Usama Sarwar

FA17-BCS-090

Question 1

(a)

Solution:

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 2$$

$$f(x_0) = 0, \; f(x_1) = y, \; f(x_2) = 3, \; f(x_3) = 2$$

The Lagrange polynomial of order 3, connecting the four points, is given by

$$P_3(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3(x) f(x_3),$$

where

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

After putting the values, we get;

$$L_0(x) = -x^3 + \frac{7}{2}x^2 - 7/2\, x + 1$$
$$L_1(x) = 8/3\, x^3 - 8x^2 + 16/3\, x$$
$$L_2(x) = -2x^3 + 5x^2 - 2x$$
$$L_3(x) = 1/3\, x^3 - 1/2\, x^2 + 1/6\, x$$

Thus,

$$P_3(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3(x) f(x_3)$$

After calculation

$$= \left(\frac{8y-16}{3}\right) x^3 + \left(-8y+14\right) x^2 + \left(\frac{16y-17}{3}\right) x$$

Since we want the coefficient of x^3 to be equal to 6, we need;

$$\frac{8y-16}{3} = 6$$

or $\quad y = 17/4 = 4.25$

with such y, the polynomial becomes

$$P_3(x) = 6x^3 - 20x^2 + 17x$$

We can check whether this polynomial function f, that is, whether we got the correct answer. Note that

$$P_3(0) = 0$$
$$P_3(0.5) = 4.25$$
$$P_3(1) = 3$$
$$P_3(2) = 2$$

(b)

Conditions for the construction of clamped cubic spline interpolant

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3, \; i = 0,1,\dots,n-1$$

Conditions:

1. $S_i(x_i) = y_i$, $i = 0,1,2,\dots,n$ - interpolating data (x_i, y_i)
2. $S_i(x_i+1) = S_{i+1}(x_i+1)$, $i = 0,1,2,\dots,n-2$ - continuity at interpior points
3. $S_i'(x_i+1) = S_{i+1}'(x_{i+1})$, $i = 0,1,\dots,n-2$, - continuous slope at interior points
4. $S_i''(x_i+1) = S_{i+1}''(x_{i+1})$, $i = 0,1,\dots n-2$ - continuous curvature at interior points.
5. $S_{i_0}''(x_0) = 0$, and $S_{n-1}(x_n) = 0$ - free spline or natural spline

   $S_0'(x_0) = \alpha$ and $S_{n-1}(x_n) = \beta$ - clamped spline

(c)

Taylor Method of order n

$$y_0 = a$$
$$y_{i+1} = y_i + hT_n(t_i, y_i), \; i = 0, 1, 2, \ldots, N-1$$

Using Taylor's Theorem to derive Euler's method - Suppose that $y(t)$, the unique solution to, has continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N-1$

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\varepsilon_i)$$

for some number ε_i in (t_i, t_{i+1})
Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\varepsilon_i)$$

and, because $y(t)$ satisfies the differential equation

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\varepsilon_i)$$

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \dots, N$, by deleting the remainder term. Thus Euler's method is

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h f(t_i, w_i), \text{ for each } i = 0, 1, \dots, N-1$$

For Euler's Method, we just take first 2 terms only.

$$y(x+h) \approx y(x) + h y'(x)$$

The last term is just h times our $\frac{dy}{dx}$ expression, so we can write Euler's Method as follows

$$y(x+h) \approx y(x) + h f(x, y)$$

QUESTION 4

$$2x - 2y + 5z = 13$$
$$2x + 3y + 4z = 20$$
$$3x - y + 3z = 10$$

$$Ax = b$$

$$\begin{bmatrix} 2 & -2 & 5 \\ 2 & 3 & 4 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 10 \end{bmatrix}$$

Augmented Matrix

$$\sim \left[\begin{array}{ccc:c} 2 & -2 & 5 & 13 \\ 2 & 3 & 4 & 20 \\ 3 & -1 & 3 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1.5 & 2.5 & 6.5 \\ 2 & 3 & 4 & 20 \\ 3 & -1 & 3 & 10 \end{array}\right] \quad \tfrac{1}{2} R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2.5 & 6.5 \\ 0 & 5 & -1 & 7 \\ 3 & -1 & 3 & 10 \end{array}\right] \quad -3R_1 + R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2.5 & 6.5 \\ 0 & 5 & -1 & 7 \\ 0 & 2 & -4.5 & 9.5 \end{array}\right] \quad -3R_1 + R_3 \rightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2.5 & 6.5 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 2 & -4.5 & -9.5 \end{array}\right] \quad 1/5\, R_2 \rightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2.3 & 7.9 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 0 & -4.5 & -9.5 \end{array}\right] \quad R_1 + R_2 \rightarrow R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2.3 & 7.9 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -0.2439\, R_3 \rightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -0.2 & 1.4 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \cancel{R_1 \rightarrow} \quad R + 2.3 \times R_3 \rightarrow R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x = 1 \qquad y = 2 \qquad z = 3$

with pivoting after pivoting matrix

$$\sim \begin{bmatrix} 3 & -1 & 3 & : & 10 \\ 2 & 3 & 4 & : & 20 \\ 2 & -2 & 5 & : & 13 \end{bmatrix} \quad R_1 \longleftrightarrow R_3$$

