steps *without* the need for solving nonlinear equations. A detailed treatment of the concepts described below can be found in [10–14].

2.8 VOLTERRA SERIES

In order to understand how the Volterra series represents the time response of a system, we begin with a simple input form, $V_{in}(t) = V_0 \exp(j\omega_1 t)$. Of course, if we wish to obtain the response to a sinusoid of the form $V_0 \cos\omega_1 t = Re\{V_0 \exp(j\omega_1 t)\}$, we simply calculate the real part of the output.[26] (The use of the exponential form greatly simplifies the manipulation of the product terms.) For a linear, time-invariant system, the output is given by

$$V_{out}(t) = H(\omega_1)V_0 \exp(j\omega_1 t), \tag{2.221}$$

where $H(\omega_1)$ is the Fourier transform of the impulse response. For example, if the capacitor in Fig. 2.72 is linear, i.e., $C_1 = C_0$, then we can substitute for V_{out} and V_{in} in Eq. (2.219):

$$R_1 C_0 H(\omega_1)(j\omega_1)V_0 \exp(j\omega_1 t) + H(\omega_1)V_0 \exp(j\omega_1 t) = V_0 \exp(j\omega_1 t). \tag{2.222}$$

It follows that

$$H(\omega_1) = \frac{1}{R_1 C_0 j\omega_1 + 1}. \tag{2.223}$$

Note that the phase shift introduced by the circuit is included in $H(\omega_1)$ here.

As our next step, let us ask, how should the output response of a dynamic nonlinear system be expressed? To this end, we apply two tones to the input, $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t)$, recognizing that the output consists of both linear and nonlinear responses. The former are of the form

$$V_{out1}(t) = H(\omega_1)V_0 \exp(j\omega_1 t) + H(\omega_2)V_0 \exp(j\omega_2 t), \tag{2.224}$$

and the latter include exponentials such as $\exp[j(\omega_1 + \omega_2)t]$, etc. We expect that the coefficient of such an exponential is a function of both ω_1 and ω_2. We thus make a slight change in our notation: we denote $H(\omega_j)$ in Eq. (2.224) by $H_1(\omega_j)$ [to indicate first-order (linear) terms] and the coefficient of $\exp[j(\omega_1 + \omega_2)t]$ by $H_2(\omega_1, \omega_2)$. In other words, the overall output can be written as

$$\begin{aligned} V_{out}(t) = {} & H_1(\omega_1)V_0 \exp(j\omega_1 t) + H_1(\omega_2)V_0 \exp(j\omega_2 t) \\ & + H_2(\omega_1, \omega_2)V_0^2 \exp[j(\omega_1 + \omega_2)t] + \cdots . \end{aligned} \tag{2.225}$$

How do we determine the terms at $2\omega_1$, $2\omega_2$, and $\omega_1 - \omega_2$? If $H_2(\omega_1, \omega_2)\exp[j(\omega_1 + \omega_2)t]$ represents the component at $\omega_1 + \omega_2$, then $H_2(\omega_1, \omega_1)\exp[j(2\omega_1)t]$ must model

26. From another point of view, in $V_0 \exp(j\omega_1 t) = V_0 \cos\omega_1 t + jV_0 \sin\omega_1 t$, the first term generates its own response, as does the second term; the two responses remain distinguishable by virtue of the factor j.

that at $2\omega_1$. Similarly, $H_2(\omega_2, \omega_2)$ and $H_2(\omega_1, -\omega_2)$ serve as coefficients for $\exp[j(2\omega_2)t]$ and $\exp[j(\omega_1 - \omega_2)t]$, respectively. In other words, a more complete form of Eq. (2.225) reads

$$\begin{aligned} V_{out}(t) = {} & H_1(\omega_1)V_0\exp(j\omega_1 t) + H_1(\omega_2)V_0\exp(j\omega_2 t) + H_2(\omega_1, \omega_1)V_0^2\exp(2j\omega_1 t) \\ & + H_2(\omega_2, \omega_2)V_0^2\exp(2j\omega_2 t) + H_2(\omega_1, \omega_2)V_0^2\exp[j(\omega_1 + \omega_2)t] \\ & + H_2(\omega_1, -\omega_2)V_0^2\exp[j(\omega_1 - \omega_2)t] + \cdots . \end{aligned} \tag{2.226}$$

Thus, our task is simply to compute $H_2(\omega_1, \omega_2)$.

