Theory of pth-Order Inverses of Nonlinear Systems

MARTIN SCHETZEN, MEMBER, IEEE

***Abstract*—The concept and theory of the pth-order inverse of a nonlinear system is developed in this article. The pth-order inverse, $K_{(p)}^{-1}$, of a system H is defined as a system for which the Volterra series of the system Q formed by the tandem connection of $K_{(p)}^{-1}$ and H is**

$$Q[x]=x+\sum_{n=p+1}^{\infty} Q_n[x]$$

so that the 2nd through the pth-order Volterra operators of Q are zero. The necessary and sufficient conditions for the existence of $K_{(p)}^{-1}$ are determined. It is shown that the pth-order pre-inverse of a system H is identical to its pth-order post-inverse. In addition, a synthesis of $K_{(p)}^{-1}$ is obtained.

The pth-order inverse offers an approach to the study of the system inverse, H^{-1}, since, as $p\to\infty$, $K_{(p)}^{-1}$ becomes the Volterra series of H^{-1}. This approach is discussed and some applications with regard to nonlinear differential equations and nonlinear feedback systems are presented.

I. INTRODUCTION

THE RELATION between the output, $y(t)$, and the input, $x(t)$, of a large class of physical nonlinear systems can be expressed in the Volterra series

$$y(t)=H[x(t)]=\sum_{n=1}^{\infty} H_n[x(t)] \tag{1}$$

in which

$$H_n[x(t)]=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)x(t-\tau_1)\cdots x(t-\tau_n)\,d\tau_1\cdots d\tau_n \tag{2}$$

and

$$h_n(\tau_1,\cdots,\tau_n)=0, \qquad \text{for any } \tau_i<0, \qquad i=1,2,\cdots,n. \tag{3}$$

In this form of representation, H is called the system operator, H_n is called the nth-order Volterra operator, and h_n are the Volterra kernels. A discussion of the Volterra series and its use in the analysis of nonlinear systems has been given by Barrett [1], George [2], and others. We shall restrict our discussion in this article to bounded input, bounded output (BIBO) stable systems for which the system operator, H, can be represented in the Volterra series, (1). In essence, the Volterra series is a power series with memory. It is a power series since if the input is changed by a gain factor c so that the new input is $cx(t)$, then the new output is

$$y(t)=\sum_{n=1}^{\infty} c^n H_n[x(t)] \tag{4}$$

which is a power series in the amplitude c. It is a series with memory since the integrals for $H_n[x(t)]$ are convolutions.

A basic problem of practical importance is the determination of the inverse of a given system, H. Not all nonlinear systems, H, possess an inverse. Also, many systems, H, possess an inverse only for a restricted range of the input amplitude. Even so, we shall see that in such cases, there exists a pth-order inverse in which the input amplitude range is not restricted. We define a pth-order inverse of a given nonlinear system, H, as one, when connected in tandem with H, results in a system in which the second through the pth-order Volterra kernels are zero. Thus, if Q is the system operator of the two systems connected in tandem, then

$$Q[x(t)]=x(t)+\sum_{n=p+1}^{\infty} Q_n[x(t)] \tag{5}$$

in which Q_n is the nth-order Volterra operator of the system Q.

In this article, we shall obtain the necessary and sufficient conditions for the existence of the pth-order inverse of a nonlinear system H represented in the form of (1). We also shall present a method for the synthesis of the pth-order inverse. To explain the theory, the pth-order post-inverse will be discussed first. Then it will be shown that the pth-order post-inverse is also a pth-order pre-inverse so that it is indeed a pth-order inverse.

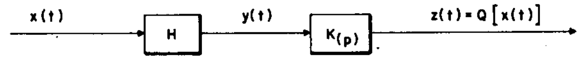

Fig. 1. System Q formed by tandem connection of $K_{(p)}$ and H.

II. TANDEM CONNECTION OF NONLINEAR SYSTEMS

Before presenting the theory of the pth-order post-inverse of a system, we shall derive the Volterra series of two nonlinear systems connected in tandem as shown in Fig. 1. In our analysis, we shall only consider the case in which the two systems do not interact so that there are no loading effects of one system on the other. The relation between the input, $x(t)$, and response, $y(t)$, of the system

Manuscript received January 29, 1975; revised July 24, 1975, and December 21, 1975.
The author is with the Department of Electrical Engineering, Northeastern University, Boston, MA.

