+1)$

Show that T is linear.

Sol. Given transformation is

$$T(p(x)) = p(x+1)$$

Let $p_1(x) = a_0 + a_1x + a_2x^2 + \cdots\cdots + a_nx^n$

& $p_2(x) = b_0 + b_1x + b_2x^2 + \cdots\cdots + b_nx^n \in V$

then we prove $T(p_1(x) + p_2(x)) = T(p_1(x)) + T(p_2(x))$

Now

$$\begin{aligned}
T(p_1(x)+p_2(x)) &= T\left((a_0+a_1x+\cdots\cdots+a_nx^n)+(b_0+b_1x+\cdots\cdots+b_nx^n)\right)\\
&= T\left((a_0+b_0)+(a_1+b_1)x+\cdots\cdots+(a_n+b_n)x^n\right)\\
&= (a_0+b_0)+(a_1+b_1)(x+1)+\cdots\cdots+(a_n+b_n)(x+1)^n\\
&= \left[a_0+a_1(x+1)+\cdots\cdots+a_n(x+1)^n\right]+\left[b_0+b_1(x+1)+\cdots\cdots+b_n(x+1)^n\right]\\
&= T(a_0+a_1x+\cdots\cdots+a_nx^n)+T(b_0+b_1x+\cdots\cdots+b_nx^n)\\
&= T(p_1(x))+T(p_2(x))
\end{aligned}$$

(ii) Let $a \in R$ & $p_1(x) = a_0 + a_1x + \cdots\cdots + a_nx^n$

then we prove $T(ap_1(x)) = aT(p_1(x))$

Now

$$\begin{aligned}
T(ap_1(x)) &= T\left(a(a_0+a_1x+\cdots\cdots+a_nx^n)\right)\\
&= T(aa_0+aa_1x+\cdots\cdots+aa_nx^n)\\
&= aa_0+aa_1(x+1)+\cdots\cdots+aa_n(x+1)^n\\
&= a\left(a_0+a_1(x+1)+\cdots\cdots+a_n(x+1)^n\right)\\
&= aT\left(a_0+a_1x+\cdots\cdots+a_nx^n\right)\\
&= aT(p_1(x))
\end{aligned}$$

Hence T is a linear transformation from V to V.

Q.9 Let $V_1 = (1,1,1)$, $V_2 = (1,1,0)$ & $V_3 = (1,0,0)$ be a basis for R^3. Find a linear transformation $T: R^3 \longrightarrow R^2$ s.t. $T(V_1) = (1,0)$, $T(V_2) = (2,-1)$ & $T(V_3) = (4,3)$

Sol. Let $x = (x_1, x_2, x_3)$ be any vector of R^3 then for scalars a_1, a_2, a_3

$$x = a_1 V_1 + a_2 V_2 + a_3 V_3$$

$$= a_1(1,1,1) + a_2(1,1,0) + a_3(1,0,0)$$

$$(x_1, x_2, x_3) = (a_1 + a_2 + a_3 ,\ a_1 + a_2 ,\ a_1)$$

$$\Rightarrow \quad a_1 + a_2 + a_3 = x_1 \quad \text{——— (1)}$$

$$a_1 + a_2 = x_2 \quad \text{——— (2)}$$

$$a_1 = x_3 \quad \text{——— (3)}$$

(3) $\Rightarrow$ $\boxed{a_1 = x_3}$

Put in (2)

$$x_3 + a_2 = x_2 \Rightarrow \boxed{a_2 = x_2 - x_3}$$

Put in (1)

$$\cancel{x_3} + x_2 - \cancel{x_3} + a_3 = x_1$$

$$\boxed{a_3 = x_1 - x_2}$$

So

$$x = x_3 V_1 + (x_2 - x_3) V_2 + (x_1 - x_2) V_3$$

Applying T on both sides

$$T(x) = T(x_3 V_1 + (x_2 - x_3) V_2 + (x_1 - x_2) V_3$$

$$= x_3 T(V_1) + (x_2 - x_3) T(V_2) + (x_1 - x_2) T(V_3) \qquad (\text{As } T \text{ is linear})$$

$$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

$$= (x_3, 0) + (2x_2 - 2x_3 , -x_2 + x_3) + (4x_1 - 4x_2 , 3x_1 - 3x_2)$$

$$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2 ,\ -x_2 + x_3 + 3x_1 - 3x_2)$$

$$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3 ,\ 3x_1 - 4x_2 + x_3)$$

Which is req. linear transformation from R^3 to R^2

Q10 Let $T : R^2 \to R$ be the linear transformation for which $T(1,1) = 3$ & $T(0,1) = -2$. Find $T(x_1, x_2)$

Sol. First we prove that the vectors $(1,1)$ & $(0,1)$ form a basis for R^2.

