Taylor's theorem with Lagrange's form of remainder after n terms :-

Statement If f is a function s.that

(i) $f, f', f'', \ldots\ldots, f^{n-1}$ are continous on $[a, a+h]$

(ii) f^n exists in $]a, a+h[$.

Then there exists a real no. θ where $0<\theta<1$ s.that
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h).$$

Proof :- Define a function :
$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{(a+h-x)^n}{n!}A$$

where A is a const. to be determined s.t. $\phi(a) = \phi(a+h)$.

From $\phi(a) = \phi(a+h)$, we have
$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}A = f(a+h) \quad \text{---} (1)$$

Obviously the function $\phi(x)$ satisfies all the conditions of Rolle's theoram on $[a, a+h]$.

There exists a real no. θ, $0<\theta<1$ s. that
$$\phi'(a+\theta h) = 0$$

Now.
$$\phi'(x) = \cancel{f'(x)} - \cancel{f'(x)} + (a+h-x)f''(x) - (a+h-x)f''(x) + \frac{(a+h-x)^2}{2!}f'''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} - \frac{(a+h-x)^{n-1}}{(n-1)!}A.$$

or $\phi'(x) = (a+h-x)f''(x) - (a+h-x)f''(x) + \frac{(a+h-x)^2}{2!}f'''(a) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}\left[f^n(x) - A\right]$

Now

$$\phi'(a+\theta h) = \frac{(h-\theta h)^{n-1}}{(n-1)!}\left[f^n(a+\theta h) - A\right] = 0$$

or $$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}\left[f^n(a+\theta h) - A\right] = 0$$

Since $h \neq 0$, $1-\theta \neq 0$

$\Rightarrow f^n(a+\theta h) - A = 0 \Rightarrow A = f^n(a+\theta h)$.

So from (i), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h)$$

Corollary: Maclaurin's Development with Lagrange's form of remainder :-

If we take interval $[0, x]$ instead of $[a, a+h]$ then above eq becomes :-

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$$

where $0 < \theta < 1$

Taylor's theorem with Cauchy's form of remainder

Statement :-

If f is a function s. that

① $f, f', f'', \ldots, f^{n-1}$ are continous on $[a, a+h]$

② f^n exists in $]a, a+h[$.

then there exists a real no θ, $0 < \theta < 1$

s.that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

Proof:- Define a function

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots\cdots + (a+h-x)^{n-1}f^{n-1}(x) + (a+h-x)A$$

where A is a const. to be determined s.that $\phi(a) = \phi(a+h)$

from $\phi(a) = \phi(a+h)$ we have

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1} + hA = f(a+h) \quad \text{--- (1)}$$

obviously the function $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Hence there exists a real no θ, $0<\theta<1$ s.that $\phi'(a+\theta h) = 0$ now

$$\phi'(x) = \cancel{f'(x)} - \cancel{f'(x)} + \cancel{(a+h-x)f''(x)} - \cancel{(a+h-x)f''(x)} + \frac{(a+h-x)^2}{2!}f'''(x) + \cdots$$

$$\cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - A$$

now $\phi'(x) = \dfrac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - A$

so $\phi'(a+\theta h) = \dfrac{(h-\theta h)^{n-1}}{(n-1)!}f^n(a+\theta h)$

put value in (1), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f(a+\theta h) \quad \text{--- (2)}$$

where $0<\theta<1$

Corollary :- Maclaurins theoram with Cauchy's form of remainder after n term:-

Taking the interval as $[0, x]$ instead of $[a, a+h]$ then eq. (2) becomes

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

where $0 < \theta < 1$

Taylor's Infinite Series :-

Let function f has continous derivative of every order in $[a, a+h]$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a).$$

then

$$\therefore f(a+h) = S_n + R_n$$

If $R_n \to 0$ as $n \to \infty$, then

$$f(a+h) = \lim_{n \to \infty} S_n$$

So that the infinite series

$$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \cdots\cdots$$ converges

and its sum is equal to $f(a+h)$.

Maclaurin's infinite Series :-

If a function f has continous derivatives of every order in $[0, x]$ & $R_n \to 0$ as $n \to \infty$ then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \cdots\cdots$$

Exercise 3.2

(1) write the Maclaurin formula for the fn $f(x) = \sqrt{1+x}$ with remainder after two terms.