H in the figure is given by (1). The system $K_{(p)}$ is a pth-order system with the response $z(t)$ for the input $y(t)$ so that

$$z(t)=K_{(p)}[y(t)]=\sum_{n=1}^{p} K_n[y(t)] \tag{6}$$

in which K_n is an nth-order Volterra operator with the kernel $k_n(\tau_1,\cdots,\tau_n)$. The system Q formed by the tandem connection of the system H and $K_{(p)}$ then can be characterized by the Volterra series

$$z(t)=Q[x(t)]=\sum_{n=1}^{\infty} Q_n[x(t)] \tag{7}$$

in which Q_n is an nth-order Volterra operator with the kernel $q_n(\tau_1,\cdots,\tau_n)$. We shall determine the operators Q_n in terms of the operators H_n and K_n in this section. In the next section, the operators K_n will be determined so that (5) is satisfied. The system $K_{(p)}$ so determined is the pth-order post-inverse of the system H and will be denoted by $K_{(p)}^{-1}$.

To determine the Volterra operators, Q_n, in terms of H_n and K_n, we first note from our previous discussion that if $x(t)$ is replaced by $cx(t)$ in (7), then

$$z(t)=Q[cx(t)]=\sum_{n=1}^{\infty} c^n Q_n[x(t)]. \tag{8}$$

Equation (8) is a power series in c with the coefficient of c^n being $Q_n[x(t)]$. We now shall obtain another expression for $z(t)$ by replacing $x(t)$ by $cx(t)$ and substituting (4) in (6). This will result in another power series in c. Since both power series are equal, the Volterra operators Q_n will be determined by equating like powers of c in both expressions. Thus by substituting (4) in (6) there results

$$z=\sum_{m=1}^{p} K_m\left[\sum_{n=1}^{\infty} c^n y_n\right] \tag{9}$$

in which, for convenience, we have dropped the argument t and we have defined

$$y_n(t)=H_n[x(t)]. \tag{10}$$

Now, from the integral representation of the operator K_m, it is seen that

$$K_m\left[\sum_{n=1}^{\infty} c^n y_n\right]=\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty} c^{n_1}\cdots c^{n_m} K_m\{y_{n_1},\cdots,y_{n_m}\} \tag{11}$$

in which

$$K_m\{y_{n_1},\cdots,y_{n_m}\}=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} k_m(\tau_1,\cdots,\tau_m)y_{n_1}(t-\tau_1)\cdots y_{n_m}(t-\tau_m)\,d\tau_1\cdots d\tau_m. \tag{12}$$

By substituting (11) in (9) the desired second power series in c is

$$z=\sum_{m=1}^{p}\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty} c^{n_1+\cdots+n_m}K_m\{y_{n_1},\cdots,y_{n_m}\}. \tag{13}$$

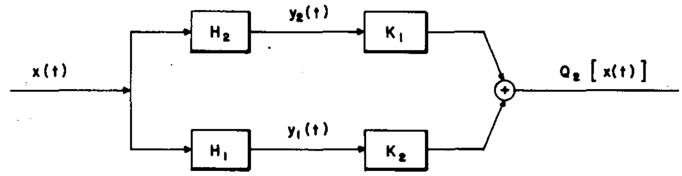

Fig. 2. Block diagram of operator Q_2.

The Volterra operators Q_n now can be obtained in terms of H_n and K_n by equating like power of c in (8) and (13). The block diagrams of the operators Q_n will be determined instead of the algebraic expressions for their kernels, q_n. The reason is that the block diagrams provide a clear view of the specific operator properties of interest which are obscure when viewing the algebraic expressions of the kernels, q_n. Of course, the expressions for the kernels q_n can be derived easily from the block diagrams.

To do this, we first observe that, in (13) the smallest value of the sum $n_1+n_2+\cdots+n_m$ is m. The first power of c in (13) thus can only come from the term with $m=1$ and $n_1=1$ for which the coefficient is $K_1[y_1]$. Consequently, equating the coefficients of the first power of c in (8) and (13) we have

$$Q_1[x]=K_1[y_1]=K_1H_1[x]. \tag{14}$$

Since, from (10), $y_1=H_1[x]$, a block diagram of the operator Q_1 is the tandem connection of the operators K_1 and H_1.

We now equate the coefficients of c^2 in (8) and (13) to determine the operator Q_2. In (13), we observe that c^2 is obtained for $m=1$ with $n_1=2$ and for $m=2$ with $n_1=n_2=1$. The coefficient of c^2 in (13) thus is $K_1[y_2]+K_2[y_1]$ so that from (8) and (13)

$$Q_2[x]=K_1[y_2]+K_2[y_1]=K_1H_2[x]+K_2H_1[x]. \tag{15}$$

By using the relation (10) for $y_1(t)$ and $y_2(t)$, we obtain the block diagram of the operator Q_2 as shown in Fig. 2.