Suppose for scalars $a, b \in R$

$$a(1,1)+b(0,1) = 0$$

$$(a,a)+(0,b) = 0$$

$$(a, a+b) = 0$$

$$\Rightarrow \quad a = 0 \quad \text{——— (1)}$$

$$a+b = 0 \quad \text{——— (2)}$$

(1) $\Rightarrow$ $\boxed{a=0}$

(2) $\Rightarrow$ $\boxed{b=0}$

Hence vectors $(1,1)$ & $(0,1)$ are linearly independent.

As there are two linearly independent vectors in R^2

So $(1,1)$ & $(0,1)$ form a basis for R^2

Suppose $(x_1,x_2) \in R^2$ be an arbitrary vector

then $(x_1,x_2) = a(1,1)+b(0,1)$ where $a,b \in R$

or $(x_1,x_2) = (a, a+b)$

$$\Rightarrow \quad a = x_1 \quad \text{——— (1)}$$

$$a+b = x_2 \quad \text{——— (2)}$$

(1) $\Rightarrow$ $\boxed{a = x_1}$

Put in (2)

$x_1 + b = x_2 \Rightarrow \boxed{b = x_2 - x_1}$

So

$$(x_1,x_2) = x_1(1,1)+(x_2-x_1)(0,1)$$

Applying T on both sides

$$T(x_1,x_2) = T\big(x_1(1,1)+(x_2-x_1)(0,1)\big)$$

$$= x_1 T(1,1) + (x_2-x_1)T(0,1) \qquad (\because T \text{ is linear})$$

$$= x_1(3) + (x_2-x_1)(-2)$$

$$= 3x_1 - 2x_2 + 2x_1$$

$T(x_1,x_2) = 5x_1 - 2x_2$ which is T in terms of co-ords.

Q11 Let $D : P_2(x) \longrightarrow P_2(x)$ be the differentiation operator & $D(p(x)) = p'(x)$ for all $p(x) \in P_2(x)$. Find $N(D)$.

Sol. Given operator is.

$D(p(x)) = p'(x)$

Here $N(D)$ will consist of those polynomials in $P_2(x)$ for which $D(p(x)) = 0$

Since we know that

$$D(p(x)) = 0 \quad \text{if} \quad p(x) = \text{Const polynomial}$$

So $N(D)$ will consist of all const. polynomials.

Q12 Define $T : R^3 \longrightarrow R^3$ by $T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$. Find $N(T)$. Is T one-to-one?

Sol. Given transformation is

$T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$

Here $N(T) = \{(x_1, x_2, x_3) \in R^3 : T(x_1, x_2, x_3) = (0, 0, 0)\}$

Now $T(x_1, x_2, x_3) = (0, 0, 0)$

$\Rightarrow (-x_3, x_1, x_1 + x_3) = (0, 0, 0)$

$$\Rightarrow \quad -x_3 = 0 \quad \text{——— (1)}$$
$$x_1 = 0 \quad \text{——— (2)}$$
$$x_1 + x_3 = 0 \quad \text{——— (3)}$$

(1) $\Rightarrow x_3 = 0$

(2) $\Rightarrow x_1 = 0$

Which shows that $N(T)$ will consist of all vectors of the form $(0, x_2, 0)$. Which is x_2-axis.

i.e., $N(T) = \{(0, x_2, 0) \in R^3 : x_2 \in R\}$

Since $N(T) = (0, x_2, 0) \neq (0, 0, 0)$. So T is not one-to-one.

Q13 Suppose U, V & W are vector spaces over the same field F. Let $T: U \to V$ & $S: V \to W$ be linear transformations. The transformation $S \circ T : U \to W$ is defined by $(S \circ T)(u) = S(T(u))$ for all $u \in U$. Show that $S \circ T$ is a linear transformation.

