Real Analysis

Chapter 1

Measurement Theory

Def. A set E is said to be bounded if it is contained in a ball with finite radius.

Def. A closed & bounded set is said to be compact

Prop. Any dense set has the Heine-Borel property

i.e. If E is a dense set, $E \subset \bigcup_{\alpha} O_{\alpha}$, where O_{α}'s are open sets, then there exist finite open sets $O_{\alpha_1}, \cdots, O_{\alpha_N}$ s.t. $E \subset \bigcup_{j=1}^{N} O_{\alpha_j}$

Thm. Every open subset O of $\mathbb{R}$ can be written as the combination of countable disjoint open intervals.

proof: $\forall x \in O$, let I_x be the biggest open interval contained in O where $x \in I_x$

If $a_x = \inf\{a < x : (a, x) \subset O\}$, $b_x = \sup\{b > x : (x, b) \subset O\}$

Then we have $a_x < x < b_x$. Let $I_x = (a_x, b_x)$

thus we get $x \in I_x$ & $I_x \subset O$. Therefore $O = \bigcup_{x \in O} I_x$

Now suppose $I_x \cap I_y \neq \emptyset$, then we must have $(I_x \cup I_y) \subset O$

Since I_x is the biggest one, then $(I_x \cup I_y) \subset I_x$

similarly, we have $(I_x \cup I_y) \subset I_y$, hence $I_x = I_y$

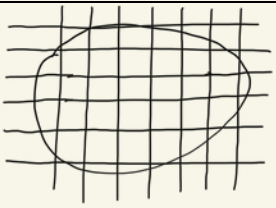
So any two distinct interval in $I = \{I_x\}_{x \in O}$ should be disjoint

Every interval contains a rational number, and since they are disjoint, then they contain distinct rational numbers

Therefore, I is countable

Lemma. if $R, R_1, \cdots, R_N$ are rectangles and $R \subset \bigcup_{k=1}^{N} R_k$

then $|R| \leq \sum_{k=1}^{N} |R_k|$



Thm. Every open subsets of $\mathbb{R}^d$, where $d \geq 1$, can be uniquely written as the combination of countable closed cubes which are almost disjoint.

Cantor Set

For $C_0 = [0,1]$, we remove the central part $(\frac{1}{3}, \frac{2}{3})$

Then we get $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, then we repeat this process

$C_2 = [0, \frac{1}{9}] \cup [\frac{1}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, repeat the process on every subinterval

then we get a dense sequence s.t. $C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_k \supset C_{k+1} \supset \cdots$

Define the Cantor Set as $C = \bigcap_{k=0}^{\infty} C_k$

Def. Suppose E is an arbitary subset of $\mathbb{R}^d$, then its outer measurement is $m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$

Where Q_j are closed cubes that $E \subset \bigcup_{j=1}^{\infty} Q_j$

Ex. The outer measurement of a closed cube equals to its volumn.

Since Q is covered by itself

then $m_*(Q) \leq |Q|$

Check. $|Q| \leq m_*(Q)$

for $\forall$ fixed $\varepsilon > 0$, we choose a open cube S_j that contains Q_j for each j

s.t. $|S_j| \leq (1+\varepsilon)|Q_j|$

from the previous lemma

$|Q| \leq \sum_{j=1}^{\infty} |S_j| \leq (1+\varepsilon) \sum_{j=1}^{\infty} |Q_j|$

Since ε is arbitary, then

$|Q| \leq \sum_{j=1}^{\infty} |Q_j|$, hence $|Q| \leq m_*(Q)$

Prop.

If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$

If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$

proof for (2): WLOG $m_*(E_j) < \infty$

$\forall \varepsilon > 0$, we can have E_j covered by closed cubes

i.e. $E_j \subset \bigcup_{k=1}^{\infty} Q_{k,j}$, $\sum_{k=1}^{\infty} |Q_{k,j}| \leq m_*(E_j) + \frac{\varepsilon}{2^j}$

And hence $E \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j,k=1}^{\infty} Q_{k,j}$

$m_*(E) \leq \sum_{j=1}^{\infty} (m_*(E_j) + \frac{\varepsilon}{2^j}) = \sum_{j=1}^{\infty} m_*(E_j) + \varepsilon$

Hence $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$

Prop. The Cantor set C is closed, $m(C) = 0$ & is uncountable

pf:) $m([0,1]) = m(O \cup C) = m(O) + m(C)$

$C_1 \supset C_2 \supset \cdots \Rightarrow \lim_{n \to \infty} m(C_n) = m(\bigcap_{n=1}^{\infty} C_n) = m(C)$

$m(C_n) = (\frac{2}{3})^{n-1}$, so $m(C) = 0$

Suppose C is countable. then $C = \{c_1, c_2, \cdots\}$

$c_1 \notin$ either $[0, \frac{1}{3}]$ or $[\frac{2}{3}, 1]$, name it as F_1

F_1 contains 2 intervals of C_2, $c_2 \notin$ either of the 2 intervals

denoted as F_2

so by ascending

$c_1, c_2, \cdots, c_n \notin F_n$

Since $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$

then $\exists\, c$ s.t. $c \in \bigcap_{n=1}^{\infty} F_n$

Clearly $c \in C$

however in this case

$c \notin \bigcap_{n=1}^{\infty} F_n \Rightarrow \Leftarrow$

※

If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(O)$, where O is any open set that covers E

proof: Since $E \subset O$, clearly $m_*(E) \leq \inf m_*(O)$

Let $\varepsilon > 0$ arbitary & Choose cubes Q_j s.t. $E \subset \bigcup_{j=1}^{\infty} Q_j$ and $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \frac{\varepsilon}{2}$. Let Q_j^o be the open cubes that covers Q_j, s.t. $|Q_j^o| \leq |Q_j| + \frac{\varepsilon}{2^{j+1}}$

$m_*(O) \leq \sum_{j=1}^{\infty} m_*(Q_j^o) \leq \sum_{j=1}^{\infty} |Q_j| + \frac{\varepsilon}{2} \leq m_*(E) + \varepsilon$

therefore $m_*(O) \leq m_*(E)$, and the proof is done

If $E = E_1 \cup E_2$, $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$

proof: Let $E \subset \bigcup_{j=1}^{\infty} Q_j$, $J_1 \cap J_2 = \emptyset$, $J_1 \cup J_2 = \mathbb{N}^+$

$E_1 \subset \bigcup_{j \in J_1} Q_j$, $E_2 \subset \bigcup_{j \in J_2} Q_j$, then $m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j|$

$= \sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \varepsilon$, Since ε is arbitary, then $m_*(E_1) + m_*(E_2) \leq m(E)$

And clearly $m_*(E) \leq m_*(E_1) + m_*(E_2)$, thus it holds

If E is the combination of countable almost distoint cubes. i.e. $E = \bigcup_{j=1}^{\infty} Q_j$, then $m_*(E) = \sum_{j=1}^{\infty} |Q_j|$

proof: Let $\widetilde{Q}_j$ be the cube that is rigorously contained in Q_j & $|Q_j| \leq |\widetilde{Q}_j| + \frac{\varepsilon}{2^j}$

Since $\widetilde{Q}_j$ are disjoint, then $m_*(\bigcup_{j=1}^{N} \widetilde{Q}_j) = \sum_{j=1}^{N} m_*(\widetilde{Q}_j) \geq \sum_{j=1}^{N} (|Q_j| - \frac{\varepsilon}{2^j})$

Since $m_*(E) \supset \bigcup_{j=1}^{N} \widetilde{Q}_j$, then $m_*(E) \geq \sum_{j=1}^{N} |Q_j| - \varepsilon$

so $m_*(E) \geq \sum_{j=1}^{\infty} |Q_j|$

Def. For a subset of $\mathbb{R}^d$, E, if $\forall \varepsilon>0$, $\exists$ open set O where $E \subset O$ & $m_*(O \setminus E) \leq \varepsilon$, then we say that E is Lebesque measurable (or measurable)

Def. If E is measurable, we define the measurement of E as : $m(E) := m_*(E)$

Prop. (1) Every open set in $\mathbb{R}^d$ is measurable

(2) If $m_*(E)=0$, then E is measurable

(3) The combination of countable measurable sets is measurable

Suppose $E=\bigcup_{j=1}^{\infty} E_j$, for given $\varepsilon>0$, we can choose O_j s.t. $E_j \subset O_j$ & $m_*(O_j \setminus E_j) \leq \frac{\varepsilon}{2^j}$ for each j, where O_j's are open sets, thus $O=\bigcup_{j=1}^{\infty} O_j$ is a open set, $E \subset O$ and $O \setminus E \subset \bigcup_{j=1}^{\infty}(O_j \setminus E_j)$, thus $m(O \setminus E) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$

(4) Every closed set is measurable

Since every closed set is a combination of countable compact sets then we only need to show that a dense set is measurable

Suppose F is dense, and we can choose O s.t. $F \subset O$

$m_*(F) = \inf m_*(O)$, where O is an open set

Clearly $O \setminus F$ is an open set, then $O \setminus F = \bigcup_{j=1}^{\infty} Q_j$

where Q_j's are almost disjoint closed cubes

for a fixed N, $K=\bigcup_{j=1}^{N} Q_j$ is also a dense set

so $d(F,K)>0$. Since $K \cup F \subset O$

then $m_*(O) \geq m_*(K) + m_*(F) = m_*(F) + \sum_{j=1}^{N} |Q_j|$

so $\sum_{j=1}^{N} |Q_j| \leq m_*(O) - m_*(F) \leq \varepsilon$, thus $m_*(O \setminus F) \leq \varepsilon$

Ex. The measurement of the set of algebraic number is 0

proof: $A=\{x \mid a_0+\cdots+a_n x^n=0, a_i \in \mathbb{Z}, n \in \mathbb{N}\}$

Clearly the set is countable

Thus $m(A)=0$

<u>Lemma.</u> If F is closed, K compact, $F\cap K=\emptyset$, then $d(F,K)>0$

Since F is closed, $\forall x\in K$, $\exists\, \delta_x>0$, s.t. $d(x,F)>3\delta_x$

Since $\bigcup_{x\in K} B(x,2\delta_x)$ covers K & K is compact, we can find a

subcover $\bigcup_{j=1}^{N} B(x_j,2\delta_j)$, Take $\delta=\min\{\delta_1,\cdots,\delta_N\}$

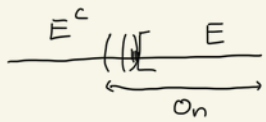
then clearly $d(K,F)\geqslant\delta>0$.

If $x\in K$ & $y\in F$, then $|x_j-x|\leq 2\delta_j$ for some j

Since $|y-x_i|\leq 2\delta_i$, then $|y-x|\geqslant |y-x_i|-|x-x_i|\geqslant\delta>0$

<u>Prop.</u>

(5) If E is measurable, then E^c is measurable



For each integer n, we can pick an open set O_n where $E\subset O_n$

And $m_*(O_n\setminus E)\leqslant\frac{1}{n}$. Since O_n is open, then O_n^c is closed, which is

measurable. then $S=\bigcup_{n=1}^{\infty}O_n^c$ is measurable

Notice that $(E^c\setminus S)\subset(O_n\setminus E)$ for $\forall n$,

Thus $m_*(E^c\setminus S)\leqslant\frac{1}{n}$ for $\forall n\in\mathbb{N}^+$, therefore $m_*(E^c\setminus S)=0$

$E^c=(E\setminus S)\cup S$, Since $E\setminus S$ & S are both measurable

therefore E^c is measurable

(6) The intersection of countable measurable sets is measurable

Since $\bigcap_{j=1}^{\infty}E=(\bigcup_{j=1}^{\infty}E^c)^c$, then it is measurable

<u>Thm.</u> If $E_1,E_2,\cdots$ are disjoint measurable sets, $E=\bigcup_{j=1}^{\infty}E_j$

then $m(E)=\sum_{j=1}^{\infty}m(E_j)$

proof: Clearly E & E_j's are measurable

Then we only need to show that $m_*(E)=\sum_{j=1}^{\infty}m_*(E_j)$

$d(E_i,E_j)\geqslant 0$ for $\forall\, i\neq j$ since E_i & E_j are disjoint

Thus $m_*(E)=\sum_{j=1}^{\infty}m_*(E_j)$

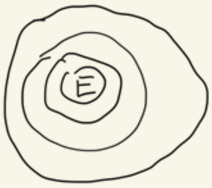
$m(E\setminus F)=m(E)-m(F)$ if $F\subset E$

this is because $E=(E\setminus F)\cup F$

Def. If $E_1, E_2, \cdots$ is a countable collection of subsets in $\mathbb{R}^d$

$E_k \uparrow E \iff E_k \subset E_{k+1}$ for $\forall k$, $E = \bigcup_{k=1}^{\infty} E_k$

$E_k \downarrow E \iff E_k \supset E_{k+1}$ for $\forall k$, $E = \bigcap_{k=1}^{\infty} E_k$



Thm. Let $E_1, E_2, \cdots$ be measurable subsets in $\mathbb{R}^d$

(1) If $E_k \uparrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$ Continuity of measurable set

Let $G_1 = E_1$, $G_2 = E_2 \setminus E_1$, $\cdots$, $G_k = E_k \setminus E_{k-1}$, $\cdots$

Clearly $E = \bigcup_{j=1}^{\infty} G_j$, thus $m(E) = m(\bigcup_{j=1}^{\infty} G_j) = \sum_{j=1}^{\infty} m(G_j) = \lim_{N\to\infty} m(E_N)$

(2) If $E_k \downarrow E$, $m(E_k) < \infty$, then $m(E) = \lim_{N\to\infty} m(E_N)$

Let $G_1 = E_1 \setminus E_2$, $\cdots$, $G_k = E_k \setminus E_{k+1}$

Clearly $E_1 = E \cup (\bigcup_{k=1}^{\infty} G_k)$

then $m(E_1) = m(E) + \sum_{k=1}^{\infty} m(G_k) = m(E) + m(E_1) + \lim_{N\to\infty} m(E_N)$

thus $m(E) = \lim_{N\to\infty} m(E_N)$

Example.

$B_k = [k, \infty)$, $k = 1, 2, \cdots$

$\bigcap_{k=1}^{\infty} B_k = \emptyset$

Thm. Suppose E is a measurable subset of $\mathbb{R}^d$, then for $\forall \varepsilon > 0$

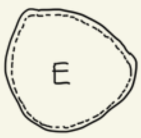
(1) $\exists$ an open set O, s.t. $E \subset O$ & $m(O \setminus E) \leq \varepsilon$

(2) $\exists$ a closed set F, s.t. $F \subset E$ & $m(E \setminus F) \leq \varepsilon$

(3) If $m(E) < \infty$, then $\exists$ a compact set K s.t. $K \subset E$ & $m(E \setminus K) \leq \varepsilon$

(4) If $m(E) < \infty$, then $\exists$ a finite combination of almost disjoint closed cubes $F = \bigcup_{j=1}^{N} Q_j$, s.t. $m(E \Delta F) \leq \varepsilon$

where $E \Delta F = (E \setminus F) \cup (F \setminus E)$ (In $\mathbb{R}^1$, change the cubes into open intervals)

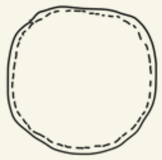


(1) This is from the definition

(2) Clearly E^c is measurable, the $\exists$ an open set O s.t. $E^c \subset O$

while $m(O / E^c) \leq \varepsilon$, thus let $F = O^c$, $E \setminus F = O \setminus E^c$, $m(E \setminus F) \leq \varepsilon$

Ex. Let C denote the Cantor set, then $m(C) = 0$



(3) Let F be a closed subset s.t. $m(E\setminus F)\leq\frac{\varepsilon}{2}$

Define dense sets $K_n=F\cap B_n$, where B_n is a ball centered at the origin with radius n, then $(E\setminus K_n)\downarrow(E\setminus F)$

Then $\forall\varepsilon>0$, $\exists N\in\mathbb{N}^+$, s.t. $m(E\setminus K_n)\leq\frac{\varepsilon}{2}+m(E\setminus F)\leq\varepsilon$ when $n>N$

where K_n is a dense set

(4) $E\subset\bigcup_{j=1}^{\infty}Q_j$, $\sum_{j=1}^{\infty}|Q_j|\leq m(E)+\frac{\varepsilon}{2}<\infty$

Thus $\forall\varepsilon>0$, $\exists N\in\mathbb{N}^+$, s.t. $\sum_{j=N+1}^{\infty}|Q_j|<\varepsilon$

Let $F=\bigcup_{j=1}^{N}Q_j$, where Q_j's are almost disjoint closed cubes

$m(E\triangle F)=m(E\setminus F)+m(F\setminus E)\leq m(\bigcup_{j=N+1}^{\infty}Q_j)+m(\bigcup_{j=1}^{\infty}Q_j\setminus E)\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Ex.

A countable point set is measurable

Def. Let $\mathcal{A}\subset\mathcal{P}(\mathbb{R}^d)$. $\mathcal{A}$ is called an algebra if

(1) if $A_1,\cdots,A_n\in\mathcal{A}$, then $\bigcup_{j=1}^{n}A_j\in\mathcal{A}$

(2) if $A\in\mathcal{A}$, then $A^c\in\mathcal{A}$

Σ (taking sum) σ (taking combination)

Def. Let Ω be a set in $\mathbb{R}^d$. A sigma-algebra (σ) $\mathcal{F}$ on Ω is a collection of subsets of Ω that satisfies:

① $\phi\in\mathcal{F}$

② If $A\in\mathcal{F}$, then the complement $A^c=\Omega\setminus A\in\mathcal{F}$

③ If $A_1,A_2,\cdots$ is countable sequence of sets in $\mathcal{F}$, then $\bigcup_{n=1}^{\infty}A_n\in\mathcal{F}$

Borel-Cantelli Lemma.

Let $\{E_k\}$ be a sequence of measurable sets.
If $\sum_{k=1}^{\infty} m(E_k) < \infty$, then almost all $x \in \mathbb{R}^n$ belongs to at most finitely many E_k's.
i.e. $m(\{x \in \mathbb{R}^n \mid x \in E_k \text{ for infinitely many } k\text{'s}\}) = 0$

pf:) Let $B_n = \bigcup_{k=n}^{\infty} E_k$, which is descending.
so $m(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} (B_n) = \lim_{n\to\infty} m(\bigcup_{k=n}^{\infty} E_k) \leq \lim_{n\to\infty} (\sum_{k=n}^{\infty} m(E_k))$

Since $x \in E_k$ for infinitely many k's, then $\exists \{k_j\}$ s.t. $x \in E_{k_j}$, where $j = 1, 2, \cdots$, $k_1 > N$

So $\{x \in \mathbb{R}^n \mid x \in E_k \text{ for } \infty \text{ many } k\text{'s}\} \subset \bigcap_{n=1}^{\infty} B_n$

Thm. (Vitali) Any set $E \subseteq \mathbb{R}$ with positive measurement has a non-negative set.

