These results can be generalized for a function of several variables. Thus if $u = f(x, y, z, \cdots)$ is a differentiable function and each of $x, y, z, \cdots$ is a differentiable function of several variables $r, s, t, \cdots$, then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} + \cdots$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} + \cdots$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \cdots$$

These equations are known as the **Chain Rule.**

Example 7. A closed box of inner dimensions 2, 3 and 4 decimetres is to be made of metal sheet 1 cm thick. Approximate the volume of the metal using differentials.

Solution. If x, y and z are dimensions of the box, then its volume is

$$V(x, y, z) = xyz$$

For $x = 2,\ y = 3,\ z = 4,\ \Delta x = \Delta y = \Delta z = \left(\frac{1}{10}\right) \times 2 = \frac{1}{2}$ decimetre, the exact volume of the metal is

$$V(x + \Delta x, y + \Delta y, z + \Delta z) - V(x, y, z) = \Delta V.$$

Since $\Delta V \approx dV$ and $\Delta x = dx$, $\Delta y = dy$, $\Delta z = dz$ we shall use differentials.

$$\Delta V \approx dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{dz}dz$$

$$= yz\,dx + zx\,dy + xy\,dz$$

$$= 3(4)\frac{1}{5} + 4(2)\frac{1}{5} + 2(3)\frac{1}{5} = \frac{26}{5} \text{ cu dm.}$$

Example 8. Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y,\ x = at^2,\ y = 2at$

Solution. $\frac{\partial z}{\partial x} = y^2 + 2xy,\ \frac{\partial z}{\partial y} = 2xy + x^2,\ \frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a$

Therefore, $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2 + 2xy)\,2at + (2xy + x^2)\,2a$$

$$= (4a^2t^2 + 4a^2t^3)\,2at + (4a^2t^3 + a^2t^4)\,2a = a^3(16t^3 + 10t^4)$$

Example 9. Let $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$. Show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution.
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$= \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$
$$= -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^v$$

Thus, $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Implicit Functions

(9.8) Theorem. Let $y = F(x)$ be a differentiable function defined by the equation $f(x, y) = 0$, where $f(x, y)$ is differentiable and $f_y(x, y) \neq 0$. Then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Proof. Let $w = f(x, y)$ and let x and y be considered as function of the new independent variable t; $x = t$, $y = F(t)$. Then w, as a function of t, is differentiable and by the chain rule

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$0 = \frac{\partial f}{\partial x}1 + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Thus, $$\frac{dy}{dt} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = -\frac{f_x}{f_y} = \frac{dy}{dx}, \quad (\text{since } x = t)$$

Example 10. Find the slope of the tangent to the hyperbola

$$x^2 - 4xy - 3y^2 = 9 \text{ at the point } (2, -1)$$

Solution. $$f(x, y) = x^2 - 4xy - 3y^2 - 9 = 0$$

Therefore, $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x - 4y}{-4x - 6y} = \frac{x - 2y}{2x + 3y} = \frac{4}{1} = 4 \text{ at } (2, -1)$$

Exercise Set 9.3

1. If $u = x - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$, find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$

2. If $z = \frac{\cos y}{x}$, $x = u^2 - v$, $y = e^v$, find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$.

Find $\frac{dy}{dx}$ **(Problems 3 – 6):**

3. $\sin xy - e^{xy} - x^2 y = 0$
4. $3(x^2 + y^2)^2 = 25(x^2 - y^2)$
5. $f(x, y) = x^y - y^x = 0$
6. $(\tan x)^y + y^{\cot x} = a$
7. If $F(x, y, z) = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
8. If $f(x, y) = 0$ and $\phi(y, z) = 0$, show that
$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$
9. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$, show that $\frac{d^2y}{dx^2} = \frac{-a}{(1 - x^2)^{3/2}}$
10. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that
$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$$
11. Find $\frac{d^2y}{dx^2}$ if $x^3 + y^3 = 3axy$.

Directional Derivatives

(9.9) Definition. Let $u = f(x, y, z)$ be defined in a region D of R^3. Let P be a point of D. Let Δs denote measure of a displacement of P in a specified direction. Let Δu denote the corresponding change in u.

Then $\lim\limits_{\Delta s \to 0} \frac{\Delta u}{\Delta s}$, if it exists, is called the **derivative** of u at P in the **specified direction** and is denoted by $\frac{du}{ds}$.

(9.10) Let u be as in the above definition. Suppose the displacement $\Delta s = [\Delta x, \Delta y, \Delta z]$ is in the direction of the vector $\mathbf{v} = [a, b, c]$. Then, for some real k,
$$[\Delta x, \Delta y, \Delta z] = k[a, b, c] \text{ and so } \Delta x = ka, \Delta y = kb, \Delta z = kc$$

Exercise Set 9.3 (Page 418)

1. If $u = x - y^2, x = 2r - 3s + 4, y = -r + 8s - 5$, find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$

Sol. We know that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = (1)(2) + (-2y)(-1) = 2(1+y)$$

Again, $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} = (1)(-3) + (-2y)(8) = -(3+16y).$

2. If $z = \frac{\cos y}{x}, x = u^2 - v, y = e^v$, find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$

Sol. We have $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$$= \frac{-\cos y}{x^2}(2u) + \frac{-\sin y}{x}\cdot 0 = \frac{-2u.\cos y}{x^2}$$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$$= \frac{-\cos y}{x^2}.(-1) + \frac{-\sin y}{x}.e^v = \frac{1}{x^2}[\cos y - x e^v \sin y]$$

$$= \frac{1}{x^2}[\cos y - xy \sin y], \quad \text{since } y = e^v.$$

Find $\frac{dy}{dx}$ (Problems 3 – 6):

3. $\sin xy - e^{xy} - x^2y = 0$

Sol. Here $f(x, y) = \sin xy - e^{xy} - x^2y = 0$

$$f_x = y\cos xy - ye^{yx} - 2xy$$

$$f_y = x\cos xy - xe^{xy} - x^2$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{(y\cos xy - ye^{xy} - 2xy)}{x\cos xy - xe^{xy} - x^2} = \frac{y(\cos xy - e^{xy} - 2x)}{x(x + e^{xy} - \cos xy)}$$

4. $3(x^2+y^2)^2 = 25(x^2-y^2)$

Sol. $f(x, y) = 3(x^2+y^2)^2 - 25(x^2-y^2) = 0$

$$f_x = 6(x^2+y^2).2x - 50x$$

$$f_x = 6(x^2+y^2).2y + 50y$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{12x(x^2+y^2) - 50x}{12y(x^2+y^2) + 50y} = \frac{25x - 6x(x^2+y^2)}{25y + 6y(x^2+y^2)}$$

5. $f(x, y) = x^y - y^x = 0$