Sol:

Here $S \circ T : U \to W$ be defined as

$$(S \circ T)(u) = S(T(u)) \quad \text{for all } u \in U$$

Let $u_1, u_2 \in U$ then we prove

$$(S \circ T)(u_1 + u_2) = (S \circ T)(u_1) + (S \circ T)(u_2)$$

Now

$$\begin{aligned}
(S \circ T)(u_1 + u_2) &= S(T(u_1 + u_2)) && \text{By def. of } S \circ T \\
&= S(T(u_1) + T(u_2)) && \because T \text{ is linear} \\
&= S(T(u_1)) + S(T(u_2)) && \because S \text{ is linear} \\
&= (S \circ T)(u_1) + (S \circ T)(u_2)
\end{aligned}$$

(ii) Let $a \in F$ & $u \in U$ then we prove

$$(S \circ T)(au) = a(S \circ T)(u)$$

Now

$$\begin{aligned}
(S \circ T)(au) &= S(T(au)) && \text{By def of } S \circ T \\
&= S(aT(u)) && \because T \text{ is linear} \\
&= aS(T(u)) && \because S \text{ is linear} \\
&= a(S \circ T)(u)
\end{aligned}$$

Hence $S \circ T$ is a linear transformation from U to W.

Q14 Let U & V be two vector spaces over the same field F. Denote the set of all linear transformations from U into V by $L(U,V)$. Show that $L(U,V)$ is a vector space over F with vector space operations as defined in example 31

Sol. Consider the set $L(U,V)$. Let $S,T \in L(U,V)$ then $S: U \to V$ & $T: U \to V$ be two linear transformations. Define

$S+T: U \to V$ & $aS: U \to V$ by

$$(S+T)(u) = S(u) + T(u)$$

$$(aS)(u) = aS(u) \qquad \text{for all } u \in U \text{ \& } a \in F$$

First we show that $L(U,V)$ is an abelian gr. under $+$.

(i) Closure law

Let $S,T \in L(U,V)$, then we show $S+T \in L(U,V)$.

Now $(S+T)(u_1+u_2) = S(u_1+u_2) + T(u_1+u_2)$ By def. of $S+T$

$= S(u_1) + S(u_2) + T(u_1) + T(u_2)$ $\because S,T$ are linear

$= S(u_1) + T(u_1) + S(u_2) + T(u_2)$

$= (S+T)(u_1) + (S+T)(u_2)$

Let $K \in F$ & $u \in U$

$(S+T)(Ku) = S(Ku) + T(Ku)$

$= KS(u) + KT(u)$ $\because S,T$ are linear

$= K\big(S(u) + T(u)\big)$

$= K(S+T)(u)$

Hence $S+T$ is linear & so $S+T \in L(U,V)$.

(ii) Associative law

Let $R,S,T \in L(U,V)$ then we prove

$$R+(S+T) = (R+S)+T$$

Now Consider for $u \in U$

$[R+(S+T)](u) = R(u) + (S+T)(u)$ (By def. of sum)

$= R(u) + [S(u) + T(u)]$ " " "

$= [R(u) + S(u)] + T(u)$ ∵ $R(u), S(u), T(u) \in F$

$= [(R+S)(u)] + T(u)$

$= [(R+S)(u) + T(u)]$

$= [(R+S)+T](u)$

$\Rightarrow R+(S+T) = (R+S)+T$

So $+$ is associative in $L(U,V)$.

(iii) <u>Identity law</u>

Clearly the zero transformation $\underline{O}$ defined by

$\underline{O}(u) = 0$ for all $u \in U$

is a linear transformation from U to V & it is the additive identity in $L(U,V)$

(iv) <u>Inverse law</u>

For each $T \in L(U,V)$, we define

$-T \in L(U,V)$ by

$(-T)(u) = -T(u)$

then $-T$ is the additive inverse of T.

(v) <u>Commutative law</u>

Let $S, T \in L(U,V)$ then we show $S+T = T+S$

Now consider

$(S+T)(u) = S(u) + T(u)$ By def. of sum

$= T(u) + S(u)$ ∵ $S(u), T(u) \in F$

$= (T+S)(u)$

$\Rightarrow S+T = T+S$

Hence $+$ is commutative in $L(U,V)$

So $L(U,V)$ is an abelian gr. under $+$.

Now we check scalar multiplication axioms.

(i) Let $a \in F$ & $S \in L(U,V)$ then we prove $aS \in L(U,V)$.