Def. We say that $a, b \in \mathbb{R}$ are rationally equi. if $a - b \in \mathbb{Q}$

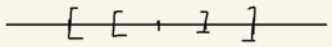
从E的每一个等价集中选择一个元素(不必可数)

For $E \subseteq \mathbb{R}$, Let $C_E = \{$Exactly one element from every rationally equivalence class of $E\}$

Here we let $C_E \subset E$

(1) The difference of between two element in C_E is NOT rational

(2) $\forall x \in E$, $\exists! c \in C_E$, s.t. $x - c \in \mathbb{Q}$



Proof of the theorem:

W.L.O.G, assume $E \subseteq [-b, b]$, $b>0$

Then $E = \bigcup_{k=-\infty}^{\infty} (E \cap [k, k+1])$. Let $A = \mathbb{Q} \cap [-2b, 2b]$

(1) $E \subset \bigcup_{\lambda \in A} (\lambda + C_E)$ (This is a countable union)

(2) $\bigcup_{\lambda \in A} (\lambda + C_E)$ is a disjoint union

(3) C_E is not measurable.

Suppose C_E is measurable, then

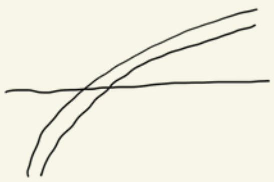
$0 < m^*(E) \leq m\left(\bigcup_{\lambda \in A} (\lambda + C_E)\right) = \sum_{\lambda \in A} m(\lambda + C_E) \leq 6b$

So we have $m(C_E) = 0$, which is impossible

Since $m(\lambda + C_E) = m(C_E)$ for each λ

And $\bigcup_{\lambda \in A} (\lambda + C_E)$ is a countable union, this implies

that $m(E) = 0$ X



Def. The characteristic function of E is defined as

$$\chi_E(x)=\begin{cases}1, & x\in E\\ 0, & x\notin E\end{cases}$$

Def. A function $f:\mathbb{R}^d\to\mathbb{R}$ is said to be measurable if for $\forall\, a\in\mathbb{R}$, $f^{-1}([-\infty,a))=\{x\in E: f(x)<a\}$ is measurable

Prop.

(1) A finite valued function f is measurable if and only if for every open set O, $f^{-1}(O)$ is measurable

(2) If f is continuous on $\mathbb{R}^d$, then f is measurable.

If f is measurable and finite valued, Φ is continuous, then $\Phi\circ f$ is measurable

(3) Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, then $\sup\limits_n f_n(x)$, $\inf\limits_n f_n(x)$, $\limsup\limits_{n\to\infty} f_n(x)$, $\liminf\limits_{n\to\infty} f_n(x)$ are measurable

Notice that $\{\sup\limits_n f_n>a\}=\bigcup\limits_{n=1}^{\infty}\{f_n>a\}$, $\{\inf\limits_n f_n<a\}=\bigcup\limits_{n=1}^{\infty}\{f_n<a\}$

so $\sup\limits_n f_n(x)$, $\inf\limits_n f_n(x)$ are measurable

$\{\limsup\limits_{n\to\infty} f_n(x)>a\}=\bigcup\limits_{n=1}^{\infty}\{f_{k(n)}(x)>a-\frac{1}{n}\}$, $\{\liminf\limits_{n\to\infty} f_n(x)<a\}=\bigcup\limits_{n=1}^{\infty}\{f_{k(n)}(x)<a+\frac{1}{n}\}$

so $\limsup\limits_{n\to\infty} f_n(x)$, $\liminf\limits_{n\to\infty} f_n(x)$ are measurable

(4) If $f_n(x)$ are measurable $(f_n(x)\to f(x))$, then $f(x)$ is measurable

(5) If f & g are measurable:

① f^k, where $k\in\mathbb{N}^+$ is measurable

② If f, g are finite valued, then $f+g$, $f\cdot g$ are measurable

① if k is odd, $\{f^k>a\}=\{f>a^{\frac{1}{k}}\}$

if f is even and $a\geq 0$, then $\{f^k>a\}=\{f<-a^{\frac{1}{k}}\}\cup\{f>a^{\frac{1}{k}}\}$

② $f\cdot g=\frac{1}{4}[(f+g)^2-(f-g)^2]$

(6) Suppose f is measurable, $f(x)=g(x)$ almost everywhere (a.e.) then g is measurable $(m(\{x: f(x)\neq g(x)\})=0)$

Ex. Let E be a given set, O_n be a open set

$O_n=\{x: d(x,E)<\frac{1}{n}\}$.

(a) If E is compact, then $m(E)=\lim\limits_{n\to\infty} m(O_n)$

proof: $m(E)$, $m(O_n)<\infty$

Notice that $\bigcap\limits_{n=1}^{\infty} O_n=E$

$O_n\downarrow E$, thus $m(E)=m(\bigcap\limits_{n=1}^{\infty} O_n)=\lim\limits_{n\to\infty} m(O_n)$

(b) If E if closed but unbounded, this might not hold

Counterexample: $E=\mathbb{Z}$, then $m(E)=0$, $\lim\limits_{n\to\infty} m(O_n)=\infty$

(c) If E is open and bounded

Counterexample: Let C be the open Cantor set

Then $m(C)=0$. $\lim\limits_{n\to\infty} m(O_n)>0$

proof for 6.

Set $E_0 = \{x : f(x) = g(x)\}$, then $m(E \setminus E_0) = 0$

$$\{x \in E : g(x) > c\} = \underbrace{\{x \in E_0 : g(x) > c\}}_{A} \cup \underbrace{\{x \in E \setminus E_0 : g(x) > c\}}_{B}$$

we see that A, B are both measurable $(m(B) \leq m(E \setminus E_0) = 0)$

Prop. Let $D \subseteq E$ be measurable. If f is measurable on E, then $f|_D$ & $f|_{E \setminus D}$ are both measurable.

proof: $\{x \in D : f(x) > c\} = \{x \in E : f(x) > c\} \cap D$

$\{x \in E \setminus D : f(x) > 0\} = \{x \in E : f(x) > c\} \setminus D$ ✓

Prop. If f, g are measurable, then $f+g$ is measurable

$$\{x \in E : f(x) + g(x) > c\} = \bigcup_{q \in \mathbb{Q}} \left(\{x \in E : f(x) > q\} \cap \{x \in E : g(x) < c - q\} \right)$$

which is a countable combination of measurable sets

Question: If f, g are measurable, can we say that $f \circ g$ is measurable?

NO!!! 絶対にない!!!

Recall. $\psi : [0,1] \to [0,2]$ cont. strictly $\uparrow$. onto

and it maps a measurable set $A \subseteq [0,1]$ to a non-measurable set.

Consider $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, χ_A is measurable

$(\chi_A \circ \psi^{-1})^{-1}((\frac{1}{2}, \frac{3}{2})) = (\psi^{-1})^{-1}(\chi_A^{-1}(\frac{1}{2}, \frac{3}{2})) = \psi(A)$ not measurable.

(1) f cont. & g measurable $\Rightarrow$ $f \circ g$ measurable

pf.) we see that $E = \{t : f(t) > c\}$ is open, so $\{x : f \circ g(x) > c\} = g^{-1}(E)$ measurable

Thm. Let $\{f_n\}$ be the sequence of measurable functions that pointwise converges to f. Then f is measurable

Def. A simple function is a linear combination of finitely many measurable characteristic functions.

Simple function approximation

(1) Let f be a bounded measurable function on E.
Then $\forall \varepsilon > 0$, $\exists$ simple functions ψ_ε & φ_ε s.t. $\varphi_\varepsilon \le f \le \psi_\varepsilon$ & $0 \le \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$

(2) A real-valued function on a measurable set E is measurable
$\Leftrightarrow$ $\exists$ a sequence of simple functions φ_n on E converge pointwisely to f with $|\varphi_n| \le |f|$

Thm. Suppose f is a non-negative measurable function on $\mathbb{R}^d$, then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that converge pointwise to f

For $N>1$, let Q_N be the cube centered at the origin with side length N. Then we define $F_N(x)=\begin{cases} f(x), & \text{if } x\in Q_N \text{ and } f(x)\le N \\ N, & \text{if } x\in Q_N \text{ and } f(x)>N \\ 0, & \text{otherwise} \end{cases}$

Then $F_N(x)\to f(x)$ as $N\to\infty$. Now, we partition the range of F_N as follow. For fixed $M, N\geq 1$, we define

$E_{\ell,M}=\{x\in Q_N : \frac{\ell}{M}<F_N(x)\le\frac{\ell+1}{M}\}$ for $0\le\ell\le NM$

Then we may form $F_{N,M}(x)=\sum_{\ell}\frac{\ell}{M}\chi_{E_{\ell,M}}(x)$

Each $F_{N,M}$ is simple function that satisfies $0\le F_N-F_{N,M}\le\frac{1}{M}$ for all x

Let $\varphi_k=F_{2^k,2^k}$, then $0\le F_{2^k}(x)-\varphi_k(x)\le\frac{1}{2^k}$ for all x

then $\{\varphi_k\}$ is increasing and converges to f pointwisely

Thm. Suppose f is measurable on $\mathbb{R}^d$. Then there exist a sequence of simple function $\{\varphi_k\}_{k=1}^{\infty}$ that satisfies $|\varphi_k(x)|\le|\varphi_{k+1}(x)|$ and $\lim_{k\to\infty}\varphi_k(x)=f(x)$

Simply let $f(x)=f_+(x)+f_-(x)$, where

$f_+(x)=\begin{cases} f(x), & f(x)\geq 0 \\ 0, & \text{otherwise}\end{cases}$ $\quad f_-(x)=\begin{cases} f(x), & f(x)\le 0 \\ 0, & \text{otherwise}\end{cases}$

Then repeat the proof

Thm. Suppose f is measurable, then $\exists$ a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges to $f(x)$ a.e. x

Difference: the domain of a step function is devided into intervals, while the domain of a simple function is devided into measurable sets

Ex. If $\delta=(\delta_1,\cdots,\delta_d)$ is a d-dimentional tuple, $\delta_i>0$

E is a subset of $\mathbb{R}^d$, $\delta E=\{(\delta_1 x_1,\cdots,\delta_d x_d),\ x_1,\cdots,x_d\in E\}$

then δE is measurable & $m(\delta E)=\delta_1\cdots\delta_d\, m(E)$

proof: Let O be a open set containing E

s.t. $m_*(O)\le m_*(E)+\varepsilon$, then $O\setminus E$ is also open

$\delta(O\setminus E)=\delta O\setminus\delta E$, $O\setminus E=\bigcup_{j=1}^{\infty}Q_j$, then $\delta(O\setminus E)=\delta(\bigcup_{j=1}^{\infty}Q_j)$

$m(\delta O\setminus\delta E)=m(\delta\bigcup_{j=1}^{\infty}Q_j)=\sum_{j=1}^{\infty}m(\delta Q_j)=\delta_1\cdots\delta_d\sum_{j=1}^{\infty}Q_j=\delta_1\cdots\delta_d\, m(O\setminus E)\le\varepsilon$

thus δE is measuratbe

$m(\delta O)=\delta_1\cdots\delta_d\, m(O)=\delta_1\cdots\delta_d\, m(E)$, $m(\delta O)\le\varepsilon+m(\delta E)$, so $m(\delta E)=\delta_1\cdots\delta_d\, m(E)$

We consider approximating a simple function with a step function.

If $m(E)<\infty$, then for $\forall \varepsilon>0$, $\exists$ cubes $Q_1, \cdots, Q_N$, s.t. $m(E \triangle \bigcup_{j=1}^{N} Q_j) \le \varepsilon$.

Consider the grid formed by these cubes, then we can find almost disjoint rectangles $\hat{R}_1, \cdots, \hat{R}_M$. Take the rectangle R_j with a size slightly smaller than $\hat{R}_j$. And we find a collection of dijoint closed rectangles s.t. $m(E \triangle \bigcup_{j=1}^{M} R_j) \le 2\varepsilon$.

Hence for each k, $\exists$ a step function ψ_k and a measurable set F_k s.t. $m(F_k)<2^{-k}$, $\varphi_k(x)=\psi_k(x)$ for $\forall x \notin F_k$

Define $F=\bigcap_{l=1}^{\infty}\bigcup_{k\ge l}^{\infty} F_k$ $\qquad m(\bigcup_{k\ge l}^{\infty} F_k) \le \sum_{k=l+1}^{\infty} m(F_k) = \frac{1}{2^l}$

thus $m(F)=0$, for $\forall x \notin F$, $\exists k_0$ s.t. $x \in \bigcap_{k>k_0} F_k^c$

Thus for all $k>k_0$, we have

$|f(x)-\psi_k(x)| \le |f(x)-\varphi_k(x)| + |\varphi_k(x)-\psi_k(x)|$. Since $\lim_{k\to\infty}\varphi_k(x)=f(x)$

then we have $\lim_{k\to\infty}\psi_k(x)=f(x)$ pointwisely

Littlewood's three principles

∀ measurable sets E in $\mathbb{R}$ with $m(E)<\infty$

(1) Every set is nearly a fininte union of intervals

(2) Every function is nearly continuous

(3) Every convergent sequence is nearly uniformly convergent

The set, function, sequence referred are measurable

$E=\bigcap_{n=1}^{\infty}\bigcup_{k\ge n} E_k$, $m(E)=m(\bigcap_{n=1}^{\infty}\bigcup_{k\ge n} E_k) \le m(\bigcup_{k\ge N+1} E_k) \le \sum_{k=N+1}^{\infty} m(E_k) \le \varepsilon$

Thm. Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E)<\infty$, and assume that $f_k \to f$ a.e. on E. Given $\varepsilon>0$, we can find a closed set $A_\varepsilon \subset E$, s.t. $m(E \setminus A_\varepsilon) \leq \varepsilon$ & $f_k \to f$ uniformly on A_ε

(The Egorov Theorem)

WLOG. $\forall x \in E$, $f_k(x) \to f(x)$

For every n & k, let $E_k^n = \{x \in E : |f_j(x) - f_k(x)| < \frac{1}{n}, \forall j > k\}$

$= \bigcap_{j=k}^{\infty} \{x \in E : |f_j(x) - f_k(x)| < \frac{1}{n}\}$. $E_k^n \subseteq E_{k+1}^n \subseteq \cdots$, $m(E)<\infty$

then $\lim_{k\to\infty} m(E_k^n) = m(E)$. $\exists k_n$ s.t. $m(E \setminus E_{k_n}^n) < \frac{1}{2^n}$

Choose $N \in \mathbb{N}^+$ s.t. $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$, let $\hat{A}_\varepsilon = \bigcap_{n \geq N} E_{k_n}^n$

$m(E \setminus \hat{A}_\varepsilon) = m(E \setminus \bigcap_{n\geq N} E_{k_n}^n) \leq \sum_{n=N}^{\infty} m(E \setminus E_{k_n}^n) < \sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$

Clearly $x \in \hat{A}_\varepsilon \Rightarrow x \in E_{k_n}^n$, $\forall \delta > 0$, take N s.t. $\frac{1}{n} < \delta$ when $n \geq \frac{1}{N}$

Thus $\forall \delta > 0$, $\exists k_n$. $\forall j, k > k_n$ $|f_j(x) - f_k(x)| < \delta$, so $f_k \to f$ uniformly

Choose A_ε as a closed subset of $\hat{A}_\varepsilon$ s.t. $m(\hat{A}_\varepsilon / A_\varepsilon) < \frac{\varepsilon}{2}$

then $m(E \setminus A_\varepsilon) \leq m(E / \hat{A}_\varepsilon) + m(\hat{A}_\varepsilon / A_\varepsilon) < \varepsilon$

Thm. Suppose f is measurable and finite valued on E, $m(E)<\infty$

Then $\forall \varepsilon > 0$, $\exists$ a closed set F_ε & $m(E \setminus F_\varepsilon) \leq \varepsilon$

s.t. $f|_{F_\varepsilon}$ is continuous (Lusin's Lemma)

Lemma. Under the hypothesis of Egorov's Th'm, we have $\forall \eta>0, \delta>0, \exists$ measurable set $A\subseteq E$ & $N\in\mathbb{N}$ s.t. $|f_n-f|<\eta$ on A for $\forall n\geq N$ and $m(E\setminus A)<\delta$

proof:) Set $E_n=\{x\in E \mid |(f_k-f)(x)|<\eta, \forall k\geq n\}=\bigcap_{k=n}^{\infty}\{x\in E \mid |f_k(x)-f(x)|<\eta\}$

so $E_n\subseteq E_{n+1}\subseteq\cdots\uparrow$, $m(E)=\lim_{n\to\infty}m(E_n) \Rightarrow \exists N$, s.t. $m(E_n)>m(E)-\delta\ \forall n\geq N$

And so we let $A=E_N$

Lemma. Let f be a simple function on E. Then $\forall\varepsilon>0, \exists$ a cont. fcn. g on $\mathbb{R}$ and a closed set $F\subseteq E$ s.t. $f|_F=g$ and $m(E\setminus F)<\varepsilon$

pf:) Write $f(x)=\sum_{k=1}^{n}a_k\chi_{E_k}(x)$, where a_k's are distinct and E_k's are disjoint.
For each E_k, $\exists F_k$ closed s.t. $m(E_k\setminus F_k)<\frac{\varepsilon}{n}$, let $F=\bigcup_{k=1}^{n}F_k$
then define $g(x)=\sum_{k=1}^{n}a_k\chi_{F_k}(x)$ so $m(E\setminus F)\leq\sum_{k=1}^{n}m(E_k\setminus F_k)<\varepsilon$, $f=g$

proof of Lusin:
Assume $m(E)<\infty$, we choose a sequence of simple functions $\{f_n\}$ s.t. $f_n\to f$ as $n\to\infty$
$\forall n, \exists$ cont. fcn. g_n on $\mathbb{R}$ & a closed set $F_n\subseteq E$ s.t. $f_n=g_n$ on F_n & $m(E\setminus F_n)\leq\frac{\varepsilon}{2^{n+1}}$
$g_n\to f$ pointwise on $\bigcap_{n=1}^{\infty}F_n$, then $\exists F_\varepsilon\subseteq\bigcap_{n=1}^{\infty}F_n$ that $g_n\to f$ uniformly and $m(\bigcap_{n=1}^{\infty}F_n/F_\varepsilon)<\frac{\varepsilon}{2}$
We see that $m(E\setminus F_\varepsilon)\leq m(E\setminus\bigcap_{n=1}^{\infty}F_n)+m(\bigcap_{n=1}^{\infty}F_n\setminus F)<\sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}}+\frac{\varepsilon}{2}=\varepsilon$

Tietze's Extension Theorem
A bdd continuous function on a closed subset of a metric space S can be extended to a bounded cont. fcn. on S.

