MULTIVARIABLE CALCULUS 6.2.18

CHAPTER # 14 966

Domain & Range

$f(x) = y = x$
$= x^2$
$= \sqrt{x}$
$= 1/x$

Example # 02

a) $f(x,y) = w = \sqrt{y-x^2}$

Domain: $y \geq x^2$

Range: $[0, \infty)$

Domain:

$y - x^2 \geq 0$

$-x^2 \geq -y$

$x^2 \leq y$

$x \leq \sqrt{y}$

Range: $[0, \infty)$

b) $w = 1/xy$

Domain: $xy \neq 0$

Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $xy \neq 0$
$xy > 0$
$y > 0$
$x > 0$

Range: $\mathbb{R} - \{0\}$

Example # 02

c) $w = \sin(xy)$ ⇒ odd function

Domain: Whole xy-Plane / Entire Plane

Range: $[-1, 1]$

d) $w = \sqrt{x^2 + y^2 + z^2}$

(7) $xy \ln(z)$

Domain: Entire Space

Range: $[0, \infty)$

e) $w = 1/x^2+y^2+z^2$

Domain: $(x, y, z) \neq (0, 0, 0)$

Range: $(0, \infty)$

INTERIOR POINT:

A point (x_0, y_0) in a region R in xy-plane is an interior point, if it is center of a disk of positive radius lies entirely in R.

BOUNDARY POINT:

A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contain points, that lie outside of R as well as points that lie in R.

INTERIOR OF A REGION:

The interior points of a region, as a set," make up interior of a region.

BOUNDARY:

The region's boundary points make up its boundary.

OPEN REGION:

A region is open, if it consists entirely of its interior points.

CLOSED REGION: A region is closed, if it contains all boundary points.

Example:

General equation of circle

$(x-h)^2+(y-k)^2=r^2$

$x^2+y^2=1$: Unit circle with center at origin

Example 3: Consider $f(x,y)=\sqrt{y-x^2}$

Describe:

(a) Domain of a function

(b) Draw the function's domain 18

(c) Find interior points, exterior points and boundary points.

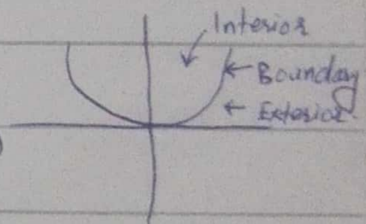
(d) Wether region is bounded or unbounded

Solution:

(a) Domain: $y-x^2 \geq 0$

(b) Graph: $y-x^2=0$

$y=x^2$



(Inside)

(c) Interior points : $y-x^2>0$

Boundary points: $y-x^2=0$

(Outside)

Exterior points: $y-x^2<0$

(d) Boundery: Un-bounded

Ex # 14.9

(8) $\sqrt{9-x^2-y^2} = f(x,y)$

(a) Domain:

$$9-x^2-y^2 \geq 0$$

$$-x^2-y^2 \geq -9$$

$$x^2+y^2 \leq 9$$

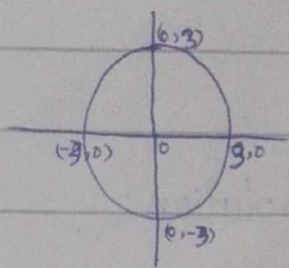
Range:

$$= [0, +\infty)$$

(b) Graph:

$$-9+x^2+y^2 = 0$$

$$x^2+y^2 = 9$$



(c) $x^2+y^2 < 9$ Interior

$x^2+y^2 = 9$ Boundary

$x^2+y > 9$ Exterior

(d) Bounded

(2) $f(x,y) = \sqrt{y-x}$

(a) Domain:

$$y-x \geq 0$$

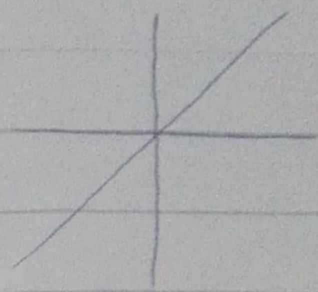
$$y \geq x$$

Range:

$$= [0, \infty)$$

(b) Graph:

$y = x$



(c) Interior :

Boundary: $y = x$

Exterior :

(d) Un bounded

LEVEL CURVE

12.02.18

968

"The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$, is called a level curve of f."

Example no. 4

Graph $f(x,y) = 100 - x^2 - y^2$ and plot the level curves $f(x,y) = 0$, $f(x,y) = 51$ and $f(x,y) = 75$ in the domain of f in plane.

Solution:

Domain of $f(x,y)$ is entire plane.

