妖怪の山-mysterious mountain

Mathematical Analysis

Lec. 1

Set-theory

Notations.

$\mathbb{N}$: natural numbers $\{1, 2, 3, \cdots\}$

$\mathbb{Z}$: integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$\mathbb{Q}$: rational numbers $\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0, \gcd(p, q) = 1\}$

$\mathbb{R}$: real numbers

$\mathbb{R} \setminus \mathbb{Q}$: irrational numbers

Algebratic numbers : roots of polynomials of integers coefficients.

· Transcedental numbers : $\mathbb{R} / \{\text{Algebratic \#s}\}$

Set. A collection of some object with certain properties

Power set 2^A : collection of all subsets of A

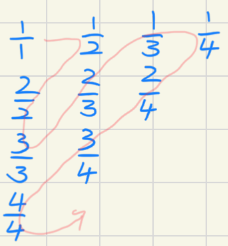
Def. Two sets A, B are said to be equivalent if there exist a bijective map $\varphi : A \to B$. This is denoted as $A \sim B$

In this case, we also say that A & B has the same cardinal number

Def. A set is called countable if either A is finite or $A \sim \mathbb{N}$

Otherwise we say A is uncountable.

Eg. $\mathbb{Q}$ is countable, {Algebratic #s} is countable



$\aleph_0^2 = \aleph_0$

The combination of two countable sets is also countable

This is from the axiom of choice $\{a_1, b_1, a_2, b_2, \cdots\}$

The countable combination of countable sets is countable

$A_1 = \{a_{11}, a_{12}, \cdots\}$

$A_2 = \{a_{21}, a_{22}, \cdots\}$

$A_3 = \{a_{31}, a_{32}, \cdots\}$

$\cdots$

<u>Thm.</u> If $A \sim B$, then $|A| = |B|$

(This is trivial)

<u>Def.</u> The Aleph-zero $\aleph_0 = |\mathbb{N}|$

Aleph-1 $\aleph_1 = |\mathbb{R}|$

<u>Eg.</u> $2^{\mathbb{N}} \sim (0,1)$, since there exist a bijective map between a real number and a binary fraction.

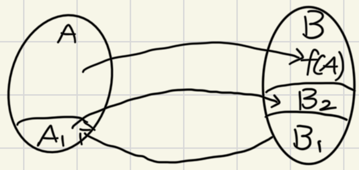
<u>Thm.</u> (Schauder-Bernstein Thm) If $\exists f: A \to B$, $g: B \to A$ 1-1, then $A \sim B$

proof: let $A_1 = g(B_1)$ $\quad B_1 = B \setminus f(A)$

$A_2 = g(B_2)$ $\quad B_2 = f(A_1)$

$\cdots$

$A_n = g(B_n)$ $\quad B_n = f(A_{n-1})$



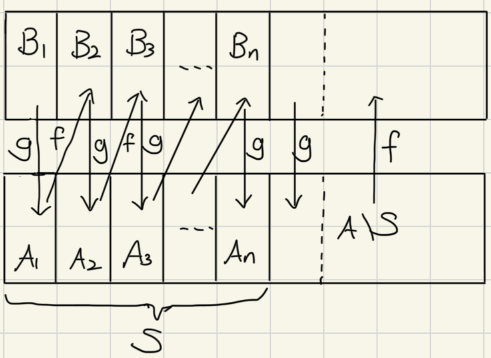
$S = \bigcup_{n=1}^{\infty} A_n$, let $F(a) = \begin{cases} f(a), & a \in A \setminus S \\ g^{-1}(a), & a \in S \end{cases}$

also $g^{-1}(S) = (B \setminus f(A)) \cup \left(\bigcup_{n=1}^{\infty} f(A_n)\right) = (B \setminus f(A)) \cup f(S)$

$f(A \setminus S) = f(A) \setminus f(S)$

Then $F(A) = g^{-1}(S) \cup f(A \setminus S) = (B \setminus f(A)) \cup f(S) \cup (f(A) \setminus f(S))$

Since $S \subseteq A$, then $F(A) = B$



Lec 2.

· $\mathbb{R}$ is uncountale

Step 1. $\mathbb{R} \sim (0,1)$

Step 2. $(0,1) \sim 2^{\mathbb{N}}$

Step 3. $2^{\mathbb{N}}$ is uncountable

Suppose $2^{\mathbb{N}}$ is countable, then $2^{\mathbb{N}} = \{A_1, A_2, \cdots, A_n, \cdots\}$

Define $B = \{k \in \mathbb{N},\ k \notin A_k\}$. Note that $B \subseteq \mathbb{N}$, so $B \in 2^{\mathbb{N}}$

However, $B = A_m$ for some $m \in \mathbb{N}$, so every element in B lies in A_m

This is impassible since $\forall k \in B$, $k \notin A_k$. which contradicts

Therefore $2^{\mathbb{N}}$ is not countable

Prop. {Argebraic numbers} is countable

The set of polynomials of integer coefficients is countable

let $P_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_j \in \mathbb{Z}, a_n \neq 0\}$

We see that P_0 is clearly countable, & $\mathbb{Z}^n \sim P_n$

Mathematical induction, suppose P_n is countable

$\mathbb{Z}^{n+1} = \mathbb{Z}^n \times \mathbb{Z} = \{(x, y) : x \in \mathbb{Z}^n, y \in \mathbb{Z}\}$

Lemma. The Cartesian product of two countable set is countable

$$\begin{tikzcd}
(x_1, y_1) \arrow[r] & (x_1, y_2) \arrow[dl] & (x_1, y_3) \arrow[d] & \\
(x_2, y_1) \arrow[d] & (x_2\ y_2) \arrow[ur] & (x_3, y_3) \arrow[dl] & \cdots \\
(x_3, y_1) \arrow[ur] & (x_3, y_2) & (x_3, y_3) &
\end{tikzcd}$$

So we see that $\mathbb{Z}^{n+1}$ is countable

Therefore $\forall n \in \mathbb{N}$, $\mathbb{Z}^n$ is countable, so each P_n is countable

Let α be an algebraic number, and we see that $m_{\mathbb{Z}}(\alpha) \in P_n$

Lemma. If $Y \sim Z$, then $X \times Y \sim X \times Z$

proof: let $g: X \times Y \to X \times Z$ be defined as $g(x, y) = (x, f(y))$

where $f: Y \xrightarrow{bij} Z$, so g is bijective

Diophantine approximation.

How well can a number be approximated by rational numbers?

Def. A real number $r \in \mathbb{R}$ is said to be approximated by rationals to order n if $\exists k = k(r)$, such that

$|r - \frac{p}{q}| < \frac{k}{q^n}$ has infinitely many sets $\frac{p}{q} \in \mathbb{Q}$

Lemma. Any rational number is approximated by rationals of order 1 and no higher.

pf:) $|\frac{p}{q} - \frac{p_k}{q_k}| = |\frac{pq_k - p_k q}{q \cdot q_k}| \geq \frac{1}{q \cdot q_k}$

so $\frac{1}{q \cdot q_k} \leq \frac{K}{q_k^n}$, if $n > 1$, then $q_k^{n-1} \leq q \cdot K$

Hence we have only finitely q_k's s.t. $|\frac{p}{q} - \frac{p_k}{q_k}| \leq \frac{K}{q_k^n}$ when $n > 1$

Thm. (Liouville 1853) A real algebraic number θ of order n is not approximated by any other order higher than n.

pf:) Suppose θ is a root of $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

let $\delta > 0$, where $f(x)$ has no other roots in $(\theta - \delta, \theta + \delta)$

let $\frac{p}{q} \in (\theta - \delta, \theta + \delta)$, $\frac{p}{q} \neq \theta$

then $|f(\frac{p}{q}) - f(\theta)| = |f'(\xi)| \cdot |\theta - \frac{p}{q}| = |f(\frac{p}{q})| \geq \frac{1}{q^n}$

let $M = \max_{\xi \in (\theta - \delta, \theta + \delta)} f'(\xi)$, so $|\theta - \frac{p}{q}| \geq \frac{1}{Mq^n}$

Suppose ξ is approximated by order $n+k$, then

$\frac{1}{M \cdot q^n} \leq |\theta - \frac{p}{q}| \leq \frac{K}{q^{n+k}}$, so $q^k \leq KM$, where there are only finitely many q's

Eg. (Liouville number) $L = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots$ is transcedentat

proof: Let $L_k = \sum_{i=1}^{k} \frac{1}{10^{i!}}$, $|L - L_k| = \sum_{i=k+1}^{\infty} \frac{1}{10^{i!}} \leq \frac{1}{10^{(k+1)!}} \leq \frac{1}{(10^{k!})^{k+1}} = \frac{2}{q_k^{k+1}} \leq \frac{2}{q_k^n}$ $(k+1 > n)$

where $q_k = (10)^{k!}$, so $\forall$ Given n, $\exists$ infinitely many q_k's where

$|L - \frac{p_k}{q_k}| \leq \frac{M}{q^n}$ for each given n

Hence it is transcedentat

Lec 3

<u>Def.</u> (1) A sequence of rationals $\{q_1, q_2, \cdots\}$ is called a Cauchy sequence if for any $\varepsilon>0$, there exist $N(\varepsilon)>0$ s.t.
$|q_n - q_m| < \varepsilon$, $\forall m, n > N$

(2) Two sequence is said to be equivalent if $\forall \varepsilon>0$, $\exists N=N(\varepsilon)>0$ s.t.
$|P_n - q_n| < \varepsilon$ when $n>N$
In this case, we say $\{q_1, q_2, \cdots\} \sim \{P_1, P_2, \cdots\}$

(3) $\mathbb{Q}$: The set of all Cauchy sequence of rationals with the equivalent relation defined in (2)

<u>Def.</u> A real number r is an equivalence class of Cauchy sequences in $\mathbb{Q}$

Question: Is $\mathbb{R} = \mathbb{Q}/\sim$ well defined ?
- $r \sim r$ ✓
- if $x \sim y$, then $y \sim x$ ✓
- if $x \sim y$, $y \sim z$, then $x \sim z$

<u>Check.</u> $|x-z| \leq |x-y| + |y-z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ✓

<u>Def.</u> Given two real numbers x & y
$x \sim \{x_1, x_2, \cdots\}$ $y \sim \{y_1, y_2, \cdots\}$
- Addition: $x+y \sim \{x_1+y_1, x_2+y_2, \cdots\}$
- Multiplication: $xy \sim \{x_1y_1, x_2y_2, \cdots\}$
- Order: We say that $x<y$ if $\exists$ a positive rational number $\varepsilon>0$ and $N \in \mathbb{N}$ such that for $\forall n>N$, $x_n + \varepsilon < y_n$

<u>Lemma.</u> $\mathbb{Q}$ is dense in $\mathbb{R}$ $\forall x \in \mathbb{R}, y>x, [x,y] \cap \mathbb{Q} \neq \emptyset$
$x = a_0 . a_1 a_2 \cdots a_n \cdots$
$y = b_0 . b_1 b_2 \cdots b_n \cdots$, then $\exists n \in \mathbb{N}$ s.t. $a_k = b_k$ for $k<n$ and $a_n < b_n$
then set $r = a_0 . a_1 a_2 \cdots a_{n-1} b_n 000 \cdots$

<u>Def.</u> In a topological space X, $\mathcal{F}$ is called a open cover if $\mathcal{F} = \{U_\alpha | \alpha \in I\}$ $U_\alpha \in X$ open and $X \subseteq \bigcup_{\alpha \in I} U_\alpha$. $\mathcal{O}$ is a finite subcover if $\mathcal{O} \subseteq \mathcal{F}$ is finite and $X \subseteq \bigcup_{j=1}^{n} V_j$, where $\mathcal{O} = \{V_j | V_j \in \mathcal{F}, 1 \leq j \leq n\}$

<u>Def.</u> A space X is said to be compact if every open cover of X has a finite subcover.

