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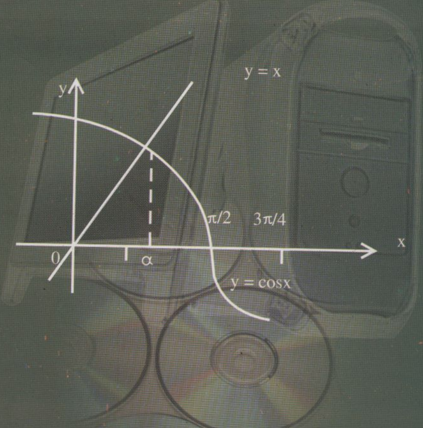
A First Course in NUMERICAL ANALYSIS With C++



Fifth Edition

Numerical Analysis is a Science

– computation is an art.

Fifth Enlarged Edition

A First Course in
NUMERICAL ANALYSIS
With C++

Prof. Saeed Akhter Bhatti
Mr. Naeem Akhtar Bhatti

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In the loving memory of our parents,

Mr and Begum Sana Ullah Sufi

[illegible] can be avoided [illegible] We shall primarily [illegible]

Rounding Errors

When a numerical method is actually run on a digital computer after transcription to computer program, some kind of error called round-off error is introduced.

[illegible] by rounding off numbers to a limited number of decimal [illegible] In simple words, the error in the result that is caused by rounding is called [illegible] For example, it would be impracticable to [illegible] the distance between two points on the earth as 15.2967 metres; it would be more [illegible] 15 metres. Thus, the error introduced by rounding is 0.2967 metres. Another example is the value of π [illegible] 3.142.

[illegible] play an important role in numerical analysis. [illegible] of round-off, we may apply the following rules when [illegible] (these rules are not normally applied when performing [illegible] computer calculations).

[illegible] we are given a number and we want to round it to the first decimal place. [illegible] after the first decimal place and proceed as follows:

(a) If the first discarded digit is less than 5, the previous digit is unchanged. For example, the number 56.44, when rounded to the first decimal place, becomes [illegible]

(b) If the discarded digit is greater than 5, the previous digit is increased by 1. For example, the number 56.46, when rounded to first decimal, it becomes [illegible]

(c) If the discarded digit is exactly 5, the previous digit is unchanged if it is even, [illegible] increased by 1, if it is odd. For example, the number 56.45 becomes [illegible] 56.75 becomes 56.8.

However, the most commonly used rule (we are familiar with) for rounding-off the number is: if the discarded digit exceeds or equals 5, we add 1 to the previous digit.

A quantitative analysis of the round-off errors present in the final result of a numerical computation, usually termed the accumulated rounded off error, is difficult, particularly when the algorithm used is of some complexity. Except in very simple cases, the accumulated error is not simply the sum of the local round-off errors due to individual rounding or truncating operations. The local error [illegible] is propagated throughout the remaining part of the computation. In order to obtain a rounded-off error bound, we may assume the worst possible [illegible] and follow the propagation [illegible] calculations.

Preface To The Fifth Edition

The Goal of this enlarged edition of our book on **Numerical Analysis** remains the same as for the previous editions: to give a comprehensive and state-of-the-art treatment of all the important aspects of the subject. In this, we have made modifications in all the first eight chapters and added extra problems at the end of each chapter. A new chapter, Chapter 9, based on eigenvalues and eigenvectors has been included. We have tried to cover all the basic and important procedures to compute eigenvalues and eigenvectors of a matrix. This chapter has been written especially on the request of users of the subject in various engineering universities.

We gratefully thank the users and the reviewers of the previous editions who provided valuable suggestions and ideas for the improvement of this book. Their feedback is valuable in our efforts for continuous improving this book. We are also thankful to our various teaching assistants both at BIIT and FUIMCS who checked the references and exercises in many chapters.

The authors would also like to thank Professor Akram Javed, Faculty of Science, University of Engineering and Technology, Taxila, for his many useful comments. We are thankful to Prof. Aftab Ahmad, Director, Institute of Management and Computer Sciences, Foundation University, Rawalpindi, for providing us the necessary infrastructure to complete this project. Thank you all.

The Bhattis
Islamabad,

Preface To The Fourth Edition

The Fourth Edition of this book on numerical analysis is in your hands now. It is geared specifically to the needs and background of our students. During this period, we received several comments from the users. In reviewing their comments, we have made modifications in some chapters of the book to sharpen the reader's understanding of the material presented. The plan of presentation of all chapters has been that of step by step. We start with an elementary method and then proceed to develop this or alternative, more sophisticated methods. The presentation just given is, of course, much over-simplified. In practice, a combination of conventional mathematical analysis and numerical analysis is likely to be used. Proofs of formulas are given where these are reasonably easy to follow but have been omitted in the more difficult cases.

A major change has been made in computer programs that implement the use of numerical methods presented in the book for solving problems. This edition contains computer programs written in C++. They have deliberately been kept as straightforward as possible so that the reader should understand the precise function of every step in each program. While the programs are intended primarily for educational-purposes, they can, of course, be used for solving some simple practical problems. However, for more complex practical problems, they do not offer any guarantee regarding the accuracy, adequacy or completeness of any information herein. Therefore, the user should make use of the excellent software packages now available. Hopefully the reader will appreciate this edition. We recommend them to learn and make more substantial use of their computers. We have benefited much by sitting at the feet of the wise, and we hope that, through this book, it may be possible to transmit a spark from their fire to all our readers. Good luck!

