Final Project of Numerical Methods for PDE

Finite Volume Method for Euler Equation

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May 3, 2022

1 Problem Description

- Mass Conservation

$$\rho_t + (\rho u)_x = 0.$$

- Momentum Conservation

$$(\rho u)_t + (\rho u^2 + p)_x = 0.$$

- Energy Conservation

$$E = \rho e + \frac{1}{2}\rho u^2.$$

$$E_t + (u(E+p))_x = 0.$$

To close the equation system, we need one more equation
- Equation of State

$$p = \rho e(\gamma - 1)$$

1.1 Hyperbolic structure of the 1D Euler equations

The governing equation could be written into a system of hyperbolic equations:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial x} = 0 \quad \text{or}$$

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{F}(\mathbf{q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}$$

1.2 Finite Volume Method

Based on Green's formula, we have

$$\nabla \cdot \mathbf{F}(\mathbf{V}) = \oint_{\partial T_j} \mathbf{F}(\mathbf{V}) \cdot \mathbf{n} ds$$

For finite volume discretization, we have[1]

$$\oint_{\partial T_j} \mathbf{F}(\mathbf{V}) \cdot \mathbf{n} ds \approx \sum_{e_{ik} \in \partial T_j} \oint_{e_{jk}} \bar{\mathbf{F}}(\mathbf{V}_j, \mathbf{V}_k) \cdot \mathbf{n}_{jk} dl$$

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t_n}{\Delta x_j} (F_{j+1/2}^n - F_{j-1/2}^n) \tag{1}$$

1.3 Eigenvalue and Eigenvector of Euler Flux Jacobian

Flux of variable $\mathbf{q}$ equal to

$$\bar{\mathbf{F}}(\mathbf{q}) = \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + \left(q_3 - \frac{q_2^2}{2q_1}\right)(\gamma - 1) \\ \frac{\gamma q_2 q_3}{q_1} - \frac{q_2^3}{2q_1^2}(\gamma - 1) \end{bmatrix}$$

Flux Jacobian of variable $\mathbf{V}$ equal to [1]

$$\frac{\partial \bar{\mathbf{F}}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} u & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ \frac{\gamma-1}{2}u^3 - uH & H - (\gamma-1)u^2 & \gamma u \end{bmatrix}$$

Then the Euler Equation becomes

$$\mathbf{q}_t + \frac{\partial \bar{\mathbf{F}}(\mathbf{q})}{\partial \mathbf{q}} \mathbf{q}_x = 0$$

The eigenvalues and corresponding eigenvectors of Jacobian matrix is

$$\lambda_1 = u - c, \qquad \lambda_2 = u, \qquad \lambda_3 = u + c$$

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ u - c \\ H - uc \end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} \quad \mathbf{r}_3 = \begin{bmatrix} 1 \\ u + c \\ H + uc \end{bmatrix}$$

Here $c = \sqrt{(\gamma - 1)(H - \frac{1}{2}u^2)}$. [2, 3]

2 Riemann Problem

2.1 Linearized Riemann Solver

Besides the exact Riemann solution, we could also obtain an approximate solution by linearizing the Euler Equation and replacing Jacobian of flux $\frac{\partial \mathbf{F}(\mathbf{q})}{\partial \mathbf{q}}$ by a linear operator $\hat{\mathbf{A}}(q_l, q_r)$ depending on the left and right status [3]. Furthermore, this approximate linear operator should satisfy:

[1] with $p = (E - \frac{1}{2}\rho u^2)(\gamma - 1)$, and $H = \frac{E+p}{\rho}$

1. Consistency: $\hat{\mathbf{A}}(\mathbf{q}_l, \mathbf{q}_r) \to \mathbf{F}'(\mathbf{q})$ as $\mathbf{q}_l, \mathbf{q}_r \to \mathbf{q}$
2. Hyperbolicity: $\hat{\mathbf{A}}$ must be diagonalizable with real eigenvalues, so that we can define the waves and speeds needed in the approximate solution.
3. Conservation:

$$\hat{\mathbf{A}}(\mathbf{q}_l, \mathbf{q}_r)(\mathbf{q}_r - \mathbf{q}_l) = \mathbf{F}(\mathbf{q}_r) - \mathbf{F}(\mathbf{q}_l) \tag{2}$$

One common linear operator is the flux Jacobian of an average state:

$$\hat{\mathbf{A}}(\mathbf{q}_l, \mathbf{q}_r) = \mathbf{F}'(\hat{\mathbf{q}}), \quad \hat{\mathbf{q}} \in [\mathbf{q}_l, \mathbf{q}_r] \tag{3}$$

Plugged Eq.(3) back into Eq.(2), we have

$$\mathbf{F}'(\hat{\mathbf{q}})(\mathbf{q}_r - \mathbf{q}_l) = \mathbf{F}(\mathbf{q}_r) - \mathbf{F}(\mathbf{q}_l) \tag{4}$$

Solve Eq.(4), we could obtain the Roe average state expression

$$\bar{u} = \frac{\sqrt{\rho_r}u_r + \sqrt{\rho_l}u_l}{\sqrt{\rho_r} + \sqrt{\rho_l}}, \quad \bar{H} = \frac{\sqrt{\rho_r}H_r + \sqrt{\rho_l}H_l}{\sqrt{\rho_r} + \sqrt{\rho_l}} \tag{5}$$