Sol :-

Here $f(x) = (1+x)^{1/2}$ $\Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

or $f'(x) = \frac{1}{2\sqrt{1+x}}$ $\Rightarrow f'(0) = \frac{1}{2}$

Now

$f''(x) = -\frac{1}{4(1+x)^{3/2}}$ $\Rightarrow f''(\theta x) = \frac{-1}{4(1+\theta x)^{3/2}}$

we know that Maclaurin theoram with remainder after two terms is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$$

or $$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{x^2}{2!}\left\{\frac{-1}{4(1+\theta x)^{3/2}}\right\}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{x^2}{8(1+\theta x)^{3/2}}$$

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(2) Find by Maclaurins theoram, the first four terms of the expansion of $f(x) = e^{ax}\cos bx$ and write the remainder after n terms.

Sol :-

Here $f(x) = e^{ax}\cos bx$ $\Rightarrow f(0) = 1$

Then $f^n(x) = (a^2+b^2)^{n/2} e^{ax}\cos(bx + n\theta)$ where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\Rightarrow \tan\theta = \frac{b}{a}$

Now put $n=1$

So $f'(x) = (a^2+b^2)^{1/2} e^{ax}\cos(bx+\theta)$

$\Rightarrow f'(0) = (a^2+b^2)^{1/2}.\cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{\sqrt{a^2+b^2}}}{\frac{a}{\sqrt{a^2+b^2}}}$

$$f'(0) = (a^2+b^2)^{1/2} \frac{a}{\sqrt{a^2+b^2}} = a$$

$$\Rightarrow \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\& \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

For $n=2$

$$f''(x) = (a^2+b^2) e^{ax} \cos(bx+2\theta)$$

$$\Rightarrow \quad f''(0) = (a^2+b^2) \cdot \cos 2\theta$$

$$= (a^2+b^2)\left[\cos^2\theta - \sin^2\theta\right]$$

$$= (a^2+b^2)\left[\frac{a^2}{(\sqrt{a^2+b^2})^2} - \frac{b^2}{(\sqrt{a^2+b^2})^2}\right]$$

$$f''(0) = (a^2+b^2)\left[\frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2}\right]$$

$$\text{so} \quad = (a^2+b^2)\left(\frac{a^2-b^2}{a^2+b^2}\right) = a^2-b^2$$

For $n=3$

$$f'''(x) = (a^2+b^2)^{3/2} e^{ax} \cos(bx+3\theta)$$

$$\text{so} \Rightarrow f'''(0) = (a^2+b^2)^{3/2} \cos 3\theta$$

$$= (a^2+b^2)^{3/2}\left[4\cos^3\theta - 3\cos\theta\right]$$

$$= (a^2+b^2)^{3/2}\left[4\cdot\frac{a^3}{(a^2+b^2)^{3/2}} - \frac{3a}{(a^2+b^2)^{1/2}}\right]$$

$$= (a^2+b^2)^{3/2}\left[\frac{4a^3 - 3a(a^2+b^2)}{(a^2+b^2)^{3/2}}\right]$$

$$= (a^2+b^2)^{3/2}\left[\frac{a^3-3ab^2}{(a^2+b^2)^{3/2}}\right]$$

$$f'''(0) = a^3-3ab^2$$

For $n=n$

$$f^n(x) = (a^2+b^2)^{n/2} \cdot e^{ax} \cos(bx+n\theta)$$

$$f^n(0x) = (a^2+b^2)^{n/2} \cdot e^{a0x} \cos\left(b0x + n\tan^{-1}\left(\frac{b}{a}\right)\right)$$

now we know that Maclaurins theoram with remainder after n terms is 39

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(\theta x)$$

So

$$e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \cdots + \frac{x^n}{n!}(a^2+b^2)^{n/2} e^{a\theta x} \cdot \cos\left(b\theta x + n\tan^{-1}\frac{b}{a}\right)$$

---

(3) Find the expansion of given functions.

(i) $\sin x$

Sol:-

Here $f(x) = \sin x$ $\Rightarrow f(0) = 0$

$f'(x) = \cos x$ $\Rightarrow f'(0) = 1$

$f''(x) = -\sin x$ $\Rightarrow f''(0) = 0$

$f'''(x) = -\cos x$ $\Rightarrow f'''(0) = -1$

$f''''(x) = \sin x$ $\Rightarrow f''''(0) = 0$

$f'''''(x) = \cos x$ $\Rightarrow f'''''(0) = 1$

By Maclaurin's infinite series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \frac{x^5}{5!} f'''''(0) + \cdots$$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