We would like to thank the users and reviewers of the previous editions whose comments and suggestions have enormously proved to be valuable in updating the material of the book. However, comments and suggestions for further improvements to the book and supporting software are welcome and can be communicated to us through the publisher. The authors would also like to express their gratitude to Prof. Akram laved, Dean, Faculty of Science, UET, Taxila, for his many useful comments received to improve the quality of this book and particularly to Dr. Jamil Sarwar, Director, BIIT, Rawalpindi, for providing necessary facilities to accomplish this reviewing exercise.

In closing, we are also grateful to our families for their continued patience and understanding during the review effort.

The Bhattis
Islamabad
May, 2002

Preface To The First Edition

The importance of Numerical Analysis to the scientists and engineers is now widely acknowledged. In the book world, there is no dearth of good books on numerical analysis written by foreign authors but the majority of these books are not available in this country. I have written this book to meet the long-felt need of indigenous students.

The main feature of the present text is to introduce numerical methods – covering the syllabi of various universities, colleges and other institutes, where this subject is being taught as a first course. In writing such an elementary book, I have inevitably been confronted by the problem of selection of material, which covers to a great extent the syllabi of the concerned institutes. Naturally, some will disagree with me over this choice of selection. I respect their prerogative. However, I shall be relieved if it is felt that the topics included do provide a reasonably solid background to the student's training and one from which he can easily proceed to further advanced courses in the subject.

The book is designed for a one-semester course in numerical analysis and consists of eight chapters. Each chapter includes a large number of thoroughly explained examples and problems of various complexity. These problems are very necessary and the students should work them out carefully. Each question has been designed to test the student's understanding of a particular formula. The answers of these problems are given at the end of the book. Proofs of formulas are included only where these are reasonably easy to follow, but the formulas are mentioned without proofs in the more difficult cases. It has been tried to keep the explanation straightforward and practically-oriented. The minimum prerequisite for using this book is elementary calculus (including some exposure to series and partial derivatives), linear algebra (determinant an matrices) and differential equations. It is also assumed that the student has taken a programming course in one of the computer languages. Fortran 77, which continues to be an excellent computer language for a wide variety of mathematical problems, is used in this book. Computer programs are given at appropriate places in the text.

No book emerges fully formed from an author's forehead. I would like to acknowledge the inspiration and encouragement I received from my colleagues and the help of many students who worked with early versions of the manuscript and checked exercise solutions and text examples.

The responsibility for any errors, omissions or lack of clarity naturally remains with me. I would appreciate having any such omissions, oversights or needed corrections called to my attention so that they can be implemented for improving the quality of this book. I would also like to thank Mr. Ghulam Shabir Qureshi and Syed Akbar Shah for their help in turning rough drafts into a beautifully prepared final manuscript.

I would like to express my gratitude to the National Book Council of Pakistan for accepting the manuscript of this book under the Creative Writer's Scheme. I also wish to thank the anonymous referees who reviewed the manuscript.

Above all, I wish to thank my family, without whose encouragement, patience and sacrifice this book would not have been completed.

Saeed A. Bhatt
Islamabad
May, 1990

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Chapter 2

Finite Differences

2.1 DIFFERENCE TABLE

Suppose we have a function f(x) which is tabulated over a range of values (called **tabular points**) of the independent variable x. Let us denote the uniform difference (constant spacing or step-size) between any two successive values by h so that,

$$x_1 - x_0 = h = x_2 - x_1 = \ldots = x_n - x_{n-1}$$

or $x_1 = x_0 + h$

$$x_2 = x_1 + h = x_0 + 2h$$

$$\ldots$$

$$x_p = x_0 + ph$$

$$\ldots$$

$$x_n = x_0 + nh$$

and $f(x_p) = f_p = f(x_0 + ph)$

In many numerical processes concerned with tabulated functions certain quantities called **finite differences** are important. A finite difference is a mathematical expression of the form $f(x + b) - f(x + a)$. The procedure to compute differences is explained below.

To build up the difference table, we first write down the values of x_1 's as well as the corresponding values of f_1 's as shown below:

x_i	f_i	1st	2nd	3rd
x_0	f_0			
		$f_1 - f_0$		
x_1	f_1		$f_2 - 2f_1 + f_0$	
		$f_2 - f_1$		$f_3 - 3f_2 + 3f_1 - f_0$
x_2	f_2		$f_3 - 2f_2 + f_1$	
		$f_3 - f_2$		$f_4 - 3f_3 + 3f_2 - f_1$
x_3	f_3		$f_4 - 2f_3 + f_2$	
		$f_4 - f_3$		
x_4	f_4			

The first-order differences are obtained from the second column by subtracting each value from the next below and placing the differences to the right but halfway between the two values from which they have been obtained. In this way, the column containing all the first-order differences is formed, but each difference column contains one entry less than its predecessor column.

We are now in a position to produce a column of second-order differences from the column of the first-order differences in a similar way. In computing differences, great care should be exercised to avoid arithmetic errors in the subtractions – the fact that we subtract the upper value from the lower causes a real source of confusion. The sign of the differences is important and shows whether the function is increasing or decreasing in the range of the values obtained.

There are several uses of a difference table; a few of which are as follows:

i) A difference table provides a convenient way for examining at a glance how a particular function behaves. It is particularly applicable in determining the behaviour of the derivatives of a given function.

ii) If there are some errors in the data, the differences will also contain errors. By inspecting the difference table, often the error (or errors) can be detected and corrected.

iii) It helps in filling missing values.

iv) It helps in extending the list of values.

