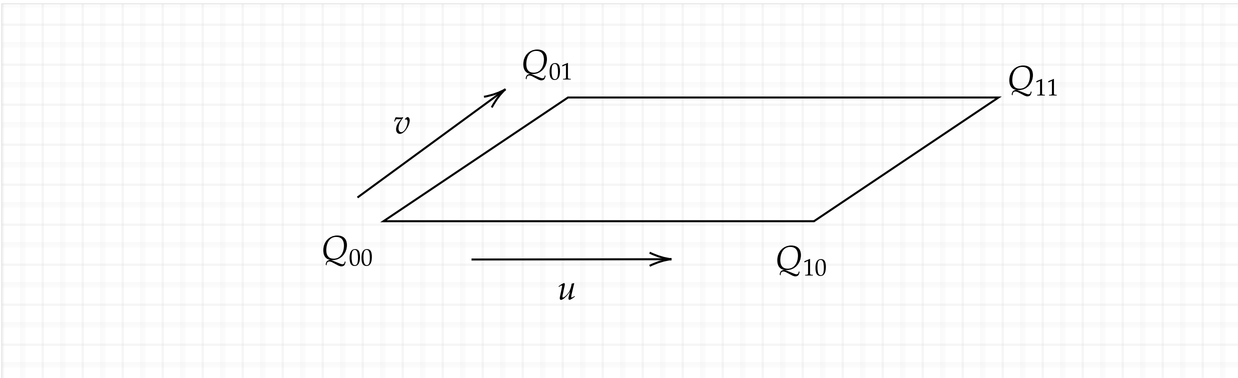
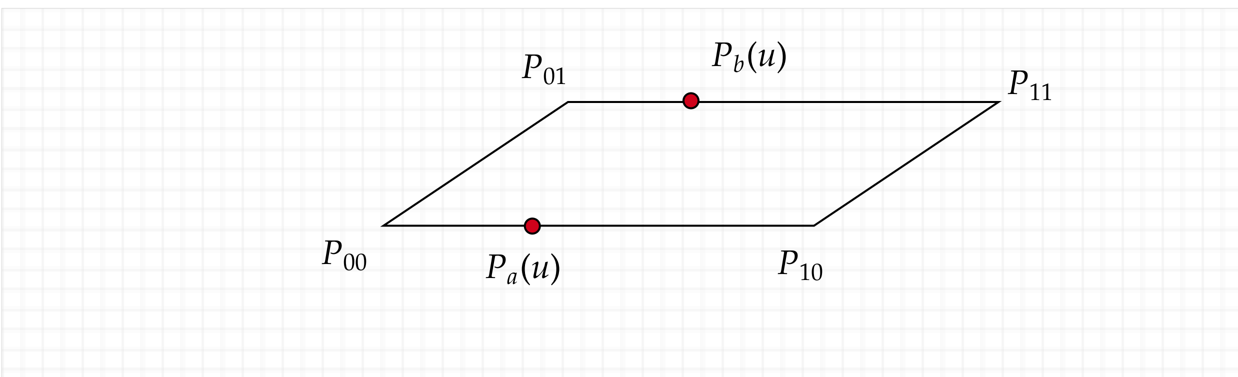
1. Surface definition

