Ordinary Differential Equations

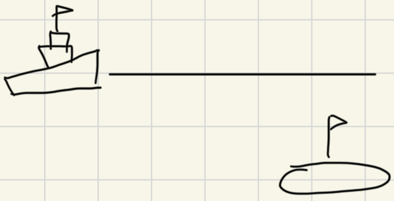
Examples

1. A , B

Newton's Cooling Law

2. Baseball

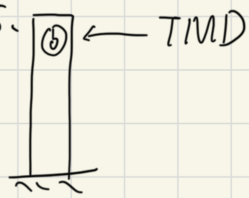
3.



4. 1940. 8. XX

Tacoma Narrows Bridge

5.



First Order Equations

$t: y$, $y' = \frac{dy}{dt}$

$F(t, y, y') = 0$

$\hookrightarrow y' = f(t, y)$ at some neighbourhood of (t_0, y_0)

(This is from the implicit function theorem)

We say the equation is linear if f is linear for y

$$\boxed{y' + a(t)y = b(t)}$$

(i) $b = 0$, then $\frac{y'}{y} = a(t)$, $\ln|y| = -\int a(t)dt + \hat{C}$, $|y| = Ce^{-\int a(t)dt}$, $C \geq 0$

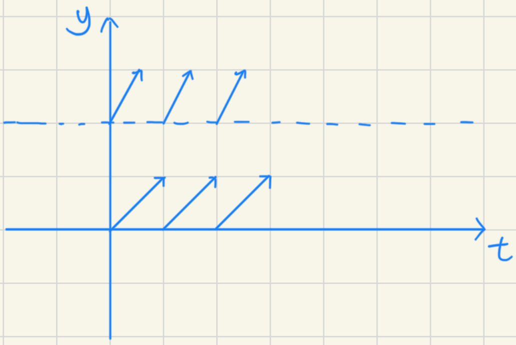
(ii) $b \neq 0$, then consider the variation of parameters

Let $y = c(t)e^{-\int a(t)dt}$

$y' + a(t)y = c'(t)e^{-\int a(t)dt} - a(t)c(t)e^{-\int a(t)dt} + a(t)c(t)e^{-\int a(t)dt} = b(t)$

so $c'(t)e^{-\int a(t)dt} = b(t)$, $y = e^{-\int a(t)dt} \cdot \int b(t)e^{\int a(t)dt} dt$

Example. $y' = y^2+1$.



$\frac{dy}{dt} = y^2+1$, $\frac{dy}{y^2+1} = dt$, $\arctan y = t+C$

$y = \tan(t+C)$

if $y(0)=0$, then $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

malthus.

$P(t)$: population of a species at time t

$\frac{dp}{dt} = r \cdot p$ $\quad$ r: birth rate - death rate

Venhulst (1837)

Logistic eqation.

$\frac{dp}{dt} = rp\left(1-\frac{p}{k}\right)$

k: carry capacity

$\frac{dp}{dt} = p(a-bp)$, $a, b>0$

$\frac{dp}{p(a-bp)} = dt$

$\frac{b}{a}\int \frac{1}{bp-a} + \frac{1}{bp} dp = t$

$t = \frac{1}{a}\ln\frac{p}{|a-bp|} + \hat{c}$, $P(0) = P_0$

$\frac{p}{|a-bp|} = C \cdot e^{at}$, so $C = \frac{P_0}{|a-bP_0|}$

if $a-bP_0 = 0$, then $P(t) \equiv \frac{a}{b}$

if $a-bP_0>0$, $\frac{p}{a-bp} = C \cdot e^{at}$

$P = \frac{aCe^{at}}{1+bCe^{at}} = \frac{a}{b+\frac{1}{C}e^{at}} = \frac{aP_0}{bP_0+(a-bP_0)e^{-at}}$

Example. A commercial fisherman is estimated to have carrying capacity 10000kg of a certain kind of fish. Suppose the annual growth of the total fish p is gorerned by the equation $\frac{dp}{dt} = p\left(1-\frac{p}{10000}\right)$. and initially there are 2000kg of fish.

(a) What is the fish population after one year

Suppose after waiting for a certain period of time, the owner of the fishery decides to harvest 2400kg of fish annually at a constant rate. Then

(b) What is the optimal waiting period

a) $P(1)=\dfrac{C}{\frac{C}{10000}+e^{-a}}$

$$C=\frac{P_0}{|a-bP_0|}=\frac{2000}{1-\frac{2000}{10000}}=2500$$

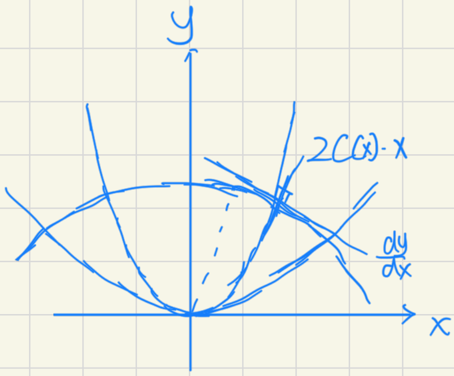
$$P(1)=\frac{2500}{\frac{1}{4}+\frac{1}{e}}\approx 4046 \text{ kg}$$

(2)

$$\frac{dp}{dt}=P\left(1-\frac{P}{10000}\right)-2400$$

$$\frac{dp}{dt}=-\frac{1}{10000}(P-6000)(P-4000)$$

Here we only require to harvest 2400kg of fish within a year, rather than harvest them in a single moment.



Example: Given a family of curves $y=Cx^2$, $C\in\mathbb{R}$

Find the orthogonal trajectories

$$\frac{dy}{dx}\cdot 2C(x)\,x=-1\ ,\quad y=C(x)\cdot x^2$$

$$2\frac{dy}{dx}\frac{y}{x^2}\cdot x=-1\ ,\quad 2y\,dy=-x\,dx\ ,\quad y^2=-\frac{1}{2}x^2+C\ ,\quad C>0$$

$$\Leftrightarrow\ \frac{x^2}{2}+y^2=C\ ,\ \text{WLOG } C=1\ ,\quad \frac{x^2}{2}+y^2=1$$

we see that it's an ellipse

Newton's Cooling Law

$T(t)$: the temperature of a body immersed in a medium.
A: the temperature of the medium
The temperature changing rate is propotional to $A-T$
$\frac{dT}{dt}=k(A-T)$, $k>0$

then $\frac{dT}{T-A}=-kdt$, $\ln|T-A|=-kt+\tilde{C}$ $\quad |T-A|=Ce^{-kt}$, $C>0$

if $T_0>A$, then $T=(T_0-A)e^{-kt}+A$
if $T_0<A$, then $T=A-(A-T_0)^{-kt}=(T_0-A)e^{-kt}+A$
if $T_0=A$, then $T\equiv A$

Need rigorous proof

You can actually find many examples in PHY1001

Thm. (Uniqueness) Let $\begin{cases} y'=f(y,t) \\ y(t_0)=y_0 \end{cases}$ be an initial value problem,
Suppose f & $\frac{\partial f}{\partial y}$ are continuous on a compact neighborhood of (t_0, y_0)
Then the (IVP) (*) has at most one solution

pf:) Suppose we have two solution y_1, y_2
i.e. $\frac{dy_1}{dt}=f(y_1,t)$, $y_1(t_0)=y_0$
$y_1(t)=y_0+\int_{t_0}^{t}f(y_1(s),s)ds \quad y_2(t)=y_0+\int_{t_0}^{t}f(y_2(s),s)ds$
$|y_1(t)-y_2(t)|=\left|\int_{t_0}^{t}f(y_1(s),s)-f(y_2(s),s)ds\right| \le \int_{t_0}^{t}|f(y_1(s),s)-f(y_2(s),s)|ds$
$=\int_{t_0}^{t}\left|\frac{\partial f}{\partial y}(s,\tilde{y}(s))\right||y_1(s)-y_2(s)|ds \le M\int_{t_0}^{t}|y_1(s)-y_2(s)|ds$

let $w(t)=|y_1(t)-y_2(t)| \Rightarrow w(t)\le M\underbrace{\int_{t_0}^{t}w(s)ds}_{z(t)}$

so $z'(t)\le Mz(t)$ $\quad \frac{d}{dt}(z(t)e^{-Mt})=e^{-Mt}(z'(t)-Mz(t))\le 0$
$z(t_0)=0$, $z(t)e^{-M(t)}\geq 0$, so $z(t)\equiv 0$
therefore we have $y_1(t)\equiv y_2(t)$