Then Solve Matrix

$$\sim \begin{bmatrix} 1 & -0.333 & 1 & : & 3.33 \\ 2 & 3 & 4 & : & 20 \\ 2 & -2 & 5\cancel{3} & : & 13 \end{bmatrix} \quad 1/3\, R_1 \rightarrow R_1$$

$$-2R_1 + R_2 \rightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -0.33 & 1 & : & 3.33 \\ 0 & 3.6667 & 2 & : & 13.33 \\ 2 & -2 & 5 & : & 13 \end{bmatrix} \quad 0.2727\, R_2 \rightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -0.33 & 1 & : & 3.33 \\ 0 & 1 & 0.54 & : & 3.63 \\ 0 & -1.33 & 3 & : & 6.33 \end{bmatrix}$$

$$0.33\, R_2 + R_1 \rightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1.1818 & : & 4.54 \\ 0 & 1 & 0.5455 & : & 3.63 \\ 0 & -1.33 & 3 & : & 6.33 \end{bmatrix}$$

$$1.33 R_2 + R_3 \rightarrow R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1.1818 & 4.545 \\ 0 & 1 & 0.54 & 3.636 \\ 0 & 0 & 3.7273 & 11.1181 \end{array}\right]$$

$$0.2683 R_3 \rightarrow R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1.1818 & 4.5455 \\ 0 & 1 & 0.5455 & 3.6364 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \leftarrow R_1 - 1.1818 \times R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0.5455 & 3.6364 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$-0.5455 R_3 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x = 1$

$y = 2$

$z = 3$

QUESTION 3

(a)

Composite trapezoid rule

$$f(x) = \int_0^{\pi} \sin x \, dx$$

within 10^{-4}

$n = ?$ & $h = ?$

Error term $\frac{b-a}{12} h^2 f''(u)$

$$\left| \frac{b-a}{12} h^2 f''(u) \right| \rightarrow (1)$$

$$\frac{\pi - 0}{12} h^2$$

$f'(u) = -\cos u$

$f''(u) = -\sin u$

put in eq (1)

$$\left| \frac{\pi - 0}{12} h^2 (-\sin u) \right|$$

$$\frac{\pi - 0}{2} h^2 (\overset{\cos}{\sin} u)$$

Maximum value of $\sin(u)$ is $\frac{\pi}{12} h^2 (1) < 0.0001$

$$\pi h^2 < 0.0001 \times 12$$

$$h^2 < \frac{0.0001 \times 12}{\pi}$$

$$h = \frac{b-a}{n}$$

$$h = \frac{\pi - 0}{n}$$

$$\boxed{h = \frac{\pi}{n}}$$

$$\left(\frac{\pi}{n}\right)^2 < \frac{0.0001 \times 12}{\pi}$$

Taking reciprocal

$$\frac{n^2}{\pi^2} > \frac{\pi}{0.0001 \times 12}$$

$$n^2 > \frac{\pi^3}{0.0001 \times 12}$$

$$n^2 > \frac{\pi^3}{1.2 \times 10^{-3}}$$

$n^2 > 25838.5639$

$n \geq 160.7437834$

$\boxed{n = 161} \Rightarrow h = \frac{\pi}{n}$

$\Rightarrow h = \frac{1}{161}\pi$

Not solve composite trapzoid rule because n=161. If n less than 14 then solve.

(b)

Composite Simpson's Rule

$$\left| \frac{\pi h^4}{180} f^{iv}(u) \right| \text{ (formula)}$$

$$f(u) = \sin u$$
$$f'(u) = -\cos u$$
$$f''(u) = -\sin u$$
$$f'''(u) = \cos u$$
$$f^{iv}(u) = -\sin u$$

$$\left| \frac{\pi h^4}{180} (-\sin u) \right|$$

Maximum value of sin is 1

$$\left| \frac{\pi h^4}{180} (1) \right|$$

$$\frac{\pi h^4}{180} < 0.0001$$

$$\pi h^4 < 0.0001 \times 180$$

$$\boxed{h = \frac{\pi}{h}}$$

$$\pi\left(\frac{\pi}{h}\right)^4 < 0.0001 \times 180$$

$$\frac{\pi^5}{h^4} < 0.018$$

$$\pi^5 < 0.018 \times h^4$$

$$h > \left(\frac{\pi^5}{0.018}\right)^{1/4}$$

$$n > (17001.0936)^{1/4}$$

$$n > 11.41876709$$

$$\boxed{n = 12}$$

If $n=12$ then apply sinspons rule

$$n = \frac{b-a}{n}$$

$$= \frac{\pi - 0}{12}$$

$$\boxed{n = \frac{\pi}{12}}$$

Simpson's rule $\approx \frac{h}{3}\left(f(x_0) + 4f(x_1)\right.$
$+2f(x_2), \dots f(x_n)$

(c)

Simpsons rule $\approx \frac{h}{3}\left(f(x_0)+4f(x_1)+2f(x_2),\dots,f(x_n)\right)$

