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CH#05
INNER PRODUCT SPACES.

INNER PRODUCT :-

Let, V be a vector space over the field F of real or complex numbers. A mapping $\langle .,. \rangle : V \times V \to F$ is said to be an inner product on V if the following conditions are satisfied.

i) $\langle v,v \rangle \geq 0$ and $\langle v,v \rangle = 0$ iff $v=0 \ \forall \ v \in V$.

ii) $\langle u,v \rangle = \overline{\langle v,u \rangle} \ \forall \ u,v \in V$.

Here, $\overline{\langle v,u \rangle}$ denotes the conjugate of $\langle u,v \rangle$.

iii) $\langle \alpha u + \beta v, w \rangle = \alpha \langle u,w \rangle + \beta \langle v,w \rangle$
$\forall \ \alpha, \beta \in F$ and $u,v,w \in V$.

If F is a field of real numbers only, then condition ii) is $\langle u,v \rangle = \langle v,u \rangle$.

EXAMPLE :-

Let $u,v \in R^2$ such that $u=(x_1,x_2)$ and $v=(y_1,y_2)$ then $\langle u,v \rangle = \langle (x_1,x_2),(y_1,y_2) \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ is an inner product on R^2.

Solution :-

i) $\langle u,u \rangle = \langle (x_1,x_2),(x_1,x_2) \rangle$
$= x_1x_1 - x_1x_2 - x_2x_1 + 3x_2x_2$
$= x_1^2 - 2x_1x_2 + 3x_2^2$
$= x_1^2 - 2x_1x_2 + x_2^2 + 2x_2^2$
$= (x_1 - x_2)^2 + 2x_2^2$

$\langle u,u \rangle \geq 0$

Now,

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$\langle u,u\rangle=0$

$(x_1-x_2)^2+2x_2^2=0^2+2(0)^2$

$(x_1-x_2)^2=0 \quad$ or $\quad x_2^2=0$

$(x_1-x_2)=0 \quad ; \quad x_2=0$

$x_1-x_2=0$

$x_1-0=0$

$x_1=0$

$u=(0,0)$

$u=0$

ii) Let, $u=(x_1,x_2)$, $v=(y_1,y_2)\in\mathbb{R}^2$ then to show that $\langle u,v\rangle=\langle v,u\rangle$.

L.H.S $\Rightarrow$

$\langle u,v\rangle=x_1y_1-x_1y_2-x_2y_1+3x_2y_2$

$=y_1x_1-y_2x_1-y_1x_2+3y_2x_2$

$\langle u,v\rangle=\langle v,u\rangle$

$\Rightarrow$ R.H.S

iii) Let, $\alpha,\beta\in F$ and $u,v,w\in\mathbb{R}^2$ such that $u=(x_1,x_2)$, $v=(y_1,y_2)$, $w=(z_1,z_2)$ to show that $\langle\alpha u+\beta v,w\rangle=\alpha\langle u,w\rangle+\beta\langle v,w\rangle$

L.H.S $\Rightarrow$

$\langle\alpha u+\beta v,w\rangle=\langle\alpha(x_1,x_2)+\beta(y_1,y_2),(z_1,z_2)\rangle$

$=\langle(\alpha x_1,\alpha x_2)+(\beta y_1,\beta y_2)+(z_1,z_2)\rangle$

$=\langle(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2),(z_1,z_2)\rangle$

$=(\alpha x_1+\beta y_1)z_1-(\alpha x_1+\beta y_1)z_2-(\alpha x_2+\beta y_2)z_1+3(\alpha x_2+\beta$

$=\alpha x_1z_1+\beta y_1z_1-\alpha x_1z_2-\beta y_1z_2-\alpha x_2z_1-\beta y_2z_1+3\alpha x_1z_2+3\beta y$

$=(\alpha x_1z_1-\alpha x_1z_2-\alpha x_2z_1+3\alpha x_2z_2)$

$+(\beta y_1z_1-\beta y_1z_2-\beta y_2z_1+3\beta y_2z_2)$

$=\alpha(x_1z_1-x_1z_2-x_2z_1+3x_2z_2)$

$+\beta(y_1z_1-y_1z_2-y_2z_1+3y_2z_2)$

$=\alpha\langle u,w\rangle+\beta\langle v,w\rangle$

Norm (Or Length Of A Vector) :-

Let, V be an inner product space and $v \in V$ then the real number $\sqrt{\langle v,v \rangle}$ is called the norm of v and is denoted by the symbol $\|v\| = \sqrt{\langle v,v \rangle}$ or $\|v\|^2 = \langle v,v \rangle$. If norm of $\|v\| = 1$ i.e $\langle v,v \rangle = 1$ then v is called a unit vector or is said to be normalized vector.

