Formulation

It doesn't use any of auto-diff tools, here is the analytical form of several derivatives.

1. 2d splat

```
struct Splat
{
    glm::vec2 pos;
    float sx;
    float sy;
    float rot;
    glm::vec3 color;
    float opacity;
};
```

2. The covariance matrix

$$\Sigma = VLV^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

3. Inverse of the covariance matrix

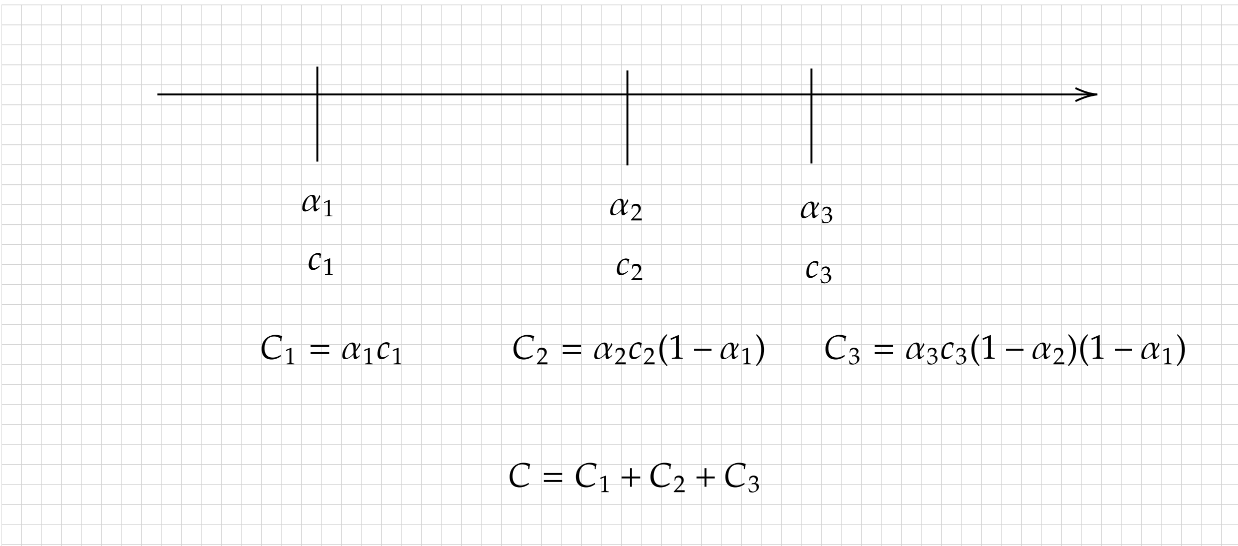
$$\Sigma^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s_y^{-2}\sin^2\theta + s_x^{-2}\cos^2\theta & s_x^{-2}\sin\theta\cos\theta - s_y^{-2}\sin\theta\cos\theta \\ s_x^{-2}\sin\theta\cos\theta - s_y^{-2}\sin\theta\cos\theta & s_y^{-2}\cos^2\theta + s_x^{-2}\sin^2\theta \end{bmatrix}$$

4. Alpha

$$\begin{aligned} \alpha_i(x) &= o_i \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \\ &= o_i \exp\left(-\frac{1}{2}\vec{v}^T\Sigma^{-1}\vec{v}\right) \end{aligned}$$