Example 2.31

Determine $H_2(\omega_1, \omega_2)$ for the circuit of Fig. 2.72.

Solution:

We apply the input $V_{in}(t) = V_0\exp(j\omega_1 t) + V_0\exp(j\omega_2 t)$ and assume the output is of the form $V_{out}(t) = H_1(\omega_1)V_0\exp(j\omega_1 t) + H_1(\omega_2)V_0\exp(j\omega_2 t) + H_2(\omega_1, \omega_2)V_0^2\exp[j(\omega_1 + \omega_2)t]$. We substitute for V_{out} and V_{in} in Eq. (2.219):

$$\begin{aligned} & R_1C_0[1 + \alpha H_1(\omega_1)V_0e^{j\omega_1 t} + \alpha H_1(\omega_2)V_0e^{j\omega_2 t} + \alpha H_2(\omega_1, \omega_2)V_0^2e^{j(\omega_1+\omega_2)t}] \\ & \quad \times [H_1(\omega_1)j\omega_1V_0e^{j\omega_1 t} + H_1(\omega_2)j\omega_2V_0e^{j\omega_2 t} + H_2(\omega_1, \omega_2)j(\omega_1 + \omega_2) \\ & \quad \times V_0^2e^{j(\omega_1+\omega_2)t}] + H_1(\omega_1)e^{j\omega_1 t} + H_1(\omega_2)e^{j\omega_2 t} + H_2(\omega_1, \omega_2)V_0^2e^{j(\omega_1+\omega_2)t} \\ & \quad = V_0e^{j\omega_1 t} + V_0e^{j\omega_2 t}. \end{aligned} \tag{2.227}$$

To obtain H_2, we only consider the terms containing $\omega_1 + \omega_2$:

$$\begin{aligned} & R_1C_0[\alpha H_1(\omega_1)H_1(\omega_2)j\omega_1V_0^2e^{j(\omega_1+\omega_2)t} + \alpha H_1(\omega_2)H_1(\omega_1)j\omega_2V_0^2e^{j(\omega_1+\omega_2)t} \\ & \quad + H_2(\omega_1, \omega_2)j(\omega_1 + \omega_2)V_0^2e^{j(\omega_1+\omega_2)t}] + H_2(\omega_1, \omega_2) \\ & \quad \times V_0^2e^{j(\omega_1+\omega_2)t} = 0 \end{aligned} \tag{2.228}$$

That is,

$$H_2(\omega_1, \omega_2) = -\frac{\alpha R_1C_0 j(\omega_1 + \omega_2)H_1(\omega_1)H_1(\omega_2)}{R_1C_0 j(\omega_1 + \omega_2) + 1}. \tag{2.229}$$

Noting that the denominator resembles that of (2.223) but with ω_1 replaced by $\omega_1 + \omega_2$, we simplify $H_2(\omega_1, \omega_2)$ to

$$H_2(\omega_1, \omega_2) = -\alpha R_1C_0 j(\omega_1 + \omega_2)H_1(\omega_1)H_1(\omega_2)H_1(\omega_1 + \omega_2). \tag{2.230}$$

Why did we assume $V_{out}(t) = H_1(\omega_1)V_0\exp(j\omega_1 t) + H_1(\omega_2)V_0\exp(j\omega_2 t) + H_2V_0^2(\omega_1, \omega_2)\exp[j(\omega_1 + \omega_2)t]$ while we know that $V_{out}(t)$ also contains terms at $2\omega_1$, $2\omega_2$, and $\omega_1 - \omega_2$? This is because these other exponentials do not yield terms of the form $\exp[j(\omega_1 + \omega_2)t]$.