(ii) $\cos x$

Sol:-

Here
$$f(x) = \cos x \Rightarrow f(0) = 1$$
$$f'(x) = -\sin x \Rightarrow f'(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$
$$f'''(x) = \sin x \Rightarrow f'''(0) = 0$$
$$f''''(x) = \cos x \Rightarrow f''''(0) = 1$$

By Maclaurin's infinite series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \cdots$$

$$\cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

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(iii) $\tan x$

Sol:- Here $f(x) = \tan x \quad \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x \quad \Rightarrow f'(0) = 1$

$f'(x) = 1 + \tan^2 x$

$f''(x) = 2\tan x \sec^2 x \qquad f''(0) = 0$

$= 2\tan x(1+\tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 2\tan^2 x + 6\tan^2 x + 6\tan^4 x$

$= 2 + 8\tan^2 x + 6\tan^4 x \quad \Rightarrow f'''(0) = 2$

$f''''(x) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x$

$= 16\tan x(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$

$= 16\tan x + 16\tan^3 x + 24\tan^3 x + 24\tan^5 x$

$= 16\tan x + 40\tan^3 x + 24\tan^5 x \quad \Rightarrow f''''(0) = 0$

& so on

By Maclaurin's infinite series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$\therefore \tan x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \ldots\ldots$$

$$\tan x = x + \frac{x^3}{3} + \ldots\ldots$$

(iv) $\sec x$

Here $f(x) = \sec x \quad \Rightarrow f(0) = 1$

$f'(x) = \sec x \tan x \quad \Rightarrow f'(0) = 0$

$f''(x) = \sec x \sec^2 x + \tan x \sec x \tan x$

$= \sec^3 x + \sec x \tan^2 x$

$= \sec^3 x + \sec x (\sec^2 x - 1)$

$f''(x) = 2\sec^3 x - \sec x \quad \Rightarrow f''(0) = 2 - 1 = 1$

$f'''(x) = 6\sec^2 x \sec x \tan x - \sec x \tan x$

$f'''(x) = 6f^2(x) f'(x) - f'(x) \quad \Rightarrow f'''(0) = 0$

$f''''(x) = 12 f(x) \cdot f'(x) \cdot f'(x) + 6 f^2(x) f''(x) - f''(x)$

$f''''(x) = 12 f(x) (f'(x))^2 + 6f^2(x) f''(x) - f''(x) \quad \Rightarrow f''''(0) = 0 + 6 - 1 = 5$

So on

By Maclaurin's infinite series.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots$

$= 1 + x(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(5) + \cdots\cdots$

$f(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots\cdots$

(v) $e^{\sin x}$

Sol:- Here $f(x) = e^{\sin x} \quad \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^{\sin x} \cos x \quad \Rightarrow f'(0) = 1$

$f''(x) = e^{\sin x} \cos^2 x + e^{\sin x}(-\sin x)$

$f''(x) = e^{\sin x} \cos^2 x - e^{\sin x} \sin x \quad \Rightarrow f''(0) = 1 - 0 = 1$

$f'''(x) = e^{\sin x} \cos x \cdot \cos^2 x + e^{\sin x}(-2\cos x \sin x) - \{e^{\sin x} \cos x \sin x + e^{\sin x} \cos x\}$

$= e^{\sin x} \cos^3 x - 2e^{\sin x} \sin x \cos x - e^{\sin x} \sin x \cos x - e^{\sin x} \cos x$

$$f'''(x) = e^{\sin x}\cos^3 x - 3e^{\sin x}\sin x\cos x - e^{\sin x}\cos x$$

$$\Rightarrow f'''(x) = e^{\sin x}\cos^3 x - \frac{3}{2}e^{\sin x}\sin 2x - e^{\sin x}\cos x \implies f'''(0) = 1 - 1 = 0$$

$$f''''(x) = e^{\sin x}\cos^4 x + e^{\sin x}(-3\cos^2 x\sin x) - \frac{3}{2}\left[e^{\sin x}\cos x\sin 2x + e^{\sin x}\cos 2x \cdot 2\right] - \left[e^{\sin x}\cos^2 x + e^{\sin x}(-\sin x)\right]$$

$$f''''(x) = e^{\sin x}\cos^4 x - 3e^{\sin x}\cos^2 x\sin x - \frac{3}{2}e^{\sin x}\cos x\sin 2x - 3e^{\sin x}\cos 2x - e^{\sin x}\cos^2 x + e^{\sin x}\sin x$$

$$\implies f''''(0) = 1 - 0 - 0 - 3 - 1 + 0 = -3 \quad \text{\& so on}$$

By Maclaurin's infinite series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \ldots\ldots$$