Def. A set $U \subseteq \mathbb{R}$ is said to be an open set if $\forall x \in U$, $\exists r>0$ s.t. $(x-r, x+r) \subseteq U$

Topological sight: For a topological space $(X, \mathscr{T})$

Thm. $\mathbb{Q}$ is dense in $\mathbb{R}$. i.e. $\forall x \in \mathbb{R}$. $\exists \{x_k\} \subset \mathbb{Q}$, $x_k \to x$

Def. Given $\{x_n\}$, we say that $x_n \to x$ if $\forall \varepsilon>0, \exists N(\varepsilon)$, s.t. $|x_n - x| < \varepsilon$, $\forall n > N$

Thm. Every Cauchy sequence in $\mathbb{R}$ converges.

pf:) Check. $\{a_n\}$ is bounded, take $\varepsilon=1$, $\exists N_0 \in \mathbb{N}$, $|a_m - a_n| < 1$ when $m, n > N_0$

So clearly $|a_n| \leq |a_{N_0+1}| + 1$ for $\forall n > N_0$

Take $M = \max\{|a_1|, \cdots, |a_{N_0}|, |a_{N_0+1}|+1\}$, so $|a_n| \leq M$ for $\forall n \in \mathbb{N}$

$\alpha = \limsup_{n\to\infty} a_n$, we see that $\forall \varepsilon>0$, $\exists N_1 \in \mathbb{N}$, s.t. $a_n < \alpha + \varepsilon$

So choose $N = \max\{N_0, N_1\}$, $|a_n - \alpha| \leq |a_m - \alpha| + |a_m - a_n| < 2\varepsilon$, $\forall m, n > N$

so we see that $\lim_{n\to\infty} a_n = \alpha$

Thm1 (Bolzano-Weierstrass) Every bounded sequence has a convergent subsequence

Thm2 (Heine-Borel) Every open cover of a bounded closed interval has a finite subcover.

Thm3 (Nested interval) If $a_1 \leq a_2 \leq \cdots$, $b_1 \geq b_2 \geq \cdots$, $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty}(a_n, b_n) = \{c\}$ for some $c \in \mathbb{R}$

Lec. 5

Continuity.

Def. $f: D \to \mathbb{R}$, where D is an open interval. Let $c \in D$, we say f is continuous at $x=c$ if $\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-f(c)|<\varepsilon, \forall x \in (c-\delta, c+\delta)$

If f is continuous at each point of D, then we say f is cont. on D.

Limits.

· Given $f: I \to \mathbb{R}$. $a \in I$, $L \in \mathbb{R}$, if $\forall \varepsilon>0, \exists \delta>0$, s.t. $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$, then we say $\lim_{x\to a} f(x) = L$

Fact. $\{a_n\} \subset D$, $\lim_{n\to\infty} a_n = a \in D$, then $\lim_{n\to\infty} f(a_n) = f(a)$

Thm. (Intermediate value theorem.)

Suppose f is continuous on $[a,b]$, $f(a) \neq f(b)$, then for every α lying between $f(a)$ & $f(b)$, $\exists\ c \in [a,b]$ s.t. $f(c)=\alpha$

proof: Consider $f(\frac{a+b}{2})$. if $f(\frac{a+b}{2}) = \alpha$, then we've done

w.l.o.g. let $f(a)<f(b)$, if $f(\frac{a+b}{2})<\alpha$, let $I_1 = [\frac{a+b}{2}, b]$

if $f(\frac{a+b}{2})>\alpha$, then let $I_2 = [a, \frac{a+b}{2}]$

Repeat such process, then we have $\bigcap_{n=1}^{\infty} I_n = \{c\}$

we see that $\exists \{x_n\}$, $x_n \to c$, also $\lim_{n\to\infty} f(x_n) = \alpha$

Since $[a,b]$ is closed, then $c \in [a,b]$. also f is cont.

so $\lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) = f(c) = \alpha$

Thm. (Boundedness) Suppose f is continuous on $[a,b]$, then its range is bdd.

proof: Suppose f is unbounded, then $\exists \{x_n\} \subset [a,b]$, $\forall M>0, \exists N>0$, s.t. $|f(x_n)|>M$ as $n>N$

Since $\{x_n\}$ is bounded, then it has a convergent subsequence $\{x_{n_j}\}$

$\lim_{n\to\infty} f(x_n) = \lim_{j\to\infty} f(x_{n_j}) = f(x_0)$. this implies that for n big enough

$|f(x_n)| \leq |f(x_0)|+1$. which contradicts.

Another proof. by Borel-Henine Thm'

we see that $\mathscr{G}_\varepsilon = \{(x-\delta_x, x+\delta_x) \mid x\in[a,b]\}$ is an open cover of $[a,b]$

where $|f(y)-f(x)|<\varepsilon$ whenever $|y-x|<\delta_x$, then we have a finite subcover

that is, $[a,b]\subset\bigcup_{j=1}^{N}(x_j-\delta_j, x_j+\delta_j)$, so $\forall y\in[a,b]$, $y\in(x_j-\delta_j, x_j+\delta_j)$ for some j

Take $M_\varepsilon = \max\{f(x_1),\cdots,f(x_N)\}$, so $|f(y)|\leqslant|f(y)-f(x_j)|+|f(x_j)|<\varepsilon+M_\varepsilon$

<u>Thm.</u> (Extreme Value Theorem)

Suppose f is cont. on $[a,b]$, then $M=\max\limits_{x\in[a,b]}f(x)$ & $m=\min\limits_{x\in[a,b]}f(x)$ exists.

proof: From Boundedness Theorem. $\sup\limits_{x\in[a,b]}f(x)=M$ exists.

so $\exists\{x_n\}\subset[a,b]$, $\lim\limits_{n\to\infty}f(x_n)=M$

By Bolzano-Weierstrass Thm., $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$

$\lim\limits_{j\to\infty}x_{n_j}=x_0$, since f is cont.. then $\lim\limits_{j\to\infty}f(x_{n_j})=f(x_0)$

$\lim\limits_{n\to\infty}f(x_n)=\lim\limits_{j\to\infty}f(x_{n_j})=f(x_0)=M$. since $[a,b]$ is closed, then $x_0\in[a,b]$

Q.E.D.

<u>Def.</u> $f: D\to\mathbb{R}$ is said to be uniformly continuous if $\forall\varepsilon>0, \exists\delta>0$

s.t. $\forall x,y\in D$, $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$

Note that δ is independent of x & y

<u>Thm.</u> A continuous function is uniformly continuous on a closed interval.

proof:) (Based on Henie-Borel Th'm)

Let f be cont. on $[a,b]$, then $\forall\varepsilon>0$, $x\in[a,b]$, $\exists\delta_x>0$. $|f(y)-f(x)|<\varepsilon$

whenever $|y-x|<\delta_x$

Then $\mathscr{G}_\varepsilon=\{(x-\frac{1}{2}\delta_j, x+\frac{1}{2}\delta_j): x\in[a,b]\}$ forms and open cover of x

We can find a finite subcover, i.e. $[a,b]\subset\bigcup_{j=1}^{n}(x_j-\frac{1}{2}\delta_j, x_j+\frac{1}{2}\delta_j)$

$\forall y\in[a,b]$. y lies in some $(x_i-\frac{1}{2}\delta_i, x_i+\frac{1}{2}\delta_i)$

Choose $\delta=\min\{\frac{1}{2}\delta_1,\cdots,\frac{1}{2}\delta_n\}$. then if $|y-z|<\delta$, we have

$|z-x_j|=|z-y|+|y-x_j|\leq\delta+\frac{1}{2}\delta_j<\delta_j$, then $|f(y)-f(z)|\leqslant|f(y)-f(x_j)|+|f(z)-f(x_j)|<2\varepsilon$

Lec. 6

Def. Let S be a set $d: S \to \mathbb{R}$ be a function, and

1、$d(x,y) \geq 0$, equation holds when $x=y$

2、$d(x,y) = d(y,x)$

3、$d(x,z) \leq d(x,y) + d(y,z)$

In this case we say (S,d) is a metric space, d is a metric.

Def. We say that two metrics are equivalent if $\exists c_1, c_2$ s.t.
$c_1 d_2(x,y) \leq d_1(x,y) \leq c_2 d_2(x,y)$, $\forall x, y \in S$, denoted $d_1 \sim d_2$

Remark. $d_1(\vec{x},\vec{y}) = |x_1-y_1| + \cdots + |x_n - y_n|$, $d_2(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$
are equivalent in $\mathbb{R}^n$
we can check that $d_2(\vec{x},\vec{y}) \leq d_1(\vec{x},\vec{y}) \leq \sqrt{n}\, d_2(\vec{x},\vec{y})$

Point set topology (点集拓扑)

Def. Let (S,d) be a metric space

(1) A sequence $\{P_n\} \subset S$ is said to be convergent to $P \in S$ if
$d(P_n, P) \to 0$ as $n \to \infty$

(2) The open ball $B_r(P_0) = B(P_0, r)$ centered at $P_0 \in S$ with radius r is
$B(P_0, r) := \{x \in S : \|x - P_0\| < r\}$

(3) The closed ball $\overline{B(P_0, r)} := \{x \in S : \|x - P_0\| \leq r\}$

(4) Let $A \subseteq S$, $P_0 \in A$, P_0 is said to be a isolated point of A if $\exists \delta > 0$
s.t. $B(P_0, \delta) \cap A = \{P_0\}$

(5) Let $A \subseteq S$, $P_0 \in S$, P_0 is said to be a limit point of A if $\forall \delta > 0$
$B(P_0, \delta) \cap A \neq \emptyset$

(6) A set is called open if $\forall P \in A$, $\exists \delta > 0$ s.t. $B_\delta(P) \subset A$

(7) A set is called closed if it contains all its limit points.