Range: $(-\infty, 100]$ "The set of real numbers less than or equal to 100."

i. For $f(x,y)=0$

$0 = 100 - x^2 - y^2$

$x^2+y^2=100$

Eq. of circle centered at origin with radius 10.

ii. For $f(x,y)=51$

$51 = 100 - x^2 - y^2$

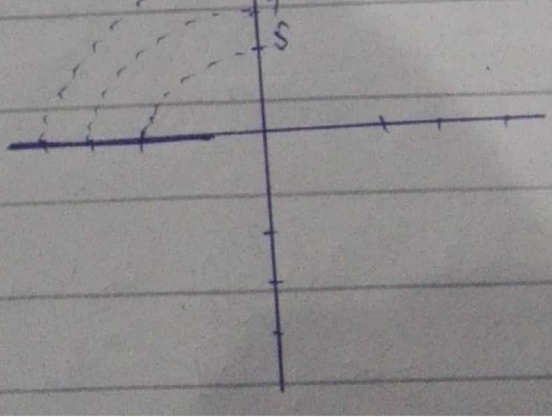
$x^2+y^2=49$

Eq. of circle centered at origin with radius 7

iii. For $f(x,y)=75$

$x^2+y^2=25$

Eq. of circle centered at origin with radius 5.



Level Surface

"The set of points (x, y, z) in space, where a function (is)* of three independent variables has a constant value $f(x,y,z) = c$ is called a level surface of f."

Example # 05

Describe the level surface of

$$f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \text{for}$$

$$f(x,y,z) = 1, \; f(x,y,z) = 2 \;\&\; f(x,y,z) = 3$$

Solution:

Domain is entire space

Range : $[0, \infty)$

i. for $f(x,y,z) = 1$

$$\sqrt{x^2+y^2+z^2} = 1$$

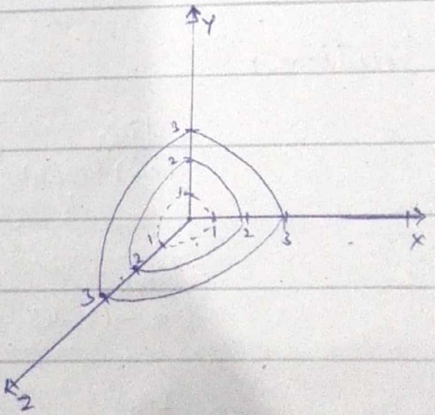
$$x^2+y^2+z^2 = 1$$

ii. for $f(x,y,z) = 2$

$$x^2+y^2+z^2 = 4$$

iii. for $f(x,y,z) = 3$

$$x^2+y^2+z^2 = 9$$



C.T P# 974

Level Surface Q# 41

Level Curve Q# 29 & 30

Example # 2 (978)

Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ $\left(\frac{0}{0} \text{ form}\right)$

Solution:

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}}$$

$\because a^2-b^2=(a+b)(a-b)$

$(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$

$= x-y$

$$= \lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y})$$

$$= 0$$

EXERCISE # 14.2
982

Q # 16

$$\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y-yx+4x^2-4x} \quad \left(\frac{0}{0} \text{ form}\right)$$

Solution:

$$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2(y+4)-x(y+4)}$$

$$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{(y+4)(x^2-x)}$$

$$= 1/2$$

Q # 17

$$\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

Solution:

$$= \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})+2(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}-\sqrt{y})[\sqrt{x}+\sqrt{y}+2]}{\sqrt{x}-\sqrt{y}}$$

$$= 2$$

Q # 19

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$$

Solution:

$$= \lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y}-2)(\sqrt{2x-y}+2)}$$

$$= \frac{1}{\sqrt{4-0}+2} = 1/4$$

Pg# 982
Q # 20
Q # 15
Q # 22

*. Limit of a function is always unique.

CONTINUOUS FUNCTIONS OF TWO VARIABLES

979

"A function $f(x,y)$ is continuous at the pt (x_0,y_0); if

1. f is defined at (x_0,y_0)
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Theorem:

980 "Two path test for non-existance of a limit."

✓ Statement: "If a function $f(x,y)$ has different limits along two different paths as (x,y) approaches to (x_0,y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exists."

EXAMPLE # 4

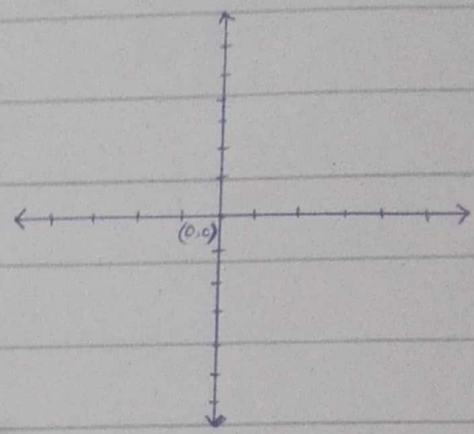
979

Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at every point except $(0,0)$

GRAPH:



Solution:

The limit can't be found by direct substitution which gives $(0/0)$. We examine the values of f along the curves that end at $(0,0)$.

⇒ Consider Paths
Two functions:

Either $Y = mx$
or $Y = mx^2$

Along the curve $y = mx$, $x \neq 0$ the function has a constant value.