We now shall determine the operator Q_3 since its determination involves one additional aspect not encountered in the determination of Q_1 and Q_2. To determine the operator Q_3, we proceed, as before, by equating the coefficients of c^3 in (8) and (13). Note that in (13) that c^3 is obtained for $m=1$ with $n_1=3$, for $m=2$ with $n_1=1$ and $n_2=2$ and also with $n_1=2$ and $n_2=1$, and finally for $m=3$ with $n_1=n_2=n_3=1$. Thus, from (8) and (13), we obtain

$$Q_3[x]=K_1[y_3]+K_2\{y_1,y_2\}+K_2\{y_2,y_1\}+K[y_1]. \tag{16}$$

Since a Volterra operator can be considered symmetric with no loss of generality, we shall consider the operators

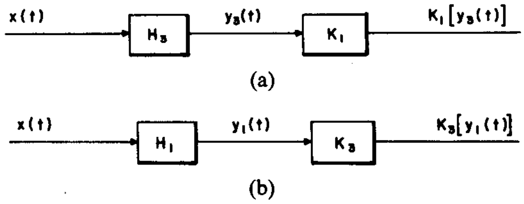

Fig. 3. (a) Synthesis of $K_1[y_3]$. (b) Synthesis of $K_3[y_1]$.

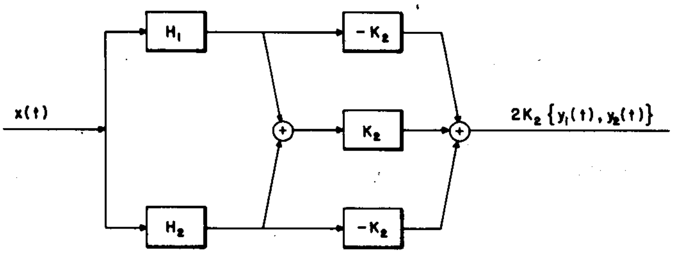

Fig. 4. Synthesis of $2K_2\{y_1,y_2\}$.

K_n to be symmetric so that

$$K_2\{y_1,y_2\}=K_2\{y_2,y_1\}. \tag{17}$$

Equation (16) then can be expressed in the form

$$Q_3[x]=K_1[y_3]+2K_2\{y_1,y_2\}+K_3[y_1]. \tag{18}$$

With the use of (10), systems with the input $x(t)$ and the outputs $K_1[y_3]$ and $K_3[y_1]$ can be synthesized in the block diagram form shown in Fig. 3. However to synthesize a block diagram for the operator Q_3 in accordance with (18) also requires the determination of a block diagram for a system with the input $x(t)$ and the output $2K_2\{y_1,y_2\}$. This is the additional aspect mentioned previously. The determination is made using the identities developed in [3] for the measurement of an nth-order Volterra kernel as an n-dimensional impulse response. The identity developed there for a second-order kernel is

$$2K_2\{y_1,y_2\}=K_2[y_1+y_2]-K_2[y_1]-K_2[y_2]. \tag{19}$$

Using (10), the system with the output $2K_2\{y_1,y_2\}$ for the input $x(t)$ thus is as shown in Fig. 4. Combining the systems of Figs. 3 and 4 in accordance with (18) we obtain the block diagram of the operator Q_3 as shown in Fig. 5. Substituting (19) into (18), $Q_3[x]$ can be expressed as

$$Q_3[x]=K_1[y_3]+K_2[y_1+y_2] \\ -K_2[y_1]-K_2[y_2]+K_3[y_1]. \tag{20}$$

Thus, with the use of (10), the expression for Q_3 in operator notation is

$$Q_3=K_1H_3+K_2[H_1+H_2]-K_2H_1-K_2H_2+K_3H_1. \tag{21}$$

The procedure presented can be used to determine each of the operators, Q_n. As a last illustration, by equating the coefficients of c^4 in (8) and (13), we obtain

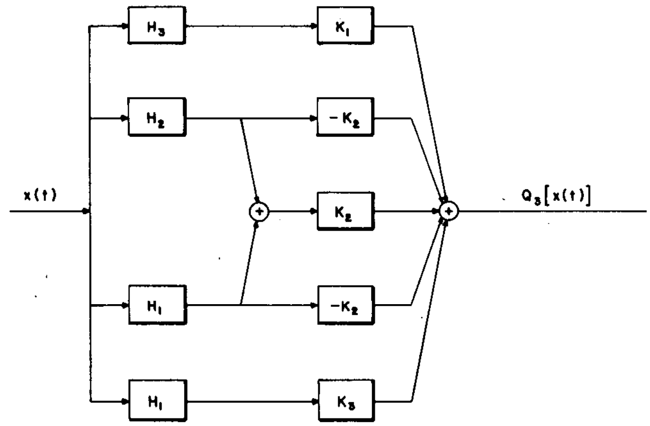

Fig. 5. Block diagram of operator Q_3.

