Enhancing Supply Chain Efficiency through Quantum Annealing Optimization of the Generalized Assignment Problem

Joseph George, Clinton Morimoto, Devin Park

QUBO Formulation

A QUBO (quadratic unconstrained binary optimization) formulation expresses a quadratic optimization as an objective function which assigns a value to be minimized to each binary vector *x*. All QUBO formulations take the following form. where Q is a matrix of coefficients:

$$f_Q(x) = x^\top Q x = \sum_{i=1}^{n} \sum_{j=i}^{n} Q_{ij} x_i x_j$$

For constrained problems such as the GAP, each constraint must be encoded as a penalty weight in the objective function.

$$f_Q(\mathbf{x}) = \mu \sum_i \overbrace{\left(\sum_j w_{ij} x_{ij} - t_i \right)^2}^{\text{Budget constraint}} + \lambda \sum_j \overbrace{\left(\sum_i x_{ij} - 1 \right)^2}^{\text{Assignment Constraint}} + \overbrace{\sum_i \sum_j p_{ij} x_{ij}}^{\text{Total Profit}}$$

Methods

In order to implement quantum annealing, we rely on the D-Wave quantum-hybrid constrained quadratic model (CQM) which minimizes the objective function under the specified constraints.

D-Wave allows users to implement customizable soft constraints: versus the hard constraints given by the GAP, these softer constraints are prioritized by a selected weighting scheme, yet possible to be broken as a secondary goal to maximizing GAP profit. Thus, in addition to our starter quantum annealer, we tack on an additional model which adds the following greedy soft constraints: prioritize utilizing 80% of each agent's budget and that each agent, across all of its tasks, averages a task-based profit: cost ratio greater than the mean across all agent-task assignments. Further, we also tested each test case on the classical CBC algorithm which uses heuristics to efficiently find a solution.

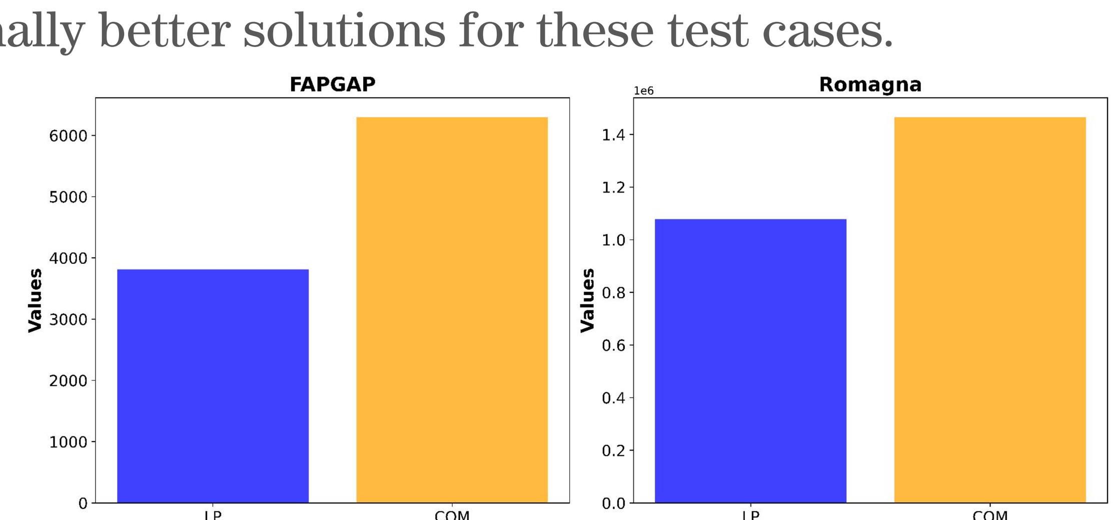
We tested our models based on GAP datasets compiled by J. E. Beasley in the OR-Library to find a minimum profit and corresponding agent-task assignment. Particularly, we used the home-brewed GIS-based and Beasley-like datasets and Yagiura set D and set E (Beasley, n.d.). We also calculated lower bounds for each test case using LP-relaxation.