Lec7.

Def. (Interior points) Let $A \subset S$, $p \in A$ is called an interior point in A if $\exists r>0$, s.t. $B(p,r) = B_r(p) \subset A$
A collection of all interior points in A will be denoted as A^o

Def. A' = the set of all limit points of A in S
$\bar{A} = A \cup A'$, the closure of A
$\partial A = \bar{A} \setminus A^o$, the boundary of A

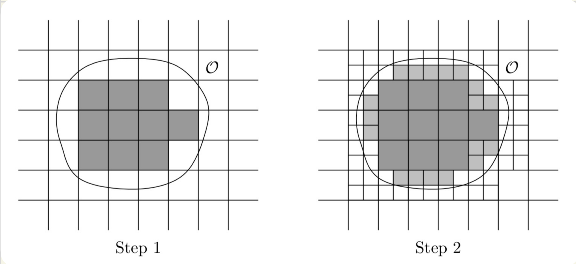
Prop. Let $A \subseteq S$, then
(1) A', $\bar{A}$, ∂A are all closed in S
(2) A^o is open in S

proof of (1): Let x_0 be a limit point of $\bar{A}$, then $\forall \varepsilon>0$, $B(x_0, \frac{\varepsilon}{2}) \cap (\bar{A} \setminus \{x_0\}) \neq \phi$
then $\exists\ x_1 \in B(x_0, \frac{\varepsilon}{2}) \cap (A \setminus \{x_1\})$, $B(x_1, \frac{\varepsilon}{2}) \cap (A \setminus \{x_1\}) \neq \phi$
therefore $\forall \varepsilon>0$, $B(x_0, \varepsilon) \cap (A \setminus \{x_0\}) \neq \phi$, so $x_0 \in \bar{A}$

(2) Suppose A^o is not open, then $\forall r>0$, s.t. $B(x,r) \setminus A^o \neq \phi$
so we can choose $y \in B(x,r) \setminus A^o$, since $A^o \subset A$, $\exists r>0$, s.t. $B(x,r) \subset A$
we see that $B(x,r)$ is also open, so $\exists \delta>0$, $B(y,\delta) \subset B(x,r) \subset A$
Therefore $y \in A^o$, this is a contradiction

Def. (1) A open cell in $\mathbb{R}^N$ is a set of the form
$\{(x_1, \cdots, x_N) \mid a_i < x_i < b_i,\ i=1, \cdots, N\}$
(2) A closed cell in $\mathbb{R}^N$ is a set of the form
$\{(x_1, \cdots, x_N) \mid a_i \leq x_i \leq b_i,\ i=1, \cdots, N\}$

Thm. Any open set in $\mathbb{R}^N$ can be written as the union of almost disjoint closed cells (or cubes).



Thm. Each open set in $\mathbb{R}$ can be written as countable union of disjoint open intervals

proof:) let A be an open set in $\mathbb{R}$, then $\forall x \in A$ ☆☆☆

we have $(x-c, x+c) \in A$ for some $c>0$ important in real analysis

let $a_x = \inf_a (a, x) \subseteq A$, $b_x = \sup_b (x, b) \subseteq A$

Then $A = \bigcup_{x \in A} (a_x, b_x)$, let $I_x = (a_x, b_x)$. If $I_x \cap I_y \neq \phi$

we see that $(I_x \cup I_y) \subset I_x$ & $(I_x \cup I_y) \subset I_y$, so $I_x = I_y$

Hence it is a disjoint union of open intervals.

And in each open interval, there is at least one rational number

Hence the union is countable

Def. Let S be a metric space

(1) $A \subseteq S$ is said to be sequential compact if every sequence in A has a convergent subsequence

(2) A space A is bounded if $A \subseteq B_r(x)$ for some $x \in S$, $r>0$

Topological definition.

A is said to be compact if any open cover of A has a finite subcover.

Lemma. (S, d) is a metric space.

(1) A sequential compact set is closed and bounded

proof: Suppose A is not bounded, then $\exists \{x_n\} \subset A$, $d(x_n, x) > n$ for some x

then we can find a convergent subsequence $\{x_{n_j}\}$ that converges to x_0

so $n < n_j < d(x_{n_j}, x) \leq d(x_{n_j}, x_0) + d(x, x_0) < d(x, x_0) + \varepsilon$

This contradicts ※

Check. A is closed, let x be a limit point of A

then choose $x_1 \in A \cap B(x, 1)$, $x_2 \in A \cap B(x, \frac{1}{2})$, $\cdots$, $x_n \in A \cap B(x, \frac{1}{n})$, $\cdots$

Thus $\forall \varepsilon > 0$, $B(x, \varepsilon)$ contains infinitely many terms of $\{x_n\}$

This implies $\exists x_{n_j}$, $x_{n_j} \to x$, hence $x \in A$

<u>Def.</u> A space is complete if every Cauchy sequence converges.

<u>Lemma.</u> A sequential compact set is complete.
proof: $d(x_n, x) \leq d(x_{n_j}, x) + d(x_n, x_{n_j}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

<u>Thm.</u> (Cantor's Lemma) Let $S_1 \supseteq \cdots \supseteq S_n \supseteq \cdots$ be a descending sequence of non-empty closed subsets of a compact set $A \subseteq S$, then $\bigcap_{k=1}^{\infty} S_k \neq \varnothing$
proof: Since S is nonempty, there exist a point $P_n \in S_n$
Denote the sequence $\{P_n\}$. and $S_n \subset S_{n-1} \Rightarrow P_n \in S_k$ for $\forall\, k \leq n$
Since $\{P_n\} \subset A$, it has a convergent subsequence $\{P_{n_k}\} \subset S_k$
<u>fact:</u> A closed subset of a compact set is compact
so $P_{n_k} \to p^* \in S_k$ for $\forall\, k \in \mathbb{N}$, thuse $p^* \in \bigcap_{k=1}^{\infty} S_k$,

<u>Lemma.</u> A sequential compact set is totally bounded
pf: Pick $P_1 \in A$, if $A \setminus B_\delta(P_1) \neq \varnothing$, then pick $P_2 \in A \setminus B_\delta(P_1)$
if $A \setminus (B_\delta(P_1) \cup B_\delta(P_2)) \neq \varnothing$, then pick $P_3 \in A \setminus (B_\delta(P_1) \cup B_\delta(P_2))$
By repeating, we can find an $\{P_k\}$, and $d(P_i, P_j) \geq \delta$ for $\forall\, i, j$
However every infinite sequence has a convergent subsequence in A
therefore $\{P_k\}$ is finite

Proof of fact:
Let $A \underset{\text{closed}}{\subseteq} S$, then every sequence has a conv. subsequence, also A is closed so it contains all it limit points.

<u>Thm.</u> A sequential compact set is compact.
proof: Suppose A is not compact, then we suppose $\mathcal{F}$ doesn't have a finite subcover. Since A is sequential compact, then $A \subseteq \bigcup_{j=1}^{k} B(P_j^1, 1)$
so $\exists\, j$ s.t. $B(P_j^1, 1)$ can't be finitely covered by $\mathcal{F}$, w.l.o.g. $B(P_1^1, 1)$
Then we let $A_1 = A \cap \bar{B}(P_1^1, 1)$, which is also seq. cpt.
$A_1 \subseteq \bigcup_{j=1}^{k_2} B_{\frac{1}{2}}(P_j^2)$, w.l.o.g. $B_{\frac{1}{2}}(P_1^2)$ can't be finite covered by $\mathcal{F}$
so denote $A_2 = A_1 \cap \bar{B}_{\frac{1}{2}}(P_1^2)$ ...
By repeating this process, we can find $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$
Here each $A_n = A_{n-1} \cap \bar{B}_{\frac{1}{n}}(P_1^n) \subseteq A_{n-1}$
By Cantor's Lemma $\bigcap_{k=1}^{\infty} A_n \neq \varnothing$, so $\exists\, P_0 \in A_k$ for $\forall\, k \in \mathbb{N}$
Since $\mathcal{F}$ covers A, then $\exists\, U_\alpha$ open s.t. $P_0 \in U_\alpha$
then $\exists\, \delta > 0$, $B(P_0, \delta) \subseteq U_\alpha$, and $\exists\, n = n(\delta) \in \mathbb{N}$, $A_n \subseteq B(P_0, \delta) \subseteq U_\alpha$
However, A_n can't be finitely covered by $\mathcal{F}$, this contradicts ※

Lec 8.

Cor. (Lebesgue's Lemma) Let $\mathcal{F}$ be an open cover for a compact set A. Then $\exists \rho>0$, s.t. $\forall p\in A$, $B_\rho(p)$ lies in an element of $\mathcal{F}$

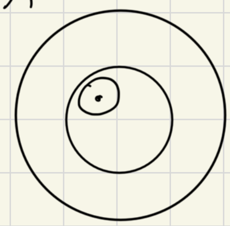
proof: For any $p\in A$, $\exists U_p\in\mathcal{F}$, s.t. $p\in U_p$, where U_p is open

$\exists \delta_p>0$. s.t. $B(p,2\delta_p)\subseteq U_p$. Then $\{B(p,\delta_p) \mid p\in A\}$ forms an open cover of A

And we can find a finite subcover $\{B(p_1,\delta_1),\cdots,B(p_n,\delta_n)\}$ of A

Let $\rho=\min\{\delta_1,\cdots,\delta_n\}$

Claim. $\forall p\in A$, $B_\rho(p)\subseteq B(p_i,2\delta_i)\subseteq U_{p_i}$



$\forall p\in A$, $p\in B(p_i,\delta_i)$ for some i

then $B_\rho(p)\subseteq B(p_i,\delta_i+\rho)\subseteq B(p_i,2\delta_i)\subseteq U_{p_i}$

Consider $f:(S_1,d_1)\longrightarrow(S_2,d_2)$

Def. We say f is continuous at $p_0\in S_1$, if $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $d_2(f(p),f(p_0))<\varepsilon$ whenever $d_1(p,p_0)<\delta$

Lemma. (Topological definition) f is continuous on S_1 $\Leftrightarrow$ $f^{-1}(U)$ is open for any open set U in S_2.

proof: $\Rightarrow$ let $U\subseteq S_2$, pick $p_0\in f^{-1}(U)$, then $f(p_0)\in U$

Since U is open, then $\exists\varepsilon>0$ s.t. $B_\varepsilon(f(p_0))\subseteq U$

Also from the continuity, $\exists\delta=\delta(\varepsilon,p_0)$ s.t. $d(f(p),f(p_0))<\varepsilon$ when $d(p,p_0)<\delta$

so $\forall p_0\in f^{-1}(U)$, $\exists\delta>0$, s.t. $B(p_0,\delta)\subseteq f^{-1}(B_\varepsilon(f(p)))\subseteq f^{-1}(U)$

$\Leftarrow$ Choose $p_0\in f^{-1}(U)$. clealy $f(p_0)\in U$, so for $\forall\varepsilon>0$

① If $B(f(p_0),\varepsilon)\setminus U\neq\emptyset$, then choose $\delta>0$ s.t. $B(p_0,\delta)\subseteq U$ arbitarily

② If $B(f(p_0),\varepsilon)\subseteq U$, then, $f^{-1}(B(f(p_0),\varepsilon))$ is open, so $\exists\delta>0$ s.t. $B(p_0,\delta)\subseteq f^{-1}(B(f(p_0),\varepsilon))$, then f is continuous.