Example 2.32

If an input $V_0 \exp(j\omega_1 t)$ is applied to the circuit of Fig. 2.72, determine the amplitude of the second harmonic at the output.

Solution:

As mentioned earlier, the component at $2\omega_1$ is obtained as $H_2(\omega_1, \omega_1)V_0^2 \exp[j(\omega_1 + \omega_1)t]$. Thus, the amplitude is equal to

$$|A_{2\omega 1}| = |\alpha R_1 C_0 (2\omega_1) H_1^2(\omega_1) H_1(2\omega_1)| V_0^2 \tag{2.231}$$

$$= \frac{2|\alpha| R_1 C_0 \omega_1 V_0^2}{(R_1^2 C_0^2 \omega_1^2 + 1)\sqrt{4R_1^2 C_0^2 \omega_1^2 + 1}}. \tag{2.232}$$

We observe that $A_{2\omega 1}$ falls to zero as ω_1 approaches zero because C_1 draws little current, and also as ω_1 goes to infinity because the second harmonic is suppressed by the low-pass nature of the circuit.

Example 2.33

If two tones of equal amplitude are applied to the circuit of Fig. 2.72, determine the ratio of the amplitudes of the components at $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$. Recall that $H_1(\omega) = (R_1 C_0 j\omega + 1)^{-1}$.

Solution:

From Eq. (2.230), the ratio is given by

$$\left|\frac{A_{\omega 1+\omega 2}}{A_{\omega 1-\omega 2}}\right| = \left|\frac{H_2(\omega_1, \omega_2)}{H_2(\omega_1, -\omega_2)}\right| \tag{2.233}$$

$$= \left|\frac{(\omega_1 + \omega_2) H_1(\omega_2) H_1(\omega_1 + \omega_2)}{(\omega_1 - \omega_2) H_1(-\omega_2) H_1(\omega_1 - \omega_2)}\right|. \tag{2.234}$$

Since $|H_1(\omega_2)| = |H_1(-\omega_2)|$, we have

$$\left|\frac{A_{\omega 1+\omega 2}}{A_{\omega 1-\omega 2}}\right| = \frac{(\omega_1 + \omega_2)\sqrt{R_1^2 C_0^2 (\omega_1 - \omega_2)^2 + 1}}{|\omega_1 - \omega_2|\sqrt{R_1^2 C_0^2 (\omega_1 + \omega_2)^2 + 1}}. \tag{2.235}$$

The foregoing examples point to a methodical approach that allows us to compute the second harmonic or second-order IM components with a moderate amount of algebra. But how about higher-order harmonics or IM products? We surmise that for Nth-order terms, we must apply the input $V_{in}(t) = V_0 \exp(j\omega_1 t) + \cdots + V_0 \exp(j\omega_N t)$ and compute $H_n(\omega_1, \ldots, \omega_n)$ as the coefficient of the $\exp[j(\omega_1 + \cdots + \omega_n)t]$ terms in the output. The output can therefore be expressed as

$$V_{out}(t) = \sum_{k=1}^{N} H_1(\omega_k)V_0 \exp(j\omega_k t) + \sum_{m=1}^{N}\sum_{k=1}^{N} H_2(\omega_m, \pm\omega_k)V_0^2 \exp[j(\omega_m \pm \omega_k)t]$$
$$+ \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{k=1}^{N} H_3(\omega_n, \pm\omega_m, \pm\omega_k)V_0^3 \exp[j(\omega_n \pm \omega_m \pm \omega_k)t] + \cdots . \quad (2.236)$$

The above representation of the output is called the Volterra series. As exemplified by (2.230), $H_m(\omega_1, \ldots, \omega_m)$ can be computed in terms of $H_1, \ldots, H_{m-1}$ with no need to solve nonlinear equations. We call H_m the m-th "Volterra kernel."

Example 2.34

Determine the third Volterra kernel for the circuit of Fig. 2.72.