Analysis

Summary Statistics : Error of CQM outputs from LP Lower Bound

n	mean	SD	min	Q_1	med	Q_3	Max
107	0.034	0.036	0	0.007	0.017	0.055	0.13

Originally, we had a sample size of 109, but two outliers were removed and are discussed below. From this data, we created a t-interval with a confidence level of 99%. Although all of the sample cases were aggregated from GAPLIB, due to the, credibility and well distributed types of data, we can generalize the results to all possible errors. Using $t^* = 2.622$, we get a confidence interval of (0.0251, 0.0429). So we can say with 99% confidence that the true mean error of CQM outputs from LP Lower Bound is between 0.0251 and 0.0429. Therefore, quantum annealing is almost always able to find solutions to the GAP that are close to optimal. However, the classic CBC algorithm was always able to find marginally better solutions for these test cases.



The error from the lower bound for the FAPGAP and ramagna were significantly higher than the other test cases and were thus deemed as outliers. For both of these test cases, the solution found by the CQM were significantly higher than the lower bound. The errors were 0.361 and 0.653 for Romagna and FAPGAP respectively . Additionally, the CBC algorithm was unable to find a solution to both of these problems after running for 6 hours each.

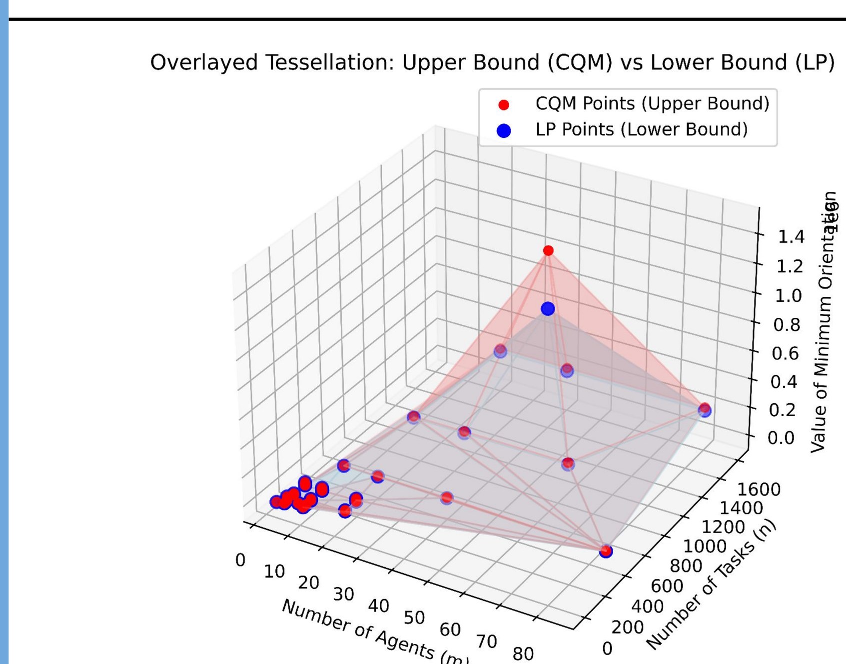
Results

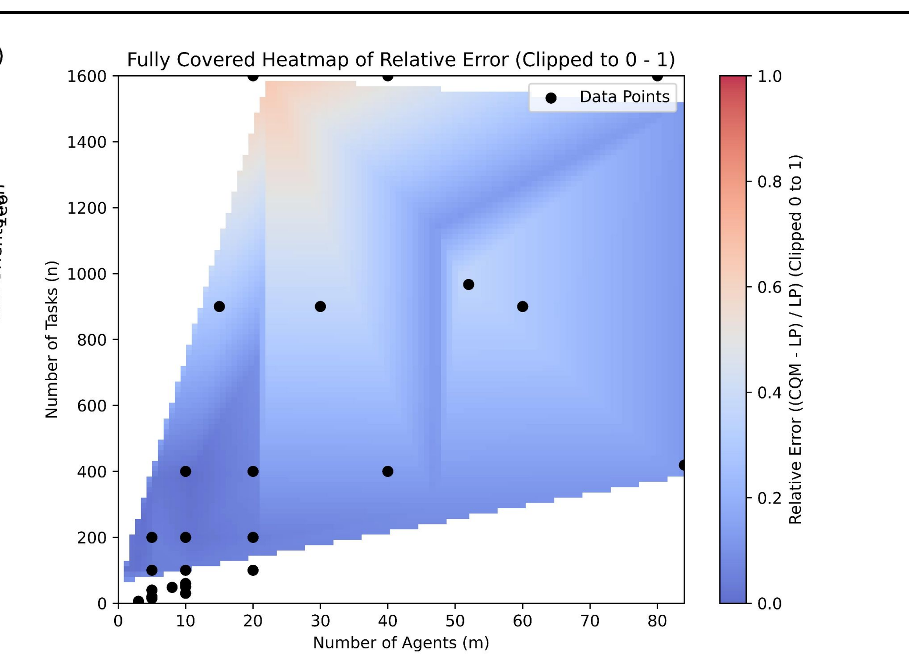
Mean Percent Error of Baseline CQM Across Datasets

Dataset	Instance Count	Mean % Error
Home-brewed, GIS-based	13	0.0031%
Home-brewed, Beasley-like	15	0.0487%
Yagiura, Set D	6	6.5867%
Yagiura, Set E	15	8.2253%

Comparing CQM and Heuristics-Based CQM in Yagiura, Set E

n	m	LB	CQM	CQM-H
100	5	12642	12882	12831
200	5	24922	25715	25780
100	10	11544	12088	12092
200	10	23294	24372	24521
400	10	45740	48061	48924
900	15	102417	112295	112741
100	20	8360	8973	9088
200	20	22356	23905	23812
400	02	44862	48201	48689
1600	20	180640	203578	213962
900	30	100413	112605	115552
400	40	44524	49111	49576
900	60	100103	112808	115260
1600	40	178283	200719	209464
1600	80	176781	199683	206337





Discussion

With a confidence interval of (0.0251, 0.0429) for true mean error of CQM profits compared to the lower bound, the CQM solver was able to consistently find close to optimal solutions for the non-outlier cases. All the CQM solutions were also completed within 60 seconds.

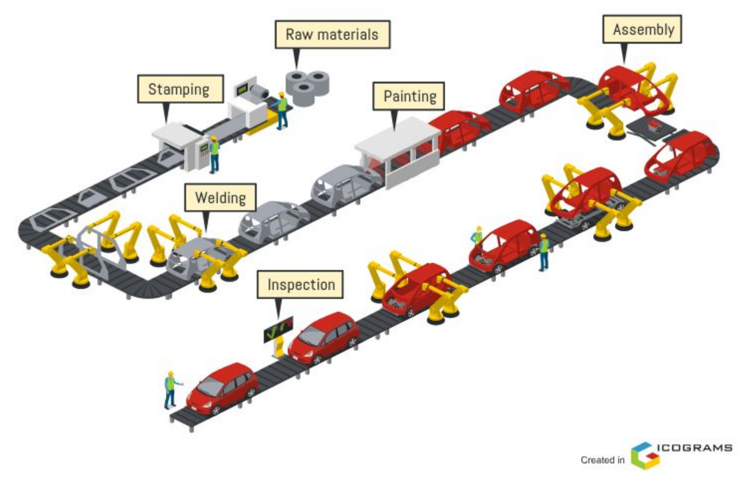
Although the classical CBC algorithm almost always outperformed the CQM algorithm by a small margin, it completely failed to even find a solution for the two outlier cases given several hours, while the CQM found a solution for each within 120 seconds.

