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Definability of Leibniz Equality

Abstract. Given a structure for a first-order language L, two objects of its domain can be indiscernible relative to the properties expressible in L, without using the equality symbol, and without actually being the same. It is this relation that interests us in this paper. It is called Leibniz equality. In the paper we study systematically the problem of its definibility mainly for classes of structures that are the models of some equality-free universal Horn class in an infinitary language $L_{\kappa\kappa}$, where κ is an infinite regular cardinal.

Key words: Leibniz equality, Universal Horn theory, Equality-free logic, Definability.

1. Introduction

Equality in first-order logic is represented in the language by means of a logical symbol and so is interpreted uniformly; the semantics says that its interpretation in any structure is the identity relation. This is the absolute notion of identity, however, and in fact the only one deserving this name. But there is a related relation: the relation of indiscernibility. Given a structure for a first-order language L, two objects of its domain can be indiscernible relative to the properties expressible in L without using the equality symbol, without actually being the same. It is this last relation that interests us in this paper. Its definition can be seen as a relativization to a first-order language of the essentially second-order Leibniz principle of the identity of indiscernibles. Namely, if $\mathfrak{A}$ is a structure for L and a and b are two members of its domain, a and b are related if the condition

$$\mathfrak{A} \models \forall\, \overline{z}(\varphi(x, \overline{z}) \leftrightarrow \varphi(y, \overline{z}))[a, b]$$

holds for any formula $\varphi(x, \overline{z})$ of L without equality. We follow [2] and name this relation *Leibniz equality.*

The use and study of the indiscernibility relation in logic is not recent. For first-order structures it was considered a long time ago, see for instance [17], and it underlies the well-known notion of Lindenbaum-Tarski algebra, which goes back to the 1920's at least; see [23]. More recently, it also appeared as a natural extension of the classical notion of identity in [24], where some basic concepts of model theory are reformulated for many-valued logics. But what has really strengthened its consideration is the development of an algebraic model theory for sentential logics based on the concept of

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logical matrix; see [16], [25], and also [8]. Roughly speaking, once a sentential logic is translated into an equality-free universal Horn theory following the idea of [4], the behavior of the Leibniz equality in its class of models is directly related to the algebraic properties of the class, in such a way that the nicer the properties of Leibniz equality we have, the richer the algebraic character of the class of models we can obtain.

Our purpose here is to answer some questions concerning the definability of the Leibniz equality. We fix a language L and a class $\mathcal{K}$ of structures, and under suitable assumptions on $\mathcal{K}$ we provide order-theoretical and algebraic characterizations of the fact that the Leibniz equalities of the members of $\mathcal{K}$ are definable uniformly by certain kinds of sets of first-order formulas. The order-theoretical characterizations relate this fact with the properties of what is known as the *Leibniz operator*, which for any algebra $\mathbb{A}$ of the algebraic similarity type of L, maps the poset of members of $\mathcal{K}$ with underlying algebra $\mathbb{A}$ into the lattice of congruence relations of $\mathbb{A}$ (Theorems 7, 16). On the other hand, the algebraic characterizations state its connection with the closure under certain operations of the class $\mathcal{K}^*$, whose members are those of $\mathcal{K}$ for which Leibniz equality is the identity (Theorems 6, 9, 10, 13, 14). The assumptions on the class $\mathcal{K}$ of structures that figure in the theorems of the paper are minimum assumptions for carrying through their proofs. Any one of these assumptions, except the ones of being closed under κ-reduced products or ultraproducts, is satisfied by any equality-free universal Horn class in the infinitary language $L_{\infty\infty}$; in particular they hold for any first-order and equality-free universal Horn class. Moreover, the assumption of being closed under κ-reduced products, for κ regular, holds for the equality-free universal Horn classes in the infinitary language $L_{\kappa\kappa}$. Therefore the paper may be seen as a study of the definability of Leibniz equality for these classes of structures.

Problems of this nature, but in a more restricted context, are studied in [3]. At the level of generality of the present paper they appeared as open problems in [11].

2. Preliminaries

Let us fix a similarity type L, whose cardinality can be greater than ω, with at least one relation symbol. Throughout the paper, κ is an arbitrary but fixed infinite regular cardinal $\leq \max(|L|, \omega)^+$. We consider the first-order language of type L. Unless otherwise specified, structures will be structures over L. We fix an enumeration $\langle v_n : n \in \omega \rangle$ of the variables and accept that 'x' and 'y' refer respectively to the first and second variables in the enumeration. For expository reasons we distinguish between *equations*, that is, formulas of the form $t = t'$ for some terms t and t', and the remaining atomic formulas. Thus, from now on the expression 'atomic formula' refers to atomic formulas that are not equations. We will also consider the infinitary language $L_{\kappa\kappa}$, that is, the infinitary language of type L that allows conjunctions of less than κ formulas and quantification of sets of variables of cardinality $< \kappa$, assuming in this case that we have κ variables. Moreover, we will consider the languaje $L_{\infty\infty}$ that allows conjunctions of arbitrary sets of formulas and quantifications of arbitrary sets of variables, assuming in this case that we have a proper class of variables.

When we write $\varphi(x, y, \overline{z})$ we make two assumptions: first, that the free variables in the formula φ are among x, y and the variables in the sequence $\overline{z}$, and second, that neither x nor y occur in $\overline{z}$. Analogous conventions apply when we write $\varphi(x, \overline{z})$ or $\varphi(y, \overline{z})$ with the proviso that neither x nor y occur in $\overline{z}$. These conventions extend in the obvious way to sets of formulas.

Given a formula φ, variables $x_0, \ldots, x_n$ and terms $t_0, \ldots, t_n$, the expression $\varphi(x_0/t_0, \ldots, x_n/t_n)$ denotes the formula obtained from φ by simultaneously replacing every free occurrence of x_i by t_i, for $i \leq n$. For the sake of convenience, we simply write $\varphi(t_0, \ldots, t_n)$ to mean $\varphi(x_0/t_0, \ldots, x_n/t_n)$. So, for instance, given a formula $\varphi(x, y, \overline{z})$, $\varphi(x, x, \overline{z})$ denotes the formula $\varphi(x, y/x, \overline{z})$, and $\varphi(y, x, \overline{z})$ the formula $\varphi(x/y, y/x, \overline{z})$. The same conventions also extend to sets of formulas.

Given an assignment s in a structure $\mathfrak{A}$, we denote by $s(x_0/a_0, \ldots, x_n/a_n)$ the assignment that sends x_i to a_i, for $i \leq n$, and coincides with s otherwise. We also say that a formula holds in a structure (or in a class of structures) if it is satisfied in the structure (resp. in any member of the class) for all assignments.

Given two structures $\mathfrak{A}$ and $\mathfrak{B}$ we say (following [11]) that $\mathfrak{B}$ is a *filter extension* of $\mathfrak{A}$, in symbols $\mathfrak{A} \sqsubseteq \mathfrak{B}$, if the underlying algebras are the same and for any relation symbol R in L, $R^{\mathfrak{A}} \subseteq R^{\mathfrak{B}}$. A function h from A into B is said to be a *strict homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ if it is a homomorphism for which the following additional condition holds: for any relation symbol R in L and any $a_1, \ldots, a_n \in A$,

$$\langle a_1, \ldots, a_n \rangle \in R^{\mathfrak{A}} \quad \text{iff} \quad \langle h(a_1), \ldots, h(a_n) \rangle \in R^{\mathfrak{B}}.$$

Strict homomorphisms are called in [17] two-way homomorphisms and they must not be confused with the strong homomorphisms of [5].

If there is a strict homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, we say that $\mathfrak{A}$ is a *strict inverse image of* $\mathfrak{B}$ and that $\mathfrak{B}$ is a *strict homomorphic image of* $\mathfrak{A}$. The following lemma will be used later on; it says that formulas without equality are preserved upwards and downwards by strict homomorphisms.

LEMMA 1. *If h is a strict homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ then, for any formula $\varphi(\overline{z})$ without the equality symbol and any sequence $\overline{a}$ of elements of A of the appropriate length,*

$$\mathfrak{A} \models \varphi(\overline{z})[\overline{a}] \quad \textit{iff} \quad \mathfrak{B} \models \varphi(\overline{z})[h(\overline{a})].$$

2.1. The consequence $\models_{\mathcal{K}}$

Given a class of structures $\mathcal{K}$, a set Γ of formulas and a formula φ, we say that φ is a *$\mathcal{K}$-consequence* of Γ, in symbols $\Gamma \models_{\mathcal{K}} \varphi$, if for any structure $\mathfrak{A} \in \mathcal{K}$ and any assignment s on $\mathfrak{A}$, $\mathfrak{A} \models \varphi[s]$ holds whenever $\mathfrak{A} \models \psi[s]$ holds for each formula $\psi \in \Gamma$.