(IVP) $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

On $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$, $M = \max_{(t,y)\in R} |f(t,y)|$

$L = \max_R |\frac{\partial}{\partial y} f(t,y)|$. (H) $f, \frac{\partial f}{\partial y}$ are continuous on R

<u>Thm.</u> (IVP) has a solution provided that (H) holds.
(i.e. $\exists \alpha > 0$, s.t. (IVP) has a solution for $(t_0 - \alpha, t_0 + \alpha)$)

proof: $\boxed{y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds}$ (*) suffice to solve (*)

$y_0(t) \equiv y_0$, $y_1(t) \equiv y_0 + \int_{t_0}^{t} f(s, y_0(s))\, ds$

Set $y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))\, ds$

...

$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\, ds$

...

(Picard iteration) <u>Claim.</u> $\{y_k(t)\}_{k=0}^{\infty}$ converges on $(t_0 - \alpha, t_0 + \alpha)$ for some $\alpha > 0$

$y_{k+1}(t) = y_0(t) + (y_1(t) - y_0(t)) + \cdots + (y_k(t) - y_{k-1}(t)) + (y_{k+1}(t) - y_k(t))$ (Telescope series)

<u>WTS.</u> $\sum_{n=1}^{\infty} (y_n(t) - y_{n-1}(t))$ converges uniformly on $(t_0 - \alpha, t_0 + \alpha)$

$|y_{n+1}(t) - y_n(t)| = |\int_{t_0}^{t} f(s, y_n(s)) - f(s, y_{n-1}(s))\, ds| \leq \int_{t_0}^{t} |f(s, y_n(s)) - f(s, y_{n-1}(s))|\, ds$
$= \int_{t_0}^{t} |\frac{\partial f}{\partial y}(s, \tilde{y})| \cdot |y_n(s) - y_{n-1}(s)|\, ds$

$|y_1(t) - y_0| \leq \int_{t_0}^{t} |f(s, y_0)| ds \leq M|t - t_0| \leq M\alpha \leq b$
Here we take $\alpha = \min\{a, \frac{b}{M}\}$, so in this case $(t, y_1(t)) \in R$
Suppose $y_n(t)$ is defined on $B_\alpha(t_0)$ for $n \leq k-1$, then when $n = k$
$|y_k(t) - y_0| \leq \int_{t_0}^{t} |f(s, y_{k-1}(s))| ds \leq M|t - t_0| \leq b$. so $\forall n$, $y_n(t)$ is defined on $B_\alpha(t_0)$

$|y_2(t) - y_1(t)| \leq \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_0)| ds \leq LM \int_{t_0}^{t} |s - t_0| ds = \frac{LM}{2!} |t - t_0|^2$

so $|y_{n+1}(t) - y_n(t)| \leq \frac{M}{L} \cdot \frac{|t - t_0|^n \cdot L^n}{n!}$

<u>Claim.</u> $\{y_n(t)\}$ is Cauchy. w.l.o.g. suppose $n > m$
$|f_n(t) - f_m(t)| \leq |f_n(t) - f_{n-1}(t)| + \cdots + |f_{m+1}(t) - f_m(t)| \leq \frac{M}{L} \left(\frac{(\alpha L)^n}{n!} + \cdots + \frac{(\alpha L)^{m+1}}{m!}\right)$

$\leq \frac{M}{L} \left(e^{\alpha L} - (1 + \frac{\alpha L}{1!} + \cdots + \frac{(\alpha L)^m}{m!}\right) < \varepsilon$ $\quad B_k$: $\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$

so $\{f_n(t)\}$ converges

n-dimensional case:

$F(t, y, y', \cdots, y^{(n)}) = 0$, $y(x_0)=y_0$, $y'(x_0)=y_0'$, $\cdots$, $y^{(n-1)}(x_0)=y_0^{(n-1)}$

$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})$

Then let $\vec{y} = (y, y', \cdots, y^{(n-1)})^T$ $\quad \vec{F}(t, y) = (y', \cdots, y^{(n-1)}, f(t, y, \cdots, y^{(n-1)}))^T$

Hence we have $\vec{y}\,' = \vec{F}(t, \vec{y})$

So, if $\vec{F}$ satisfies the Lipschitz condition, i.e. $\|\vec{F}(x, \vec{y}_1) - \vec{F}(x, \vec{y}_2)\| \leq L\|\vec{y}_1 - \vec{y}_2\|$

we can repeat the Picard iteration.

Hint: The norm in $\mathbb{R}^n$ is given by $(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$

Here we have $R = [t - a_0, t + a_0] \times \cdots \times [y_0^{(n-1)} - a_{n-1}, y_0^{(n-1)} + a_{n-1}]$, $M = \max\limits_{(t, \vec{y}) \in R} \|\vec{F}(t, y)\|$

$\vec{y}_0 = \vec{y}(t_0)$

$\vec{y}_1 = \vec{y}_0 + \int_{t_0}^{t} \vec{F}(s, \vec{y}_0(s))\,ds$

$\cdots$

$\vec{y}_n = \vec{y}_0 + \int_{t_0}^{t} \vec{F}(s, \vec{y}_{n-1}(s))\,ds$

$\vec{y}_n - \vec{y}_{n-1} = \int_{t_0}^{t} \vec{F}(s, \vec{y}_n(s)) - F(s, \vec{y}_{n-1}(s))\,ds$

$$\|\vec{y}_n - \vec{y}_{n-1}\| \leq \int_{t_0}^{t} \|F(s, y_{n-1}) - F(s, y_{n-2})\|\,ds \leq L \int_{t_0}^{t} \|y_{n-1} - y_{n-2}\|\,ds$$

$$\leq \cdots \leq L^{n-1} \int_{[t_0, t]}^{n-1} \|\vec{y}_1 - \vec{y}_0\|\,ds^{n-1} \leq M L^{n-1} \int_{[t_0, t]}^{n-1} |s - t_0|\,ds^{n-1} = M L^{n-1} \cdot \frac{|t - t_0|^n}{n!}$$

Choose $\alpha = \min\{a_0, \frac{M}{a_1}, \cdots, \frac{M}{a_{n-1}}\}$, so $\|\vec{y}_n - \vec{y}_{n-1}\| \leq \frac{M}{L} \frac{(\alpha L)^n}{n!}$

It is eazy to check that $\{\vec{y}_n\}$ is Cauchy.

