

Formulation

It doesn't use any of auto-diff tools, here is the analytical form of several derivatives.

1. 2d splat

```
struct Splat
{
    glm::vec2 pos;
    float sx;
    float sy;
    float rot;
    glm::vec3 color;
    float opacity;
};
```

2. The covariance matrix

$$\Sigma = VLV^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

3. Inverse of the covariance matrix

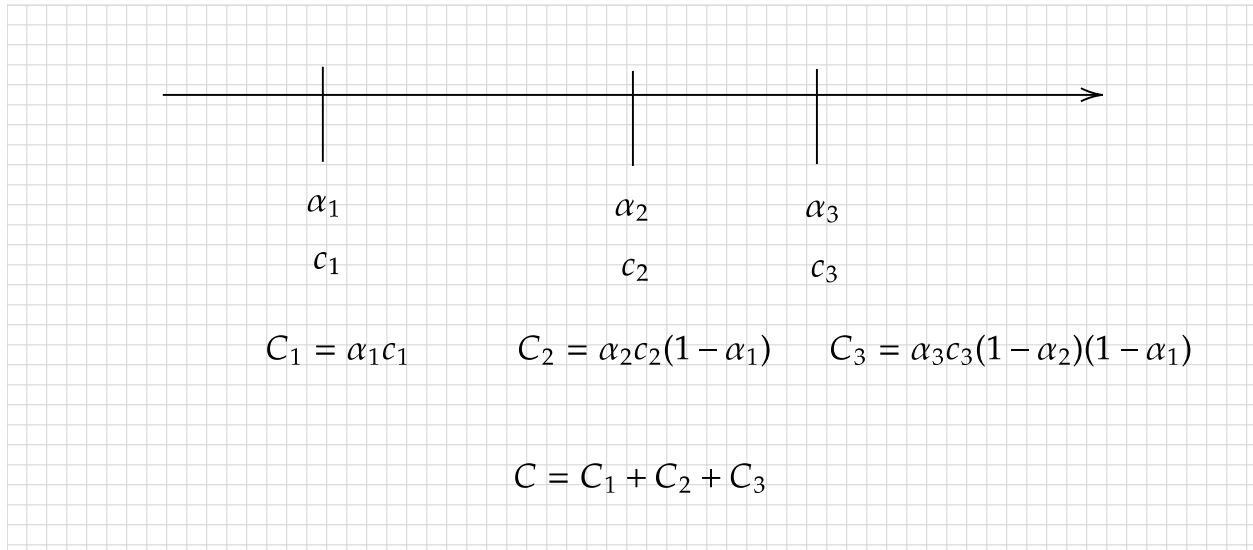
$$\Sigma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s_y^{-2} \sin^2 \theta + s_x^{-2} \cos^2 \theta & s_x^{-2} \sin \theta \cos \theta - s_y^{-2} \sin \theta \cos \theta \\ s_x^{-2} \sin \theta \cos \theta - s_y^{-2} \sin \theta \cos \theta & s_y^{-2} \cos^2 \theta + s_x^{-2} \sin^2 \theta \end{bmatrix}$$

4. Alpha

$$\begin{aligned} \alpha_i(x) &= o_i \exp \left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \\ &= o_i \exp \left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v} \right) \end{aligned}$$

5. Pixel Color

$$C = \sum_{i \in N} c_i \alpha_i \prod_{j=1}^{i-1} (1 - \alpha_j)$$



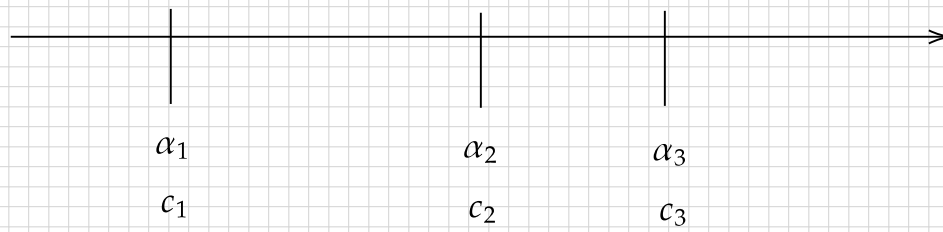
Note that this is 2d so order is pre-defined. Also, there is no reorder, cloning etc.