5. Pixel Color

$$C = \sum_{i \in N} c_i \alpha_i \prod_{j=1}^{i-1} (1 - \alpha_j)$$



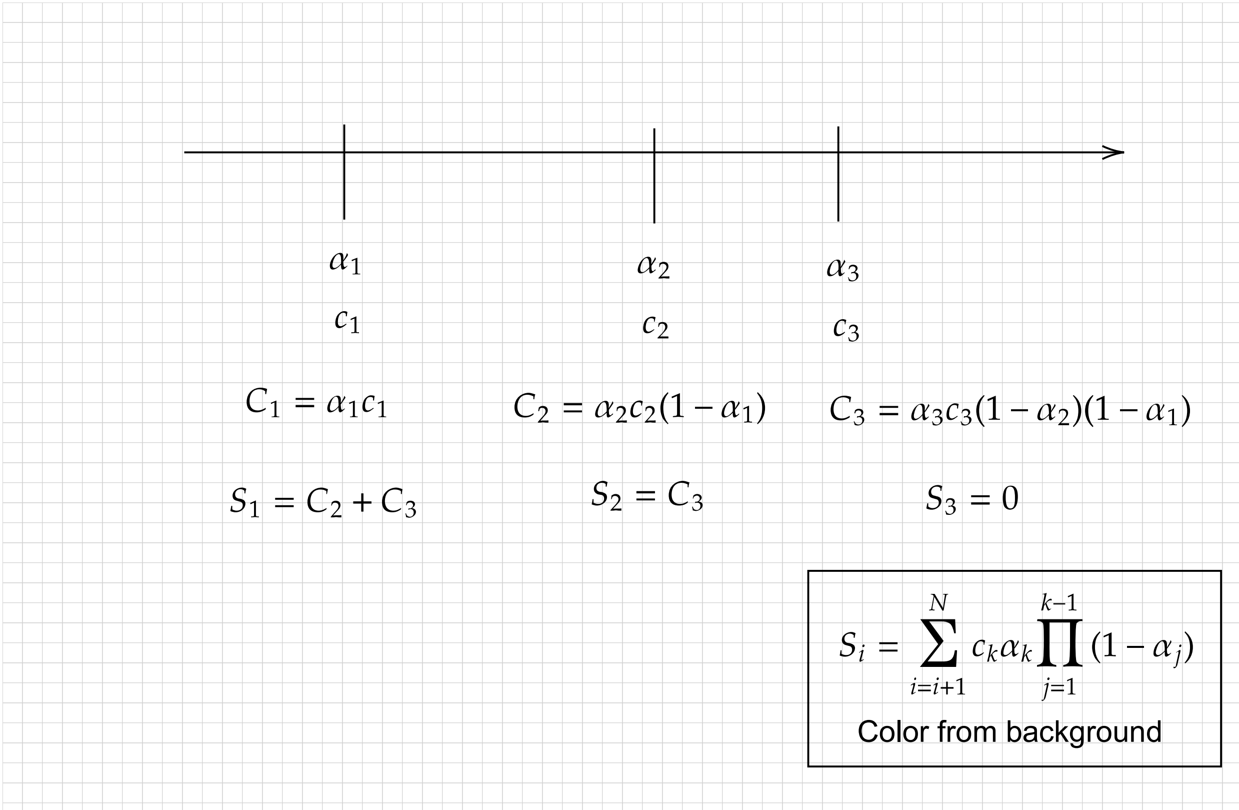
Note that this is 2d so order is pre-defined. Also, there is no reorder, cloning etc.

6. Derivative of color

$$\frac{\partial C_{r,g,b}}{\partial c_i} = \sum_{i \in N} \alpha_i \prod_{j=1}^{i-1} (1 - \alpha_j)$$

7. Derivative of alpha

Let's use an idea of color from the back S_i:



$$
\begin{aligned}
\frac{\partial C}{\partial \alpha_i} &= \frac{\partial C_i}{\partial \alpha_i} + \frac{\partial S_i}{\partial \alpha_i} \\
&= c_i \prod_{j=1}^{i-1} (1 - \alpha_j) - \frac{S_i}{(1 - \alpha_i)}
\end{aligned}
$$

Note that $\frac{S_i}{(1-\alpha_i)}$ is a constant.

