Linear Algebra

(Week 16-20)

Lecture 1

Exercise No. 5.2

Q1 Solve for x, each of the following equations:

(i) $$\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$$

Sol. Given

$$\begin{vmatrix} 1 & 2+x & 3 \\ 2 & 1 & 3+x \\ 3 & 2+x & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2+x & 3 \\ 0 & -3-2x & -3+x \\ 0 & -4-2x & -8 \end{vmatrix} = 0 \qquad \text{By } R_2 - 2R_1,\ R_3 - 3R_1$$

Expanding from C_1

$$\begin{vmatrix} -3-2x & -3+x \\ -4-2x & -8 \end{vmatrix} = 0$$

$$-8(-3-2x) - (-3+x)(-4-2x) = 0$$

$$24 + 16x - (12 + 6x - 4x - 2x^2) = 0$$

$$24 + 16x - 12 - 2x + 2x^2 = 0$$

$$2x^2 + 14x + 12 = 0$$

$$x^2 + 7x + 6 = 0$$

$$x^2 + 6x + x + 6 = 0$$

$$x(x+6) + 1(x+6) = 0$$

$$(x+6)(x+1) = 0$$

$$\Rightarrow \boxed{x = -6, -1}$$

(iii)
$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$$

Sol.

Given
$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 9 & 9-x^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1-x^2 & 0 & 0 \\ 2 & 1 & -3 & -1 \\ 2 & 1 & 5 & 3-x^2 \end{vmatrix} = 0 \qquad \text{By } \begin{matrix} C_2 - C_1 \\ C_3 - 2C_1 \\ C_4 - 3C_1 \end{matrix}$$

Expanding from R_1

$$\begin{vmatrix} 1-x^2 & 0 & 0 \\ 1 & -3 & -1 \\ 1 & 5 & 3-x^2 \end{vmatrix} = 0$$

Expanding from R_1

$$(1-x^2)\begin{vmatrix} -3 & -1 \\ 5 & 3-x^2 \end{vmatrix} = 0$$

$$(1-x^2)(-9+3x^2+5) = 0$$
$$(1-x^2)(3x^2-4) = 0$$
$$1-x^2 = 0 \quad , \quad 3x^2-4 = 0$$
$$x^2 = 1 \quad , \quad x^2 = 4/3$$
$$x = \pm 1, \pm \frac{2}{\sqrt{3}}$$

65

(iii)
$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$$

Soln. Given

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 2-x & 2^2-x^2 & 2^3-x^3 \\ 0 & 3-x & 3^2-x^2 & 3^3-x^3 \\ 0 & 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} = 0$$

R_2-R_1
R_3-R_1
R_4-R_1

Expanding from C_1

$$\begin{vmatrix} 2-x & 2^2-x^2 & 2^3-x^3 \\ 3-x & 3^2-x^2 & 3^3-x^3 \\ 4-x & 4^2-x^2 & 4^3-x^3 \end{vmatrix} = 0$$

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 1 & 3+x & 9+3x+x^2 \\ 1 & 4+x & 16+4x+x^2 \end{vmatrix} = 0$$

taking $(2-x),(3-x),(4-x)$ common from R_1, R_2, R_3 resp.

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 2+x & 4+2x+x^2 \\ 0 & 1 & 5+x \\ 0 & 2 & 12+2x \end{vmatrix} = 0$$

R_2-R_1
R_3-R_1

Expanding from C_1

$$(2-x)(3-x)(4-x)\begin{vmatrix} 1 & 5+x \\ 2 & 12+2x \end{vmatrix} = 0$$

$$(2-x)(3-x)(4-x)(12+\cancel{2x}-10-\cancel{2x}) = 0$$

$$(2-x)(3-x)(4-x)(2) = 0$$

$$(2-x)(3-x)(4-x) = 0$$

$$\Rightarrow \boxed{x = 2, 3, 4}$$

(iv)

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x+1 & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$$

Sol.-

Given

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x+1 & 3 & 4 & 5 \\ 1 & 2 & x+1 & 4 & 5 \\ 1 & 2 & 3 & x+1 & 5 \\ 1 & 2 & 3 & 4 & x+1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & x-1 & 0 & 0 & 0 \\ 0 & 0 & x-2 & 0 & 0 \\ 0 & 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & 0 & x-4 \end{vmatrix} = 0 \qquad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \\ R_5-R_1 \end{matrix}$$

Expanding from C_1

$$\begin{vmatrix} x-1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-4 \end{vmatrix} = 0$$

$(x-1)(x-2)(x-3)(x-4) = 0$ ($\because$ det. of a diagonal matrix is equal to the product of diagonal elements.)

