9

FUNCTIONS OF SEVERAL VARIABLES

The idea of a function of several variables was introduced in Chapter 2. The concepts of limit and continuity of functions of two variables were studied there. Definitions of partial derivatives of first and higher orders were also discussed.

In this chapter, we shall study further properties and some applications of partial derivatives. Volumes and surfaces are other topics of this chapter.

Homogeneous Functions

(9.1) **Definition.** A function f defined by

$$u = f(x, y, z, \ldots)$$

of any number of variables is said to be **homogeneous of degree** n in these variables if multiplication of these variables by any number t $(\neq 0)$ results in the multiplication of the function by t^n, i.e.,

$$f(tx, ty, tz, \ldots) = t^n f(x, y, z, \ldots) \tag{1}$$

provided that $(tx, ty, tz, \ldots)$ is in the domain of f.

Taking $t = \dfrac{1}{x}$, $(x \neq 0)$ the equation (1) becomes

$$f\left(1, \frac{y}{x}, \frac{z}{x} \ldots\right) = \frac{1}{x^n} f(x, y, z, \ldots)$$

or

$$f(x, y, z, \ldots) = x^n f\left(1, \frac{y}{x}, \frac{z}{x} \ldots\right)$$

$$= x^n g\left(\frac{y}{x}, \frac{z}{x} \ldots\right)$$

which is another criterion for a function to be homogeneous.

- Homogeneous Function:

A function $f(x,y)$ is called homogeneous of degree n if

$$f(tx, ty) = t^n f(x,y) \quad \forall \; t \in \mathbb{R}^+ - \{0\}$$

Example:

$$f(x,y) = \frac{x^2+y^2}{x^2-y^2}$$

$$f(tx, ty) = \frac{t^2x^2 + t^2y^2}{t^2x^2 - t^2y^2}$$

$$= \frac{\cancel{t^2}(x^2+y^2)}{\cancel{t^2}(x^2-y^2)}$$

$$= t^0 f(x,y)$$

Date:

$\Rightarrow$ $f(x,y)$ is homogeneous function of degree zero.

Example:

$$f(x,y) = x^3 + y^3$$

$$f(tx, ty) = t^3x^3 + t^3y^3$$

$$= t^3(x^3+y^3)$$

$$= t^3 f(x,y)$$

$\Rightarrow$ $f(x,y)$ is homogeneous function of degree 3.

Example:

$$f(x,y) = \frac{x}{y} + \frac{3}{4}\frac{y}{x} + \cos\sqrt{\frac{y}{x}} + \ln x - \ln y$$

$$f(tx, ty) = \frac{tx}{ty} + \frac{3}{4}\frac{ty}{tx} + \cos\sqrt{\frac{ty}{tx}} + \ln tx - \ln ty$$

$$= \frac{\cancel{t}x}{\cancel{t}y} + \frac{3}{4}\frac{\cancel{t}y}{\cancel{t}x} + \cos\sqrt{\frac{\cancel{t}y}{\cancel{t}x}} + \ln tx - \ln ty$$

$$= \frac{x}{y} + \frac{3}{4}\frac{y}{x} + \cos\sqrt{\frac{y}{x}} + \ln tx - \ln ty$$

$$= \frac{x}{y} + \frac{3}{4}\frac{y}{x} + \cos\sqrt{\frac{y}{x}} + \ln t + \ln x - [\ln t + \ln y]$$

$$= \frac{x}{y} + \frac{3}{4}\frac{y}{x} + \cos\sqrt{\frac{y}{x}} + \cancel{\ln t} + \ln x - \cancel{\ln t} - \ln y$$

$$= \frac{x}{y} + \frac{3}{4}\frac{y}{x} + \cos\sqrt{\frac{y}{x}} + \ln x - \ln y$$

$f(x,y)$ is homogeneous function of degree zero.