6. Derivative of color

$$\frac{\partial C_{r,g,b}}{\partial c_i} = \sum_{i \in N} \alpha_i \prod_{j=1}^{i-1} (1 - \alpha_j)$$

7. Derivative of alpha

Let's use an idea of color from the back S_i :



$$C_1 = \alpha_1 c_1$$

$$C_2 = \alpha_2 c_2 (1 - \alpha_1)$$

$$C_3 = \alpha_3 c_3 (1 - \alpha_2)(1 - \alpha_1)$$

$$S_1 = C_2 + C_3$$

$$S_2 = C_3$$

$$S_3 = 0$$

$$S_i = \sum_{k=i+1}^N c_k \alpha_k \prod_{j=1}^{k-1} (1 - \alpha_j)$$

Color from background

$$\begin{aligned} \frac{\partial C}{\partial \alpha_i} &= \frac{\partial C_i}{\partial \alpha_i} + \frac{\partial S_i}{\partial \alpha_i} \\ &= c_i \prod_{j=1}^{i-1} (1 - \alpha_j) - \frac{S_i}{(1 - \alpha_i)} \end{aligned}$$

Note that $\frac{S_i}{(1 - \alpha_i)}$ is a constant.

8. Derivatives of position

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial \vec{v}_x} &= \frac{\partial}{\partial \vec{v}_x} o_i \exp\left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= \alpha_i \frac{\partial}{\partial \vec{v}_x} \left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial \vec{v}_x} (\vec{v}^T \Sigma^{-1} \vec{v}) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial \vec{v}_x} \left(\vec{v}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{v}\right) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial p_x} \left(\begin{bmatrix} \vec{v}_x & \vec{v}_y \end{bmatrix} \begin{bmatrix} a\vec{v}_x + b\vec{v}_y \\ c\vec{v}_x + d\vec{v}_y \end{bmatrix} \right) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial p_x} (a\vec{v}_x^2 + b\vec{v}_x \vec{v}_y + c\vec{v}_x \vec{v}_y + d\vec{v}_y^2) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial p_x} (a\vec{v}_x^2 + (b+c)\vec{v}_x \vec{v}_y + d\vec{v}_y^2) \\
&= -\frac{1}{2} \alpha_i (2a\vec{v}_x + (b+c)\vec{v}_y) \\
\\
\frac{\partial \alpha_i}{\partial \vec{v}_y} &= -\frac{1}{2} \alpha_i \frac{\partial}{\partial p_y} (a\vec{v}_x^2 + (b+c)\vec{v}_x \vec{v}_y + d\vec{v}_y^2) \\
&= -\frac{1}{2} \alpha_i (2d\vec{v}_y + (b+c)\vec{v}_x)
\end{aligned}$$

Thus, derivatives for the position of gaussian are:

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial \vec{\mu}_x} &= -\frac{\partial \alpha_i}{\partial \vec{v}_x} = \frac{1}{2} \alpha_i (2a\vec{v}_x + (b+c)\vec{v}_y) \\
\frac{\partial \alpha_i}{\partial \vec{\mu}_y} &= -\frac{\partial \alpha_i}{\partial \vec{v}_y} = \frac{1}{2} \alpha_i (2d\vec{v}_y + (b+c)\vec{v}_x)
\end{aligned}$$

9. Derivatives of scaling x,y

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial s_x} &= \frac{\partial}{\partial \vec{s}_x} o_i \exp\left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= \alpha_i \frac{\partial}{\partial s_x} \left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial s_x} \left(\vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial s_x} \left(\vec{v}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{v}\right)
\end{aligned}$$

Need derivatives for each components in Σ^{-1} :

$$\begin{aligned}
\frac{\partial a}{\partial s_x} &= -2s_x^{-3} \cos^2 \theta \\
\frac{\partial a}{\partial s_y} &= -2s_y^{-3} \sin^2 \theta \\
\frac{\partial b}{\partial s_x} &= \frac{\partial c}{\partial s_x} = -2s_x^{-3} \sin \theta \cos \theta \\
\frac{\partial b}{\partial s_y} &= \frac{\partial c}{\partial s_y} = 2s_y^{-3} \sin \theta \cos \theta \\
\frac{\partial d}{\partial s_x} &= -2s_x^{-3} \sin^2 \theta \\
\frac{\partial d}{\partial s_y} &= -2s_y^{-3} \cos^2 \theta
\end{aligned}$$