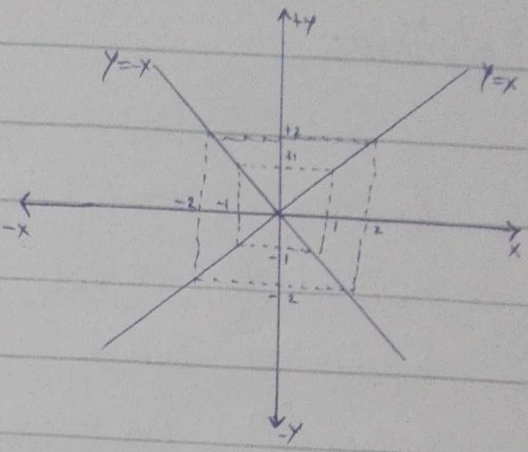
$$f(x,y)\Big|_{Y=mx} = \frac{2xy}{x^2+y^2}\Big|_{Y=mx} = \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}$$

Therefore, f has this number as its limit as $(x,y) \to (0,0)$ along the line

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{(x,y)\to(0,0)} \left[f(x,y) \Big|_{y=mx} \right] = \frac{2m}{1+m^2}$$

This limit changes with m, therefore function isn't continuous at $(x,y)\to(0,0)$.

GRAPH:



EXAMPLE # 5

Show that

$$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

∵ Now, we'll take path $y = mx^2$ (Parabola)

is continuous at every point except (0,0)

Solution:

The limit can not be found by direct substitution, which gives (0/0). We examine the values of f along the curves that end at (0,0).

Along the curve $y = mx^2,\ x \neq 0$ the function has a constant value.

$$f(x,y)\Big|_{y=mx^2} = \frac{2x^2y}{x^4+y^2}\Bigg|_{y=mx^2} = \frac{2mx^4}{x^4+m^2x^4} = \frac{2m}{1+m^2}$$

Therefore, f has this number as its limit as $(x,y) \to (0,0)$ along the line

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left[f(x,y)\Big|_{y=mx^2} \right] = \frac{2m}{1+m^2}$$

This limit changes with m, therefore function is not continuous at $(x,y) \to (0,0)$.

Q # 35

By considering different paths approach show that function has no limit as $(x,y) \to (0,0)$

$$f(x,y) = \frac{-x}{\sqrt{x^2+y^2}}$$

Solution:

The limit can not be found by direct substitution which gives $(0/0)$. We examine the values of f along the curves that end at $(0,0)$.

Along the curve $y = mx$, $x \neq 0$ the function has a constant value

$$f(x,y)\Big|_{y=mx} = \frac{-x}{\sqrt{x^2+y^2}}\Bigg|_{y=mx} = \frac{-x}{\sqrt{x^2+m^2x^2}}$$

$$= \frac{-x}{\sqrt{x^2(1+m^2)}} = \frac{-x}{x\sqrt{1+m^2}}$$

$$= \frac{-1}{\sqrt{1+m^2}}$$

at $m=0$ $\quad f(x,y)\Big|_{y=mx} = \frac{-1}{\sqrt{1}} = -1$

$m = +1$ $\quad = \frac{-1}{\sqrt{2}}$

$m = 1$ $\quad = -1/\sqrt{2}$

$m = 2$ $\quad = -1/\sqrt{5}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left[f(x,y)\Big|_{y=mx} \right] = \frac{-1}{\sqrt{1+m^2}}$$

This limit changes with m, therefore function isn't continuous at $(x,y) \to (0,0)$

21.2.18

PARTIAL DERIVATIVES

987

Example 1

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

at Pt. $(4,-5)$, if

$f(x,y) = x^2 + 3xy + y - 1$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial x}(-1)$$

$$= 2x + 3y + 0 + 0$$

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2(4) + 3(-5) = -7 \quad \rightarrow (A)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1)$$

$$= 0 + 3x + 1 - 0$$

$$\frac{\partial f}{\partial y} = 3x + 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3(4) + 1 = 13 \quad \rightarrow (B)$$

Example 2

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, if $f(x,y) = y\sin(xy)$

$$\frac{\partial f}{\partial x} = y.\frac{\partial}{\partial x}(\sin(xy))$$

$$= y.\cos(xy).\frac{\partial}{\partial x}(xy)$$

$$\frac{\partial f}{\partial x} = y^2\cos(xy) \rightarrow (A)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[y.\sin(xy)\right]$$

$$= y.\frac{\partial}{\partial y}(\sin(xy)) + \sin(xy).\frac{\partial}{\partial y}(y)$$

$$= y.\cos(xy).x + \sin(xy).(1)$$

$$= xy\cos(xy) + \sin(xy) \rightarrow (B)$$

Example. 3

Find f_x and f_y, if

$$f(x,y) = \frac{2y}{y+\cos x}$$

$$f_y = \frac{(y+\cos x)\cdot\frac{\partial}{\partial y}(2y) - \frac{\partial}{\partial y}(y+\cos x)\cdot 2y}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)2 - (1+\overset{0}{\cancel{\cos x}}).2y}{(y+\cos x)^2}$$