• Euler's Theorem:

If $u = f(x,y)$ is homogeneous function of x, y of degree n, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu.$$

★ Example:

Verify Euler's theorem for

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y} - \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{y}{x^2}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} - \frac{\cancel{x^2}}{x^2+y^2} \cdot \frac{y}{\cancel{x^2}}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2} \longrightarrow ①$$

$$= \frac{x^2+y^2 - x^2(\sqrt{y^2-x^2})}{(x^2+y^2)\sqrt{y^2-x^2}}$$

$$\frac{\partial U}{\partial y} = \frac{-x}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y^2} + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{-x\cancel{y}}{y^{\cancel{2}}\sqrt{y^2-x^2}} + \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x^{\cancel{2}}}{x^2+y^2} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{-x}{y(\sqrt{y^2-x^2})} + \frac{x}{x^2+y^2} \rightarrow ②$$

Using ① & ② in ③

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = nu$$

$$x\left(\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}\right) + y\left(\frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}\right) = nu.$$

$$\cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{xy}{y\sqrt{y^2-x^2}}} + \cancel{\frac{yx}{x^2+y^2}} = nu.$$

Now

nu.

Date:

$$f(x,y) = \sin^{-1}(x/y) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$f(tx,ty) = \sin^{-1}\left(\frac{tx}{ty}\right) + \tan^{-1}\left(\frac{ty}{tx}\right)$$

$$= \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$$

$$= t^{0}$$

By use homogeneous of degree 0.

So

L.H.S = R.H.S

0 = 0 Proved.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$0 = 0$$

Example 1. Consider the function f defined by

$$f(x, y) = \frac{x}{y} + \frac{3}{4}\frac{y}{x} + \cos\sqrt{\frac{y}{x}} + \ln x - \ln y.$$

Here,
$$\begin{aligned} f(tx, ty) &= \frac{tx}{ty} + \frac{3}{4}\frac{ty}{tx} + \cos\sqrt{\frac{ty}{tx}} + \ln t + \ln x - \ln t - \ln y \\ &= f(x, y) = t^0 f(x, y) \end{aligned}$$

Thus f is a homogeneous function of degree zero.

Example 2. Let $f(x, y) = \dfrac{\sqrt{y} + \sqrt{x}}{y + x}$.

Here,
$$\frac{\sqrt{y}+\sqrt{x}}{y+x} = \frac{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}{x\left[1+\frac{y}{x}\right]} = x^{-1/2}\,\frac{1+\sqrt{\frac{y}{x}}}{1+\frac{y}{x}}.$$

Thus $f(x, y)$ is a homogeneous function of degree $-\dfrac{1}{2}$.

(9.2) Theorem (*Euler's). If $u = f(x, y)$ is a homogeneous function of x, y of degree n, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu.$

Proof. We have $u = f(x, y) = x^n g\left(\dfrac{y}{x}\right)$. Therefore,

$$\begin{aligned} \frac{\partial u}{\partial x} &= n x^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \\ &= n x^{n-1} g\left(\frac{y}{x}\right) - y x^{n-2} g'\left(\frac{y}{x}\right) \qquad (1) \end{aligned}$$

and
$$\frac{\partial u}{\partial y} = x^n g'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} g'\left(\frac{y}{x}\right) \qquad (2)$$

Thus from (1) and (2), we have

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right) \\ &= n x^n g\left(\frac{y}{x}\right) = nu = nf(x, y). \end{aligned}$$

Example 3. If $u = \arctan\left(\dfrac{x^3 + y^3}{x - y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

---

*Pronounced 'Oiler'. A Swiss mathematician (1707 – 1783)

Solution. We have $z = \tan u = \dfrac{x^3+y^3}{x-y} = x^2\dfrac{1+\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}$

Thus z is a homogeneous function of x, y of degree 2.

Therefore, $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z, \text{ by (9.2)} \tag{1}$$

But $$\frac{\partial z}{\partial x} = \sec^2 u\,\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \sec^2 u\,\frac{\partial u}{\partial y}$$

Substituting into (1), we get

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2z = 2\tan u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u}\cos^2 u = 2\sin u\cos u = \sin 2u.$$

Exercise Set 9.1

1. Verify Euler's Theorem for

(a) $u = \arcsin\left(\dfrac{x}{y}\right) + \arctan\left(\dfrac{y}{x}\right)$

(b) $u = x^n \ln\left(\dfrac{y}{x}\right)$ (c) $u = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

2. If $u = f\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

3. If $u = xy\,f\left(\dfrac{x}{y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

4. If $z = \arctan\left(\dfrac{y}{x}\right)$, verify that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

5. If $u = \arcsin\left(\dfrac{x^2+y^2}{x+y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$