Exact equation.

intuition: Consider function $F(t, y(t)) = C$

then $\frac{d}{dt}F = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} = 0$

let $\frac{\partial F}{\partial t} = M(t,y)$, $\frac{\partial F}{\partial y} = N(t,y)$, i.e. $M(t,y) + N(t,y)\frac{dy}{dt} = 0$

<u>Thm.</u> Given $M(t,y)$ & $N(t,y)$. Then

$\exists \phi(t,y)$, s.t. $\begin{cases} \frac{\partial \phi}{\partial t} = M \\ \frac{\partial \phi}{\partial y} = N \end{cases}$ $\Longleftrightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$

proof: $\Rightarrow$ ✓

$\Leftarrow$ Set $\phi(t,y) = \int M(t,y)dt + \int [N(t,y) - \int \frac{\partial}{\partial y} M(t,y)dt]\,dy$

<u>Example</u> $\overbrace{\frac{1}{2}y^2 + 2ye^t}^{M} + \overbrace{(y+e^t)}^{N}\frac{dy}{dt} = 0$

$M_y = y + 2e^t$, $N_t = e^t$

Suppose $\exists\, \mu = \mu(t)$, s.t. $(\mu M)_y = (\mu N)_t$

$\mu M_y = \mu_t N + \mu N_t$

$\mu_t = \frac{1}{N}\mu(M_y - N_t)$

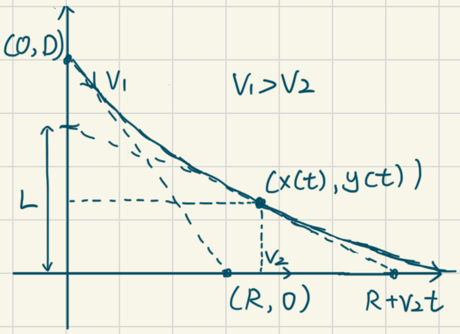
$\frac{1}{\mu}\frac{d\mu}{dt} = \frac{M_y - N_t}{N} = \frac{y+e^t}{y+e^t} = 1$, $\frac{d\mu}{\mu} = dt$

Therefore we take $\mu = e^t$

$F(t,y) = \frac{1}{2}y^2e^t + ye^{2t}$, then $\frac{dF}{dt} = \frac{1}{2}y^2e^t + 2ye^{2t} + (ye^t + e^{2t})\frac{dy}{dt} = 0$

so $\frac{1}{2}y^2e^t + ye^{2t} = C$ for some constant C

Pursuit problem.



$$\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2=V_1^2$$

$$y(t)\cdot\frac{dy}{dx}=R+V_2t$$

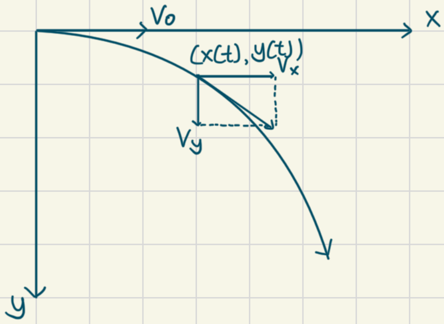
$$\frac{dy}{dx}=\frac{R+V_2t}{y}$$

$$\frac{dy}{dt}=\frac{R+V_2t}{y}\cdot\frac{dx}{dt}=\frac{R+V_2t}{y}\cdot\sqrt{V_1^2-\left(\frac{dy}{dt}\right)^2}$$

$$\left(\frac{dy}{dt}\right)^2=\left(\frac{R+V_2t}{y}\right)^2\cdot\left(V_1^2-\left(\frac{dy}{dt}\right)^2\right)$$

$$\left(1+\left(\frac{R+V_2t}{y}\right)^2\right)\left(\frac{dy}{dt}\right)^2=\frac{R+V_2t}{y}$$

Projectile Motion (with air friction)



Lec. 6

2^{nd} order Linear Equations

$$\begin{cases} y'' = f(t, y, y') \\ y(0) = y_0 \\ y'(0) = y_1 \end{cases}$$

$$\underbrace{y'' + p(t)y' + q(t)y}_{L[y] \equiv} = r(t)$$

$L[y] = y'' + p(t)y' + q(t)y = 0$

$$\text{(IVP)} \begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = \alpha \\ y'(t_0) = \beta \end{cases}$$

Thm. (IVP) has a unique solution if p, q are continuous.

V = the set of all solutions to $L[y] = 0$

Prop. V is a vector space of dimension 2

pf:) ① $0_V \in V$

② If y_1, y_2 are solutions for $L[y] = 0$, then so is $\alpha y_1 + \beta y_2$

We say that $V = Ker(L)$ ↳ V is a vector space

$$\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0 \\ y_1(t_0) = 1 \\ y_1'(t_0) = 0 \end{cases} \qquad \begin{cases} y_2'' + p(t)y_2' + q(t)y_2 = 0 \\ y_2(t_0) = 0 \\ y_2'(t_0) = 1 \end{cases}$$

Claim. $\{y_1, y_2\}$ is a basis for V

i.e. $y \in V$, let $\alpha = y(t_0)$ $\beta = y'(t_0)$

Since $(\alpha y_1 + \beta y_2)'(t_0) = \alpha$, $(\alpha y_1 + \beta y_2)''(t_0) = \beta$, then $y = \alpha y_1 + \beta t_2$

Check. $\{y_1, y_2\}$ is linearly independent

$a_1 y_1 + a_2 y_2 \equiv 0$, so $a_1 y_1(t_0) = 0$, so $a_1 = 0$

$a_1 y_1' + a_2 y_2' \equiv 0$, so $a_2 y_2'(t_0) = 0$, so $a_2 = 0$

we see that $span\{y_1, y_2\} = V$ and $\{y_1, y_2\}$ is linearly independent

so it is a basis for V, $\dim(V) = 2$

Wronskian $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$. a function of t

Prop. If y_1, y_2 are 2 solutions to $y'' + p(t)y' + q(t) = 0$, then $W' + p(t)W = 0$, where $W = W[y_1, y_2]$, i.e.
$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}$

See that $W' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' = y_1y_2'' - y_1''y_2$

$W' + p(t)W = y_1y_2'' - y_1''y_2 + p(t)(y_1y_2' - y_1'y_2) = y_1(y_2'' - p(t)y_2') - y_2(y_1'' - p(t)y_1')$

$= -q(t)y_1y_2 + q(t)y_1y_2 = 0$

Prop(2) $W \equiv 0 \Leftrightarrow \{y_1, y_2\}$ is linearly dependent.

$(\Leftarrow)$: $y_1 = -\frac{c_2}{c_1}y_2$, so $W[y_1, y_2] \equiv 0$

$(\Rightarrow)$: Suppose $y_2(\bar{t}) \neq 0$ as some $\bar{t}$, so set $y_3 = \frac{y_1(\bar{t})}{y_2(\bar{t})}y_2(t)$

we see that $y_3(\bar{t}) = y_1(\bar{t})$ & $y_3'(\bar{t}) = y_1'(\bar{t})$, so

(3) $W[y_1, y_2] \equiv 0 \Leftrightarrow W[y_1, y_2](t_0) = 0$ (y_1, y_2 are 2 solutions)

$(\Rightarrow)$: ✓

$(\Leftarrow)$: $(y_1y_2' - y_1'y_2)(t_0) = 0$

$$\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0 \\ y_2'' + p(t)y_2' + q(t)y_2 = 0 \end{cases} \Rightarrow \begin{cases} y_1''(t_0) + p(t_0)y_1'(t_0) + q(t_0)y_1(t_0) = 0 \\ y_2''(t_0) + p(t_0)y_2'(t_0) + q(t_0)y_2(t_0) = 0 \end{cases}$$

Since $W[y_1, y_2] = 0$, then $\begin{cases} y_2''(t_0) = cy_1''(t_0) \\ y_2'(t_0) = cy_1'(t_0) \\ y_2(t_0) = cy_1(t_0) \end{cases}$

We notice that $y_2 = cy_1$ is a solution, since this is an IVP then $y_2 = cy_1$ is unique, so $\{y_1, y_2\}$ is linearly dependent, i.e. $W[y_1, y_2] \equiv 0$

Another proof

$$\begin{cases} y_1''y_2 + p(t)y_1'y_2 + q(t)y_1y_2 = 0 \\ y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 = 0 \end{cases}$$

so

$(y_1''y_2 - y_1y_2'') + p(t)(y_1'y_2 - y_1y_2') = 0$

i.e. $\frac{dW}{dt} + p(t)W = 0$

so see $W(t_0) = 0$, and $W(t) \equiv 0$ is a solution

by the uniqueness theorem, $W(t) \equiv 0$

Lec 7.