Solution:

We assume $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t) + V_0 \exp(j\omega_3 t)$. Since the output contains many components, we introduce the short hands $H_{1(1)} = H_1(\omega_1)V_0 \exp(j\omega_1 t)$, $H_{1(2)} = H_1(\omega_2)V_0 \exp(j\omega_2 t)$, etc., $H_{2(1,2)} = H_2(\omega_1, \omega_2)V_0^2 \exp[j(\omega_1 + \omega_2)t]$, etc., and $H_{3(1,2,3)} = H_3(\omega_1, \omega_2, \omega_3)V_0^3 \exp[j(\omega_1 + \omega_2 + \omega_3)t]$. We express the output as

$$V_{out}(t) = H_{1(1)} + H_{1(2)} + H_{1(3)} + H_{2(1,2)} + H_{2(1,3)} + H_{2(2,3)} + H_{2(1,1)}$$
$$+ H_{2(2,2)} + H_{2(3,3)} + H_{3(1,2,3)} + \cdots . \quad (2.237)$$

We must substitute for V_{out} and V_{in} in Eq. (2.219) and group all of the terms that contain $\omega_1 + \omega_2 + \omega_3$. To obtain such terms in the product of αV_{out} and dV_{out}/dt, we note that $\alpha H_{2(1,2)} j\omega_3 H_{1(3)}$ and $\alpha H_{1(3)} j(\omega_1 + \omega_2)H_{2(1,2)}$ produce an exponential of the form $\exp[j(\omega_1 + \omega_2)t]\exp(j\omega_3)$. Similarly, $\alpha H_{2(2,3)} j\omega_1 H_{1(1)}$, $\alpha H_{1(1)} j(\omega_2 + \omega_3)H_{2(2,3)}$, $\alpha H_{2(1,3)} j\omega_2 H_{1(2)}$, and $\alpha H_{1(2)} j(\omega_1 + \omega_3)H_{2(1,3)}$ result in $\omega_1 + \omega_2 + \omega_3$. Finally, the product of αV_{out} and dV_{out}/dt also contains $1 \times j(\omega_1 + \omega_2 + \omega_3)H_{3(1,2,3)}$. Grouping all of the terms, we have

$$H_3(\omega_1, \omega_2, \omega_3)$$
$$= -j\alpha R_1 C_0 \frac{H_2(\omega_1, \omega_2)\omega_3 H_1(\omega_3) + H_2(\omega_2, \omega_3)\omega_1 H_1(\omega_1) + H_2(\omega_1, \omega_3)\omega_2 H_1(\omega_2)}{R_1 C_0 j(\omega_1 + \omega_2 + \omega_3) + 1}$$
$$- j\alpha R_1 C_0 \frac{H_1(\omega_1)(\omega_2 + \omega_3)H_2(\omega_2, \omega_3) + H_1(\omega_2)(\omega_1 + \omega_3)H_2(\omega_1, \omega_3)}{R_1 C_0 j(\omega_1 + \omega_2 + \omega_3) + 1}$$
$$- j\alpha R_1 C_0 \frac{H_1(\omega_3)(\omega_1 + \omega_2)H_2(\omega_1, \omega_2)}{R_1 C_0 j(\omega_1 + \omega_2 + \omega_3) + 1}. \quad (2.238)$$

Note that $H_{2(1,1)}$, etc., do not appear here and could have been omitted from Eq. (2.237). With the third Volterra kernel available, we can compute the amplitude of critical terms. For example, the third-order IM components in a two-tone test are obtained by substituting ω_1 for ω_3 and $-\omega_2$ for ω_2.

The reader may wonder if the Volterra series can be used with inputs other than exponentials. This is indeed possible [14] but beyond the scope of this book.

