Linear Algebra

(Week 09-13)

Lecture 2

15

Let $\Delta = \begin{vmatrix} 9 & 93 & 12 & -6 \\ 1 & 92 & 84 & -6 \\ 2 & 185 & 100 & -12 \\ 4 & 270 & 196 & -24 \end{vmatrix}$

taking -6 Common from C_4

$$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ 1 & 92 & 84 & 1 \\ 2 & 185 & 100 & 2 \\ 4 & 270 & 196 & 4 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 9 & 93 & 12 & 1 \\ -8 & -1 & 72 & 0 \\ -16 & -1 & 76 & 0 \\ -32 & -102 & 148 & 0 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix}$$

Expanding from C_4

$$= -6 \begin{vmatrix} -8 & -1 & 72 \\ -16 & -1 & 76 \\ -32 & -102 & 148 \end{vmatrix}$$

taking $-8, -1, 4$ Common from C_1, C_2, C_3

$$= (-6)(-8)(-1)(4) \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$$

$$= -192 \begin{vmatrix} 1 & 1 & 18 \\ 2 & 1 & 19 \\ 4 & 102 & 37 \end{vmatrix}$$

$$= -192 \begin{vmatrix} 1 & 1 & 18 \\ 0 & -1 & -17 \\ 0 & 98 & -35 \end{vmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

Expanding from C_1

$$= -192 \begin{vmatrix} -1 & -17 \\ 98 & -35 \end{vmatrix}$$

$$\Delta = -192(35+1666)$$

$$= -192(1701)$$

$$\Delta = -326592$$

(v) $$\begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$$

Sol.

Let $$\Delta = \begin{vmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 \\ -1 & 2 & -2 & 0 & -2 \\ 1 & -2 & 0 & 0 & 2 \\ -1 & 0 & -2 & 0 & 0 \end{vmatrix} \qquad \begin{matrix} C_2 - C_1 \\ C_3 + C_1 \\ C_4 + C_1 \\ C_5 + C_1 \end{matrix}$$

$$= 0 \qquad (\because C_4 = 0)$$

So $\Delta = 0$

Q5 Without expanding, show that

(i) $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$$

Sol.

Sol:

Consider $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$

$= \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix}$ R_{12}

$= \begin{vmatrix} e & d & f \\ b & a & c \\ h & g & k \end{vmatrix}$ C_{12}

$= \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$ By taking transpose

So $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

(ii) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

Sol:

Let $\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$ By taking transpose

$= (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ By taking -1 Common from R_1, R_2, R_3

$\Delta = -\Delta$

$\Delta + \Delta = 0$

$$2\Delta = 0$$

$$\Delta = 0$$

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & c & 1 \\ b+c+a & a & 1 \\ c+a+b & b & 1 \end{vmatrix} \qquad C_1 + C_2$$

$$= \begin{vmatrix} a+b+c & c & 1 \\ a+b+c & a & 1 \\ a+b+c & b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & c & 1 \\ 1 & a & 1 \\ 1 & b & 1 \end{vmatrix} \qquad \text{taking } a+b+c \text{ common from } C_1$$

$$= (a+b+c)(0) \qquad \because C_1 = C_3$$

$$\Delta = 0$$

or $$\begin{vmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{vmatrix} = 0$$

Q6 Prove that

(i) $$\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix} \qquad \text{Multiplying } R_2 \text{ by } abc$$

$$= \frac{1}{abc}(0) \qquad \because R_1 = R_2$$

$$\Delta = 0$$

So $$\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Sol:-

$$\text{Let } \Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

2.