For the system of hyperbolic equations, we could solve the system by decomposing the linear operator $\hat{\mathbf{A}}$ into the eigenvectors $\mathbf{R}$ and the matrix that eigenvalues on the diagonal $\mathbf{\Lambda}$ so that

$$\begin{aligned}
\mathbf{q_t} + \hat{\mathbf{A}}\mathbf{q_x} &= 0 \\
\mathbf{q_t} + \mathbf{R\Lambda R^{-1}}\mathbf{q_x} &= 0 \\
\mathbf{R^{-1}}\mathbf{q_t} + \mathbf{\Lambda R^{-1}}\mathbf{q_x} &= 0 \\
\mathbf{w_t} + \mathbf{\Lambda w_x} &= 0 \\
w_k(x,t) &= w_k(x - \lambda_k t, 0) \\
\mathbf{q}(\mathbf{x}, \mathbf{t}) &= \sum_k w_k(x - \lambda_k t, 0)\mathbf{r_k}
\end{aligned}$$

2.2 Two-wave solver

Instead of linearization, we could also approximate the solution to be one wave propagating in each direction based on previous analysis of Euler Equations. Thus, there will be only one intermediate state q_m in this situation, which satisfies $\mathcal{W}_1 = \mathbf{q}_m - \mathbf{q}_l$ and $\mathcal{W}_2 = \mathbf{q}_r - \mathbf{q}_m$. Thus,

$$\mathbf{F}(\mathbf{q}_r) - \mathbf{F}(\mathbf{q}_l) = s_1\mathcal{W}_1 + s_2\mathcal{W}_2 \tag{6}$$

in which s_1 and s_2 are the corresponding wave speeds.

From Eq.(6), we could obtain the expression for the intermediate state

$$q_m = \frac{\mathbf{F}(\mathbf{q}_r) - \mathbf{F}(\mathbf{q}_l) + s_1\mathbf{q}_l - s_2\mathbf{q}_r}{s_1 - s_2} \tag{7}$$

2.2.1 Lax-Friedrichs Numerical Flux

The simpliest method is Lax-Friedrichs method. This method assumes that the wave speeds have the same value but different signs. $s_1 = -s_2 = a$. Furthermore, the wave speed is chosen by the maximum of flux for q over $\mathbf{q}_l$ and $\mathbf{q}_r$:

$$a = max(|\mathbf{F}'(\mathbf{q})|) \tag{8}$$

2.2.2 Harten-Lax-van Leer (HLL)

This method assumes that wave speeds are different but still based on the maximum and minimum of the flux:

$$s_1 = \min(\mathbf{F}'(\mathbf{q})) \quad \text{and} \quad s_2 = \max(\mathbf{F}'(\mathbf{q})) \tag{9}$$

3 Error Analysis

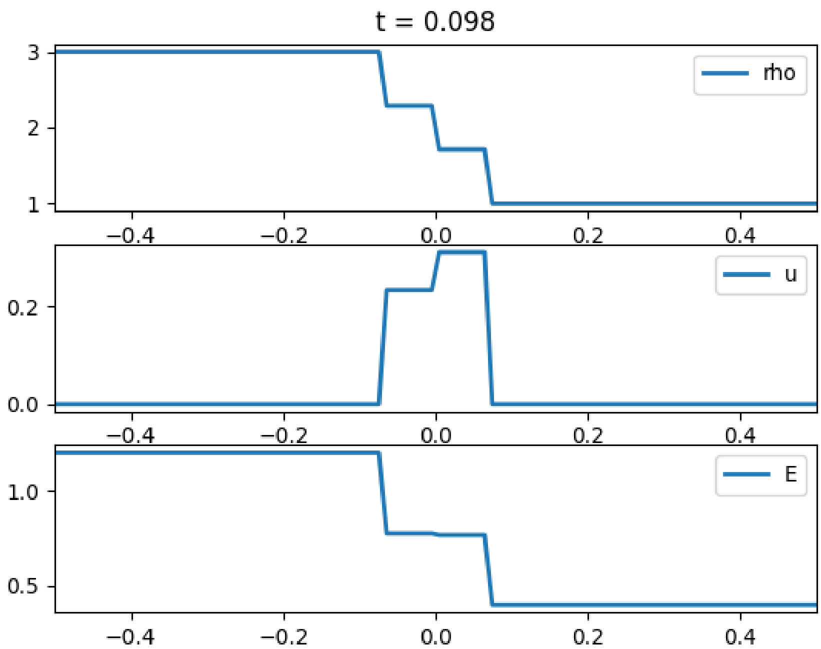

Figure 1: Riemann Solution of Shock Tube

4 Conclusion

References

1. Ruo Li, Xin Wang, and Weibo Zhao. A multigrid block LU-SGS algorithm for euler equations on unstructured grids. page 26.

2. David I. Ketcheson, Randall J. LeVeque, and Mauricio J. del Razo. *Chapter 13: Approximate solvers for the Euler equations of gas dynamics*. SIAM. https://epubs.siam.org/doi/pdf/10.1137/1.9781611976212.ch13.

3. P.L Roe. Approximate riemann solvers, parameter vectors, and difference schemes. 43(2):357–372.