Parallel programming. Task 10.

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1 Task statement

Consider boundary value problem:

$$y'' = f(x, y); \quad y(x_0) = a; \quad y(x_1) = b \tag{1}$$

The task is to obtain solution via Numerov approximation:

$$y_{m+1} - 2y_m + y_{m-1} = h^2 \cdot [f_m + \frac{1}{12}(f_{m+1} - 2f_m + f_{m-1})] \tag{2}$$

2 Obtaining solution

2.1 Newton's linearization

First of all, using Numerov approximation we can construct the system of $N-2$ non-linear equations.

$$\begin{cases} g_1 = y_2 - 2y_1 + a - h^2 \cdot [f_1 + \frac{1}{12}(f_2 - 2f_1 + f(a))] = 0 \\ \dots \\ g_m = y_{m+1} - 2y_m + y_{m-1} - h^2 \cdot [f_m + \frac{1}{12}(f_{m+1} - 2f_m + f_{m-1})] = 0 \\ \dots \\ g_{N-2} = b - 2y_{N-2} + y_{N-3} - h^2 \cdot [f_{N-2} + \frac{1}{12}(f(b) - 2f_{N-2} + f_{N-3})] = 0 \end{cases} \tag{3}$$

The solution can be found using Newton's linearization:

$$\mathbf{y}^{k+1} = \mathbf{y}^k - [G'(\mathbf{y}^k)]^{-1} \cdot \mathbf{g}(\mathbf{y}^k) = \mathbf{y}^k - \mathbf{p}(\mathbf{y}^k) \tag{4}$$

Where G is Jacobi matrix and $\mathbf{p}(\mathbf{y}^k)$ can be obtained from:

$$G(\mathbf{y}^k) \cdot p = \mathbf{g}(\mathbf{y}^k) \tag{5}$$

The matrix G has the form:

$$\begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \cdots & \frac{\partial g_1}{\partial y_{N-2}} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \cdots & \frac{\partial g_2}{\partial y_{N-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_{N-2}}{\partial y_1} & \frac{\partial g_{N-2}}{\partial y_2} & \cdots & \frac{\partial g_{N-2}}{\partial y_{N-2}} \end{pmatrix} \tag{6}$$

With Numerov approximation this matrix become tridiagonal.

$$\frac{\partial g_i}{\partial y_i} = -2 - h^2 \cdot \frac{5}{6} f'_y(x_i, y_i), \quad i \in [1, N-2] \tag{7}$$

$$\frac{\partial g_i}{\partial y_{i-1}} = 1 - h^2 \cdot \frac{1}{12} f'_y(x_{i-1}, y_{i-1}), \quad i \in [2, N-2] \tag{8}$$

$$\frac{\partial g_i}{\partial y_{i+1}} = 1 - h^2 \cdot \frac{1}{12} f'_y(x_{i+1}, y_{i+1}), \quad i \in [1, N-3] \tag{9}$$

Others 0.

2.2 Solving linear system of equations

In order to obtain the solution of linear system of equations the reduction algorithm is used. The main idea in this algorithm is to consequently getting rid of each second parameter. You can read more about it here `https://intuit.ru/studies/courses/4447/983/lecture/14931?page=4`. Also it is given in code.