The word **finite** refers to the finite-size of the interval (increment) used in the table as opposed to the infinitesimal interval, which are met in infinitesimal calculus. For this reason, the theory and application of finite differences is sometimes referred to as **Finite Calculation**. It plays an important role in interpolation, numerical differentiation, numerical integration, numerical solutions of difference, ordinary and partial differential equations and time series analysis.

A numerical example at this stage should help clarify some basic concepts for constructing a difference table.

Example 1 Construct the difference table for the function $f(x) = x^4$ for $x = -2$ to $x = 4$, at the interval of 1. [Usually written as $x = -2(1)4$; the figure in brackets being the constant increment.]

Solution The values of f_i and the differences are shown in the table below:

x_i	$f_i = x_i^4$	1st	2nd	3rd	4th	5th
–2	16					
		–15				
–1	1		14			
		–1		–12		
0	0		2		24	
		1		12		0
1	1		14		24	
		15		36		0
2	16		50		24	
		65		60		
3	81		110			
		175				
2	256					

An examination of the difference table reveals that all fourth-order differences are constant and thus the fifth and all higher-order differences would be zero, which is the peculiar property of an exact polynomial (i.e., when all entries in the table are exact and not rounded).

Some obvious results

a) The nth-differences of an exact polynomial of degree n are constant.

b) The $(n + 1)$st differences of that polynomial are zero.

c) The above values are only true of polynomials when they are tabulated at equal intervals.

If the function does not represent an exact polynomial, the above results will not hold. In practice, we always deal with rounded numbers, where we seldom come across a column with all its differences zeros. The differences of rounded numbers are irregular and thus give rise to the irregular part of the table. In that case, the nth-order differences due to the rounding errors oscillate between $\pm 2^{n-1}$.

The reason for this is that when the tabulated values are rounded, each value has an error usually lying in the range $\pm \frac{1}{2}$, if we work in units of the last place. These errors will build-up in the differences just as do mistakes, and eventually, if the true values have convergent differences, they will become greater than the true differences. In the worst case, the rounded-off errors will be alternately $+\frac{1}{2}$ and $-\frac{1}{2}$ and their contribution to any nth difference will be,

$$\pm\frac{1}{2}\cdot\left\{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}\right\}=\pm 2^{n-1}$$

Example 2 Construct the difference table for the function $f(x)=\sqrt{x^2+x+1}$, rounded to 4 dp, for x = 10 (1)16.

x	$f(x)=\sqrt{x^2+x+1}$	1st	2nd	3rd	4th
10	10.5375				
		.9969			
11	11.5326		5		
		.9974		−2	
12	12.5300		3		3
		.9977		1	
13	13.5275		4		−2
		.9981		−1	
14	14.5258		3		−1
		.9984		−2	
15	15.5242		1		
		.9985			
16	16.5227				

Since the function is tabulated at 4 dp, each difference is also to 4 dp. Because of this, the decimal point and the leading zeros may be omitted in the formation of a difference table and they may then be written as integers. This makes the table easier to construct and much neater too. For instance, the first entry in the column of fourth differences is an abbreviation of 0.0003. The table shows that the fourth-order differences oscillate and are all within the range $\pm 2^{4-1} = \pm 8$.

2.2 DETECTION AND CORRECTION OF ERRORS IN A DIFFERENCE TABLE

It is likely that an error (errors) may show up while constructing differences. We observe a very peculiar kind of error propagation, which we shall illustrate in this section. An error caused by reversing the order of a pair of digits in a number is commonly made in copying down the number from the given data. It affects the other differences in the table. We may denote the error in a single entry in the difference table by the symbol, ε, which can be negative, positive, small or large. Its effect on the differences spreads out fan-wise as shown in the table below:

x	f	1st	2nd	3rd	4th
0	0				
		0			
1	0		0		ε
		0		ε	
2	0		ε		-4ε
		ε		-3ε	
3	ε		-2ε		$+6\varepsilon$
		$-\varepsilon$		3ε	
4	0		ε		-4ε
		0		$-\varepsilon$	
5	0		0		ε
		0			
6	0				

This fan-wise (triangular patterns) propagation of ε in the difference table grows quickly and makes it possible in certain cases to locate an error and also to find its numerical value, thus enabling us to rectify it with the help of tabular values. A glance at the table reveals that the coefficients of ε in the nth-order differences are binomial coefficients of x, which occur in the expansion of $(1 - x)^n$. For example, the coefficients in the third-order difference column are 1, –3, 3, –1, which occur in the expansion of $(1 - x)^3$ in the increasing powers of x, i.e., $1 - 3x + 4x^2 - x^3$. The corresponding coefficients for the fourth-order differences of $(1 - x)^3$ are 1, –4, 6, –4, 1. The binomial coefficients in the fifth and sixth difference columns are 1, –5, 10, –10, 5, –1 and 1, –6, 15, –20, 15, –6, 1, respectively. The table shows that the higher-order differences are very sensitive to slight changes in any of the ordinates or lower-order differences. Relatively small input changes generate relatively large output changes. For the identification of gross errors, the above picture should be kept in mind.

We illustrate the procedure by means of the following example.