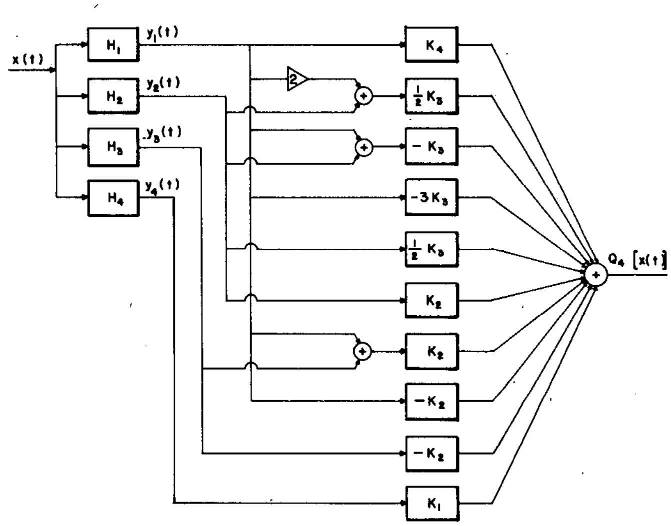

Fig. 6. Block diagram of operator Q_4.

$$Q_4[x]=K_1[y_4]+2K_2\{y_1,y_3\}+K_2[y_2] \\ +3K_3\{y_1,y_1,y_2\}+K_4[y_1]. \tag{22}$$

The identity developed in [3] for the measurement of a third-order Volterra kernel as a three-dimensional impulse response is

$$\begin{aligned}3!\,K_3\{y_1,y_1,y_2\}&=K_3[2y_1+y_2]-2K_3[y_1+y_2]\\&\quad-K_3[2y_1]+2K_3[y_1]+K_3[y_2]\\&=K_3[2y_1+y_2]-2K_3[y_1+y_2]\\&\quad-6K_3[y_1]+K_3[y_2].\end{aligned} \tag{23}$$

Using (19) and (23) a block diagram for the operator Q_4 is shown in Fig. 6. We observe that, as expected, the larger the order of Q_n, the more complicated is its block diagram. An important point to observe in the expression for $Q_n[x]$ is that the only term which involves the operator K_n is $K_n[y_1]$; all other terms involve the operators K_j with

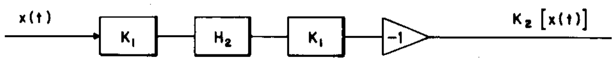

Fig. 7. Operator K_2 of $K_{(p)}^{-1}$.

$j<n$. That is

$$Q_n = K_n H_1 + \text{terms involving only } K_j \text{ for } j \leqslant n-1. \quad (24)$$

This will be needed in our discussion of the pth-order post-inverse, $K_{(p)}^{-1}$.

III. The pth-Order Post-Inverse

We shall now determine the Volterra operators, K_n, in order that the system $K_{(p)}$ be a pth-order post-inverse of the system H. This is accomplished by determining the Volterra operators, K_n, so that the tandem system Q satisfies the conditions given by (5).

The first condition to be satisfied from (5) is

$$Q_1[x(t)] = x(t). \quad (25)$$

It is clear from (14) that this condition is satisfied by a linear operator K_1 which is the inverse of the causal linear operator H_1. The requirement thus is that $K_1 = H_1^{-1}$ or

$$\mathcal{K}_1(s) = \frac{1}{\mathcal{H}_1(s)} \quad (26)$$

in which $\mathcal{K}_1(s)$ and $\mathcal{H}_1(s)$ are the Laplace transforms of the Volterra kernels, $k_1(\tau)$ and $h_1(\tau)$, respectively. It is not always possible to satisfy (26) with a BIBO stable and causal linear operator, K_1. If $\mathcal{H}_1(s)$ is a rational function of s, the requirement is that $\mathcal{H}_1(\infty) \neq 0$ and that all the zeros of $\mathcal{H}_1(s)$ be in the left-half of the s-plane in order that the operator K_1 which satisfies (26) be causal and stable. We thus observe that a BIBO stable and causal pth-order inverse of the system H does not exist if (26) cannot be satisfied by a stable and causal linear operator, K_1. We shall assume for the rest of this discussion that the linear operator which satisfies (26) is stable and causal.