Suppose $ay''+by'+c=0$, let $D=\frac{d}{dt}$

so $(aD^2+bD+c)y=0$

$(D-r_1)(D-r_2)y=0$, $\quad r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

First, suppose $\Delta=b^2-4ac>0$

then $\underset{y_1}{e^{r_1t}}$, $\underset{y_2}{e^{r_2t}}$ are two solutions, and $W[y_1,y_2]=(r_2-r_1)e^{r_1t}e^{r_2t}\neq 0$

so $\{y_1, y_2\}$ forms a basis

if $\Delta<0$, we consider complex roots, so $r_1=\alpha+\beta i$, $r_2=\alpha-\beta i$

we have $y_1=e^{\alpha t}(\cos(\beta t)+i\sin(\beta t))$, $\quad y_2=e^{\alpha t}(\cos(\beta t)-i\sin(\beta t))$

Since we are more used to real solutions, we change the basis into $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$

if $\Delta=0$, $(D-r)^2y=0$

Consider the Jordan Conanical

if $(D-r)y=0$, then we find a solution $y_1=e^{rt}$

if $(D-r)y=e^{rt}$, then we find another solution $y_2=te^{rt}$

$W[y_1,y_2]=\begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & rte^{rt}+e^{rt} \end{vmatrix}=e^{2rt}\neq 0$

<u>Check.</u> $(D-r_1)(D-r_2)y=(D-r_2)(D-r_1)y$

$(D-r_1)(\frac{dy}{dt}-r_2y)=\frac{d^2y}{dt}-(r_1+r_2)\frac{dy}{dt}+r_1r_2y$

$(D-r_2)(\frac{dy}{dt}-r_1y)=\frac{d^2y}{dt}-(r_1+r_2)\frac{dy}{dt}+r_1r_2y$

<u>Def.</u> D is said to be a differential operator on a field K if for any $a, b\in K$

$D: K\to K$
- $D(a+b)=D(a)+D(b)$
- $D(ab)=aD(b)+bD(a)$

Then for the n'th order case

$L_n[y]:=a_0y+a_1y^{(1)}+\cdots+a_ny^{(n)}=0$

$\mathbb{C}$ is an algebraic closed field, any thus we can write the equation as

$(D-\lambda_1)^{m_1}(D-\lambda_2)^{m_2}\cdots(D-\lambda_k)^{m_k}y=0 \quad (m_1+\cdots+m_k=n)$

<u>Claim.</u> $\{e^{\lambda_1t},\cdots,t^{m_1-1}e^{\lambda_1t}, e^{\lambda_2t},\cdots,t^{m_2-1}e^{\lambda_2t},\cdots,\cdots, e^{\lambda_kt},\cdots,t^{m_k-1}e^{\lambda_kt}\}$ forms a basis of solutions to $L_n[y]=0$

<u>Check.</u> $\{e^{\lambda_1t},\cdots,t^{m_1-1}e^{\lambda_1t}\}$ is linearly independent sets of solution

Lec. 8

Variation of parameters

Example. $(1-t^2)y''+2ty'-2y=0 \quad -1<t<1$

Notice that $y=t$ is a solution

Consider $\frac{y}{t}$. $\frac{d}{dt}\left(\frac{y}{t}\right)=\frac{1}{t}y'-\frac{1}{t^2}y \qquad \frac{d^2}{dt^2}\left(\frac{y}{t}\right)=\frac{1}{t}y''-\frac{2}{t^2}y'+\frac{2}{t^3}y=\frac{1}{t^3}(t^2y''-2ty'+2y)=\frac{y''}{t^3}$

$\frac{d}{dt}\left(\frac{y}{t}\right)=\frac{1}{t^2}(ty'-y)=\frac{t^2-1}{2t^2}y''$, so $\frac{d}{dt}\left(\frac{y}{t}\right)=\frac{1}{2}t(t^2-1)\frac{d^2}{dt^2}\left(\frac{y}{t}\right)$

$y=t^2+1$ is also a solution

$W[t,t^2+1]=\begin{vmatrix} t & t^2+1 \\ 1 & 2t \end{vmatrix}=-t^2-1<0$ ✓

Suppose we have 2 linearly independent solutions to $y''+P(t)y'+q(t)y=0$, denoted $y_1(t), y_2(t)$

Then we consider the equation $y''+P(t)y'+q(t)y=r(t)$

$y(t)=c_1(t)y_1(t)+c_2(t)y_2(t)$

$y'=(c_1'y_1+c_2'y_2)+(c_1y_1'+c_2y_2')$

To simplify the equation, let $c_1'y_1+c_2'y_2=0$

so $y'=c_1y_1'+c_2y_2'$, $y''=c_1y_1''+c_1'y_1'+c_2'y_2'+c_2y_2''$

$y''+P(t)y'+q(t)y=c_1(y_1''+P(t)y_1'+q(t)y_1)+c_2(y_2''+P(t)y_2'+q(t)y_2)+c_1'y_1'+c_2'y_2'=r(t)$

$$\begin{cases} c_1'y_1+c_2'y_2=0 \\ c_1'y_1'+c_2'y_2'=r \end{cases}$$

$$c_1'=\frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{W[y_1,y_2]}=\frac{-ry_2}{W[y_1,y_2]} \qquad c_2'=\frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{W[y_1,y_2]}=\frac{ry_1}{W[y_1,y_2]}$$

$$\left(D^n+P_1(t)D^{n-1}+\cdots+P_{n-1}(t)D+P_n\right)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}=0$$

$$W[y_1,\cdots,y_n]=\det\begin{pmatrix} y_1, & \cdots, & y_n \\ y_1^{(1)}, & \cdots, & y_n^{(1)} \\ \vdots & & \vdots \\ y_1^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{pmatrix}\neq 0 \iff \{y_1,\cdots,y_n\} \text{ linearly indep.}$$

Lec 11.

Series solutions

(*) $P(t)y'' + Q(t)y' + R(t)y = 0$

<u>Thm.</u> If $\frac{Q(t)}{P(t)}$ & $\frac{R(t)}{P(t)}$ have Taylor series expansions converges on $(t_0-\rho, t_0+\rho)$, then every solution of (*) must have a Taylor series expansion which converges on $(t_0-\rho, t_0+\rho)$

<u>Euler's Equation.</u> $t^2y'' + \alpha t y' + \beta y = 0$

Bessel's equation of order $\frac{1}{2}$

$t^2y'' + ty + (t^2 - \frac{1}{4})y = 0$

$y = \sum_{n=0}^{\infty} a_n t^{n+r}$. $(a_0 \neq 0)$

$y' = \sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2}$

$t^2y'' + ty' + (t^2 - \frac{1}{4})y = \sum_{n=0}^{\infty} [(n+r)(n+r-1)a_n + (n+r)a_n - \frac{1}{4}a_n] t^{n+r} + \sum_{n=2}^{\infty} a_{n-2} t^{n+r}$

$= t^r [\sum_{n=0}^{\infty} t^n [(n+r)^2 - \frac{1}{4}]a_n + \sum_{n=2}^{\infty} a_{n-2} t^n] = \{a_0(r^2 - \frac{1}{4}) + a_1[(r+1)^2 - \frac{1}{4}]t + \sum_{n=2}^{\infty} t^n\{[(n+r)^2 - \frac{1}{4}]a_n + a_{n-2}]\}\} t^r = 0$

so $r = \pm\frac{1}{2}$

① $r = \frac{1}{2}$, $[(1+\frac{1}{2})^2 - \frac{1}{4}]a_1 = 0$, so $a_1 = 0$

For $n \geq 2$. $[(n+\frac{1}{2})^2 - \frac{1}{4}]a_n + a_{n-2} = 0$. $a_n = \frac{(-1)a_{n-2}}{n(n+1)}$

w.l.o.g. $a_0 = 1$, then $y = \frac{1}{\sqrt{t}}(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) = \frac{\sin t}{\sqrt{t}}$

O

Def. Suppose $f(t,y)$ is continuous on G, if $\forall (t,y_1).(t,y_2) \in G$
. $|f(x,y_1)-f(x,y_2)| \leq F(|y_1-y_2|)$
Where $F(r)>0$ is a continuous function $(r>0)$. with
$\int_0^{\varepsilon} \frac{1}{F(r)} dr = +\infty$ for $\forall \varepsilon > 0$
Then say $f(t,y)$ is of Osgood condition about y