$$f_y = \frac{2y+2\cos x-2y}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}$$ Answer. 1

$$\frac{\partial f}{\partial x} = f_x = \frac{2y}{y+\cos x}$$

$$= \frac{(y+\cos x)\cdot\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial x}(y+\cos x)\cdot 2y}{(y+\cos x)^2}$$

$$= \left[(y+\cos x)\cdot 0 - (0+(-\sin x))\cdot 2y\right] \div \left[(y+\cos x)^2\right]$$

$$f_x = \frac{2y\sin x}{(y+\cos x)^2}$$ *Answer.2*

Example#9

17 $f(x,y) = x\cos y + ye^x$

Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x)$$

$$f_x = \cos y + ye^x$$

i $\leftarrow$ Ans $\quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^x) = 0 + y\cancel{e^x} = ye^x$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x$$

ii $\leftarrow$ Ans $\quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y$

iii $\leftarrow$ Ans $\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y\partial x} = -\sin y + e^x = f_{xy}$

iv $\leftarrow$ Ans $\quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2}{\partial x\partial y} = -\sin y + e^x = f_{yx}$

Example # 10

Find $\frac{\partial^2 w}{\partial x \partial y}$, if

$$w = xy + \frac{e^y}{y^2+1}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1}\right)$$

$$= y + 0 = y$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}(y) = 1$$

NOTE: $\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

Example # 11

Find f_{yxyz}, if

$f(x,y,z) = 1 - 2x^2yz + x^2y$

$f_y = 0 + (-2x^2z) + x^2 = x^2 - 2x^2z$

$f_{yx} = -4xz + 2x$

$f_{yxy} = 0 + 0$

$f_{yxyz} = 0$

EXERCISE # 14.3
994

Q.52

Ex # 14.3
Pg # 994
Q # 51 (d,e)
52
49
25
26
31
29
11
08
18

Find $\frac{\partial^5 f}{\partial x^2 \partial y^3}$, if

a) $f(x,y) = y^2 x^4 e^x + 2$

$$\frac{\partial f}{\partial y} = 2y x^4 e^x$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^4 e^x$$

$$\frac{\partial^3 f}{\partial y^3} = 0$$

$$\frac{\partial^4 f}{\partial x \partial y^3} = 0 \quad ; \quad \frac{\partial^5 f}{\partial x^2 \partial y^3} = 0$$

Laplace Equation:

Two dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Three dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Example# 65

Show that function satisfies Laplace equation

$$f(x,y) = e^{-2y} \cdot \cos(2x)$$

$$\frac{\partial f}{\partial x} = -2e^{-2y} \sin(2x)$$

$$\frac{\partial^2 f}{\partial x^2} = -4e^{-2y} \cos(2x)$$

$$\frac{\partial f}{\partial y} = -2e^{-2y} \cdot \cos(2x)$$

$$\frac{\partial^2 f}{\partial y^2} = 4e^{-2y} \cos(2x)$$

Now

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4e^{-2y} \cos(2x) + 4e^{-2y} \cos(2x)$$

$$= 0$$

Example # 68

$$f(x,y,z) = e^{3x+4y}\cos(5z)$$

$$\frac{\partial f}{\partial x} = 3e^{3x+4y}\cdot\cos(5z)$$

$$\frac{\partial f}{\partial y} = 4e^{3x+4y}\cos(5z)$$

$$\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y}\cdot\cos(5z)$$

$$\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y}\cos(5z)$$

$$\frac{\partial f}{\partial z} = -5e^{3x+4y}\cdot\sin(5z)$$

$$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y}\cos(5z)$$

Now,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (9+16-25)\,e^{3x+4y}\cos(5z)$$
$$= 0$$

996
Questions
6.3
⋮
68

One-Dimensional Wave Equation:

Pg# 996
Q(69-74)

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

or

$$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = 0$$

Q # 74

Show that the function is a solution of wave equation.

$$w = 5 \cos(3x + 3ct) + e^{x+ct}$$

$$\frac{\partial w}{\partial x} = -15 \sin(3x+3ct) + e^{x+ct}$$

$$\frac{\partial^2 w}{\partial x^2} = -45 \cos(3x+3ct) + e^{x+ct}$$

$$\frac{\partial w}{\partial t} = -15c \sin(3x+3ct) + ce^{x+ct}$$

$$\frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x+3ct) + c^2 e^{x+ct}$$

Now

$$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = -45c^2 \cos(3x+3ct) + c^2 e^{x+ct} + 45c^2 \cos(3x+3ct) - c^2 e^{x+ct}$$

$$= 0$$

Implicit Partial Derivatives:

988

Example # 4

Example # 6

Find $\frac{\partial z}{\partial x}$ if

$$yz - \ln z = x + y$$

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)$$

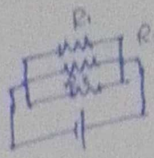
$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y)$$

$$y \cdot \frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1$$

$$\frac{\partial z}{\partial x}\left(\frac{yz-1}{z}\right) = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}$$



Q (36 & 39)

Example # 7

imp.