Thus, combining aboves:

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial s_x} &= -\frac{1}{2} \alpha_i \frac{\partial}{\partial s_x} \left(a \vec{v}_x^2 + b \vec{v}_x \vec{v}_y + c \vec{v}_x \vec{v}_y + d \vec{v}_y^2 \right) \\
&= -\frac{1}{2} \alpha_i \left\{ \frac{\partial}{\partial s_x} a \vec{v}_x^2 + \frac{\partial}{\partial s_x} b \vec{v}_x \vec{v}_y + \frac{\partial}{\partial s_x} c \vec{v}_x \vec{v}_y + \frac{\partial}{\partial s_x} d \vec{v}_y^2 \right\} \\
&= -\frac{1}{2} \alpha_i \left\{ (-2s_x^{-3} \cos^2 \theta) \vec{v}_x^2 + 2(-2s_x^{-3} \sin \theta \cos \theta) \vec{v}_x \vec{v}_y + (-2s_x^{-3} \sin^2 \theta) \vec{v}_y^2 \right\} \\
&= -\alpha_i \left\{ (-s_x^{-3} \cos^2 \theta) \vec{v}_x^2 + (-2s_x^{-3} \sin \theta \cos \theta) \vec{v}_x \vec{v}_y + (-s_x^{-3} \sin^2 \theta) \vec{v}_y^2 \right\} \\
&= \alpha_i \left\{ (s_x^{-3} \cos^2 \theta) \vec{v}_x^2 + (2s_x^{-3} \sin \theta \cos \theta) \vec{v}_x \vec{v}_y + (s_x^{-3} \sin^2 \theta) \vec{v}_y^2 \right\} \\
&= \alpha_i \frac{1}{s_x^3} \left\{ \cos^2 \theta \vec{v}_x^2 + 2 \sin \theta \cos \theta \vec{v}_x \vec{v}_y + \sin^2 \theta \vec{v}_y^2 \right\} \\
&= \frac{\alpha_i}{s_x^3} \begin{bmatrix} \cos^2 \theta \\ 2 \sin \theta \cos \theta \\ \sin^2 \theta \end{bmatrix}^T \begin{bmatrix} \vec{v}_x^2 \\ \vec{v}_x \vec{v}_y \\ \vec{v}_y^2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial s_y} &= -\frac{1}{2} \alpha_i \left\{ \frac{\partial}{\partial s_y} a \vec{v}_x^2 + \frac{\partial}{\partial s_y} b \vec{v}_x \vec{v}_y + \frac{\partial}{\partial s_y} c \vec{v}_x \vec{v}_y + \frac{\partial}{\partial s_y} d \vec{v}_y^2 \right\} \\
&= -\frac{1}{2} \alpha_i \left\{ (-2s_y^{-3} \sin^2 \theta) \vec{v}_x^2 + 2(2s_y^{-3} \sin \theta \cos \theta) \vec{v}_x \vec{v}_y + (-2s_y^{-3} \cos^2 \theta) \vec{v}_y^2 \right\} \\
&= \frac{\alpha_i}{s_y^3} \left\{ \sin^2 \theta \vec{v}_x^2 - 2 \sin \theta \cos \theta \vec{v}_x \vec{v}_y + \cos^2 \theta \vec{v}_y^2 \right\} \\
&= \frac{\alpha_i}{s_y^3} \begin{bmatrix} \sin^2 \theta \\ -2 \sin \theta \cos \theta \\ \cos^2 \theta \end{bmatrix}^T \begin{bmatrix} \vec{v}_x^2 \\ \vec{v}_x \vec{v}_y \\ \vec{v}_y^2 \end{bmatrix}
\end{aligned}$$