The next lemma establishes a property of compactness of $\models_{\mathcal{K}}$; it is interesting only when the similarity type L is of cardinality $\mu \geq \lambda$, for in this case the set of universal Horn formulas has cardinality $\geq \mu$.

LEMMA 2. *Let λ be an infinite regular cardinal and let $\mathcal{K}$ be a class of structures closed under λ-reduced products. Then $\models_{\mathcal{K}}$ is λ-compact in the following sense: If $\Sigma \cup \{\varphi\}$ is a set of universal Horn formulas,*

$$\Sigma \models_{\mathcal{K}} \varphi \textit{ iff there is } \Sigma' \subseteq \Sigma \textit{ with } |\Sigma'| < \lambda \textit{ such that } \Sigma' \models_{\mathcal{K}} \varphi.$$

PROOF. Let Σ be a set of universal Horn formulas and let φ be a universal Horn formula. Suppose $\Sigma \models_{\mathcal{K}} \varphi$ and that for all $\Sigma' \subseteq \Sigma$ with $|\Sigma'| < \lambda$, $\Sigma' \not\models_{\mathcal{K}} \varphi$. If φ is of the form $\forall \overline{z}(\bigwedge_{\alpha<\beta} \psi_\alpha \to \delta)$, we can assume without loss of generality that the variables in $\overline{z}$ do not appear in Σ and argue as follows. Let $I = \{\Delta \subseteq \Sigma : |\Delta| < \lambda\}$. For any $\Delta \in I$, take a structure $\mathfrak{A}_\Delta \in \mathcal{K}$ and an assignment s_Δ on it such that $\mathfrak{A}_\Delta \models \Delta[s_\Delta]$ and $\mathfrak{A}_\Delta \models \bigwedge_{\alpha<\beta} \psi_\alpha[s_\Delta]$ but $\mathfrak{A}_\Delta \not\models \delta[s_\Delta]$. Put, for any $\Delta \in I$, $J_\Delta = \{\Delta' \in I : \Delta \subseteq \Delta'\}$. Since λ is regular, the set $\{J_\Delta : \Delta \in I\}$ has the λ-intersection property (i.e. the intersection of fewer than λ of its elements is non-empty), and therefore can be extended to a λ-complete filter F. Consider the λ-reduced product $\mathfrak{B} = \prod_{\Delta \in I} \mathfrak{A}_\Delta / F$. Clearly $\mathfrak{B} \in \mathcal{K}$ for $\mathcal{K}$ is closed under λ-reduced products. Now we define the assignment s on $\mathfrak{B}$ as follows: For any $n \in \omega$, let $f_n \in \prod_{\Delta \in I} A_\Delta$ be given by $f_n(\Delta) = s_\Delta(x_n)$, for each $\Delta \in I$, and then define s by setting $s(x_n) = f_n/F$ for each $n \in \omega$. Since $J_{\{\psi\}} \subseteq \{\Delta \in I : \mathfrak{A}_\Delta \models \psi[s_\Delta]\} \in F$, for any $\psi \in \Sigma \cup \{\psi_\alpha : \alpha < \beta\}$, and $\mathfrak{B} \in \mathcal{K}$, we have that $\mathfrak{B} \models \Sigma[s]$. But $\mathfrak{B} \not\models \varphi[s]$ because the set $\{\Delta \in I : \mathfrak{A}_\Delta \models \varphi[s_\Delta]\}$ is empty. So, this contradicts the assumption and completes the argument. Now, if φ is of the form $\forall \overline{z}(\bigvee_{\alpha<\beta} \neg\psi_\alpha)$ we argue in a similar way. ∎

Given a class of structures $\mathcal{K}$, we say that a set Σ of atomic formulas is an *atomic theory* of $\mathcal{K}$ if for any atomic formula φ the condition $\Sigma \models_{\mathcal{K}} \varphi$

entails that $\varphi \in \Sigma$. An atomic theory of $\mathcal{K}$ is *consistent* if it is not the set of all atomic formulas.

To any set Γ of atomic formulas we associate the structure $\mathfrak{A}_\Gamma$ on the algebra of terms, called the *term structure of* Γ, defined by the next two conditions:

(i) The algebraic part of $\mathfrak{A}_\Gamma$ is the algebra of terms.

(ii) For any relation symbol R of arity n and any terms $t_1, \ldots, t_n$,

$$\langle t_1, \ldots, t_n \rangle \in R^{\mathfrak{A}_\Gamma} \quad \text{iff} \quad Rt_1 \ldots t_n \in \Gamma.$$

Note that if Σ and Σ' are sets of atomic formulas, then $\Sigma \subseteq \Sigma'$ iff $\mathfrak{A}_\Sigma \sqsubseteq \mathfrak{A}_{\Sigma'}$. A class $\mathcal{K}$ of structures is said to be *atomic theory closed* if for any consistent atomic theory Γ of $\mathcal{K}$ the structure $\mathfrak{A}_\Gamma$ belongs to $\mathcal{K}$.

A class of structures is an $L_{\kappa\kappa}$*-universal Horn class* if it is the class of models of some set of universal Horn sentences of $L_{\kappa\kappa}$, and is an $L_{\kappa\kappa}$*-strict universal Horn class* if it is the class of models of some set of strict universal Horn sentences of $L_{\kappa\kappa}$. We will also speak of an $L_{\kappa\kappa}$-equality-free (strict) universal Horn class when the class is axiomatizable by $L_{\kappa\kappa}$-equality-free (strict) universal Horn sentences. Analogous notions will be considered for the languaje $L_{\infty\infty}$.

LEMMA 3. *Any $L_{\kappa\kappa}$-equality-free universal Horn class is atomic theory closed.*

PROOF. Let $\mathcal{K}$ be an $L_{\kappa\kappa}$-equality-free universal Horn class and let Γ be a consistent atomic theory of $\mathcal{K}$. Let $\sigma = \forall \overline{x} \bigvee_{\xi<\alpha} \neg\psi_\xi(\overline{x})$, with $\psi_\xi(\overline{x})$ atomic formulas, $\alpha < \kappa$ and $\overline{x}$ a possibly infinite sequence of variables of length less than κ. Suppose that σ holds in $\mathcal{K}$ and $\mathfrak{A}_\Gamma \nvDash \sigma$. Then there is a sequence of terms $\overline{t}$ such that $\mathfrak{A}_\Gamma \nvDash \bigvee_{\xi<\alpha} \neg\psi_\xi(\overline{x})[\overline{t}]$. So, for each $\xi < \alpha$, $\mathfrak{A}_\Gamma \vDash \psi_\xi(\overline{x})[\overline{t}]$. Thus, $\psi_\xi(\overline{x}/\overline{t}) \in \Gamma$, for each $\xi < \alpha$. But since the models of $\{\psi_\xi(\overline{x}/\overline{t}) : \xi < \alpha\}$ cannot belong to $\mathcal{K}$, $\{\psi_\xi(\overline{x}/\overline{t}) : \xi < \alpha\} \vDash_{\mathcal{K}} \varphi$ for any atomic φ, and we obtain that Γ must be inconsistent, contrary to our assumption. In conclusion, $\mathfrak{A}_\Gamma$ must be a model of σ. If $\sigma = \forall \overline{x}(\psi(\overline{x}) \vee \bigvee_{\xi<\alpha} \neg\psi_\xi(\overline{x}))$ or $\sigma = \forall \overline{x}\, \psi(\overline{x})$ we can argue in a similar way and conclude that $\mathfrak{A}_\Gamma$ belongs to $\mathcal{K}$. Hence $\mathcal{K}$ is atomic theory closed. ∎

2.2. Leibniz equalities

Given a structure $\mathfrak{A}$ we define the *Leibniz equality* of $\mathfrak{A}$ as the following binary relation $\mathbf{\Omega}(\mathfrak{A})$ on A: if $a, b \in A$ then

$$(*) \quad \langle a, b \rangle \in \mathbf{\Omega}(\mathfrak{A}) \quad \text{iff} \quad \begin{array}{l} \text{for any atomic formula } \varphi\text{, assignment } s\text{, and} \\ \text{variable } v\text{: } \mathfrak{A} \vDash \varphi[s(v/a)] \text{ iff } \mathfrak{A} \vDash \varphi[s(v/b)]. \end{array}$$

An easy induction on the complexity of the formulas allows us to prove that the atomic formulas φ can be replaced by arbitrary formulas without equality. It is also straightforward to see that $\mathbf{\Omega}(\mathfrak{A})$ is the maximum congruence of $\mathfrak{A}$, where a congruence of a structure $\mathfrak{A}$ is an equivalence relation θ on its universe that is a congruence of the algebraic part of $\mathfrak{A}$ and for any relation symbol R, if $\langle a_1, b_1\rangle \in \theta$, $\ldots \langle a_n, b_n\rangle \in \theta$ and $\langle a_1, \ldots, a_n\rangle \in R^{\mathfrak{A}}$, then $\langle b_1, \ldots, b_n\rangle \in R^{\mathfrak{A}}$.

