# Surrogate model constructed using neural networks for the forward and inverse problems

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## Surrogate Model

**Problem:**
The mapping between input and output spaces is not achievable, but can be approximated.

**Goal:**
Analyzes the convergence of probability densities solving uncertainty quantification problems using surrogate model.

## Example

Consider the following ODE:

$$y' = -\lambda x y$$
$$y(0) = 1$$

where $0.3 \leq x \leq 0.7$, $\lambda$ is an uncertainty parameter.

**Exact Solution:**

$$y(x, \lambda) = e^{-\lambda x^2/2}$$

## Forward UQ Problem

Assume $\lambda \sim Beta(2, 2)$,
QoI is $Q(\lambda) = y(0.5; \lambda)$.

*For demonstration purpose, assume we do not know the exact solution describing the relation from $x$, $\lambda$ to $y$.*

**Surrogate Model:**
Neural Network: 400 training data for $x$ is from $U[0.3, 0.7]$; 400 training data for $\lambda$ is from $Beta(2, 2)$.
The neural network is trained under different numbers of epochs, 1000, 2000, 5000, 10000, to create surrogate models [**1**].

### Convergence of surrogate model

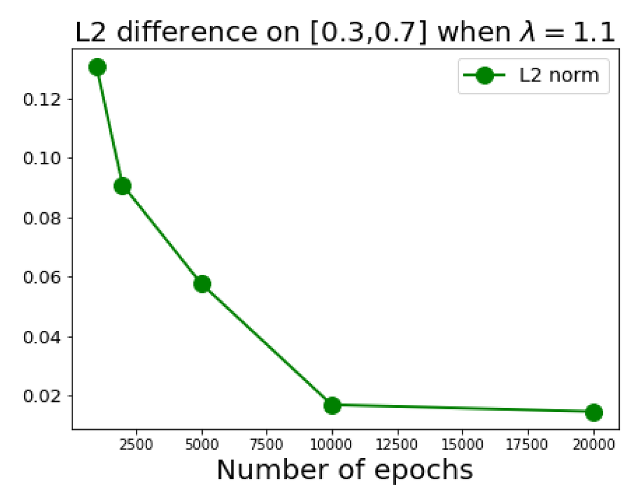


Fig. 1: L2 difference between true solution and surrogate model.

*Since our quantity of interest is uncertain due to the randomness from $\lambda$, to verify the convergence of push-forward densities using approximate maps, we create the error plot by using the $L^2$ distance on $[0.3, 0.7]$ for a fixed $\lambda$ value 1.1.*

**Conclusion:**
Surrogate model converges in the $L^2$ sense.

## Forward UQ Problem (Continued)

### Convergence of push-forward density

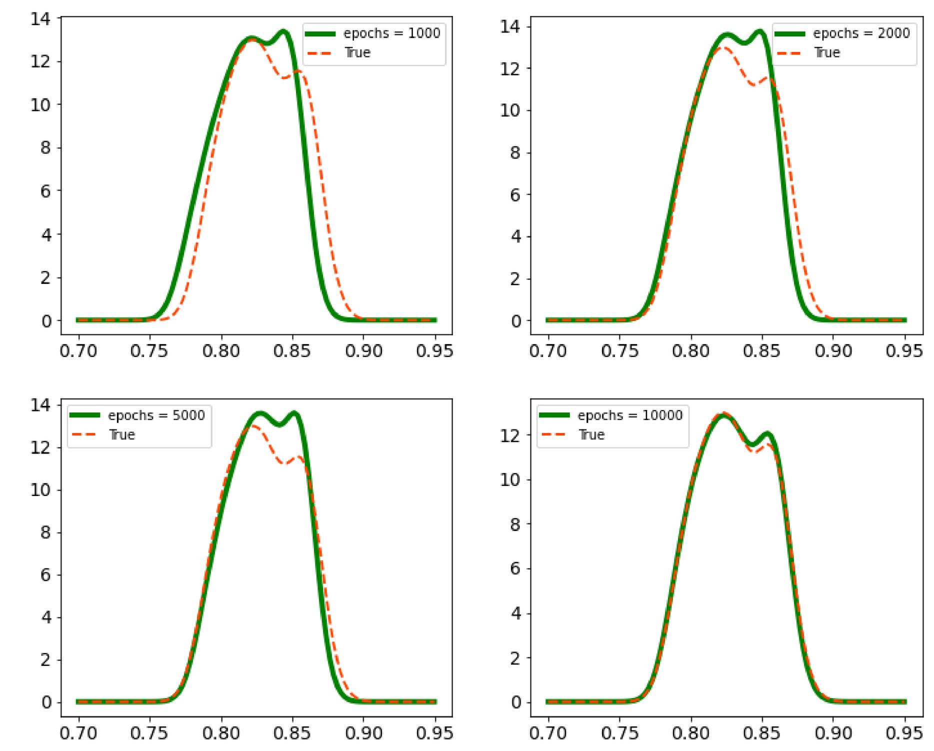


Fig. 2: Push-forward density.