Let's define bilinear surface as same as the article

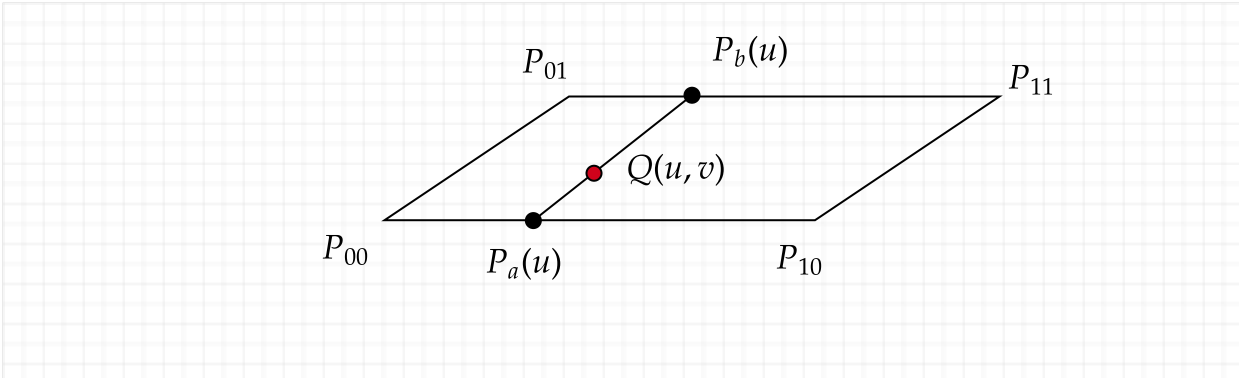


let's define
P_a on the line between Q_{00} and Q_{10}
P_b on the line between Q_{01} and Q_{11}



$$P_a(u) = (1-u)P_{00} + uP_{10}$$
$$P_b(u) = (1-u)P_{01} + uP_{11}$$

So Q on the surface is defined as



$$Q(u,v) = (1-v)P_a(u) + P_b(u)$$

So

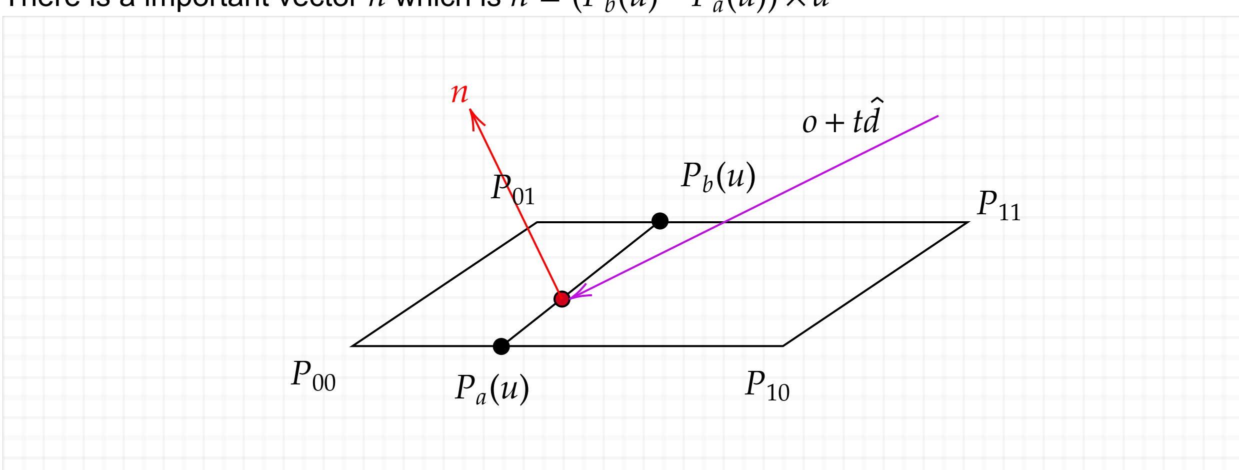
$$\begin{aligned} Q(u,v) &= (1-v)P_a(u) + P_b(u) \\ &= (1-v)((1-u)Q_{00} + uQ_{10}) + v((1-u)Q_{01} + uQ_{11}) \\ &= (1-v)(1-u)Q_{00} + u(1-v)P_{10} + (1-u)vQ_{01} + uvQ_{11} \\ &= (1-v)(1-u)Q_{00} + (1-u)vQ_{01} + u(1-v)P_{10} + uvQ_{11} \end{aligned}$$

We got the surface equation.