We say that a structure is *reduced* if its Leibniz equality is the identity. Thus, given a structure $\mathfrak{A}$, the quotient structure $\mathfrak{A}/\mathbf{\Omega}(\mathfrak{A})$ is reduced; it is called the *reduction of* $\mathfrak{A}$. The equivalence class of $a \in A$ modulo $\mathbf{\Omega}(\mathfrak{A})$ is denoted by a^* or by $a/\mathfrak{A}$, if some confusion may arise. We sometimes write $\mathfrak{A}^*$ instead of $\mathfrak{A}/\mathbf{\Omega}(\mathfrak{A})$. Finally, given a class of structures $\mathcal{K}$ we denote by $\mathcal{K}^*$ the class of reductions of members of $\mathcal{K}$. Note that the canonical homomorphism $*: a \longmapsto a^*$ from $\mathfrak{A}$ onto $\mathfrak{A}^*$ is strict and thus $\mathfrak{A}$ and $\mathfrak{A}^*$ satisfy the same sentences without equality.

Let us consider the formulas of the following form:

$$\forall \overline{z}(\varphi(x, \overline{z}) \leftrightarrow \varphi(y, \overline{z})),$$

where φ is an atomic formula. The formulas of this form are the *Leibniz formulas* (in the variables x and y). If $\Lambda(x, y)$ denotes the set of all such formulas, then $(*)$ says that the following holds for any structure $\mathfrak{A}$:

$$\mathbf{\Omega}(\mathfrak{A}) = \{\langle a, b\rangle \in A^2 : \mathfrak{A} \vDash \Lambda(x, y)[a, b]\}.$$

In general, given a class of structures $\mathcal{K}$ and a set $\Delta(x, y)$ of first-order formulas in at most two free variables x, y, we say that *Leibniz equalities are definable in* $\mathcal{K}$ *by* Δ if for any $\mathfrak{A} \in \mathcal{K}$,

$$\mathbf{\Omega}(\mathfrak{A}) = \{\langle a, b\rangle : \mathfrak{A} \vDash \Delta(x, y)[a, b]\}.$$

Thus, Leibniz equalities are definable in the class of all structures by the set of Leibniz formulas. As stated in the Introduction, the paper is devoted to studying the properties of a given class of structures according to the definability of the Leibniz equalities of its members in this sense.

We single out for future use the following two facts that follow at once from Lemma 1:

- Leibniz equalities are definable in $\mathcal{K}$ by a set $\Delta(x, y)$ of formulas without equality iff the identity relation is definable in K^* by $\Delta(x, y)$.
- If $\mathcal{K}$ is any class of structures, the consequences $\vDash_{\mathcal{K}}$ and $\vDash_{\mathcal{K}^*}$ restricted to formulas without equality coincide.

2.3. The set $\Sigma(\mathcal{K})$

Let $\mathcal{K}$ be any class of structures. We define the following set of atomic formulas associated with $\mathcal{K}$, which plays a fundamental role in this paper:

$$\Sigma(\mathcal{K}) = \{\varphi : \varphi \text{ is atomic and } \varphi(y/x) \text{ holds in } \mathcal{K}\}.$$

(cf. [3], [15].) Notice that, of course, $\Sigma(\mathcal{K})$ can be the empty set. We will focus especially in the variables x and y that may appear in the formulas in $\Sigma(\mathcal{K})$ leaving the others as parameters.

LEMMA 4. (i) *For any atomic formula φ and any variable v,*

$$\varphi(v/x) \in \Sigma(\mathcal{K}) \quad \textit{iff} \quad \varphi(v/y) \in \Sigma(\mathcal{K}).$$

(ii) *For any atomic formula $\varphi(x, y, \overline{z})$,*

$$\varphi(x, y, \overline{z}) \in \Sigma(\mathcal{K}) \quad \textit{iff} \quad \varphi(y, x, \overline{z}) \in \Sigma(\mathcal{K}).$$

Thus, $\Sigma(\mathcal{K})(x, y, \overline{z}) = \Sigma(\mathcal{K})(y, x, \overline{z})$.

(iii) $\Sigma(\mathcal{K})$ *is an atomic theory of* $\mathcal{K}$.

PROOF. (i) Since $\varphi(v/x)(y/x)$ is the formula $\varphi(v/y)(y/x)$.

(ii) Since the formulas $\varphi(x, y, \overline{z})(y/x)$ and $\varphi(y, x, \overline{z})(y/x)$ are equal.

(iii) If $\Sigma(\mathcal{K}) \vDash_{\mathcal{K}} \varphi$, then $\{\psi(y/x) : \psi \in \Sigma(\mathcal{K})\} \vDash_{\mathcal{K}} \varphi(y/x)$. But the elements of $\{\psi(y/x) : \psi \in \Sigma(\mathcal{K})\}$ hold in $\mathcal{K}$. Therefore $\varphi(y/x)$ holds in $\mathcal{K}$ and hence $\varphi \in \Sigma(\mathcal{K})$. ∎

The importance of the set $\Sigma(\mathcal{K})$ defined above mainly rests on the following property, which will be used later on.

LEMMA 5. *Given any class $\mathcal{K}$ of structures, $\langle x, y\rangle \in \boldsymbol{\Omega}(\mathfrak{A}_{\Sigma(\mathcal{K})})$.*

PROOF. The proof uses the fact that for any set of atomic formulas and any terms t and t', $\langle t, t'\rangle \in \boldsymbol{\Omega}(\mathfrak{A}_{\Gamma})$ iff for any atomic formula φ and any variable v, $\varphi(v/t) \in \Gamma$ iff $\varphi(v/t') \in \Gamma$. Then by (i) of Lemma 4 we obtain the desired result. ∎

3. Elementary definability of Leibniz equalities

We say that *Leibniz equalities are finitely elementarily definable* in a class of structures $\mathcal{K}$ if they are definable in $\mathcal{K}$ by a first-order formula, possibly with equality, in at most the variables x and y. The following result generalizes a result of [19].

THEOREM 6. *Let $\mathcal{K}$ be a class of structures closed under ultraproducts and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Then the following statements are equivalent:*

(i) *Leibniz equalities are finitely elementarily definable in $\mathcal{K}$.*

(ii) *Leibniz equalities are definable in $\mathcal{K}$ by a finite set of first-order universal Horn formulas without equality.*

(iii) *$\mathcal{K}^*$ is closed under ultraproducts.*

PROOF. Clearly (ii) implies (i). Also, if $\varphi(x, y)$ is a first-order formula that defines Leibniz equalities in $\mathcal{K}$, then the assumption $\mathcal{K}^* \subseteq \mathcal{K}$ entails that $\mathfrak{A} \in \mathcal{K}^*$ iff $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{A} \models \forall xy(\varphi(x, y) \leftrightarrow x = y)$. This proves the implication from (i) to (iii). To see (iii) implies (ii) we use that $\Lambda(x, y)$ defines Leibniz equalities in $\mathcal{K}$. Then $\Lambda(x, y) \models_{\mathcal{K}^*} x = y$. Now, using that $\mathcal{K}^*$ is closed under ultraproducts, by an argument similar to the one used in the proof of Lemma 2 we conclude that there exists a finite subset Λ_0 of Λ such that $\Lambda_0(x, y) \models_{\mathcal{K}^*} x = y$. Thus $\Lambda_0(x, y)$ defines Leibniz equalities in $\mathcal{K}^*$ and therefore does so in $\mathcal{K}$, for it is a set of formulas without equality. ∎

Notice that the assumptions $\mathcal{K}$ is closed under ultraproducts and $K^* \subseteq K$ do not entail in general that $\mathcal{K}$ is elementary. It is true, however, that, if $\mathcal{K}$ is elementary then the previous statements are equivalent to the fact that K^* is elementary; the proof is straightforward using the fact that an elementary substructure of a reduced one is reduced. We can also follow a compactness argument to conclude from the theorem that for any class $\mathcal{K}$ of structures closed under ultraproducts and such that $\mathcal{K}^* \subseteq \mathcal{K}$, if Leibniz equalities are definable in $\mathcal{K}$ by a set of first-order universal Horn formulas $\Delta(x, y)$, then there is a finite subset Δ_0 of Δ that defines Leibniz equalities in $\mathcal{K}$ iff $\mathcal{K}^*$ is closed under ultraproducts.

