Surrogate model constructed using neural networks for the forward and inverse problems

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Surrogate Model

Problem:
How to quantify the uncertainties when the mapping between the input and output spaces is not achievable, but can be approximated?

Goal:
Analyzes the convergence of probability densities solving uncertainty quantification problems using surrogate model.

Example

Consider the following ODE:

$$y' = -\lambda x y$$
$$y(0) = 1$$

where $0.3 \leq x \leq 0.7$, λ is an uncertainty parameter.

Exact Solution:

$$y(x, \lambda) = e^{-\lambda x^2/2}$$

Forward UQ Problem

Assume $\lambda \sim Beta(2, 2)$,
QoI is $Q(\lambda) = y(0.5; \lambda)$.

For demonstration purpose, assume we do not know the exact solution describing the relation from x, λ to y.

Surrogate Model:
Neural Network: 400 training data for x is from $U[0.3, 0.7]$; 400 training data for λ is from $Beta(2, 2)$.
The neural network is trained under different numbers of epochs, 1000, 2000, 5000, 10000, to create surrogate models [**1**].

Convergence of L_2 distance

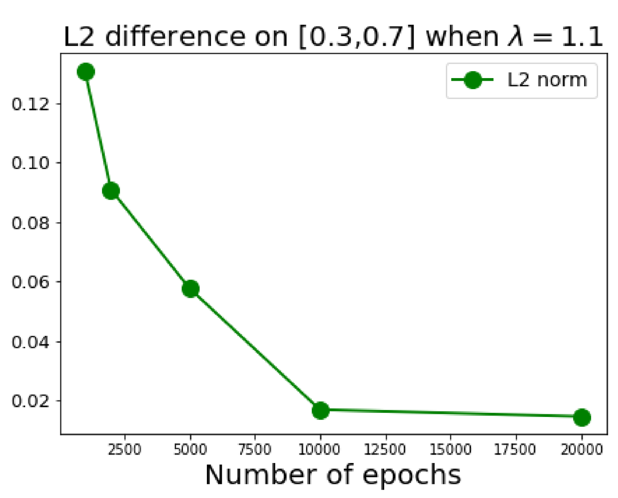

Fig. 1: L2 difference by using surrogate model.

Since our quantity of interest is uncertain due to the randomness from λ, to verify the convergence of push-forward densities using approximate maps, we create the error plot by using the L^2 distance on $[0.3, 0.7]$ for a fixed λ value 1.1.

Forward UQ Problem (Continued)

Convergence of push-forward density

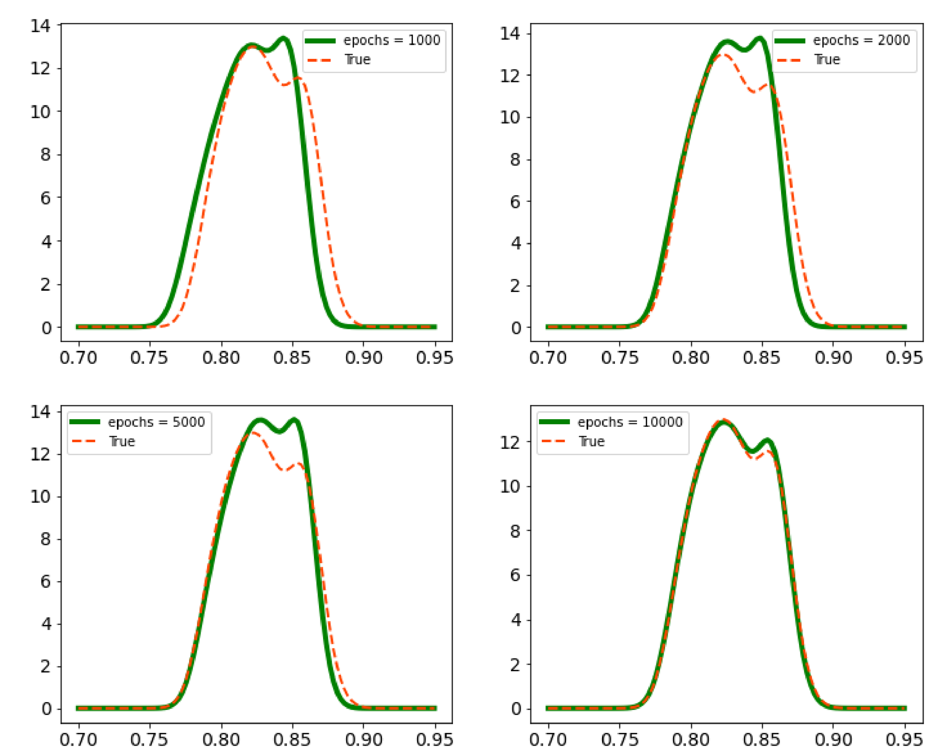

Fig. 2: Push-forward density.

Conclusion:
Surrogate model constructed using neural networks helps approximate true push-forward density.