Chapter 2

Integration Theory

Def. The conanical form of a simple function φ is a unique decomposition in the form
$\varphi(x)=\sum_{k=1}^{N} a_k \chi_{E_k}(x)$

Def. Let $F_k=\{x:\varphi(x)=c_k\}$. Clearly F_k's are disjoint.
If φ is a simple function with the conanical form
$\varphi(x)=\sum_{k=1}^{M} c_k \chi_{F_k}(x)$, then we define the Lebesgue integral of φ by:
$\int_{\mathbb{R}^d}\varphi(x)dx=\sum_{k=1}^{M} c_k m(F_k)$ (Here each c_k's are distinct)
If E is a measurable subset of $\mathbb{R}^d$ & $m(E)<\infty$, then $\varphi(x)\chi_E(x)$ is also a simple function, then we define
$\int_E \varphi(x)dx=\int_{\mathbb{R}^d}\varphi(x)\chi_E(x)dx$

Prop.
(1) If $\varphi=\sum_{k=1}^{N} a_k \chi_{E_k}$ is an arbitary representation of φ, then
$\int\varphi=\sum_{k=1}^{N} a_k m(E_k)$
(2) If φ and ψ are simple functions, $a,b\in\mathbb{R}$, then
$\int a\varphi+b\psi=a\int\varphi+b\int\psi$
(3) If $E\cap F=\phi$ & $m(E),m(F)<\infty$, then
$\int_{E\cup F}\varphi=\int_E\varphi+\int_F\varphi$
(4) If φ & ψ are simple functions, $\varphi\leq\psi$, then
$\int\varphi\leq\int\psi$
(5) If φ is a simple function, then so do $|\varphi|$, and
$|\int\varphi|\leq\int|\varphi|$

Def. Let f be a bounded function on a measurable set E with $m(E)<\infty$. Then we define

$\underline{\int_E} f = \sup\{\int_E \varphi \mid \varphi \le f,\ \varphi \text{ is simple}\}$

$\overline{\int_E} f = \inf\{\int_E \psi \mid \psi \ge f,\ \psi \text{ is simple}\}$

And we say that f is integrable if $\underline{\int_E} f = \overline{\int_E} f$

Thm. A bounded function on a measurable set E with $m(E)<\infty$ is Lebesgue integrable $\Leftrightarrow$ f is measurable.

proof: ($\Leftarrow$): If f is measurable on E, then for every fcn., $\exists$ simple functions $\varphi_n \le f \le \psi_n$ on E, s.t. $\psi_n - \varphi_n < \frac{1}{n}$

Then $0 \le \overline{\int_E} f - \underline{\int_E} f \le \int_E \psi_n - \int_E \varphi_n = \int_E \psi_n - \varphi_n \le \frac{1}{n} m(E)$

($\Rightarrow$) is directly from the definition.

Def. The support of a measurable function f is defined to be the set of all points where f does not vanish

i.e. $supp(f) = \{x : f(x) \neq 0\}$

f is supported on E if $f(x) = 0$ whenever $x \notin E$

Lemma. Let f be a bounded function supported on E where $m(E) < \infty$. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple function bounded by M, supported on E, $\varphi_n \to f$ for a.e. x, then:

(1) $\lim_{n\to\infty} \int \varphi_n$ exists

(2) If $f = 0$ a.e., then $\lim_{n\to\infty} \int \varphi_n = 0$

(1) Since $m(E) < \infty$. $\forall \varepsilon > 0$, the Egorov Theorem ensures that $\exists$ a measurable set A_ε s.t. $m(E \setminus A_\varepsilon) \leq \varepsilon$, where $\varphi_n \to f$ uniformly

Thus, let $I_n = \int \varphi_n$, then

$|I_n - I_m| \leq \int_E |\varphi_n(x) - \varphi_m(x)| dx = \int_{A_\varepsilon} |\varphi_n(x) - \varphi_m(x)| dx + \int_{E \setminus A_\varepsilon} |\varphi_n(x) - \varphi_m(x)| dx$

$\leq \int_{A_\varepsilon} |\varphi_n(x) - \varphi_m(x)| dx + 2M \cdot m(E \setminus A_\varepsilon) \leq \int_{A_\varepsilon} |\varphi_n(x) - \varphi_m(x)| dx + 2M \cdot \varepsilon$

Since $\varphi_n \to f$ uniformly on A_ε, then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^+$

$\sup_{x \in A_\varepsilon} |\varphi_m(x) - \varphi_n(x)| < \varepsilon$ when $m, n > N$

thus $|I_n - I_m| \leq (m(E) + 2M) \cdot \varepsilon$. by Cauchy Criterion, $\lim_{n\to\infty} I_n$ exists

(2) Since $f = 0$. $\varphi_n \to f$, then $|I_n| \leq \int_E |\varphi_n(x)| dx \leq \int_E |\varphi_n(x) - f(x)| dx$

$\leq (m(E) + 2M)\varepsilon$, thus $\lim_{n\to\infty} \int \varphi_n(x) = 0$

(From this Lemma, we define)

Def. The Lebesgue intergral of a function supported on E is defined as $\int f(x) dx = \lim_{n\to\infty} \int \varphi_n(x) dx$

Where $\{\varphi_n(x)\}$ is any sequence of simple functions that φ_n's are supported on E, $|\varphi_n(x)| \leq M$, $\varphi_n \to f$ as $n \to \infty$ a.e.

Def. If $m(E)<\infty$, f is bounded and $m(supp(f))<\infty$, then
$\int_E f(x)dx = \int f(x)\chi_E(x)dx$

Prop.
(1) $\int(af+bg) = a\int f + b\int g$
(2) $\int_{E\cup F} f = \int_E f + \int_F f$, where $E\cap F=\emptyset$
(3) If $f\leq g$, then $\int f\leq \int g$
(4) $|\int f|\leq \int|f|$

Thm. (Bounded convergence theorem) Suppose $\{f_n\}$ bounded by M is a sequence of measurable functions that is supported on E. $f_n(x)\to f(x)$ as $n\to\infty$ a.e. x. Then for a.e. x, f is measurable. bounded, supported on E, $\int|f_n-f|\to 0$ as $n\to\infty$ ($\lim_{n\to\infty}\int f_n=\int f$)
For a given $\varepsilon>0$, by Egrov Theorem, we can find a measurable subset A_ε s.t. $m(E\setminus A_\varepsilon)\leq\varepsilon$. $f_n\longrightarrow f$ uniformly on A_ε.
Then for $\forall x\in A_\varepsilon$, $\exists$ n big enough s.t. $|f_n(x)-f(x)|\leq\varepsilon$
therefore. $\int_E|f_n-f| = \int_{E\setminus A_\varepsilon}|f_n-f| + \int_{A_\varepsilon}|f_n-f|\leq 2M\varepsilon + m(E)\varepsilon$
Hence $\int_E|f_n-f|\to 0$ as $n\to\infty$

Recall. In the Riemman integral, a function is integrable if and only if the Darboux Sum tends to zero.
Riemman Integral is convinient as it can deal with those common functions. However, it is not complete.
Counterexample: $f_n(x)=\begin{cases}0, & x \text{ is irrational}\\ \sqrt[n]{\frac{1}{q}}, & x \text{ is rational in the irreducible form } \frac{p}{q}\end{cases}$
$f_n(x)$ is integrable for each fixed n, yet $\lim_{n\to\infty}f_n(x)$ is not integrable

Thm. If f is Riemman integrable on $[a,b]$, then f is measurable and $\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx$

By definition. f is bounded on $[a,b]$, $|f(x)| \leq M$

Also from the definition, we construct two step function $\{\varphi_k\}$ & $\{\psi_k\}$, $\forall x \in [a,b]$, $k \geq 1$, we have $|\varphi_k(x)|, |\psi_k(x)| \leq M$

$\varphi_1(x) \leq \cdots \leq f(x) \leq \cdots \leq \psi_1(x)$. $\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \varphi_k(x)dx = \lim_{k\to\infty} \int_{[a,b]}^{\mathcal{R}} \psi_k(x)dx = \int_{[a,b]}^{\mathcal{R}} f(x)dx$

Since φ_k, ψ_k are step functions, then

$\int_{[a,b]}^{\mathcal{R}} \varphi_k(x)dx = \int_{[a,b]}^{\mathcal{L}} \varphi_k(x)dx$, $\int_{[a,b]}^{\mathcal{R}} \psi_k(x)dx = \int_{[a,b]}^{\mathcal{L}} \psi_k(x)dx$

Let $\tilde{\varphi}(x) = \lim_{k\to\infty} \varphi_k(x)$, $\tilde{\psi}(x) = \lim_{k\to\infty} \psi_k(x)$

By the bounded convergence theorem,

$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \varphi_k(x) = \int_{[a,b]}^{\mathcal{L}} \tilde{\varphi}(x)dx$, $\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \psi_k(x)dx = \int_{[a,b]}^{\mathcal{L}} \tilde{\psi}(x)dx$

then $\int_{[a,b]}^{\mathcal{L}} (\tilde{\psi} - \tilde{\varphi})(x)dx = 0$. Since $\psi_k \geq \varphi_k$, then $\tilde{\psi} - \tilde{\varphi} \geq 0$

And since $\int_{[a,b]}^{\mathcal{L}} (\tilde{\psi} - \tilde{\varphi})(x)dx = 0$, then $\tilde{\psi} = \tilde{\varphi}$ a.e.x.

this show that f is measurable and $\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx$

Lemma. (Fatou) If f is a measurable function supported on a measurable set E, $\{f_n\}$ is a sequence of function that $f_n \to f$, then $\int f \leq \liminf_{n\to\infty} \int f_n$ $(f_n \geq 0)$

Let $G = \{\int g \mid g$ measurable, $0 \leq g \leq f$, finite support, bdd.$\}$

and take $g_n = \min\{f_n, g\}$, we see g_n is bounded and finite supported. so $\int g_n \to \int g$

Also, $g_n \leq f_n$, so $\lim_{n\to\infty} \int g_n = \int g \leq \liminf_{n\to\infty} \int f_n$

Take supremum of G, then $\int f \leq \liminf_{n\to\infty} \int f_n$

Bounded Convergence Theorem

Let $\{f_n\}$ be a bounded sequence of measurable function on E with $m(E)<\infty$

Then $\{f_n\}\to f$ pointwise $\Rightarrow \int_E f_n \to \int_E f$

proof: By the Egorov's Thm. $\exists$ a closed set $F\subseteq E$ s.t. $m(E\setminus F)<\frac{\varepsilon}{4M}$

where $f_n\to f$ uniformly on F

so $|\int_E f_n - \int_E f| \le \int_E |f_n-f| = \int_F |f_n-f| + \int_{E\setminus F}|f_n-f| \le m(E)\cdot\frac{\varepsilon}{2m(E)} + 2M\cdot\frac{\varepsilon}{4M} = \varepsilon$

Def. Let f be nonnegative measurable function on a measurable set E

define $\int_E f = \sup\{\int_E h \mid h$ is bounded & finitely supported, $0\le h\le f\}$

Prop. $f\ge 0$, measurable. let $E_\lambda = \{x\in E \mid f(x)\ge\lambda\}$, $\lambda\ge 0$

(1) (Chebychev inequality)

$m(E_\lambda)\le\frac{1}{\lambda}\int_E f$, $\forall\lambda>0$

(2) $\int_E f=0 \Rightarrow f=0$ a.e. on E

(3) (Linearity) $f, g\ge 0$ measurable, $\alpha,\beta\ge 0$

$\int_E \alpha f+\beta g = \alpha\int_E f + \beta\int_E g$

If $0\le f\le g$, $\int_E f\le\int_E g$

(4) If A, B are measurable & $A\cap B=\emptyset$, then $\int_{A\cup B} f = \int_A f + \int_B f$

pf:) (1) $\lambda m(E_\lambda) = \int_E \lambda\chi_{E_\lambda}$

Since $\lambda\chi_{E_\lambda}\le f$, while its support is finite, then $\int_E \lambda\chi_{E_\lambda}\le\int_E f$

(2) $m(E_{\frac{1}{n}})\le n\int_E f = 0$, and $m(E_0) = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} = 0$

so $m(\{x\in E \mid f(x)>0\})=0$, i.e. $f=0$ a.e. on E

(3) $\int_E(\alpha f) = \alpha\int_E f$ $\quad 0\le\alpha h\le\alpha f \Leftrightarrow 0\le h\le f$

For linearity, use the fact that $\sup(A+B)=\sup A+\sup B$

This is from MAT1012 or MAT2060

For monotonicity, since $0\le f\le g$

$\int_E f = \sup\{\int_E h \mid 0\le h\le f$, h bounded, measurable, finite supported$\}$

we see $\{\int_E h \mid 0\le h\le f, \sim\}\subseteq\{\int_E k \mid 0\le k\le g, \sim\}$

so $\int_E f\le\int_E g$

Cor. Suppose f is a non-negative measurable function, $\{f_n\}$ is a sequence of non-negative measurable functions that $f_n(x) \le f(x)$ & $f_n(x) \to f(x)$ for a.e.x., then $\lim_{n\to\infty} \int f_n = \int f$

Since $f_n(x) \le f(x)$, then $\int f_n \le \int f$, so clearly we have $\limsup_{n\to\infty} \int f_n \le \int f$, by Fatou's lemma,
$\limsup_{n\to\infty} \int f_n \le \int f \le \liminf_{n\to\infty} \int f_n$, so $\int f = \lim_{n\to\infty} \int f_n$

Cor. (Monotone Convergence Theorem) Suppose $\{f_n\}$ is a sequence of non-negative measurable functions, $f_n \uparrow f$ then $\lim_{n\to\infty} \int f_n = f$ ✓

Cor. Consider the series $\sum_{k=1}^{\infty} a_k(x)$, $\forall k \ge 1$, $a_k(x) \ge 0$ is measurable, then $\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x) dx$

If $\sum_{k=1}^{\infty} \int a_k(x) dx$ is finite, then $\sum_{k=1}^{\infty} a_k(x)$ converges for a.e.x.

let $f_n(x) = \sum_{k=1}^{n} a_k(x)$, $f(x) = \sum_{k=1}^{\infty} a_k(x)$

Since $a_k(x) \ge 0$, then $f_n \uparrow f$, then $\lim_{n\to\infty} \int \sum_{k=1}^{n} a_k(x) dx = \int \sum_{k=1}^{\infty} a_k(x) dx$

$\lim_{n\to\infty} \sum_{k=1}^{n} \int a_k(x) dx = \int \sum_{k=1}^{\infty} a_k(x) dx$, therefore $\sum_{k=1}^{\infty} \int a_k(x) dx = \int \sum_{k=1}^{\infty} a_k(x) dx$

Def. A non-negative measurable function f is integrable $\Leftrightarrow \int_E f < \infty$

Now we consider the most common situation

Let $f^+(x) = \max(f(x), 0)$, $f^-(x) = \min(-f(x), 0)$

Hence $f = f^+ - f^-$, where f^+ & f^- are non-negative

Thus we define the Lebesgue Integral of f as

$\int f = \int f^+ - \int f^-$

Def. A general measurable function f is integrable if $\int_E |f| < \infty$

Prop. (1) f is integrable $\Rightarrow$ f is finite a.e.

(2) $\int_E f = \int_{E \setminus F} f$ is $m(F) = 0$

(3) $|f| \leq g$ & g is integrable $\Rightarrow$ f is integrable

(4) Linearity

(5) Monotonicity.

(6) $\int_{A \cup B} f = \int_A f + \int_B f$ if $A \cap B = \emptyset$

pf:) (1) $m(\{x \in E \mid f(x) = \infty\}) \leq m(\{x \in E \mid f(x) \geq n\}) \leq \frac{1}{n}\int_E f \to 0$

so $m(\{x \in E \mid f(x) = \infty\}) = m(\bigcap_{n=1}^{\infty} \{x \in E \mid f(x) \geq n\}) = 0$

(4) $\int_E (\alpha f) = \alpha \int_E f$ $\quad |f+g| \leq |f| + |g|$, integrable

so $\int_E (f+g) = \int_E (f^+ + g^+) - (f^- + g^-) = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^- = \int_E f^+ - f^- + \int_E g^+ - g^- = \int_E f + \int_E g$

Lebesgue Dominated Convergence Theorem (Another proof)

pf:) Since $f_n \to f$ a.e. & f_n measurable, then f is integrable.