4. Pseudo-atomic definability of Leibniz equalities

A *pseudo-atomic formula* in the free variables x and y is a first-order formula of the form $\forall \overline{z}\, \varphi(x, y, \overline{z})$, where $\varphi(x, y, \overline{z})$ is atomic.

Leibniz equalities are *pseudo-atomically definable* in $\mathcal{K}$ if they are definable in $\mathcal{K}$ by some set of pseudo-atomic formulas. The next result contains a characterization of pseudo-atomic definability of Leibniz congruences in a class of structures in terms of a property of the $\boldsymbol{\Omega}$ operator as a mapping between posets. Namely, we say that $\boldsymbol{\Omega}$ is $\sqsubseteq$-*monotone in* $\mathcal{K}$ if for any $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$, $\boldsymbol{\Omega}(\mathfrak{A}) \subseteq \boldsymbol{\Omega}(\mathfrak{B})$. Then we have the next theorem. The idea of the proof can be found in [3], in the proof of Theorem 13.2.

THEOREM 7. *Let $\mathcal{K}$ be an atomic theory closed class of structures. The following statements are equivalent:*

(i) *Leibniz equalities are pseudo-atomically definable in $\mathcal{K}$.*

(ii) $\mathbf{\Omega}$ *is* $\sqsubseteq$-monotone *in $\mathcal{K}$.*

PROOF. (i) $\Rightarrow$ (ii) Let $\Delta(x, y)$ be a set of pseudo-atomic formulas that defines Leibniz equalities in $\mathcal{K}$. Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ be such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$. If $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$ then $\mathfrak{A} \models \Delta[a, b]$. Thus, since Δ is a set of pseudo-atomic formulas, $\mathfrak{B} \models \Delta[a, b]$. Therefore $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{B})$.

(ii) $\Rightarrow$ (i) Assume that $\mathbf{\Omega}$ is monotone in $\mathcal{K}$. We claim that for any atomic formula $\varphi(x, \overline{z})$,

$$\Sigma(\mathcal{K}) \cup \{\varphi(x, \overline{z})\} \models_{\mathcal{K}} \varphi(y, \overline{z}).$$

Indeed let $\varphi(x, \overline{z})$ be an atomic formula and define

$$T := \{\psi : \psi \text{ is atomic and } \Sigma(\mathcal{K}) \cup \{\varphi(x, \overline{z})\} \models_{\mathcal{K}} \psi\}.$$

Clearly T is an atomic theory of $\mathcal{K}$. If T is inconsistent the claim is clear. If T is consistent then, since $\mathcal{K}$ is atomic theory closed, $\mathfrak{A}_T \in \mathcal{K}$. Also, since T includes $\Sigma(\mathcal{K})$, we have $\mathfrak{A}_{\Sigma(\mathcal{K})} \sqsubseteq \mathfrak{A}_T$. Therefore, by (ii) and Lemma 5, $\langle x, y\rangle \in \mathbf{\Omega}(\mathfrak{A}_T)$. Thus, as $\varphi(x, \overline{z}) \in T$, we obtain that $\varphi(y, \overline{z}) \in T$, as desired. This completes the proof of the claim.

Now, since $\Sigma(\mathcal{K})(x/y, y/x, \overline{z}) = \Sigma(\mathcal{K})$, we do have that for any atomic formula $\varphi(x, \overline{z})$, $\Sigma(\mathcal{K}) \cup \{\varphi(y, \overline{z})\} \models_{\mathcal{K}} \varphi(x, \overline{z})$. So take

$$\Delta := \{\forall \overline{z}\, \varphi(x, y, \overline{z}) : \varphi(x, y, \overline{z}) \in \Sigma(\mathcal{K})\}$$

and let us see that Leibniz equalities are definable in $\mathcal{K}$ by Δ. Using the claim and the fact just stated, we obtain that for any $\mathfrak{A} \in \mathcal{K}$ and any $a, b \in A$, if $\mathfrak{A} \models \Delta[a, b]$ then $\mathfrak{A} \models \forall \overline{z}(\varphi(x, \overline{z}) \leftrightarrow \varphi(y, \overline{z}))$ for all atomic formulas $\varphi(x, \overline{z})$. Thus, $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$. Conversely, assume $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$. Then, for all $\overline{c} \in A$ and all $\varphi(x, y, \overline{z}) \in \Sigma(\mathcal{K})$, $\mathfrak{A} \models \varphi(x, y, \overline{z})[a, a, \overline{c}]$ implies that $\mathfrak{A} \models \varphi(x, y, \overline{z})[a, b, \overline{c}]$. Hence, since $\models_{\mathcal{K}} \Sigma(\mathcal{K})(x, x, \overline{z})$, we conclude that $\mathfrak{A} \models \Sigma(\mathcal{K})[a, b, \overline{c}]$ for all $\overline{c} \in A$. So $\mathfrak{A} \models \Delta[a, b]$. ■

The proof of (ii) $\Rightarrow$ (i) above suggests that the existence of a set of atomic formulas satisfying the property stated by the claim suffices to conclude that Leibniz equalities are pseudo-atomically definable. Indeed, we just need to add the reflexivity property that defines $\Sigma(\mathcal{K})$ to obtain the following purely syntactical characterization of the pseudo-atomic definability of Leibniz equalities. Notice that the symmetry of $\Sigma(\mathcal{K})$ stated in Lemma 4(ii) and used in the previous proof is missing.

THEOREM 8. *Let $\mathcal{K}$ be an atomic theory closed class of structures. Then Leibniz equalities are pseudo-atomically definable in $\mathcal{K}$ iff there exists a set of atomic formulas $\Sigma := \Sigma(x, y, \overline{z})$ such that*

(i) $\models_{\mathcal{K}} \Sigma(x, x, \overline{z})$;

(ii) $\Sigma(x, y, \overline{z}) \cup \{\varphi(x, \overline{z})\} \models_{\mathcal{K}} \varphi(y, \overline{z})$, *for any atomic formula* $\varphi(x, \overline{z})$.

PROOF. $\Rightarrow$) This implication follows immediately from the definition of $\Sigma(\mathcal{K})$ and the proof of Theorem 7.

$\Leftarrow$) We apply Theorem 7 again. Let Σ be any set of atomic formulas that verifies conditions (i) and (ii) and let us show that $\mathbf{\Omega}$ is $\sqsubseteq$-monotone in $\mathcal{K}$. Take $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$ and suppose $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$. By condition (i) we have that $\mathfrak{A} \models \Sigma(x, x, \overline{z})[a, \overline{c}]$ for all $\overline{c} \in A$. Hence, as $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$, we obtain both

$$\mathfrak{A} \models \Sigma(x, y, \overline{z})[a, b, \overline{c}] \text{ and } \mathfrak{A} \models \Sigma(x, y, \overline{z})[b, a, \overline{c}]$$

for all $\overline{c} \in A$. Now $\mathfrak{B}$ is a filter extension of $\mathfrak{A}$ and hence, given any $\overline{c} \in B$,

$$\mathfrak{B} \models \Sigma(x, y, \overline{z})[a, b, \overline{c}] \text{ and } \mathfrak{B} \models \Sigma(x, y, \overline{z})[b, a, \overline{c}].$$

Consequently, condition (ii) entails that for all atomic formulas $\varphi(x, \overline{z})$,

$$\mathfrak{B} \models \forall \overline{z}(\varphi(x, \overline{z}) \leftrightarrow \varphi(y, \overline{z}))[a, b]$$

and this proves $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{B})$, as desired. ■

The argument that proves the implication from (ii) to (i) of Theorem 7 is based on the fact that $\Sigma(\mathcal{K})(y, x)$ is $\Sigma(\mathcal{K})$. The same argument applies in the previous theorem by taking the symmetrization of Σ, that is, the set $\Sigma \cup \Sigma(x/y, y/x)$ can be used to define Leibniz equalities in $\mathcal{K}$. An analog of the theorem was obtained independently by K. Pałasińska in [18].