Example 3 The following table contains an incorrect value of f(x). Locate the error, suggest a possible cause and a suitable correction:

x	1	2	3	4	5	6	7	8	9	10
f(x)	37	74	135	226	353	531	739	1010	1341	1738

Solution **Difference Table**

x	f	1st	2nd	3rd	4th
1	37				
		37			
2	74		24		
		61		6	
3	135		30		0
		91		6	
4	226		36		9
		127		15	
5	353		51		–36
		178		–21	
6	531		30		54
		208		33	
7	739		63		–36
		271		–3	
8	1010		60		9
		331		6	
9	1341		66		
		397			
10	1738				

In the above table, f(x) seems to represent an exact polynomial; thus all fourth-order differences should be zero. The error seems to have appeared in the fourth-order differences with coefficients: 1, – 4, 6, – 4, 1. The incorrect difference may be written as:

$$1(9), -4(9), 6(9), -4(9), 1(9)$$

This indicates that the error is 9. The next step is to locate the incorrect functional value. This can be moving backward to the second column. It shows that the term in error is 531 and the correct value is 531 – 9 = 522. The likely cause of the error may be due to wrongly copying the digits. The result can be checked by correcting the wrongly-placed entry and reconstructing the difference table. If the function is known analytically, it would be preferable to recalculate it at x = 6, so that the correction can be made with certainty rather just estimated.

In the above example, the functional values are exact and it was fairly easy to locate and correct the error with certainty, but this is not always the case especially when the values of f(x) have been rounded, since the errors will not then be exact multiples of the binomial coefficients. In such a case, we can only make an estimate of the error. Moreover, in a difference table in which there are two or more errors, their fans will eventually overlap, making it more difficult to discover the errors. Some more care is necessary to find out a reasonable pattern to locate the error(s) in such cases.

A table in which two errors have been made is more difficult to analyze since the binomial coefficients overlap. The following pattern shows a possible example.

Solution **Difference Table**

f	1st	2nd	3rd	4th
0				
	0			
0		0		
	0		ε_1	
0		ε_1		$-4\varepsilon_1$
	ε_1		$-3\varepsilon_1$	
ε_1		$-2\varepsilon_1$		$6\varepsilon_1$
	$-\varepsilon_1$		$3\varepsilon_1$	
0		ε_1		$\varepsilon_2-4\varepsilon_1$
	0		$\varepsilon_2-4\varepsilon_1$	
0		ε_2		$-4\varepsilon_1+4\varepsilon_1$
	ε_2		$-3\varepsilon_2$	
ε_2		$-2\varepsilon_2$		$6\varepsilon_2$
	$-\varepsilon_2$		$3\varepsilon_2$	
0		ε_2		
	0			
0				

It may be possible to identify the error pattern in the third-order differences column but the confusion in the fourth-order difference would probably be too great to give an opportunity to detect the error. We, therefore, concentrate on the problems where only one mistake is made.

Example 4 It is suspected that the following table contains an error. By differencing, locate any probable error and correct it. Check by re-differencing if any correction is made. The values of f(x) are rounded to 3 dp.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
f(x)	0.905	0.819	0.741	0.677	0.607	0.549	0.497

Solution **Difference Table**

x	f (x)	1st	2nd	3rd	4th
0.1	0.905				
		– 86			
0.2	0.819		8		
		– 78		6	
0.3	0.741		14		– 26
		– 64		– 20	
0.4	0.677		– 6		38
		– 70		18	
0.5	0.607		12		– 24
		– 58		– 6	
0.6	0.549		6		
		– 52			
0.7	0.497				

Comparing the third-differences with the coefficients of x, we get

$$\begin{array}{cccc} 1 & -3 & 3 & -1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & -20 & 18 & -6 \end{array}$$

We may deduce that $\varepsilon = 6$, i.e., 0.006 and the corrected value of f(0.4) = 0.677 – 0.006 = 0.671. We can check by reconstructing the difference table, but we are not 100% sure that it is the correct value, we just estimated. However, it may prove to be a reasonable estimate. If the entries are more in the data and the above mentioned options fail to give a reasonable clue to pick-up the error, we should use of the fifth or sixth difference column and then try again.

If there is no obvious pattern for locating an error in the difference table, we use the following formula for finding the error.

$$\text{Error} = \frac{\text{Largest value in a column}}{\text{Corresponding coefficient of } \varepsilon \text{ in that column}}$$

In this section, we have studied how to locate and correct a single error in a difference table. If there are two or more errors in the entries, it is usually not easy to separate their overlapping effects and thus locations and corrections of such errors become extremely difficult. In some cases, irregular behaviour of the differences may be caused not by errors but by irregularities in the functions.

2.3 DIFFERENCE OPERATIONS

To refer to specific entries in a difference table we use some operators, called **difference operators**. An operator is not a number but it is an operation, which when applied to a function changes it to some other function. The operator technique proves to be a most useful tool when we wish to construct formulas for interpolation, numerical differentiation, numerical integration, etc. One of the biggest advantages is that we can fix the type of formula desired in advance and then proceed directly toward the goal.

The following operators are commonly used:

Δ Forward-difference operator (usually read as delta)

∇ Backward-difference operator (usually read as del or nebla)

δ Central difference operator (read as sigma)

μ Average (mean) operator (read as mu)

E Shift operator

Let us define these operators one by one. It must be emphasized that these operators assume equally-spaced data points.

2.3.1 Forward Difference Operator

The forward difference operator Δ is defined by the following relation:

$$\Delta f_r = f_{r+1} - f_r$$

where r is an integer, and $\Delta f_r = \Delta f(x_r)$.

Also, $\Delta f_{r+1} = \Delta f(x_r + h)$ and $\Delta f_{r+\frac{1}{2}} = \Delta f\left(x_r + \frac{h}{2}\right)$.