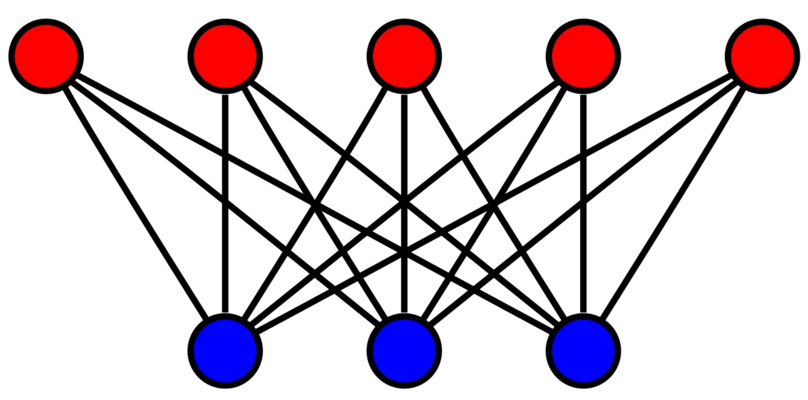
The two outlier test cases were both real world assignment problems, with romagna being GIS-based and FAPGAP being taken from machine scheduling. Thus these test cases are most similar to real world supply chain optimization problems.

The modified CQM with heuristics did not yield significantly different profits. Often, the CQM with heuristics gave slightly worse results, and sometimes slightly improved the minimum profit.

Motivation

Efficient global supply chains have become increasingly important as a result of rising geopolitical tensions, and increasing digitalization. Global supply chains have faced numerous challenges—rising delivery costs, failing collaboration with foreign markets, extreme weather events, inflation, and the longstanding impact of COVID-19—all of which jeopardize the smooth flow of products. It is paramount to optimize the core mathematical dilemmas at the heart of the system.





The core task of supply chain management reduces to the Generalized Assignment Problem (GAP): finding the optimal assignment of a set of tasks to a set of agents to maximize global profit while adhering to each agent's budget. GAP is notoriously difficult to optimize since it belongs to the class of NP-hard problems, so its solution space grows exponentially with the number of tasks and agents.

Generalized Assignment Problem

Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} p_{i,j}x_{i,j}$

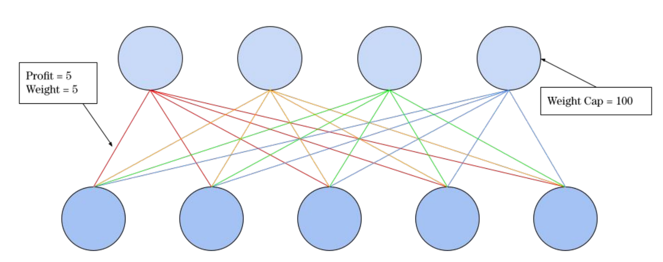
Subject to $\sum_{j=1}^{n} w_{i,j}x_{i,j} \le t_i \qquad i = 1, \cdots, m$

$\sum_{i=1}^{m} x_{i,j} = 1 \qquad j = 1, \cdots, n$

$x_{i,j} \in \{0, 1\}$

Let m be the number of agents and n be the number of tasks. **x**, **p**, and **w**, are each vectors of length m*n describing the assignment (boolean), profit, and weight (cost) of each agent task assignment.

Originally, the goal of the generalized assignment problem (GAP) is to *maximize* profits under the specified constraints, but we chose to minimize profits as researchers tend to use this variation of the problem. Albeit, the two problems are essentially equivalent. Additionally, we also require all tasks to be completed by exactly one agent.