Also $|f_n| \leq g$ for $\forall n$ $\Rightarrow$ $|f| \leq g$, thus f is integrable

By Fatou's Lemma, $\int g - f \leq \liminf_{n\to\infty} \int g - f_n$. $\int g + f \leq \liminf_{n\to\infty} \int g + f_n$

This implies that $-f \leq \liminf_{n\to\infty} \int(-f_n)$, $f \leq \liminf_{n\to\infty} \int f_n$

so $-\liminf_{n\to\infty} (-\int f_n) \leq \int f \leq \liminf_{n\to\infty} (\int f_n)$, i.e. $\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n$

so $\int f_n \to \int f$

Continuity of Integration

Prop. Let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union of measurable sets & f be integrable over E, then

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

pf:) $\sum_{n=1}^{N} \int_{E_n} f_n = \sum_{n=1}^{N} \int_E f \chi_{E_n} = \int_E f \chi_{\bigcup_{n=1}^{N} E_n} \to \int_E f$ as $N \to \infty$

Thm. Let f be integrable over E

(i) Let $\{E_n\}_{n=1}^{\infty}$ be an ascending sequence of measurable subsets of E. Then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$$

pf:) $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots$

let $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $\cdots$, $F_n = E_n \setminus E_{n-1}$, $\cdots$

so $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$

Since each F_n are disjoint, then $\int_{\bigcup_{n=1}^{\infty} F_n} f = \sum_{n=1}^{\infty} \int_{F_n} f = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{E_n \setminus E_{n-1}} f$

$= \lim_{N\to\infty} \int_{E_N} f$

(ii) Let $\{E_n\}$ be a descending sequence of measurable subsets of E. Then

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$$

pf:) $E_1 \supseteq E_2 \supseteq \cdots E_n \supseteq \cdots$, let $F_n = E_n \setminus E_{n+1}$

$\bigcup_{n=1}^{\infty} F_n = E_1 \setminus \bigcap_{n=1}^{\infty} E_n$, so $\int_{E_1} f - \int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{E_n \setminus E_{n+1}} f = \int_{E_1} f - \lim_{N\to\infty} \int_{E_N} f_N$

$\int_{\bigcap_{n=1}^{\infty} E_n} = \lim_{n\to\infty} \int_{E_n} f$

Equi-integrability $[m(E)<\infty]$

Prop. Let f be a measurable & $m(E)<\infty$, then f is integrable $\Leftrightarrow$ $\forall \varepsilon>0$, $\exists \delta>0$. s.t. $^{(\dagger)}$ if $A\subseteq E$ measurable and $m(A)<\delta$, then $\int_A |f|<\varepsilon$

proof: w.l.o.g. assume $f>0$

$(\Rightarrow)$ $\forall \varepsilon>0$, $\exists\, g\le f$, s.t. $\int_A g \geq \int_A f - \frac{\varepsilon}{2}$

Set $M=\sup\int_A g$, for each $\varepsilon>0$, set $\delta=\frac{\varepsilon}{2M}$

then $\int_A f=\int_A (f-g)+\int_A g \le \frac{\varepsilon}{2}+M\cdot m(A) \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

$(\Leftarrow)$ $m(E)<\infty$, set $E_n=E\cap[-n,n]$, $E_n\to E$

so $\exists N>0$, s.t. $m(E\setminus E_N)<\frac{\delta}{2}$, so $\int_{E\setminus E_N} f<\varepsilon$

For $[-N,N]$, let $[-N,N]=\bigcup_{j=1}^{k} I_j$, where each $I_i\cap I_j=\emptyset$, $m(I_j)<\delta$

Hence $\int_E f=\int_{E\setminus E_N} f+\int_{E_N} f \le (k+1)\varepsilon$

Take $\varepsilon=1$, then $\int_E f\le k+1<\infty$

<u>Def.</u> A family of integrable functions $\mathcal{F}$ on E is said to be equi-integrable if $\forall\varepsilon>0$, $\exists\delta>0$, s.t. $(\dagger)$ holds $\forall f\in\mathcal{F}$

<u>Remark.</u> (1) Any family of finitely many integrable functions on E is equi-integrable.

(2) Any equi-integrable family of integrable functions must have bounded integrals, i.e. $\exists M>0$, s.t. $\int|f|<M$ for $\forall f\in\mathcal{F}$

<u>Vitali Conv. Theorem.</u>

Let E be a <u>finite</u> measurable set, & $\{f_n\}$ be an equi-integrable family of integrable functions. If $\{f_n\}\to f$ pointwisely, then $\int_E f_n\to\int_E f$

proof: $|f_n|\to|f|$ pointwisely, then $\int|f|\le\liminf_{n\to\infty}\int|f_n|$, so f is integrable

Also, $m(E)<\infty$, so $\exists A\subseteq E$, s.t. $m(E\setminus A)<\delta$, $f_n\to f$ uniformly on A

$\forall\varepsilon>0$, $\exists\delta>0$, s.t. $\int_B|f-f_n|\le\int_B|f|+\int_B|f_n|\le\liminf\int|f_n|+\int_B|f_n|<2\varepsilon$

Then $\int_E|f-f_n|=\int_{E\setminus A}|f-f_n|+\int_A|f-f_n|\le(m(A)+2)\varepsilon$

<u>Collary.</u> Let $m(E)<\infty$, suppose $\{h_n\}$ is a sequence of nonnegative integrable function s.t. $h_n\to 0$ pointwisely. Then $\int_E h_n\to 0$ $\Leftrightarrow$ $\{h_n\}$ is equi-integrable

proof: $(\Leftarrow)$ ✓

$(\Rightarrow)$ $\forall\varepsilon>0$, $\exists N$, s.t. $\int_E h_n<\varepsilon$ for $n>N$

For $\{h_1,\cdots,h_N\}$, clearly it holds

Lemma. Let f be integrable on E. Then $\forall \varepsilon>0$, $\exists E_0 \subseteq E$, where $m(E_0)<\infty$, s.t. $\int_{E \setminus E_0}|f| < \varepsilon$

proof: f is integrable, then $\exists\, 0 \le g \le |f|$ s.t. $\int_E g \geq \int_E |f| - \varepsilon$ with g bounded and finitely supported.

$\int_{E\setminus E_0}|f| = \int_{E\setminus E_0}(|f|-g) \leq \int_E (|f|-g) \leq \varepsilon$, here E_0 is the support of g

Def. A family of measurable functions is called tight over E if $\forall \varepsilon>0$, $\exists E_0 \subseteq E$, s.t. $\int_{E\setminus E_0}|f| \le \varepsilon$, $\forall f \in \mathcal{F}$, where $m(E_0)<\infty$

General Vitali Convergence Theorem.

- Let $\{f_n\}$ be a tight and equi-integrable family of functions.

If $\{f_n\} \to f$ pointwisely a.e., then f is integrable & $\int_E f_n \to \int_E f$

proof: $\forall \varepsilon>0$, $\exists E_0 \subseteq E$, s.t. $\int_{E\setminus E_0} f_n < \frac{\varepsilon}{4}$, $m(E_0)<\infty$

$\int_E |f-f_n| \le \int_{E_0}|f-f_n| + \int_{E\setminus E_0}|f-f_n| \le \frac{\varepsilon}{2} + \int_{E\setminus E_0}|f| + \int_{E\setminus E_0}|f_n| < \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \varepsilon$

($\int_{E\setminus E_0}|f| \le \liminf_{n\to\infty}\int_{E\setminus E_0}|f_n| \le \frac{\varepsilon}{4}$

Def. Let $\{f_n\}$ & f be measurable and finite a.e. on E. Then we say that $\{f_n\} \to f$ in measure if $\forall \eta>0$, $m\{x\in E \mid |f_n - f| > \eta\} \to 0$

Prop. $m(E)<\infty$. If $\{f_n\} \to f$ pointwise a.e. & f is finite a.e., then $\{f_n\} \to f$ in measure.

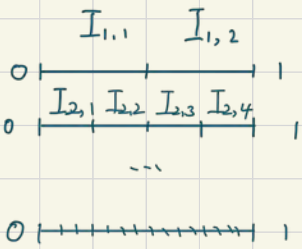
proof: Given $\eta>0$. claim: $\forall \varepsilon>0$, $\exists N$ s.t. $m\{x\in E \mid |f_n - f|>\eta\} < \varepsilon$ for $\forall n \ge N$

$\{f_n\} \to f$ pointwise a.e., Egorov's Theorem implies that $\exists F \subseteq E$ closed

$\{f_n\} \to f$ uniformly on F, i.e. $\forall \eta>0$, $\exists N$, s.t. $|f_n - f| \le \eta$ for $x \in F$, $m(E\setminus F)<\varepsilon$

Therefore $\forall \varepsilon>0$, $\eta>0$, $\exists N = N(\varepsilon, \eta)$, $m\{x\in E \mid |f_n - f| < \eta\} \le m(E\setminus F) < \varepsilon$

Example.



Let $f_1 = \chi_{I_{1,1}}$, $f_2 = \chi_{I_{1,2}}$, $f_3 = \chi_{I_{2,1}}$, $f_4 = \chi_{I_{2,2}}$ ...

Then $m\{x\in[0,1] \mid |f_n|>0\} \to 0$ as $n\to\infty$

However, for $\forall x\in[0,1]$, $\{f_n(x)\}$ diverges

Thm. (Riesz)

$\{f_n\} \to f$ in measure $\Rightarrow$ $\{f_n\}$ has a subsequence $\{f_{n_k}\} \to f$ pointwise a.e.

proof: $\{f_n\} \to f$ in measure $\Rightarrow$ for $\eta = \frac{1}{k}$, $\varepsilon = \frac{1}{2^k}$. $\exists N_k$ s.t.

$m\{x \in E \mid |f_n - f|(x) > \frac{1}{k}\} < \frac{1}{2^k}$ for $\forall n \geqslant N_k$

Set $E_k = \{x \in E \mid |f_{N_k} - f|(x) > \frac{1}{k}\}$, $m(E_k) < \frac{1}{2^k}$.

let $F = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$, for $x \in F$, $|f_{N_k} - f|(x) > \frac{1}{k}$ for infinitely many k's

$m(F) = \lim_{m\to\infty} \frac{1}{2^m} = 0$,

WTS. $f_{N_k} \to f$ point wisely, $\forall x \in E \setminus F$, $\exists K > 0$, s.t.

$|f_{N_k} - f|(x) < \frac{1}{2^k}$ for $k > K$, Therefore $f_{N_k} \to f$ pointwisely

Corollary. Let $\{f_n\}$ be a sequence of nonnegative integrable functions.

Then $\int_E |f_n - f| \to 0 \Leftrightarrow \{f_n\}$ is equi-integrable & tight & $\{f_n\} \to f$ in measure.

proof: ($\Leftarrow$) For each $\varepsilon > 0$, $\exists E_0 \subseteq E$, $m(E_0) < \infty$, $\int_{E \setminus E_0} |f_n| < \frac{\varepsilon}{5}$

$\int_E |f_n - f| \leqslant \int_{E_0} |f - f_n| + \int_{E \setminus E_0} |f - f_n| \leqslant \int_{E_0} |f - f_n| + \int_{E \setminus E_0} |f| + \int_{E \setminus E_0} |f_n|$

$\{f_n\} \to f$ in measure, so $\exists \{f_{n_k}\} \to f$ pointwisely, then

$\int_{E \setminus E_0} |f| \leqslant \liminf_{k\to\infty} \int_{E \setminus E_0} |f_{n_k}| \leqslant \liminf_{n\to\infty} \int_{E \setminus E_0} |f_n| \leqslant \frac{\varepsilon}{5}$

$\{f_n\}$ is equi-integrable, i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$. $\int_A |f_n| < \frac{\varepsilon}{5}$ whenever $m(A) < \delta$

Let $F_n = \{x \in E_0 \mid |f - f_n|(x) > \frac{\varepsilon}{5m(E_0)}\}$, then $\exists N$, s.t. $\forall n \geqslant N$, $m(F_n) < \delta$

Then $\int_{E_0} |f - f_n| = \int_{F_n} |f - f_n| + \int_{E_0 \setminus F_n} |f - f_n| \leqslant m(E_0) \cdot \frac{\varepsilon}{5m(E_0)} + \int_{F_n} |f| + \int_{F_n} |f_n| < \frac{3}{5}\varepsilon$

so $\int_E |f_n - f| < \frac{2}{5}\varepsilon + \frac{3}{5}\varepsilon = \varepsilon$

($\Rightarrow$) Clearly $\{f_n\}$ is tight, so $\forall n$, $\varepsilon > 0$, $\exists E_0 \subseteq E$, $m(E_0) < \infty$

$\int_{E \setminus E_0} |f_n| < \frac{\varepsilon}{2}$, Take $\delta = \frac{\varepsilon}{2m(E_0)M}$, $M = \sup_{x \in E_0} |f_n|$, $m(F) < \delta$

then $\int_F |f_n| = \int_{F \cap E_0} |f_n| + \int_{F \setminus E_0} |f_n| < \frac{m(E_0)M}{2m(E_0)M}\varepsilon + \frac{\varepsilon}{2} = \varepsilon$. Hence $\{f_n\}$ is equi-integrable

For each $\eta > 0$, $m(\{x \in E \mid |f_n(x) - f(x)| > \eta\}) \leqslant \frac{1}{\eta} \int_E |f_n - f| \to 0$

Thm.

f is integrable on $\mathbb{R}^d$, then for $\forall \varepsilon>0$

(1) $\exists$ a finite-measurement set B s.t. $\int_{B^c}|f|<\varepsilon$

(2) $\exists\ \delta>0$, s.t. $\int_E |f|<\varepsilon$ whenever $m(E)<\delta$

(1) Let B_N be a ball centered at the origin, where the radius is N. Notice that if $f_N(x)=f(x)\chi_{B_N}(x)$, then $\lim_{N\to\infty} f_N(x)=f(x)$

By the monotone convergence theorem, we have

$\lim_{N\to\infty}\int f_N=\int f$, thus for $\forall\varepsilon>0$, $\exists N>0$, $0\le\int f-\int f\chi_{B_N}<\varepsilon$

$0\le\int(1-\chi_{B_N})f<\varepsilon$, so $\int_{B_N^c} f<\varepsilon$

(2) Suppose $f\ge 0$, let $f_N(x)=f(x)\chi_{E_N}(x)$, where $E_N=\{x: f(x)\le N\}$

Clearly $f_N\uparrow f$, so $\forall\varepsilon>0, \exists N$, $\int(f-f_N)<\frac{\varepsilon}{2}$

Choose $\delta>0$ s.t. $N\delta>\frac{\varepsilon}{2}$, then we have

$\int_E f=\int_E(f-f_N)+\int_E f_N\le\int(f-f_N)+\int_E f_N<\frac{\varepsilon}{2}+N\delta<\varepsilon$

Thm. Suppose $\{f_n\}$ is a sequence of measurable functions $f_n(x)\to f(x)$ a.e. x, if $|f_n(x)|\le g(x)$, g is integrable.

then $\int|f_n-f|\to 0$ as $n\to\infty$, i.e. $\int f_n\to\int f$

$\forall N>0$, let $E_N=\{x: |x|\le N, g(x)\le N\}$.

Then for $\forall\varepsilon>0$, $\exists N\ge 0$ s.t. $\int_{E_N^c} g<\frac{\varepsilon}{3}$

then $f_n\chi_{E_N}$ is bounded and supported on a finite measured set

$\int|f-f_n|=\int|f\chi_{E_N}-f_n\chi_{E_N}|+\int_{E_N^c}|f-f_n|\le\frac{\varepsilon}{3}+2\cdot\int_{E_N^c} g<\varepsilon$

so $\int|f-f_n|\to 0$ as $n\to\infty$, i.e. $\lim_{n\to\infty}\int f_n=\int f$

Thm. Let f be bounded on a compact set D. Then f is Riemman integrable $\Leftrightarrow$ it's continuous a.e. on D.

($\Rightarrow$) Let $\{\mathcal{P}_n\}$ be a sequence of partitions s.t. $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$

$\|\mathcal{P}_n\| \to 0$ as $n \to \infty$. Let φ_n & ψ_n be the corresbonding step functions.

$\int_D \varphi_n \to \int_D f$. $\int_D \psi_n \to \int_D f$ as $n \to \infty$ $\quad \int_D(\varphi_n - \psi_n) \to 0$ as $n \to \infty$

As $\varphi_n \uparrow$ $\quad \psi_n \downarrow$, so $\exists \varphi^*, \psi^*$ $\quad \int_D(\psi^* - \varphi^*) = 0$, so $\psi^* = \varphi^*$ a.e.

Let $E_0 = \{x \in D \mid$ either $\varphi_n(x) \not\to f(x)$ or $\psi_n(x) \not\to f(x)\}$

Claim. If $x \notin E_0$, then f is cont. at x

Fix $x_0 \notin E_0$. $\varphi_n(x_0) \to f(x_0)$ $\quad \psi_n(x_0) \to f(x_0)$

WTS. $\forall \varepsilon > 0, \exists \delta > 0$, $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$

$\exists N \in \mathbb{N}$, s.t. $f(x_0) - \varepsilon < \varphi_n(x_0) \leq f(x_0) \leq \psi_n(x_0) < f(x_0) + \varepsilon$ when $n \geq N$

Note that $x_0 \notin \{\mathcal{P}_n\}$, $\exists$ a cube I_N s.t. $x_0 \in I_N$

$\forall x \in I_n$, $f(x_0) - \varepsilon < \varphi_N(x_0) = \varphi_N(x) \leq f(x) \leq \psi_N(x) = \psi_N(x_0) < f(x_0) + \varepsilon$

And clearly $m(E_0) = 0$ since $\varphi_n, \psi_n \to f$ a.e.

($\Leftarrow$) WTS. For any partition $\{\mathcal{P}_n\}$ with $\|\mathcal{P}_n\| \to 0$, $\Delta(\mathcal{P}_n) \to 0$ as $n \to \infty$.

Let φ_n, ψ_n be corresbonding step functions s.t. $\varphi_n \leq f \leq \psi_n$

Claim. $\varphi_n \to f$ a.e. & $\psi_n \to f$ a.e.

Let $E = \{x \in D \mid f$ is cont. at $x\}$, Fix $x_0 \in E \setminus \bigcup_{n=1}^{\infty} \{\mathcal{P}_n\}$, $m(E \setminus \bigcup_{n=1}^{\infty} \{\mathcal{P}_n\}) = m(E) = m(D)$

$\forall \varepsilon > 0, \exists \delta > 0$. s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ (This is from Lebesgue lemma)

So $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$, $\exists$ a cube I_N determined by $\mathcal{P}_n$ s.t. $\operatorname{diam}(I_N) < \delta$

Also $\begin{cases} f(x_0) - \varepsilon < f(x) < \psi_n(x) = \psi_n(x_0) < f(x_0) + \varepsilon \\ f(x_0) - \varepsilon < \varphi_n(x_0) = \varphi_n(x) < f(x) < f(x_0) + \varepsilon \end{cases}$ for $n \geq N$

Therefore $\varphi_n \to f$ a.e. & $\psi_n \to f$ a.e.