$a=0, \quad b=3.14285$

$\boxed{h=\pi/12}$

$x_0=0, \quad x_2=\frac{3142857143}{12000000000},$

$, x_3=\frac{3142857143}{6000000000}$

$, x, \dots, \frac{3142857143}{120000000000}, \frac{34571428573}{12000000000},$

$, \frac{3142.857143}{10000000000}$

$f(x_0)=0$

$4f(x_1)=1.03568330898776$

$2f(x_2)=2f\left(\frac{3142857143}{600000000}\right)=1.00036500044457$

$\vdots$

$4f(x_{11})=1.0003029700835385$

$$f(x_{12}) = \sin\left(\frac{3142857143}{1000000000}\right)$$

$$= 0.001264489073234 38$$

Now just sum all the values;

$$\boxed{\text{Simpson rule} \approx 2.000051909978 19}$$

Exact value of integral - 1.9999

Apply composite simpsons rule to this problem subdivide $[0, \pi]$

$[0, 1.5714][1.5714, 3.142857143]$

$$\int_0^{1.5714} \sin x \, dx + \int_{1.5714}^{3.14285} \sin x \, dx$$

$n = 24$

$$\int_0^{1.5714} \sin x dx$$

$$a = 0, \frac{2619}{40000}, \frac{2619}{20000}, \ldots, \frac{28809}{20000}$$

$$, \ldots, \frac{60237}{40000}, \frac{7857}{5000} = b$$

Now we just evaluate the function those points

$f(x_0) = 0$

$4f(x_1) = 4f\left(\frac{2619}{40000}\right) = 0.261712913626209$

$2f(x_2) = 0.26115213555 9263$

$\vdots$

$f(x_{23}) = 3.99158637371 2184$

$f(x_{24}) = 0.999999 1.7789336$

finally just Esump above the value

1.00060377538313

$$\int_0^{1.5714} \sin x\,dx = 1.000603775 3813$$

now

$$h = \int_{1.5714}^{3.142} \frac{b-a}{n}$$

$$h = \frac{31429}{480000}$$

subintervals

$\frac{78517}{5000}, \frac{785701}{480000}, \frac{81713}{48000}, \ldots \frac{144541}{48000}$

$, \frac{62857}{2000}$

$f(x_0) = 0.999999817789$
$4f(x_1) = 3.99126984493$
$2f(x_2) = 1.9827174827$
$\vdots$
$4f(x_{23}) = 0.5267024151599 81$
$f(x_{24}) = -0.00125734607891277$

sum all values

$$\int_{1.57}^{3.142} \sin x \, dx = 0.99939563847593 8$$

Now

$$\int_0^{1.57} \sin^4 dx + \int_{1.57}^{3.142} \sin x \, dx$$

$= \cancel{1.00} \, 1.99999414$

Error $= -1.99999 + 1.999999414$

Error $= 0.00000\boxed{414}$ closely to first value

QUESTION 2

(a)

Modified Euler Method

$$y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, p_{k+1})\right]$$

$y_k + hf(t_k, y_k)$; $h = 0.1$ $y(0) = 1$

$$N = \frac{1-0}{0.1} = 10$$

$$f(t_i, w_i) = -t_i w_i + \frac{4t_i}{w_i}$$

$$f(t_{i+1}, w_i + hf(t_i, w_i)) = \left[-t_i + f(w_i + h\right)$$

$$\left(-t_i w + \frac{4t_i}{w_i}\right]\Big) + \frac{4t_{i+1}}{w_i + h\left(-t_i w_i + \frac{4t_i}{w_i}\right)}\Bigg]$$

$$w_{i+1} - w_i + h/2\left[\left(-t_i w_i + \frac{4t_i}{w_i}\right) + \right.$$

$$\left[-t_{i+1} w_i + h\left(-t_i + f\left(w_i + h\right) + \frac{4t_{i+1}}{\text{with}\left(-t_i w_i \frac{4t_i}{w_i}\right.}\right.\right.$$

$$t_{w_i+1} - w_i + h/2\left[\left(-t_i w_i + h\left(-t_i w_i + \frac{4t_i}{w_i}\right)\right.\right.$$

$$+ \frac{4t_{i+1}}{w_i + h\left(-t_i w_i + \frac{4t_i}{w_i}\right)}$$

$w_1 = 1.015$

$w_2 = 1.059247$

$w_4 = 1.21122$

$w_5 = 1.30414$

$w_6 = 1.399716$

$w_7 = 1.4929$

$w_8 = 1.58045$

$w_9 = 1.6598144$

$w_{10} = 1.729739474$

i	$a + ih$
0	0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1.10

(b)

$y(t) = \sqrt{4-3e^{-t^2}}$

$y(0) = \sqrt{4-3e^{-(0)^2}} = 1$

$y(0.1) = 1.13379$

$y(0.2) = 1.2425006$

$y(0.3) = 1.33324$

$y(0.4) = 1.426620473$

$y(0.5) = 1.4766204$

$y(0.6) = 1.53133336$

$y(0.7) = 1.584374$

$y(0.8) = 1.6228506$

$y(0.9) = 1.667424$

$y(1.10) = 1.70187005$