Def. Given $I \subseteq \mathbb{R}$, $|A| = \infty$, the $\{f_\alpha\}_{\alpha\in A}$ is uniformly bounded if $\exists M>0$, s.t. $|f_\alpha(x)| \leq M$ for $\forall \alpha \in A$, $x \in I$

Def. $\{f_\alpha\}_{\alpha\in A}$ is equi-continuous if $\forall \varepsilon>0, \exists \delta>0$, s.t.
$|f_\alpha(x)-f_\alpha(y)|<\varepsilon$ whenever $\delta>0$ for $\forall \alpha \in A$

Sturm Comparison Th'm Let y, z be solutions to
$y''(t)+q(t)y=0$ & $z''(t)+r(t)z=0$, where $q(t)>r(t)>0$
Then y vanishes at least once between any 2 consecutive zeros of z
proof: $W(t)=yz'-y'z$, $W'(t)=yz''-y''z$
Suppose y has no zeros in (α, β)
$W'(t) = zqy - yrz = yz(q-r) \neq 0$, so w.l.o.g. $W'(t)>0$
$W(t) \nearrow$ $W(\alpha)=y(\alpha)z'(\alpha)$. $W(\beta)=y(\beta)z'(\beta)$
Since y has no zeros, then w.l.o.g. $y(t)>0$ on (α,β)
Also $W'(t)>0$, therefore $W(\beta)>0$ strictly, $z'(t)>0$
This is impossible since $z(\alpha)=z(\beta)=0$

Examples

1. $y''+3y'+2y=e^t$

$$\begin{cases} x_1=y \\ x_2=y' \end{cases} \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}=\begin{pmatrix} x_2 \\ -2x_1-3x_2+e^t \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'=\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} 0 \\ e^t \end{pmatrix}, \quad X'=AX+b$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix}=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)$$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} c_1+c_2 \\ -2c_1-2c_2 \end{pmatrix}=0 \qquad \text{. so } A=PDP^{-1}.\ P=\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\begin{pmatrix} 2c_1+c_2 \\ -2c_1-c_2 \end{pmatrix}=0$$

let $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $\begin{pmatrix} z_1' \\ z_2' \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}+\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ e^t \end{pmatrix}$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix}=\begin{pmatrix} -z_1+e^t \\ -2z_2-e^t \end{pmatrix} \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}e^t \\ -\frac{1}{3}e^{2t} \end{pmatrix}+\begin{pmatrix} C_1e^{-t} \\ C_2e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}e^t-\frac{1}{3}e^{2t} \\ -\frac{1}{2}e^t+\frac{2}{3}e^{2t} \end{pmatrix}+\begin{pmatrix} C_1e^{-t}+C_2e^{-2t} \\ -C_1e^{-t}-2C_2e^{-2t} \end{pmatrix}$$

Let $\mathbb{X}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. $A(t)=\begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$ $\quad b(t)=\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$

$$(*)\ \begin{cases} \mathbb{X}'=A\mathbb{X}+b \\ \mathbb{X}(t_0)=\mathbb{X}_0 \end{cases}$$

Thm. $(*)$ has exactly one solution in the entire interval.
(Use Picard iteration directly)

$(\dagger)$ $\mathbb{X}'=A\mathbb{X}$, where $\mathbb{X}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Thm. Let V be the set of all solutions to $(\dagger)$.
Then V is a n-dimensional vector space.

pf:) $\mathbb{X}_1'=A\mathbb{X}_1$, $\mathbb{X}_2'=A\mathbb{X}_2$, then $(\alpha\mathbb{X}_1+\beta\mathbb{X}_2)'=\alpha\mathbb{X}_1'+\beta\mathbb{X}_2'=\alpha A\mathbb{X}_1+\beta A\mathbb{X}_2=A(\alpha\mathbb{X}_1+\beta\mathbb{X}_2)$

Claim. $\{\mathbb{X}_1,\cdots,\mathbb{X}_n\}$ forms a basis for V, $\mathbb{X}_i$ is given by $\mathbb{X}_i(t_0)=e_i$, unique

Check. $V=\operatorname{span}\{\mathbb{X}_1,\cdots,\mathbb{X}_n\}$, $\forall x\in V$, $\mathbb{X}(t_0)=\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}=c_1e_1+\cdots+c_ne_n=c_1\mathbb{X}_1(t_0)+\cdots+c_n\mathbb{X}_n(t_0)$

$\mathbb{X}=c_1\mathbb{X}_1+\cdots+c_n\mathbb{X}_n$ is a unique solution

Check. Linear independence, let $(c_1\mathbb{X}_1+\cdots+c_n\mathbb{X}_n)(t_0)=0$, then $c_1=\cdots=c_n=0$

Def. The wronskian of $\{\mathbb{X}_1, \cdots, \mathbb{X}_n\}$ is given by
$W[\mathbb{X}_1, \cdots, \mathbb{X}_n] = \det(\mathbb{X}_1, \cdots, \mathbb{X}_n)$

Prop. $W(t) = W(0)e^{tr(A)t}$

$$\frac{d}{dt}W(t) = \frac{d}{dt}\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{i1}' & \cdots & x_{in}' \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

proof: $(v_1 \wedge \cdots \wedge v_n) = \det(v_1, \cdots, v_n)(e_1 \wedge \cdots \wedge e_n)$

$\frac{d}{dt}(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^{n}(v_1 \wedge \cdots \wedge \frac{dv_i}{dt} \wedge \cdots \wedge v_n) = \frac{d}{dt}\det(v_1, \cdots, v_n)(e_1 \wedge \cdots \wedge e_n)$

Also, $v_1 \wedge \cdots \wedge \frac{dv_i}{dt} \wedge \cdots \wedge v_n = \det(v_1, \cdots, \frac{dv_i}{dt}, \cdots, v_n)(e_1 \wedge \cdots \wedge e_n)$

so $\frac{d}{dt}\det(v_1, \cdots, v_n)(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{n}\det(v_1, \cdots, \frac{dv_i}{dt}, \cdots, v_n)(e_1 \wedge \cdots \wedge e_n)$

Since $\dim(\Lambda^n V) = 1$, then $\frac{d}{dt}\det(v_1, \cdots, v_n) = \sum_{i=1}^{n}\det(v_1, \cdots, \frac{dv_i}{dt}, \cdots, v_n)$

$\frac{dW}{dt} = \sum_{i=1}^{n}\det(v_1, \cdots, \frac{dv_i}{dt}, \cdots, v_n) = \sum_{i=1}^{n}\det(v_1, \cdots, Av_i, \cdots, v_n) = \sum_{i=1}^{n} a_{ii} W = tr(A) \cdot W$

so $W(t) = W(0)e^{tr(A)t}$

Suppose A has eigenvalue λ with corresponding eigenvector v

i.e. $Av = \lambda v$

$\mathbb{X} = e^{\lambda t}\vec{v} \implies \mathbb{X}' = \lambda e^{\lambda t}\vec{v} = \lambda \mathbb{X}$, which is a solution.

$\{e^{\lambda_1 t}v_1, \cdots, e^{\lambda_n t}v_n\}$

Consider matrix with 1 real eigenvalue and 2 complex conjugate eigenvectors. λ, $\alpha \pm \beta i$

$\dot{x} = Ax$, $\hat{x}_1 = e^{(\alpha+\beta i)t}(w_1 + iw_2) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(w_1 + iw_2)$

$= e^{\alpha t}(\cos\beta t \cdot w_1 - \sin\beta t \cdot w_2 + i(\cos\beta t \cdot w_2 + \sin\beta t \cdot w_1))$

$\hat{x}_2 = e^{\alpha t}(\cos\beta t w_1 - \sin\beta t w_2 - i(\cos\beta t w_2 + \sin\beta t w_1))$

Then $\frac{\hat{x}_1 + \hat{x}_2}{2}$ & $\frac{\hat{x}_1 - \hat{x}_2}{2i}$ are solutions

i.e. $\mathrm{Re}(\hat{x}_1) \to x_1$ & $\mathrm{Im}(\hat{x}_1) \to x_2$

Here we see that $x_3(t) = e^{\lambda t} w_3$

Since $W(t) = W(0)e^{tr(A)t}$, we only need to check that $\{x_1(0), x_2(0), x_3(0)\}$ is linearly independent.