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} \qquad C_1 + (C_2+C_3)$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$$\Delta = 0 \qquad \because C_1 = 0$$

So $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

(iii) $\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = 0$

Sol.

let $\Delta = \begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a & a^2 & a^2 \\ b & b^2 & b^2 \\ c & c^2 & c^2 \end{vmatrix} \qquad \text{Multiplying } C_3 \text{ by } abc$$

$$\Delta = \frac{1}{abc}(0)$$

$$\Delta = 0$$

S.
$$\begin{vmatrix} a & a^2 & a/bc \\ b & b^2 & b/ca \\ c & c^2 & c/ab \end{vmatrix} = 0$$

(iv)
$$\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = 0$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2\theta+\cos^2\theta & 1 & \cos^2\theta \\ \sin^2\phi+\cos^2\phi & 1 & \cos^2\phi \\ \sin^2\psi+\cos^2\psi & 1 & \cos^2\psi \end{vmatrix} \qquad R_1+R_2$$

$$= \begin{vmatrix} 1 & 1 & \cos^2\theta \\ 1 & 1 & \cos^2\phi \\ 1 & 1 & \cos^2\psi \end{vmatrix}$$

$$\Delta = 0 \qquad \because C_1 = C_2$$

or
$$\begin{vmatrix} \sin^2\theta & 1 & \cos^2\theta \\ \sin^2\phi & 1 & \cos^2\phi \\ \sin^2\psi & 1 & \cos^2\psi \end{vmatrix} = 0$$

(v)
$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} = 0$$

Sol.

Let $\Delta = \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$

$$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha - \sin^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta - \sin^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma - \sin^2\gamma \end{vmatrix} \qquad C_3 - C_1$$

$$= \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos 2\alpha \\ \sin^2\beta & \cos 2\beta & \cos 2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos 2\gamma \end{vmatrix} \qquad \because \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$\Delta = 0 \qquad \because C_2 = C_3$

So

$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} = 0$$

(vi)

$$\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

<u>Sol.</u>

Let $\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix}$

or

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha\cos\delta \\ \cos\beta & \sin\beta & \sin\beta\cos\delta \\ \cos\gamma & \sin\gamma & \sin\gamma\cos\delta \end{vmatrix} + \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha\sin\delta \\ \cos\beta & \sin\beta & \cos\beta\sin\delta \\ \cos\gamma & \sin\gamma & \cos\gamma\sin\delta \end{vmatrix}$$

$$= \cos\delta \begin{vmatrix} \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & \sin\beta & \sin\beta \\ \cos\gamma & \sin\gamma & \sin\gamma \end{vmatrix} + \sin\delta \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \cos\beta & \sin\beta & \cos\beta \\ \cos\gamma & \sin\gamma & \cos\gamma \end{vmatrix}$$

$= \cos\delta(0) + \sin\delta(0)$ (∵ two Clumns are identical)

$= 0 + 0$

$\Delta = 0$

So

$$\begin{vmatrix} \cos\alpha & \sin\alpha & \sin(\alpha+\delta) \\ \cos\beta & \sin\beta & \sin(\beta+\delta) \\ \cos\gamma & \sin\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

(vii)

$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix} = 0$$

Sol.

$$\Delta = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & 1-\cos^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & 1-\cos^2\beta \end{vmatrix} \qquad \begin{matrix} C_2 - (\cos\alpha)C_1 \\ C_3 - (\cos\beta)C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin^2\alpha & -\sin\alpha\sin\beta \\ \cos\beta & -\sin\alpha\sin\beta & \sin^2\beta \end{vmatrix}$$

$$= (\sin\alpha)(-\sin\beta)\begin{vmatrix} 1 & 0 & 0 \\ \cos\alpha & \sin\alpha & \sin\alpha \\ \cos\beta & -\sin\beta & -\sin\beta \end{vmatrix}$$ taking $\sin\alpha$ common from C_2 & $-\sin\beta$ common from C_3

$$= -\sin\alpha\sin\beta(0) \qquad \because C_2 = C_3$$

$$\Delta = 0$$

So $$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos(\alpha+\beta) \\ \cos\beta & \cos(\alpha+\beta) & 1 \end{vmatrix} = 0$$

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(viii) $$\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = 0$$

Sol.