Quantum Annealing

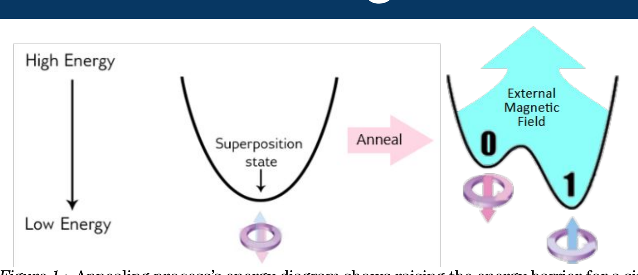


Figure 1 : Annealing process's energy diagram shows raising the energy barrier for a single qubit, resulting in a 50/50 probability of ending in a classical state of 0 or 1. , dwave, https://docs.dwavesys.com/docs/latest/_images/simple_anneal_one_qubit_without_h.png

In addition the external magnetic field, a coupler can be used to create entangled states. Each coupler can be assigned a bias, such that if the state meets the conditions for the coupler, it becomes a more favorable solution. The combined energies from the coupler and the magnetic field determines the energy level of the solution. The entangled state with the lowest energy is given the highest probability to be measured.

Quantum states are in a superposition between 0 and 1. The probability of measuring either a 0 or 1 can be set by applying an external magnetic field.

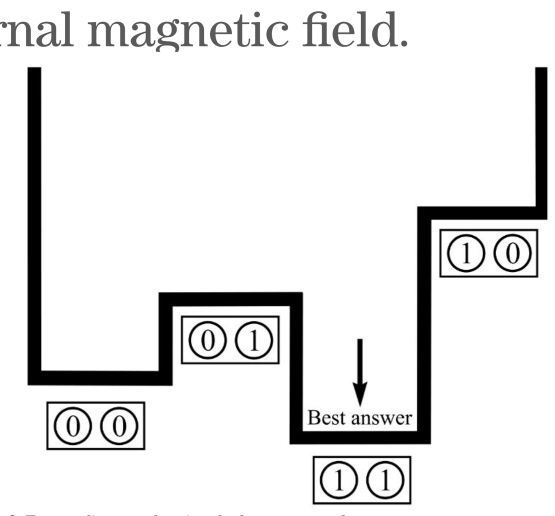


Figure 2: Energy diagram showing the best answer, dwve, https://docs.dwavesys.com/docs/latest/_images/simple_anneal_one_qubit_without_h.png

The value of these biases/energy can be assigned to each qubit via. the Hamiltonian, which can either be in the form of the Ising Model or the QUBO model.

$$f(x) = \sum_i Q_{i,i}x_i = \sum_{i<j} Q_{i,j}x_ix_j \qquad E_{ising}(s) = \sum_{i=1}^{N} h_is_i + \sum_{i=1}^{N}\sum_{j=i+1}^{N} J_{i,j}s_is_j$$

Research Questions

1. Could quantum annealing be effective in finding solution to the Generalized Assignment Problem for large m and n?
2. How much of an improvement in profit minimization and time depletion does quantum annealing provide over classical methods?
3. Can any heuristics be embedded into the QUBO to improve performance?
4. Can quantum annealing be used to solve GAP problems that classical algorithms cannot in a reasonable amount of time?
5. Can quantum annealing be applied to a dataset which is similar to a real world supply chain?

QUBO Coefficients

The coefficients of the objective function of the QUBO formulation are encoded in the Q matrix of size (mn) x (mn). They can be calculated by simplifying the objective function.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}\,x_{ij} \;+\; \lambda\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} x_{ij} - 1\Big)^2 \;+\; \mu\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} w_{ij}\,x_{ij} - t_i\Big)^2$$

where the constants λ, μ > 0 are sufficiently large penalty coefficients such that selecting a configuration that violates the constraints costs more than any valid configuration.

$$\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} x_{ij} - 1\Big)^2 = \sum_{j=1}^{n}\Big[\Big(\sum_{i=1}^{m} x_{ij}\Big)^2 - 2\sum_{i=1}^{m} x_{ij} + 1\Big]$$

