# Final Project of Numerical Methods for PDE

*Finite Volume Method for Euler Equation*

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## 1 Problem Description

- Mass Conservation

$$\rho_t + (\rho u)_x = 0.$$

- Momentum Conservation

$$(\rho u)_t + (\rho u^2 + p)_x = 0.$$

- Energy Conservation

$$E = \rho e + \frac{1}{2}\rho u^2.$$

$$E_t + (u(E+p))_x = 0.$$

To close the equation system, we need one more equation
- Equation of State

$$p = \rho e(\gamma - 1)$$

### 1.1 Hyperbolic structure of the 1D Euler equations

The governing equation could be written into a system of hyperbolic equations:

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{V})}{\partial x} = 0 \quad \text{or}$$

$$\mathbf{V} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{F}(\mathbf{V}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}$$

Introduce *primitive variable* $\mathbf{q} = [\rho, u, p]^T$ and write above equation in quasi-linear form with primitive variable:

$$\begin{bmatrix}\rho\\u\\p\end{bmatrix}_t + \begin{bmatrix}u & \rho & 0\\0 & u & 1/\rho\\0 & \gamma p & u\end{bmatrix}\begin{bmatrix}\rho\\u\\p\end{bmatrix}_x = 0$$

$$\mathbf{q}_t + \frac{\partial \bar{\mathbf{F}}(\mathbf{q})}{\partial \mathbf{q}}\mathbf{q}_x = 0$$

## 1.2 Finite Volume Method

Based on Green's formula, we have

$$\nabla \cdot \mathbf{F}(\mathbf{V}) = \oint_{\partial T_j} \mathbf{F}(\mathbf{V}) \cdot \mathbf{n} ds$$

For finite volume discretization, we have[1]

$$\oint_{\partial T_j} \mathbf{F}(\mathbf{V}) \cdot \mathbf{n} ds \approx \sum_{e_{ik} \in \partial T_j} \oint_{e_{jk}} \bar{\mathbf{F}}(\mathbf{V}_j, \mathbf{V}_k) \cdot \mathbf{n}_{jk} dl$$

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t_n}{\Delta x_j}(F_{j+1/2}^n - F_{j-1/2}^n) \tag{1}$$

## 1.3 Eigenvalue and Eigenvector of Euler Flux Jacobian

1. Flux Jacobian of primitive variable is equal to

$$\frac{\partial \bar{\mathbf{F}}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix}u & \rho & 0\\0 & u & 1/\rho\\0 & \gamma p & u\end{bmatrix}$$

The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = u - c, \qquad \lambda_2 = u, \qquad \lambda_3 = u + c$$

$$\mathbf{r}_1 = \begin{bmatrix}-\rho/c\\1\\-\rho c\end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \mathbf{r}_3 = \begin{bmatrix}\rho/c\\1\\\rho c\end{bmatrix}$$

2. Flux of variable $\mathbf{V}$ equal to

$$\bar{\mathbf{F}}(\mathbf{V}) = \begin{bmatrix} V_2 \\ \frac{V_2^2}{V_1} + \left(V_3 - \frac{V_2^2}{2V_1}\right)(\gamma - 1) \\ \frac{\gamma V_2 V_3}{V_1} - \frac{V_2^3}{2V_1^2}(\gamma - 1)\end{bmatrix}$$

Flux Jacobian of variable $\mathbf{V}$ equal to [1]

$$\frac{\partial \bar{\mathbf{F}}(\mathbf{V})}{\partial \mathbf{V}} = \begin{bmatrix} u & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ \frac{\gamma-1}{2}u^3 - uH & H - (\gamma-1)u^2 & \gamma u\end{bmatrix}$$

Then the Euler Equation becomes

[1] with $p = (E - \frac{1}{2}\rho u^2)(\gamma - 1)$, and $H = \frac{E+p}{\rho}$

$$\mathbf{V}_t + \frac{\partial \bar{\mathbf{F}}(\mathbf{V})}{\partial \mathbf{V}} \mathbf{V}_x = 0$$

The eigenvalues and corresponding eigenvectors of Jacobian matrix is

$$\lambda_1 = u - c, \qquad \lambda_2 = u, \qquad \lambda_3 = u + c$$

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ u - c \\ H - uc \end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} \quad \mathbf{r}_3 = \begin{bmatrix} 1 \\ u + c \\ H + uc \end{bmatrix}$$

Here $c = \sqrt{(\gamma - 1)(H - \frac{1}{2}u^2)}$. [2]

- **linear degenerate**

## 1.4 Conservative Flux

### 1.4.1 Lax-Friedrichs Numerical Flux

### 1.4.2 Refined Lax-Friedrichs Numerical Flux

### 1.4.3 Roe Numerical Flux

[3]

- **Roe Average**

$$\bar{u} = \frac{\sqrt{\rho_r}u_r + \sqrt{\rho_l}u_l}{\sqrt{\rho_r} + \sqrt{\rho_l}}, \quad \bar{H} = \frac{\sqrt{\rho_r}H_r + \sqrt{\rho_l}H_l}{\sqrt{\rho_r} + \sqrt{\rho_l}} \tag{2}$$