Prop. Let A be a compact space in S, B be a closed set in S.
$A \cap B = \emptyset$, then $\exists \delta > 0$, $\forall p \in A$, $q \in B$, $d(p,q) > \delta$

proof: Since B is closed, then B^c is open and clearly $A \subseteq B^c$
Then $\forall x \in A$, $\exists \delta_x > 0$, s.t. $B(x, 2\delta_x) \subseteq B^c$, and $\mathcal{F} = \{B(x, 2\delta_x) \mid x \in A\}$ forms an open cover of A. Since A is compact, then $\exists x_1, \cdots, x_n$, s.t. $A \subseteq \bigcup_{j=1}^{n} B(x_j, 2\delta_{x_j})$

Def. Let $A \subseteq S$, S be a metric space
(1) $A \subseteq S$ is dence if $\bar{A} = S$
(2) $A \subseteq S$ is nowhere dense if $\bar{A}$ does not contain any open ball in S

Lemma. The following statements are equivallent.
(1) A is nowhere dense in S
(2) Every open ball B of S contains an open ball $B' \subseteq B$ s.t. $B' \cap A = \emptyset$
(3) Every open ball B of S contains an open ball $B' \subseteq B$ s.t. $\overline{B'} \cap \bar{A} = \emptyset$

proof: (3) => (2) ✓
(1) => (3): $\forall$ open ball $B \subseteq S$, since A is nowhere dense in S, $\bar{A}$ does not contain any open ball in S.
so $B \cap \bar{A}^c \neq \emptyset$. Since $\bar{A}$ is closed, then $\bar{A}^c$ is open
Also B is open, therefore $B \cap \bar{A}^c$ is open, take $x \in B \cap \bar{A}^c$
Thus $\exists \delta > 0$, $B(x, \delta) \subseteq B \cap \bar{A}^c$, let $B' = B(x, \frac{1}{114514}\delta)$
So $\overline{B'} \subset B \cap \bar{A}^c$, i.e. $\overline{B'} \cap \bar{A} = \emptyset$
(2) => (1): Suppose $\bar{A}$ contains an open ball B
So $\exists B' \subseteq B$, s.t. $B' \cap A = \emptyset$,
Therefore we have $B' \subseteq \bar{A} \setminus A$, let $B' = B(x, \delta)$
Since $x \in A'$, then $\forall \varepsilon > 0$, $B(x, \varepsilon) \cap (A \setminus \{x\}) \neq \emptyset$
However $B(x, \delta) \cap A = \emptyset$, so this is a contradiction.

Lemma. A is dense in S <=> Any open ball in S intersect with A

proof: (=>) $S = A \cup A'$, for $\forall B(x, r)$, if $x \in A$, ✓
if $x \in A'$, $B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$
(<=)

Lec. 10

Def. A set E in a metric space S is of the first category if E is a countable union of nowhere dense sets

· E is of the second category if is no of the first category

Example. · $\mathbb{Q}$ is countable, if of the 1st category

· Any countable set is of the 1st category.

Thm. (Baire's) Let (S, d) be a complete metric space.

(1) S itself is of the 2nd category

(2) A nonempty open set in S is of the 2nd category

(3) Let A be a set in S. If A is of the 1st category, then A^c is dense

proof of (3): Let $A \subseteq S$ be of the 1st category, $\exists A_k$ nowhere dense $A = \bigcup_{k=1}^{\infty} A_k$. It suffices to show that $\forall$ open ball $B \subseteq S$, $B \cap A^c \neq \emptyset$

· Since A_1 is nowhere dense in S, $\exists B_1 = B(p_1, r_1) \subseteq \bar{B}_1 \subseteq B = B_0$ s.t. $\bar{B}_1 \cap \bar{A}_1 = \emptyset$, w.l.o.g. $r_1 < \frac{r_0}{2}$

· A_2 is nowhere dense in S, Choose $B_2 \subseteq \bar{B}_2 \subseteq B_1$

$\bar{B}_2 \cap \bar{A}_2 = \emptyset$, $B_2 = B(p_2, r_2)$, $r_2 \leq \frac{r_1}{2}$

...

· A_n is nowhere dense in S. $B_n \subseteq \bar{B}_n \subseteq B_{n-1}$

$\bar{B}_n \cap \bar{A}_n = \emptyset$. $B_n = B(p_n, r_n)$, $r_n \leq \frac{r_{n-1}}{2}$

Since $p_n, p_{n-1} \in B_{n-1}$, $d(p_n, p_{n-1}) \leq \frac{r_0}{2^{n-1}}$

Hence $\{p_n\}$ is a Cauchy sequence, $\exists p^*$ s.t. $p_n \to p^*$

Claim. $p^* \notin A_n$, $\forall n \in \mathbb{N}$

Since $\bar{A}_n \cap \bar{B}_n = \emptyset$, then $p^* \notin A$. Hence A^c is dense

Def. For $f: S \to \mathbb{R}$, $\omega(f, E) := \sup_{x,y \in E} |f(y) - f(x)|$

$\omega(f, x_0) := \lim_{\delta \to 0^+} \omega(f, B(x_0, \delta))$

Lemma. $\omega(f, x_0) = 0 \iff f$ is continuous at x_0

Thm. Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous on a dense set of $\mathbb{R}$, then the set of all discontinuous points is of 1st category.

proof: let $D_n = \{x \in \mathbb{R} \mid \omega(f,x) \geq \frac{1}{n}\}$, $D = \bigcup_{n=1}^{\infty} D_n$

Suppose $\exists D_N$, s.t. D_N is not nowhere dense, then

$\exists$ an open interval $I \subseteq \overline{D_N}$, so $\exists a \in I$ s.t. f is cont. at a

Take $\varepsilon = \frac{1}{4N}$, $\exists \delta > 0$, s.t. $|f(x) - f(a)| < \frac{1}{4N}$, $\forall x \in (a-\delta, a+\delta)$

Since $a \in \overline{D_N}$, $\exists \{b_k\} \subset D_N$, $b_k \to a$ as $k \to \infty$

$\limsup\limits_{\substack{\delta \to 0^+ \\ x,y \in B_\delta(b_k)}} |f(x) - f(y)| \geq \frac{1}{N}$, so we can take $\{b_k^j, \hat{b}_k^j\} \subset D_N$, $b_k^j, \hat{b}_k^j \to b_k$

s.t. for large j, $b_k^j, \hat{b}_k^j \in (a-\delta, a+\delta)$, and $|f(b_k^j) - f(\hat{b}_k^j)| \geq \frac{1}{N} - \eta \geq \frac{1}{2N}$

However, $|f(b_k^j) - f(\hat{b}_k^j)| \leq |f(b_k^j) - f(a)| + |f(\hat{b}_k^j) - f(a)| < \frac{1}{2N}$. ※

So D is of 1st category

Lec. 12

Goal: Approximate functions by smooth functions

- Extend $\mathcal{C}(A) \longrightarrow \mathcal{C}(\mathbb{R})$ $(A \subseteq \mathbb{R})$
- Find smooth functions on $\mathbb{R}$

Def. Let (S, d) be a metric space, $A \subseteq S$, then $\forall p \in S$

$d(p, A) := \inf\{d(p, a), a \in A\}$

Prop. (1) For fixed A $d(\cdot, A): S \to \mathbb{R}$ is Lipschitz continuous on S, where $|d(p,A) - d(p,A)| \leq d(p,q)$

proof: For $\forall \varepsilon > 0$, $\exists a \in A$ s.t. $d(p,a) \leq d(p,A) + \varepsilon$

$d(q,A) \leq d(p,q) + d(p,a) \leq d(p,q) + d(p,A) + \varepsilon$

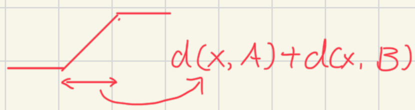
Conversely we can choose $b \in A$, s.t. $d(q,b) \leq d(q,A) + \varepsilon$

Hence $d(p,A) \leq d(p,q) + d(q,A) + \varepsilon$, then $|d(q,A) - d(p,A)| \leq d(p,q)$

(2) Suppose A, B are two disjoint closed set in S, then there exist a continuous function $\psi: S \to [0,1]$ s.t.