This is clear since $x_1(0) = w_1$, $x_2(0) = w_2$, $x_3(0) = w_3$

Consider Jordan Normal Form

Example.

$$\dot{x} = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} x \qquad \det(A - \lambda I) = -(\lambda - 2)^3$$

$$(A - 2I)(v) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \quad \Rightarrow \quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Fundamental Matrix Solution

For $\dot{X} = AX$, $A \in M_{n\times n}(\mathbb{R})$

$\Phi(t)$ is a FMS if $\Phi(t)$ is a $n\times n$ matrix where each column is a solution while n columns are linearly independent.

Thm. Let $\Phi(t)$ be a FMS to $\dot{X} = AX$. Then $\Phi(t)\Phi(t_0)^{-1} = e^{A(t-t_0)}$

(This can be used to deal with nonhomogeneous linear systems)

pf:) Consider $Y(t) = X(t)X^{-1}(t_0)$, this is also a solution, $Y(t_0) = I$

So $X(t)X^{-1}(t_0) = e^{A(t-t_0)}$

Let $X_1(t), \cdots, X_n(t)$ be n linearly independent solutions to $\dot{X} = AX$

$\Rightarrow c_1X_1(t) + \cdots + c_nX_n(t)$ is a solution to $\dot{X} = AX$

Suppose $X(t) = u_1(t)X_1(t) + \cdots + u_n(t)X_n(t)$ is a solution to $\dot{X} = AX + \vec{f}$

$\Phi(t) = \begin{pmatrix} | & & | \\ x_1(t) & \cdots & x_n(t) \\ | & & | \end{pmatrix}$, $X(t) = \Phi(t)\begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} = \Phi(t)u(t)$

$X(t_0) = \Phi(t_0)u(t_0)$

$X'(t) = \Phi'(t)u(t) + \Phi(t)u'(t) = A\Phi(t)u(t) + \Phi(t)u'(t) = AX + \Phi(t)u'(t) = AX + \vec{f}$

so $\vec{f}(t) = \Phi(t)u'(t)$, $u'(t) = \Phi^{-1}(t)f(t)$

$X(t) = \Phi(t)u(t) = \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)f(s)\,ds + \Phi(t)u(t_0) = \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)f(s)\,ds + \Phi(t)\Phi^{-1}(t_0)X(t_0)$

$= e^{A(t-t_0)}X(t_0) + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)f(s)\,ds = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t}e^{A(t-s)}f(s)\,ds$

$$X(t) = e^{A(t-t_0)}X(t_0) + \int_{t_0}^{t}e^{A(t-s)}f(s)\,ds$$

If $\Phi(t)$ is given & we hope to find A

we see that $e^{At} = \Phi(t)\Phi^{-1}(0)$

$\frac{d}{dt}e^{At}\Big|_{t=0} = \Phi'(0)\Phi^{-1}(0) = A\Phi(0)\Phi^{-1}(0)$

i.e. $A = \Phi'(0)\Phi^{-1}(0)$

(*) $X' = \vec{f}(X)$, $X \in m$

<u>Def.</u> If $\vec{f}(X_0) = 0$, then $X(t) \equiv X_0$ is a solution to (*) & is called an equilibrim (Steady State)

<u>Def.</u> A solution $\vec{X}(t) = \vec{\Phi}(t)$ of (*) is said to be stable if $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t. for any solution $\vec{\Psi}(t)$ of (*) with $\|\psi(0) - \varphi(0)\| < \delta$ has the property that $\|\psi(t) - \varphi(t)\| < \varepsilon$ for $\forall t > 0$

Continuous dependence on initial value
$\forall \varepsilon > 0, T > 0$, $\exists \delta = \delta(\varepsilon, T)$, s.t. $|\tilde{y}_0 - y_0| < \delta \Rightarrow |y_2(t) - y_1(t)| < \varepsilon$ for $\forall 0 < t < T$

Stability for linear systems (+) $\dot{X} = AX$

<u>Prop.</u> (Reduction) All solutions to (+) are stable $\Leftrightarrow$ $X(t) \equiv \vec{0}$ is stable

proof: ($\Rightarrow$) ✓

($\Leftarrow$) Let $\vec{\varphi}(t)$ be a solution of (+)

Since $X(t) \equiv 0$ is stable, $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $\|\psi(0)\| < \delta \Rightarrow \|\psi(t)\| < \varepsilon$, $\forall t > 0$

Then let $\rho(t)$ be an arbitary solution s.t. $\|\vec{\rho}(0) - \vec{\varphi}(0)\| < \delta$

Set $\vec{\psi} = \vec{\rho} - \vec{\varphi}$. Then $\vec{\psi}$ is a solution (By linearity of (+))

Therefore $\|\vec{\rho}(t) - \vec{\varphi}(t)\| < \varepsilon$ for $\forall t > 0$. i.e. $\vec{\varphi}$ is stable

$X = AX$, A: $n \times n$ constant matrix

(1) At least 1 eigenvalue of A has positive real parts

(2) All eigenvalues of A have negative real parts

(3) All eigenvalues of A have non-positive real parts, with some of them being 0 or purely imaginary

<u>Prop.</u> In case (1), zero solution is unstable

pf:) Suppose $\lambda > 0$ is an eigenvalue of A

Let $\vec{v}$ be a corresponding eigenvector of λ, then $\varphi(t) = ce^{\lambda t} \cdot \vec{v}$ is a solution to (+)

$\forall \delta > 0$, pick $c > 0$ s.t. $\|\varphi(0)\| = c\|\vec{v}\| = \frac{1}{2}\delta < \delta$, $\|\varphi(t)\| = \frac{1}{2}\delta e^{\lambda t} \to \infty$

Now consider $\lambda = \alpha + \beta i$, $\alpha > 0$

$\vec{\psi}(t) = ce^{\alpha t}(\cos\beta t\, \vec{v} - \sin\beta t\, \vec{w})$. Take $t_k = \frac{2k\pi}{\beta}$, then $\|\psi(t)\| \to \infty$ as $k \to \infty$

In case (2), all solutions tends to 0 as $t \to \infty$

pf:) Let $\lambda_1, \cdots, \lambda_k$ be negative eigenvalues of A

$\lambda_{k+1}, \cdots, \lambda_n$ be eigenvalues with negative real parts

$\lambda_j = \alpha_j + i\beta_j$ for $j = k+1, \cdots, m$, $\alpha_j < 0$. $V_j = w_j + i u_j$

Then the solution must be a linear combination of the following:

(1) $e^{\lambda_l t}(\vec{v}_l + t\vec{v}_l + \cdots + t^p \vec{v}_l)$. $l = 1, \cdots, k$

(2) $e^{\alpha_j t}(\cos\beta_j t\, \vec{w}_j - \sin\beta_j t\, \vec{u}_j)(1 + t + \cdots + t^q)$, $j = k+1, \cdots, m$

Then all solution tends to 0 as $t \to \infty$

(3) Assume A has

$\lambda_1 = i\beta_1, \cdots, \lambda_k = i\beta_k$, $\beta_j \in \mathbb{R}$,

$\lambda_{k+1}, \cdots, \lambda_m$ having negative real parts

<u>Prop.</u> Suppose λ_j, $j = 1, \cdots, k$ has multiplicity s_j

Then all solution of $\dot{X} = AX$ are stable $\Leftrightarrow$ A has s_j linearly independent eigenvectors for $\lambda_j = i\beta_j$, $j = 1, \cdots, k$

proof: ($\Rightarrow$) Suppose $\lambda_1 = i\beta_1$ doesn't have s_1 linearly independent eigenvectors, then $te^{i\beta_1 t}$ is a solution