The second condition we need to satisfy from (5) is that $Q_2[x(t)]=0$. Using (15) and Fig. 2, this requires

$$K_2[y_1] + K_1[y_2] = K_2 H_1[x] + K_1 H_2[x] = 0. \quad (27)$$

This will be satisfied for all x only if the operator K_2 satisfies

$$\begin{aligned} K_2 H_1 &= -K_1 H_2 \\ K_2 &= -K_1 H_2 H_1^{-1} = -K_1 H_2 K_1. \end{aligned} \quad (28)$$

The operator K_2 thus must be as shown in Fig. 7. From the block diagram of Fig. 7 it is clear that the operator K_2 is causal and stable since the operators K_1 and H_2 are causal and stable.

We proceed in the same manner to determine each of the operators of the system $K_{(p)}^{-1}$. For example, to determine the operator K_3, we precede the system shown in Fig. 6 by $K_1 = H_1^{-1}$. Then, for the response of the system to be zero, the output of the operator K_3 must be the negative of the sum of the outputs of all the other branches. Consequently, a block diagram for the operator K_3 is as shown in Fig. 8. The block diagram of Fig. 8 can be simplified by substituting the operator K_2 given in Fig. 7. The result is shown in Fig. 9.

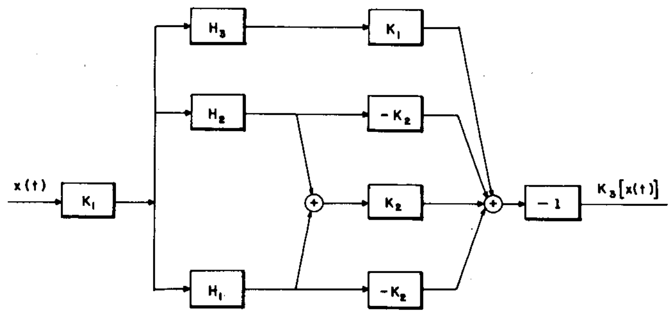

Fig. 8. Operator K_3 of $K_{(p)}^{-1}$

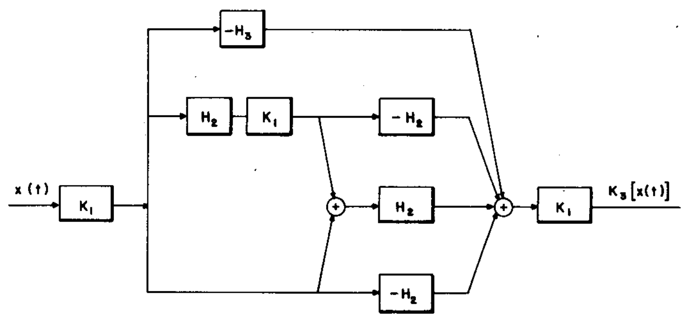

Fig. 9. Simplified form of operator K_3 of $K_{(p)}^{-1}$.

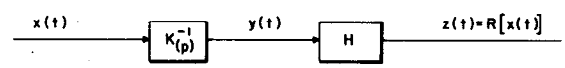

Fig. 10. System R.

Note that it always is possible to isolate the operator K_n as we have done by preceding the operator Q_n with K_1. This is so since the only term in the expression for $Q_n[x(t)]$ which involves the operator K_n is $K_n[y_1]$. Also, when determining the operator K_n from Q_n, all other terms involve the operators K_j for $j<n$ which have been determined. Also observe from our construction, that each of the operators K_n is causal and stable since the system H and the operator K_1 is causal and stable. We thus note that a stable and causal pth-order post-inverse, $K_{(p)}^{-1}$, of a system H exists if and only if the inverse of the linear operator H_1 is stable and causal.

IV. The pth-Order Pre-Inverse

In this section, we shall show that the pth-order pre-inverse of a system H is identical to the pth-order post-inverse of H. Consider the system R shown in Fig. 10. As shown, the system R is composed of the system H con-

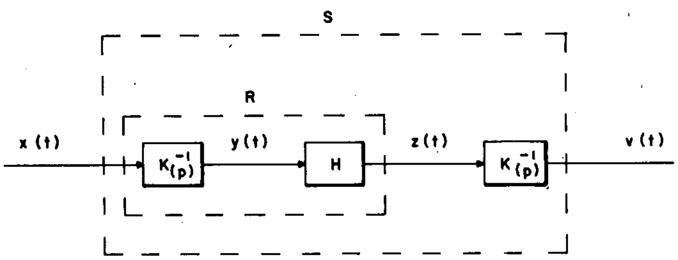

Fig. 11. System S.