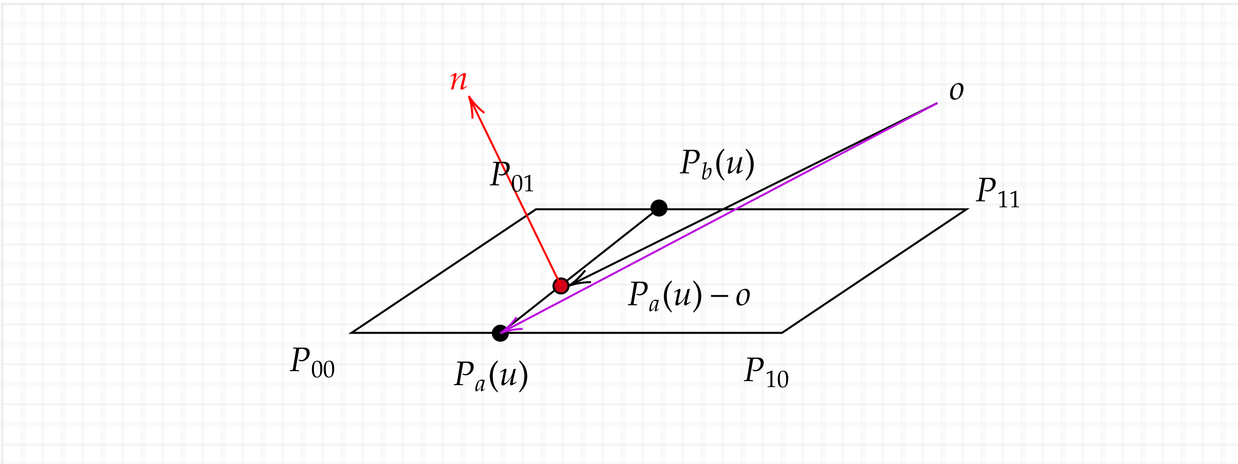
$$Q(u,v) = (1-u)(1-v)Q_{00} + (1-u)vQ_{01} + u(1-v)Q_{01} + uvQ_{11}$$

2. Build u formula

There is a important vector n which is $n = (P_b(u) - P_a(u)) \times \hat{d}$



of course, where $n \perp (P_b(u) - P_a(u))$ and $n \perp \hat{d}$
An important notice is that a vector from o to P_a is perpendicular to n



Thus we got

$$Q(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)Q_{01} + uvP_{11}$$

$$P_a(u) = (1-u)P_{00} + uP_{10} = P_{00} + u(P_{10} - P_{00})$$
$$P_b(u) = (1-u)P_{01} + uP_{11} = P_{01} + u(P_{11} - P_{10})$$

$$n = (P_b(u) - P_a(u)) \times \hat{d}$$
$$n \cdot (P_a(u) - o) = 0$$

So

$$n \cdot (P_a(u) - o) = 0$$
$$(P_b(u) - P_a(u)) \times \hat{d}) \cdot (P_a(u) - o) = 0$$
$$(P_{01} - P_{00} + u(P_{11} - P_{10}) - u(P_{01} - P_{00})) \times \hat{d}) \cdot (P_{00} + u(P_{10} - P_{00}) - o) = 0$$
$$(P_{01} - P_{00} + u(P_{11} - P_{10} - P_{01} + P_{00})) \times \hat{d}) \cdot (P_{00} + u(P_{10} - P_{00}) - o) = 0$$
$$((P_{01} - P_{00}) \times \hat{d} + u(P_{11} - P_{10} - P_{01} + P_{00}) \times \hat{d}) \cdot (P_{00} + u(P_{10} - P_{00}) - o) = 0$$
$$((P_{01} - P_{00}) \times \hat{d} + u(P_{11} - P_{10} - P_{01} + P_{00}) \times \hat{d}) \cdot (u(P_{10} - P_{00}) + P_{00} - o) = 0$$