It is still possible to obtain a third characterization of pseudo-atomic definability of Leibniz equalities in a given class of structures $\mathcal{K}$, this time involving an algebraic property of the corresponding class of reduced structures K^*.

THEOREM 9. *Let $\mathcal{K}$ be an atomic theory closed class of structures closed under subdirect products and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Then Leibniz equalities are pseudo-atomically definable in $\mathcal{K}$ iff $\mathcal{K}^*$ is closed under subdirect products.*

PROOF. $\Rightarrow$) Suppose that Leibniz equalities are definable in $\mathcal{K}$ by a set Δ of pseudo-atomic formulas. Let $\mathfrak{A} \subseteq_{\mathrm{SD}} \Pi_{i\in I}\mathfrak{A}_i$, where $\mathfrak{A}_i \in \mathcal{K}^*$ for all $i \in I$, and assume that $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$. Since $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ is a surjective homomorphism for all $i \in I$, $\mathfrak{A} \models \Delta[a,b]$ entails that for all $i \in I$, $\mathfrak{A}_i \models \Delta[a(i), b(i)]$. Also $\mathfrak{A}_i \in \mathcal{K}$, for $\mathcal{K}^* \subseteq \mathcal{K}$ by assumption. Hence, $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$ implies $\langle a(i), b(i)\rangle \in \mathbf{\Omega}(\mathfrak{A}_i)$ for all $i \in I$. So, since the $\mathfrak{A}_i$'s are reduced, we have that $a(i) = b(i)$ for all $i \in I$, i.e. $a = b$. This proves that $\mathbf{\Omega}(\mathfrak{A})$ is the identity and hence that $\mathfrak{A}$ is reduced.

$\Leftarrow$) By Theorem 7 it suffices to see that $\mathbf{\Omega}$ is $\sqsubseteq$-monotone in $\mathcal{K}$. So let $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ be such that $\mathfrak{A} \sqsubseteq \mathfrak{B}$. Define $\theta = \mathbf{\Omega}(\mathfrak{A}) \cap \mathbf{\Omega}(\mathfrak{B})$; it is a congruence both of $\mathfrak{A}$ and $\mathfrak{B}$. Let us consider the quotient structures $\mathfrak{A}^*$, $\mathfrak{B}^*$ and $\mathfrak{A}/\theta$. Clearly $\mathfrak{A}/\theta$ is isomorphic to a subdirect product of $\mathfrak{A}^*$ and $\mathfrak{B}^*$ via the mapping $a/\theta \to \langle a/\mathbf{\Omega}(\mathfrak{A}), a/\mathbf{\Omega}(\mathfrak{B})\rangle$. Therefore, since $\mathfrak{A}^*, \mathfrak{B}^* \in \mathcal{K}^*$ and, by assumption, $\mathcal{K}^*$ is closed under subdirect products, $\mathfrak{A}/\theta$ is reduced. But then $\theta = \mathbf{\Omega}(\mathfrak{A})$ and consequently $\mathbf{\Omega}(\mathfrak{A}) \subseteq \mathbf{\Omega}(\mathfrak{B})$. ∎

One can consider restricting the cardinality of the defining set Δ. These restrictions have a natural counterpart from the preceding algebraic point of view. Namely, we have the following theorem.

THEOREM 10. *Let $\mathcal{K}$ be any class of structures closed under κ-reduced products and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Assume that Leibniz equalities are pseudo-atomically definable in $\mathcal{K}$. Then Leibniz equalities are definable in $\mathcal{K}$ by a set $\Delta(x, y)$ of pseudo-atomic formulas of cardinality less than κ iff $\mathcal{K}^*$ is closed under κ-reduced products.*

PROOF. Let $\Delta(x, y)$ be a set of pseudo-atomic formulas that defines Leibniz equalities in $\mathcal{K}$.

$\Leftarrow$) Assume $\mathcal{K}^*$ is closed under κ-reduced products so that $\models_{\mathcal{K}^*}$ is κ-compact (for universal Horn formulas, see Lemma 2). Since $\Delta(x, y)$ defines Leibniz equalities in $\mathcal{K}$, it does so in $\mathcal{K}^*$, and therefore $\Delta(x, y) \models_{\mathcal{K}^*} x = y$. So, by κ-compactness, let Δ_0 be a subset of Δ such that $|\Delta_0| < \kappa$ and $\Delta_0(x, y) \models_{\mathcal{K}^*} x = y$. Then $\Delta_0(x, y)$ defines Leibniz equalities in $\mathcal{K}$ for it does so in the reduced class $\mathcal{K}^*$.

$\Rightarrow$) Suppose now that $|\Delta| < \kappa$. Let us see that $\mathcal{K}^*$ is closed under κ-reduced products. Indeed, for some $\beta \in \kappa$, the (infinite) conjunction of $\Delta(x, y)$ is equivalent to a formula of the infinitary language $L_{\kappa\kappa}$ of the form $\forall \overline{z} \bigwedge_{\alpha<\beta} \varphi_\alpha(x, y, \overline{z})$, where $\varphi_\alpha(x, y, \overline{z})$ is atomic for all $\alpha < \beta$ and $\overline{z}$ is an appropriate, possibly infinite, sequence of variables. So, let σ be the following sentence of $L_{\kappa\kappa}$:

$$\forall xy(\forall \overline{z} \bigwedge_{\alpha<\beta} \varphi_\alpha(x, y, \overline{z}) \to x = y).$$

Then, for any $\mathfrak{A} \in \mathcal{K}$,

$$\mathfrak{A} \models \sigma \quad \text{iff} \quad \mathfrak{A} \in \mathcal{K}^*.$$

But clearly σ is (equivalent to) a Horn sentence of $L_{\kappa\kappa}$, and hence it is preserved by κ-reduced products. Therefore, $\mathcal{K}^*$ is closed under κ-reduced products. ∎

Note that it follows from the preceding proof that the set $\Delta(x,y)$ of pseudo-atomic formulas of cardinality less than κ in the preceding theorem can be taken to be a subset of the initial set. Therefore we can put together the last two theorems to obtain the following result.

COROLLARY 11. *Let $\mathcal{K}$ be an atomic theory closed class of structures closed under subdirect products and κ-reduced products, and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Then Leibniz equalities are definable in $\mathcal{K}$ by a set of pseudo-atomic formulas of cardinality less than κ iff $\mathcal{K}^*$ is closed under subdirect products and κ-reduced products.*

To conclude this section we state a corollary that shows an application of the preceding results to equality-free universal Horn classes; it relates the pseudo-atomic definability of the Leibniz equalities in an equality-free universal Horn class $\mathcal{K}$ to the way the class $\mathcal{K}^*$ can be axiomatized. From the proof it is clear that we just need to add to a set of axioms for $\mathcal{K}$ a sentence that defines the equality relation in $\mathcal{K}^*$

COROLLARY 12. *Let $\mathcal{K}$ be an $L_{\kappa\kappa}$-equality-free universal Horn class. Then the following statements are equivalent:*

(i) *Leibniz equalities are pseudo-atomically definable in $\mathcal{K}$ by a set of pseudo-atomic formulas of cardinality less than κ.*

(ii) $\mathbf{\Omega}$ *is* $\sqsubseteq$-monotone *in* $\mathcal{K}$.

(iii) *$\mathcal{K}^*$ is closed under subdirect products and κ-reduced products.*

(iv) *$\mathcal{K}^*$ is axiomatizable by sentences of $L_{\kappa\kappa}$, possibly with equality, of the form*

$$\forall \overline{z}(\textstyle\bigwedge_{\alpha<\beta} \varphi_\alpha \to \psi) \ \text{or} \ \forall \overline{z} \textstyle\bigvee_{\alpha<\beta} \neg\varphi_\alpha,$$

where the φ_α's are atomic or pseudo-atomic and ψ is atomic or an equation.