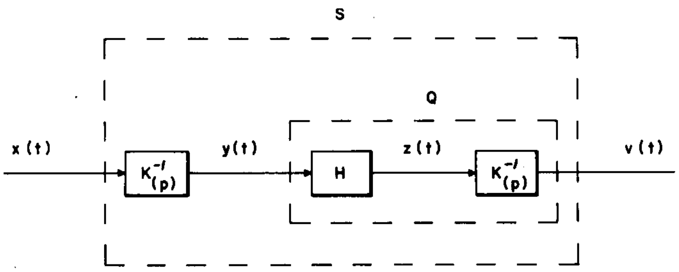

Fig. 12. Alternate form of system S.

nected in tandem with the system $K_{(p)}^{-1}$. The system H is characterized by the Volterra series given by (1), and the system $K_{(p)}^{-1}$ is the pth-order post-inverse of the system H. The Volterra series of the system R is

$$z(t)=R[x(t)]=\sum_{n=1}^{\infty} R_n[x(t)] \tag{29}$$

in which R_n is the nth-order Volterra operator of R. We shall show that the pth-order post-inverse of H, $K_{(p)}^{-1}$, also is a pth-order pre-inverse of H by showing that

$$R_n[x(t)]=\begin{cases} x(t), & \text{for } n=1 \\ 0, & \text{for } n=2,\cdots,p. \end{cases} \tag{30}$$

To show this, we connect the system $K_{(p)}^{-1}$ in tandem with the system R to form the system S as shown in Fig. 11. The output $v(t)$ of the system S for the input $x(t)$ is expressed by the Volterra series

$$v(t)=S[x(t)]=\sum_{n=1}^{\infty} S_n[x(t)]. \tag{31}$$

The system S also can be viewed as shown in Fig. 12 in which the system Q is connected in tandem with the system $K_{(p)}^{-1}$. Since $K_{(p)}^{-1}$ is the pth-order post-inverse of H, we have from (5)

$$v(t)=Q[y(t)]=y(t)+\sum_{n=p+1}^{\infty} Q_n[y(t)]. \tag{32}$$

However, we also have from Fig. 12

$$y(t)=K_{(p)}^{-1}[x(t)]=\sum_{n=1}^{p} K_n[x(t)] \tag{33}$$

so that

$$v(t)=\sum_{n=1}^{p} K_n[x(t)]+\sum_{n=p+1}^{\infty} Q_n[y(t)]. \tag{34}$$

Thus we have shown by means of Fig. 12 that

$$S_n[x(t)]=K_n[x(t)], \qquad \text{for } n\leqslant p. \tag{35}$$

Let us now return to the system S as viewed in Fig. 11. For convenience, we shall define

$$z_n(t)=R_n[x(t)]. \tag{36}$$

We first shall show that $R_1[x(t)]=x(t)$ in accordance with (30). Using the techniques we developed for determining the operators of two systems connected in tandem, we obtain for Fig. 11

$$S_1[x(t)]=K_1[z_1(t)]. \tag{37}$$

Thus, from (32) and (34),

$$K_1[x(t)]=K_1[z_1(t)]. \tag{38}$$

Now, from (26) $H_1=K_1^{-1}$. Thus operating on the time functions in (38) by H_1, we obtain

$$x(t)=z_1(t) \tag{39}$$

so that, from (36), $R_1[x(t)]=x(t)$. To show that $R_2[x(t)]=0$, consider $S_2[x(t)]$ which is, from Fig. 11,

$$S_2[x(t)]=K_1[z_2(t)]+K_2[z_1(t)]. \tag{40}$$

Substituting (35) and (39) in (40) we obtain

$$K_1[z_2(t)]=0. \tag{41}$$

Thus, operating on the time function of (41) with $H_1=K_1^{-1}$, we obtain

$$z_2(t)=0 \tag{42}$$

so that from (36), $R_2[x(t)]=0$. We can continue in this manner to demonstrate the validity of (30).