Let $$\Delta = \begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+2ab & a^2+b^2 & ab \\ c^2+d^2+2cd & c^2+d^2 & cd \\ g^2+h^2+2gh & g^2+h^2 & gh \end{vmatrix}$$

$$= \begin{vmatrix} 2ab & a^2+b^2 & ab \\ 2cd & c^2+d^2 & cd \\ 2gh & g^2+h^2 & gh \end{vmatrix} \qquad C_1 - C_2$$

$$= 2\begin{vmatrix} ab & a^2+b^2 & ab \\ cd & c^2+d^2 & cd \\ gh & g^2+h^2 & gh \end{vmatrix} \qquad \text{taking 2 common from } C_1$$

$$= 2(0) \qquad (\because C_1 = C_3)$$

$$\Delta = 0$$

So $$\begin{vmatrix} (a+b)^2 & a^2+b^2 & ab \\ (c+d)^2 & c^2+d^2 & cd \\ (g+h)^2 & g^2+h^2 & gh \end{vmatrix} = 0$$

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(ix) $$\begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix} = 0$$

Soln

Let $$\Delta = \begin{vmatrix} (a^m+a^{-m})^2 & (a^m-a^{-m})^2 & abc \\ (b^n+b^{-n})^2 & (b^n-b^{-n})^2 & abc \\ (c^p+c^{-p})^2 & (c^p-c^{-p})^2 & abc \end{vmatrix}$$

$$= abc\begin{vmatrix} a^{2m}+a^{-2m}+2 & a^{2m}+a^{-2m}-2 & 1 \\ b^{2n}+b^{-2n}+2 & a^{2m}+a^{-2m}-2 & 1 \\ c^{2p}+c^{-2p}+2 & c^{2p}+c^{-2p}-2 & 1 \end{vmatrix}$$

$$= abc\begin{vmatrix} a^{2m}+a^{-2m} & a^{2m}+a^{-2m} & 1 \\ b^{2n}+b^{-2n} & b^{2n}+b^{-2n} & 1 \\ c^{2p}+c^{-2p} & c^{2p}+c^{-2p} & 1 \end{vmatrix} \qquad \begin{matrix} C_1-2C_3 \\ C_2+2C_3 \end{matrix}$$

$$= (abc)(0) \qquad \because C_1 = C_2$$

$$\Delta = 0$$

(x) $$\begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} = 0$$

Soln

let $$\Delta = \begin{vmatrix} \frac{1}{2!} & 1 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} \end{vmatrix}$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{24}\right) \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$$ taking $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}$ common from R_1, R_2, R_3

$$= \frac{1}{288} \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 6 \\ 1 & 4 & 12 \end{vmatrix}$$

$$= \frac{1}{288} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 1 & 2 & 12 \end{vmatrix}$$

$$= \frac{6}{288} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$ taking 6 common from C_3

$$= \frac{1}{48}(0)$$ $\because C_2 = C_3$

$$\Delta = 0$$

(xi)

$$\begin{vmatrix} a^2 & b\sin\alpha & c\sin\alpha \\ b\sin\alpha & 1 & \cos\alpha \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix} = 0$$

where a, b, c are the magnitudes of the sides of a triangle & α is the measure of the angle opposite to the side with magnitude a

$$= \begin{vmatrix} a^2 - c^2 \sin^2\alpha & b\sin\alpha - c\sin\alpha\cos\alpha & 0 \\ b\sin\alpha - c\sin\alpha\cos\alpha & 1 - \cos^2\alpha & 0 \\ c\sin\alpha & \cos\alpha & 1 \end{vmatrix} \qquad \begin{matrix} R_1 - c\sin\alpha\, R_3 \\ R_2 - \cos\alpha\, R_3 \end{matrix}$$

Expanding from C_3

$$= \begin{vmatrix} a^2 - c^2\sin^2\alpha & \sin\alpha(b - c\cos\alpha) \\ \sin\alpha(b - \cos\alpha) & \sin^2\alpha \end{vmatrix}$$

$$= \sin^2\alpha(a^2 - c^2\sin^2\alpha) - \sin^2\alpha(b - c\cos\alpha)^2$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - \sin^2\alpha(b^2 + c^2\cos^2\alpha - 2bc\cos\alpha)$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - b^2\sin^2\alpha - c^2\sin^2\alpha\cos^2\alpha + 2bc\sin^2\alpha\cos\alpha$$