$$= 1 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \ldots\ldots$$

$$e^{\sin x} = 1 + x - \frac{3x^4}{4!} + \ldots\ldots$$

$$e^{\sin x} = 1 + x - \frac{x^4}{8} + \ldots\ldots$$

(vi) $\ln(1-x)$

Sol: Here $f(x) = \ln(1-x) \implies f(0) = 0$

Then we know that

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!\cdot(-1)^n}{(1-x)^n}$$

$$= \frac{(-1)^{2n-1}(n-1)!}{(1-x)^n}$$

So $$f^n(x) = -\frac{(n-1)!}{(1-x)^n} \quad \left(\because 2n-1 \text{ is odd for } \forall n \in Z^+\right)$$

For

$n=1$, $\therefore f'(0) = -1$

$n=2$, $f''(0) = -1$

$n=3$, $f'''(0) = -2$

$n=4$, $f''''(0) = -6$ & so on

By Maclaurin's infinite series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots\dots$$

$$\Rightarrow \ln(1-x) = 0 + x(-1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(-6) + \dots\dots$$

$$\therefore \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots\dots\dots$$

(iii) a^x

Here $f(x) = a^x \Rightarrow f(0) = 1$

$f'(x) = a^x \ln a \Rightarrow f'(0) = \ln a$

$f''(x) = a^x (\ln a)^2 \Rightarrow f''(0) = (\ln a)^2$

$f'''(x) = a^x (\ln a)^3 \Rightarrow f'''(0) = (\ln a)^3$

$f''''(x) = a^x (\ln a)^4 \Rightarrow f''''(0) = (\ln a)^4$

& so on

By Maclaurin's infinite series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \dots\dots$$

$$a^x = 1 + x \ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \frac{x^4}{4!}(\ln a)^4 + \dots\dots$$

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(4) Apply Taylor's theorem to prove that

$(a+b)^m = a^m + \frac{m}{1!}a^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots\cdots$

for all real m, $a>0$ & $-a<b<a$.

Sol :-

By Taylor's theorem $f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!}f''(x) + \cdots\cdots$ —— (1)

Consider $f(x+b) = (x+b)^m$

$\Longrightarrow \quad f(x) = x^m$

$f'(x) = mx^{m-1}$

$f''(x) = m(m-1)x^{m-2}$

$f'''(x) = m(m-1)(m-2)x^{m-3}$

$f^n(x) = m(m-1)(m-2)\cdots\cdots\cdots(m-(n-1))x^{m-n}$

or $f^n(x) = m(m-1)(m-2)\cdots\cdots\cdots(m-n+1)x^{m-n}$

So from (1)

$(x+b)^m = x^m + bmx^{m-1} + \frac{b^2}{2!}m(m-1)x^{m-2} + \cdots\cdots\cdots$

Put $x=a$, so

$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots\cdots$

Here remainder after n terms in Cauchy's form is

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$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h).$$

$$= \frac{b^n}{(n-1)!}(1-\theta)^{n-1}\left[m(m-1)(m-2)\ldots\ldots(m-n+1)(a+\theta h)^{m-n}\right]$$

$$= \frac{b^n}{(n-1)!}(1-\theta)^{n-1}\frac{m(m-1)(m-2)\ldots\ldots(m-n+1)(m-n)\ldots\ldots 3.2.1.(a+\theta h)^{m-n}}{(m-n)\ldots\ldots 3.2.1}$$

$$\therefore R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1}\frac{m!}{(m-n)!}$$

Now $\lim_{n\to\infty} R_n = \dfrac{b^n(1-\theta)^{n-1}\cdot m!}{(n-1)!\cdot(m-n)!} \longrightarrow 0$ for all $m, a>0, -a<b<a$

Hence above series can be expanded into an infinite series as

$$(a+b)^m = a^m + \frac{m}{1!}a^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \ldots\ldots$$

(5) Prove that

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$+ \frac{(x-a)^n}{n!}f^n(a+\overline{x-a}\,\theta)$$

Stating the conditions under which it holds.

Sol :-

Then there exists a real no. θ, $0<\theta<1$ s.t

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(x-a)^n}{n!}f^n(a+\overline{x-a}\theta)$$

Now we prove above theorem.

Let $f(x)=(a+\overline{x-a})$

Expand by Taylor's theoram with Lagranges form of remainder after n terms.