$\psi|_A = 0$, $\psi|_B = 1$

proof: let $\psi(x) = \dfrac{d(x,A)}{d(x,A) + d(x,B)}$ intuition:



Cor. $\forall a \leq b$, $\exists \varphi: S \to [0,1]$, $\varphi|_A = a$, $\varphi|_B = b$

$\varphi(x) = a + (b-a)\psi(x)$

<u>Thm.</u> (Trieze Extension) Let A be a closed subset in (S, d) $f: A \to \mathbb{R}$ be a continuous bounded function, then $\exists g: S \to \mathbb{R}$ continuous s.t. $g|_A = f$, $|g| \leqslant \sup_A |f|$ on S

proof: Assume $|f| \leqslant M$ on A, then consider

$A_1 = \{x \in A \mid f(x) \leqslant -\frac{1}{3}M\}$, $B_1 = \{x \in A \mid f(x) \geqslant \frac{1}{3}M\}$

So $\exists \varphi_1$ cont. s.t. $\varphi_1 = \begin{cases} -\frac{1}{3}M & \text{on } A_1 \\ \frac{1}{3}M & \text{on } B_1 \end{cases}$, and let $f_2 = f - \varphi_1$

We see that $|\varphi_1| \leqslant \frac{1}{3}M$, $|\varphi_1|_{A \setminus (A_1 \cup B_1)}| \leqslant \frac{1}{3}M$, so $|f_2| \leqslant \frac{2}{3}M$

let $A_2 = \{x \in A \mid f_2(x) \leqslant -\frac{1}{3}(\frac{2}{3}M)\}$, $B_2 = \{x \in A \mid f_2(x) \geqslant \frac{1}{3}(\frac{2}{3}M)\}$

$\exists \varphi_2$ cont. s.t. $\varphi_2 = \begin{cases} -\frac{1}{3}(\frac{2}{3}M) & \text{on } A_2 \\ \frac{1}{3}(\frac{2}{3}M) & \text{on } B_2 \end{cases}$

Then let $f_3 = f_2 - \varphi_2$, then repeat this process

we have $f_{n+1} = f_n - \varphi_n$, $|f_{n+1}| \leqslant (\frac{2}{3})^n M$

- Let $g_n = \varphi_1 + \varphi_2 + \cdots + \varphi_n \in \mathcal{C}(S, \mathbb{R})$

<u>Fact 1</u> $\{g_n\}$ is a Cauchy sequence on $\mathcal{C}(S, \mathbb{R})$

$|g_m - g_n| \leqslant |\varphi_{n+1}| + \cdots + |\varphi_m| \leqslant \frac{1}{3}[(\frac{2}{3})^n + \cdots + (\frac{2}{3})^{n-1}] \longrightarrow 0$

<u>Fact 2</u> $\{g_n|_A\} \to f$

$g_n|_A = f_1 - f_2 + f_2 - f_3 + \cdots + f_n - f_{n+1} = f_1 - f_{n+1}$, $\lim_{n\to\infty} |f_{n+1}| \leqslant \lim_{n\to\infty} (\frac{2}{3})^n M = 0$

so $\lim_{n\to\infty} g_n|_A = f$

<u>Fact 3.</u> $|g_n| \leqslant |\varphi_1| + \cdots + |\varphi_n| \leqslant \frac{1}{3}(1 + (\frac{2}{3}) + \cdots + (\frac{2}{3})^{n-1})M \leqslant M$

<u>Mollifier</u>

$\varphi(x) = \begin{cases} 0, & x \leqslant 0 \\ e^{-\frac{1}{x^2}}, & x > 0 \end{cases}$, $\varphi(x) \in C^\infty(\mathbb{R})$. $\varphi^{(n)}(0) = 0$, $\forall n \in \mathbb{N}$

Set $\check{\varphi}(x) = \varphi(x)\varphi(1-x)$, $\operatorname{supp}(\check{\varphi}) = [0, 1]$

$\cdot\ \psi(x) = \begin{cases} 1 - \varphi^*(2x-1), & x \geqslant 0 \\ \psi(-x), & x < 0 \end{cases}$, $\operatorname{supp}(\psi) = [-1, 1]$. $\psi \equiv 1$ on $(-\frac{1}{2}, \frac{1}{2})$

<u>Def.</u> $\rho(\vec{x}) = \dfrac{\psi_N(\vec{x})}{\int_{\mathbb{R}^N} \psi_N(\vec{x}) d\vec{x}}$, $\psi_N(\vec{x}) = \psi(\|\vec{x}\|)$

$\operatorname{supp}(\rho) \subseteq \overline{B_1(0)}$, $\rho \equiv 1$ over $B_{\frac{1}{2}}(0)$

Def. A nonnegative function $\rho \in C_c^\infty(\mathbb{R}^N)$ is a mollifier if $\operatorname{supp}(\rho) \subseteq \overline{B_1(0)}$ & $\int_{\mathbb{R}^N} \rho(x)dx = 1$

Let $\rho_\delta = \frac{1}{\delta^N}\rho(\frac{\vec{x}}{\delta})$, $\forall \delta \in (0,1)$, is also a mollifier

Thm. Suppose $f: \mathbb{R}^N \to \mathbb{R}$ is continuous, and ρ_δ is a mollifier, then $\forall \delta > 0$, consider the convolution

$f_\delta(\vec{x}) := \rho_\delta * f = \int_{\mathbb{R}^N} \rho_\delta(\vec{x}-\vec{y}) f(\vec{y}) d\vec{y}$

while $f_\delta \to f$ uniformly on any compact subset of $\mathbb{R}^N$

proof: $f_\delta - f = \int_{\mathbb{R}^N} \rho_\delta(\vec{x}-\vec{y}) f(\vec{y}) d\vec{y} - f(\vec{x}) \int_{\mathbb{R}^N} \rho_\delta(\vec{x}-\vec{y}) d\vec{y}$

$= \int_{\mathbb{R}^N} \rho_\delta(\vec{y})(f(\vec{x}-\vec{y}) - f(\vec{x})) d\vec{y}$

$= \int_{B_\delta(0)} \rho_\delta(\vec{y})(f(\vec{x}-\vec{y}) - f(\vec{x})) d\vec{y}$

On a compact subset of $\mathbb{R}^N$, f is uniformly continuous

let A be a compact subset, then for $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$

s.t. $|f(x) - f(y)| < \varepsilon$ whenever $\|x-y\| < \delta$

Therefore $|f_\delta - f| < \varepsilon \cdot \int_{B_\delta(0)} \rho_\delta(y)dy = \varepsilon$. $\delta = \delta(\varepsilon)$

Hence $f_\delta \to f$ uniformly

Then we show that f_δ is smooth

$f_\delta(x) = \int_{\mathbb{R}^N} \rho_\delta(x-y)f(y)dy$, by Lebniz's rule $\frac{\partial^k \rho_\delta}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}$ is continuous

so $\partial_x f_\delta(x) = \int_{\mathbb{R}^N} \partial_x f(x-y) f(y) dy$, which is smooth

Suppose X is of the 1st category, then $X = \bigcup_{n=1}^\infty A_n$

where each A_n is nowhere dense.

Since $\overline{A_1} \neq X$, then $\exists B(x_1, R_1) \subset X_1 \setminus \overline{A_1}$, Take $\overline{B(x_1, r_1)} \subset B(x_1, R_1)$

s.t. $\overline{B(x_1, r_1)} \cap \overline{A_1} = \emptyset$

$\overline{A_2}$ contains no open ball, then $\exists B(x_2, R_2) \subset B(x_1, r_1) \setminus \overline{A_1}$

Take $\overline{B(x_2, r_2)} \subset B(x_2, R_2)$, $r_2 = \min\{\frac{1}{2}R_2, \frac{1}{2}r_1\}$, $\overline{B(x_2, r_2)} \cap \overline{A_2} = \emptyset$

...

$\overline{B(x_1, r_1)} \supset \overline{B(x_2, r_2)} \supset \cdots \supset \overline{B(x_n, r_n)} \supset \cdots$, $r_n \leq \frac{1}{2} r_{n-1} \leq \cdots \leq \frac{1}{2^{n-1}} r_1$

so $\exists x_0 \in \bigcap_{n=1}^\infty B(x_n, r_n)$, so $x_0 \notin X$

However, X is a complete metric space, so $x_0 \in X$ ※

Thm. $K \subseteq \mathbb{R}^n$ compact, $f: K \to \mathbb{R}$ continuous, then f can be approximated uniformly by C^∞-functions on K

i.e. $\exists \{f_n\}$, s.t. $d(f_n, f) = \sup_{x\in K} |f_n(x) - f(x)| \to 0$ as $n \to \infty$

proof: By Tieze's extension, $\exists g: \mathbb{R} \to \mathbb{R}$, $g \equiv f$ on K, $|g| \leq \sup_K |f|$

Then define $f_n := \rho_{\frac{1}{n}} * g$. so $f_n \to g$ on any compact set of $\mathbb{R}$

i.e. $f_n \to f$ uniformly on K

Cor. If $f \in C^m(K)$, then $f_n \to f$ in $C^m(K)$, i.e.

$D^\alpha f_n \to D^\alpha f$ in $C(K)$, $\alpha = (\alpha_1, \cdots, \alpha_m)$, $|\alpha| \leq m$

Def. Let $\mathcal{L}$ be a family of cont. functions on S, i.e. $\mathcal{L} \subseteq \mathcal{C}(S)$

(1) $\mathcal{L}$ seperates points in S if

$\forall p, q \in S, p \neq q, \exists f \in \mathcal{L}$, s.t. $f(p) \neq f(q)$

(2) $\mathcal{L}$ seperates points in S & $\mathbb{R}$ if

$\forall p, q \in S\ (p \neq q), \forall a, b \in \mathbb{R}, \exists f \in \mathcal{L}$, s.t. $f(p) = a$. $f(q) = b$

Stone's approximation th'm

$\mathcal{L}$ is dense in $\mathcal{C}(S)$ if (S is compact)

(1) $\mathcal{L}$ seperates points in S & $\mathbb{R}$

(2) $\forall f, g \in \mathcal{L}$, $\max\{f, g\}, \min\{f, g\} \in \mathcal{L}$

Proof: $\forall F \in \mathcal{C}(S)$ WTS. $\forall \varepsilon > 0, \exists g \in \mathcal{L}$, s.t. $|F - g|(x) < \varepsilon$ for $\forall x \in S$

Suppose $p, q \in S$, $p \neq q$, $F(p) = a$, $F(q) = b$

Claim. Fix p, $\forall \varepsilon > 0$, $\exists g_p \in \mathcal{L}$. s.t. $g_p(p) = a = F(p)$, $g_p(s) \geq F(s) - \varepsilon$, $\forall s \in S$

Since $\mathcal{L}$ seperates points in S & $\mathbb{R}$, $\exists g_{p,q} \in \mathcal{L}$ s.t.