(Electrical resistors in parallel)

If resistors R_1, R_2 & R_3 ohms are connected in parallel to make R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find the value of $\frac{\partial R}{\partial R_1}, \frac{\partial R}{\partial R_2}$ & $\frac{\partial R}{\partial R_3}$, when

$R_1 = 30$ ohm, $R_2 = 45$ ohms & $R_3 = 90$ ohms

Solution:

To find $\frac{\partial R}{\partial R_2}$, we treat R_1 & R_3 as constants and use implicit differentiation.

Now

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\frac{\partial}{\partial R_2}(R^{-1}) = 0 + \frac{\partial}{\partial R_2}(R_2^{-1}) + 0$$

$$-R^{-2} \cdot \frac{\partial R}{\partial R_2} = -R_2^{-2} \cdot \frac{\partial R_2}{\partial R_2}$$

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_2} = \frac{-1}{R_2^2} \Rightarrow \frac{\partial R}{\partial R_2} = \left(\frac{R}{R_2}\right)^2 = 1/9$$

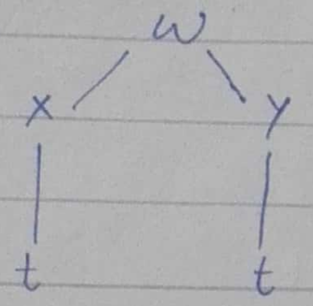
Chain Rule:

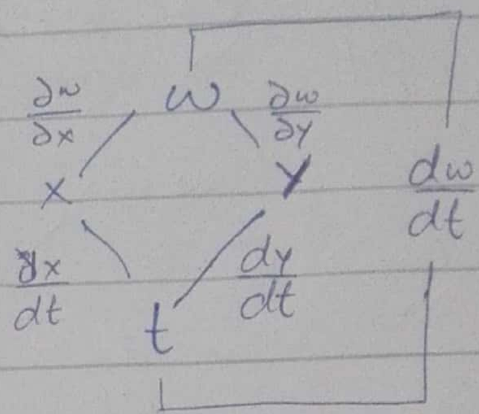
998

Example 1: Use the chain rule to find the derivative of $w = xy$ with respect to t along $x = \cos t$, $y = \sin t$.

What is derivative value at $t = \pi/2$

Solution:





$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dw}{dt} = -y \sin t + x \cos t$$

$$= -(\sin t)\sin t + (\cos t)\cos t$$

$$\left.\frac{dw}{dt}\right|_{\pi/2} = -\sin^2 t + \cos^2 t$$

$$\Rightarrow \cos(\pi) = -1$$

Example #2 999

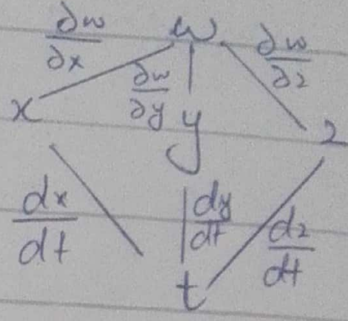
Find $\left.\frac{dw}{dt}\right|_{t=0}$, if

$w = xy + z$, $y = \sin t$

$x = \cos t$

$z = t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



$w = xy + z$

$\frac{\partial w}{\partial x} = y$

$\frac{dx}{dt} = -\sin t$

$\frac{\partial w}{\partial y} = x$

$\frac{dy}{dt} = \cos t$

$\frac{\partial w}{\partial z} = 1$

$\frac{dz}{dt} = 1$

$$\left.\frac{dw}{dt}\right|_{t=0} = -y \sin t + x \cos t + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \cos(2t) + 1$$

$$\frac{dw}{dt} = \cos(0) + 1$$

$$= 1 + 1$$

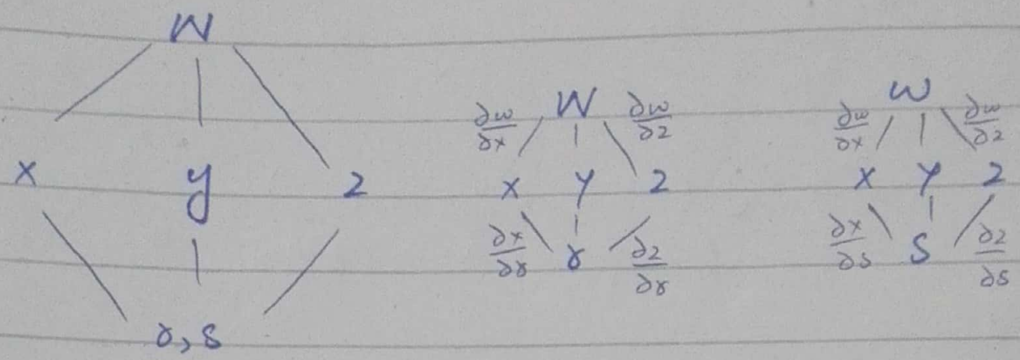
$$\frac{dw}{dt} = 2.$$

Ex:-

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r & s, if

$w = x + \cancel{yz}\, 2y + z^2,$

$x = \frac{r}{s}$, $y = r^2 + \ln s$ & $z = 2r$



Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$\Rightarrow \frac{\partial w}{\partial x} = 1 + 0 + 0 = 1$