So

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(x-a)^n}{n!}f^n(a+(x-a)\theta)$$

Q) Use Taylor's theorem to prove that

$$\ln\sin(x+h)=\ln\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x+\frac{h^3}{3}\cot x\operatorname{cosec}^2 x+\cdots$$

Sol:- By Taylors theoram

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\quad ①$$

Here $f(x+h)=\ln(\sin(x+h))$

$\Rightarrow f(x)=\ln\sin x$

$\Rightarrow f'(x)=\frac{1}{\sin x}\cdot\cos x=\cot x$

$f''(x)=-\operatorname{cosec}^2 x$

$f'''(x)=-2\operatorname{cosec} x(-\operatorname{cosec} x\cot x)=2\operatorname{cosec}^2 x\cot x$

& so on

Putting these values in eq ①

$$\ln\sin(x+h)=\ln\sin x+h\cot x+\frac{h^2}{2!}(-\operatorname{cosec}^2 x)+\frac{h^3}{3!}(2\operatorname{cosec}^2 x\cot x)+\cdots$$

$\ln \sin(x+h) = \ln \sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3}\cot x \operatorname{cosec}^2 x + \cdots$ 48

Q. prove that under certain conditions (to be stated) $f(a+h) = f(a) + hf'(a+\theta h)$ where $0<\theta<1$. prove also that the limiting value of θ when h decreases indefinitely is $\frac{1}{2}$.

Sol:- If f is continuous in $[a, a+h]$ & f' exists in $]a, a+h[$ then by taylor's theorem with Lagrange's form of remainder after one term

$f(a+h) = f(a) + hf'(a+\theta h)$ —— (1) where $0<\theta<1$

Also by Taylor's theorem with Lagranges form of remainder after two terms

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta_1 h)$ —— (2)

where $0<\theta_1<1$

from (1) & (2)

$\cancel{f(a)} + hf'(a+\theta h) = \cancel{f(a)} + hf'(a) + \frac{h^2}{2!}f''(a+\theta_1 h)$

$\Rightarrow hf'(a+\theta h) = hf'(a) + \frac{h^2}{2!}f''(a+\theta_1 h)$

$\Rightarrow f'(a+\theta h) = f'(a) + \frac{h}{2}f''(a+\theta_1 h)$

or $f'(a+\theta h) - f'(a) = \frac{h}{2}f''(a+\theta_1 h)$

$\theta h f''(a+\theta\theta_2 h) = \frac{h}{2}f''(a+\theta_1 h)$ (By Lag. M.V.T)

where $0<\theta_2<1$

$\Rightarrow \theta f''(a+\theta\theta_2 h) = \frac{1}{2}f''(a+\theta_1 h)$

Let $h\to 0$, then in limiting case

$\theta f''(a) = \frac{1}{2}f''(a)$

$\Rightarrow \boxed{\theta = \frac{1}{2}}$

(ii) If the function f, ϕ and ψ are continous on $[a,b]$ & derivable in $]a,b[$; Show that there exist a pt $\xi \in]a,b[$ s. that

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0$$

Hence deduce the M.V.T & Cauchy's M.V.T.

Sol. Define a new function

$$F(x) = f(x) + A\phi(x) + B\psi(x)$$

where A & B are constt to be chosen s. that

$$F(a) = 0 = F(b)$$

From $F(a) = 0$ & $F(b) = 0$ we have

$$f(a) + A\phi(a) + B\psi(a) = 0 \quad \text{———} \quad (1)$$

$$f(b) + A\phi(b) + B\psi(b) = 0 \quad \text{———} \quad (2)$$

Obviously the function $F(x)$, satisfies all the Conditions of Rolle's Theoram. Hence there exists a pt. $\xi \in]a,b[$ s. that $F'(\xi) = 0$

$$f'(\xi) + A\phi'(\xi) + B\psi'(\xi) = 0 \quad \text{———} \quad (3)$$

Eliminating A & B from eqs (1), (2) & (3) we have

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0 \quad \text{——} \quad (4)$$

as desired

Deductions :-

(i) put $\psi(x) = c$ where c is a Const.

then $\psi(a) = c$

& $\psi(b) = c$ so $\psi(a) = \psi(b)$

& $\psi'(x) = 0$

or $\psi'(f) = 0$

so from eq (4), we have

$$\begin{vmatrix} f'(f) & \phi'(f) & 0 \\ f(a) & \phi(a) & c \\ f(b) & \phi(b) & c \end{vmatrix} = 0$$

Taking c as common

$$c \cdot \begin{vmatrix} f'(f) & \phi'(f) & 0 \\ f(a) & \phi(a) & 1 \\ f(b) & \phi(b) & 1 \end{vmatrix}$$