$g_{p,q}(p) = a$, $g_{p,q}(q) = b$

Note that $g_{p,q}$, F are both cont. at q, so $\exists B_{\delta_q}(q)$

s.t. $g_{p,q} > F - \varepsilon$ on $B_{\delta_q}(q)$

Then $\{B_{\delta_q}(q) \mid q \in S\}$ forms an open cover of S

Since S is compact, then it has a finite subcover

i.e. $S \subseteq \bigcup_{j=1}^{n} B_{\delta_{q_j}}(q_j)$, Take $g_p = \max\{g_{p,q_1}, \cdots, g_{p,q_n}\}$

Therefore $g_p(s) > F(s) - \varepsilon$ for $\forall s \in S$

For each g_p, $g_p(p) = F(p) = a$, so $\forall \varepsilon > 0, \exists B_{\delta_p}(p)$

s.t. $g_p < F + \varepsilon$ on $B_{\delta_p}(p)$, so $\{B_{\delta_p}(p) \mid p \in S\}$ is an open cover of S

then $S \subseteq \bigcup_{j=1}^{m} B_{\delta_{p_j}}(p_j)$, take $g = \min\{g_{p_1}, \cdots, g_{p_m}\}$

Then $g(s) < F(s) + \varepsilon$ on S, i.e. $|F - g| < \varepsilon$

Def. We say a set $\mathcal{A} \in \mathcal{C}(S)$ forms an algebra of functions in $\mathcal{C}(S)$
if · $\mathcal{A}$ is a vector space
· $\forall f, g \in \mathcal{A}$, $fg \in \mathcal{A}$

Stone-Weierstrass Thm
Let S be compact metric space, $\mathcal{A}$ is an algebra in S,
which (i) seperates points in S
(ii) contains function $f \equiv 1$
Then $\mathcal{A}$ is dense in $\mathcal{C}(S)$
proof: Let $\mathcal{L} = \bar{\mathcal{A}}$
Claim. $\forall p, q \in S, p \neq q$. $\forall a, b \in \mathbb{R}$, $\exists \tilde{f} \in \mathcal{L}$ s.t. $\tilde{f}(p) = a$, $\tilde{f}(q) = b$
Wee see that $\exists f \in \mathcal{A}$, $f(p) \neq f(q)$
Let
$$\tilde{f}(s) = b\frac{f(s) - f(p)}{f(q) - f(p)} + a\frac{f(s) - f(q)}{f(p) - f(q)}$$
, then $\tilde{f}(p) = a$, $\tilde{f}(q) = b$

Claim. $\forall f, g \in \mathcal{L}$, $\max\{f, g\}$, $\min\{f, g\} \in \mathcal{L}$
- $\max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$
- $\min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

WTS. $|x|$ can be approximated by polynomials
$|x| = M\sqrt{\frac{x^2}{M^2}} = M\sqrt{1 + [(\frac{x}{M})^2 - 1]}$, then use Taylor series
$\forall \varepsilon > 0$, $\exists P$, s.t. $\big||f-g| - p(f-g)\big| < \varepsilon$
$p(f-g) \in \mathcal{L}$, so $\max\{f, g\}, \min\{f, g\} \in \bar{\mathcal{L}} = \mathcal{L}$
Then by Stone-approximation, $\bar{\mathcal{L}} = \mathcal{L} = \bar{\mathcal{A}} = \mathcal{C}(S)$
i.e. $\mathcal{A}$ is dense in $\mathcal{C}(S)$

Def. A sequence $\{f_k\}$ is rapidly Cauchy if $\exists$ a convergent series $\sum_{k=1}^{\infty} \varepsilon_k$ s.t. $\|f_{k+1} - f_k\| < \varepsilon_k^2$, $\forall k$

Prop. Every Cauchy sequence has a rapidly Cauchy subsequence.
proof: Let $\{f_n\}$ be Cauchy, then $\exists n_k$ s.t. $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k}$ ($n_{k+1} > n_k$)
Here $\sum_{k=1}^{\infty} (\sqrt{2})^{-k}$ converges

This lecture is mainly about differentiation, which was already learned in Calculus, so just check the previous notes.

L-Hôpital's rule

(1) Suppose $f, g \in \mathcal{C}[a,b]$, f', g' exists in (a,b), $g'(x) \neq 0$
$f(a) = g(a) = 0$, then if $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, $\lim_{x\to a^+}\frac{f(x)}{g(x)} = L$

(2) Suppose $f, g \in \mathcal{C}(a, +\infty)$, f', g' exists in $(a, +\infty)$, with $g' \neq 0$
If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$

(3) Suppose $f, g \in \mathcal{C}[a,b]$, f', g' exists in (a,b), $g'(x) \neq 0$,
$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$, $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a^+}\frac{f(x)}{g(x)} = L$

Differentiation from $\mathbb{R}^N$ to $\mathbb{R}$

$f: \mathbb{R}^N \longrightarrow \mathbb{R}$, $x = (x_1, \cdots, x_N)$, $a = (a_1, \cdots, a_n)$

<u>Lemma</u>. Suppose $\frac{\partial f}{\partial x_i}$ exist in a ball $B(a)$ centered at a
If $\frac{\partial f}{\partial x_i}$ are all cont. at a, then
$f(a+h) - f(a) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(a) h_i + o(\|h\|)$, $h = (h_1, \cdots, h_N)$

Proof: When $N=2$, $f(a_1+h_1, a_2+h_2) - f(a_1, a_2) = f(a_1+h_1, a_2+h_2) - f(a_1, a_2+h_2)$
$+ f(a_1, a_2+h_2) - f(a_1, a_2)$

By MVT. $f(a_1+h_1, a_2+h_2) = \frac{\partial f}{\partial x_1}(a_1+\xi_1(h_1), a_2+h_2)\cdot h_1 + \frac{\partial f}{\partial x_2}(a_1, a_2+\xi_2(h_2))\cdot h_2$
$= (\frac{\partial f}{\partial x_1}(a) + O(\|h\|))\cdot h_1 + (\frac{\partial f}{\partial x_2}(a) + O(\|h\|))\cdot h_2 = \frac{\partial f}{\partial x_1}(a) h_1 + \frac{\partial f}{\partial x_2}(a) h_2 + o(\|h\|)$

<u>Thm.</u> Suppose $f: \mathbb{R}^N \to \mathbb{R}$ with $\frac{\partial^2 f}{\partial x_i \partial x_j}$ being continuous at $x=a$
for any $i, j \in \{1, \cdots, N\}$, then $\frac{\partial^2}{\partial x_i \partial x_j} f = \frac{\partial^2}{\partial x_j \partial x_i} f$

Proof: $N=2$

Define $\Delta_2(f) = f(a_1+h, a_2+k) - f(a_1+h, a_2) - (f(a_1, a_2+k) - f(a_1, a_2))$
$\varphi(s) = f(a_1+sh, a_2+k) - f(a_1+sh, a_2)$
$\Delta_2 f = \varphi(1) - \varphi(0) = \varphi'(\theta_1) = h\frac{\partial f}{\partial x_1}(a_1+\theta_1 h, a_2+k)$
$\psi(t) = f(a_1+h, a_2+tk) - f(a_1, a_2+tk)$, $\varphi(1) - \varphi(0) = \psi(1) - \psi(0) = \Delta_2 f$
$\psi(1) - \psi(0) = \psi'(\theta_2) = k\frac{\partial f}{\partial x_2}(a_1+h, a_2+\theta_2 k) = h\frac{\partial f}{\partial x_1}(a_1+\theta_1 h, a_2+k)$

Def. $f: \mathbb{R}^N \to \mathbb{R}$, $a, h \in \mathbb{R}^N$, $\|h\|=1$. The directional derivative of f at a is given by

$$D_h(f) = \lim_{t\to 0} \frac{f(a+th)-f(a)}{t}$$

Lemma. $D=(D_1, \cdots, D_N)$, $a \in \mathbb{R}^N$ $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_N!$

$$D_h^n(f) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} h^\alpha D^\alpha f(a)$$, where $D^\alpha = D_1^{\alpha_1}\cdots D_N^{\alpha_N}$, $h^\alpha = h_1^{\alpha_1}\cdots h_N^{\alpha_N}$

Taylor's Thm. $f: \mathbb{R}^N \to \mathbb{R}$. Suppose f & all its derivatives of order $\le n$ are cont., it's $(n+1)$-th order derivative exists, then

$$f(x) = \sum_{|\alpha|\le n} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=n+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha$$

proof: $h = \frac{x-a}{\|x-a\|}$, let $\varphi: \mathbb{R}\to\mathbb{R}$ be given by $\varphi(t) := f(a+th)$, $t=\|x-a\|$

$$\varphi(t) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(0)}{k!} t^k + \frac{\varphi^{(n+1)}(\theta)}{(n+1)!} t^{n+1}$$

$$\varphi^{(k)}(0) = D_h^k f = \sum_{|\alpha|=k} \frac{k!}{\alpha!} h^\alpha D^\alpha(a)$$, so $$f(x) = \sum_{k=0}^{n} \frac{t^k}{k!} \cdot \sum_{|\alpha|=k} \frac{k!}{\alpha!} \cdot h^\alpha D^\alpha(a) + \sum_{|\alpha|=n+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha$$

$$= \sum_{|\alpha|\le n} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=n+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha$$

Def. Suppose $f: \mathbb{R}^N \to \mathbb{R}$ has continuous partial derivative up to order 2 in $B_r(a)$, where a is a critical point.

Let $Q(h) = \sum_{|\alpha|=2} \frac{D^\alpha f(\alpha)}{\alpha!} h^\alpha$ be the quatic form where $h=(h_1, \cdots, h_N)$

(1) $Q(h)>0$, $\forall h$, then f has local maximum at a

proof: $f(x) = f(a) + \sum_{|\alpha|=2} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + o(\|x-a\|^2)$

Def. Two continuous functions $f, g: \mathbb{R}^N \to \mathbb{R}$ are tangent at $\vec{a}$ if $\lim_{x\to a} \frac{|f(x)-g(x)|}{\|x-a\|} = 0$

Def. Let A be open in $\mathbb{R}^N$, $f: A\to \mathbb{R}$, f is said to be differentiable if $\exists$ $L(x) = f(a) + \sum_{k=1}^{N} C_k(x_k - a_k)$ s.t. f is tangent to L

Here L is called the derivative of f at a

Prop. If f is differentiable at $a \in \mathbb{R}^N$, then

(1) The derivative is unique

(2) f is continuous at a

(3) All partial derivative exists

proof: (1) Suppose $L_1(x) = f(a) + \sum_{k=1}^{N} C_k(x_k - a_k)$, $L_2(x) = f(a) + \sum_{k=1}^{N} d_k(x_k - a_k)$

$\frac{|L_1(x)-L_2(x)|}{\|x-a\|} \leq \frac{|f(x)-L_1(x)|}{\|x-a\|} + \frac{|f(x)-L_2(x)|}{\|x-a\|} < \varepsilon$, then $\left|\sum_{k=1}^{N} (C_k - d_k)(x_k - a_k)\right| < \varepsilon\|x-a\|$

Let $x = (a_1, \cdots, x_k, \cdots, a_N)$, so $|C_k - d_k| < \varepsilon$.