(v) *$\mathcal{K}^*$ is axiomatizable by special Horn sentences of $L_{\kappa\kappa}$, possibly with equality.*

PROOF. Any $L_{\kappa\kappa}$-equality-free universal Horn class $\mathcal{K}$ is atomic theory closed, is closed under subdirect products, κ-reduced products and has the property that $\mathcal{K}^* \subseteq \mathcal{K}$. Therefore by the previous results we have the equivalence between (i), (ii) and (iii). Moreover, the implications (iv) to (v) and (v) to (iii) are clear, since special Horn sentences of $L_{\kappa\kappa}$ are preserved by subdirect products. Thus, we need only to prove that (i) implies (iv). Let Δ be a set of pseudo-atomic formulas of cardinality less than κ that defines Leibniz equalities in $\mathcal{K}$. Let σ be the sentence

$$\forall xy(\bigwedge \Delta \rightarrow x = y)$$

and let Σ be a set of equality-free universal Horn sentences of $L_{\kappa\kappa}$ that axiomatizes $\mathcal{K}$. Then, for any structure $\mathfrak{A}$,

$$\mathfrak{A} \in \mathcal{K}^* \text{ iff } \mathfrak{A} \models \Sigma \cup \{\sigma\}.$$

Thus, (iv) holds. ■

An analogous corollary holds for $L_{\infty\infty}$-equality-free universal Horn classes. It suffices to remove the restriction to a set of cardinality less than κ in (i), the condition that $\mathcal{K}^*$ be closed under κ-reduced products in (iii) and to substitute $L_{\infty\infty}$ for $L_{\kappa\kappa}$ in (iv) and (v).

5. Atomic definability of Leibniz equalities

Let $\mathcal{K}$ be an arbitrary class of structures. We say that Leibniz equalities are *atomically definable in* $\mathcal{K}$ if there exists a set Δ of atomic formulas in the free variables x and y that defines Leibniz equalities in $\mathcal{K}$. If Δ is in addition of cardinality $< \kappa$ for a regular infinite cardinal $\kappa \leq \max(|L|, \omega)^+$, then we say that Leibniz equalities are κ-*atomically definable in* $\mathcal{K}$. Notice that κ-atomic definability amounts to atomic definability when $\kappa = \max(|L|, \omega)^+$. Our first characterization of atomic definability is a natural companion to Theorem 9.

THEOREM 13. *Let $\mathcal{K}$ be an atomic theory closed class of structures closed under direct products, substructures and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Then, Leibniz equalities are atomically definable in $\mathcal{K}$ iff $\mathcal{K}^*$ is closed under direct products and substructures.*

PROOF. $\Rightarrow$) Since Leibniz equalities are atomically definable in $\mathcal{K}$, they are pseudo-atomically definable in $\mathcal{K}$. Thus, by Theorem 9, $\mathcal{K}^*$ is closed under direct products. To see that it is closed under substructures, let $\Delta(x, y)$ be a set of atomic formulas that defines Leibniz equalities in $\mathcal{K}$ and let $\mathfrak{A} \subseteq \mathfrak{B}$ with $\mathfrak{B} \in \mathcal{K}^*$. Then, if $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{A})$ we have $\mathfrak{A} \models \Delta[a, b]$. Thus, $\mathfrak{B} \models \Delta[a, b]$, for Δ is a set of atomic formulas. Therefore $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{B})$. This proves that $a = b$ and hence $\mathfrak{A} \in \mathcal{K}^*$.

$\Leftarrow$) Assume that $\mathcal{K}^*$ is closed under direct products and substructures. Then it is closed under subdirect products. By Theorem 9, let Δ be a set of pseudo-atomic formulas that defines Leibniz equalities in $\mathcal{K}$. Take the set

$$\Sigma := \{\varphi(x, y, t_1, \ldots, t_n) : \forall \overline{z}\, \varphi(x, y, \overline{z}) \in \Delta \text{ and } t_1, \ldots, t_n \text{ are terms in } x, y\}.$$

We claim that Σ defines Leibniz equalities in $\mathcal{K}$. Let $\mathfrak{A} \in \mathcal{K}$. Clearly $\langle a, b\rangle \in \boldsymbol{\Omega}(\mathfrak{A})$ implies that $\mathfrak{A} \vDash \Sigma[a, b]$, since Δ defines Leibniz equalities in $\mathcal{K}$. Conversely, assume $\mathfrak{A} \vDash \Sigma[a, b]$. Let $\mathfrak{B}$ be the substructure of $\mathfrak{A}^*$ generated by a^* and b^*. Then $\mathfrak{B} \vDash \Sigma[a^*, b^*]$ and hence, since $\mathfrak{B}$ is generated by a^* and b^*, we have that $\mathfrak{B} \vDash \Delta[a^*, b^*]$. Also, $\mathfrak{B} \in \mathcal{K}^*$, for K^* is closed under substructures by assumption. So $a^* = b^*$, i.e. $\langle a, b\rangle \in \boldsymbol{\Omega}(\mathfrak{A})$. ■

A more general form of the previous theorem is provided by the next result (cf. Corollary 11 above).

THEOREM 14. *Let $\mathcal{K}$ be an atomic theory closed class of structures closed under κ-reduced products, substructures and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Then, Leibniz equalities are κ-atomically definable in $\mathcal{K}$ iff $\mathcal{K}^*$ is closed under κ-reduced products and substructures.*

PROOF. $\Rightarrow$) From Theorem 10 it follows that $\mathcal{K}^*$ is closed under κ-reduced products, whereas Theorem 13 entails that it is closed under substructures.

$\Leftarrow$) Assume now that $\mathcal{K}^*$ is closed under κ-reduced products and substructures. In particular, it is closed under direct products. Therefore, by Theorem 13, Leibniz equalities are definable by a set $\Delta(x, y)$ of atomic formulas. So by an easy compactness argument there also exists a subset of Δ of cardinality less than κ that defines Leibniz equalities in $\mathcal{K}$, as desired. ■

In order to state the characterization of (κ-)atomic definability of Leibniz equalities corresponding to Theorem 7 we need some definitions and a lemma.

Given a non-empty family of structures $\{\mathfrak{A}_i : i \in I\}$ with the same underlying algebra $\mathbb{A}$, we define their union as the structure $\bigcup_{i\in I} \mathfrak{A}_i$, with underlying algebra $\mathbb{A}$, such that for any relation symbol R of L, $R^{\bigcup_{i\in I} \mathfrak{A}_i} = \bigcup_{i\in I} R^{\mathfrak{A}_i}$. We also define their intersection as the structure $\bigcap_{i\in I} \mathfrak{A}_i$, with underlying algebra $\mathbb{A}$, such that for any relation symbol R of L, $R^{\bigcap_{i\in I} \mathfrak{A}_i} = \bigcap_{i\in I} R^{\mathfrak{A}_i}$.

Then the operator $\boldsymbol{\Omega}$ is said to be *$(\kappa, \sqsubseteq)$-continuous* on a class of structures $\mathcal{K}$ if for any algebra $\mathbb{A}$ of type the algebraic subtype of L and any non-empty family $\{\mathfrak{A}_i : i \in I\} \subseteq \mathcal{K}$ of structures on $\mathbb{A}$ κ-directed by $\sqsubseteq$, if

$\bigcup_{i\in I}\mathfrak{A}_i \in \mathcal{K}$ then

$$\mathbf{\Omega}(\bigcup_{i\in I}\mathfrak{A}_i) = \bigcup_{i\in I}\mathbf{\Omega}(\mathfrak{A}_i).$$

Recall that a family of structures on the same algebra is called κ-directed by $\sqsubseteq$ if for any subfamily X of cardinality less than κ, there exists a structure in the family that is a filter extension of every member of X. Note that if $\mathbf{\Omega}$ is $(\kappa, \sqsubseteq)$-continuous in $\mathcal{K}$ and $\kappa \geq 2$, then $\mathbf{\Omega}$ also preserves intersections; namely, if $\{\mathfrak{A}_i : i \in I\}$ is a non-empty family of members of $\mathcal{K}$ on the same algebra, then $\mathbf{\Omega}(\bigcap_{i\in I}\mathfrak{A}_i) = \bigcap_{i\in I}\mathbf{\Omega}(\mathfrak{A}_i)$. The point is that $(\kappa, \sqsubseteq)$-continuity of $\mathbf{\Omega}$ entails $\sqsubseteq$-monotonicity of $\mathbf{\Omega}$.