Expand from C_3

$$0 - 1 \begin{vmatrix} f'(f) & \phi'(f) \\ f(b) & \phi(b) \end{vmatrix} + 1 \begin{vmatrix} f'(f) & \phi'(f) \\ f(a) & \phi(a) \end{vmatrix} = 0$$

or $-\left[f'(f)\phi(b) - f(b)\phi'(f)\right] + \left[f'(f)\phi a - f(a)\phi'(f)\right] = 0$

$-f'(f)\phi(b) + f(b)\phi'(f) + f'(f)\phi(a) - f(a)\phi'(f) = 0$

$-\left[\phi(b) - \phi(a)\right]f'(f) + \left[f(b) \; f(a)\right]\phi'(f) = 0$

$\Rightarrow \left[\phi(b) - \phi(a)\right]f'(f) = \left[f(b) - f(a)\right]\phi'(f)$

$\Rightarrow \boxed{\dfrac{f(b)-f(a)}{\phi(b)-\phi(a)} = \dfrac{f'(f)}{\phi'(f)}}$ — (5) for some $f \in]a,b[$ which is Cauchy's M.V.T.

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Now put $\phi(x) = x$

So $\phi(a) = a$

& $\phi(b) = b$

& $\phi'(x) = 1$

$\Rightarrow \phi'(\xi) = 1$

Putting these values in (5), we have

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(\xi)}{1}$$

$$\Rightarrow \boxed{\frac{f(b)-f(a)}{b-a} = f'(\xi)} \quad \text{for some } \xi \in]a,b[$$

which is Lagranges M.V.T

---

(1) Assuming f'' continuous on $[a,b]$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where c & ξ both belong to $]a,b[$.

Sol:- Define a function

$$\phi(x) = f(x) + Ax + Bx^2 \quad \text{———— (1)}$$

where A & B are consts. to be determined s.that

$$\phi(a) = \phi(b) = \phi(c)$$

From eqs $\phi(a) = \phi(b) = \phi(c)$, we have

$$f(a) + Aa + Ba^2 = f(b) + Ab + Bb^2 = f(c) + Ac + Bc^2$$

or $f(a) + Aa + Ba^2 = f(b) + Ab + Bb^2$

& $f(b) + Ab + Bb^2 = f(c) + Ac + Bc^2$

or

$$(a-b)A + (a^2-b^2)B + [f(a)-f(b)] = 0$$

$$(b-c)A + (b^2-c^2)B + [f(b)-f(c)] = 0$$

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Solving

$$\frac{A}{(a^2-b^2)\{f(b)-f(c)\}-(b^2-c^2)\{f(a)-f(b)\}} = \frac{B}{(b-c)\{f(a)-f(b)\}-(a-b)\{f(b)-f(c)\}} = \frac{1}{(a-b)(b^2-c^2)-(b-c)(a^2-b^2)}$$

or

$$\frac{A}{(a^2-b^2)\{f(b)-f(c)\}-(b^2-c^2)\{f(a)-f(b)\}} = \frac{B}{(b-c)\{f(a)-f(b)\}-(a-b)\{f(b)-f(c)\}} = \frac{1}{(a-b)(b-c)\{\cancel{b}+c-a-\cancel{b}\}}$$

$$\Rightarrow A = \frac{(a^2-b^2)\{f(b)-f(c)\}-(b^2-c^2)\{f(a)-f(b)\}}{(a-b)(b-c)(c-a)} \quad \text{--- (2)}$$

& $$B = \frac{(b-c)\{f(a)-f(b)\}-(a-b)\{f(b)-f(c)\}}{(a-b)(b-c)(c-a)} \quad \text{--- (3)}$$

Since f'' is continous on $[a,b]$, so $f(x)$ is derivable in $]a,b[$. Also obviously $Ax+Bx^2$ is derivable in $]a,b[$. So from (1) we conclude that $\phi(x)$ is derivable in $]a,b[$.

If we consider the interval $[a,c]$. Then $\phi(x)$ is derivable in $]a,c[$ $\quad (\because [a,c] \subset [a,b])$

Also $\phi(a) = \phi(c)$

Hence by Rolles theoram

$\phi'(\alpha) = 0$ --- (4) $\quad$ for som $\alpha \in]a,c[$

Similarly if we consider interval $[c,b]$. Then $\phi(x)$ is derivable in $]c,b[$ $\quad (\because [c,b] \subset [a,b])$

Also $\phi(c) = \phi(b)$

Hence by Rolles theoram

$\phi'(\beta) = 0$ --- (5) for some $\beta \in]c,b[$.