$\Rightarrow \frac{\partial x}{\partial r} = \frac{1}{s}$

$\Rightarrow \frac{\partial w}{\partial y} = 0 + 2 + 0 = 2$

$\Rightarrow \frac{\partial x}{\partial s} = r$

$\Rightarrow \frac{\partial y}{\partial r} = 2r + 0$

$\Rightarrow \frac{\partial w}{\partial z} = 0 + 0 + 2z = 2z$

$\Rightarrow \frac{\partial y}{\partial s} = \frac{1}{s}$

$\Rightarrow \frac{\partial z}{\partial r} = 2$

$\Rightarrow \frac{\partial z}{\partial s} = 0$

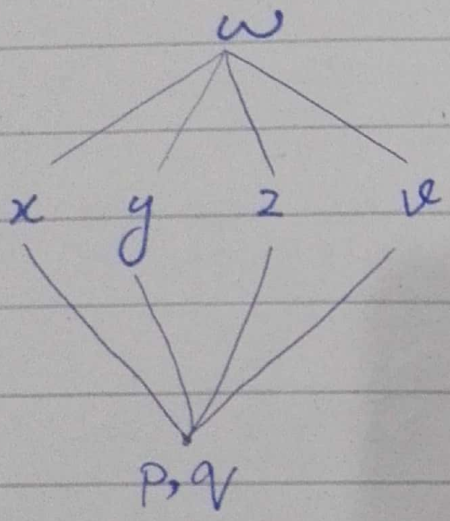
EXERCISE 14.4

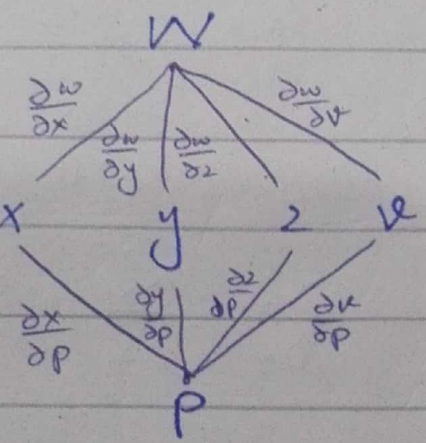
Draw a tree diagram and write a chain rule formula

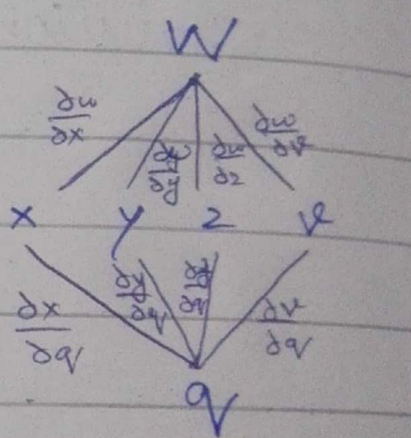
Q # 22 $\qquad$ Q (39, 40, 43, ~~44~~)

(i) $\dfrac{\partial w}{\partial q}$ (ii) $\dfrac{\partial w}{\partial p}$ for $w = f(x, y, z, v)$

$$x = g(p, q), \quad y = h(p, q)$$
$$z = j(p, q) \quad v = k(p, q)$$







(i) $$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial q} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial q} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial q}$$

(ii) $$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial p} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p}$$

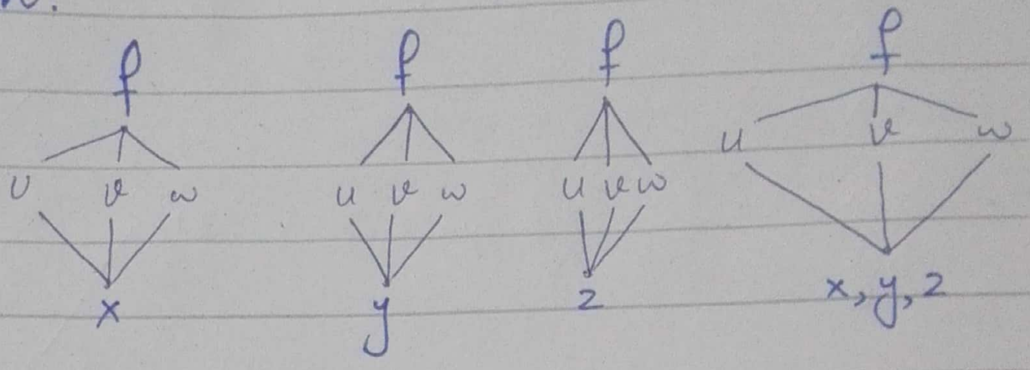
Q(33,36,04
39,05,07,11
40,18,12)