So $\int_D(\psi_n - \varphi_n) \to 0$ as $n \to \infty$

Prop. Let $\mathcal{C}$ be an arbitary countable subset of (a, b). Then $\exists$ an increasing function f on (a, b) which is cont. exactly on $(a, b)/\mathcal{C}$

proof: Let $\mathcal{C} = \{a_1, a_2, \cdots\}$

Define $f(x) = \sum_{\{n \mid a_n < x\}} \frac{1}{2^n}$, when $x_0 = a_N$: for $x < x_0$, $f(x_0) - f(x) = \sum_{\{n \mid x < a_n \leq x_0\}} \frac{1}{2^n} \geq \frac{1}{2^N}$

For $x_0 \notin \mathcal{C}$, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ $\forall$ $|x - x_0| < \delta$

$\exists N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$. Pick $\delta > 0$ s.t. $\{a_1, \cdots, a_N\} \cap \{x \mid |x - x_0| < \delta\} \neq \emptyset$

w.l.o.g. $x < x_0$, $f(x_0) - f(x) = \sum_{\{n \mid x < a_n \leq x_0\}} \frac{1}{2^n} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \varepsilon$

Def. For any integrable function f, we define the norm of f as:

$$\|f\| = \|f\|_{L^1} = \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|\,dx$$

The space of all function with this norm is $\mathcal{L}^1(\mathbb{R}^d)$ space.

Thm. The $\mathcal{L}^1$ is complete under its metric.

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^1$, then

$\|f_m - f_n\| \to 0$ as $m, n \to \infty$

($\exists \{f_n\}$ s.t. $\|f - f_n\| \to 0$ but exist no x s.t. $f_n(x) \to f(x)$)

Hence we consider a subsequence since $\|f\|$ is bounded

Consider a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$:

$\forall k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\| \leq \frac{1}{2^k}$, this is from $\|f_m - f_n\| < \varepsilon$

Consider $f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$

$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$

Notice that: $\int |f_{n_1}(x)| + \sum_{k=1}^{N} \int |f_{n_{k+1}} - f_{n_k}| \leq \int |f_{n_1}(x)| + \sum_{k=1}^{N} \frac{1}{2^k} < \infty$

Let $g_N(x) = |f_{n_1}(x)| + \sum_{k=1}^{N} |f_{n_{k+1}}(x) - f_k(x)|$, $g_N \uparrow g$

Since g_N is integrable, then g is also integrable

$|f| \leq g$, then f is also integrable.

And clearly $f_{n_k} \to f$ as $k \to \infty$

So $\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon$

This tells us that the Lebesgue integral is complete

Chapter 3

Differentiation
&
Intergration

Hardy-Littlewood maximal function

If f is integrable on $\mathbb{R}^d$, then its maximal functin f^*:

$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| dy$, $x \in \mathbb{R}^d$

<u>Thm.</u> Suppose f is integrable on $\mathbb{R}^d$, then

(i) f^* is measurable

(ii) for a.e. x, $f^*(x) < \infty$

(iii) $\forall \alpha > 0$, $m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$

<u>Lemma.</u> Suppose $\mathbb{B} = \{B_1, \cdots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^d$, then $\exists$ disjoint sub-collection of $\mathbb{B}$, $B_{i_1}, \cdots, B_{i_k}$, s.t. $m(\bigcup_{l=1}^N B_l) \leq 3^d \sum_{j=1}^k m(B_{i_j})$

We first pick the one with biggest radius B_{i_1}

then we remove B_{i_1} and any other who intersect with it.

So the balls removed are contained in $\widehat{B_{i_1}}$

where $\widehat{B_{i_1}}$ centered at the center of B_{i_1} while its radius is three times as B_{i_1}'s.

The balls left form a new collection $\mathbb{B}'$, and we repeat the process of removing balls.

Eventually, $\bigcup_{l=1}^N B_l \subset \bigcup_{j=1}^k \widehat{B_{i_K}}$

So $m(\bigcup_{l=1}^N B_l) \leq m(\bigcup_{j=1}^k \widehat{B_{i_K}}) = \sum_{j=1}^k m(\widehat{B_{i_j}}) = 3^d \sum_{j=1}^k (B_{i_j})$

Let $E_\alpha = \{x : f^*(x) > \alpha\}$. $\forall x \in E_\alpha$, $\exists$ a ball containing x

s.t. $\frac{1}{m(B_x)} \int_{B_x} |f(y)| dy > \alpha$

so for each B_x, $m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)| dy$

Fix a compact subset K, since $K \subset \bigcup_{x \in E_\alpha} B_x$, then we can find a finite subcover of K. s.t. $K \subset \bigcup_{i=1}^{N} B_i$

$m(K) \le m(\bigcup_{i=1}^{N} B_i) \le 3^d \sum_{j=1}^{k} m(B_{i_j}) \le \frac{3^d}{\alpha} \sum_{j=1}^{k} \int_{B_{i_j}} |f(y)| dy = \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)| dy$

$\le \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| dy$

And thus it holds for any compact subset of E_α

Hence $m(E_\alpha) \le \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| dy$

Thm. If f is integrable on $\mathbb{R}^d$, then for a.e. x,

$\lim_{\substack{m(B) \to 0 \\ x \in B}} \frac{1}{m(B)} \int_B |f(y)| dy = f(x)$

We show that for $\forall \alpha > 0$, the measurement of $E_\alpha = \{x : \limsup_{\substack{m(B) \to 0 \\ x \in B}} |\frac{1}{m(B)} \int_B f(y) dy - f(x)| > 2\alpha\}$ is zero

$\forall \varepsilon > 0$, we can choose a compact-supported continuous function g s.t. $\|f - g\|_{L^1(\mathbb{R}^d)} < \varepsilon$

Since g is continuous, then $\lim_{\substack{m(B) \to 0 \\ x \in B}} \frac{1}{m(B)} \int_B g(y) dy = g(x)$

(This is from Mathematics Analysis)

$\frac{1}{m(B)} \int_B f(y) dy - f(x) = \frac{1}{m(B)} \int_B (f(y) - g(y)) dy + (\frac{1}{m(B)} \int_B g(y) dy - g(x)) + g(x) - f(x)$

then $\limsup_{\substack{m(B) \to 0 \\ x \in B}} |\frac{1}{m(B)} \int_B f(y) dy - f(x)| \le |f(x) - g(x)| + (f - g)^*(x)$

Let $F_\alpha = \{x : |f(x) - g(x)| > \alpha\}$, $G = \{x : (f - g)^*(x) > \alpha\}$

$E_\alpha \subset (F_\alpha \cup G_\alpha)$ (Since if $u_1, u_2 \ge 0$, $u_1 + u_2 > 2\alpha$, $\exists u_i$ s.t. $u_i > \alpha$)

$m(F_\alpha) \le \frac{1}{\alpha} \|f - g\|_{L^1(\mathbb{R}^d)}$, $m(G_\alpha) \le \frac{3^d}{\alpha} \|f - g\|_{L^1(\mathbb{R}^d)}$

So $m(E_\alpha) \le m(F_\alpha) + m(G_\alpha) \le \frac{1 + 3^d}{\alpha} \cdot \varepsilon$

Therefore by the arbitariness of ε, $m(E_\alpha) = 0$

Thm. If $f \in L^1_{loc}(\mathbb{R}^d)$, then $\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x)$

Def. If E is a measurable set and $x \in \mathbb{R}^d$, we say that x is a point of Lebesgue density of E if:

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{m(B\cap E)}{m(B)} = 1$$

Cor. Let E be a measurable subset of $\mathbb{R}^d$. then

(i) Almost every $x \in E$ is a density point of E

(ii) Almost every $x \notin E$ is not a density point of E

Def. If f is locally integrable on $\mathbb{R}^d$, then the Lebesgue set of f is the set of all $\bar{x} \in \mathbb{R}^d$ where $f(\bar{x})$ is finite and $\lim_{\substack{m(B)\to 0 \\ \bar{x}\in B}} \frac{1}{m(B)} \int_B |f(y) - f(\bar{x})|\,dy = 0$

If $\bar{x}$ belongs to the Lebesgue set of f, then:

$$\lim_{\substack{m(B)\to 0 \\ \bar{x}\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(\bar{x})$$

Cor. If f is locally integrable on $\mathbb{R}^d$, then almost every point belongs to the Lebesgue set of f.

Def. Let f be a parameterized curve in the plane given by $z(t)=(x(t),y(t))$, where $a\le t\le b$. Here $x(t), y(t)$ are continuous real-valued functions on $[a,b]$.

The curve γ is rectifiable if there exists $M<\infty$ s.t. for any partition $a=t_0<\cdots<t_N=b$ of $[a,b]$

$$\sum_{j=1}^{N}|z(t_j)-z(t_{j-1})|\le M$$

The length of γ is given by $L(\gamma)=\sup_{a=t_0<\cdots<t_N=b}\sum_{j=1}^{N}|z(t_j)-z(t_{j-1})|$

Def. Let $F(t)$ be a complex-valued continuous function defined on $[a,b]$. The variation of F on this patition is given by $\sum_{j=1}^{N}|F(t_j)-F(t_{j-1})|$

If $\exists\, M<\infty$ s.t. for any partition $a=t_0<\cdots<t_N=b$

$$\sum_{j=1}^{N}|F(t_j)-F(t_{j-1})|\le M$$

Then F is said to be of bounded variation.

Thm. A curve parameterized by $(x(t),y(t))$, $a\le t\le b$ is rectifiable if and only if both $x(t)$ and $y(t)$ are both of bounded variations.

pf:) We observe that if $F(t)=x(t)+iy(t)$

then $F(t_j)-F(t_{j-1})=(x(t_j)-x(t_{j-1}))+i(y(t_j)-y(t_{j-1}))$

if $a,b\in\mathbb{R}$, then $|a+bi|\le|a|+|b|\le 2|a+bi|$

Thus the proof is done.

Eg. If f is real-valued, monotone and bounded then f is of bounded variation

pf:) WLOG $F\uparrow$, $|F|\le M$

then $\sum_{j=1}^{N}|F(t_j)-F(t_{j-1})|=\sum_{j=1}^{N}(F(t_j)-F(t_{j-1}))=F(b)-F(a)\le 2M$

Prop.

$F(x) + T_F(a,x) \nearrow$

pf:) Let $y > x$

$F(y) + T_F(a,y) - [F(x) + T_F(a,x)]$

$= (F(y) - F(x)) + T_F(x,y) \geq 0$

Def. The total variation of F on $[a,x]$ is given by

$T_F(a,x) = \sup \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|$

where we take supremum on all the partition.

$P_F(a,x) = \sup \sum_{(+)} F(t_j) - F(t_{j-1})$, is the positive variation

$N_F(a,x) = \sup \sum_{(-)} -(F(t_j) - F(t_{j-1}))$, is the negative variation.

Lemma. If F is real-valued and of bounded variation on $[a, b]$, then $\forall\ a \leq x \leq b$

$F(x) - F(a) = P_F(a,x) - N_F(a,x)$

$T_F(a,x) = P_F(a,x) + N_F(a,x)$

pf:) $F(x) - F(a) = \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -[F(t_j) - F(t_{j-1})]$

So $|F(x) - F(a) - (P_F - N_F)| \leq |P_F - \sum_{(+)} F(t_j) - F(t_{j-1})| + |N_F - \sum_{(-)} -[F(t_j) - F(t_{j-1})]| < 2\varepsilon$

Also $\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| = \sum_{(+)} F(t_j) - F(t_{j-1}) + \sum_{(-)} -[F(t_j) - F(t_{j-1})] \leq T_F$

Taking supremum from both side, then $T_F = P_F + N_F$

Thm. A real-valued function on $[a, b]$ if of bounded variation if and only if F is the difference of two bounded increasing function.

pf:) Clearly if $F = F_1 - F_2$, F_1, F_2 are of bounded variation

Therefore F is also of bounded variation.

Conversely, let $F_1(x) = P_F(a,x) + F(a)$ $\quad F_2 = N_F(a,x)$

And $F = F_1 - F_2$, F_1 & F_2 are clearly increasing.

Thm. If f is of bounded variation on $[a,b]$, then F is differentiable almost everywhere.

i.e. a.e. $x \in [a,b]$, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists

(To prove this, we need more tools)

Lemma. Suppose G is a continuous real-valued function on $\mathbb{R}$, let E be the set of points x s.t.

$G(x+h) > G(x)$ for some $h = h_x > 0$

If E is non-empty, then it must be open, then it can be written as a countable disjoint union of open intervals

$E = \bigcup (a_k, b_k)$. If (a_k, b_k) is a finite interval in this union

then we have $G(a_k) = G(b_k)$ (Riesz Lemma)

pf:) Since G is continuous, then $\forall$ open set B, $G^{-1}(B)$ is open in $\mathbb{R}$, therefore E is open, and clearly it can be written as disjoint union of open intervals.

Let (a_k, b_k) be a member of E, then $a_k \notin E$

Therefore we can't have $G(b_k) > G(a_k)$

Suppose $G(a_k) > G(b_k)$

Since G is continuous, then $\exists c$, $G(c) = \frac{G(a_k)+G(b_k)}{2}$

Let $c = \sup\{x : x \in (a_k, b_k), G(x) = \frac{G(a_k)+G(b_k)}{2}\}$

Since $c \in E$, $\exists d > c$ s.t. $G(d) > G(c) > G(b_k)$

Since $b_k \notin E$, then $G(x) \leq G(b_k)$ for $\forall x \geq b_k$, so $d < b_k$

Then $\exists c' \in (d, b_k)$, s.t. $G(c) = G(c')$, where $c' > c$

This is impossible since c is already the supremum

Therefore $G(a_k) = G(b_k)$

Def. Dini Number Define $\Delta_h(F)(x) = \frac{F(x+h)-F(x)}{h}$

$D^+(F)(x) = \limsup_{h\to 0^+} \Delta_h(F)(x)$ $\quad D_+(F)(x) = \liminf_{h\to 0^+} \Delta_h(F)(x)$

$D^-(F)(x) = \limsup_{h\to 0^-} \Delta_h(F)(x)$ $\quad D_-(F)(x) = \liminf_{h\to 0^-} \Delta_h(F)(x)$

Here we need to show that

(i) $D^+(F)(x) < \infty$ a.e.x. (ii) $D^+(F)(x) \leq D_-(F)(x)$ a.e.x.

Since F is of bounded variation, then it can be written as the difference between two increasing function

In the following proof we assume F is increasing.

(i) Let $E = \{x \in [a,b] : D^+(F)(x) > R > r > D_-(F)(x)\}$

$= [a,b] \cap \{x : D^+(F)(x) > R\} \cap \{x : D_-(F)(x) < r\}$

Let $H^+ = \{x : D^+(F)(x) > R\}$, then $\exists$ a sequence $\{h_n\}$, $h_n \to 0$

$\lim_{n\to\infty} \frac{F(x+h_n)-F(x)}{h_n} = D^+(F)(x)$, $H = \bigcap_{n=1}^{\infty} \{x : \frac{F(x+h_n)-F(x)}{h_n} > c\}$

Therefore H^+ is measurable, similarly so do $\{x : D_-(F)(x) < r\}$

$\exists$ O with $E \subset O \subset [a,b]$ & $m(O) \leq \frac{R}{r} m(E)$

Let $H_R = \{x : \frac{F(x+h)-F(h)}{h} > R$ for some $h>0\}$, which is open

Let $G(x) = F(x) - Rx$, then $E \subset H \subset H_R = \bigcup (a_k, b_k)$

Then $F(a_k) - Ra_k \leq F(b_k) - Rb_k$, $b_k - a_k \leq \frac{F(b_k)-F(a_k)}{R}$

Hence $m(H_R) \leq \sum_{k=1}^{N} (b_k - a_k) \leq \frac{F(b)-F(a)}{R}$. as $R \to \infty$, $m(H_R) \to 0$

this implies that $D^+(F)(x) < \infty$ a.e.x.

(ii) Suppose $m(E) > 0$

O can be written as the union of disjoint intervals

i.e. $O = \bigcup I_n$, we apply the lemma on $-F(-x)+rx$ on each I_n

so we have $F(b_k) - F(a_k) \leq r(b_k - a_k)$,

On each (a_k, b_k), apply the lemma on $G(x) = F(x) - Rx$

we get $O_n = \bigcup (a_{k,j}, b_{k,j})$, $F(b_{k,j}) - F(a_{k,j}) \geq R(b_{k,j} - a_{k,j})$

$m(O_n) = \sum_{k,j} (b_{k,j} - a_{k,j}) \leq \underbrace{\frac{1}{R} \sum_{k,j} F(b_{k,j}) - F(a_{k,j}) \leq \frac{1}{R} \sum_{k} F(b_k) - F(a_k)}_{\text{This is because } F \text{ is increasing}}$

$\leq \frac{r}{R}\sum_k (b_k - a_k) \leq \frac{r}{R} I_n$

Notice that $O_n \supset E \cap I_n$, so

$m(E) \leq \sum_n m(E \cap I_n) \leq \sum_n m(O_n) \leq \frac{r}{R}\sum_n I_n = \frac{r}{R} m(O) < m(E)$

Which contradicts, so $m(E) = 0$,

$m\{x : D^+(F)(x) > D_-(F)(x)\} = m(\bigcup_{\substack{R,r\in\mathbb{Q} \\ R>r}} E) = 0$

Hence, by the proof, we have $D_- = D^- = D_+ = D^+$ a.e.x.

So we get a theorem that.

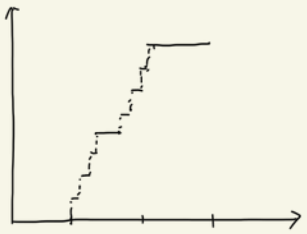
<u>Thm.</u> If F is continuous and of bounded variation, then F is differentiable almost everywhere.

<u>Cor.</u> If F is increasing and continuous, then F' exists almost everywhere, F' is measurable, non-negative and $\int_a^b F'(x)\,dx \leq F(b) - F(a)$.

Especially if F is bounded on $\mathbb{R}$, then F' is integrable on $\mathbb{R}$.

pf: Take $G_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{n}$, $\lim_{n\to\infty} G_n(x) = F'(x)$

Each G_n is cont. and measurable. therefore F' is measurable.



By the Fatou's Lemma, $\int_a^b F'(x)\,dx \leq \liminf_{n\to\infty} \int_a^b G_n(x)\,dx$

$\int_a^b G_n(x)\,dx = \frac{1}{n}\int_a^b F(x+\frac{1}{n})\,dx - \frac{1}{n}\int_a^b F(x)\,dx = \frac{1}{n}\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} F(x)\,dx - \frac{1}{n}\int_a^b F(x)\,dx$

$= \frac{1}{n}\int_b^{b+\frac{1}{n}} F(x)\,dx - \frac{1}{n}\int_a^{a+\frac{1}{n}} F(x)\,dx$

Since F is cont., then it is locally integrable on $\mathbb{R}$

therefore $\lim_{n\to\infty}\int_a^b G_n(x)\,dx = F(b) - F(a)$

Hence $\int_a^b F'(x)\,dx \leq F(b) - F(a)$

$\int_a^b F'(x)\,dx$ and $F(b) - F(a)$ are not strictly equal.