Now let

$F(x) = \phi'(x) = f'(x) + A + 2Bx$

$\Rightarrow F'(x) = f''(x) + 2B$

Since $f''(x)$ exists in $[a,b]$ so $F(x)$ also exists in $[a,b]$. Hence $F(x)$ is derivable in $]a,b[$

$\Rightarrow F(x)$ is also derivable in $]\alpha,\beta[$ $(\because [\alpha,\beta] \subset [a,b])$

& $F(\alpha) = \phi'(\alpha) = 0$

& $F(\beta) = \phi'(\beta) = 0$

so

$F(\alpha) = F(\beta)$

Hence by Rolles theoram

$F'(\xi) = 0$, for some $\xi \in]\alpha,\beta[$.

i.e., $f''(\xi) + 2B = 0$ for some $\xi \in]a,b[$.

$$\Rightarrow f''(\xi) + 2\, \frac{(b-c)[f(a)-f(b)] - (a-b)[f(b)-f(c)]}{(a-b)(b-c)(c-a)} = 0$$

$$\Rightarrow (a-b)(b-c)(c-a)f''(\xi) + 2\left[(b-c)[f(a)-f(b)] - (a-b)[f(b)-f(c)]\right] = 0$$

$$\Rightarrow (a-b)(b-c)(c-a)f''(\xi) = 2\left[(b-c)[f(a)-f(b)] - (a-b)[f(b)-f(c)]\right]$$

$$\Rightarrow -\frac{1}{2}(c-a)(c-b)f''(\xi) = \frac{(b-c)[f(a)-f(b)] - (a-b)[f(b)-f(c)]}{(a-b)}$$

$$= \frac{(b-c)f(a) - (b-c)f(b) - (a-b)f(b) + (a-b)f(c)}{(a-b)}$$

$$= \frac{(b-c)f(a) - [b-c+a-b]f(b) + (a-b)f(c)}{(a-b)}$$

$$= \frac{(b-c)f(a) - (a-c)f(b) + (a-b)f(c)}{(a-b)}$$

$$= \frac{(b-c)f(a)}{a-b} - \frac{(a-c)}{a-b} f(b) + f(c)$$

$$\Rightarrow \frac{1}{2}(c-a)(c-b) f''(\xi) = f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a}$$

(iv) Show that the no. θ which occurres in the Taylor's theoram with Lagranges form of remainder after n terms approaches the limit $\frac{1}{n+1}$ as $h \to 0$ provided that $f^{n+1}(x)$ is Continous & different from zero at $x = a$

Sol:-

we know that Taylors theoram with Lagranges form of remainder after n terms is

$$f(a+h) = f(a) + hf'(a) + \dots\dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h) \quad \text{--- (1)}$$

Also Taylors theoram with Lagranges form of remainder after $(n+1)$ terms is — where $0 < \theta < 1$

$$f(a+h) = f(a) + hf'(a) + \dots\dots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta_1 h) \quad \text{--- (2)}$$

where $0 < \theta_1 < 1$

From (1) & (2) we get

$$\frac{h^n}{n!} f^n(a+\theta h) = \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta_1 h)$$

or $f^n(a+\theta h) = f^n(a) + \frac{h}{n+1} f^{n+1}(a+\theta_1 h)$

$$\Rightarrow f^n(a+\theta h) - f^n(a) = \frac{h}{n+1} f^{n+1}(a+\theta_1 h)$$

$$\Rightarrow \theta h f^{n+1}(a+\theta_2 \theta h) = \frac{h}{n+1} f^{n+1}(a+\theta_1 h) \quad \text{(By M.V.T)}$$

$0 < \theta_2 < 1$

$$\Rightarrow \theta f^{n+1}(a+\theta_2 \theta h) = \frac{1}{n+1} f^{n+1}(a+\theta_1 h)$$

Let $h \to 0$ so in Limiting case

$\theta f^{n+1}(a) = \frac{1}{n+1} f^{n+1}(a) \Rightarrow \theta = \frac{1}{n+1}$ as $h \to 0$.

Solved Examples

① Find Maclaurins development with Lagranges form of remainder after n terms of function $f(x) = e^x$

Sol:- we know that Maclaurins development with Lagranges form of remainder after n terms is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x) \quad \text{--- ①}$$

where $0 < \theta < 1$

Here $f(x) = e^x$

Then $f^n(x) = e^x$

$\Rightarrow f^n(0) = 1$

so from ①

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}$$

where $0 < \theta < 1$

---

② Expand e^x by Maclaurins theoram.