10. Derivatives of rotation

Starts from derivatives of each component in Σ^{-1} :

$$\begin{aligned}
\frac{\partial a}{\partial \theta} &= 2(s_y^{-2} - s_x^{-2}) \sin \theta \cos \theta \\
&= 2 \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2} \right) \sin \theta \cos \theta \\
\frac{\partial b}{\partial \theta} &= \frac{\partial c}{\partial \theta} = - \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2} \right) (\cos^2 \theta - \sin^2 \theta)
\end{aligned}$$

$$\frac{\partial d}{\partial \theta} = -2(s_y^{-2} - s_x^{-2}) \sin \theta \cos \theta = -\frac{\partial a}{\partial \theta}$$

Combining the aboves:

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial \theta} &= \frac{\partial}{\partial \theta} o_i \exp \left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \\
&= -\frac{1}{2} \alpha_i \frac{\partial}{\partial \theta} \left(\vec{v}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{v} \right) \\
&= -\frac{1}{2} \alpha_i \left(\frac{\partial a}{\partial \theta} \vec{v}_x^2 + \frac{\partial b}{\partial \theta} \vec{v}_x \vec{v}_y + \frac{\partial c}{\partial \theta} \vec{v}_x \vec{v}_y + \frac{\partial d}{\partial \theta} \vec{v}_y^2 \right) \\
&= -\frac{1}{2} \alpha_i \left(\frac{\partial a}{\partial \theta} \vec{v}_x^2 + 2 \frac{\partial b}{\partial \theta} \vec{v}_x \vec{v}_y + \frac{\partial d}{\partial \theta} \vec{v}_y^2 \right) \\
&= -\frac{1}{2} \alpha_i \left(\frac{\partial a}{\partial \theta} \vec{v}_x^2 + 2 \frac{\partial b}{\partial \theta} \vec{v}_x \vec{v}_y - \frac{\partial a}{\partial \theta} \vec{v}_y^2 \right) \\
&= -\frac{1}{2} \alpha_i \left(\frac{\partial a}{\partial \theta} (\vec{v}_x^2 - \vec{v}_y^2) + 2 \frac{\partial b}{\partial \theta} \vec{v}_x \vec{v}_y \right) \\
&= -\frac{1}{2} \alpha_i \left(2 \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2} \right) \sin \theta \cos \theta (\vec{v}_x^2 - \vec{v}_y^2) + 2 \left(- \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2} \right) (\cos^2 \theta - \sin^2 \theta) \right) \vec{v}_x \vec{v}_y \right) \\
&= -\alpha_i \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2} \right) \left(\sin \theta \cos \theta (\vec{v}_x^2 - \vec{v}_y^2) - (\cos^2 \theta - \sin^2 \theta) \vec{v}_x \vec{v}_y \right) \\
&= \alpha_i \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2} \right) \left((\cos^2 \theta - \sin^2 \theta) \vec{v}_x \vec{v}_y - \sin \theta \cos \theta (\vec{v}_x^2 - \vec{v}_y^2) \right)
\end{aligned}$$

11. Derivatives of opacity

$$\begin{aligned}\frac{\partial \alpha_i}{\partial o_i} &= \frac{\partial \alpha_i}{\partial o_i} o_i \exp\left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v}\right) \\ &= \exp\left(-\frac{1}{2} \vec{v}^T \Sigma^{-1} \vec{v}\right)\end{aligned}$$

12. Bounding box of the covariance matrix

You can define an ellipse by Mahalanobis' Distance

$$\begin{aligned}(x - \mu)^T \Sigma^{-1} (x - \mu) &= k^2 \\ a\vec{v}_x^2 + (b + c)\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2 &= 0\end{aligned}$$

