Towards MCDA based decision support system addressing sustainable assessment - assumptions and fundamentals for the TOPSIS and COMET methods

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1. The TOPSIS Method

Step 1. Decision matrix is normalized.

Step 1. The decision matrix with *m* alternatives and *n* criteria represented as $X = (x_{ij})_{m \times n}$ is normalized. In this research, the Minimum-Maximum normalization method was used. In the case of this procedure, the normalized values of r_{ij} are determined for the profit and cost criteria using the formulas (1), while for the cost criteria using the formula (2).

$$r_{ij} = \frac{x_{ij} - min_j(x_{ij})}{max_j(x_{ij}) - X_{min}} \tag{1}$$

$$r_{ij} = \frac{max_j(x_{ij}) - x_{ij}}{max_j(x_{ij}) - min_j(x_{ij})} \tag{2}$$

Step 2. Weighted values of the normalized decision matrix v_{ij} are determined according to the Equation (3).

$$v_{ij} = w_i r_{ij} \tag{3}$$

Step 3. Calculate the positive ideal solution (PIS) values and negative anti-ideal solution (NIS) vectors. The PIS represented by the vector (4) expresses the maximum values for each criterion, and the NIS represented by the vector (5) minimum values. It is unnecessary to divide the criteria into cost and profit criteria in this step because the cost criteria were transformed to profit criteria in the normalization step.

$$v_j^+ = \left\{v_1^+, v_2^+, \ldots, v_n^+\right\} = \left\{\max_j(v_{ij})\right\} \tag{4}$$

$$v_j^- = \left\{v_1^-, v_2^-, \ldots, v_n^-\right\} = \left\{\min_j(v_{ij})\right\} \tag{5}$$

Step 4. Calculate distance from PIS according to the Equation (6) and NIS, using the Equation (7) for each of the alternatives considered.

$$D_i^+ = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2} \tag{6}$$

$$D_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2} \tag{7}$$

Step 5. Calculate the outcome for each of the respected alternatives according to Equation (8). This score takes values between 0 and 1. The closer the value of a given alternative is to 1, the better is the alternative.

$$C_i = \frac{D_i^-}{D_i^- + D_i^+} \tag{8}$$

2. The COMET Method

The stages of the COMET method are described below to understand the operation of the algorithm better.

Step 1. Definition of the space of the problem. The expert determines the dimensionality of the problem with the selection *r* criteria, $C_1, C_2, ..., C_r$. Then a set of fuzzy numbers is selected for each criterion $C_i, e.g., \left\{\tilde{C}_{i1}, \tilde{C}_{i2}, ..., \tilde{C}_{ic_i}\right\}$ according to the Equation (9).

$$\begin{array}{c} C_1 = \left\{\tilde{C}_{11}, \tilde{C}_{12}, ..., \tilde{C}_{1c_1}\right\} \\ C_2 = \left\{\tilde{C}_{21}, \tilde{C}_{22}, ..., \tilde{C}_{2c_2}\right\} \\ \dots \\ C_r = \left\{\tilde{C}_{r1}, \tilde{C}_{r2}, ..., \tilde{C}_{rc_r}\right\} \end{array} \tag{9}$$

where $C_1, C_2, ..., C_r$ are the ordinals of the fuzzy numbers for all criteria.

Step 2. The generation of characteristic objects (COs) with the usage of the Cartesian product of the fuzzy numbers' cores of all the criteria according to the Equation (10).

$$CO = \langle C(C_1) \times C(C_2) \times ...C(C_r)\rangle \tag{10}$$

As a result, an ordered set of all COs is obtained (11).

$$\begin{array}{c} CO_1 = \langle C(\tilde{C}_{11}), C(\tilde{C}_{21}), ..., C(\tilde{C}_{r1})\rangle \\ CO_2 = \langle C(\tilde{C}_{11}), C(\tilde{C}_{21}), ..., C(\tilde{C}_{r2})\rangle \\ \dots \\ CO_t = \langle C(\tilde{C}_{1c_1}), C(\tilde{C}_{2c_2}), ..., C(\tilde{C}_{rc_r})\rangle \end{array} \tag{11}$$

where *t* is the count of COs and is equal to Equation (12).

$$t = \prod_{i=1}^{r} c_i \tag{12}$$

Step 3. Assessment of characteristic objects by identifying the Matrix of Expert Judgment MEJ by comparing pairwise objects COs by the expert. The MEJ matrix is presented as Equation (13).

$$MEJ = \begin{pmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1t} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2t} \\ ... & ... & ... & ... \\ \alpha_{t1} & \alpha_{t2} & ... & \alpha_{tt} \end{pmatrix} \tag{13}$$

where α_{ij} is the outcome of comparing CO_i and CO_j by the expert. The more preferred characteristic object receives a value of 1, and the less preferred object receives a value of 0. If the preferences are equal, both objects get a value of half. This step depends totally on the expert's knowledge and can be represented as (14).

$$\alpha_{ij} = \begin{cases} 0.0, & f_{exp}(CO_i) < f_{exp}(CO_j) \\ 0.5, & f_{exp}(CO_i) = f_{exp}(CO_j) \\ 1.0, & f_{exp}(CO_i) > f_{exp}(CO_j) \end{cases} \tag{14}$$

where the expert function f_{exp} denotes the empirical preferences of the expert.

After the MEJ matrix is provided, a vertical vector of the Summed Judgments SJ is obtained as shown by Equation 15.

$$SJ_i = \sum_{j=1}^{t} \alpha_{ij} \tag{15}$$

Finally, preference values are determined for each characteristic object. As a result, a vertical vector P is obtained, where the i-th row contains the approximate value of preference for CO_i.

Step 4. Each CO and its preference value is converted to a fuzzy rule by using the following Equation (16)

$$IF \quad C\left(\tilde{C}_{1i}\right) \quad AND \quad C\left(\tilde{C}_{2i}\right) \quad AND \quad ... \quad THEN \quad P_i \tag{16}$$

In this procedure, a complete fuzzy rule base is prepared.

Step 5. Inference and getting the final ranking. Each alternative is represented as a set of values, e.g.
$A_i = \{\alpha_{i1}, \alpha_{i2}, \alpha_{ri}\}$. This set refers to the criteria $C_1, C_2, ..., C_r$. Mamdani's fuzzy inference technique is used to determine the preference of the i-th decision variant. The constant rule base guarantees that the results obtained are unequivocal, which makes COMET completely resistant to the rank reversal paradox.

3. Weighting methods

3.1. Equal weigths

According to Equal Weight procedure, all criteria's weights are equal and calculated by formula (17), where *n* is the number of criteria.

$$w_j = 1/n \tag{17}$$

3.2. Entropy weights

The entropy method used for determining criteria weights is based on a measure of information uncertainty. It is calculated using the following formulas (18), (19) and (20)

$$p_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \; i = 1, \ldots, m; \; j = 1, \ldots, n \tag{18}$$

where x_{ij} symbolizes the individual values from the decision matrix.

$$E_j = -\frac{\sum_{i=1}^{m} p_{ij} ln(p_{ij})}{ln(m)}; \; j = 1, \ldots, n \tag{19}$$

$$w_j = \frac{1 - E_j}{\sum_{i=1}^{n}(1 - E_i)}; \; j = 1, \ldots, n \tag{20}$$