Here $f(x) = e^x$

Then $f^n(x) = e^x$

& $f^n(0) = 1$

& $f^n(\theta x) = e^{\theta x}$

Now remainder after n terms in Lagranges form is

$$R_n = \frac{x^n}{n!}f^n(\theta x)$$

$$R_n = \frac{x^n}{n!}e^{\theta x}$$

Now $R_n = \left|\frac{x^n}{n!}e^{\theta x}\right|$

since $e^{\theta x} < e^{x}$ if $x > 0$

& $e^{\theta x} < 1$ if $x < 0$

so

$$|R_n| = \left|\frac{x^n}{n!} e^{\theta x}\right| < \frac{x^n}{n!} e^{x} \quad \text{if } x > 0$$

$$\& \ |R_n| = \left|\frac{x^n e^{\theta x}}{n!}\right| < \frac{x^n}{n!} \quad \text{if } x < 0$$

Now $\lim_{n \to \infty} \frac{x^n}{n!} \longrightarrow 0$

so $\lim_{n \to \infty} R_n \longrightarrow 0$, Hence $f(x) = e^{x}$ Canbe expanded into an infinite series.

By Maclaurins theoram

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots$$

$\Longrightarrow$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \quad \text{— Ans}$$

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③. Expand $\ln(1+x)$ by Maclaurins theoram

Sol:- Here $f(x) = \ln(1+x)$

Then we know that

$$f^{n}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

$$\Longrightarrow f^{n}(0) = (-1)^{n-1}(n-1)!$$

For $n = 1$ $\quad f'(0) = 1$

$n = 2$ $\quad f''(0) = -1$

$n = 3$ $\quad f'''(0) = 2$

$n = 4$ $\quad f''''(0) = -6$ & so on.

Now remainder after n terms in Lagranges form is

$$R_n = \frac{x^n}{n!} f^n(\theta x)$$

$$= \frac{x^n}{n!} \cdot \frac{(-1)^{n-1} \cdot (n-1)!}{(1+\theta x)^n}$$

$$R_n = \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x}\right)^n$$

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x}\right)^n$$

If $0 \le x \le 1$, Then $\left|\frac{x}{1+\theta x}\right|$ is +ve & less than 1.

Hence,

$$\lim_{n\to\infty} |R_n| \to 0$$

But if $-1 < x < 0$ Then $\left|\frac{x}{1+\theta x}\right|$ may not be less than 1.

Now taking remainder after n terms in Cauchys form is

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x) \quad \text{where } 0<\theta<1$$

$$= \frac{x^n}{\cancel{(n-1)!}} (1-\theta)^{n-1} \cdot \frac{(-1)^{n-1} \cdot \cancel{(n-1)!}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} \cdot x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{1}{1+\theta x}$$

Now

$\frac{1-\theta}{1+\theta x}$ is +ve & less than 1 $(\because |x|<1)$

So $$0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1$$

also $x^n \to 0$ as $n \to \infty$ because $|x|<1$

So $R_n \to 0$ as $n \to \infty$

Hence $f(x) = \ln(1+x)$ canbe expanded into an infinite series.

By Maclaurins theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots\dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots$$

[Indeterminate Forms]

The form $\left(\frac{0}{0}\right)$:- suppose that two functions f & ϕ satisfy the conditions of couchys M.V.T on same interval. If $f(a) = \phi(a) = 0$

Then the expression $\frac{f(a)}{\phi(a)}$ is meaningless. But $\lim_{x\to a}\frac{f(x)}{\phi(x)}$ may exist.

The calculation of limits of this type is known as evaluating the indeterminate form $\left(\frac{0}{0}\right)$. e.g

$$\frac{\sin x}{x} \text{ is meaningless at } x = 0$$

But $\frac{\sin x}{x} \longrightarrow 1$ as $x \longrightarrow 0$

L' Hospitals' Rule :-

Statement :-

(1) Let the functions f & ϕ are continuous on $[a,b]$

(2) f & ϕ are derivable in $]a,b[$

(3) $f(a) = 0 = \phi(a)$ & $\phi'(x) \neq 0 \quad \forall x \in\,]a,b[$

(4) $\lim_{x\to a}\frac{f(x)}{\phi(x)} = l$

Then $\lim_{x\to a}\frac{f(x)}{\phi(x)} = l$