where $b = c$, we get

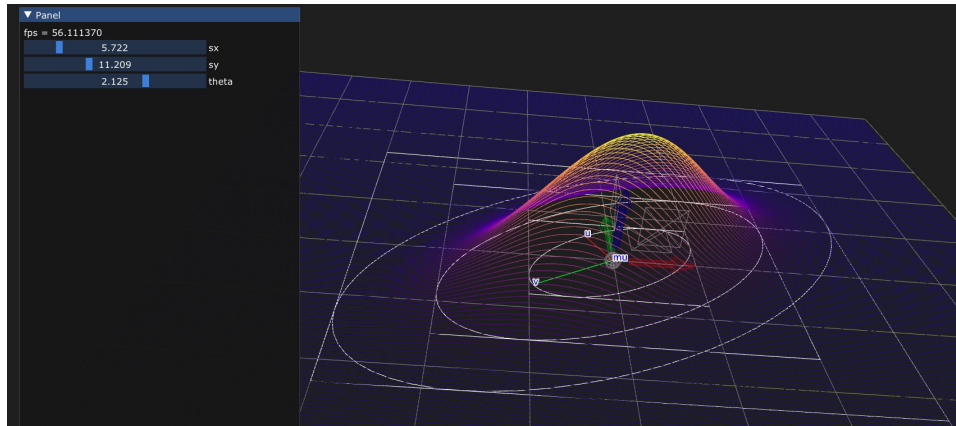
$$a\vec{v}_x^2 + 2b\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2 = 0$$

Let's use lagrange multiplier to get min max of \vec{v}_x, \vec{v}_y

$$\begin{aligned}L(\vec{v}_x, \vec{v}_y, \lambda) &= \vec{v}_x - \lambda \left(a\vec{v}_x^2 + 2b\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2 \right) \\ L(\vec{v}_x, \vec{v}_y, \lambda) &= \vec{v}_y - \lambda \left(a\vec{v}_x^2 + 2b\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2 \right)\end{aligned}$$

Then, we get

$$\begin{aligned}\vec{v}_x &= \pm k \sqrt{\frac{d}{ad - b^2}} = \pm k \sqrt{\frac{d}{\det(\Sigma^{-1})}} \\ \vec{v}_y^2 &= \pm k \sqrt{\frac{a}{ad - b^2}} = \pm k \sqrt{\frac{a}{\det(\Sigma^{-1})}}\end{aligned}$$



13. The exact range of \vec{v}_x

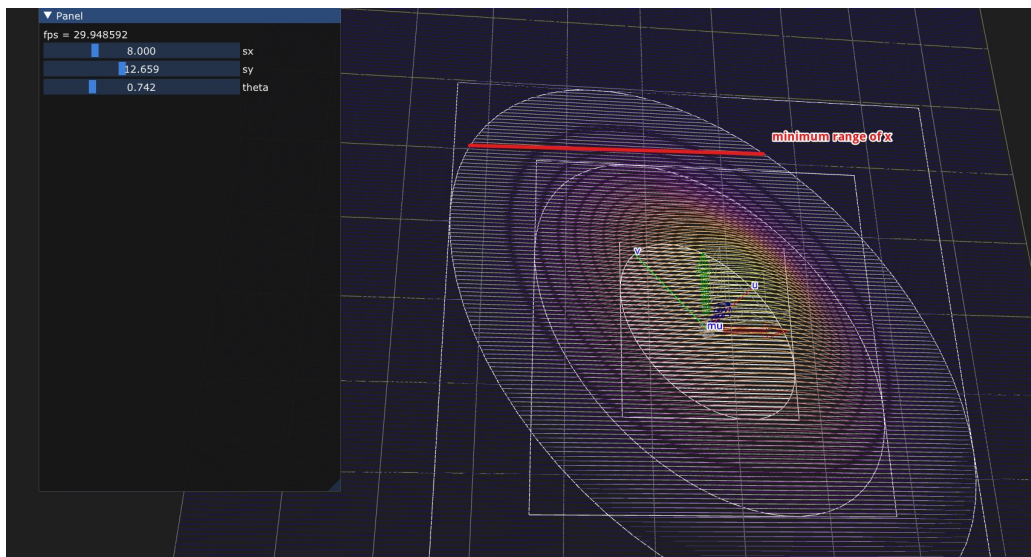
if \vec{v}_y is known, then you can know 2 points of the ellipse

$$g(\vec{v}_x) = a\vec{v}_x^2 + 2b\vec{v}_x\vec{v}_y + d\vec{v}_y^2 - k^2 = 0$$

$$Ax^2 + Bx + C = 0$$

$$x = \frac{-B \pm \sqrt{b^2 - 4AC}}{2A}$$

where:



$$A = a$$

$$B = 2b\vec{v}_y$$

$$C = d\vec{v}_y^2 - k^2$$