$\Rightarrow \boxed{x = 1, 2, 3, 4}$

(v)

$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$$

__Sol.__

Given

$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & a & x \end{vmatrix} = 0$$

$$\begin{vmatrix} x+4a & a & a & a & a \\ x+4a & x & a & a & a \\ x+4a & a & x & a & a \\ x+4a & a & a & x & a \\ x+4a & a & a & a & x \end{vmatrix} = 0 \qquad C_1 + (C_2+C_3+C_4+C_5)$$

$$(x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 1 & x & a & a & a \\ 1 & a & x & a & a \\ 1 & a & a & x & a \\ 1 & a & a & a & x \end{vmatrix} = 0.$$ taking $x+4a$ Common from C_1

$$(x+4a)\begin{vmatrix} 1 & a & a & a & a \\ 0 & x-a & 0 & 0 & 0 \\ 0 & 0 & x-a & 0 & 0 \\ 0 & 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & 0 & x-a \end{vmatrix} = 0$$

R_2-R_1
R_3-R_1
R_4-R_1
R_5-R_1

Expanding from C_1

$$(x+4a)\begin{vmatrix} x-a & 0 & 0 & 0 \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} = 0$$

$$(x+4a)(x-a)(x-a)(x-a)(x-a) = 0$$

$$\Rightarrow \boxed{x = -4a, a, a, a, a}$$

(vi) $$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$$

Sol. Given

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0$$

$$\begin{vmatrix} 4+x & 1 & 1 & 1 \\ 4+x & 1+x & 1 & 1 \\ 4+x & 1 & 1+x & 1 \\ 4+x & 1 & 1 & 1+x \end{vmatrix} = 0 \qquad C_1+(C_2+C_3+C_4)$$

$$(4+x)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+x & 1 \\ 1 & 1 & 1 & 1+x \end{vmatrix} = 0 \qquad \text{taking } 4+x \text{ common from } C_1$$

$$(4+x)\cdot\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} = 0 \qquad \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \end{matrix}$$

Expanding from C_1

$$(4+x)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = 0$$

$$\Rightarrow \quad (4+x)\cdot x^3 = 0$$

$$\Rightarrow$$

$$\boxed{x = -4, 0, 0, 0}$$

Q2 Evaluate each of the following determinants:

(i)

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$$

Sol.

Let $\Delta = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix}$

$= \begin{vmatrix} a+(n-1)d & b & b & \cdots & b \\ a+(n-1)d & a & b & \cdots & b \\ a+(n-1)d & b & a & \cdots & b \\ \vdots & & & & \\ a+(n-1)d & b & b & \cdots & b \end{vmatrix}$ By $C_1 + (C_2 + C_3 + \cdots + C_n)$

$= [a+(n-1)d] \begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & b & \cdots & b \\ 1 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & b & b & \cdots & b \end{vmatrix}$ taking $a+(n-1)d$ common from C_1

$= [a+(n-1)d] \begin{vmatrix} 1 & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a-b \end{vmatrix}$ $R_2 - R_1$, $R_3 - R_1$, $\vdots$, $R_n - R_1$

Expanding from C_1

$$\Delta = [a+(n-1)d]\begin{vmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & a-b \end{vmatrix}$$

$$(a+(n-1)d)(a-b)^{n-1}$$

(ii)

$$\begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1-n & 1 & \cdots & 0 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix} \qquad R_1+(R_2+R_3+\cdots+R_n)$$

$$\Delta = 0 \qquad \because R_1 = 0$$

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(iii) $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$

$= \begin{vmatrix} \Sigma n & \Sigma n & \Sigma n & \cdots & \Sigma n & \Sigma n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & & & & & \\ n-2 & n-1 & n & \cdots & n-4 & n-3 \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix}$ Adding R_2 to R_n in R_1 & denoting $1+2+\cdots+n$ by Σn

$= \Sigma n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-2 & n-1 & n & & n-4 & n-3 \\ n-1 & n & 1 & & n-3 & n-2 \\ n & 1 & 2 & & n-2 & n-1 \end{vmatrix}$ taking Σn Common from C_1