Linear Algebra

(Week 14-18(December)

Lecture 2

Linear Combination Of Vectors:-

Let, V be a vector space over field F and $x_1, x_2, \ldots, x_n \in V$, then any element $x = a_1x_1 + a_2x_2 + \ldots$ $\in V$ is said to be a linear combination of the vect $x_1, x_2, \ldots, x_n$ over field F where $a_1, a_2, \ldots, a_n \in F$ (scala

Example:-

Determine whether the first vector can be expressed as a linear combination of other two vecto

i) $(-2,0,3), (1,3,0), (2,4,-1)$

ii) $(3,4,1), (1,-2,1), (-2,-1,1)$

Solution:-

i) Let, $u=(-2,0,3)$, $v=(1,3,0)$, $w=(2,4,-1)$

To show that u is a linear combination of v and w. By definition of linear combination:-

$$u = av + bw$$

$$(-2,0,3) = a(1,3,0) + b(2,4,-1)$$

$$= (a, 3a, 0) + (2b, 4b, -b)$$

$$= (a+2b, 3a+4b, -b)$$

Comparing:-

$$a+2b = -2 \quad \rightarrow ①$$

$$3a+4b = 0 \quad \rightarrow ②$$

$$-b = 3 \quad \rightarrow ③$$

From eq. ③ :-

$$b = -3$$

Using the value of $b=-3$ into eq. ① :-

$$a + 2(-3) = -2$$

$$a - 6 = -2$$

$$a = -2 + 6$$

$$a = 4$$

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① ⇒
$$a+2b=-2$$
$$(4)+2(-3)=-2$$
$$4-6=-2$$
$$-2=-2$$
So, eq. ① is satisfied.

② ⇒
$$3a+4b=0$$
$$3(4)+4(-3)=0$$
$$12-12=0$$
$$0=0$$
So, eq. ② is satisfied.

③ ⇒
$$-b=3$$
$$-(-3)=3$$
$$3=3$$
So, eq. ③ is satisfied.

Since the values of both a and b satisfy the eq. ①–③, so the vector u can be written as a linear combination of vectors v and w.

> Let, $u=(3,4,1)$, $v=(1,-2,1)$, $w=(-2,-1,1)$

To show that u is a linear combination of v and w. By definition of linear combination:-
$$u=av+bw$$
$$(3,4,1)=a(1,-2,1)+b(-2,-1,1)$$
$$=(a,-2a,a)+(-2b,-b,b)$$
$$=(a-2b,-2a-b,a+b)$$

Comparing:-
$$a-2b=3 \rightarrow ①$$
$$-2a-b=4 \rightarrow ②$$
$$a+b=1 \rightarrow ③$$

Adding eq. ② and eq. ③:-

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$$-2a-\cancel{b}=4$$
$$a+\cancel{b}=1$$
$$-a\quad=5$$
$$\Rightarrow\quad a=-5$$

Using the value of $a=-5$ into eq. ①:-

$$a-2b=3$$
$$(-5)-2b=3$$
$$-2b=3+5$$
$$-2b=8$$
$$b=\frac{\cancel{8}^{4}}{-\cancel{2}}$$
$$b=-4$$

① $\Rightarrow$
$$a-2b=3$$
$$(-5)-2(-4)=3$$
$$-5+8=3$$
$$3=3$$

So, eq. ① is satisfied.

② $\Rightarrow$
$$-2a-b=4$$
$$-2(-5)-(-4)=4$$
$$10+4=4$$
$$14\neq4$$

So, eq. ② is not satisfied.

Since eq. ② is not satisfied for the values of and b, so the vector u can not be written as a linear combination of vectors v and w.

EXAMPLE:-

Determine whether the first polynomial can be expressed as a linear combination of other two polynomials

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$2x^3-2x^2+12x-6$; x^3-2x^2-5x-3 ;
$3x^3-5x^2-4x-9$

Solution:-

Let, $P(x) = 2x^3-2x^2+12x-6$
$Q(x) = x^3-2x^2-5x-3$
$R(x) = 3x^3-5x^2-4x-9$

To show that $P(x)$ is a linear combination of $Q(x)$ and $R(x)$. By definition of linear combination:-

$$P(x) = aQ(x) + bR(x)$$

$$2x^3-2x^2+12x-6 = a[x^3-2x^2-5x-3]+b[3x^3-5x^2-4x-9]$$
$$= [ax^3-2ax^2-5ax-3a]+[3bx^3-5bx^2-4bx-9b]$$
$$= (ax^3+3bx^3)+(-2ax^2-5bx^2)$$
$$+(-5ax-4bx)+(-3a-9b)$$
$$= (a+3b)x^3+(-2a-5b)x^2$$
$$+(-5a-4b)x+(-3a-9b)$$

Comparing the coefficients:-

$x^3 \Rightarrow \quad 2 = a+3b \quad \rightarrow (1)$
$x^2 \Rightarrow \quad -2 = -2a-5b \quad \rightarrow (2)$
$x \Rightarrow \quad 12 = -5a-4b \quad \rightarrow (3)$
$x^0 \Rightarrow \quad -6 = -3a-9b \quad \rightarrow (4)$

Multiplying eq.(1) with 2:-
$4 = 2a+6b \quad \rightarrow (5)$

Adding eq.(2) and eq.(5):-

$$-2 = -\cancel{2a}-5b$$
$$4 = \cancel{2a}+6b$$
$$2 = b$$
$$\Rightarrow b = 2$$

Using the value of $b=2$ into eq.(1):-

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$$2 = a + 3b$$
$$2 = a + 3(2)$$
$$2 = a + 6$$
$$2 - 6 = a$$
$$-4 = a$$
$$\Rightarrow \quad a = -4$$

① $\Rightarrow$
$$2 = a + 3b$$
$$2 = (-4) + 3(2)$$
$$2 = -4 + 6$$
$$2 = 2$$

So, eq. ① is satisfied.

② $\Rightarrow$
$$-2 = -2a - 5b$$
$$-2 = -2(-4) - 5(2)$$
$$-2 = 8 - 10$$
$$-2 = -2$$

So, eq. ② is satisfied.

③ $\Rightarrow$
$$12 = -5a - 4b$$
$$12 = -5(-4) - 4(2)$$
$$12 = 20 - 8$$
$$12 = 12$$

So, eq. ③ is satisfied.

④ $\Rightarrow$
$$-6 = -3a - 9b$$
$$-6 = -3(-4) - 9(2)$$
$$-6 = 12 - 18$$
$$-6 = -6$$

So, eq. ④ is satisfied.

Since the values of both a and b satisfy the eq. ① - ④, so the polynomial $P(x)$ can b expressed as a linear combination of $Q(x)$ and R