The last required definition and the next lemma are due to J. Czelakowski. If $\{\mathfrak{A}_i : I \in I\}$ is a family of structures, a structure $\mathfrak{A}$ is said to be a *κ-reduced subdirect product* of $\{\mathfrak{A}_i : I \in I\}$ if there is a subdirect product $\mathfrak{B}$ of $\{\mathfrak{A}_i : I \in I\}$ and a κ-complete filter F on I such that $\mathfrak{A}$ is the quotient $\mathfrak{B}/F$ of $\mathfrak{B}$ modulo the congruence relation $\theta(F)$ on $\mathbf{B}$ defined as follows: for every $f, g \in B$,

$$\langle f, g\rangle \in \theta(F) \text{ iff } \{i \in I : f(i) = g(i)\} \in F.$$

Namely, the underlying algebra of $\mathfrak{B}/F$ is $\mathbf{B}/\theta(F)$ and, for any relational symbol R of L, if $f_1, \dots, f_n \in B$ then

$$\langle f_1/F, \dots, f_n/F\rangle \in R^{\mathfrak{B}/F} \text{ iff } \{i \in I : \langle f_1(i), \dots, f_n(i)\rangle \in R^{\mathfrak{A}_i}\} \in F,$$

where f/F is the equivalence class of f modulo $\theta(F)$.

Notice that, by definition, any κ-reduced subdirect product is isomorphic to a substructure of a κ-reduced product. Actually, subdirect products are a special case of the above construction because they are obtained by taking as filter the set $\{I\}$.

The required lemma concerns κ-reduced subdirect products. Let us denote by $\mathbf{P}_{\kappa\text{-RSD}}(\mathcal{K})$ the class of structures obtained by closing $\mathcal{K}$ under κ-reduced subdirect products and isomorphic copies. Then we have the next result.

LEMMA 15. *For any class $\mathcal{K}$ of structures, if κ is a regular cardinal then*

$$\mathbf{SP}_{\kappa\text{-R}}(\mathcal{K}) = \mathbf{P}_{\kappa\text{-RSD}}(\mathcal{K}).$$

PROOF. Clearly $\mathbf{P}_{\kappa\text{-RSD}}(\mathcal{K}) \subseteq \mathbf{SP}_{\kappa\text{-R}}(\mathcal{K})$, since any κ-reduced subdirect product is isomorphic to a substructure of a κ-reduced product. To see the other inclusion one must first prove that $\mathbf{P}_{\kappa\text{-RSD}}\mathbf{P}_{\kappa\text{-RSD}}(\mathcal{K}) \subseteq \mathbf{P}_{\kappa\text{-RSD}}(\mathcal{K})$. This is done in a similar way as the one used to prove that $\mathbf{P}_{\kappa\text{-R}}\mathbf{P}_{\kappa\text{-R}}(\mathcal{K}) \subseteq$

$\mathbf{P}_{\kappa\text{-R}}(\mathcal{K})$ but with a little more involved argument. Then, to obtain the lemma, it is enough to show that $\mathbf{S}(\mathcal{K}) \subseteq \mathbf{P}_{\kappa\text{-RSD}}(\mathcal{K})$. So let $\mathfrak{A}$ be a substructure of some $\mathfrak{B} \in \mathcal{K}$. Define

$$C = \{f \in \prod_{\alpha\in\kappa} B : \exists\alpha \in \kappa\, \forall\beta,\gamma \geq \alpha\; f(\beta) = f(\gamma) \in A\}.$$

If $f \in C$ there exists a least α such that $f(\beta) = f(\gamma)$ for all $\beta, \gamma \geq \alpha$; let us call it α_f and put $a_f = f(\alpha_f + 1)$. It is easy to check that C is the universe of a substructure of $\prod_{\alpha\in\kappa}\mathfrak{B}$; rather it is the universe of a subdirect product of $\prod_{\alpha\in\kappa}\mathfrak{B}$, for the projection of C into each component is clearly surjective. So let $\mathfrak{C}$ be the corresponding substructure and consider the filter $F = \{X \subseteq \kappa : |\kappa - X| < \kappa\}$. This filter is κ-complete because κ is regular. Thus the mapping h from $\mathfrak{C}/F$ into $\mathfrak{A}$ defined by $h(f/F) = a_f$ is the required isomorphism, since $a_{t^{\mathfrak{C}}(f_1,\ldots,f_n)} = t^{\mathfrak{B}}(a_{f_1},\ldots,a_{f_n})$ for any function symbol t of L and $f/F = g/F$ iff $a_f = a_g$. ■

The desired theorem can now be stated as follows.

THEOREM 16. *Let $\mathcal{K}$ be an atomic theory closed class of structures closed under κ-reduced products, substructures and such that $\mathcal{K}^* \subseteq \mathcal{K}$. Also assume that $\mathcal{K}$ is closed under intersections, unions of κ-directed families by $\sqsubseteq$ and strict homomorphic images and strict inverse images. Then Leibniz equalities are κ-atomically definable in $\mathcal{K}$ iff $\mathbf{\Omega}$ is $(\kappa, \sqsubseteq)$-continuous on $\mathcal{K}$.*

PROOF. $\Rightarrow$) Suppose Leibniz equalities are definable in $\mathcal{K}$ by a set Δ of atomic formulas in the variables x and y whose cardinality is less than κ. Let $\{\mathfrak{A}_i : i \in I\}$ be a non-empty family of members of $\mathcal{K}$ on the same underlying algebra $\mathbb{A}$, κ-directed by filter extension. Then we have the following equivalences, the second one holds by the assumption that the family is κ-directed and $|\Delta| < \kappa$:

$$\begin{aligned}
\langle a,b\rangle \in \mathbf{\Omega}(\textstyle\bigcup_{i\in I}\mathfrak{A}_i) \quad &\text{iff} \quad \textstyle\bigcup_{i\in I}\mathfrak{A}_i \models \Delta(x,y)[a,b]\\
&\text{iff} \quad \mathfrak{A}_i \models \Delta(x,y)[a,b], \text{ for some } i \in I\\
&\text{iff} \quad \langle a,b\rangle \in \mathbf{\Omega}(\mathfrak{A}_i), \text{ for some } i \in I\\
&\text{iff} \quad \langle a,b\rangle \in \textstyle\bigcup_{i\in I}\mathbf{\Omega}(\mathfrak{A}_i).
\end{aligned}$$

$\Leftarrow$) We apply the preceding theorem. We will show that if the $\mathbf{\Omega}$ operator is $(\kappa, \sqsubseteq)$-continuous then $\mathcal{K}^*$ is closed under κ-reduced products and substructures, by proving that it is closed under $\mathbf{P}_{\kappa-\mathbf{RSD}}$ and applying Lemma 15. So let $\{\mathfrak{A}_i : i \in I\}$ be a non-empty family of members of $\mathcal{K}^*$, and let F be any κ-complete filter on I. We first need to verify that if $\mathfrak{A} \subseteq_{\text{SD}} \Pi_{i\in I}\mathfrak{A}_i$ then $\mathfrak{A}/F$ is reduced, for clearly it belongs to $\mathcal{K}$ because is a strict homomorphic image of $\mathfrak{A}$. Indeed, let π_i be the projection from $\mathfrak{A}$ onto $\mathfrak{A}_i$, for each $i \in I$, and let

$$\mathfrak{C} = \bigcup_{X \in F} \bigcap_{i \in X} \pi_i^{-1}[\mathfrak{A}_i],$$

where, for any $i \in I$, $\pi_i^{-1}[\mathfrak{A}_i]$ is the structure on the underlying algebra of $\mathfrak{A}$ whose relational part is defined as follows: if R is a relational symbol of L,

$$R^{\pi_i^{-1}[\mathfrak{A}_i]} = \{\langle a_1, \dots, a_n\rangle \in A^n : \langle \pi_i(a_1) \dots, \pi_i(a_n)\rangle \in R^{\mathfrak{A}_i}\}.$$

Note that π_i is a strict homomorphism from $\pi_i^{-1}[\mathfrak{A}_i]$ onto $\mathfrak{A}_i$. Moreover, since F is κ-complete, the family of structures $\{\bigcap_{i \in X} \pi_i^{-1}[\mathfrak{A}_i] : X \in F\}$ is κ-directed by $\sqsubseteq$. Therefore, by the assumptions on $\mathcal{K}$, $\mathfrak{C} \in \mathcal{K}$. Now, let

$$\theta_F = \{\langle f, g\rangle : f, g \in A \text{ and } \{i \in I : f(i) = g(i)\} \in F\}.$$

It is easy to check that the quotient of $\mathfrak{C}$ modulo θ_F is $\mathfrak{A}/F$. Hence the proof will be finished if we see that $\boldsymbol{\Omega}(\mathfrak{C}) = \theta_F$. To this end, we use the continuity property of $\boldsymbol{\Omega}$ in $\mathcal{K}$; we obtain that

$$\boldsymbol{\Omega}(\mathfrak{C}) = \bigcup_{X \in F} \bigcap_{i \in X} \boldsymbol{\Omega}(\pi_i^{-1}[\mathfrak{A}_i]).$$

Moreover, as $\mathfrak{A}_i$ is a reduced structure for each $i \in I$,

$$\boldsymbol{\Omega}(\pi_i^{-1}[\mathfrak{A}_i]) = \pi_i^{-1}[\boldsymbol{\Omega}([\mathfrak{A}_i])] = \pi_i^{-1}[\mathrm{Id}_{A_i}] = ker(\pi_i).$$

Consequently, $\langle f, g\rangle \in \boldsymbol{\Omega}(\mathfrak{C})$ iff $\{i \in I : f(i) = g(i)\} \in F$, for all $f, g \in A$. ■

To conclude the section we state for the case of atomic definability an analogous corollary to Corollary 12.