This implies that 0 is not stable, ※

($\Leftarrow$) trivial

<u>Example</u> $\dot{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X$

$\det(A - \lambda I) = \lambda^2 + 1 = 0$, $\lambda = \pm i$

$(A - iI)(v) = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\tilde{X}(t) = e^{it} v = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

so $X(t) = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $\|X(0)\| = \sqrt{c_1^2 + c_2^2}$

$\|X(t)\| = \sqrt{c_1^2 + c_2^2} = \|X(0)\|$

Take $\delta = \varepsilon$, then $\forall$ solution that $\|X(0)\| < \delta$, we have $\|X(t)\| < \varepsilon$

<u>Def.</u> A solution $\vec{\Phi}(t)$ is said to be asymptotically stable if $\exists\, \delta > 0$, s.t. as $t \to \infty$, $\|\vec{\Phi}(t) - \vec{\Psi}(t)\| \to 0$, $\forall$ solution $\vec{\Psi}(t)$ s.t. $\|\vec{\Psi}(0) - \vec{\Phi}(0)\| < \delta$

Stability for Nonlinear systems $\dot{X} = AX + \vec{g}(\vec{x})$

where A: $n\times n$ constant matrix, $\vec{g}(\vec{X}) = o(\|\vec{X}\|)$ near $\vec{X}=0$

Thm.

(1) All eigenvalues of A have negative real parts

(2) At least 1 eigenvalue of A has positive real parts

(3) Otherwise

For nonlinear system $\dot{x} = AX + \vec{g}(x)$, where $\vec{g}(x) = o(\|x\|)$ at $x = \vec{0}$

Then consider the stability of $X\equiv 0$, one has

(1) Asympotically stable

(2) unstable

proof: (1) $x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}\vec{g}(x(t))\,dt$

Claim. $\exists\ \alpha>0,\ K>0$, s.t. $\|e^{At}\vec{c}\| \leq Ke^{-\alpha t}\|\vec{c}\|,\ \forall \vec{c},\ t>0$

$e^{-\lambda t}(c_1\vec{v_1} + c_2 t\vec{v_2} + \cdots + c_p t^{p-1}\vec{v_p}) = O(e^{-\lambda t}t^{p-1}) = O(e^{-\lambda t})$

Thus take $\alpha = \max\{\lambda_1, \cdots, \lambda_k\}$

So one has $\|e^{At}\vec{c}\| \leq Ke^{-\alpha t}\|\vec{c}\|$ for some $K>0$

$\forall\ \varepsilon>0,\ \exists\ \delta>0$, s.t. $\|\vec{g}(x)\| \leq \varepsilon\|X\|$ when $\|X\|<\delta$

Goal. $\exists\ \eta>0$, s.t. if $\|x(0)\|<\eta$, then $X(t)\to\vec{0}$ as $t\to\infty$

Stragegy. Pick $\varepsilon>0$, $\delta=\delta(\varepsilon)$, & $\eta>0$. s.t. $\|x(0)\|<\eta \Rightarrow \|x(t)\|<\delta$

$\|x(t)\| \leq \|e^{At}x(0)\| + \int_0^t \|e^{A(t-s)}\vec{g}(x(s))\|ds \leq Ke^{-\alpha t}\|x(0)\| + K\varepsilon\int_0^t e^{-\alpha(t-s)}\|x(s)\|ds$

$e^{\alpha t}\|x(t)\| \leq K\|x(0)\| + K\varepsilon\int_0^t e^{-\alpha s}\|x(s)\|ds \quad (*)$

let $u(t) = \int_0^t e^{-\alpha s}\|x(s)\|\,ds$, then $u'(t) = e^{\alpha t}\|x(t)\| \leq K\|x(0)\| + K\varepsilon u(t)$

$u'(t) - \varepsilon K u(t) \leq K\|x(0)\|$, i.e. $(u(t)e^{-\varepsilon Kt})' \leq K\|x(0)\|e^{-\varepsilon Kt}$

Hence $u(t) \leq \frac{1}{\varepsilon}\|x(0)\|(e^{\varepsilon Kt}-1)$

plug in $(*)$, then $e^{\alpha t}\|x(t)\| \leq K\|x(0)\| + K\|x(0)\|(e^{\varepsilon Kt}-1)$

$\|x(t)\| \leq Ke^{(K\varepsilon-\alpha)t}\|x(0)\|$

Here, set $\varepsilon = \frac{\alpha}{2K}$, then $K\varepsilon - \alpha = -\frac{1}{2}\alpha < 0$

Choose $\eta = \frac{\delta}{K}$, then $\|x(t)\| < \delta e^{-\frac{1}{2}\alpha t}$ whenever $\|x(0)\|<\eta$

Hence $\lim_{t\to\infty}\|x(t)\| = 0$

General Nonlinear System.

(†) $\dot{X} = f(\vec{X})$, $f(X_0) = 0$. i.e. $X(t) \equiv X_0$ is a solution

Set $y = X - X_0$, then $y' = X'$

$\vec{f}(X) = \vec{f}(X_0) + D\vec{f}(X_0)(X - X_0) + o(\|X - X_0\|^2)$

$y' = D\vec{f}(X_0)y + o(\|y\|)$

$$D\vec{f}(X_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \cdots, & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}, & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{X = \vec{X}_0}$$

where $D\vec{f}(X_0)$ is the Jacobean matrix

Phase Plane (2-dimensional) $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

(x,y) : phase plane

$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ => $\frac{dy}{dx} = h(x,y)$

Def. An orbit (trajectory) is the solution curve $\{(x(t), y(t)) \mid t \in \mathbb{R}\}$ on the phase plane.

Example. (1) $\begin{cases} x' = y \\ y' = -x \end{cases}$

so $\frac{dy}{dx} = -\frac{x}{y}$ $\quad y\,dy = -x\,dx$, $x^2 + y^2 = C^2$

$\vec{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$, then $\vec{x}_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ $\quad \vec{x}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

(2) $\begin{cases} x' = y(1 + x^2 + y^2) \\ y' = -x(1 + x^2 + y^2) \end{cases}$ => $x^2 + y^2 = C^2$. $C > 0$

Then $\vec{x}' = \begin{pmatrix} 0 & (1+C^2) \\ -(1+C^2) & 0 \end{pmatrix} \vec{x}$. all solutions has period $\frac{2\pi}{1+C^2}$

Qualitative Properties of Orbits

$$\vec{x}' = f(\vec{x}) = \begin{pmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_n(x_1, \cdots, x_n) \end{pmatrix} \quad (*)$$

Prop. 1. Suppose $f_1, \cdots, f_n$ are continuously differentiable. Then for every $X_0 \in \mathbb{R}^n$. (*) has a unique orbit going through X_0

Cor. No two orbits can intersect with each other

Prop. 2 Let $\vec{\varphi}(t) = \vec{X}(t)$ be a solution of (*). If for some t_0, $\exists\, T > 0$ s.t. $\vec{\varphi}(t_0 + T) = \vec{\varphi}(t_0)$, then $\forall\, t > 0$, $\vec{\varphi}(t) = \vec{\varphi}(t + T)$

Cor. Every closed orbit is periodic

(*) $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

Poincaré - Bendixson Thm.

Suppose Ω is a compact domain which contains no equilibrium point of (*). If (*) has one solution that completely stays in Ω for all time t large (i.e. $\exists t_0>0$, s.t. $(x(t), y(t)) \in \Omega$ for $t \geq t_0$), then either the solution $(x(t), y(t))$ is periodic or it spirals into a periodic solution.