8. Derivatives of position

$$
\begin{aligned}
\frac{\partial \alpha_i}{\partial \vec{v}_x} &= \frac{\partial}{\partial \vec{v}_x} o_i \exp\left(-\frac{1}{2}\vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= \alpha_i \frac{\partial}{\partial \vec{v}_x}\left(-\frac{1}{2}\vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial \vec{v}_x}\left(\vec{v}^T \Sigma^{-1} \vec{v}\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial \vec{v}_x}\left(\vec{v}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{v}\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial p_x}\left(\begin{bmatrix} \vec{v}_x & \vec{v}_y \end{bmatrix} \begin{bmatrix} a\vec{v}_x + b\vec{v}_y \\ c\vec{v}_x + d\vec{v}_y \end{bmatrix}\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial p_x}\left(a\vec{v}_x^2 + b\vec{v}_x\vec{v}_y + c\vec{v}_x\vec{v}_y + d\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial p_x}\left(a\vec{v}_x^2 + (b+c)\vec{v}_x\vec{v}_y + d\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i (2a\vec{v}_x + (b+c)\vec{v}_y)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \alpha_i}{\partial \vec{v}_y} &= -\frac{1}{2}\alpha_i \frac{\partial}{\partial p_y}\left(a\vec{v}_x^2 + (b+c)\vec{v}_x\vec{v}_y + d\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i (2d\vec{v}_y + (b+c)\vec{v}_x)
\end{aligned}
$$

Thus, derivatives for the position of gaussian are:

$$
\begin{aligned}
\frac{\partial \alpha_i}{\partial \vec{\mu}_x} &= -\frac{\partial \alpha_i}{\partial \vec{v}_x} = \frac{1}{2}\alpha_i (2a\vec{v}_x + (b+c)\vec{v}_y) \\
\frac{\partial \alpha_i}{\partial \vec{\mu}_x} &= -\frac{\partial \alpha_i}{\partial \vec{v}_y} = \frac{1}{2}\alpha_i (2d\vec{v}_y + (b+c)\vec{v}_x)
\end{aligned}
$$

9. Derivatives of scaling x,y

$$
\begin{aligned}
\frac{\partial \alpha_i}{\partial s_x} &= \frac{\partial}{\partial \vec{s}_x} o_i \exp\left(-\frac{1}{2}\vec{v}^T \Sigma^{-1}\vec{v}\right) \\
&= \alpha_i \frac{\partial}{\partial s_x}\left(-\frac{1}{2}\vec{v}^T \Sigma^{-1}\vec{v}\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial s_x}\left(\vec{v}^T \Sigma^{-1}\vec{v}\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial s_x}\left(\vec{v}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{v}\right)
\end{aligned}
$$

Need derivatives for each compornents in Σ^{-1}:

$$
\begin{aligned}
\frac{\partial a}{\partial s_x} &= -2s_x^{-3}\cos^2\theta \\
\frac{\partial a}{\overrightarrow{\partial s_y}} &= -2s_y^{-3}\sin^2\theta \\
\frac{\partial b}{\partial s_x} &= \frac{\partial c}{\partial s_x} = -2s_x^{-3}\sin\theta\cos\theta \\
\frac{\partial b}{\partial s_y} &= \frac{\partial c}{\partial s_y} = 2s_y^{-3}\sin\theta\cos\theta \\
\frac{\partial d}{\partial s_x} &= -2s_x^{-3}\sin^2\theta \\
\frac{\partial d}{\partial s_y} &= -2s_y^{-3}\sin^2\theta
\end{aligned}
$$