COROLLARY 17. *Let $\mathcal{K}$ be an $L_{\kappa\kappa}$-equality-free universal Horn class. Then the following statements are equivalent:*

(i) *Leibniz equalities are atomically definable in $\mathcal{K}$ by a set of atomic formulas of cardinality less than κ.*

(ii) *$\mathcal{K}^*$ is closed under substructures and κ-reduced products.*

(iii) *$\mathcal{K}^*$ is an $L_{\kappa\kappa}$-universal Horn class.*

(iv) *$\boldsymbol{\Omega}$ is $(\kappa, \sqsubseteq)$-continuous on $\mathcal{K}$.*

PROOF. Similar to the one of Corollary 12 taking into account that the conditions of Theorem 16 hold for any $L_{\kappa\kappa}$-equality-free universal Horn class, as it is not difficult to check. ■

As in Corollary 12 we have an analog for $L_{\infty\infty}$-equality-free universal Horn classes. We need only to delete in condition (i) the restriction to sets of cardinality less than κ, in condition (ii) the condition that $\mathcal{K}^*$ be under κ-reduced products, and substitute in condition (iii) $L_{\infty\infty}$ for $L_{\kappa\kappa}$.

6. Some examples

To conclude the paper we present several examples where our results apply.

6.1. Algebras with an equivalence relation

Let L be an arbitrary language with possibly some function symbols and only one binary relation symbol R, and let $\mathcal{K}$ be the class of L-structures where R is interpreted as an equivalence relation. By applying the ideas of the proof of Theorem 7, it is easy to see that Leibniz equalities are definable in $\mathcal{K}$ by the set

$$\begin{aligned} \Delta(x,y) &= \{\forall \overline{z}\, Rt(x,y,\overline{z})t'(x,y,\overline{z}) : t \text{ and } t' \text{ are terms} \\ &\qquad \text{that unify by the substitution } x \to y\}. \end{aligned}$$

Thus, in general, Leibniz equalities are pseudo-atomically definable in $\mathcal{K}$. For particular languages of the type considered they are also atomically definable. For instance, if L has no function symbols, then Leibniz equalities are trivially definable in $\mathcal{K}$ by the single atomic formula Rxy. Also, if L has only unary function symbols, then Leibniz equalities are definable in $\mathcal{K}$ by the set of atomic formulas

$$\Delta(x,y) = \{Rf^n x f^n y : f \text{ is a fuction symbol, } n \geq 0\}.$$

We are going to see, however, that if L has an n-ary function symbol f with $n > 1$, then Leibniz equalities are not atomically definable in $\mathcal{K}$. For this, suppose that L just contains the binary relation symbol R and a binary function symbol f; in the more general case we can argue similarly. Note that for any L-structure $\mathfrak{A}$ such that $R^{\mathfrak{A}}$ is reflexive, $\mathbf{\Omega}(\mathfrak{A}) \subseteq R^{\mathfrak{A}}$. So consider the structure $\mathfrak{A} = \langle A, f^{\mathfrak{A}}, R^{\mathfrak{A}}\rangle$ where $A = \{1,2,3\}$, $R^{\mathfrak{A}} = \mathrm{Id}_A \cup \{\langle 1,2\rangle, \langle 2,1\rangle\}$, and $f^{\mathfrak{A}}(n,m) = \max(n,m)$ if n and m are not 2, and $f^{\mathfrak{A}}(n,m) = 2$ otherwise. Since $\langle f^{\mathfrak{A}}(1,3), f^{\mathfrak{A}}(2,3)\rangle \notin R^{\mathfrak{A}}$, we have that $\langle 1,2\rangle \notin \mathbf{\Omega}(\mathfrak{A})$, and hence $\mathfrak{A}$ is reduced. On the other hand, the substructure of $\mathfrak{A}$ generated by $\{1,2\}$ is not reduced. Therefore, by Theorem 13, we conclude that Leibniz equalities are not atomically definable in $\mathcal{K}$.

6.2. Algebras with a quasi-ordering

Let L be as before and consider the class, also denoted by $\mathcal{K}$, of all those structures where R is interpreted as a quasi-ordering (i.e., a reflexive and transitive relation). Once again we have trivially that Leibniz equalities are ω-atomically definable in $\mathcal{K}$ by the set $\Delta(x,y) = \{Rxy, Ryx\}$ if L has no function symbols, whereas if L has only unary function symbols then Leibniz equalities are atomically definable by the set

$$\begin{aligned}\Delta(x,y) \;=\; & \{Rf^nxf^ny : f \text{ a function symbol, } n \geq 0\} \cup \\ & \{Rf^nyf^nx : f \text{ a function symbol, } n \geq 0\}.\end{aligned}$$

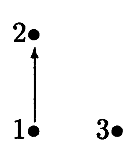
In general, however, Leibniz equalities are pseudo-atomically definable by the set

$$\begin{aligned}\Delta(x,y) \;=\; & \{\forall \overline{z}\, Rt(x,y,\overline{z})t'(x,y,\overline{z}) : t \text{ and } t' \text{ are terms} \\ & \quad \text{that unify by the substitution } x \to y\}.\end{aligned}$$

As in the example of equivalence relations, if L has an n-ary function symbol, with $n > 1$, then Leibniz equalities are not atomically definable.

6.3. Sets with a strict partial ordering

Let L be the language with a binary relation symbol R, and consider the class $\mathcal{K}$ of all the structures where R is interpreted as a strict partial order. This class is clearly an equality-free universal Horn class and, as we will justify, Leibniz equalities are not pseudo-atomically definable in it. In any strict partial order two objects are related by the Leibniz equality if both have the same set of objects below them and the same set of objects above them. Therefore, any two isolated objects are related by the Leibniz equality. In order to prove that Leibniz equalities are not pseudo-atomically definable in $\mathcal{K}$, by Theorem 9 it is enough to see that $\mathcal{K}^*$ is not closed under direct products. So consider the partial ordering on $\{1,2,3\}$ with diagram



This partial ordering is reduced but the direct product of it by itself is clearly non reduced, because it has several isolated points.

6.4. Examples arising from logical systems

As mentioned in the Introduction, any propositional calculus is equivalent (from the point of view of matrix theory) to a strict and equality-free universal Horn theory, so that all the theorems in the paper apply to the study of the properties of Leibniz equality in their classes of models, that is, their matrices. Blok and Pigozzi's paper [3] contains some examples of this nature. The classes where Leibniz equalities are pseudo-atomically definable are the classes of matrix models of their protoalgebraic deductive systems, whereas those classes where Leibniz equalities are atomically definable correspond to their (weakly) congruential deductive systems also known as (finitely) equivalential in the literature (see [6] and [7]). But maybe our main immediate contribution lies on the fact that the generalization to arbitrary first-order languages worked out in this paper makes it possible to apply the results to the study of more general kinds of logical systems, viz. the k-dimensional deductive systems of [3], the Gentzen systems treated as a "consequence" relation between sequents, as in [20], and the multidimensional Gentzen systems of [22]. The point is that the interpretation as strict universal Horn theories over appropriate languages can also be carried out for a rather general sort of formal systems that include the preceding ones and even some other consequences naturally associated to classes of algebras, such as quasi-equational consequences. In this sense, this paper is a contribution to the general theory started in [9], [10], [11], [12] and [13] which aims to encompass the investigation of algebraic semantics of particular types of logical systems.

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