A counter-example is the Cantor-Lebesgue function where $F'(x) = 0$ a.e.x. but $F(1) - F(0) = 1$

let $\varphi(x)$ be the Cantor-Lebesgue function

Define $\psi(x) = \varphi(x) + x$, which is 1-1, onto, strictly $\nearrow$

Prop: ψ maps a measurable set to a non-measurable set

(pf:) Observe that $m(\psi(C)) = 1$, $m(\psi([0,1])) = m(\psi(C \cup O))$

$= m(\psi(C) \cup \psi(O)) = m(\psi(C)) + m(\psi(O)) = 1 + m(\psi(O)) = 2$

Since $m(\psi(C)) > 0$, then $\psi(C)$ contains a non-measurable set D

However $m(C) = 0$, then $m(\psi^{-1}(D)) = 0$

Def. For F defined on $[a,b]$, if $\forall \varepsilon>0$, $\exists \delta(\varepsilon)>0$ s.t. $\sum_{k=1}^{N}|F(b_k)-F(a_k)|<\varepsilon$ whenever $\sum_{k=1}^{N}(b_k-a_k)<\delta$

where (a_k, b_k), $k=1,\cdots,N$ are disjoint intervals

then we say that F is absolutely continuous.

Prop. If F is absolutely continuous, then

(i) F is uniformly continuous

(ii) F is of bounded variation

(iii) If $F(x)=\int_a^x f(y)dy$, f integrable, then F is AC

Thm. If F is absolutely continuous on $[a,b]$, then $F'(x)$ exists a.e. If $F'(x)=0$ a.e., then F is a constant.

Def. A collection of balls $\mathcal{B}=\{B\}$ is said to be a Vitali covering if $\forall x\in E$, $\eta>0$, $\exists B\in\mathcal{B}$, s.t. $x\in B$ and $m(B)<\eta$

Lemma. Suppose E is a set of finite measure, $\mathcal{B}$ is a Vitali covering of E. $\forall \delta>0$, we can find finite disjoint balls s.t. $\sum_{i=1}^{N} m(B_i)\geq m(E)-\delta$

pf:) Take δ small enough that $m(E)>\delta$

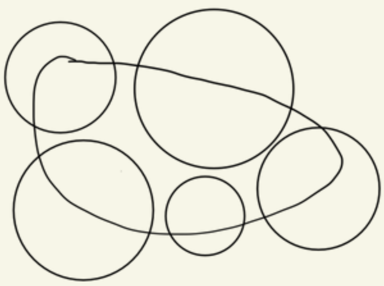
Then we choose a compact set $E'\subsetneqq E$ s.t. $m(E)>m(E')\geq\delta$

We choose a Vitali covering that covers E'

Since it is compact, then $\exists$ finite subcover

then $\exists$ disjoint balls where $\sum_{i=1}^{N_1} m(B_i)\geq 3^{-d}m(E')\geq 3^{-d}\delta$

① If $3^{-d}\delta\geq m(E)-\delta$, then it is true



② Otherwise, $\sum_{i=1}^{N_1} m(B_i) < m(E) - \delta$, then let $E_2 = E \setminus (\bigcup_{i=1}^{N} \overline{B}_i)$

Therefore $m(E_2) \geq m(E) - \sum_{i=1}^{N} m(B_i) > \delta$

Choose a compact subset E_2', $m(E_2) > m(E_2') \geq \delta$

for every $y \in E_2$, $y \in E$ & $y \notin B_i$ for some i

then $\exists\ r_i$ s.t. $B(y, r_i) \cap \overline{B}_i = \emptyset$

Therefore we can find a Vitali covering where the balls in the covering are disjoint with $\overline{B}_i$ for E_2

Chose a finite disjoint collection of balls B_i, $N_1 < i \leq N_2$

$\sum_{i=N_1+1}^{N_2} m(B_i) \geq 3^{-d} m(E_2') \geq 3^{-d}\delta$, $\sum_{i=1}^{N_2} m(B_i) \geq 2 \cdot 3^{-d}\delta$

$\{ \sum_{i=1}^{N_2} m(B_i) \geq m(E) - \delta$ ✓

otherwise: repeat this for k times, so $\sum_{i=1}^{N_k} m(B_i) \geq k \cdot 3^{-d} \cdot \delta$

Then when $k \geq \frac{3^d (m(E) - \delta)}{\delta}$, we have $\sum_{i=1}^{N_k} m(B_i) \geq m(E) - \delta$

Now we look back to $\mathbb{R}$.

Let E be the set where $F'(x) = 0$ and $x \in (a, b)$

$\forall x \in E$, fix $\varepsilon > 0$, $\lim_{h \to 0} \left| \frac{F(x+h) - F(x)}{h} \right| = 0$, $\exists\ \hat{\delta} > 0$

$|F(x+h) - F(x)| < \frac{\varepsilon}{2}|h|$ whenever $|h| < \hat{\delta}$

So $\forall \eta \in (0, \hat{\delta})$, $\exists$ an open interval $I_x = (a_x, b_x)$

where $|F(b_x) - F(a_x)| \leq \varepsilon (b_x - a_x)$ and $b_x - a_x < \eta$

thus $\{(b_x - a_x) : x \in E\}$ forms a Vitali covering of E

and we can find disjoint intervals where $I_i = (a_i, b_i) \subset [a, b]$

$1 \leq i \leq N$, $\sum_{i=1}^{N} m(I_i) \geq m(E) - \delta = (b - a) - \delta$

However, $|F(b_i) - F(a_i)| \leq \varepsilon (b_i - a_i)$, so $\sum_{i=1}^{N} |F(b_i) - F(a_i)| \leq \varepsilon (b - a)$

Now, we consider $[a, b] \setminus \bigcup_{i=1}^{N} (a_i, b_i)$, with the total length less than δ, formed by finite closed intervals $\bigcup_{k=1}^{M} [\alpha_k, \beta_k]$

Since F is absolutely cont., then $\sum_{k=1}^{M} |F(\alpha_k) - F(\beta_k)| \leq \varepsilon$

Hence $|F(b)-F(a)| \leq \sum_{i=1}^{N}|F(a_i)-F(b_i)|+\sum_{k=1}^{M}|F(\alpha_k)+F(\beta_k)|$

$\leq (b-a+1)\varepsilon$

By the arbitariness of ε, $F(b)=F(a)$. F remain constant

<u>Thm.</u> Suppose F is absolutely continuous on $[a,b]$. then F' exists a.e.x. and it's integrable. Also, $\forall x\in[a,b]$

$F(x)-F(a)=\int_a^x F'(y)dy$, $F(b)-F(a)=\int_a^b F'(x)dx$

pf:) F is AC, so it is of BV, therefore it can be written as the difference of two increasing function.

WLOG F is increasing on $[a,b]$

Then F' is integrable on $[a,b]$, let $G(x)=\int_a^x F'(y)dy$

Since F is cont., then it is intergrable on $[a,b]$

therefore $G'(x)=F'(x)$ a.e.x. let $H(x)=G(x)-F(x)$

$H'(x)=0$ a.e.x. Clearly H is also AC, so H remain constant

$G(x)-F(x)=G(a)-F(a)=0$, so $\int_a^x F'(y)dy=F(x)-F(a)$

<u>Lemma.</u> An bounded increasing function F on $[a,b]$ has at most countable numbers of discontinuous points.

Here we define $F(x^-)=\lim_{y\to x^-}F(y)$, $F(x^+)=\lim_{y\to x^+}F(y)$

pf:) Suppose F is discontinuous at x, then we can find a rational number r_x s.t. $F(x^-)<r_x<F(x^+)$

Suppose F is also discontinuous at z, $x<z$

then $r_x<F(x^+)\leq F(z^-)<r_z$

Thus every rational number is onto at most one discont. point. Hence we're done.

<u>Lebesgue Thm</u> If $f \uparrow$ in $[a,b]$, then f is differentiable a.e.

Recall: $\overline{D}f(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x)}{h}$, $\underline{D}f(x) = \liminf_{h\to 0} \frac{f(x+h)-f(x)}{h}$

pf:) $E = \{x\in[a,b] \mid \overline{D}f(x) > \underline{D}f(x)\} = \bigcup_{\alpha,\beta\in\mathbb{Q}} \overbrace{\{x\in[a,b] \mid \overline{D}f(x) > \alpha > \beta > \underline{D}f(x)\}}^{E_{\alpha,\beta}}$

$\forall \varepsilon > 0$, $\exists$ openset $O \supset E_{\alpha,\beta}$ s.t. $m(O) < m(E_{\alpha,\beta}) + \varepsilon$

Let $\mathcal{F}$ be the family of all closed bounded subintervals of O $[c,d]$ s.t. $\frac{f(d)-f(c)}{d-c} < \beta$. Here $\mathcal{F}$ forms a Vitali covering of $E_{\alpha,\beta}$

Then $\exists$ finite collection $\{I_k\}_{k=1}^n$ s.t. $m(E_{\alpha,\beta} / \bigcup_{k=1}^n I_k) < \varepsilon$

$m(E_{\alpha,\beta}) \le m(E_{\alpha,\beta} \cap (\bigcup_{k=1}^n I_k)) + m(E_{\alpha,\beta} \setminus \bigcup_{k=1}^n I_k)$

Also, $m(\{x\in[c_k,d_k] \mid \overline{D}f(x) > \alpha\}) \le \frac{1}{\alpha}(f(d_k) - f(c_k))$

so $m(E_{\alpha,\beta}) \le \frac{1}{\alpha}\sum_{k=1}^n (f(d_k)-f(c_k)) + \varepsilon \le \frac{\beta}{\alpha}\sum_{k=1}^n (m(E_{\alpha,\beta}) + \varepsilon) + \varepsilon$

$(1-\frac{\beta}{\alpha})\, m(E_{\alpha,\beta}) < (1+\frac{\beta}{\alpha})\,\varepsilon \quad \Rightarrow \quad m(E_{\alpha,\beta}) = 0$

<u>Prop.</u>

(1) If F is AC, then it is of BV

proof: $\forall \{(c_k,d_k)\}_1^n$ disjoint, we have $\sum_{k=1}^n |f(c_k)-f(d_k)| < \varepsilon$ when $\sum_{k=1}^n |c_k - d_k| < \delta$

Take $\varepsilon = 1$, then we have a fixed δ

let $\mathcal{P} = \{a = x_0 < \cdots < x_m = b\}$, with $|\mathcal{P}| < \delta$

By refining every $[x_{k-1}, x_k]$, we have $T_F(x_{k-1}, x_k) \le 1$

Therefore $T_F(a,b) \le m$

(2) $T_F(a,x)$ is AC if F is AC

proof: For any partition P_k of $[c_k, d_k]$

$\sum_{k=1}^n V(f|_{[c_k,d_k]}, P_k) < \frac{\varepsilon}{2}$, then $\sum_{k=1}^n T_F(c_k,d_k) \le \frac{\varepsilon}{2}$

(3) Suppose f is cont. on $[a,b]$, then f is AC $\Leftrightarrow \{D_n f\}_{0<h\le 1}$ is equi-integrable.

proof: ($\Leftarrow$) $\{D_n f\}_{0<h\le 1}$ is equi-integrable.

i.e. $\forall \varepsilon>0, \exists \delta>0$, if $m(E)<\delta$, then $\int_E |D_n f| < \frac{\varepsilon}{2}$, $\forall\, 0<h\le 1$

For $\{(c_k,d_k)\}_{k=1}^n$ disjoint, with $\sum_{k=1}^n |c_k-d_k|<\delta$

Take $E=\bigcup_{k=1}^n (c_k,d_k)$, $\int_{c_k}^{d_k} D_n f = AV_n f(d_k) - AV_n f(c_k) \longrightarrow f(d_k)-f(c_k)$ as $h\to 0$

This implies that $\sum_{k=1}^n |f(c_k)-f(d_k)| \le \sum_{k=1}^n \left|\int_{c_k}^{d_k} D_n f\right| + \frac{\varepsilon}{2} \le \sum_{k=1}^n \int_{c_k}^{d_k} |D_n f| + \frac{\varepsilon}{2} = \varepsilon$

($\Rightarrow$) WTS. $\forall \varepsilon>0, \exists \delta>0$, s.t. for $m(E)<\delta$, $\int_E |D_n f|<\varepsilon$, $\forall\, 0<h\le 1$

Claim. $\forall \varepsilon>0, \exists \delta>0$, s.t. if $\sum_{k=1}^n |c_k-d_k|<\delta$, then $\int_{\bigcup_{k=1}^n (c_k,d_k)} |D_n f| < \frac{\varepsilon}{2}$, $\forall\, 0<h\le 1$

pf) f is AC, then f is of BV, w.l.o.g. f increasing.

$\forall \varepsilon>0, \exists \delta>0$, s.t. if $\sum_{k=1}^n (d_k-c_k)<\delta$, then $\sum_{k=1}^n |f(c_k)-f(d_k)|<\frac{\varepsilon}{2}$

$\int_{c_k}^{d_k} D_n f = AV_n f(d_k) - AV_n f(c_k) = \frac{1}{h}\int_0^h (f(d_k+t)-f(c_k+t))\,dt$

Since $\sum_{k=1}^n [(d_k+t)-(c_k+t)] = \sum_{k=1}^n (d_k-c_k) < \delta$

Then $\int_{\bigcup_{k=1}^n (c_k,d_k)} D_n f = \frac{1}{h}\int_0^h \sum_{k=1}^n (f(d_k+t)-f(c_k+t))\,dt \le \frac{1}{h}\int_0^h \frac{\varepsilon}{2}\,dt = \frac{\varepsilon}{2}$

Fundmental Theorem of Calculus.

f is AC on $[a,b] \Leftrightarrow f'$ is integrable & $f(x)=\int_a^x f'(t)\,dt + f(a)$

proof: ($\Rightarrow$) $\int_a^x D_n(f)(t)\,dt = A_n f(x) - A_n f(a)$

Since f is AC, then $D_n(f)$ is equi-integrable

By Vitali's Convergence thm, $\lim_{h\to 0}\int_a^x D_n(f)(t)\,dt = \int_a^x f'(t)\,dt$

i.e. $f(x)=\int_a^x f'(t)\,dt + f(a)$

($\Leftarrow$) $\sum_{k=1}^N |f(d_k)-f(c_k)| \le \sum_{k=1}^N \int_{c_k}^{d_k} |f'(t)|\,dt = \int_{\bigcup (c_k,d_k)} |f'(t)|\,dt$

Since f' is integrable, then it is equi-integrable

then $\sum_{k=1}^N |f(d_k)-f(c_k)|<\varepsilon$ when $\sum_{k=1}^N |c_k-d_k|<\delta$

Lebesgue Decomposition Lemma

Let f be of BV, then $f=g+h$, where g is AC & h is singular

Def. h is said to be singular if $h'=0$ a.e.

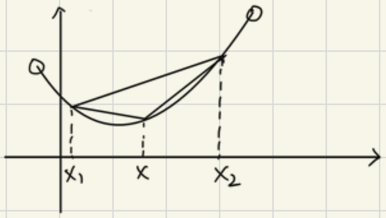
proof of Lemma.

Set $g(x)=\int_a^x f'(t)\,dt$, which is AC

$g'(x)=f'(x)$ a.e. since f' is integrable

<u>Def.</u> A real valued function φ on (a,b) is convex if
$\forall x_1, x_2 \in (a,b), \forall \lambda \in [0,1], \varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$

Observe: φ must be continuous on (a,b)



$\forall x_1 < x < x_2, \quad \frac{f(x)-f(x_1)}{x-x_1} \leq \frac{f(x_2)-f(x)}{x_2-x}$

<u>Prop.</u> Let φ be convex on (a,b). then
(1) φ is differentiable a.e. on (a,b)
(2) φ has left-handed & right-handed derivative for $\forall x \in (a,b)$, &
$\varphi'(x^-) \leq \varphi'(x^+) \leq \frac{\varphi(y)-\varphi(x)}{y-x} \leq \varphi'(y^-) \leq \varphi'(y^+)$. where $a<x<y<b$
(3) φ is Lipschitz on $[c,d] \subset (a,b)$ for $\forall\ a<c<d<b$
proof: (1) For $\forall [c,d] \subset (a,b)$, $\sum_{i=1}^{n} |\varphi(x_i)-\varphi(x_{i-1})| \leq L \sum_{i=1}^{n}(x_i - x_{i-1}) = L(d-c)$
So φ is of BV. and therefore φ' exists a.e.
(2) $\forall\ x \in (x_1, x_2)$, $\frac{\varphi(x)-\varphi(x_1)}{x-x_1} \leq \frac{\varphi(x_2)-\varphi(x)}{x_2-x}$
Take $x_1 \to x^-$, $x_2 \to x^+$, then it's done
(3) $|\varphi(x_1)-\varphi(x_2)| \leq \max_{x\in[c,d]} \left\{ \left|\frac{\varphi(x)-\varphi(c)}{x-c}\right|, \left|\frac{\varphi(x)-\varphi(d)}{x-d}\right| \right\} |x_1-x_2| = L|x_1-x_2|$

(More exactly. φ has at most countably many indifferentiable points)

<u>Def.</u> A line $y=\varphi(x_0)+m(x-x_0)$ is called a supporting line if the line lies below the graph of φ, i.e. $\varphi(x) \geq \varphi(x_0)+m(x-x_0)$, $\forall\ x \in (a,b)$

<u>Jensen's inequality</u> Suppose φ is convex on $\mathbb{R}$. f is integrable on $(0,1)$. with $\varphi\circ f$ also integrable, then

$$\varphi\left(\int_0^1 f(x)\,dx\right) \leq \int_0^1 \varphi\circ f(x)\,dx$$

pf:) set $\alpha = \int_0^1 f(x)dx$
Then $\exists$ a supporting line $y=\varphi(\alpha)+m(x-\alpha)$
So $\varphi\circ f(x) \geq \varphi(f(\alpha)) + m(f(x)-\alpha)$
$\int_0^1 (\varphi\circ f(x))\,dx \geq \varphi\left(\int_0^1 f(x)\,dx\right)$

Thm. Suppose $(x(t), y(t))$ is a curve where $a \le t \le b$. If $x(t)$, $y(t)$ are absolutely continuous, then the curve is rectifiable. while its length $L = \int_a^b (x'(t)^2 + y'(t)^2)^{\frac{1}{2}} dt$

Thm. Suppose F is a complex valued absolutely function $[a, b]$. then $T_F(a, b) = \int_a^b |F'(t)| dt$

pf: $\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| = \sum_{j=1}^{N} \left| \int_{t_{j-1}}^{t_j} F'(t) dt \right| \le \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} |F'(t)| dt = \int_a^b |F'(t)| dt$

where $a = t_0 < \cdots < t_N = b$ is any partition of $[a, b]$

Therefore we have $T_F(a, b) \le \int_a^b |F'(t)| dt$

Conversely, we use a step function to approximate F'

let $F' = g + h$, where $\int_a^b |h(t)| dt < \varepsilon$, let $G(t) = \int_a^t g(x) dx$

g is a step function. Let $H(t) = \int_a^t h(x) dx$

then $F = G + H$. $T_F(a, b) \ge T_G(a, b) - T_H(a, b) \ge T_G(a, b) - \varepsilon$

Also $T_G(a, b) \ge \sum_{j=1}^{N} |G(t_j) - G(t_{j-1})| = \sum_{j=1}^{N} \left| \int_{t_{j-1}}^{t_j} g(t) dt \right| \ge \int_a^b |g(t)| dt \ge \int_a^b |F'(t)| dt - \varepsilon$

Thus $T_F(a, b) \ge \int_a^b |F'(t)| dt - 2\varepsilon$

Hence we have $T_F(a, b) = \int_a^b |F'(t)| dt$

Construction of a increasing function which is only discontinuous at rational points:

let $f_p(x) = \begin{cases} 0, & x = 0 \\ \frac{p}{q}, & x \in \left(\frac{p}{q+1}, \frac{p}{q}\right], q = p, p+1, \cdots \end{cases}$

$g_p(x) = \min\{f_1(x), \cdots, f_p(x)\}$

$g(x) = \inf_{p \in \mathbb{N}^+} g_p(x)$

<u>Def.</u> The convolution of f and g is given by
$(f*g)(x)=\int_{\mathbb{R}^d} f(x-y)g(y)\,dy$

<u>Prop.</u>
(1) If f, g are integrable, then $f*g$ is integrable
(2) $\|f*g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \cdot \|g\|_{L^1(\mathbb{R}^d)}$
(Cauchy Swarchz inequality)

<u>Thm.</u> (Stone-Wierstrass) Any continuous function on $\mathbb{R}$ can be approximated by a polynomial.