In words, when Δ operates on a function, we first shift r by r + 1 and then subtract the original function from the shifted function. This produces the difference function Δf_r.

$$\begin{aligned}\text{Thus, } \Delta f_0 &= f_1 - f_0\\ \Delta f_1 &= f_2 - f_1\\ &\vdots\\ \Delta f_{-1} &= f_0 - f_{-1}, \text{ etc.}\end{aligned}$$

$\Delta f_{-1}, \Delta f_0, \Delta f_1$, are called **first-order forward differences**. The differences of the first-order differences are called **second-order differences** and are computed as follows:

Thus, $\Delta^2 f_r = \Delta(\Delta f_r)$

$$= \Delta(f_{r+1} - f_r)$$
$$= \Delta f_{r+1} - \Delta f_r$$
$$= (f_{r+2} - f_{r+1}) - (f_{r+1} - f_r)$$
$$= f_{r+2} - 2f_{r+1} + f_r$$

The higher-order differences are obtained in the same way.

Thus, $\Delta^3 f_r = \Delta(\Delta^2 f_r)$

$$= \Delta\{f_{r+2} - 2f_{r+1} + f_r\}$$
$$= \Delta f_{r+2} - 2\Delta f_{r+1} + \Delta f_r$$
$$= (f_{r+3} - f_{r+2}) - 2(f_{r+2} - f_{r+1}) + (f_{r+1} - f_r)$$
$$= f_{r+3} - 3f_{r+2} + 3f_{r+1} - f_r$$

$$\Delta^4 f_r = f_{r+4} - 4f_{r+3} + 6f_{r+2} - 4f_{r+1} + f_r$$

In general, nth-order differences are given by:

$$\Delta^n f_r = \Delta^{n-1} f_{r+1} - \Delta^{n-1} f_r$$

where $\Delta^n f_r \neq (\Delta f_r)^n$, and $n \geq 1$.

The following difference table shows how the forward differences of all orders can be formed:

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

We observe from the above table that differences with the same subscripts all lie on a downward sloping diagonal.

While experimenting with differences, we observe that if x^n is a polynomial of degree n, then Δx^n is a polynomial of degree (n – 1). In other words, differencing behaves like differentiation in the sense of reducing the degree of a polynomial.

Thus, $\Delta x^n = (x+1)^n - x^n$

$$= nx^{n-1} + n(n-1)x^{n-2} + \ldots$$

If the above process is continued for n times, the polynomial x^n is reduced to degree zero, i.e., constant. This is exactly what was shown by Example 1, that the nth-order differences of a polynomial of degree n are constant and all higher-order differences are zero.

Algorithm for Generating Differences Using Forward Scheme

In general, for a function tabulated at n points, the corresponding forward difference table can be represented by a matrix of size (n – 1) * (n – 1). Note that only the elements in the columns from 1 to n – i, where the row i = 1, 3, ..., n – 1, are of interest.

The algorithm to generate forward differences table may look like the following:

Steps

```
For   J = 1 TO n – 1 by 1 DO
      FOR   I = 1 TO n – J by 1 DO
            IF (J = 1) THEN
      SET   D_IJ = F(X_IJ) – D_I,J–1
            ELSE
      SET   D_IJ = D_I+1,J–1 – D_I,J–1
      PRINT "all differences, D_I,J"
```

This algorithm will compute the forward differences of all orders that can be computed from the given function table. The data with equi-spaced abscissas are initialized in the program.

Example 5 Computerize the algorithm for generating forward differences. Use the following test data:

x	1.2	1.4	1.6	1.8	2.0
y	5.64642	6.44218	7.17356	7.83327	8.41471

Solution

Program No. 1: Difference Table

```
# include<iostream.h>
# include<stdio.h>
# include<conio.h>

void main(void)
  {
    clrser( );
    float interval, array[20][20]={0.0};
    int no, col, x, y;

    cout<<"\tDIFFERENCE TABLE";
    cout<<"\n\n\tENTER THE FIRST VALUE :  "; cin>>array[0][0];
    cout<<"\n\tENTER THE INTERVAL :  "; cin>>interval;
    cout<<"\n\tENTER TOTAL NO. OF X :  "; cin>>no;

    for(int i=1; i<no; i++)
       array[i][0]=array[i-1][0]+interval;

    cout<<"\n\n\tENTER FUNCTIONAL VALUES :  \n";
    for(i=0;i<no;i++)

       {
         cout<<"\tX("<<i<<") =  ";cin>>array[i][1];
       }

    cout<<"\n\tHOW MANY COLUMNS ARE REQUIRED :  "; cin>>col;
    for(i=1; i<=(col+2);i++)
       array[j][i]=array[j+1][i-1]-array[j][i-1];

    clrscr( );
    cout<<"\t\tDIFFERENCE TABLE\n\n";
    cout<<" X          F(X)";
    for(i=1;i<col;i++)
       cout<<"     col     "<<;

    cout<<"\n\n";

    for(i=0;i<no;i++)
       cout<<""<<array[i][0]<<"\n\n";
```

```
x=8; y=5;
for(i=1;i<=(col+1);i++)
  {
    gotoxy(x,y);
    for(int j=0;j<=(no-i);j++)
      {
        cout<array[j][i];
        y+=2;
        gotoxy(x,y);
      }
    x+=9; y=i+5;
  }
}
```

DIFFERENCE TABLE

ENTER THE FIRST VALUE : 1.2

ENTER THE INTERVAL : 0.2

ENTER TOTAL NO. OF X : 5

ENTER FUNCTIONAL VALUES:
X(0) = 5.64642
X(1) = 6.44218
X(2) = 7.17356
X(3) = 7.83327
X(4) = 8.41471

HOW MANY COLUMNS ARE REQUIRED : 4

Computer Program

DIFFERENCE TABLE

X	F(X)	col 1	col 2	col 3	col 4
1.2	5.64642				
		0.79576			
1.4	644218		– 0.06438		
		0.73138		– 0.07167	
1.6	7.17356		– 0.07167		0.00069
		0.65971		– 0.00660	
1.8	7.83327		– 0.07827		
		0.58144			
2.0	8.41471				

2.3.2 Backward Difference Operator

The backward difference operator ∇ is defined by the following relation:

$$\nabla f_r = f_r - f_{r-1}$$

Hence, we shift r backward by one step, the function becomes f_{r-1} and subtract this function from the original f_r.