The approach described in this section is called the "harmonic" method of kernel calculation. In summary, this method proceeds as follows:

1. Assume $V_{in}(t) = V_0 \exp(j\omega_1 t)$ and $V_{out}(t) = H_1(\omega_1)V_0 \exp(j\omega_1 t)$. Substitute for V_{out} and V_{in} in the system's differential equation, group the terms that contain $\exp(j\omega_1 t)$, and compute the first (linear) kernel, $H_1(\omega_1)$.
2. Assume $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t)$ and $V_{out}(t) = H_1(\omega_1)V_0 \exp(j\omega_1 t) + H_1(\omega_2)V_0 \exp(j\omega_2 t) + H_2(\omega_1, \omega_2)V_0^2 \exp[j(\omega_1 + \omega_2)t]$. Make substitutions in the differential equation, group the terms that contain $\exp[j(\omega_1 + \omega_2)t]$, and determine the second kernel, $H_2(\omega_1, \omega_2)$.
3. Assume $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t) + V_0 \exp(j\omega_3 t)$ and $V_{out}(t)$ is given by Eq. (2.237). Make substitutions, group the terms that contain $\exp[j(\omega_1 + \omega_2 + \omega_3)t]$, and calculate the third kernel, $H_3(\omega_1, \omega_2, \omega_3)$.
4. To compute the amplitude of harmonics and IM components, choose $\omega_1, \omega_2, \ldots$ properly. For example, $H_2(\omega_1, \omega_1)$ yields the transfer function for $2\omega_1$ and $H_3(\omega_1, -\omega_2, \omega_1)$ the transfer function for $2\omega_1 - \omega_2$.

2.8.1 Method of Nonlinear Currents

As seen in Example 2.34, the harmonic method becomes rapidly more complex as n increases. An alternative approach called the method of "nonlinear currents" is sometimes preferred as it reduces the algebra to some extent. We describe the method itself here and refer the reader to [13] for a formal proof of its validity.

The method of nonlinear currents proceeds as follows for a circuit that contains a two-terminal nonlinear device [13]:

1. Assume $V_{in}(t) = V_0 \exp(j\omega_1 t)$ and determine the linear response of the circuit by ignoring the nonlinearity. The "response" includes both the output of interest *and* the voltage across the nonlinear device.
2. Assume $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t)$ and calculate the voltage across the nonlinear device, assuming it is linear. Now, compute the *nonlinear* component of the current flowing through the device, assuming the device is nonlinear.
3. Set the main input to *zero* and place a current source equal to the nonlinear component found in Step 2 in parallel with the nonlinear device.
4. Ignoring the nonlinearity of the device again, determine the circuit's response to the current source applied in Step 3. Again, the response includes the output of interest and the voltage across the nonlinear device.
5. Repeat Steps 2, 3, and 4 for higher-order responses. The overall response is equal to the output components found in Steps 1, 4, etc.

The following example illustrates the procedure.

Example 2.35

Determine $H_3(\omega_1, \omega_2, \omega_3)$ for the circuit of Fig. 2.72.

Solution:

In this case, the output voltage also appears across the nonlinear device. We know that $H_1(\omega_1) = (R_1 C_0 j\omega_1 + 1)^{-1}$. Thus, with $V_{in}(t) = V_0 \exp(j\omega_1 t)$, the voltage across the capacitor is equal to

$$V_{C1}(t) = \frac{V_0}{R_1 C_0 j\omega_1 + 1} e^{j\omega_1 t}. \tag{2.239}$$

In the second step, we apply $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t)$, obtaining the linear voltage across C_1 as

$$V_{C1}(t) = \frac{V_0 e^{j\omega_1 t}}{R_1 C_0 j\omega_1 + 1} + \frac{V_0 e^{j\omega_2 t}}{R_1 C_0 j\omega_2 + 1}. \tag{2.240}$$

With this voltage, we compute the nonlinear current flowing through C_1:

$$I_{C1,non}(t) = \alpha C_0 V_{C1} \frac{dV_{C1}}{dt} \tag{2.241}$$

$$= \alpha C_0 \left(\frac{V_0 e^{j\omega_1 t}}{R_1 C_0 j\omega_1 + 1} + \frac{V_0 e^{j\omega_2 t}}{R_1 C_0 j\omega_2 + 1} \right) \times \left(\frac{j\omega_1 V_0 e^{j\omega_1 t}}{R_1 C_0 j\omega_1 + 1} + \frac{j\omega_2 V_0 e^{j\omega_2 t}}{R_1 C_0 j\omega_2 + 1} \right). \tag{2.242}$$