**Conclusion:**
Surrogate model constructed using neural networks helps approximate true push-forward density.

## Intro to Data Consistent Inversion

**Data Consistent Inversion** is a novel framework that uses push-forward and pullback measures to ensure solutons are consistent with the observed distribution of data.

**Data Consistent Inversion Approach**

**Using the exact model** [**2**]:

$$\pi_\Lambda^u(\lambda) = \pi_\Lambda^i(\lambda)\,\frac{\pi_\mathcal{D}(Q(\lambda))}{\pi_\mathcal{D}^Q(Q(\lambda))}$$

**Using the surrogate model** [**3**]:

$$\pi_\Lambda^{u,n}(\lambda) = \pi_\Lambda^i(\lambda)\,\frac{\pi_\mathcal{D}(Q_n(\lambda))}{\pi_\mathcal{D}^{Q_n}(Q_n(\lambda))}$$

### Demonstration

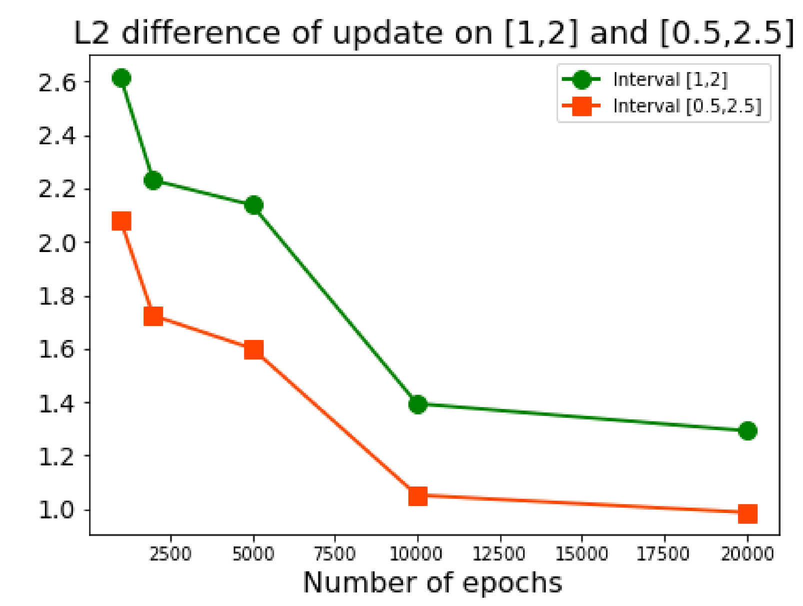


Fig. 3: Data Consistent Inversion Method.

## Inverse UQ Problem

### Convergence of updated density

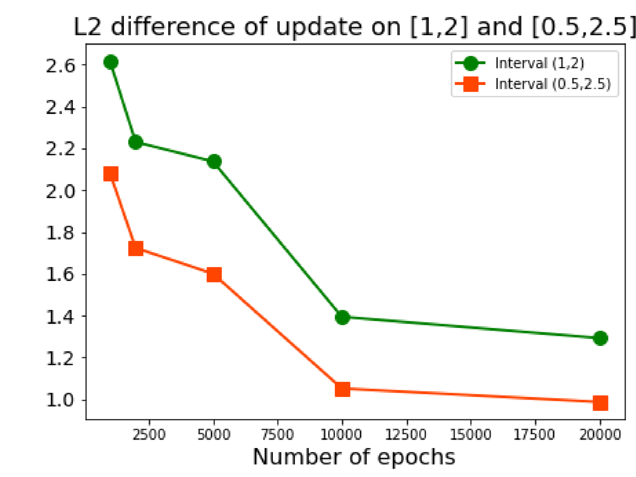


Fig. 4: L2 difference between true update and approximate update.

**Conclusion:**
Surrogate model constructed using neural networks helps approximate true updated density.

## Notation

| Notation | Description |
|---|---|
| $\lambda \in \Lambda$ | Parameter Space |
| $\mathcal{D}$ | Observable Space |
| $Q$ | Exact model |
| $Q_n$ | $n$-th surrogate model |
| $\pi_\Lambda^i$ | Initial density guess of $\lambda$ |
| $\pi_\Lambda^u$ | Update pullback density |
| $\pi_\Lambda^{u,n}$ | Approximate update pullback density using $Q_n$ |
| $\pi_\mathcal{D}$ | Observed density |
| $\pi_\mathcal{D}^Q$ | Push-forward density |
| $\pi_\mathcal{D}^{Q_n}$ | Approximate push-forward density |

## References

### References

[1] Lagaris, I., Likas, A., and Fotiadis, D., *Artificial Neural Networks for Solving Ordinary and Partial Differential Equations*, *IEEE Transactions on Neural Networks*, 9(5):987-1000, 1998.

[2] Butler, T., Jakeman, J., and Wildey, T., *Combining push-forward measures and bayes' rule to construct consistent solutions to stochastic inverse problems*, *SIAM Journal on Scientific Computing*, 40(2):A984-A1011, 2018.

[3] Butler, T., Jakeman, J., and Wildey, T., *Convergence of probability densities using approximate models for forward and inverse problems in uncertainty quantification*, *SIAM Journal on Scientific Computing*, 40(5):A3523–A3548, 2018.