Now $(aS)(u_1+u_2) = a[S(u_1+u_2)]$

$= a[S(u_1)+S(u_2)]$ ∵ S is linear

$= aS(u_1) + aS(u_2)$

Suppose $K \in F$ & $u \in U$ then

$(aS)(Ku) = a[S(Ku)]$

$= a[KS(u)]$ ∵ S is linear

$= (aK)S(u)$

$= (Ka)S(u)$

$= K(aS)(u)$

Hence aS is linear & so $aS \in L(U,V)$.

(ii) Let $a,b \in F$ & $S \in L(U,V)$ then we prove $a(bS) = (ab)S$

Now $[a(bS)](u) = a.(bS)(u)$

$= a[b.S(u)]$

$= (ab).S(u)$

$= [(ab)S](u)$

$\Rightarrow \quad a(bS) = (ab)S$

(iii) Let $a,b \in F$ & $S \in L(U,V)$ then we prove $(a+b)S = aS+bS$

Now $[(a+b)S](u) = (a+b).S(u)$

$= a.S(u) + b.S(u)$

$= (aS)(u) + (bS)(u)$

$= [aS+bS](u)$

(iv) Let $a \in F$ & $S,T \in L(U,V)$ then we prove $a(S+T) = aS+aT$

Now $[a(S+T)](u) = a[(S+T)(u)]$

$= a[S(u) + T(u)]$

$= a.S(u) + a.T(u)$

$= (aS)(u) + (aT)(u)$

So $[\alpha(S+T)](u) = [\alpha S + \alpha T](u)$

$\Rightarrow \quad \alpha(S+T) = \alpha S + \alpha T$

(v) Let $1 \in F$ & $S \in L(U,V)$ then we prove $1.S = S$

Now. $(1.S)(u) = 1.S(u)$

$= S(u)$

$\Rightarrow \quad 1.S = S$

Since all the Conditions are satisfied. So $L(U,V)$ is a vector space over F.

Q15 Find a basis & dimension of each of $R(T)$ & $N(T)$, where

(i) $T: R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3)$$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3)$$

Since R^3 is generated by $(1,0,0), (0,1,0)$ & $(0,0,1)$. So $R(T)$ will be generated by $T(1,0,0), T(0,1,0)$ & $T(0,0,1)$

Here $T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

& $T(0,0,1) = (-1,1,-2)$

Hence $R(T)$ is generated by $(1,0,1), (2,1,2)$ & $(-1,1,-2)$

Since $(2,1,1) = 3(1,0,1) + 1(-1,1,-2)$

So casting out the vector $(2,1,1)$, the set $\{(1,0,1), (-1,1,-2)\}$ also spans $R(T)$. Since none of the two vectors is a multiple of other, so the set $\{(1,0,1), (-1,1,-2)\}$ is linearly independent & so forms a basis for $R(T)$.

Hence $\dim R(T) = 2$

Now we find $\dim N(T)$.

A vector $(x_1, x_2, x_3) \in N(T)$ if $T(x_1, x_2, x_3) = 0$

i.e., if $(x_1 + 2x_2 - x_3,\ x_2 + x_3,\ x_1 + x_2 - 2x_3) = (0,0,0)$

$$\Rightarrow \quad x_1 + 2x_2 - x_3 = 0 \quad \text{———} (1)$$

$$x_2 + x_3 = 0 \quad \text{———} (2)$$

$$x_1 + x_2 - 2x_3 = 0 \quad \text{———} (3)$$

Adding (1) & (2)

$$x_1 + 3x_2 = 0$$

or $\boxed{x_1 = -3x_2}$

Put in (3)

$$-3x_2 + x_2 - 2x_3 = 0$$

$$-2x_2 - 2x_3 = 0$$

$$x_2 + x_3 = 0$$

or $\boxed{x_3 = -x_2}$

If $x_2 = 1$

then $x_1 = -3,\ x_2 = 1,\ x_3 = -1$

So the vector $(-3, 1, -1)$ spans $N(T)$. Also $(-3,1,-1)$ is linearly independent. So $\{(-3,1,-1)\}$ forms a basis for $N(T)$.