Thus, combining aboves:

$$
\begin{aligned}
\frac{\partial \alpha_i}{\partial s_x} &= -\frac{1}{2}\alpha_i \frac{\partial}{\partial s_x}\left(a\vec{v}_x^2 + b\vec{v}_x\vec{v}_y + c\vec{v}_x\vec{v}_y + d\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i\left\{\frac{\partial}{\partial s_x}a\vec{v}_x^2 + \frac{\partial}{\partial s_x}b\vec{v}_x\vec{v}_y + \frac{\partial}{\partial s_x}c\vec{v}_x\vec{v}_y + \frac{\partial}{\partial s_x}d\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i\left\{\left(-2s_x^{-3}\cos^2\theta\right)\vec{v}_x^2 + 2\left(-2s_x^{-3}\sin\theta\cos\theta\right)\vec{v}_x\vec{v}_y + \left(-2s_x^{-3}\sin^2\theta\right)\vec{v}_y^2\right) \\
&= -\alpha_i\left\{\left(-s_x^{-3}\cos^2\theta\right)\vec{v}_x^2 + \left(-2s_x^{-3}\sin\theta\cos\theta\right)\vec{v}_x\vec{v}_y + \left(-s_x^{-3}\sin^2\theta\right)\vec{v}_y^2\right) \\
&= \alpha_i\left\{\left(s_x^{-3}\cos^2\theta\right)\vec{v}_x^2 + \left(2s_x^{-3}\sin\theta\cos\theta\right)\vec{v}_x\vec{v}_y + \left(s_x^{-3}\sin^2\theta\right)\vec{v}_y^2\right) \\
&= \alpha_i\frac{1}{s_x^3}\left\{\cos^2\theta\vec{v}_x^2 + 2\sin\theta\cos\theta\vec{v}_x\vec{v}_y + \sin^2\theta\vec{v}_y^2\right) \\
&= \frac{\alpha_i}{s_x^3}\begin{bmatrix}\cos^2\theta \\ 2\sin\theta\cos\theta \\ \sin^2\theta\end{bmatrix}^T\begin{bmatrix}\vec{v}_x^2 \\ \vec{v}_x\vec{v}_y \\ \vec{v}_y^2\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \alpha_i}{\partial s_y} &= -\frac{1}{2}\alpha_i\left\{\frac{\partial}{\partial s_y}a\vec{v}_x^2 + \frac{\partial}{\partial s_y}b\vec{v}_x\vec{v}_y + \frac{\partial}{\partial s_y}c\vec{v}_x\vec{v}_y + \frac{\partial}{\partial s_y}d\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i\left\{\left(-2s_y^{-3}\sin^2\theta\right)\vec{v}_x^2 + 2\left(2s_y^{-3}\sin\theta\cos\theta\right)\vec{v}_x\vec{v}_y + \left(-2s_y^{-3}\sin^2\theta\right)\vec{v}_y^2\right) \\
&= \frac{\alpha_i}{s_y^3}\left\{\sin^2\theta\vec{v}_x^2 - 2\sin\theta\cos\theta\vec{v}_x\vec{v}_y + \sin^2\theta\vec{v}_y^2\right) \\
&= \frac{\alpha_i}{s_y^3}\begin{bmatrix}\sin^2\theta \\ -2\sin\theta\cos\theta \\ \cos^2\theta\end{bmatrix}^T\begin{bmatrix}\vec{v}_x^2 \\ \vec{v}_x\vec{v}_y \\ \vec{v}_y^2\end{bmatrix}
\end{aligned}
$$

10. Derivatives of rotation

Starts from derivatives of each compornent in Σ^{-1}:

$$\frac{\partial a}{\partial \theta} = 2\left(s_y^{-2} - s_x^{-2}\right)\sin\theta\cos\theta$$

$$= 2\left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2}\right)\sin\theta\cos\theta$$

$$\frac{\partial b}{\partial \theta} = \frac{\partial c}{\partial \theta} = -\left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2}\right)\left(\cos^2\theta - \sin^2\theta\right)$$

$$\frac{\partial d}{\partial \theta} = -2\left(s_y^{-2} - s_x^{-2}\right)\sin\theta\cos\theta = -\frac{\partial a}{\partial \theta}$$

Combining the aboves:

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial \theta} &= \frac{\partial}{\partial \theta} o_i \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) \\
&= -\frac{1}{2}\alpha_i \frac{\partial}{\partial \theta}\left(\vec{v}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{v}\right) \\
&= -\frac{1}{2}\alpha_i \left(\frac{\partial a}{\partial \theta}\vec{v}_x^2 + \frac{\partial b}{\partial \theta}\vec{v}_x\vec{v}_y + \frac{\partial c}{\partial \theta}\vec{v}_x\vec{v}_y + \frac{\partial d}{\partial \theta}\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i \left(\frac{\partial a}{\partial \theta}\vec{v}_x^2 + 2\frac{\partial b}{\partial \theta}\vec{v}_x\vec{v}_y + \frac{\partial d}{\partial \theta}\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i \left(\frac{\partial a}{\partial \theta}\vec{v}_x^2 + 2\frac{\partial b}{\partial \theta}\vec{v}_x\vec{v}_y - \frac{\partial a}{\partial \theta}\vec{v}_y^2\right) \\
&= -\frac{1}{2}\alpha_i \left(\frac{\partial a}{\partial \theta}\left(\vec{v}_x^2 - \vec{v}_y^2\right) + 2\frac{\partial b}{\partial \theta}\vec{v}_x\vec{v}_y\right) \\
&= -\frac{1}{2}\alpha_i \left(2\left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2}\right)\sin\theta\cos\theta\left(\vec{v}_x^2 - \vec{v}_y^2\right) + 2\left(-\left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2}\right)\left(\cos^2\theta - \sin^2\theta\right)\right)\vec{v}_x\vec{v}_y\right) \\
&= -\alpha_i \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2}\right)\left(\sin\theta\cos\theta\left(\vec{v}_x^2 - \vec{v}_y^2\right) - \left(\cos^2\theta - \sin^2\theta\right)\vec{v}_x\vec{v}_y\right) \\
&= \alpha_i \left(\frac{s_x^2 - s_y^2}{s_x^2 s_y^2}\right)\left(\left(\cos^2\theta - \sin^2\theta\right)\vec{v}_x\vec{v}_y - \sin\theta\cos\theta\left(\vec{v}_x^2 - \vec{v}_y^2\right)\right)
\end{aligned}$$

11. Derivaties of opacity

$$\frac{\partial \alpha_i}{\partial o_i} = \frac{\partial \alpha_i}{\partial o_i} o_i \exp\left(-\frac{1}{2}\vec{v}^T \Sigma^{-1} \vec{v}\right)$$
$$= \exp\left(-\frac{1}{2}\vec{v}^T \Sigma^{-1} \vec{v}\right)$$

12. Bounding box of the covariance matrix

You can define an ellipse by Mahalanobis' Distance

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = k^2$$
$$a\vec{v}_x^2 + (b+c)\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2 = 0$$

where $b = c$, we get

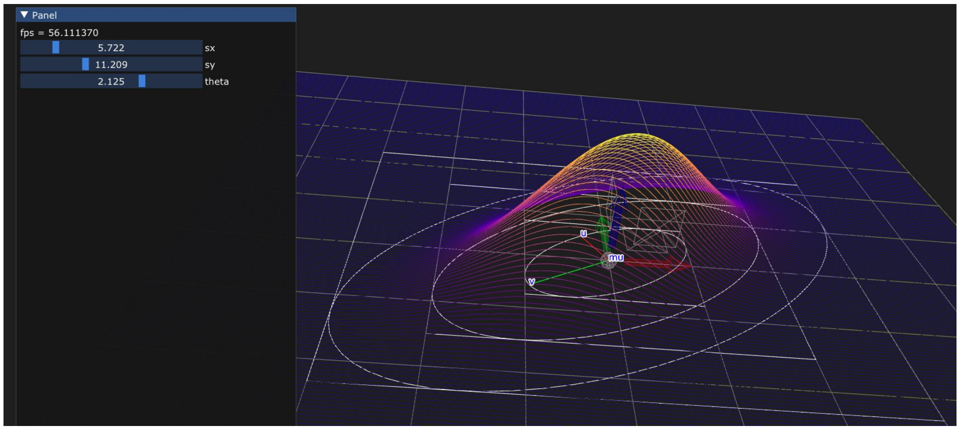
$$a\vec{v}_x^2 + 2b\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2 = 0$$