for simplicity,

$$(P_{11} - P_{10} - P_{01} + P_{00}) \times \hat{d} = K$$

then

$$((P_{01}-P_{00})\times\hat{d}+uK)\cdot(u(P_{10}-P_{00})+P_{00}-o)=0$$
$$((P_{01}-P_{00})\times\hat{d})\cdot u(P_{10}-P_{00})+((P_{01}-P_{00})\times\hat{d})\cdot(P_{00}-o)+(uK)\cdot u(P_{10}-P_{00})+(uK)\cdot(P_{00}-o)=0$$
$$u((P_{01}-P_{00})\times\hat{d})\cdot(P_{10}-P_{00})+((P_{01}-P_{00})\times\hat{d})\cdot(P_{00}-o)+u^2(K\cdot(P_{10}-P_{00}))+u(K)\cdot(P_{00}-o)=0$$
$$u^2(K\cdot(P_{10}-P_{00}))+u((P_{01}-P_{00})\times\hat{d})\cdot(P_{10}-P_{00})+(K)\cdot(P_{00}-o))+((P_{01}-P_{00})\times\hat{d})\cdot(P_{00}-o)=0$$

This is obviously a quadaric equation.

$$a=K\cdot(P_{10}-P_{00})$$
$$b=(P_{01}-P_{00})\times\hat{d})\cdot(P_{10}-P_{00})+(K)\cdot(P_{00}-o)$$
$$c=((P_{01}-P_{00})\times\hat{d})\cdot(P_{00}-o)$$
$$au^2+bu+c=0$$

note: a and c is opposite on the original article.

We can simplify a bit more.

2.1. Term a

$$a=(K\cdot(P_{10}-P_{00}))$$
$$=((P_{11}-P_{10}-P_{01}+P_{00})\times\hat{d})\cdot(P_{10}-P_{00})$$
$$=(P_{10}-P_{00})\cdot((P_{11}-P_{10}-P_{01}+P_{00})\times\hat{d})$$

with scalar triple rule,

$$=\hat{d}\cdot((P_{10}-P_{00})\times(P_{11}-P_{10}-P_{01}+P_{00}))$$
$$=\hat{d}\cdot((P_{10}-P_{00})\times((P_{11}-P_{01})-(P_{10}-P_{00})))$$
$$=\hat{d}\cdot((P_{10}-P_{00})\times(P_{11}-P_{01})-(P_{10}-P_{00})\times(P_{10}-P_{00}))$$

where $(P_{10}-P_{00})\times(P_{10}-P_{00})=0$ because it's just $a\times a$ so

$$a=\hat{d}\cdot((P_{10}-P_{00})\times(P_{11}-P_{01}))$$

note: its sign is opposite from the article

2.2. Term c

$$
\begin{aligned}
c &= ((P_{01} - P_{00}) \times \hat{d}) \cdot (P_{00} - o) \\
c &= (P_{00} - o) \cdot ((P_{01} - P_{00}) \times \hat{d}) \\
c &= (P_{01} - P_{00}) \cdot (\hat{d} \times (P_{00} - o))
\end{aligned}
$$

note: its sign is opposite from the article

2.3. Term b

let's deform to $\hat{d} \cdot (...)$ style

$$
\begin{gathered}
b = (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + (K) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{10} - P_{01} + P_{00}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + (((P_{11} - P_{01}) - (P_{10} - P_{00})) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d} - (P_{10} - P_{00}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) - ((P_{10} - P_{00}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) - ((P_{10} - P_{00}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) - (P_{00} - o) \cdot ((P_{10} - P_{00}) \times \hat{d}) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) - (P_{10} - P_{00}) \cdot (\hat{d} \times (P_{00} - o)) \\
= (P_{01} - P_{00}) \times \hat{d}) \cdot (P_{10} - P_{00}) - (P_{10} - P_{00}) \cdot (\hat{d} \times (P_{00} - o)) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d} - \hat{d} \times (P_{00} - o)) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00}) \times \hat{d} + (P_{00} - o) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - P_{00} + P_{00} - o) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{01} - o) \times \hat{d}) \cdot (P_{10} - P_{00}) + ((P_{11} - P_{01}) \times \hat{d}) \cdot (P_{00} - o) \\
= (P_{10} - P_{00}) \cdot (P_{01} - o) \times \hat{d}) + (P_{00} - o) \cdot ((P_{11} - P_{01}) \times \hat{d}) \\
= \hat{d} \cdot ((P_{10} - P_{00}) \times (P_{01} - o)) + \hat{d} \cdot ((P_{00} - o) \times (P_{11} - P_{01})) \\
= \hat{d} \cdot ((P_{10} - P_{00}) \times (P_{01} - o) + (P_{00} - o) \times (P_{11} - P_{01}))
\end{gathered}
$$