Hence $C_k = d_k$ for $\forall$ $1 \leq k \leq N$, i.e. $L_1 \equiv L_2$

(2) $|f(x) - f(a)| \leq |f(x) - L(x)| + |L(x) - f(a)| \leq |f(x) - L(x)| + \sum_{k=1}^{n} |C_k(x_k - a_k)| \leq |f(x) - L(x)| + M\|x-a\|$

Here $|f(x) - L(x)| \leq \|x-a\|$ when $\|x-a\| < \eta$

Take $\delta = \min\{\eta, \frac{\varepsilon}{M+1}\}$, then $|f(x) - f(a)| < (M+1)\frac{\varepsilon}{M+1} = \varepsilon$ when $\|x-a\| < \delta$

(3) $\lim_{h\to 0} \frac{|f(a+he_k) - f(a)|}{h}$

<u>Thm.</u> (Implicit function theorem)

Suppose $D \subseteq \mathbb{R}^{N+1}$, $F: D \to \mathbb{R}$, $(x, y) = (x_1, \cdots, x_N, y)$

(1) F is continuous in D

(2) $\frac{\partial F}{\partial y}$ is continuous in D

(3) $F(\bar{x}, \bar{y}) = 0$, $\frac{\partial F}{\partial y}(\bar{x}, \bar{y}) \neq 0$

Then $\exists \alpha, \beta > 0$, s.t. in $B_\alpha(\bar{x}) \times (\bar{y} - \beta, \bar{y} + \beta)$, $\exists!\, f \in \mathcal{C}^0(B_\alpha(\bar{x}))$ s.t.

$\bar{y} = f(\bar{x})$, $F(x, y) = 0 \iff y = f(x)$

proof: Since $F_y(\bar{x}, \bar{y}) \neq 0$, w.l.o.g. $F_y(\bar{x}, \bar{y}) > 0$

Then by the continuity of F_y, $\exists\ \beta > 0$, s.t. $F_y(\bar{x}, y) > 0$ on $(\bar{y} - \beta, \bar{y} + \beta)$

Since $F(\bar{x}, \bar{y}) = 0$, then $F(\bar{x}, \bar{y} - \beta) < 0$, $F(\bar{x}, \bar{y} + \beta) > 0$

Since continuous in D, then $\exists \alpha > 0$, s.t. for $x \in B_\alpha(\bar{x})$

$F(x, \bar{y} - \beta) < 0$, $F(x, \bar{y} + \beta) > 0$.

Fix $x \in B_\alpha(\bar{x})$. set $g(y) = F(x, y)$, we see that $g \uparrow$ on $(\bar{y} - \beta, \bar{y} + \beta)$

Then $\exists!\, \check{y} \in (\bar{y} - \beta, \bar{y} + \beta)$, s.t. $F(x, \check{y}) = 0$, denote $\check{y} = f(x)$

By the uniqueness of $\check{y}$, $\exists f: B_\alpha(\bar{x}) \to (\bar{y} - \beta, \bar{y} + \beta)$ s.t. $f(x) = y$

<u>Prop.</u> If $\frac{\partial F}{\partial x_i}(\bar{x}, \bar{y})$ is continuous for $\forall\ 1 \leq i \leq N$, then $f \in C^1$

Consider $f: \mathbb{R}^N \to \mathbb{R}^M$

$F = (F_1, \cdots, F_M)$, $x = (x_1, \cdots, x_N)$, $y = (y_1, \cdots, y_M)$

$$\frac{\partial F}{\partial x} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}, & \cdots, & \frac{\partial F_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_M}{\partial x_1}, & \cdots, & \frac{\partial F_M}{\partial x_N} \end{pmatrix}_{M \times N} \qquad \frac{\partial F}{\partial y} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}, & \cdots, & \frac{\partial F_1}{\partial y_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_M}{\partial y_1}, & \cdots & \frac{\partial F_M}{\partial y_M} \end{pmatrix}_{M \times M}$$

$$D_x f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \cdots, & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1}, & \cdots, & \frac{\partial f_M}{\partial x_N} \end{pmatrix}_{M \times N} \qquad f = (f_1, \cdots, f_M)$$

<u>Thm.</u> $F: D (\subseteq \mathbb{R}^{M+N}) \longrightarrow \mathbb{R}^M$, $(x_0, y_0) \in D$

(1) F is continuous in D

(2) F has continuous partial derivative $\frac{\partial}{\partial y} F$ in D

(3) $F(x_0, y_0) = 0$, $\frac{\partial}{\partial y} F(x_0, y_0)$ is invertible

$\exists\, \alpha, \beta > 0$, s.t. in $B_\alpha(x_0) \times B_\alpha(y_0)$, $\exists!\ f (\in C^0): B_\alpha(x_0) \to B_\alpha(y_0)$

s.t. $f(x_0) = y_0$ & $F(x, y) \Longleftrightarrow y = f(x)$

Moreover if $\frac{\partial}{\partial x} F$ is continuous , then $[D_x f(x)] = [\frac{\partial}{\partial y} F]^{-1} [\partial_x F]$

<u>Integration</u> (Riemman)

Let $\mathcal{P}$ be an arbitary partition of $[a, b]$

$L(f, \mathcal{P}) := \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$, where $a = x_0 < x_1 < \cdots < x_n = b$

Define $\underline{\int_a^b} f(x)dx := \inf_{\mathcal{P}} L(f, \mathcal{P})$. $\overline{\int_a^b} f(x)dx := \sup_{\mathcal{P}} L(f, P)$

<u>Def.</u> A function is <u>Riemman</u> integrable if $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$
while $|\underline{\int_a^b} f(x)dx|$, $|\overline{\int_a^b} f(x)dx| < \infty$

The Riemman integration of f is defined as $\int_a^b f(x)dx$

<u>Fact.</u> f is Riemman integrable $\Longleftrightarrow \Delta(P) = U(f, P) - L(f, P) = 0$

From real analysis, we know that f is Riemman integrable
$\Longleftrightarrow$ f is continuous a.e.

Prop. $\{f_n\}$ are integrable on $[a,b]$. If $f_n \rightrightarrows f$ uniformly on $[a,b]$, then f is also integrable.

proof: $\forall \varepsilon>0, \exists N>0, \quad |f_n(x)-f(x)|<\varepsilon$ when $n \geq N$

Then $L(f_N, P)-(b-a)\varepsilon \leq L(f,P) \leq U(f,P) \leq U(f_N,P)+(b-a)\varepsilon$

So $\quad U(f,P)-L(f,P) \leq U(f_N,P)-L(f_N,P)+2\varepsilon(b-a)$ ✓

Prop. $\{f_n\} \to f$ pointwise, then $D(f)$ must be of the 1st category.

proof: $D(f)=\bigcup_{n=1}^{\infty} F_{\frac{1}{n}}$, $\quad F_{\frac{1}{n}} := \{x\in\mathbb{R} \mid \omega(x,f)>\frac{1}{n}\}$, fix $\varepsilon>0$

WTS. $\forall$ Ball $B\subseteq\mathbb{R}$, $\exists B_0 \subseteq B$ s.t. $B_0 \cap F^{\frac{1}{n}}=\emptyset$

① Consider $E_N=\{x \mid |f_i(x)-f_j(x)|\leq\varepsilon, \forall i,j\geq N\} = \bigcap_{i,j\geq N}\{x \mid |f_i(x)-f_j(x)|\leq\varepsilon\}$

Here each E_N is closed, $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_N \subseteq \cdots$

Since $f_n \to f$ pointwisely, then $\bigcup_{k=1}^{\infty} E_k=\mathbb{R}$

$\forall B\subseteq\mathbb{R}$, $\bar{B}=\bar{B}\cap(\bigcup_{k=1}^{\infty}E_k)=\bigcup_{k=1}^{\infty}(\bar{B}\cap E_k)$.

Since $\bar{B}$ is of 2nd category, $\exists E_N$ s.t. $\bar{B}\cap E_N$ is not nowhere dense.

$\exists \bar{B}_1 \subseteq \overline{E_N\cap\bar{B}} = E_N\cap\bar{B}$, $\quad \forall x\in B_1$, $|f_i(x)-f_j(x)|\leq\varepsilon$ for $i,j>N$

Therefore $|f(x)-f_i(x)|\leq\varepsilon$ for $i>N$, $x\in B_1$

② Fix $x_0\in B_1$, WTS. $\omega(x_0,f)<\varepsilon$, $\quad x_0\notin F_{\frac{1}{n}}$ for $\forall n\in\mathbb{N}$

Since f_N is continuous at x_0, then $|f_N(x)-f_N(x_0)|<\varepsilon$ for $x\in B_\delta(x_0)\subseteq B_1$

Then $\forall x\in B_\delta(x_0)$, $|f(x)-f(x_0)| \leq |f(x)-f_N(x)|+|f_N(x)-f_N(x_0)|+|f_N(x_0)-f(x_0)|<3\varepsilon$

So $\forall B\subseteq\mathbb{R}$, $\exists B_\delta(x_0)\subseteq B$, $\overline{B_\delta(x_0)}\cap F_{\frac{1}{n}}=\emptyset$, so $F_{\frac{1}{n}}$ is nowhere dense

Therefore $D(f)$ is of 1st category

Def. Let $n\in\mathbb{N}$

$V_n^-(S) = (\#\text{ inner cubes})*\cdot(\frac{1}{2^n})^N$

$V_n^+(S) = (\#\text{ exterior cubues})*\cdot(\frac{1}{2^n})^N$

$V^+(S)=\inf_{n\in\mathbb{N}} V_n^+(S)$, $\quad V^-(S)=\sup_{n\in\mathbb{N}} V_n^-(S)$

Def. If $V^+(S)=V^-(S)$, then we say S is a figure

Denote $V(S)=V^+(S)$.