Thus, $\nabla f_1 = f_1 - f_0$

$$\nabla f_0 = f_0 - f_{-1}$$

$$\nabla f_2 = f_2 - f_1$$

The above differences are called first-order backward differences. In a similar manner, we can define backward differences of higher-orders. Thus, we obtain:

$$\begin{aligned} \nabla^2 f_r &= \nabla(\nabla f_r) \\ &= \nabla(f_r - f_{r-1}) \\ &= \nabla f_r - \nabla f_{r-1} \\ &= (f_r - f_{r-1}) - (f_{r-1} - f_{r-2}) \\ &= f_r - 2f_{r-1} + f_{r-2} \end{aligned}$$

Similarly, $\nabla^3 f_r = f_r - 3f_{r-1} + 3f_{r-2} - f_{r-3}$.

In general, nth-order differences are given by:

$$\nabla^n f_r = \nabla^{n-1} f_r - \nabla^{n-1} f_{r-1} \,;\, n \ge 1.$$

With the help of this operator, we can construct the table for backward differences:

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

We observe from the above table differences with the same subscripts all lie on an upward sloping diagonal.

Algorithm to Generate Differences Using Backward Scheme

In general, for a function tabulated at n points, the corresponding backward difference table can be represented by a matrix of size n × n. Note that only the elements in the columns from 1 to n – i, where the row i = 2, 3, ..., n, are of interest.

The algorithm to generate backward differences table may look like the following:

Steps

```
For   J = 1 TO n – 1 by 1 DO
      FOR   I = J + 1 TO n by 1 DO
            IF (J = 1) THEN
      SET   D_IJ = F(X_I) – F(XD_I-1, J-1)
            ELSE
      SET   D_IJ = D_I, J-1 – D_I-1, J-1
      PRINT "all differences, D_I,J"
```

23.3 Central Difference Operator

The central difference operator δ is defined as:

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}$$

Thus, $\delta f_{r+\frac{1}{2}} = f_{\left(r+\frac{1}{2}\right)+\frac{1}{2}} - f_{\left(r-\frac{1}{2}\right)-\frac{1}{2}} = f_{r+1} - f_r$

Similarly, $\delta^2 f_r = \delta(\delta f_r)$

$$= \delta\left(f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}\right)$$

$$= \delta f_{r+\frac{1}{2}} - \delta f_{r-\frac{1}{2}}$$

$$= (f_{r+1} - f_r) - (f_r - f_{r-1})$$

$$= f_{r+1} - 2f_r + f_{r-1}$$

In general, nth-order differences are given by:

$$\delta^n f_r = \delta^{n-1} f_{r+\frac{1}{2}} - \delta^{n-1} f_{r-\frac{1}{2}}$$

The difference table for δ is given below:

x	f	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
x_0	f_0				
		$\delta f_{\frac{1}{2}}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{\frac{3}{2}}$		$\delta^2 f_{\frac{3}{2}}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_4$
		$\delta f_{\frac{5}{2}}$		$\delta^3 f_{\frac{5}{2}}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{\frac{7}{2}}$			
x_4	f_4				

We note that all differences with the same subjects lie on the same horizontal line and all even-order differences have integer subscripts. The central difference notation is preferable for many purposes but has the disadvantage of requiring fractional suffixes.

It is to be kept in mind that whatever notation we use, there is only one difference table and hence each entry in the table has one of the three names, for instance,

$$f_{r+1} - f_r = \Delta f_r = \nabla f_{r+1} = \delta f_{r+\frac{1}{2}}$$

Also, $\Delta f_0 = \nabla f_1 = \delta f_{\frac{1}{2}}$

$$\Delta^2 f_0 = \nabla^2 f_2 = \delta^2 f_1$$

$$\Delta^3 f_2 = \nabla^3 f_5 = \delta^3 f_{\frac{7}{2}}$$

$$\Delta^4 f_{-2} = \nabla^4 f_2 = \delta^4 f_0, \text{ etc.}$$

2.3.4 Shift Operator

The shift operator (also called the **step operator**) E is defined by,

$$E f_r = f_{r+1}$$

$$E^{-1} f_r = f_{r-1}$$

$$E^2 f_r = E(f_r) = E f_{r+1} + f_{r+2}$$

In general, $E^n f_r = f_{r+n}$.

2.3.5 Mean Operator

The mean (or average) operator μ is defined by,

$$\mu f_r = \frac{1}{2}\left(f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}}\right)$$

Thus, $\mu f_{r+\frac{1}{2}} = \frac{1}{2}\left\{f_{r+\frac{1}{2}+\frac{1}{2}} + f_{r+\frac{1}{2}-\frac{1}{2}}\right\} = \frac{1}{2}\left(f_{r+1} + f_r\right)$.