Further more,

$$
\begin{gathered}
= \hat{d} \cdot ((P_{10} - P_{00}) \times (P_{01} - o) + (P_{00} - o) \times (P_{11} - P_{01})) \\
= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} - (P_{10} - P_{00}) \times o + (P_{00} - o) \times (P_{11} - P_{01})) \\
= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} - (P_{10} - P_{00}) \times o + P_{00} \times (P_{11} - P_{01}) - o \times (P_{11} - P_{01})) \\
= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) - (P_{10} - P_{00}) \times o - o \times (P_{11} - P_{01}))
\end{gathered}
$$

$$
\begin{aligned}
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) + o \times (P_{10} - P_{00}) - o \times (P_{11} - P_{01})) \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) + o \times (P_{10} - P_{00} - P_{11} + P_{01})) \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) + o \times (P_{10} - P_{11} + P_{01} - P_{00})) \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) + o \times (P_{10} - P_{11}) + o \times (P_{01} - P_{00}))
\end{aligned}
$$

here we already have this.

$$
\begin{aligned}
c &= (P_{01} - P_{00}) \cdot (\hat{d} \times (P_{00} - o)) \\
&= \hat{d} \cdot ((P_{00} - o) \times (P_{01} - P_{00})) \\
&= \hat{d} \cdot (P_{00} \times (P_{01} - P_{00}) - o \times (P_{01} - P_{00})) \\
&= \hat{d} \cdot (P_{00} \times P_{01} - P_{00} \times P_{00} - o \times (P_{01} - P_{00})) \\
&= \hat{d} \cdot (P_{00} \times P_{01} - o \times (P_{01} - P_{00}))
\end{aligned}
$$

We can use this to simplify.

$$
\begin{aligned}
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) + o \times (P_{10} - P_{11}) + o \times (P_{01} - P_{00})) + c - c \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times (P_{11} - P_{01}) + P_{00} \times P_{01} + o \times (P_{10} - P_{11})) - c \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times P_{11} + o \times (P_{10} - P_{11})) - c
\end{aligned}
$$

also we already have

$$
\begin{aligned}
a &= \hat{d} \cdot ((P_{10} - P_{00}) \times (P_{11} - P_{01}) \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{11} - (P_{10} - P_{00}) \times P_{01})
\end{aligned}
$$

We can use this to simplify.

$$
\begin{aligned}
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{01} + P_{00} \times P_{11} + o \times (P_{10} - P_{11})) + a - a - c \\
&= \hat{d} \cdot ((P_{10} - P_{00}) \times P_{11} + P_{00} \times P_{11} + o \times (P_{10} - P_{11})) - a - c \\
&= \hat{d} \cdot (P_{10} \times P_{11} + o \times (P_{10} - P_{11})) - a - c \\
&= \hat{d} \cdot (P_{10} \times P_{11} - P_{11} \times P_{11} + o \times (P_{10} - P_{11})) - a - c \\
&= \hat{d} \cdot ((P_{10} - P_{11}) \times P_{11} + o \times (P_{10} - P_{11})) - a - c \\
&= \hat{d} \cdot ((P_{10} - P_{11}) \times P_{11} - (P_{10} - P_{11}) \times o) - a - c \\
&= \hat{d} \cdot ((P_{10} - P_{11}) \times (P_{11} - o)) - a - c
\end{aligned}
$$