Intro to Data Consistent Inversion

Data Consistent Inversion is a novel framework that uses push-forward and pullback measures to ensure solutons are consistent with the observed distribution of data.

Data Consistent Inversion Approach

Using the exact model [**2**]:

$$\pi_\Lambda^{\text{up}}(\lambda) = \pi_\Lambda^{\text{init}}(\lambda)\, \frac{\pi_{\mathcal{D}}(Q(\lambda))}{\pi_{\mathcal{D}}^{Q}(Q(\lambda))}$$

Using the surrogate model [**3**]:

$$\pi_\Lambda^{\text{up},n}(\lambda) = \pi_\Lambda^{\text{init}}(\lambda)\, \frac{\pi_{\mathcal{D}}(Q_n(\lambda))}{\pi_{\mathcal{D}}^{Q_n}(Q_n(\lambda))}$$

Demonstration

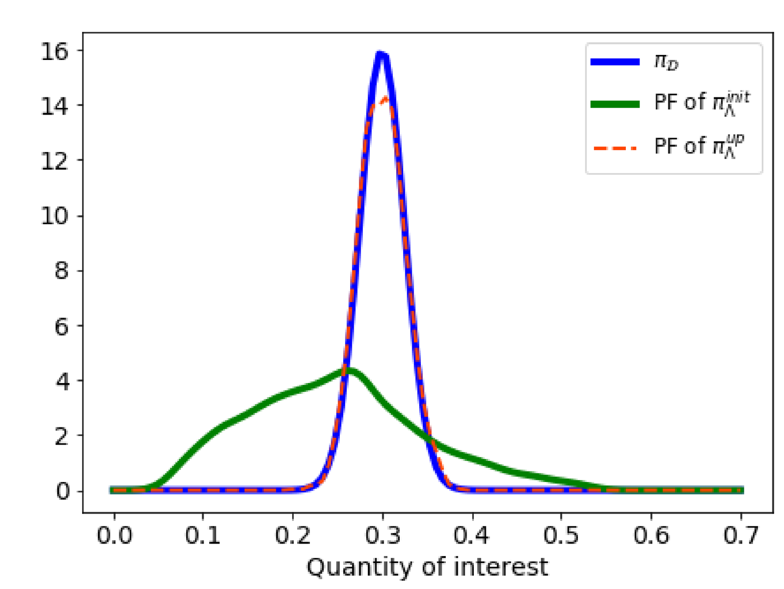

Fig. 3: Data Consistent Inversion Method.

Inverse UQ Problem

Convergence of updated density

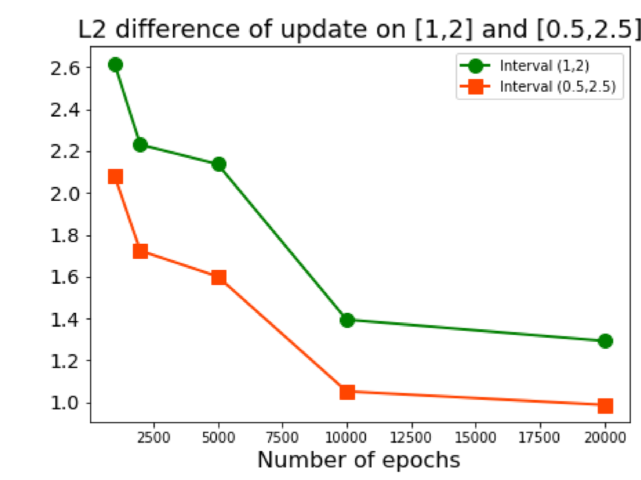

Fig. 4: L2 difference between true update and approximate update.

Conclusion:
Surrogate model constructed using neural networks helps approximate true updated density.

Notation

Notation	Description
$\lambda \in \Lambda$	Parameter Space
$\mathcal{D}$	Observable Space
Q	Exact model
Q_n	n-th surrogate model
$\pi_\Lambda^{\text{init}}$	Initial density guess of λ
π_Λ^{up}	Update pullback density
$\pi_\Lambda^{\text{up},n}$	Approximate update pullback density using Q_n
$\pi_{\mathcal{D}}$	Observed density
$\pi_{\mathcal{D}}^{Q}$	Push-forward density
$\pi_{\mathcal{D}}^{Q_n}$	Approximate push-forward density

References

References

[1] Lagaris, I., Likas, A., and Fotiadis, D., *Artificial Neural Networks for Solving Ordinary and Partial Differential Equations*, *IEEE Transactions on Neural Networks*, 9(5):987-1000, 1998.

[2] Butler, T., Jakeman, J., and Wildey, T., *Combining push-forward measures and bayes' rule to construct consistent solutions to stochastic inverse problems*, *SIAM Journal on Scientific Computing*, 40(2):A984-A1011, 2018.

[3] Butler, T., Jakeman, J., and Wildey, T., *Convergence of probability densities using approximate models for forward and inverse problems in uncertainty quantification*, *SIAM Journal on Scientific Computing*, 40(5):A3523–A3548, 2018.