Since only the component at $\omega_1 + \omega_2$ is of interest at this point, we rewrite the above expression as

$$I_{C1,non}(t) = \alpha C_0 \left[\frac{j(\omega_1 + \omega_2) V_0^2 e^{j(\omega_1 + \omega_2)t}}{(R_1 C_0 j\omega_1 + 1)(R_1 C_0 j\omega_2 + 1)} + \cdots \right] \tag{2.243}$$

$$= \alpha C_0 [j(\omega_1 + \omega_2) V_0^2 e^{j(\omega_1 + \omega_2)t} H_1(\omega_1) H_1(\omega_2) + \cdots]. \tag{2.244}$$

In the third step, we set the input to zero, assume a linear capacitor, and apply $I_{C1,non}(t)$ in parallel with C_1 (Fig. 2.73). The current component at $\omega_1 + \omega_2$ flows through the parallel combination of R_1 and C_0, producing $V_{C1,non}(t)$:

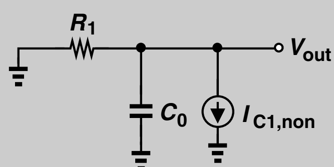

Figure 2.73 *Inclusion of nonlinear current in RC section.*

Example 2.35 (*Continued*)

$$V_{C1,non}(t) = -\alpha C_0 j(\omega_1 + \omega_2)V_0^2 e^{j(\omega_1+\omega_2)t} H_1(\omega_1) \times H_1(\omega_2)\frac{R_1}{R_1 C_0 j(\omega_1 + \omega_2) + 1} \tag{2.245}$$

$$= -\alpha R_1 C_0 j(\omega_1 + \omega_2) H_1(\omega_1) H_1(\omega_2) H_1(\omega_1 + \omega_2) V_0^2 e^{j(\omega_1+\omega_2)t}. \tag{2.246}$$

We note that the coefficient of $V_0^2 \exp[j(\omega_1 + \omega_2)t]$ in these two equations is the same as $H_2(\omega_1, \omega_2)$ in (2.229).

To determine $H_3(\omega_1, \omega_2, \omega_3)$, we must assume an input of the form $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t) + V_0 \exp(j\omega_3 t)$ and write the voltage across C_1 as

$$V_{C1}(t) = H_1(\omega_1)V_0 e^{j\omega_1 t} + H_1(\omega_2)V_0 e^{j\omega_2 t} + H_1(\omega_3)V_0 e^{j\omega_3 t} + H_2(\omega_1, \omega_2)V_0^2 e^{j(\omega_1+\omega_2)t} + H_2(\omega_1, \omega_3)V_0^2 e^{j(\omega_1+\omega_3)t} + H_2(\omega_2, \omega_3)V_0^2 e^{j(\omega_2+\omega_3)t}. \tag{2.247}$$

Note that, in contrast to Eq. (2.240), we have included the second-order nonlinear terms in the voltage so as to calculate the third-order terms.[27] The nonlinear current through C_1 is thus equal to

$$I_{C1,non}(t) = \alpha C_0 V_{C1} \frac{dV_{C1}}{dt}. \tag{2.248}$$

We substitute for V_{C1} and group the terms containing $\omega_1 + \omega_2 + \omega_3$:

$$\begin{aligned} I_{C1,non}(t) = \alpha C_0[&H_1(\omega_1)H_2(\omega_2, \omega_3)j(\omega_2 + \omega_3) + H_2(\omega_2, \omega_3)j\omega_1 H_1(\omega_1) \\ &+ H_1(\omega_2)H_2(\omega_1, \omega_3)j(\omega_1 + \omega_3) + H_2(\omega_1, \omega_3)j\omega_2 H_1(\omega_2) \\ &+ H_1(\omega_3)H_2(\omega_1, \omega_2)j(\omega_1 + \omega_2) + H_2(\omega_1, \omega_2)j\omega_3 H_1(\omega_3)]V_0^3 e^{j(\omega_1+\omega_2+\omega_3)t} \\ &+ \cdots . \end{aligned} \tag{2.249}$$

This current flows through the parallel combination of R_1 and C_0, yielding $V_{C1,non}(t)$. The reader can readily show that the coefficient of $\exp[j(\omega_1 + \omega_2 + \omega_3)t]$ in $V_{C1,non}(t)$ is the same as the third kernel expressed by Eq. (2.238).