Finally, we got a simple form

$$a = \hat{d} \cdot ((P_{10} - P_{00}) \times (P_{11} - P_{01}))$$
$$c = (P_{01} - P_{00}) \cdot (\hat{d} \times (P_{00} - o))$$
$$b = \hat{d} \cdot ((P_{10} - P_{11}) \times (P_{11} - o)) - a - c$$

$$ax^2 + bx + c = 0$$

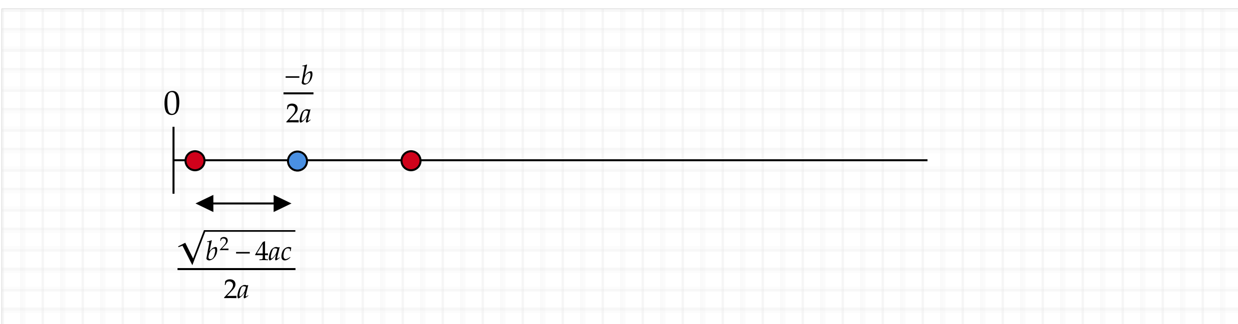
3. Stable quadratic formula solver

classic form is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In general, it might cause large cancellation in some cases, for example

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$



This case

$$x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

is quite close to zero. This cause large cancellation error.

This can be avoided by Vieta's formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \frac{-b \mp \sqrt{b^2 - 4ac}}{-b \mp \sqrt{b^2 - 4ac}}$$
$$= \frac{b^2 - \left(b^2 - 4ac\right)}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)}$$
$$= \frac{4ac}{2a\left(-b \mp \sqrt{b^2 - 4ac}\right)}$$
$$= \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

so

$$x_1 x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \frac{2c}{-b \pm \sqrt{b^2 - 4ac}} = \frac{c}{a}$$

$$x_1 x_2 = \frac{c}{a}$$
$$x_2 = \frac{c}{a} \frac{1}{x_1}$$
$$x_1 = \frac{c}{a} \frac{1}{x_2}$$

Thus, we can align sign $-b$ and $\sqrt{b^2 - 4ac}$. so you can always get more far x from zero.

$$x_1 = \frac{-b - sign(b)\sqrt{b^2 - 4ac}}{2a}$$
$$x_2 = \frac{c}{ax_1}$$

for simplicity,

$$k = \frac{-b - sign(b)\sqrt{b^2 - 4ac}}{2}$$

then

$$x_1 = \frac{k}{a}$$

$$x_2 = \frac{c}{ax_1} = \frac{c}{a}\frac{a}{k} = \frac{c}{k}$$

Note, Another benefit of this form is x_2 is vaid even if $a = 0$ case, which is $bx + c = 0$
However, original article recommend to use explicit handling for $a = 0$ case.

4. Calculation of t, v

Once you find two solusion for u. Then you have to find closest point line and line.
There is an another technique. please check here.
https://github.com/Ushio/ClosestPointLineLine

$$L_0(u) = P_0 + ud_0$$
$$L_1(v) = P_1 + vd_1$$

$$n = d_0 \times d_1$$
$$u = d_1 \cdot ((P_0 - P_1) \times n)\frac{1}{n \cdot n}$$
$$v = d_2 \cdot ((P_0 - P_1) \times n)\frac{1}{n \cdot n}$$

```
glm::vec3 n = glm::cross( d0, d1 );
float length2n = glm::dot( n, n );
glm::vec3 n2 = glm::cross( P0 - P1, n );
float u = glm::dot(n2, d1) / length2n;
float v = glm::dot(n2, d0) / length2n;
```