Q # 41

If $f(u,v,w)$ is differentiable
and $u = x-y$, $v = y-z$ & $w = z-x$
show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

Solution:



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v} \cdot (0) + \frac{\partial f}{\partial w} \cdot (-1)$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = \frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}$$

Similarly

$$\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

Now

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} + \frac{\partial f}{\partial v} - \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} = 0$$

Importa

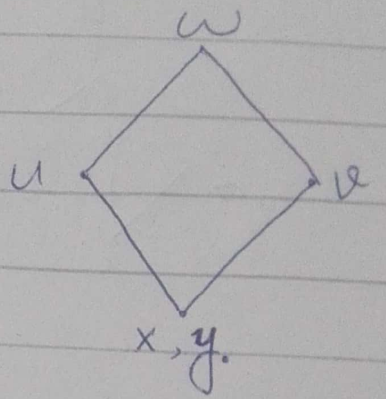
7.3.18

v.v. imp.

Q # 43 Show that if $w = f(u, v)$ satisfies Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies Laplace equation.

$$w_{xx} + w_{yy} = 0$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}$$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot x + \frac{\partial w}{\partial v} \cdot y$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(x \cdot \frac{\partial w}{\partial u}\right)$$

$$+ \frac{\partial}{\partial x}\left(y \cdot \frac{\partial w}{\partial v}\right)$$

$$w_{xx} = \frac{\partial w}{\partial u} + x \cdot \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + y \cdot \frac{\partial}{\partial w}\left(\frac{\partial w}{\partial u}\right)$$

$$= \frac{\partial w}{\partial u} + x \cdot \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right) + y \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right)$$

$$w_{xx} = \frac{\partial w}{\partial u} + x \cdot \frac{\partial}{\partial u}\left[x \cdot \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial v}\right] + y \cdot \frac{\partial}{\partial v}\left[x \cdot \frac{\partial w}{\partial u} + y \cdot \frac{\partial w}{\partial v}\right]$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= -\frac{\partial w}{\partial u} y + \frac{\partial w}{\partial v} \cdot x$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(-y \frac{\partial w}{\partial u}\right) + \frac{\partial}{\partial y}\left(x \cdot \frac{\partial w}{\partial v}\right)$$

$$= -\frac{\partial w}{\partial u} - y \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right) + x \cdot \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)$$

$$= -\frac{\partial w}{\partial u} - y \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right) + x \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right)$$

$$w_{yy} = -\frac{\partial w}{\partial u} - y \frac{\partial}{\partial u}\left(-y \frac{\partial w}{\partial u} + x \cdot \frac{\partial w}{\partial v}\right)$$

$$+ x \cdot \frac{\partial}{\partial u}\left(-y \cdot \frac{\partial w}{\partial u} + x \cdot \frac{\partial w}{\partial u}\right)$$

$$w_{xx} + w_{yy} = x^2\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right) + y^2\left(\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right)$$

$$= x^2(0) + y^2(0)$$

$$= 0$$

7.3.18

Linearization

1019

Imp

"The linearization of a function $f(x,y)$ at a point (x_0, y_0), where f is differentiable is a function $L(x,y) = f(x_0,y_0) + f_x\big|_{(x_0,y_0)}(x-x_0) + f_y\big|_{(x_0,y_0)}(y-y_0)$

The approximation

$$f(x,y) \approx L(x,y)$$

is standard linear approximation of f at (x_0,y_0)."

Example 5

Find the linearization of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point $(3,2)$

Solution:

$$\because L(x,y) = f(x_0,y_0) + f_x\big|_{(x_0,y_0)}(x-x_0) + f_y\big|_{(x_0,y_0)}(y-y_0) \rightarrow (1)$$

$$f(3,2) = (3)^2 - 3(2) + \frac{1}{2}(2)^2 + 3 = 8$$

$f_x = 2x - y$

$f_x\big|_{(3,2)} = 2(3) - 2 = 4$

$f_y = -x + y$

$f_y\big|_{(3,2)} = -3 + 2 = -1$

(1) $\Rightarrow$

$$\mathcal{L}(x,y) = 8 + 4(x-3) - (y-2)$$

$$\mathcal{L}(x,y) = 4x - y - 2$$

Ex#24.6

Q (26, 28, 29, 30)