2.4 RELATIONSHIPS BETWEEN OPERATORS

Various relationships exist between operators. For example,

$$\Delta f_r = f_{r+1} - f_r$$

$$\Delta f_r = E f_r - f_r = (E-1)\, f_r$$

or, $\Delta = E - 1$

or, $E = E - \Delta$

Similarly, $\nabla f_r = f_r - f_{r-1}$

$$= f_r - E^{-1} f_r$$

or, $\nabla = 1 - E^{-1}$

and $E = (1-\nabla)^{-1}$

$$\delta f_r = f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}} = E^{\frac{1}{2}} f_r - E^{-\frac{1}{2}} f_r$$

$$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f_r$$

or, $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Also, $\mu f_r = \frac{1}{2}\left(f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}}\right)$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}} f_r\right)$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) f_r$$

or $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

The relationships between various operators are given in the following table:

	E	Δ	∇	δ
E	E	$1+\Delta$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$
Δ	$E-1$	Δ	$\nabla(1-\nabla)^{-1}$	$\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}$
∇	$1-E^{-1}$	$\Delta(1+\Delta)^{-1}$	∇	$-\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-\frac{1}{2}}$	$\nabla(1-\nabla)^{-\frac{1}{2}}$	δ
μ	$\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$	$\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-\frac{1}{2}}$	$\left(1-\frac{\nabla}{2}\right)(1-\nabla)^{-\frac{1}{2}}$	$\sqrt{1+\frac{\delta^2}{4}}$

The above relationships can easily be proved and we leave this as an exercise to the student to fill in the details of the above results.

PROBLEMS

1. Construct the difference tables for the following functions:
   a) $f(x) = x^4 - x - 1$, over the range, $x = -3(1)5$.
   b) $f(x) = 2x^3 + 2x^2 + 2x - 1$, over the range, $x = -1(1)7$.
   c) $f(x) = 2x^3 + 2x^2 - 3x + 4$, over the range, $x = -1(.5)1$.
   d) $f(x) = 2^x$ for $x = 0(1)6$. Will there ever be a column of constant differences in this case?
   e) $f(x) = \sin x$ for $x = 1.0(0.1)1.6$.
   f) $f(x) = 2x^3 + 3x + 1$ for $x = 0.1(0.1)0.5$. What can you say about fourth-order difference column? What is the reason for your observation?
   g) $f(x) = 3x^3 + 4x^2 + 1$ and $f(x) = x^3$ for values $x = 0(1)5$. What do you conclude from the third-order differences column of the difference tables based on the above functions?

2. (a) It is suspected that there is an error in one of the values of f(x) in the following table:

x	1.0	1.52	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f(x)	−38	−46	−59	−76	−92	−118	−140	−161	−180

Construct the differences-table, detect and correct the error.

(b) Consider the following table of values:

x	1	2	3	4	5	6	7	8	9	10	11
f(x)	7	10	17	33	63	121	185	287	423	598	817

It is suspected that one of the values may have been recorded in error. Assuming that the data follow a polynomial, determine which one, if any, of the functional values is in error and what it should be?

(c) Locate the error and estimate the correct value for the following table:

x	0	1	2	3	4	5	6	7	8
f(x)	1.0000	1.1002	1.2013	1.3045	1.4105	1.5210	1.6366	1.7586	1.8881

Construct the differences-table, detect and correct the error.

3. Locate and correct mistakes in each of the following tables:

x	1	2	3	4	5	6	7	8	9	10
f(x)	7	12	21	34	51	70	97	126	159	196
z(x)	.500	.520	.540	.560	.579	.589	.618	.637	.655	.674

Construct the differences-table, detect and correct the error.

4. The table of values for two quadratic polynomials y(x) and z(x) are given to 3 sf as follows:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y(x)	1.00	1.13	1.35	1.76	2.10	2.69	3.46
z(x)	4.00	4.87	5.91	7.15	8.60	10.3	12.3

Locate and correct the errors (other than those attributed to rounding off) in each table.

5. Compute the missing entries in the following tables:

a.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x				
	x			
x		-4		
	3		-1	
9		-5		x
	x		x	
x		0		
	x			
x				

b.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x				
	x			
x		-3		
	2		x	
7		-3		x
	x		3	
x		0		
	x			
x				

c.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x				
	x			
1		12		
	x		x	
x		x		24
	60		60	
x		108		
	x			
241				

6. (a) The following table of values contains an error. Locate the incorrect value and find an estimate of correct value:

x	−1	0	1	2	3	4	5	6
f(x)	1.51	1.17	1.51	2.35	4.23	6.61	9.67	13.41

Reconstruct the difference table with the correct value. Comment on the nature of the function f(x).

(b) The table below contains an error. Locate the incorrect the error:

x	3.60	3.61	3.62	3.63	3.64	3.5	3.66	3.67	3.68
f(x)	.112046	.120204	.128350	.136462	.144600	.152702	.60788	.168857	.17690

(c) Use the difference table method to locate and correct the error in the following table of values:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
f(x)	10.30	10.70	11.04	11.26	11.30	11.01	10.60	9.74	8.46	6.70

7. (a) Form the difference table for the function given below. Find the values of a, b, and c, so that $\Delta^4 f(a) = \nabla^4 f(b) = \delta^4 f(c) = -0.0428$.

x	0	1	2	3	4	5	6
f(x)	.3679	.7358	.9197	.9810	.9963	.9994	.9999

(b) Tabulate the function $f(x) = x(x-1)(x-2)$ for $x = -0.2(0.1)0.2$, correct to 3 dp. What do you say on the value of $\delta^4 f$?

(c) Prove that the sum of the numbers in any column of a difference table is equal to the difference between the last and first numbers in the preceding column.

Set up a table showing the first and second differences for the following data to check the arithmetical work:

0.0000, − 0.0104, − 0.0206, − 0.0307, − 0.0404, − 0.0496.

