Linear Algebra

(Week 14-18(December)

Lecture 1

SUBSPACE :-

A non-empty subset W of a vector space V over field F is said to be subspace of V if and only if W itself is vector space over field F under the same operation as defined in V.

OR

Let, V be a vector space over field F and let W be a non-empty subset of V, then W is a subspace of V if and only if

1) $x, y \in W \Rightarrow x+y \in W$
2) $\alpha \in F$ and $x \in W \Rightarrow \alpha x \in W$

EXAMPLE :-

Show that $W = \{(x, y, z) \in R^3 / x+y+z=0\}$ is a subspace of R^3.

Solution :-

1) Let, $x = (x_1, y_1, z_1)$; $y = (x_2, y_2, z_2)$ in W

$\Rightarrow x_1+y_1+z_1 = 0$ and $x_2+y_2+z_2 = 0 \rightarrow (1)$

To show that $x+y \in W$ i.e $x+y=0$

$$x+y = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$
$$= (x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= x_1+x_2+y_1+y_2+z_1+z_2$$
$$= (x_1+y_1+z_1) + (x_2+y_2+z_2)$$
$$= 0 + 0 \quad \text{from } (1)$$
$$= 0$$

$x+y \in W$

2) Let, $\alpha \in F$ and $x \in W$

To show that $\alpha x \in W$ i.e $\alpha x = 0$

$$\alpha x = \alpha(x_1, y_1, z_1)$$
$$= (\alpha x_1, \alpha y_1, \alpha z_1)$$

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$= \alpha x_1 + \alpha y_1 + \alpha z_1$

$= \alpha(x_1 + y_1 + z_1)$

$= \alpha(0)$ from ①

$= 0$

$\alpha x \in W$

Therefore, W is a subspace of R^3.

EXAMPLE :-

Show that $W = \{(x, y, z) \in R^3 / x + y + z = 1\}$ is not a subspace of R^3.

Solution :-

Let, $x = (x_1, y_1, z_1)$; $y = (x_2, y_2, z_2)$ in W.

$\Rightarrow x_1 + y_1 + z_1 = 1$ and $x_2 + y_2 + z_2 = 1 \rightarrow ①$

To show that $x + y \in W$ i.e $x + y = 1$

$x + y = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= x_1 + x_2 + y_1 + y_2 + z_1 + z_2$

$= (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2)$

$= 1 + 1$

$= 2$

$x + y \notin W$

Since, the first property does not hold.

Therefore, W is not a subspace of R^3.

EXAMPLE :-

The set W of all symmetric matrices $(A^t = A)$ in $M_{n \times n}(F)$ is a subspace of $M_{n \times n}(F)$.

$W = \{A \in M_{n \times n}(F) / A^t = A\}$.

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Solution :-

1) Let, A and B belong to W.

$\Rightarrow A^t = A$ and $B^t = B$ $\rightarrow$ ①

To show that $(A+B) \in W$ ie $(A+B)^t = A+B$.

$(A+B)^t = A^t + B^t$

$= A + B$ from ①

So, $A+B$ is symmetric ie $A+B \in W$.

2) Let, $\alpha \in F$ and $A \in W$.

$\Rightarrow A^t = A$ $\rightarrow$ ①

To show that $\alpha A \in W$ ie $(\alpha A)^t = \alpha A$

$(\alpha A)^t = \alpha A^t$

$= \alpha A$ from ①

So, αA is symmetric ie $\alpha A \in W$.

Therefore, W is a subspace of $M_{n\times n}(F)$.

EXAMPLE :-

The trace of $n\times n$ matrix M denoted by $Tr(M)$ is the sum of entries of its main diagonal

$$Tr(M) = M_{11} + M_{22} + M_{33} + M_{44} + \dots + M_{nn}$$

Show that the set of $n\times n$ matrices having trace equal to 0 is a subspace of $M_{n\times n}(F)$.

$$W = \{ A \in M_{n\times n}(F) / Tr(A) = 0 \}$$

Solution :-

1) Let, A and B belong to W.

$\Rightarrow Tr(A) = 0$ and $Tr(B) = 0$ $\rightarrow$ ①

To show that $A+B \in W$ ie $Tr(A+B) = 0$.

$Tr(A+B) = Tr(A) + Tr(B)$

$= 0 + 0$ from ①

$= 0$

So, $A+B \in W$

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2) Let, $\alpha \in F$ and $A \in W$.
To show that $\alpha A \in W$ i.e $Tr(\alpha A) = 0$

$$Tr(\alpha A) = \alpha\, Tr(A)$$
$$= \alpha(0) \quad \text{from } (1)$$
$$= 0$$

So, $\alpha A \in W$

Therefore, W is a subspace of $M_{n \times n}(F)$.

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