Def. S is a figure, $\Delta:=\{S_1,\cdots,S_k\}$ is a decomposition of S where each S_i is a figure

The integral of f on a figure S is given by

$U(f,\Delta)=\sum_{i=1}^{k}\sup_{x\in S_i} f(x)V(S_i)$, $\quad L(f,\Delta)=\sum_{i=1}^{k}\inf_{x\in S_i} f(x)V(S_i)$

where $\|\Delta\|\to 0$

Def. $f: [a,b) \to \mathbb{R}$, suppose f is integrable on any $[a,c] \subset [a,b)$ with $a<c<b$, then we say f is integrable on $[a,b)$

Thm. Suppose $f \in C[a,b)$, $|f(x)| \leq g(x)$ on $[a,b)$ $\int_a^b |g(x)| dx < \infty$
then $\int_a^b f(x) dx < \infty$ and $\left|\int_a^b f(x) dx\right| \leq \int_a^b g(x) dx$
proof: Since $f \in C[a,b)$, then it is integrable on any $[a,c] \subset [a,b)$
i.e. it is integrable on $[a,b)$
1. Suppose $f(x) \geq 0$
Claim. Let $F(x) = \int_a^x f(y) dy$, then $\lim_{x \to b} F(x)$ exists
pf:) Let $\{b_n\} \to b$ be an increasing sequence on $[a,b)$
where $\exists M>0$, $|F(x)| \leq M$, so $\{F(b_n)\}$ has a convergent subsequence
$\{F(b_{n_k})\} \nearrow L$, since $F(x) \nearrow$, then $F(b_n) \nearrow L$
$\exists N \in \mathbb{N}$, s.t. $|F(b_n) - L| < \varepsilon$ for $n \geq N$
then for $x \in (b_N, b)$, $|F(x) - L| \leq |F(b_N) - L| < \varepsilon$
Hence $\lim_{x \to b} F(x)$ exists
Then clearly $\left|\int_a^b f(x) dx\right| \leq \int_a^b g(x) dx$
2. Write $f(x) = f^+(x) - f^-(x)$
Then $F^+(x)$ is increasing while $F^-(x)$ is decreasing
(Recall the real analysis, $\int_a^x f(y) dy$ is absolutely continuous)

Lebniz Rule

Suppose $T = [a,b] \times [c,d]$, $\phi(x) = \int_c^d f(x,t) dt$
If f and $\frac{\partial f}{\partial x}$ are both continuous on T, then $\phi'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t) dt$
proof: $\frac{\phi(x+h) - \phi(x)}{h} - \int_c^d \frac{\partial f}{\partial x}(x,t) dt = \int_c^d \frac{f(x+h,t) - f(x,t)}{h} - \frac{\partial f}{\partial x}(x,t) dt$
$= \int_c^d \frac{1}{h}\left(\int_x^{x+h} \frac{\partial f}{\partial x}(y,t) dy\right) - \frac{\partial f}{\partial x}(x,t) dt = \int_c^d \int_x^{x+h} \frac{1}{h} \left(\frac{\partial f}{\partial x}(y,t) - \frac{\partial f}{\partial x}(x,t)\right) dy dt$
$\leq \int_c^d \int_x^{x+h} \frac{1}{h} \varepsilon \, dy dt = (d-c)\varepsilon$
Thus $\phi'(x) = \int_c^d \frac{\partial f}{\partial x}(x,t) dt$

Arzela Ascoli Thm.

Prop.

(1) Suppose $f_n \in C'(I)$, $f_n \rightrightarrows f$, then $f \in C'(I)$

(2) Suppose $f_n \rightrightarrows f$ uniformly on I. $\{f_n\}$ is integrable then $\int_a^x f_n(y)\,dy \rightrightarrows \int_a^x f(y)\,dy$ uniformly on $[a, b]$

(3) Suppose $f_n \in C'(I)$. If $f_n \rightrightarrows f$ & $f_n' \rightrightarrows g$ uniformly, then $f \in C'(I)$ and $f' = g$

proof: (1) fix $x_0 \in [a, b]$, then $|f(x) - f(x_0)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$

(2) $|\int_a^x f(y) - f_n(y)\,dy| \le \int_a^b |f(y) - f_n(y)|\,dy \le (b-a)\varepsilon$

(3) Fix $x_0 \in [a, b]$, $f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(y)\,dy \rightrightarrows f(x_0) + \int_{x_0}^x g(y)\,dy = f(x)$

This implies that $f' = g$, $f \in C'(I)$

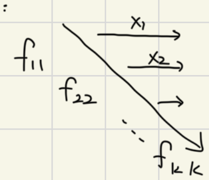
Def. Let $\mathcal{F}$ be a family of functions on I, then we say $\mathcal{F}$ is equicontinuous at $x_0 \in I$ if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall f \in \mathcal{F}$, $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$

We say $\mathcal{F}$ is equi-continuous on I if it is equicontinuous at $\forall x_0 \in I$

Def. $\mathcal{F}$ is uniformly bounded if $\exists M > 0$, s.t. $|f(x)| \le M$ for $\forall f \in \mathcal{F}$, $\forall x \in I$

Thm. (Ascoli-Arzela) $\{f_n\}$ is an equi-continuous collection of $f_n : I \to \mathbb{R}$ which is also uniformly bounded, I is seperable, then:

(1) $\exists \{f_{n_k}\} \subset \{f_n\}$, s.t. $f_{n_k} \to f$ pointwisely, $f \in C'(I)$



(2) If I is compact, then $f_{n_k} \rightrightarrows f$ uniformly

proof: (1) I seperable $\Rightarrow \exists \{x_1, x_2, \cdots, x_k, \cdots\}$ dense in I

$\{f_n(x_1)\}_{n=1}^{\infty}$ bounded. by B-W Thm, $\exists \{f_{1n}\} \subseteq \{f_n\}$, $f_{1n}(x_1)$ converges as $n \to \infty$

$\{f_{1n}(x_2)\}_{n=1}^{\infty}$ bounded, by B-W Thm, $\exists \{f_{2n}\} \subseteq \{f_{1n}\}$, $f_{2n}(x_2)$ converges as $n \to \infty$

...

$\{f_{kn}(x_{k+1})\}_{n=1}^{\infty}$ bounded, by B-W Thm, $\exists \{f_{k+1,n}\} \subseteq \{f_{kn}\}$, $f_{k+1,n}(x_{k+1})$ converges as $n \to \infty$

...

Claim. Fix k, $\{f_{nn}(x_k)\}_n^{\infty}$ converges

pf:) For $n \ge k$, $\{f_{nn}\} \subseteq \{f_{k,n}\}$, so $f_{nn}(x_k)$ converges for each fixed k

$\forall \varepsilon > 0$, $\exists \delta > 0$, $|f_{nn}(x) - f_{nn}(y)| < \varepsilon$ when $|x - y| < \delta$, Since $\{x_1, \cdots, x_k, \cdots\}$ is dense then for each fixed $x_0 \in I$, $\exists x_k$ s.t. $|x_0 - x_k| < \delta$, so $|f_{nn}(x_0) - f_{nn}(x_k)| < \varepsilon$

$|f_{nn}(x_0) - f_{mm}(x_0)| \le |f_{nn}(x_0) - f_{nn}(x_k)| + |f_{nn}(x_k) - f_{mm}(x_k)| + |f_{mm}(x_k) - f_{mm}(x_0)| < 3\varepsilon$

proof of (2)

Claim. If I is compact, $\{f_k\}$ equi-continuous on I, then $\forall \varepsilon>0, \exists \delta>0$ s.t. $|f_k(x)-f_k(y)|<\varepsilon$ whenever $|x-y|<\delta$ for $\forall k$

f_k is equi-continuous on I, so $\forall \varepsilon>0$, $x\in I$, $\exists \delta_x>0$, s.t. $|f_k(x)-f_k(y)|<\frac{\varepsilon}{2}$ when $|y-x|<\delta_x$, then $\{B_{\frac{1}{2}\delta_x}(x)\,|\,x\in I\}$ forms an open cover of I

Since I is compact, then $\exists$ a finite subcover $\{B_{\frac{1}{2}\delta_{x_j}}(x_j)\,|\,j=1,\cdots,n\}$

$\forall y\in I$, $\exists x_j$ s.t. $y\in B_{\frac{1}{2}\delta_{x_j}}(x_j)$, choose $\delta=\min\{\frac{1}{2}\delta_{x_1},\cdots,\frac{1}{2}\delta_{x_n}\}$

Then if $|y-z|<\delta$, $|z-x_j|\leq |y-z|+|y-x_j|<\delta_{x_j}$

Hence $|f_k(y)-f_k(z)|\leq |f_k(y)-f_k(x_i)|+|f_k(z)-f_k(x_i)|<\varepsilon$, $\forall k$ ✓

Now consider the open cover $\{B_\delta(x_i)\,|\,i=1,2,\cdots\}$ of I

Since I is compact, then it has a finite subcover $\{B_\delta(x_{i_\ell})\,|\,\ell=1,\cdots,n\}$

① $f_{nn}(x_k)\longrightarrow f(x_k)$, $\forall\varepsilon>0$, $\exists N_k>0$ s.t. $\forall n>N_k$, $|f_{nn}(x_k)-f(x_k)|<\frac{1}{3}\varepsilon$

② $|f_{nn}(x)-f(x)|\leq |f_{nn}(x)-f_{nn}(x_k)|+|f_{nn}(x_k)-f(x_k)|+|f(x)-f(x_k)|<\varepsilon$ ✓

Vector Valued function

· Fundamental Theorem of Calculus

$f\in C[a,b]$, then $\int_a^b f'(x)dx=F(b)-F(a)$

Remark. $\int_a^b f'(x)dx\leq f(b)-f(a)$, equation holds iff f is absolutely cont

Divergence Th'm: Given $\vec{F}: D(\subseteq\mathbb{R}^3)\longrightarrow\mathbb{R}^3$, $\vec{F}=(M,N,P)$, $M=M(x,y,z)$, so as N, P, $\operatorname{div}(\vec{F})=\nabla\cdot\vec{F}=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}$, then

$\int_D\nabla\cdot\vec{F}dV=\int_{\partial D}\vec{n}\cdot\vec{F}dS$, here $\vec{n}$ is the outer normal

proof: 证明纯硬算，不想写了，自己看 lecture note

Stokes Thm. Let $D\subseteq\mathbb{R}^N$ be a smooth domain, ω is the differentiable form on ∂D, then

$\int_{\partial D}\omega=\int_D d\omega$

For $\vec{F}=(F_1,\cdots,F_N)$, $\omega=F_1dx_1+\cdots+F_Ndx_N$

$d(f,\omega)=\sum_{i=1}^N\frac{\partial f}{\partial x_i}dx_i\wedge d\omega$, and hence $d^2\omega=0$

Green's Th'm $D\subseteq\mathbb{R}^2$, ∂D is a curve, then

$\int_{\partial D}Mdx+Ndy=\iint_D(-\frac{\partial M}{\partial y}+\frac{\partial N}{\partial x})dxdy$

By Stokes th'm, $d\omega=$

Let $u=f(x_1,x_2,\cdots,x_n)$, $\operatorname{grad} u:=(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})^T$