Example. $z'' + (z^2 + 2z'^2 - 1)z' + z = 0$

Set $\begin{cases} x = z \\ y = z' \end{cases}$, then $\begin{cases} x' = y \\ y' = -y(x^2+2y^2-1) - x \end{cases}$

$\frac{1}{2}\frac{d(x^2+y^2)}{dt} = xy - y^2(x^2+2y^2-1) - xy = y^2(1-x^2-2y^2)$

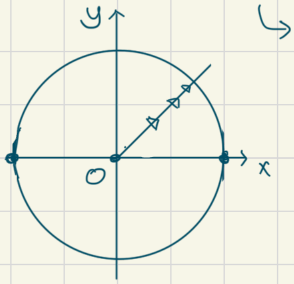
Limit Cycle.

Def. A closed orbit $\mathcal{C}$ is called a limit circle of (*) if orbits of (*) spirals into it or away from it

Example. Find all limit cycles of $\begin{cases} x' = x(1-x^2-y^2) \\ y' = y(1-x^2-y^2) \end{cases}$

$x' = 0 \Rightarrow x = 0$ or $x^2+y^2 = 1$

$\hookrightarrow y(1-y^2) = 0 \Rightarrow y = 0$ or $y = \pm 1$



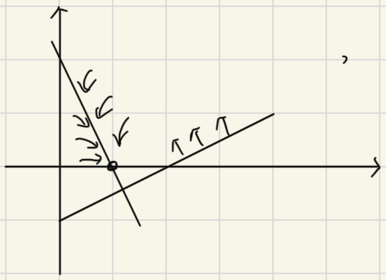
$\begin{cases} x' = xy + x\cos(x^2+y^2) \\ y' = -x^2 y + y\cos(x^2+y^2) \end{cases}$

$\begin{cases} xy + x\cos(x^2+y^2) = 0 \\ x^2 y = y\cos(x^2+y^2) \end{cases}$, so $x = 0$ or $\cos(x^2+y^2) + y = 0$

if $x = 0$, we have $y\cos(y^2) = 0$, so $y = 0$ or $y^2 = (n+\frac{1}{2})\pi$

$$\begin{cases} x' = x(a-by-ex) \\ y' = y(-c+dx-fy) \end{cases}, \quad a,b,c,d,e,f > 0$$

x: prey （猎物） y: predator （捕食者）



, $\frac{a}{e} < \frac{c}{d}$ implies that all solutions with $(x(0)>0, y(0)>0)$ will tend to $(0,$

$$\begin{cases} x' = x(a-by) \\ y' = y(-c+dx) \end{cases} \quad (*)$$

$\frac{dy}{dx} = \frac{y(-c+dx)}{x(a-by)}$. $\int \frac{a-by}{y} dy = \int \frac{-c+dx}{x} dx$

$y^a e^{-by} = x^{-c} e^{dx} \cdot C$

Set $\varphi(x,y) = \frac{y^a x^c}{e^{by} e^{dx}} = f(x)g(y) = k$

Consider the equillibrium points of $(*)$, we have $(0,0)$ & $(\frac{c}{d}, \frac{a}{b})$

$\nabla \varphi = (f'(x)g(y), f(x)g'(y))^T$

If $\|\nabla\varphi\| = 0$, we have $f'(x)=0$ & $g'(y)=0$, and therefore $x = \frac{c}{d}$, $y = \frac{a}{b}$

So for all $0 < k < \varphi(\frac{c}{d}, \frac{a}{b})$. we are getting a closed & bounded region without equillibrium point.

Therefore $(x(t), y(t))$ forms a periodic solution

Let $(x(t), y(t))$ be such periodic solution with period T

Lemma. $\frac{1}{T}\int_0^T x(t)dt = \frac{c}{d}$, $\frac{1}{T}\int_0^T y(t)dt = \frac{a}{b}$

proof: $\frac{x'}{x} = a - by$, $\frac{y'}{y} = -c+dy$

$0 = \ln x \,\big|_0^T = \int_0^T \frac{x'}{x} dt = \int_0^T a - by\, dt = aT - b\int_0^T y(t)dt$

so $\frac{1}{T}\int_0^T y(t)dt = \frac{a}{b}$; similarly $\frac{1}{T}\int_0^T x(t)dt = \frac{c}{d}$

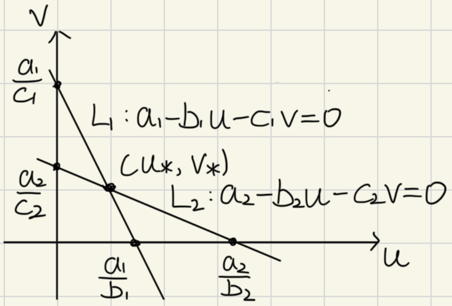
Lotka-Voltena Competition System

$$\begin{cases} u' = u(a_1 - b_1 u - c_1 v) \\ v' = v(a_2 - b_2 u - c_2 v) \end{cases} \quad (+)$$

a_i, b_i, c_i's are positive constants

Weak Competition: $\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$

Strong Competition: $\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$



Here $u_* = \frac{a_1 c_2 - a_2 c_1}{b_1 c_2 - b_2 c_1}$, $v_* = \frac{a_2 b_1 - a_1 b_2}{b_1 c_2 - b_2 c_1}$

Linearize (+) at (u_*, v_*)

$$J = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}_{(u_*, v_*)} = \begin{pmatrix} a_1 - 2b_1 u_* - c_1 v_* & -c_1 u_* \\ -b_2 v_* & a_2 - b_2 u_* - 2c_2 v_* \end{pmatrix} = \begin{pmatrix} -b_1 u_* & -c_1 u_* \\ -b_2 v_* & -c_2 v_* \end{pmatrix}$$

$$0 = |\lambda I - J| = \begin{vmatrix} \lambda + b_1 u_* & c_1 u_* \\ b_2 u_* & \lambda + c_2 v_* \end{vmatrix} = \lambda^2 + (b_1 u_* + c_2 v_*)\lambda + (b_1 c_2 - b_2 c_1) u_* v_*$$

<u>Thm.</u> (u_*, v_*) is globally asympotically stable in the weak competition case with $u(0), v(0) > 0$

proof: Let $E(u(t), v(t)) = b_2 (u(t) - u_* - u_* \ln \frac{u(t)}{u_*}) + c_1 (v(t) - v_* - v_* \ln \frac{v(t)}{v_*})$

At (u_*, v_*), $E(u_*, v_*) = 0$

$\frac{\partial E}{\partial u} = b_2 (1 - \frac{u_*}{u}) = 0 \qquad \frac{\partial E}{\partial v} = c_1 (1 - \frac{v_*}{v}) = 0$. then $(u, v) = (u_*, v_*)$

so (u_*, v_*) is the global minimal

$$\frac{dE}{dt} = b_2 (u' - u_* \frac{u'}{u}) + c_1 (v' - v_* \frac{v'}{v}) = \frac{b_2}{u} u'(u - u_*) + \frac{c_1}{v} v'(v - v_*)$$

$$= b_2 \underbrace{(a_1 - b_1 u - c_1 v)}_{\substack{b_1 u_* + c_1 v_* - b_1 u - c_1 v \\ = b_1 (u_* - u) + c_1 (v_* - v)}} (u - u_*) + c_1 \underbrace{(a_2 - b_2 u - c_2 v)}_{\substack{b_2 u_* + c_2 v_* - b_2 u - c_2 v \\ = b_2 (u_* - u) + c_2 (v_* - v)}} (v - v_*)$$

$$= b_1 b_2 (u_* - u)(u - u_*) + 2 b_2 c_1 (u_* - u)(v - v_*) + c_1 c_2 (v - v_*)(v_* - v)$$

$$= -\left[b_1 b_2 (u - u_*)^2 + 2 b_2 c_1 (u - u_*)(v - v_*) + c_1 c_2 (v - v_*)^2 \right] \leq 0$$

equality holds iff $(u, v) \equiv (u_*, v_*)$

Since $E(t)$ is strictly decreasing, then $\exists\, c \geq 0$ s.t. $\lim_{t \to \infty} E(t) = c$

Hence $\lim_{t \to \infty} E'(t) = 0$, so $\lim_{t \to \infty} (u(t), v(t)) = (u_*, v_*)$