To complete our proof, assume we have shown that $z_n(t)=R_n[x(t)]=0$ for $n=2, 3,\cdots,m-1$ in which $m\leqslant p$. Then we shall show that $z_m(t)=0$. Our proof will then be complete since we have already shown that $z_2(t)=0$ and so by induction $z_n(t)=0$ for $n=2, 3,\cdots,p$. To show that $z_m(t)=0$, we consider the time functions $S_m[x(t)]$ which is, from Fig. 11 and (11),

$$S_m[x]=K_1[z_m]+\sum_{i=1}^{m-1} K_2\{z_i,z_{m-i}\} + \sum_{i=1}^{m-2}\sum_{j=1}^{m-1-i} K_3\{z_i,z_j,z_{m-i-j}\}+\cdots+K_m[z_1]. \tag{43}$$

The sum of the subscripts of the arguments of $K_n\{\cdot\}$ is m since $S_m[x]$ is an mth-order operator of x and, from (36), z_j is a jth-order operator of x. Now since we are assuming $z_n(t)=0$ for $n=2, 3,\cdots,m-1$, only the first and last terms of (43) can be nonzero. This follows from (12) and the observation that each term in the sums in (43) contains at least one of the functions $z_n(t)$ which are zero. Thus we have for (43)

$$S_m[x(t)] = K_1[z_m(t)] + K_m[z_1(t)]. \tag{44}$$

Substituting (35) and (39) in (44) we obtain

$$K_1[z_m(t)] = 0. \tag{45}$$

Operating on the time function of (45) with $H_1 = K_1^{-1}$, we conclude that

$$z_m(t) = R_m[x(t)] = 0, \qquad m \leqslant p \tag{46}$$

in accordance with (30).

We have shown that the pth-order post-inverse of a given system H is the same as its pth-order pre-inverse. We know that since the order of cascading two nonlinear systems generally affects the overall system response, we should not expect the cascade system Q and R to be the same. All we have shown is that their first p Volterra operators are the same; the Volterra operators of orders greater than p however, generally are different. Again, it should be emphasized that we have only considered the case in which the two systems connected in tandem do not interact in that there are no loading effects of one system on the other.

V. Application to the Study of H^{-1}

As p becomes infinite, the system $K_{(\infty)}^{-1}$ obtained is H^{-1}. A problem encountered in letting p become infinite is that the Volterra series for the limit, $K_{(\infty)}^{-1}$, may converge only for a limited range of its input amplitude which generally is not easily determined [4]. However, from our results, we observe that the response of the tandem connection of H and H^{-1} is equal to its input irrespective of the order of connection of the two systems. We also note that a stable and causal inverse, H^{-1}, with the Volterra series $K_{(\infty)}^{-1}$ does not exist for any stable system H if the linear operator, H_1^{-1}, is not stable and causal. In addition, if H_1^{-1} is stable, then H^{-1} exists and can be expressed in a Volterra series which however may converge only for a limited range of its input amplitude. This last result has been obtained previously by Brilliant using a different approach [5].

As an application of the pth-order inverse, consider the nonlinear differential equation

$$H[x] = H_1[x] + N[x] = y \tag{47}$$

in which H_1 is a linear operator and N is a nonlinear operator which can be described in terms of the Volterra series

$$N[x] = \sum_{n=2}^{\infty} H_n[x]. \tag{48}$$

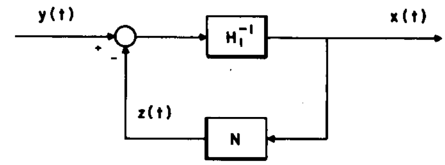

Fig. 13. Nonlinear feedback system which is H^{-1} for (44).

As an example, the equation of motion of a simple pendulum with linear damping is

$$\ddot{x} + a\dot{x} + b \sin x = y. \tag{49}$$

In (49), $x(t)$ is the angle of the swing and $y(t)$ is an externally applied torque. For this example,

$$H_1[x] = \ddot{x} + a\dot{x} + bx \tag{50a}$$

and

$$N[x] = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{50b}$$

The solution of (47) is the determination of the inverse of the system operator, H. From the theory developed in this article, the Volterra series solution of (47) exists for a range of the amplitude $|y| < M_y$, if the inverse of the linear operator, H_1^{-1}, is stable. Note that the Volterra series is an expansion of the solution of (47) about the origin. The condition $|y| < M_y$ ensures that x is in the neighborhood of the limit point at the origin of (47). The condition that H_1^{-1} is a causal operator need be imposed only in those cases in which physical considerations require it. For our example, the Laplace transform of the Volterra kernel of H_1^{-1} is, in accordance with (26) and (50a).

$$\mathcal{K}_1(s) = \frac{1}{s^2 + as + b} \tag{51}$$

Thus H_1^{-1} of the example will be a causal and stable operator if both the poles of $\mathcal{K}_1(s)$ lie in the left-half of the s-plane. A pth-order approximate solution of (47) in the form of a Volterra series is $K_{(p)}^{-1}[y]$ and the approximation error is, from (5), $\sum_{n=p+1}^{\infty} Q_n[x(t)]$. Approximate solutions of nonlinear differential equtions by means of Volterra series have been obtained by Barrett [1] and Parente [6].