Let's use lagrange multiplier to get min max of $\vec{v}_x, \vec{v}_y$

$$L(\vec{v}_x, \vec{v}_y, \lambda) = \vec{v}_x - \lambda\left(a\vec{v}_x^2 + 2b\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2\right)$$
$$L(\vec{v}_x, \vec{v}_y, \lambda) = \vec{v}_y - \lambda\left(a\vec{v}_x^2 + 2b\vec{v}_x \vec{v}_y + d\vec{v}_y^2 - k^2\right)$$

Then, we get

$$\vec{v}_x = \pm k\sqrt{\frac{d}{ad-b^2}} = \pm k\sqrt{\frac{d}{\det\left(\Sigma^{-1}\right)}}$$
$$\vec{v}_y^2 = \pm k\sqrt{\frac{a}{ad-b^2}} = \pm k\sqrt{\frac{a}{\det\left(\Sigma^{-1}\right)}}$$



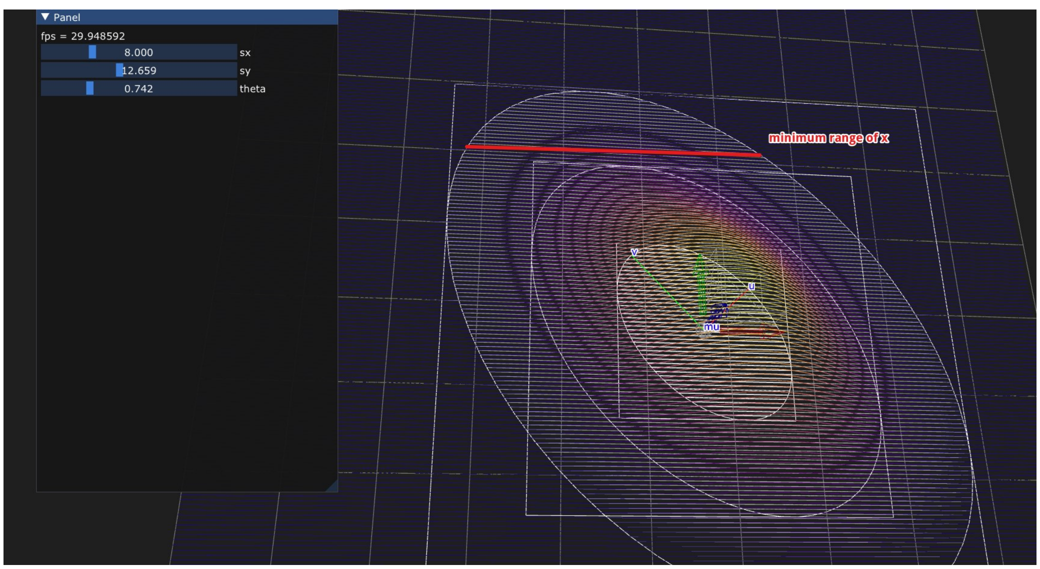
13. The exact range of $\vec{v}_x$

if $\vec{v}_y$ is known, then you can know 2 points of the ellipse

$$g(\vec{v}_x) = a\vec{v}_x^{\,2} + 2b\vec{v}_x\vec{v}_y + d\vec{v}_y^{\,2} - k^2 = 0$$

$$Ax^2 + Bx + C = 0$$
$$x = \frac{-B \pm \sqrt{b^2 - 4AC}}{2A}$$

where:



$$A = a$$
$$B = 2b\vec{v}_y$$

$$C = d\vec{v}_y^2 - k^2$$