Since $x_{ij}^2 = x_{ij}$ for $x_{ij} \in \{0, 1\}$,

$$\Big(\sum_{i=1}^{m} x_{ij}\Big)^2 = \sum_{i=1}^{m} x_{ij} + 2\sum_{1\le i<k\le m} x_{ij}\,x_{kj}$$

Thus, for each j,

$$\Big(\sum_{i=1}^{m} x_{ij} - 1\Big)^2 = -\sum_{i=1}^{m} x_{ij} + 2\sum_{i<k} x_{ij}x_{kj} + 1,$$

$$\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} w_{ij}\,x_{ij} - t_i\Big)^2 = \sum_{i=1}^{m}\Big[\Big(\sum_{j=1}^{n} w_{ij}x_{ij}\Big)^2 - 2\,t_i\sum_{j=1}^{n} w_{ij}x_{ij} + t_i^2\Big]$$

Expanding

$$\Big(\sum_{j=1}^{n} w_{ij}x_{ij}\Big)^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} w_{ij}\,w_{ik}\,x_{ij}\,x_{ik}$$

Indexing each binary variable x_ij with a single index u: $u = (i-1)\,n + j, \quad x_u \equiv x_{ij}$

$$\boxed{Q_{u,u} = p_{ij} - \lambda + \mu\,[w_{ij}^2 - 2\,t_i\,w_{ij}],}$$

$$\boxed{Q_{u,v} = \begin{cases} \lambda, & \text{if } x_u \text{ and } x_v \text{ correspond to the same item } j \text{ but different bins } (i \ne k), \\ \mu\,w_{ij}\,w_{i\ell}, & \text{if } x_u \text{ and } x_v \text{ correspond to the same bin } i \text{ but different items } (j \ne \ell), \\ 0, & \text{otherwise} \end{cases}}$$

The remaining constant terms can be dropped since they do affect which binary vector x produces the value in the objective function.

Conclusion

Our research demonstrates the potential of quantum annealing in optimizing the GAP for large datasets, a key challenge in supply chain management. We found through an implementation of the D-Wave CQM-based quantum annealer that we can achieve near-optimal results within only a few percentage points on average from theoretical bounds for the GAP on most datasets.

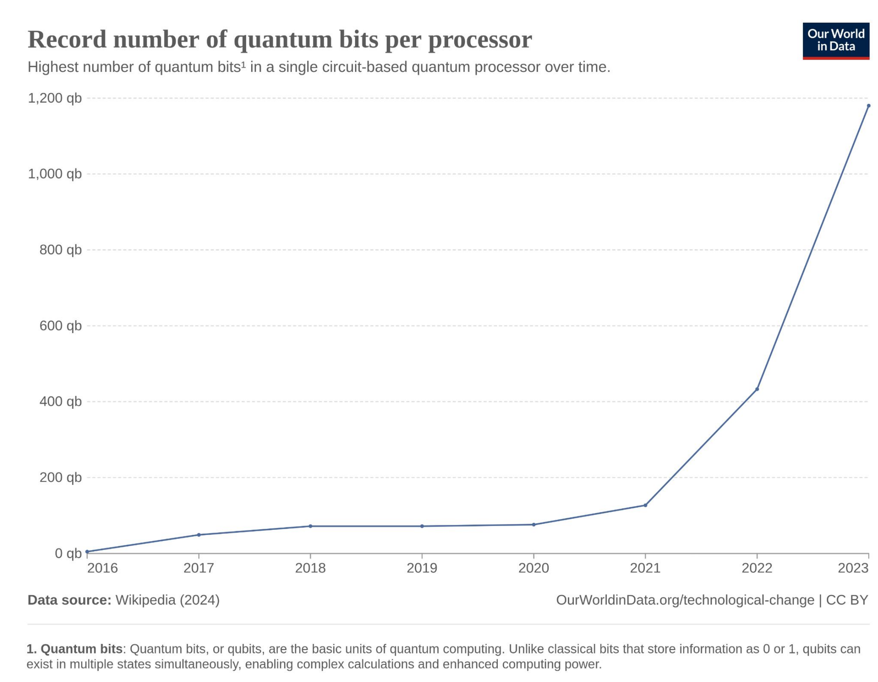


Figure 3 : Wikipedia (2024) – with minor processing by Our World in Data. "Record number of quantum bits per processor" [dataset]. Wikipedia, "List of quantum processors" [original data]. Retrieved January 31, 2025 from https://ourworldindata.org/grapher/quantum-bits-per-processor