Note that for the range $|y| < M_y$, the feedback system with the input $y(t)$ as shown in Fig. 13 has the output $x(t)$ which is the solution of (47). This is so since from the figure,

$$x = H_1^{-1}[y - z] = H_1^{-1}[y] - H_1^{-1}[z]. \tag{52}$$

By operating on both sides of (52) with H_1, we obtain

$$H_1[x] = y - N[x] \tag{53}$$

which is (47). Thus the nonlinear feedback system of Fig. 13 is stable for $|y|<M_y$ and its response, $x(t)$, for the input, $y(t)$, can be approximated by the pth-order Volterra series $K_{(p)}^{-1}[y]$ of the system H. In this manner, the theory of nonlinear feedback systems of the form shown in Fig. 13 and nonlinear differential equations of the form given by (47) are equivalent.

VI. Summary

The concept and the theory of pth-order inverses, $K_{(p)}^{-1}$, has been developed in this article. It was shown that the pth-order post-inverse of a given system, H, is identical with its pth-order pre-inverse. Thus the order of the tandem connection of $K_{(p)}^{-1}$ with H affects only the Volterra operators of the tandem connection of order greater than p. The synthesis of $K_{(p)}^{-1}$ was obtained. It should be noted that the determination of $K_{(p)}^{-1}$ requires knowledge of only the first p Volterra operators of the system H. It also was shown that $K_{(p)}^{-1}$ will be a stable and causal system if and only if the inverse of H_1, the linear part of the system H, is stable and causal. These results are similar to those obtained for the inverse of a functional series [7].

The pth-order inverse offers an approach to the study of the system inverse H^{-1} by analyzing $\lim_{p\to\infty} K_{(p)}^{-1}$. Some aspects of the system inverse were discussed by means of this approach. In specific, some applications and interpretations with regard to nonlinear differential equations and nonlinear feedback systems were presented.

Acknowledgment

The author expresses his appreciation to S. Johnson, Northeastern University, for his helpful comments concerning this paper.

References

[1] J. F. Barrett, "The use of functionals in the analysis of non-linear physical systems," *Journal of Electronics and Control*, vol. 15, pp. 567–615, 1963.

[2] D. A. George, Continuous nonlinear systems, *Tech. Rep.* 355, Research Laboratory of Electronics, M.I.T., Cambridge, MA, July, 1959.

[3] M. Schetzen, "Measurement of the kernels of a non-linear system of finite order," *International Journal of Control*, vol. 1, no. 3, pp. 251–263, Mar. 1955.

[4] G. Zames, "On the input–output stability of time-varying nonlinear feedback systems, Part I," *IEEE Transactions on Automatic Control*, vol. AC-11, no. 2, pp 228–238, Apr. 1966.

[5] M. B. Brilliant, Theory of the Analysis of Nonlinear Systems, Tech. Rep. 345, Research Laboratory of Electronics, M.I.T., Cambridge, MA, Mar. 1958.

[6] R. B. Parente, "Nonlinear differential equations and analytic system theory," *SIAM, Journal of Applied Math*, vol 18, no. 1, pp. 41–66, Jan. 1970.

[7] M. Schetzen, "Power-series equivalence of some functional series with applications," *IEEE Transactions on Circuit Theory*, vol. CT-17, no. 3, pp. 305–313, Aug. 1970.

Martin Schetzen (S'50–A'51–M'52) was born in Brooklyn, New York, NY, on February 10, 1928. He received the B.E.E. degree from New York University, New York, N. Y., in 1951, and the S.M. and Sc. D. degrees in electrical engineering from the Massachusetts Institute of Technology, Cambridge, in 1954 and 1961, respectively.

From 1951 to 1952 he was employed as an Electronic Scientist by the National Bureau of Standards, Washington, D.C. While studying for the S. M. degree, he was a research assistant in the M.I.T. Research Laboratory of Electronics. From 1954 to 1956 he was an engineer with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, MD. He was a Teaching Assistant while studying for the Sc.D. degree, appointed Instructor in 1960, and Assistant Professor in 1961 in the Department of Electrical Engineering, M. I. T. He joined the faculty of Northeastern University, Boston, MA., in 1965 where he is currently Professor of Electrical Engineering.

Dr. Schetzen is a member of Eta Kappa Nu, Tau Beta Pi, and Sigma Xi. He received the Apollo Achievement Award and the Apollo Certificat of Commendation in 1970.