Sol. $f(x, y) = x^y - y^x$

$$f_x = yx^{y-1} - y^x \ln y$$
$$= yx^{y-1} - x^y \ln y = x^{y-1}(y - x \ln y)$$

and $f_y = x^y \ln x - xy^{x-1} = x^y \ln x - \dfrac{x}{y} \cdot y^x = x^y \ln x - \dfrac{x}{y} x^y$

$$= x^y \frac{(y \ln x - x)}{y}$$

Hence $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{x^{y-1}(y - x \ln y)}{x^y (y \ln x - x)} \cdot y$

$$= \frac{y(y - x \ln y)}{x(x - y \ln x)}.$$

6. $(\tan x)^y + y^{\cot x} = a$

Sol. We have $f(x, y) = (\tan x)^y + y^{\cot x} - a = 0$

$$f_x = y(\tan x)^{y-1} \cdot \sec^2 x - y^{\cot x} \cdot \ln y\,(\csc^2 x)$$

and $f_y = (\tan x)^y \cdot \ln \tan x + (\cot x) y^{\cot x - 1}$

Hence $\dfrac{dy}{dx} = \dfrac{-f_x}{f_y} = -\dfrac{y \sec^2 x\,(\tan x)^{y-1} - \csc^2 x \cdot y^{\cot x} \ln y}{(\tan x)^y \ln \tan x + (\cot x) y^{\cot x - 1}}$

7. If $F(x, y, z) = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

Sol. We know that if $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \qquad (1)$$

Now in $F(x, y, z) = 0$ we may regard z as a function of x and y. In order to find $\dfrac{\partial z}{\partial x}$, we treat y as constant and use (1)

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad (2)$$

Similarly, we get $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Here $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$ are partial derivatives because z is a function of two variables x and y.

8. If $f(x, y, z) = 0$ and $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Sol. From $f(x, y) = 0$, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

From $\phi(y, z) = 0$, we get (1)

$$\frac{dz}{dy} = \frac{-\phi_y}{\phi_z}$$

(2)

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{\frac{\partial \phi}{\partial y} \cdot \frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y}}$$

Cross multiplying, we obtain

$$\frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y} \cdot \frac{dz}{dx} = \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

9. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that $\frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}}$.

Sol. We have $f(x, y) = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a$

$$f_x = \sqrt{1-y^2} - \frac{1 \,.\, y \,.\, 2x}{2\sqrt{1-x^2}} = \frac{\sqrt{1-y^2}\sqrt{1-x^2} - xy}{\sqrt{1-x^2}}$$

$$f_y = \frac{-x}{2\sqrt{1-y^2}} \,.\, 2y + \sqrt{1-x^2} = \frac{-xy + \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{1-x^2}\left(\frac{-1}{2\sqrt{1-y^2}}2y\right)\frac{dy}{dx} - \sqrt{1-y^2} \times \left(-\frac{1}{2}\frac{2x}{\sqrt{1-x^2}}\right)}{1-x^2}$$

$$= \frac{y\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\frac{dy}{dx} - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \,.\, x}{(1-x^2)} = \frac{-y\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\left(\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right) - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \,.\, x}{(1-x^2)}$$

$$= -\frac{(y\sqrt{1-x^2} + x\sqrt{1-y^2})}{(1-x^2)^{3/2}} = \frac{-a}{(1-x^2)^{3/2}}$$

10. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$$

Sol. Here $f_x = 2ax + 2hy + 2g$

and $f_y = 2hx + 2by + 2f$

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f} \qquad (1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{(hx + by + f)\left(a + h\frac{dy}{dx}\right) - (ax + hy + g)\left(h + b\frac{dy}{dx}\right)}{(hx + by + f)^2}$$

$$= -\frac{dy}{dx}(h^2x + hby + hf - abx - hby - gb)/(hx + by + f)^2$$

$$\frac{-(ahx + aby + af - ahx - h^2y - hg)}{(hx + by + f)^2}$$

$$= \frac{\frac{ax + hy + g}{hx + by + f}(h^2x + hf - abx - gb) - (aby + af - h^2y - hg)}{(hx + by + f)^2}$$

$$= \frac{(ax + hy + g)(h^2x + hf - abx - gb) - (hx + by + f)(aby + af - h^2y - hg)}{(hx + by + f)^3}$$

$$= \frac{h^2(ax^2 + 2hxy + by^2 + 2gx + 2fy) - ab(ax^2 + 2hxy + by^2 + 2gx + 2fy) - af^2 - bg^2 + 2fgh}{(hx + by + f)^3}$$

$$= \frac{h^2(-c) - ab(-c) - af^2 - bg^2 + 2fgh}{(hx + by + f)^3} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

11. Find $\frac{d^2y}{dx^2}$ if $x^3 + y^3 = 3axy$

Sol. Here $f(x, y) = x^3 + y^3 - 3axy = 0$

$f_x = 3x^2 - 3ay$ and $f_y = 3y^2 - 3ax$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{ay - x^2}{y^2 - ax} \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

$$= \frac{\frac{dy}{dx}(ay^2 - a^2x - 2ay^2 + 2x^2y) - (2xy^2 - 2ax^2 - a^2y + ax^2)}{(y^2 - ax)^2}$$

$$= \frac{\frac{ay - x^2}{y^2 - ax}(2x^2y - ay^2 - a^2x) - (2xy^2 - ax^2 - a^2y)}{(y^2 - ax)^2}$$

$$= \frac{2ax^2y^2 - a^2y^3 - a^3xy - 2x^4y + ax^2y^2 + a^2x^3}{(y^2 - ax)^3}$$

$$+ \frac{-(2xy^4 - ax^2y^2 - a^2y^3 - 2ax^2y^2 + a^2x^3 + a^3xy)}{(y^2 - ax)^3}$$

$$= \frac{-(2xy^4 - ax^2y^2 - a^2y^3 - 2ax^2y^2 + a^2x^3 + a^3xy)}{(y^2 - ax)^3}$$

$$+ \frac{3a^2x^2 - a^2y^3 - a^3xy - 2x^4y + a^2x^3}{(y^2 - ax)^3}$$

$$= \frac{6ax^2y^2 - 2a^3xy - 2xy\,(x^3 + y^3)}{(y^2 - ax)^3}$$

$$= \frac{6ax^2y^2 - 2a^3xy - 2xy\,(3axy)}{(y^2 - ax)^3}$$

$$= \frac{-2a^3xy}{(y^2 - ax)^3} = \frac{2a^3xy}{(ax - y^2)^3}.$$

Exercise Set 9.4 (Page 420)