The procedure described above applies to two-terminal nonlinear devices. For transistors, a similar approach can be taken. We illustrate this point with the aid of an example.

Example 2.36

Figure 2.74(a) shows the input network of a commonly-used LNA (Chapter 5). Assuming that $g_m L_1/C_{GS} = R_S$ (Chapter 5) and $I_D = \alpha(V_{GS} - V_{TH})^2$, determine the nonlinear terms in I_{out}. Neglect other capacitances, channel-length modulation, and body effect.

(*Continues*)

27. Other terms are excluded because they do not lead to a component at $\omega_1 + \omega_2 + \omega_3$.

Example 2.36 (*Continued*)

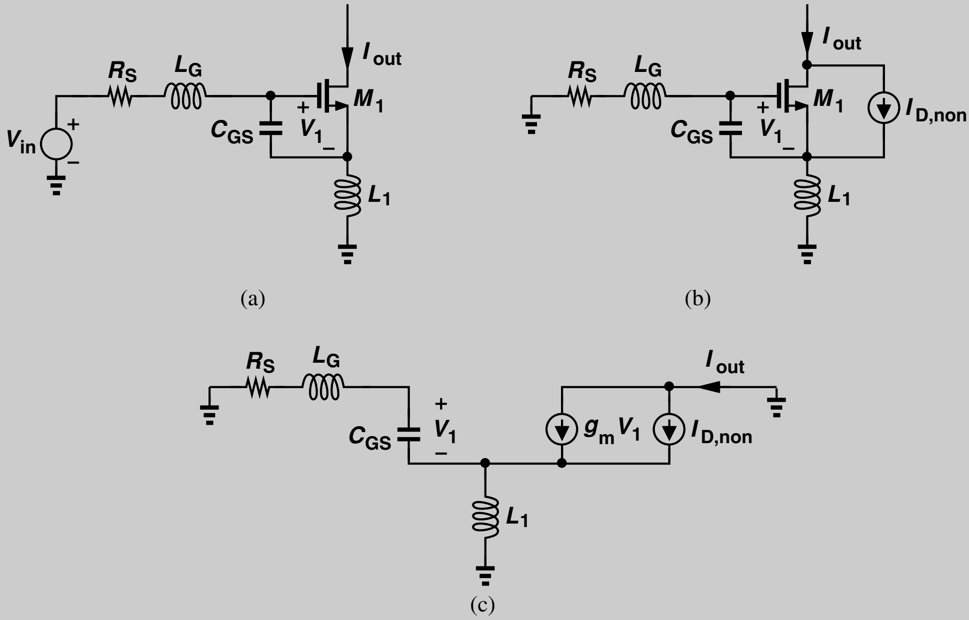

Figure 2.74 *(a) CS stage with inductors in series with source and gate, (b) inclusion of nonlinear current, (c) computation of output current.*

Solution:

In this circuit, two quantities are of interest, namely, the output current, I_{out} ($= I_D$), and the gate-source voltage, V_1; the latter must be computed each time as it determines the nonlinear component in I_D.

Let us begin with the linear response. Since the current flowing through L_1 is equal to $V_1 C_{GS} s + g_m V_1$ and that flowing through R_S and L_G equal to $V_1 C_{GS} s$, we can write a KVL around the input loop as

$$V_{in} = (R_S + L_G s) V_1 C_{GS} s + V_1 + (V_1 C_{GS} s + g_m V_1) L_1 s. \tag{2.250}$$

It follows that

$$\frac{V_1}{V_{in}} = \frac{1}{(L_1 + L_G) C_{GS} s^2 + (R_S C_{GS} + g_m L_1) s + 1}. \tag{2.251}$$

Since we have assumed $g_m L_1 / C_{GS} = R_S$, for $s = j\omega$ we obtain

$$\frac{V_1}{V_{in}}(j\omega) = \frac{1}{2 g_m L_1 j\omega + 1 - \dfrac{\omega^2}{\omega_0^2}} = H_1(\omega), \tag{2.252}$$

where $\omega_0^2 = [(L_1 + L_G) C_{GS}]^{-1}$. Note that $I_{out} = g_m V_1 = g_m H_1(\omega) V_{in}$.