Any non-zero vector $u \in V$ can be normalized by multiplying it by $\frac{1}{\|u\|}$, thus

$n = \frac{1}{\|u\|} . u = \frac{u}{\|u\|}$ is a unit vector and we say that the vector u has been normalized.

Example :-

Find the norm of $u = (3,4) \in \mathbb{R}^2$ w.r.t Euclidian Inner Product and the inner product defined by :-

$$\langle u,v \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3 x_2 y_2$$

where,

$$u = (x_1, x_2) \; ; \; v = (y_1, y_2)$$

Solution :-

$$\|u\| = \sqrt{\langle u,u \rangle}$$
$$= \sqrt{x_1^2 - 2x_1x_2 + 3x_2^2}$$
$$= \sqrt{(3)^2 - 2(3)(4) + 3(4)^2}$$
$$= \sqrt{9 - 24 + 48}$$
$$= \sqrt{33}$$

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EXAMPLE :-

Normalize $(1,4,1) \in R^3$.

Solution :-

Let, $u = (1,4,1)$

$$\|u\| = \sqrt{\langle u,u \rangle}$$
$$= \sqrt{\langle (1,4,1),(1,4,1) \rangle}$$
$$= \sqrt{1+16+1}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

The normalized vector of u is $= \dfrac{u}{\|u\|}$

$$= \frac{1}{3\sqrt{2}}(1,4,1)$$
$$= \left(\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$$

ORTHOGONAL VECTORS :-

The vectors u and v are said to be orthogonal if and only if $\langle u,v \rangle = 0$. If u is orthogonal to v then we write $u \perp v$.

EXAMPLE :-

Let $x = (1,-1,2)$, $y = (-1,1,1) \in R^3$.
Show that $x \perp y$.

Solution :-

We know that x will be orthogonal to y iff $\langle x,y \rangle = 0$.

$$\langle (1,-1,2),(-1,1,1) \rangle = 0$$
$$-1-1+2 = 0$$
$$-2+2 = 0$$

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$$0=0$$
$$\Rightarrow \langle x,y\rangle=0$$
$$\Rightarrow x\perp y$$
Proved!

EXAMPLE:-

Find the unit vector orthogonal to both $(1,1,2)$ and $(0,1,3)\in R^3$.

Solution:-

Let, $x=(x,y,z)\in R^3$ is a vector whose orthogonal to $(1,1,2)$ and $(0,1,3)$.

Suppose, $u=(1,1,2)$ and $v=(0,1,3)$.

By definition:-

$$\langle x,u\rangle=0 \Rightarrow \langle(x,y,z)(1,1,2)\rangle=0$$
$$x+y+2z=0 \rightarrow ①$$
$$\langle x,v\rangle=0 \Rightarrow \langle(x,y,z),(0,1,3)\rangle=0$$
$$y+3z=0$$
$$y=-3z$$

Using the value of y into eq. ①:-

$$x-3z+2z=0$$
$$x-z=0$$
$$x=z$$

Now,

$$x=(x,y,z)$$
$$x=(z,-3z,z)$$
$$x=z(1,-3,1)$$

Put $z=t\in R$

$$x=t(1,-3,1)$$
or
$$x=(1,-3,1)$$

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$$\|x\| = \sqrt{\langle x, x\rangle}$$
$$= \sqrt{\langle (1,-3,1),(1,-3,1)\rangle}$$
$$= \sqrt{1+9+1}$$
$$= \sqrt{11}$$

The unit vector orthogonal to both u and v is $= \dfrac{x}{\|x\|}$

$$= \frac{1}{\sqrt{11}}(1,-3,1)$$

$$= \left(\frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$$