Q # 29

$$f(x,y) = e^x \cdot \cos y \quad \text{at } (0, \pi/2)$$

Solution:

$$\mathcal{L}(x,y) = f(x_0, y_0) + f_x\Big|_{(x_0,y_0)}(x - x_0) + f_y\Big|_{(x_0,y_0)}(y - y_0)$$

$$f(0, \pi/2) = e^x \cdot \cos y$$
$$= e^0 \cdot \cos(\pi/2)$$
$$= 1 \cdot 0 = 0$$

$$f_x\Big|_{(0,\pi/2)} = \cos y \cdot e^x$$
$$= \cos(\pi/2) \cdot e^0 = 0$$

$$f_y\Big|_{(0,\pi/2)} = e^x \cdot \cos y = e^0 - \cancel{\sin}(\pi/2) = \cancel{0} \; e^0 \cdot \sin(90) = -1$$

$$\mathcal{L}(x,y) = 0 + 0(x-0) + \overset{-1}{\cancel{0}}(y - \pi/2)$$
$$= \pi/2 - y$$

Ex # 14.7

13-03-18

Extreme Values & Saddle Points

⇒ Local maximum, Local minimum:

Let $f(x,y)$ be defined on a region R containing the point (a,b). then,

max: $f(a,b)$ is local maximum value of f, if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

min: $f(a,b)$ is local minimum value of f, if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

⇒ First derivative Test for Local extreem values:

If $f(x,y)$ has a local maximum or local minimum value at an interior point (a,b) of its domain and if the first partial derivatives exists there, then $f_x(a,b) = 0$ and

$$f_y(a,b) = 0$$

→ Critical Point:

"An interior point of the domain of the function $f(x,y)$, one or both f_x and f_y are zero or where one or both f_x and f_y do not exist is a critical point of f."

EXAMPLE # 01
1029

v. Imp.

Find local extreme values of $f(x,y) = x^2 + y^2$

Solution:

The domain of f is entire plane and first partial derivative, i.e., $f_x = 2x$ & $f_y = 2y$ exist everywhere,

Critical Points:

$f_x = 0$; $f_y = 0$

$2x = 0$; $2y = 0$

$x = 0$; $y = 0$

{Therefore local extreme value occer only when}

Only possibility is origin.

Since f is never negative, we see that origin gives local minimum

→ Saddle Point:

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b), if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points where $f(x,y) < f(a,b)$.

Example # 03

1029

Find local extreme values, (if any) of $f(x,y) = y^2 - x^2$

Solution:

The domain of f is entire plane and first partial derivative $f_x = -2x$ and $f_y = 2y$ exist everywhere, so local extreme values can occur only at origion $(0,0)$.

Along positive x-axis, $y=0$, i.e.

$$f(x,0) = -x^2 < 0$$

Along positive y-axis, $x=0$, i.e.

$$f(0,y) = y^2 > 0$$

* As f changes sign, ~~do~~ so f has a saddle point at origin.

NOTE: The expression $f_{xx}f_{yy}-f_{xy}^2$ is called discriminant or Hessian of f, i.e.

$$f_{xx}f_{yy}-f_{xy}^2=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Theorem: 14.3.18

Second derivative test for local extreme values:

1030

"Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b)=0$ & $f_y(a,b)=0$. Then

i) f has local maximum at (a,b), if

$$f_{xx}<0 \text{ and } f_{xx}f_{yy}-f_{xy}^2>0 \text{ at } (a,b)$$

ii) f has local minimum at (a,b), if

$$f_{xx}>0 \text{ and } f_{xx}f_{yy}-f_{xy}^2>0 \text{ at } (a,b)$$

iii) f has Saddle point at (a,b), if

$$f_{xx}f_{yy}-f_{xy}^2<0 \text{ at } (a,b)$$

iv) The test is inconclusive at (a,b), if

$$f_{xx}f_{yy}-f_{xy}^2=0$$

The value of f at $(-2,-2)$ is:

$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$

$f(-2,-2) = 8$

Pg#1034
Ex#14.7
Q#1&5

EXAMPLE # 03

Find local extreme values of $f(x,y)$

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Solution:

The function is defined and differentiable for all x and y. The function therefore has extreme values only at the points where $f_x = 0$ and $f_y = 0$.

$y - 2x - 2 = 0 \rightarrow (1)$ & $x - 2y - 2 = 0 \rightarrow (2)$

$y = 2x + 2$ put in (2)

$x - 2(2x+2) - 2 = 0$

$x - 4x - 4 - 2 = 0$

$x = -2 \rightarrow (3)$

$(1) \Rightarrow y - 2(-2) - 2 = 0$

$y = -2 \rightarrow (4)$

So, $(-2,-2)$ is the only point, where f may take on extreme value.

Since; $f_x = y - 2x - 2$, $f_{xy} = 1$

$f_{xx} = -2 < 0$, $f_{xy}^2 = 1$

$f_y = x - 2y - 2$

$f_{yy} = -2$ Now, $f_{xx} \times f_{yy} - f_{xy}^2 = 3 > 0$

As $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$,

therefore f has local maximum at $(-2,-2)$