Although classical methods employing heuristics frequently outperformed the quantum algorithm by a small margin (<1%), the quantum algorithm was able to find solutions to the FAPGAP and romagna which the classical algorithm was not able to solve. Especially as quantum annealing is projected to improve in both speed and accuracy as quantum technologies rapidly developing every year, our quantum models will serve as exceptionally fast and reliable alternatives rivaling long-standing classical GAP approaches.

For time-sensitive consumers, businesses, organizations, and governments involved in local and global supply chains, our model provides perhaps the best tool for optimizing their systems in order to reduce resource waste, foster cooperation, and maximize productive output and economic value retention.

Future Work

Although quantum annealing is the primary quantum technique for combinatorial optimization problems, there are still many potential algorithms that could provide stronger results or quantum speedups.

- Grover's Search Algorithm: Provides quadratic speedups against classical brute-force search
  - Could offer some advantage for smaller problem sizes
- Quantum Monte Carlo: A family of computational methods favoring quantum or statistical problems
  - Potential for a hybrid model leveraging statistical sampling
- Variational Quantum Eigensolver: Commonly used in quantum chemistry
  - Frequently applied to combinatorial optimization problems

Further research could also be conducted into the usage of heuristics in quantum annealing algorithms to find more significant improvements.

References

Quantinuum. (2024, April 25). Quantinuum extends its significant lead in quantum computing, achieving historic milestones for hardware fidelity and quantum volume. Quantinuum. https://www.quantinuum.com/blog/quantinuum-extends-its-significant-lead-in-quantum-computing-achieving-historic-milestones-for-hardware-fidelity-and-quantum-volume

IBM. (2023, May 24). IBM Quantum roadmap 2025. IBM Quantum. https://www.ibm.com/quantum/blog/ibm-quantum-roadmap-2025

Beasley, J. E. (n.d.). Generalised assignment problem. OR-Library. https://people.brunel.ac.uk/~mastjjb/jeb/orlib/gapinfo.html

Born, M., & Fock, V. A. (1928). Beweis des adiabatensatzes. Zeitschrift für Physik, 51 (3–4), 165–180. https://doi.org/10.1007/BF01343193

Bożejko, W., Burduk, A., Pempera, J., Uchroński, M., & Wodecki, M. (2024). Optimal solving of a binary knapsack problem on a d-wave quantum machine and its implementation in production systems. Annals of Operations Research. https://doi.org/10.1007/s10479-024-06025-1

Business Wire. D-Wave introduces service-level agreements for leap quantum cloud customers in production. (2024). https://www.businesswire.com/news/home/20241003517817/en

Chu, P.C., & Beasley, J.C. (1997). A genetic algorithm for the generalized assignment problem. Computers & Operations Research, 24, 17-23.

Denchev, V. S., Boixo, S., Isakov, S. V., Ding, N., Babbush, R., Smelyanskiy, V., Martinis, J., & Neven, H. (2016). What is the computational value of finite-range tunneling? Physical Review X, 6(3). https://doi.org/10.1103/physrevx.6.031015

Fleischer, L., Goemans, M. X., Mirrokni, V. S., & Sviridenko, M. (2011). Tight Approximation Algorithms for Maximum Separable Assignment Problems. Mathematics of Operations Research, 36(3), 416–431.

*All uncited images were created by student/researcher