Example 2.36 (*Continued*)

Now, we assume $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t)$ and write

$$V_1(t) = H_1(\omega_1)V_0 e^{j\omega_1 t} + H_1(\omega_2)V_0 e^{j\omega_2 t}. \tag{2.253}$$

Upon experiencing the characteristic $I_D = \alpha V_1^2$, this voltage results in a nonlinear current given by

$$I_{D,non} = 2\alpha H_1(\omega_1)H_1(\omega_2)V_0^2 e^{j(\omega_1+\omega_2)t}. \tag{2.254}$$

In the next step, we set V_{in} to zero and insert a current source having the above value in parallel with the drain current source [Fig. 2.74(b)]. We must compute V_1 in response to $I_{D,non}$, assuming the circuit is *linear*. From the equivalent circuit shown in Fig. 2.74(c), we have the following KVL:

$$(R_S + L_G s)V_1 C_{GS} s + V_1 + (g_m V_1 + I_{D,non} + V_1 C_{GS} s)L_1 s = 0. \tag{2.255}$$

Thus, for $s = j\omega$

$$\frac{V_1}{I_{D,non}}(j\omega) = \frac{-jL_1\omega}{2g_m L_1 j\omega + 1 - \dfrac{\omega^2}{\omega_0^2}}. \tag{2.256}$$

Since $I_{D,non}$ contains a frequency component at $\omega_1 + \omega_2$, the above transfer function must be calculated at $\omega_1 + \omega_2$ and multiplied by $I_{D,non}$ to yield V_1. We therefore have

$$H_2(\omega_1, \omega_2) = \frac{-jL_1(\omega_1 + \omega_2)}{2g_m L_1 j(\omega_1 + \omega_2) + 1 - \dfrac{(\omega_1+\omega_2)^2}{\omega_0^2}} 2\alpha H_1(\omega_1)H_1(\omega_2). \tag{2.257}$$

In our last step, we assume $V_{in}(t) = V_0 \exp(j\omega_1 t) + V_0 \exp(j\omega_2 t) + V_0 \exp(j\omega_3 t)$ and write

$$\begin{aligned} V_1(t) = {} & H_1(\omega_1)V_0 e^{j\omega_1 t} + H_1(\omega_2)V_0 e^{j\omega_2 t} + H_1(\omega_3)V_0 e^{j\omega_3 t} + H_2(\omega_1, \omega_2)V_0^2 e^{j(\omega_1+\omega_2)t} \\ & + H_2(\omega_1, \omega_3)V_0^2 e^{j(\omega_1+\omega_3)t} + H_2(\omega_2, \omega_3)V_0^2 e^{j(\omega_2+\omega_3)t}. \end{aligned} \tag{2.258}$$

Since $I_D = \alpha V_1^2$, the nonlinear current at $\omega_1 + \omega_2 + \omega_3$ is expressed as

$$\begin{aligned} I_{D,non} = {} & 2\alpha[H_1(\omega_1)H_2(\omega_2, \omega_3) + H_1(\omega_2)H_2(\omega_1, \omega_3) \\ & + H_1(\omega_3)H_2(\omega_1, \omega_2)]V_0^3 e^{j(\omega_1+\omega_2+\omega_3)t}. \end{aligned} \tag{2.259}$$

The third-order nonlinear component in the output of interest, I_{out}, is equal to the above expression. We note that, even though the transistor exhibits only second-order nonlinearity, the degeneration (feedback) caused by L_1 results in higher-order terms.

The reader is encouraged to repeat this analysis using the harmonic method and see that it is much more complex.