$$= a^2\sin^2\alpha - c^2\sin^4\alpha - b^2\sin^2\alpha - c^2\sin^2\alpha(1 - \sin^2\alpha) + 2bc\sin^2\alpha\cos\alpha$$

$$= a^2\sin^2\alpha - \cancel{c^2\sin^4\alpha} - b^2\sin^2\alpha - c^2\sin^2\alpha + \cancel{c^2\sin^4\alpha} + 2bc\sin^2\alpha\cos\alpha$$

$$= [a^2 - b^2 - c^2 + 2bc\cos\alpha]\sin^2\alpha$$

$$= \left[a^2 - b^2 - c^2 + 2bc\left(\frac{b^2 + c^2 - a^2}{2bc}\right)\right]\sin^2\alpha$$

$$= [\cancel{a^2} - \cancel{b^2} - \cancel{c^2} + \cancel{b^2} + \cancel{c^2} - \cancel{a^2}]\sin^2\alpha$$

$$= (0)\sin^2\alpha$$

$$\Delta = 0$$

(xii)

$$\begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix} = 0$$

Sol:-

Let $\Delta = \begin{vmatrix} a & b & c & d & 1 \\ b & c & d & a & 1 \\ c & d & a & b & 1 \\ d & a & b & c & 1 \\ b & a & d & c & 1 \end{vmatrix}$

$$= \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ b+c+d+a & c & d & a & 1 \\ c+d+a+b & d & a & b & 1 \\ d+a+b+c & a & b & c & 1 \\ b+a+d+c & a & d & c & 1 \end{vmatrix} \qquad C_1 + (C_2+C_3+C_4)$$

$$= \begin{vmatrix} a+b+c+d & b & c & d & 1 \\ a+b+c+d & c & d & a & 1 \\ a+b+c+d & d & a & b & 1 \\ a+b+c+d & a & b & c & 1 \\ a+b+c+d & a & d & c & 1 \end{vmatrix}$$

$$= (a+b+c+d) \begin{vmatrix} 1 & b & c & d & 1 \\ 1 & c & d & a & 1 \\ 1 & d & a & b & 1 \\ 1 & a & b & c & 1 \\ 1 & a & d & c & 1 \end{vmatrix}$$

$= (a+b+c+d)(0) \qquad \because C_1 = C_5$

$= 0$

So $\Delta = 0$

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(xiii) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$

Sol.

Let $\Delta = \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & a^2+2a+1 & a^2+4a+4 & a^2+6a+9 \\ b^2 & b^2+2b+1 & b^2+4b+4 & b^2+6b+9 \\ c^2 & c^2+2c+1 & c^2+4c+4 & c^2+6c+9 \\ d^2 & d^2+2d+1 & d^2+4d+4 & d^2+6d+9 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - C_1 \end{matrix}$$

$$= \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} \quad \begin{matrix} C_3 - 2C_2 \\ C_4 - 3C_2 \end{matrix}$$

$$= 3\begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} \quad \text{taking 3 Common from } C_4$$

$\Delta = 0 \qquad \because C_3 = C_4$

Q7 Without expansion, prove that

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Sol.

Let $\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$

$= \dfrac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$ Multiplying R_1, R_2, R_3 by a, b, c

$= \dfrac{\cancel{abc}}{\cancel{abc}}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ taking abc common from C_1

$\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

So

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Q8 Prove that

$$\begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$$

Sol.

Let $\Delta = \begin{vmatrix} 1 & \alpha & \beta\gamma \\ 1 & \beta & \gamma\alpha \\ 1 & \gamma & \alpha\beta \end{vmatrix}$

Q9 Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

Sol.

$$\text{Let } \Delta = \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$$

$$= \frac{1}{abcd}\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ -ab & ab & -cd & cd \\ -ac & bd & ac & -bd \\ -ad & -bc & bc & ad \end{vmatrix}$$ Multiplying C_1, C_2, C_3, C_4 by a, b, c, d resp.