$w_1 f(x_1)+\cdots+w_n f(x_n)$

pf: Let $g(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$, f be a continuous function
Let the metric in the space be defined by
$\|p-q\|=\sup_{\mathbb{R}}|p(x)-q(x)|$
Let $\check{f}(x)=f*g(x)=\int_{\mathbb{R}} f(x-y)g(y)\,dy=\int_{\mathbb{R}} f(t)g(x-t)\,dt$
Clearly $\check{f}$ is smooth for each $\sigma>0$
then we can use Taylor extension to get a polynomial
as $\sigma\to 0$, $\check{f}\to f$
thus we can find a polynomial $p(x)$
where $\|p-f\|\leq\|p-\check{f}\|+\|f-\check{f}\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

<u>Def.</u> K_δ are good kernels if they are integrable and for each $\delta>0$,
(1) $\int_{\mathbb{R}^d} K_\delta(x)\,dx=1$
(2) $\exists A\geq 1$, $\int_{\mathbb{R}^d}|K_\delta(x)|\,dx\leq A$
(3) $\forall \eta>0$, $\int_{|x|\geq\eta}|K_\delta(x)|\,dx\to 0$ as $\delta\to 0$

To make $(f*K_\delta)(x) \to f(x)$ as $\delta \to 0$, we need stronger conditions.

(2') $\forall \delta > 0$, $|K_\delta(x)| \leqslant A\delta^{-d}$

(3') $\forall \delta > 0$, $x \in \mathbb{R}^d$, $|K_\delta(x)| \leqslant \frac{A\delta}{|x|^{d+1}}$

=> (2): We consider the function $f(x) = \begin{cases} \frac{1}{|x|^{d+1}}, & |x| \neq 0 \\ 0, & x = 0 \end{cases}$

we show that $\int_{|x|>\varepsilon} f(x)\,dx \leqslant \frac{C}{\varepsilon}$ for some constant C

Let $A_k = \{x \in \mathbb{R}^d : 2^k\varepsilon \leqslant |x| \leqslant 2^{k+1}\varepsilon\}$, $g(x) = \sum_{k=0}^{\infty} \frac{\chi_{A_k}(x)}{(2^k\varepsilon)^{d+1}}$

Therefore we have $f(x) \leqslant g(x)$

Since the series is uniformly convergent, then

$\int g = \sum_{k=0}^{\infty} \int_{|x|\geq\varepsilon} \frac{\chi_{A_k}(x)}{(2^k\varepsilon)^{d+1}}\,dx = \sum_{k=0}^{\infty} \frac{m(A_k)}{(2^k\varepsilon)^{d+1}}$

Let $A = \{x : 1 \leqslant |x| \leqslant 2\}$, so $m(A_k) = (2^k\varepsilon)^d m(A)$

so $\int_{|x|\geqslant\varepsilon} \frac{dx}{|x|^{d+1}} \leqslant m(A) \sum_{k=0}^{\infty} \frac{1}{2^k\varepsilon} = \frac{2m(A)}{\varepsilon}$, let $C = 2m(A)$

so $\int_{\mathbb{R}^d} |K_\delta(x)|\,dx = \int_{|x|<\delta} |K_\delta(x)|\,dx + \int_{|x|\geqslant\delta} |K_\delta(x)|\,dx \leqslant A\int_{|x|<\delta} \frac{dx}{\delta^d} + \int_{|x|\geqslant\delta} \frac{A\delta}{|x|^{d+1}}\,dx$

$\leqslant A_1 + A\delta \cdot \frac{C}{\delta} = A_1 + A_2 < \infty$

=> (3): Also $\int_{|x|\geqslant\eta} |K_\delta(x)|\,dx \leqslant A\delta \int_{|x|\geqslant\eta} \frac{dx}{|x|^{d+1}} \leqslant \frac{AC\delta}{\eta}$

thus $\forall$ fixed $\eta > 0$, $\int_{|x|\geqslant\eta} |K_\delta(x)|\,dx \to 0$ as $\delta \to 0$

We say K_δ is an identity approximation in this case

<u>Thm.</u> If f is integrable on $\mathbb{R}^d$ and $\{K_\delta\}_{\delta>0}$ is an identity approximation, then $\forall \delta > 0$, the convolution $(f*K_\delta)(x) = \int_{\mathbb{R}^d} f(x-y)K_\delta(y)\,dy$ is integrable

and $\|(f*K_\delta) - f\|_{L^1(\mathbb{R}^d)} \to 0$ as $\delta \to 0$

Chapter 4

Hilbert Space

Eg. A prime example of a Hilbert space is the collection of square integrable function on $\mathbb{R}^d$, which is denoted by $L^2(\mathbb{R}^d)$, and consists of all complex-valued measurable function f that
$\int_{\mathbb{R}^d} |f(x)|^2 dx < \infty$
where the norm is defined by
$\|f\|_{L^2(\mathbb{R}^d)} = (\int_{\mathbb{R}^d} |f(x)|^2 dx)^{\frac{1}{2}}$
the inner product is defined by
$(f, g) = \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx$

Define the metric as $d(f, g) = \|f - g\|_{L^2(\mathbb{R}^d)}$
Thm. $L^2(\mathbb{R}^d)$ is complete under this metric

Thm. $L^2(\mathbb{R}^d)$ is seperable, i.e. $\exists$ a countable collection of elements $\{f_k\}$ in $L^2(\mathbb{R}^d)$ whose linear combination is dense is dense in $L^2(\mathbb{R}^d)$
pf:) Let $f \in L^2(\mathbb{R}^d)$, $g_n(x) = \begin{cases} f(x), & |x| \le n \text{ and } |f(x)| \le n \\ 0, & \text{otherwise} \end{cases}$
$g_n \to f$ as $n \to \infty$, also we have $|f - g_n|^2 \le 4|f|^2$
therefore $\int |f - g_n|^2 dx \to \int |f - f|^2 dx = 0$ as $n \to \infty$
thus for some N, $\|f - g_N\|_{L^2(\mathbb{R}^d)} < \frac{\varepsilon}{2}$
let $g = g_N$, we see that g is supported on a finite set which is bounded, thus $g \in L^1(\mathbb{R}^d)$
Then we choose a step function φ s.t. $|\varphi| \le N$ and $\int |g - \varphi| < \frac{\varepsilon^2}{16N}$ (Since compact-supported function are dense in $L^1(\mathbb{R}^d)$)
Replace the coefficients and rectangles that

ps: the coefficient of a step function is r where $\varphi(x) = r \cdot \chi_R(x)$

appear in the conanical form of φ by complex numbers with rational real and imaginary parts, and rectangles with rational coordinates

then we find $|\psi| \le N$ s.t. $\int |\varphi - \psi| \le \frac{\varepsilon^2}{16N}$

then $\int |g-\psi| < \frac{\varepsilon^2}{8N}$, so $\int |g-\psi|^2 \le 2N \int |g-\psi| < \frac{\varepsilon^2}{4}$

so we have $\|f-\psi\| \le \|f-g\| + \|g-\psi\| < \frac{\varepsilon}{2} + \sqrt{\int |g-\psi|^2} < \varepsilon$

<u>Def.</u> A set $\mathcal{H}$ is a <u>Hilbert Space</u> if:

(i) $\mathcal{H}$ is an inner product space over $\mathbb{C}$ (or $\mathbb{R}$)

(ii) $\mathcal{H}$ is complete under the metric $d(f,g) = \|f-g\|$

(ii) $\mathcal{H}$ is seperable

<u>Def.</u> Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces. A mapping $U: \mathcal{H} \to \mathcal{H}'$ is unitary if:

(i) U is linear

(ii) U is a bijection

(iii) $\|Uf\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$

Notice that $(f,g) = \frac{1}{4}(\|f+g\|^2 - \|f-g\|^2 + i(\|f+ig\|^2 - \|f-ig\|^2))$

so $(Uf, Ug)_{\mathcal{H}'} = (f,g)_{\mathcal{H}}$

We say $\mathcal{H}$ and $\mathcal{H}'$ are unitarily equivalent or unitarily isomorphic

<u>Cor.</u> $\mathcal{H}$ and $\mathcal{H}'$ are unitarily equivalent if and only if $\dim \mathcal{H} = \dim \mathcal{H}'$

Def. $(\mathfrak{X}, \|\cdot\|)$ is a normed space if $\mathfrak{X}$ is a vector space and $\|\cdot\|: \mathfrak{X} \to \mathbb{R}^+$ has the following property.

(1) $\|f\| \geq 0$, $\forall f \in \mathfrak{X}$ & $\|f\| = 0 \Leftrightarrow f = 0$

(2) $\|\alpha f\| = |\alpha| \, \|f\|$, $\alpha \in \mathbb{R}$

(3) $\|f+g\| \leq \|f\| + \|g\|$

We see that $d(f,g) := \|f-g\|$ is a metric in $\mathfrak{X}$

$\mathfrak{X}$ is called a Banach Space if $\mathfrak{X}$ is a complete metric space.

Example. L^1 space is a Banach space

We've proved its completeness in the previous chapter

Let E be a measurable set. $\mathcal{F}$ be the collection of all extended real valued functions which are finite a.e. on E.

$L^p(E) := \{f \in \mathcal{F} \mid \int_E |f|^p < \infty\}$, with the norm $\|f\|_p = \left(\int_E |f|^p\right)^{\frac{1}{p}}$

where $1 \leq p < \infty$

$L^\infty(E) := \{g \in \mathcal{F} \mid \exists M > 0, \text{ s.t. } |f(x)| < M \text{ a.e. } E\}$. $\|g\|_\infty = \inf M$

Young's inequality.

$\forall a, b > 0$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

q is defined as the conjugate of p

proof: Use the convexity

Notice that $\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q \leq \ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)$ ($\ln x$ is convex)

so $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Hölder's inequality.

$1 \leq p < \infty$, q is the conjugate of p

then $\int_E |fg| \leq \|f\|_p \|g\|_q$, $\forall f \in L^p(E)$, $g \in L^q(E)$

proof: when $p = 1$, $\int_E |fg| \leq \|g\|_\infty \int_E |f| = \|f\|_1 \|g\|_\infty$

for $p > 0$. set $a = \frac{|f(x)|}{\|f\|_p}$, $b = \frac{|g(x)|}{\|g(x)\|_q}$

Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

$$\int_E \frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} dx \leq \frac{1}{p}\int_E \frac{|f(x)|^p}{\|f\|_p^p} dx + \frac{1}{q}\int \frac{|g(x)|^q}{\|g\|_q^q} dx = \frac{1}{p} + \frac{1}{q} = 1$$

Therefore $\int_E |f(x)g(x)| dx \leq \|f\|_p \|g\|_q$

Remark. For $f \in L^p(E)$, its conjugate $f^*(x) = (\mathrm{sgn}(f(x))\frac{|f(x)|^{p-1}}{\|f\|_p^{p-1}} \in L^q(E)$
This is true since $\|f^*\|_q^q = 1$
$\Rightarrow \int_E f f^* = \int_E |f f^*| = \int_E \frac{|f(x)|^p}{\|f\|_p^{p-1}} dx = \|f\|_p$

Minkovski inequality
$1 \le p < \infty$, $f, g \in L^p(E) \Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p$
proof: Observe $f+g \in L^p(E)$, since $|f(x)+g(x)|^p \le (2f(x))^p + (2g(x))^p$
$\|f+g\|_p = \int_E (f+g)(f+g)^* = \int_E f(f+g)^* + \int_E g(f+g)^* \le (\|f\|_p + \|g\|_p)\|(f+g)^*\| = \|f\|_p + \|g\|_p$

This inequality implies that L^p space is a linear normed space.

Prop. Suppose $m(E) < \infty$, Then $L^p(E)$ is decreasing in p, $1 \le p < \infty$
In fact, $\|f\|_{p_1} \le C\|f\|_{p_2}$ if $p_1 \le p_2$ & $f \in L^{p_2}(E)$, with $C = (m(E))^{\frac{1}{p_1} - \frac{1}{p_2}}$
proof: $\|f\|_{p_1}^{p_1} = \int_E |f| \le (\int_E (|f|^{p_1})^{\frac{p_2}{p_1}})^{\frac{p_1}{p_2}} (\int_E 1^q)^{\frac{1}{q}} = \|f\|_{p_2}^{p_1} (m(E))^{\frac{1}{q}}$
Here $\frac{1}{q} + \frac{p_1}{p_2} = 1$, so $\|f\|_{p_1} \le C\|f\|_{p_2}$

Prop. Given $p > 1$, then the family $\{f \in L^p(E) \mid \|f\|_p \le M\}$, where $M > 0$ is a constant, is equi-integrable.
pf:) let $m(A) < \infty$, then $\int_A |f| \le (m(A))^{1-\frac{1}{p}} \|f\|_p \le M(m(A))^{1-\frac{1}{p}}$
Take $\delta = (\frac{\varepsilon}{M})^{\frac{p}{p-1}}$, then $\int_A |f| < \varepsilon$ whenever $m(A) < \delta$

Thm. Then L^p spaces are complete, i.e. $\forall 1 \le p < \infty$, $\{f_n\}_{n=1}^{\infty} \subset L^p(E)$
$\forall \varepsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall m, n > N$, $\|f_m - f_n\|_p < \varepsilon \Rightarrow \exists f \in L^p(E)$. $\|f_n - f\|_p \to 0$
The proof follows directly from the proof in L^1 space.
Since the sequence is Cauchy, take $\{f_{n_k}\}$ s.t.
$\|f_{n_{k+1}} - f_{n_k}\| \le 2^{-k}$, let $g_N(x) := \sum_{k=1}^{N} |f_{n_{k+1}}(x) - f_{n_k}(x)|$, $g(x) := \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$
$\|g_N\|_p \le \sum_{k=1}^{N} \|f_{n_{k+1}} - f_{n_k}\|_p \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (Minkovski inequality)
By Fatou's Lemma, $\|g\|_p \le \liminf_{N\to\infty} \|g_N\|_p \le 1$, so $g \in L^p(E)$
Therefore a.e. x, $\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < \infty$, define $f(x) = \lim_{k\to\infty} f_{n_k}(x)$
$\|f_n - f\|_p \le \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p$
Dominated Convergence Th'm
$|f_{n_k}(x) - f(x)|^p \le (\sum_{j=k}^{\infty} |f_{n_j}(x) - f_{n_{j+1}}(x)|)^p \le g(x)^p$, so $\|f_{n_k} - f\|_p < \frac{1}{2}\varepsilon$
Then $\|f_n - f\|_p \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$

Thm. If $\{f_n\} \to f$ pointwise a.e., where $f_n, f \in L^p(E)$ $(p \geq 1)$

Then the followings are equivallent.

(1) $f_n \to f$ in $L^p(E)$, i.e. $\|f_n - f\|_p \to 0$

(2) $\|f_n\|_p \to \|f\|_p$

(3) $\{f_n\}$ is equi-integrable.

proof: (1) => (2) $\|f\|_p \leq \|f_n - f\|_p + \|f_n\|_p$, $\|f_n\|_p \leq \|f_n - f\|_p + \|f\|$

So $\big| \|f\|_p - \|f_n\|_p \big| \leq \|f_n - f_n\|_p \to 0$

(2) => (1) By convexity, $\frac{|f_n|^p + |f|^p}{2} - \left|\frac{f_n - f}{2}\right|^p \geq 0$, this converges to $|f|^p$ pointwise

$\int_E |f|^p \leq \liminf_{n\to\infty} \left(\int_E \frac{|f_n|^p + |f|^p}{2} - \left|\frac{f_n - f}{2}\right|^p \right) = \int_E |f|^p - \frac{1}{2^p} \limsup_{n\to\infty} \int_E |f_n - f|^p$

So $\limsup_{n\to\infty} \int_E |f_n - f| \leq 2^p \left| \int_E |f|^p - \int_E |f_n|^p \right|$, so $\|f_n - f\|_p \to 0$

(1) <=> (3)

Set $h_n = |f_n - f|^p \geq 0$, $\int_E h_n \to 0$

Then it is equi-integrable and tight

$|f_n|^p \leq (|f_n - f| + |f|)^p \leq 2^p (|f - f_n|^p + |f|^p)$

Thm. $L^p(E)$ is seperable for $1 \le p < \infty$

Step 1 Simple functions are dense in $L^p(E)$

proof: Let $g \in L^p(E)$. Then $\exists$ a sequence $\{\varphi_n\}$ on E s.t. $|\varphi_n| \le |g|$ and $\varphi_n \to g$ pointwise.