(d) (i) Construct the difference table for the following functional values:

x	−2	−1	0	1	2	3	4
f(x)	15	1	1	3	19	85	261

If the origin $x_0 = 1$, determine the values of Δf_0, ∇f_{-1}, $\delta f_{\frac{3}{2}}$, $\delta^2 f_1$, $\Delta^3 f_0$, $\nabla^3 f_2$, $\Delta^2 f_1$ and $\delta^4 f_0$.

(ii) Given the set of values:

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Construct the difference table and report the values of Δy_{20}, $\Delta^2 y_{10}$, $\Delta^3 y_{15}$ and $\Delta^5 y_{10}$.

(e) Given the difference table:

f	f(x)	1st	2nd	3rd	4th
–1	–14				
		14			
0	0		0		
		14		96	
2	14		96		0
		110		96	
4	124		192		
		302			
6					

If the origin $x_0 = 2$, express using forward, backward and central differences in the entries, 110, 302 and 192.

8. (a) The values of y shown in the following table are alleged to be derived from a fourth entries degree polynomial. Test this and correct the value, where necessary.

x	0	1	2	3	4	5	6	7	8	9	10
f(x)	0	2	20	90	272	605	1332	2450	4160	6642	10100

(b) Suggest appropriate correction for the following table of values:

x	10	11	12	13	14	15	16	17	18
f(x)	2.1544	2.2240	2.2894	2.3513	2.4121	2.4662	2.5198	2.5713	2.6207

(c) The following table contains an error. Identify the error and estimate the correct value of the function:

x	1	2	3	4	5	6	7	8	9
f(x)	10	12	15	21	32	50	79	115	166

(d) Locate the incorrect entry in the following tables and estimate the correct value of each function:

(i)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
f(x)	-9.800	-9.061	-8.341	-7.594	-6.671	-5.776	-4.530	-2.945	-0.899	1.736	5.100

(ii)

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
f(x)	0.905	0.819	0.741	0.677	0.607	0.549	0.497	0.449	0.407

(e) Locate the incorrect entry in the following table and find its correct value:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
f(x)	0.000	0.012	0.072	0.252	0.672	1.500	2.952	5.922

0.8	0.9	1.0	1.1	1.2
8.832	13.932	21.000	30.492	42.912

9. Prove the following relationships:

(a) $\Delta = \sqrt{E} \cdot \delta$ (b) $\Delta = E\nabla$ (c) $\delta^2 = \Delta - \nabla = \Delta\nabla - \nabla\Delta$

(d) $E = 1 + \mu\,\delta + \frac{\delta^2}{2}$ (e) $\mu^2 = 1 + \frac{\delta^2}{4}$ (f) $E = 1 + \delta\sqrt{E}$

(g) $\nabla = E^{-1}\Delta$ (h) $2\,\mu\,\delta = \Delta + \nabla$ (i) $\mu\,\delta = \frac{1}{2}(\Delta + \nabla)$

(j) $E^{\frac{1}{2}} = \mu + \frac{1}{2}\delta$ (k) $E^{-\frac{1}{2}} = \mu - \frac{1}{2}\delta$ (l) $\Delta + \nabla = \frac{\Delta}{\Delta} - \frac{\nabla}{\nabla}$

10.(a) Find Δy_n, $\Delta^2 y_n$ and $\Delta^3 y_n$ in the following cases:

(i) $y_n = n^2$

(ii) $y_n = n^3 + 3n^2$

(iii) $y_n = n^3 - n^2 + 17 - 1$

(iv) $y_n = n(n-1)(n-2)(n-3)(n-4)$

(b) Prove that $y_n = 3^n(A + Bn)$ satisfies the equation,

$$y_{n+2} - 6y_{n+1} + 9y_n = 0$$

(c) If $f(x) = \sin(\pi x)$, prove that $\Delta f = -2f$.

(d) If $f(x) = x^3$, compute the following:

(i) $\left(\dfrac{\Delta^2}{E^2}\right) f(x)$ (ii) $\dfrac{\Delta^2 f(x)}{E^2 f(x)}$

(e) Obtain the following results:

(i) $\Delta\left(\dfrac{f_n}{g_n}\right)$ (ii) $\Delta\left(\dfrac{1}{g_n}\right)$ (iii) $\Delta(\log f_n)$

(iv) $\Delta(f_n \cdot g_n) = f_n \Delta g_n + g_{n+1} \Delta f_n$ (v) $\Delta\sqrt{f_r} = \dfrac{\Delta f_r}{\sqrt{f_r} + \sqrt{f_{r+1}}}$

(f) Find $\Delta^2 x^4$.

(g) Find $x\Delta(x\Delta - 1)x^2$.

11.(a) Show that,

(i) $\Delta f_i = \nabla f_{i+1} = \delta f_{i+\frac{1}{2}}$ (ii) $\Delta^2 f_i = \nabla^2 f_{i+2} = \delta^2 f_{i+1}$

(iii) If $f(x) = x^4$, then $\Delta^2 f(x) = 12x^2 + 24x + 14$ and $\Delta^4 f(x) = 24$

(iv) If $f(x) = 2^x$, then $\Delta f(x) = f(x)$.

(b) Show that

(i) $\nabla^3 f_i = \Delta^3 f_{i-3}$ (ii) $\Delta^4 f_i = \nabla^4 f_{i+4}$ (iii) $\Delta^3 \nabla f_i = \Delta^4 f_{i-1}$

(iv) $\delta^2 f_i = \Delta^2 f_{i-1}$