$$= \frac{1}{abcd}\begin{vmatrix} a^2+b^2+c^2+d^2 & b^2 & c^2 & d^2 \\ 0 & ab & -cd & cd \\ 0 & bd & ac & -bd \\ 0 & -bc & bc & ad \end{vmatrix}$$ $C_1 + (C_2+C_3+C_4)$

Expanding from C_1

$$= \frac{(a^2+b^2+c^2+d^2)}{abcd}\begin{vmatrix} ab & -cd & cd \\ bd & ac & -bd \\ -bc & bc & ad \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2+d^2)\cancel{bcd}}{a\cancel{bcd}}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$ taking b, c, d common from C_1, C_2, C_3

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\begin{vmatrix} a & -d & c \\ d & a & -b \\ -c & b & a \end{vmatrix}$$

Expanding from R_1

$$\Delta = \frac{(a^2+b^2+c^2+d^2)}{a}\left\{a(a^2+b^2)+d(ad-bc)+c(bd+ac)\right\}$$

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\left\{a^3+ab^2+ad^2-\cancel{bc}d+\cancel{bc}d+ac^2\right\}$$

$$= (a^2+b^2+c^2+d^2)(a^2+b^2+c^2+d^2)$$

$$\Delta = (a^2+b^2+c^2+d^2)^2$$

<u>Q10</u> Prove that

(i) $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

<u>Sol.</u>

Let $$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix} \qquad \begin{matrix} C_1-C_3 \\ C_2-C_3 \end{matrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$\Delta = (a+b+c)^2\begin{vmatrix}(b+c-a) & 0 & a^2\\ 0 & (c+a-b) & b^2\\ (c-a-b) & (c-a-b) & (a+b)^2\end{vmatrix}$$ taking $(a+b+c)$ Common from C_1 & C_2

$$= (a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ -2b & -2a & 2ab\end{vmatrix}$$ $R_3-(R_1+R_2)$

$$= -2(a+b+c)^2\begin{vmatrix}b+c-a & 0 & a^2\\ 0 & c+a-b & b^2\\ b & a & -ab\end{vmatrix}$$

Expanding from R_1

$$= -2(a+b+c)^2\left[(b+c-a)\left\{(-ab)(c+a-b)-ab^2\right\}-0+a^2\left\{0-b(c+a-b)\right\}\right]$$

$$= -2(a+b+c)^2\left\{(b+c-a)(-abc-a^2b+\cancel{ab^2}-\cancel{ab^2})+a^2(-bc-ab+b^2)\right\}$$

$$= -2(a+b+c)^2\left\{(-ab)(b+c-a)(c+a)+(-ab)(ac+a^2-ab)\right\}$$

$$= -2(a+b+c)^2(-ab)\left\{(b+c-a)(c+a)+ac+a^2-ab\right\}$$

$$= 2ab(a+b+c)^2\left\{bc+\cancel{ab}+c^2+\cancel{ac}-\cancel{ac}-\cancel{a^2}+ac+\cancel{a^2}-\cancel{ab}\right\}$$

$$= 2ab(a+b+c)^2(bc+c^2+ac)$$

$$= 2abc(a+b+c)^2(b+c+a)$$

$$\Delta = 2abc(a+b+c)^3$$

(ii)
$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

Sol:

$$\text{Let } \Delta = \begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$ Multiplying R_1, R_2, R_3 by c, a, b resp.

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ b^2-c^2-a^2 & b^2-c^2-a^2 & c^2+a^2 \end{vmatrix}$$ C_1-C_3, C_2-C_3

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ -2a^2 & -2c^2 & 0 \end{vmatrix}$$ $R_3-(R_1+R_2)$

$$= \frac{-2}{abc}\begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ a^2 & c^2 & 0 \end{vmatrix}$$ taking -2 Common from R_3

Expanding from R_1

$$= -\frac{2}{abc}\left\{(a^2+b^2-c^2)(0-a^2c^2) - 0 + c^2\left(0 - a^2(b^2+c^2-a^2)\right)\right\}$$

$$-\frac{2}{abc}\left\{-a^2c^2(a^2+b^2-c^2) - a^2c^2(b^2+c^2-a^2)\right\}$$