The Gram-Schmidt Orthonormalization Process :-

Given a basis $\{v_1, v_2, v_3, \dots, v_n\}$ of an inner product space V over $\mathbb{R}$, an orthonormal basis $\{u_1, u_2, u_3, \dots, u_n\}$ of V can be constructed as follows:-

Step 1 : $u_1 = \dfrac{v_1}{\|v_1\|}$

Step 2 : $w_2 = v_2 - \langle v_2, u_1 \rangle u_1$; $u_2 = \dfrac{w_2}{\|w_2\|}$

Step 3 : $w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$; $u_3 = \dfrac{w_3}{\|w_3\|}$

Example :-

Let, $(1,1,1)$, $(0,1,1)$, $(0,0,1)$ are the basis of $\mathbb{R}^3$, using Gram-Schmidt orthonormalization process, transform the given basis into orthonormal basis.

Solution :-

Let, $v_1 = (1,1,1)$, $v_2 = (0,1,1)$, $v_3 = (0,0,1)$

Step 1 :

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\|v_1\|} v_1$$

$$\|v_1\| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$u_1 = \frac{1}{\sqrt{3}} (1,1,1)$$

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Step 2:

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$= (0,1,1) - \langle (0,1,1), \frac{1}{\sqrt{3}}(1,1,1) \rangle \frac{1}{\sqrt{3}}(1,1,1)$$

$$= (0,1,1) - \frac{1}{3}\langle (0,1,1), (1,1,1) \rangle (1,1,1)$$

$$= (0,1,1) - \frac{1}{3}(0+1+1)(1,1,1)$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \left(0-\frac{2}{3}, 1-\frac{2}{3}, 1-\frac{2}{3}\right)$$

$$= \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\|w_2\| = \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{6}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\|w_2\|} \cdot w_2$$

$$= \frac{3}{\sqrt{6}}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \frac{1}{\sqrt{6}}(-2,1,1)$$

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Step 3:

$$\omega_3 = v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_2\rangle u_2$$

$$= (0,0,1) - \left\langle (0,0,1), \frac{1}{\sqrt{3}}(1,1,1)\right\rangle \frac{1}{\sqrt{3}}(1,1,1) - \left\langle (0,0,1), \frac{1}{\sqrt{6}}(-2,1,1)\right\rangle \frac{1}{\sqrt{6}}(-2,1,1)$$

$$= (0,0,1) - \frac{1}{3}\langle (0,0,1), (1,1,1)\rangle (1,1,1) - \frac{1}{6}\langle (0,0,1), (-2,1,1)\rangle (-2,1,1)$$

$$= (0,0,1) - \frac{1}{3}(0+0+1)(1,1,1) - \frac{1}{6}(0+0+1)(-2,1,1)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{6}(-2,1,1)$$

$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(\frac{-2}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(0 - \frac{1}{3} + \frac{2}{6}, 0 - \frac{1}{3} - \frac{1}{6}, 1 - \frac{1}{2} - \frac{1}{6}\right)$$

$$= \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

$$\|\omega_3\| = \sqrt{0^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{0 + \frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{\cancel{2}^{1}}{\cancel{4}_{2}}}$$

$$= \frac{1}{\sqrt{2}}$$

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$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\|w_3\|} . w_3$$

$$= \sqrt{2} . \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

$$= \left(0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$= \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

ORTHOGONAL MATRIX :-

A square matrix A over $\mathbb{R}$ is said to be an orthogonal matrix if $A^t = A^{-1}$ or $A.A^t = A^t.A = I$

EXAMPLE :-

Show that $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

is an orthogonal matrix.

Solution :-

We know that A will be orthogonal matrix if $A.A^t = A^t A = I$.

$$A^t = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A.A^t = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta + 0 & \cancel{\cos\theta\sin\theta} - \cancel{\cos\theta\sin\theta} \\ \cancel{\cos\theta\sin\theta} - \cancel{\cos\theta\sin\theta} & \sin^2\theta + \cos^2\theta + 0 \\ 0+0+0 & 0+0+0 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A.A^t = I$

EXAMPLE:-

Find an orthogonal matrix A whose first row is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

Solution:-

Let, $v_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $w_2 = (x, y, z)$ orthogonal to v_1 i.e. $\langle w_2, v_1 \rangle = 0$.

$$\left\langle (x,y,z), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\rangle = 0.$$

$$\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = 0$$

$$\frac{x + 2y + 2z}{3} = 0$$

$$x + 2y + 2z = 0$$

Choose $x = 0$

$$2y + 2z = 0$$

$$2(y+z) = 0$$

$$y + z = 0$$

$$y = -z$$

$$w_2 = (x, y, z) = (0, -z, z) \quad (0,-1,1)$$

Let, $z = t \in R$

$$w_2 = t(0,-1,1)$$

Choose $t = 1$

$$w_2 = (0,-1,1)$$

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$$v_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\|w_2\|} \cdot w_2$$

$$\|w_2\| = \sqrt{(0)^2+(-1)^2+(1)^2}$$
$$= \sqrt{0+1+1}$$
$$= \sqrt{2}$$

$$v_2 = \frac{1}{\sqrt{2}}(0,-1,1)$$

$$v_2 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Now let $w_3 = (x, y, z)$ be orthogonal to v_1 and

$$\langle w_3, v_1 \rangle = 0$$

$$\left\langle (x,y,z), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\rangle = 0$$

$$\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = 0$$

$$\frac{x+2y+2z}{3} = 0$$

$$x+2y+2z = 0 \rightarrow ①$$

$$\left\langle (x,y,z) \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle = 0$$

$$0 - \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 0$$

$$\frac{-y}{\sqrt{2}} + \frac{z}{\sqrt{2}} = 0$$

$$\frac{-y+z}{\sqrt{2}} = 0$$

$$-y+z = 0$$

$$y = z$$

Using $y = z$ into eq. ① :-

$$x + 2(z) + 2z = 0$$

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$$x + 2z + 2z = 0$$
$$x + 4z = 0$$
$$x = -4z$$
$$w_3 = (x, y, z) = (-4z, z, z) = z(-4, 1, 1)$$
Let, $z = t \in R$
$$w_3 = t(-4, 1, 1)$$

Choose $t = 1$

$$w_3 = (-4, 1, 1)$$
$$v_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\|w_3\|} . w_3$$

$$\|w_3\| = \sqrt{(-4)^2 + (1)^2 + (1)^2}$$
$$= \sqrt{16 + 1 + 1}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

$$v_3 = \frac{1}{3\sqrt{2}} . (-4, 1, 1)$$
$$v_3 = \left(\frac{-4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$$

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -4/3\sqrt{2} & 1/3\sqrt{2} & 1/3\sqrt{2} \end{bmatrix}$$