$|\varphi_n - g|^p \le (|\varphi_n| + |g|)^p \le 2^p |g|^p$, which is dominated, then $\|\varphi_n - g\|_p \to 0$

Step.2 Step functions are dense in $L^p(E)$

Claim. we can approximate χ_A ($A \subseteq E$ measurable) by step functions

$\forall \varepsilon > 0$. $\exists$ open intervals $I_1, \cdots, I_n$ s.t.

$m(A \triangle (\bigcup_{k=1}^{n} I_k)) < \varepsilon$, denote $G = \bigcup_{k=1}^{n} I_k$

Then $\int_E |\chi_F - \chi_A|^p = m(A \triangle G) < \varepsilon$, so $\|\chi_A - \chi_F\|_p < \varepsilon^{\frac{1}{p}}$

Step 3 $L^p([a,b])$ is seperable

Set $\mathcal{F} = \{ \sum_{k=1}^{m} c_k \chi_{[a_k, b_k]} \mid a_k, b_k, c_k \in \mathbb{Q} \}$,

Then for $\forall f \in L^p[a,b]$, $\exists \varphi \in$, $\|f - \varphi\|_p < \varepsilon$

Step 4. $L^p(\mathbb{R})$ is seperable.

Set $\mathcal{F}_n = \mathcal{F} \cap [-n, n]$, clearly $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$

Let $f \in L^p(\mathbb{R})$, define $f_n : x \longmapsto \begin{cases} f(x), & x \in [-n, n] \\ 0, & |x| > n \end{cases}$

clearly $f_n \in L^p([-n, n])$

Then $\exists \hat{f}_n \in \mathcal{F}_n$, s.t. $\|f_n - \hat{f}_n\|_p < \frac{1}{n}$

Therefore $\|f - \hat{f}_n\|_p \le \|f - f_n\|_p + \|f_n - \hat{f}_n\| \to 0$

Step.5 $L^p(E)$ is seperable

Let $f \in L^p(E)$, Extend f to $\mathbb{R}$ by setting $\hat{f} = 0$ on $\mathbb{R} \setminus E$

Then it's done

Another proof. by Stone-Weierstrass Approximation.

Since f is measurable on $\mathbb{R}$, $\exists E \subseteq \mathbb{R}$ s.t. $m(\mathbb{R} \setminus E) < \varepsilon$, s.t. $\exists g$ continuous s.t. $f \equiv g$ on E

By S-W Approximation, $\exists$ polynomial r with rational coefficients s.t. $\|g - r\|_p < \frac{1}{2}\varepsilon$, then $\|f - r\|_p \le \|f - g\|_p + \|g - r\|_p < \varepsilon$

Def.

(1) Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. Then $T: \mathfrak{X} \to \mathbb{R}$ is called a linear functional if $T(\alpha f+\beta g)=\alpha T(f)+\beta T(g)$

(2) A linear functional T is said to be bounded if $\exists\ M>0$ s.t. $|T(f)| \leq M\|f\|$ for $\forall f \in \mathfrak{X}$

(3) Set $\mathfrak{X}^* = \{T: \mathfrak{X} \to \mathbb{R} \mid T \text{ is a bounded linear functional}\}$,
$\|T\|_* = \inf\{M>0 \mid |T(f)| \leq M\|f\|, \forall f \in \mathfrak{X}\}$

Here $(\mathfrak{X}^*, \|\cdot\|_*)$ is called the dual space of $\mathfrak{X}$

Prop.

(1) $\|T\|_* = \sup\{|T(f)| \mid \|f\|=1\}$

(2) A bounded functional is continuous

proof: (1) let $M = \sup_{\|f\|=1} |T(f)|$, then $M \leq \|T\|_*$

Conversely, we let $\|f\|>0$, then $\|\frac{f}{\|f\|}\|=1$, $|T(\frac{f}{\|f\|})| \leq M$

So $|T(f)| \leq M\|f\|$, this implies that $M \geq \|T\|_*$

(2) $|T(f)-T(g)| = |T(f-g)| \leq M\|f-g\|$ done

Thm. Let $p \geq 1$, q be its conjugate, then $(L^p)^* \cong L^q$

Lemma. Suppose $p \geq 1$, q be its conjugate

(1) If $g \in L^q$, then $\|g\|_q = \sup_{\|f\|_p \leq 1} |\int fg|$ (†)

(2) Suppose g is integrable on all sets of finite measure and $\sup_{\|f\|_p \leq 1, f \text{ simple}} |\int fg| = M < \infty$. Then $g \in L^q$, $\|g\|_{L^q} = M$. (*)

proof: (1) By Hölder's inequality, $\|g\|_q \geq \sup_{\|f\|_p \leq 1} |\int fg|$

Conversely, set $g^*(x) = \operatorname{sgn}(g(x)) \frac{|g(x)|^{q-1}}{\|g\|_q^{q-1}}$, $\|g^*\|_p = 1$

$|\int g g^*| = \|g\|_q$, so $\|g\|_q \leq \sup_{\|f\|_p \leq 1} |\int fg|$

(2) By previous th'm, $\exists\ g_n$ finite supported simple function that $|g_n(x)| \leq |g(x)|$ & $g_n \to g$ a.e.

Then set $f_n(x) = \operatorname{sgn} g_n(x) \frac{|g_n(x)|^{q-1}}{\|g_n\|_q^{q-1}}$, then $\|f_n\|_{L^p} = 1$

So $|\int f_n g_n| = \|g_n\|_q \leq M$

$\int |g|^q \leq \liminf_{n\to\infty} \int |g_n|^q \leq M^q$, therefore $g \in L^q$ & $\|g\|_{L^q} \leq M$

By Hölder's inequality, $\|g\|_q \geq \int |fg|$ for $\|f\|_p = 1$

Take supremum, then $\|g\|_q \geq M$, hence $\|g\|_q = M$

Lemma. For $1 \le p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\forall\ \phi \in (L^p([a,b]))^*$ with $\|\phi\|_* < \infty$, $\exists!\ g \in L^q([a,b])$ s.t. $\phi(f) = \int_a^b fg$ for $\forall f \in L^p$

proof: Set $\varphi(x) = \phi(\chi_{[a,x]})$

Claim. φ is absolutely continuous

$\sum_{k=1}^n |\varphi(c_k) - \varphi(d_k)| = \sum_{k=1}^n |\phi(c_k) - \phi(d_k)| = \sum_{k=1}^n sgn(\phi(\chi_{[c_k,d_k]}))\phi(\chi_{[c_k,d_k]})$

$= \phi(\sum_{k=1}^n sgn(\phi(\chi_{[c_k,d_k]}))\chi_{[c_k,d_k]}) \le \|\phi\|_* \|\chi_{\bigcup_{k=1}^n [c_k,d_k]}\|_p = \|\phi\|_* (\sum_{k=1}^n (d_k - c_k))^{\frac{1}{p}}$

Take $\delta = (\frac{\varepsilon}{\|\phi\|_*})^p$, then $\sum_{k=1}^n |\varphi(c_k) - \varphi(d_k)| < \varepsilon$ when $\sum_{k=1}^n (d_k - c_k) < \delta$

Hence $\varphi(x) = \int_a^x \varphi'(t)dt$

Therefore $\phi(\chi) = \int_a^b g\chi$, where $g = \varphi'$, χ is any characteristic function

So $\phi(f) = \int_a^b gf$ for any simple function f

Take $\|f\|_p \le 1$, then $|\int_a^b gf| \le \|\phi\|_*$, previous lemma implies that $g \in L^q([a,b])$

Simple function approximation & Fatou's Lemma implies that $\phi(f) = \int_a^b gf$ for any $f \in L^p([a,b])$

Claim. g is unique

If $\int_a^b gf = \int_a^b hf$, $\int_a^b f(g-h) = 0$, set $f(x) = sgn(g(x)-h(x))$, then $g \equiv h$

This lemma can be extended to $\mathbb{R}$

proof: On $L^p[-n,n]$, define $\phi_n(f) := \phi(\hat{f})$, $\hat{f}(x) := \begin{cases} f(x), & x \in [-n,n] \\ 0, & \text{otherwise} \end{cases}$

$|\phi_n(f)| = |\phi(\hat{f})| \le \|\phi\|_* \|\hat{f}\|_{L^p(\mathbb{R})} = \|\phi\|_* \|\hat{f}\|_{L^p([-n,n])}$

Clearly, for each n, $\exists g_n \in L^p([-n,n])$

By the uniqueness, $g_n = g_{n+1}$ on $[-n,n]$

Set $g(x) = \lim_{n\to\infty} g_n(x)$, monotone convergence th'm implies $\int_{\mathbb{R}} g_n \to \int_{\mathbb{R}} g$

For $f \in L^p(E)$, just set $f \equiv 0$ on $\mathbb{R} \setminus E$, done

Thm. $(L^p(E))^* \cong L^q(E)$ for $\forall E \subseteq \mathbb{R}$

proof: $\forall \phi_g \in (L^p(E))$, $\exists! g \in L^q(E)$ s.t. $\phi_g(f) = \int_E fg$ for $f \in L^p(E)$

Set $T: L^q(E) \to (L^p(E))^*$ by $T(g) := \phi_g$

The map is well defined, $|\int_E fg| \le \|f\|_p \|g\|_q < \infty$

Hence we immediately have T bijective

(This theorem holds for any measure spaces, the proof relies on Radon-Nikodym Theorem, which will be covered in the abstract measure space)

Def. A sequence $\{u_n\}$ in the normed linear space X is said to converge weakly in X to $u \in X$ provided that
$\lim_{n\to\infty} \Phi(u_n) = \Phi(u)$ for $\forall\ \Phi \in X^*$

A sequence $\{u_n\}$ is said to converge strongly if
$|\Phi(u_n) - \Phi(u)| \le \|\Phi\|_* \|u - u_n\|$ for all $\Phi \in X^*$

Prop. (1) Strong convergence implies weak convergence
(2) In $L^p(E)$, $f_n \to f$ weakly $\Leftrightarrow \int_E f_n g \to \int_E fg$, $\forall g \in L^q$
(3) A sequence in L^p can converge weakly to at most one function in $L^p(E)$. $\forall\ 1 \le p < \infty$
(4) Let $\mathcal{F} \subseteq L^q(E)$ s.t. span($\mathcal{F}$) is dense in L^q. Then a bounded sequence $\{f_n\} \subseteq L^p$ converges weakly to $f \in L^p \Leftrightarrow \int_E f_n g \to \int_E fg$, $\forall g \in \mathcal{F}$
proof: (1) trivial
(2) Riesz Representation Th'm. still trivial
(3) Suppose $f_n \to f$ & $f_n \to \hat{f}$
$(f-\hat{f})^*(x) = \mathrm{sgn}(f(x) - \hat{f}(x)) \frac{|f(x) - \hat{f}(x)|^{p-1}}{\|f - \hat{f}\|_p^{p-1}} \in L^q$, then
$\int_E f_n (f-\hat{f})^* \to \int_E f(f-\hat{f})^*$ & $\int_E \hat{f}(f-\hat{f})^*$, then $\int_E (f-\hat{f})(f-\hat{f})^* = \|f-\hat{f}\|_p^p = 0$
Therefore $f \equiv \hat{f}$
(4) ($\Leftarrow$) Claim. $\int_E (f_n - f) g \to 0$, $\forall g \in L^q$
$\forall g \in L^q$, $\exists$ a sequence $\{g_k\} \subseteq \mathrm{span}\,\mathcal{F}$ s.t. $g_k \to g$ in L^q
$\int (f_n - f) g = \int (f_n - f) g_k + \int (f_n - f)(g_k - g)$
$\int_E (f_n - f)(g_k - g) \le \|f_n - f\|_p \|g_k - g\|_q \le 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}$
Then $f_n \to f$ weakly
($\Rightarrow$) trivial

Recall.
(1) Simple functions are dense in $L^p(E)$ for $1 \le p \le \infty$
(2) Simple functions with finite support are dense in $L^p(E)$, $1 \le p < \infty$
(3) Step functions are dense in $L^p[a,b]$, $1 \le p < \infty$

Notation:
$\to$: Convergence
$\rightharpoonup$: Weak convergence

<u>Thm.</u> (1) Suppose $\{f_n\}$ is bounded in $L^1(E)$, $f \in L^1(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^1(E)$ $\Leftrightarrow$ $\int_A f_n \to \int_A f$ for every measurable $A \subseteq E$

(2) $1<p<\infty$: Suppose $\{f_n\}$ is bounded in $L^p(E)$, $f \in L^p(E)$. Then $\{f_n\} \rightharpoonup f$ in $L^p(E)$ $\Leftrightarrow$ $\int_A f_n \to \int_A f$ for every measurable $A \subseteq E$ with $m(A) < \infty$

(3) $1<p<\infty$: Suppose $\{f_n\}$ is bounded in $L^p[a,b]$, $f \in L^p[a,b]$. Then $f_n \rightharpoonup f$ in $L^p[a,b]$ $\Leftrightarrow$ $\int_a^x f_n(t)dt \to \int_a^x f(t)dt$

proof: (2) & (3) is directly the 4^{th} property in the last page.

<u>Thm.</u> $1<p<\infty$. Suppose $\{f_n\}$ is bounded in $L^p(E)$ & $f_n \to f$ pointwisely a.e. Then $f \in L^p(E)$ & $f_n \rightharpoonup f$ in $L^p(E)$

proof: $|f_n|^p \to |f|$, so $\int_E |f|^p \le \liminf_{n\to\infty} \int_E |f_n|^p < \infty$

Simple function approximation implies that $\int_E f_n g \to \int_E fg$ for $\forall g \in L^q(E)$

<u>Helly's Thm</u> $(X, \|\cdot\|)$ is a seperable normed linear space, $\{T_n\} \subseteq X^*$ is a bounded sequence. Then $\exists \{T_{n_k}\} \subset \{T_n\}$, $T \in X^*$, $T_{n_k} \rightharpoonup T$

proof: Need to show that $T_{n_k}(f) \to T_n(f)$, $\forall f \in X$

Let $\{f_k\} \subset X$ be a dense sequence.

$|T_n(f_1)| \le \|T_n\|_* \|f_1\| \le M\|f_1\|$, so $T_n(f_1)$ has a convergent subsequence

$T_{11}(f_1)$, $T_{12}(f_1)$, $\cdots \to a_1$, $\{T_{1n}(f_2)\}$ bdd., so $\exists$ conv. subseq.

$T_{21}(f_2)$, $T_{22}(f_2)$, $\cdots \to a_2$, $\{T_{2n}(f_3)\}$ bdd., so $\exists$ conv. subseq.

$\vdots \qquad \vdots$

$T_{k1}(f_k)$, $T_{k2}(f_k)$, $\cdots \to a_k$

$\vdots$

Consider $\{T_{jj}(f_l)\}_{j=1}^{\infty}$, l fixed, $T_{jj}(f_l) \to a_l$ as $j \to \infty$

<u>Claim.</u> $T_{jj}(f)$ converges for $\forall f \in X$

Then we say that $T = \lim_{j\to\infty} T_{jj}$

<u>Thm.</u> $1<p<\infty$. Then every bounded sequence in $L^p(E)$ has a weakly convergent subsequence.

proof: $L^p(E) \cong (L^q(E))^*$, then apply Helly's Th'm

<u>Example.</u> $f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$

Abstract Measure Space

Def. A measure space consists of X and two objects:
(1) A σ-algebra $\mathcal{M}$ whose members are subsets of X
(2) A measure $\mu: \mathcal{M} \to [0,\infty)$, if $E_1, E_2, \cdots$ is a countable family of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$

Def. Let X be a set. An exterior measure μ_* on X is a function μ_* from the collection of all subsets of X to $[0,\infty]$ that satisfies the following properties.
(1) $\mu(\emptyset)=0$
(2) if $E_1 \subset E_2$, then $\mu_*(E_1) \leq \mu_*(E_2)$
(3) if $E_1, E_2, \cdots$ is a countable collection, then
$\mu_*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu_*(E_n)$

Def. A set E in X is Carathéodory measurable if
$\mu_*(A) = \mu_*(E \cap A) + \mu_*(E^c \cap A)$ for every $A \subset X$

Thm. Given an exterior measure μ_* on a set X, the collection of all Caratéodory measurable sets $\mathcal{M}$ forms a σ-algebra. Moreover, $\mu_*|_{\mathcal{M}}$ is a measure.
proof: Clearly $\emptyset \in \mathcal{M}$, $X \in \mathcal{M}$, also $(E^c)^c = E$
Take $E_1, E_2 \in \mathcal{M}$. Observe that $E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1^c \cap E_2) \cup (E_1 \cap E_2^c)$
Take $A \subset X$, then since $E_1, E_2 \in \mathcal{M}$, we have
$\mu_*(A) = \mu_*(A \cap E_1^c) + \mu_*(A \cap E_1) = \mu_*(A \cap E_1 \cap E_2) + \mu_*(A \cap E_1^c \cap E_2) + \mu_*(A \cap E_1 \cap E_2^c)$
$+ \mu_*(A \cap E_1^c \cap E_2^c) \geq \mu_*(A \cap (E_1 \cup E_2)^c) + \mu_*(A \cap (E_1 \cup E_2))$
$A = (A \cap (E_1 \cup E_2)^c) \cup (A \cap (E_1 \cup E_2))$, so the converse holds
Therefore $E_1 \cup E_2$ is measurable
Then if $E_1 \cap E_2 = \emptyset$, $\mu_*(E_1 \cup E_2) = \mu_*(E_1 \cap (E_1 \cup E_2)) + \mu_*(E_1^c \cap (E_1 \cup E_2)) = \mu_*(E_1) + \mu_*(E_2)$
Let $E_1, E_2, \cdots$ be a countable collection of disjoint sets in $\mathcal{M}$
We see that $\mu_*|_{\mathcal{M}}$ is a measure