# 2 Riemann Problem

## 2.1 Exact Riemann Problem

- **Rankine-Hugoniot jump condition**
  For a shock wave connecting a known left or right state $q^*$ and an unknown middle state $q$, the moving speed $s$ of the shock should obey Eq.(3).[2]

$$s(q^* - q) = f(q^*) - f(q) \tag{3}$$

- **Hugoniot locus**
- **Lax entropy condition**
  This physical entropy condition is equivalent to the mathematical condition that for a 1-shock to be physically relevant, the 1-characteristics must impinge on the shock (the Lax entropy condition). If the entropy condition is violated, the 1-characteristics would spread out, allowing the insertion of an expansion fan (rarefaction wave).
  - Shocks appear in regions where characteristics converge, as in the traffic jam example above.
  - Rarefactions appear in regions where characteristics are spreading out, as in the green light example.[2]

- **Integral curves**
- **Riemann invariants**
- **similarity solutions**
-

[4]

### 2.2 Approximate Riemann Problem

Besides the exact Riemann solution, we could also obtain an approximate solution by linearizing the Euler Equation and replacing Jacobian of flux $\frac{\partial \mathbf{F}(\mathbf{V})}{\partial \mathbf{q}}$ by a linear operator $\hat{\mathbf{A}}(q_l, q_r)$ depending on the left and right status [3]. Furthermore, this approximate linear operator should satisfy:

1. Consistency: $\hat{\mathbf{A}}(q_l, q_r) \to f'(q)$ as $q_l, q_r \to q$
2. Hyperbolicity: $\hat{\mathbf{A}}$ must be diagonalizable with real eigenvalues, so that we can define the waves and speeds needed in the approximate solution.
3. Conservation: $\hat{\mathbf{A}}(q_l, q_r)(q_r - q_l) = f(q_r) - f(q_l)$

One common linear operator is the flux Jacobian of an average state:

$$\hat{\mathbf{A}}(q_l, q_r) = f'(\hat{q}), \quad \hat{q} \in [q_l, q_r] \tag{4}$$

For the system of hyperbolic equations, we could solve the system by decomposing the linear operator $\hat{\mathbf{A}}$ into the eigenvectors $\mathbf{R}$ and the matrix that eigenvalues on the diagonal $\mathbf{\Lambda}$ so that

$$\begin{aligned}
\mathbf{q_t} + \hat{\mathbf{A}}\mathbf{q_x} &= 0 \\
\mathbf{q_t} + \mathbf{R}\mathbf{\Lambda}\mathbf{R^{-1}}\mathbf{q_x} &= 0 \\
\mathbf{R^{-1}}\mathbf{q_t} + \mathbf{\Lambda}\mathbf{R^{-1}}\mathbf{q_x} &= 0 \\
\mathbf{w_t} + \mathbf{\Lambda}\mathbf{w_x} &= 0 \\
w_k(x,t) &= w_k(x - \lambda_k t, 0) \\
\mathbf{q}(\mathbf{x},\mathbf{t}) &= \sum_k w_k(x - \lambda_k t, 0)\mathbf{r_k}
\end{aligned}$$

### 2.3 Newton Iteration

#### 2.3.1 Stationary Solution

For the stationary solution, we would have following iteration scheme [1]

$$\alpha \|\mathbf{R}_j^{(n)}\|_{l^1} \delta \mathbf{U}_j^{(n)} + \sum_{e_j k \in \partial T_j} \int_{e_{jk}} \left( \frac{\partial \bar{\mathbf{F}}^{(n)}}{\partial \mathbf{U}_j} \delta \mathbf{U}_j^{(n)} \right) \cdot \mathbf{n}_{jk} dl + \sum_{e_j k \in \partial T_j} \int_{e_{jk}} \left( \frac{\partial \bar{\mathbf{F}}^{(n)}}{\partial \mathbf{U}_k} \delta \mathbf{U}_k^{(n)} \right) \cdot \mathbf{n}_{jk} dl = -\mathbf{R}_j^{(n)}$$

1. Input $\mathbf{U}^{(0)}$ as the initial guess, and set $n = 0$
2. Use an approximate solver for the system to get a $\delta \mathbf{U}^{(n)}$
3. update $\mathbf{U}^{(n+1)}$ by $\mathbf{U}^{(n)} + \tau \delta \mathbf{U}^{(n)}$
4. Reconstruct $\mathbf{U}^{(n+1)}$ using its cell mean values to get a piecewise polynomial expression on each cell
5. Check if the residual $\mathbf{R}^{n+1}$ is small enough

#### 2.3.2 PDE

### 2.4 Linear Multigrid Method for Jacobian Matrix

[5]

## 3 Error Analysis

## 4 Conclusion

## References


**1.** Ruo Li, Xin Wang, and Weibo Zhao. A multigrid block LU-SGS algorithm for euler equations on unstructured grids. page 26.

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**3.** P.L Roe. Approximate riemann solvers, parameter vectors, and difference schemes. 43(2):357–372.

**4.** Richard Sanders. An introduction to the finite volume method for the equations of gas dynamics. page 32.

**5.** Antony Jameson. Solution of the euler equations for two dimensional transonic flow by a multigrid method. 13(3):327–355.