Hence $\dim N(T) = 1$

(ii) $T : R^3 \longrightarrow R^4$ is defined by

$T(x_1, x_2, x_3) = (2x_1 + x_3,\ 4x_1 + x_2,\ x_1 + x_3,\ x_3 - 4x_2)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (2x_1 + x_3,\ 4x_1 + x_2,\ x_1 + x_3,\ x_3 - 4x_2)$

Since R^3 is generated by $(1,0,0), (0,1,0)$ & $(0,0,1)$. So $R(T)$ will be generated by $T(1,0,0), T(0,1,0)$ & $T(0,0,1)$

Here $T(1,0,0) = (2,4,1,0)$

$T(0,1,0) = (0,1,0,-4)$

$T(0,0,1) = (1,0,1,1)$

Hence $R(T)$ will be generated by $(2,4,1,0), (0,1,0,-4), (1,0,1,1)$

Now we check whether these vectors are linearly independent. For this let

$a(2,4,1,0) + b(0,1,0,-4) + c(1,0,1,1) = (0,0,0,0)$ where $a, b, c \in F$

or $(2a + c,\ 4a + b,\ a + c,\ -4b + c) = (0,0,0,0)$

$$\Rightarrow \quad 2a + c = 0 \quad \text{——— (1)}$$

$$4a + b = 0 \quad \text{——— (2)}$$

$$a + c = 0 \quad \text{——— (3)}$$

$$-4b + c = 0 \quad \text{——— (4)}$$

(1) − (3) $\Rightarrow \boxed{a = 0}$

(3) $\Rightarrow 0 + c = 0 \Rightarrow \boxed{c = 0}$

(2) $\Rightarrow 0 + b = 0 \Rightarrow \boxed{b = 0}$

Hence vectors $(2,4,1,0), (0,1,0,-4)$ & $(1,0,1,1)$ are linearly independent. Hence $\{(2,4,1,0), (0,1,0,-4), (1,0,1,1)\}$ form a basis for $R(T)$

Hence $\dim R(T) = 3$

Q16 Show that linear transformations preserve linear dependence.

Sol: Let $T : U \to V$ be a linear transformation, where U & V are vector spaces over the same field F.

Suppose a set $\{u_1, u_2, \ldots, u_n\}$ in U is linearly dependent. We want to show that $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ is a linearly dependent set in V.

Since $\{u_1, u_2, \ldots, u_n\}$ is linearly dependent, so there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$

Applying T on both sides

$$T(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = T(0)$$

or $a_1T(u_1) + a_2T(u_2) + \cdots + a_nT(u_n) = 0 \quad (\because T \text{ is linear})$

Since $a_1, a_2, \ldots, a_n$ are not all zero, so the above eq. shows that $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ are linearly dependent in V. Hence T preserve linear dependence.

Q17 Find the rank of each matrix in problem 8 of exercise 3.2 by the method of 6.42

(i) $\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$

Sol:

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -2-0 \\ -1-15 \\ 3+6 \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -2 \\ -16 \\ 9 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} -2 & 0 \\ -16 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & 0 \\ 9 & 0 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{rank } A = 2$$

(ii) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

Sol:

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$ then

$$\text{rank } A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} -3 & 6 \\ 3 & -3 \\ 6 & -5 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} -3 & 6 \\ 3 & -3 \end{vmatrix} \\ \begin{vmatrix} -3 & 6 \\ 6 & -5 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 9-18 \\ 15-36 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} -9 \\ -21 \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} \begin{vmatrix} -9 & 0 \\ -21 & 0 \end{vmatrix} \end{bmatrix}$$

$$= 3 + \text{rank}\begin{bmatrix} 0 \end{bmatrix}$$

$$= 3$$

(iii) $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Sol.

Let $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$ then

$$\text{rank}\, A = 1 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 1 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & -8 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{rank}\begin{bmatrix} 1 & 2 & 1 & -1 \\ -3 & -6 & -3 & 3 \\ -1 & -2 & -1 & 1 \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ -3 & -6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 2 + 0$$

$$= 2$$

(iv)
$$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$$

Sol.

Let $A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$ then

$$\operatorname{rank} A = 1 + \operatorname{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 2 & 13 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \operatorname{rank}\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 4 & -1 & -1 \\ 1 & 1 & -4 & 5 \end{bmatrix}$$

$$= 2 + \operatorname{rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \operatorname{rank}\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$$

$$= 3 + \operatorname{rank}\begin{bmatrix} \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \end{bmatrix}$$

$$= 3 + \operatorname{rank}\begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= 3 + 0$$

$$\operatorname{rank} A = 3$$