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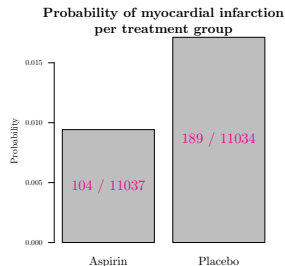
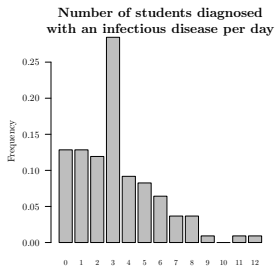
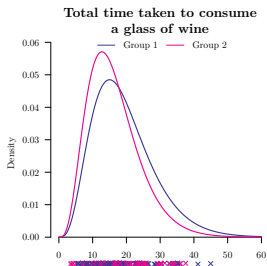
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Generalised Linear Models (GLM)

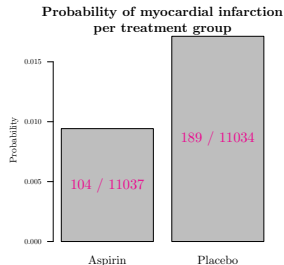
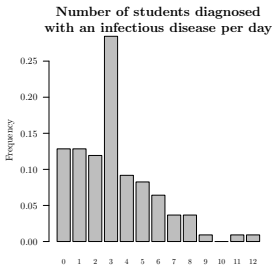
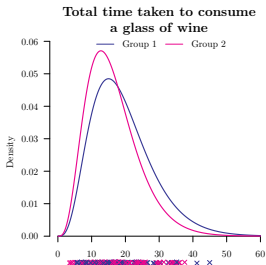
Cancer Research UK – 19th of July 2017

D.-L. Couturier / M. Dunning / R. Nicholls

Examples of data with non-normal conditional distributions



Examples of data with non-normal conditional distributions



Linear model not suitable:

► Assumed model:

$$Y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \epsilon_i \text{ where } \epsilon_i \sim N(0, \sigma^2),$$

$$Y_i | (\mathbf{x}_i, \boldsymbol{\beta}) \sim N(\mu_i, \sigma^2).$$

- ▷ theoretical range of $\epsilon_i = [-\infty, +\infty]$,
- ▷ $\mathbf{x}_i^T \boldsymbol{\beta}$ not bounded to $[0, \infty]$,
- ▷ $\text{Var}[Y_i]$ independent of $\text{E}[Y_i]$.

► Solution:

$$Y_i | (\mathbf{x}_i, \boldsymbol{\beta}) \sim \text{distribution}(\text{function}(\mathbf{x}_i^T \boldsymbol{\beta}), \phi),$$

where *distribution* belongs to the exponential family and *function* is monotonically increasing.

GLM: conditional distributions

$$Y_i | (\mathbf{x}_i, \boldsymbol{\beta}) \sim \text{distribution}(\text{function}(\mathbf{x}_i^T \boldsymbol{\beta}), \phi),$$

- Some possible conditional *distributions* :
statistical probability mass functions & density functions

- Within the exponential family ['classical' GLM framework]

normal
exponential
gamma

chi-squared
beta
Dirichlet

Bernoulli
Poisson
Wishart

Inverse Wishart
...

- Outside the exponential family ['extended' GLM framework]

Box-Cox power
exponential
exponential
Gaussian
generalized beta

generalized gamma
generalized inverse
Gaussian
inverse Gaussian
logistic

power exponential
reverse Gumbel
skew power
exponential
Weibull

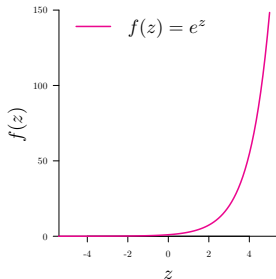
Pareto type I, II, III
Poisson inverse
Gaussian
...

GLM: link functions

$$Y_i | (\mathbf{x}_i, \boldsymbol{\beta}) \sim \text{distribution}(\text{function}(\mathbf{x}_i^T \boldsymbol{\beta}), \phi),$$

- Most used link *functions* :
connection between Y_i and $\mathbf{x}_i^T \boldsymbol{\beta}$

- to restrict $f(\mathbf{x}_i^T \boldsymbol{\beta})$ to belong to $[0, \infty[$:
 - ▷ log link: $f(z) = e^z$



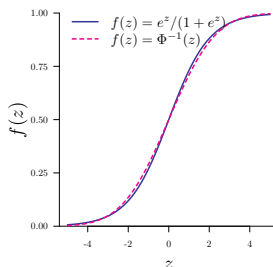
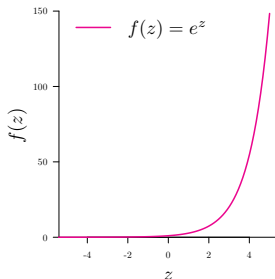
GLM: link functions

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► Most used link functions :

connection between Y_i and $\mathbf{x}_i^T \beta$

- to restrict $f(\mathbf{x}_i^T \beta)$ to belong to $[0, \infty[$:
 - ▷ log link: $f(z) = e^z$
- to restrict $f(\mathbf{x}_i^T \beta)$ to belong to $[0, 1]$:
 - ▷ logit link: $f(z) = e^z / (1 + e^z) = 1 / (1 + e^{-z})$ where z is positive
 - ▷ probit link: $f(z) = \Phi^{-1}(z)$, where Φ denotes the $N(0, 1)$.



Distribution for dichotomous variates: Bernoulli

Example:

in Jones (*Unpublished BSc dissertation, University of Southampton, 1975*), the main outcome is the presence/absence of bronchitis:

Sample of 212 men in Cardiff: $i = 1$ $i = 2$ $i = 3$ \dots $i = 212$

	B	B	B	\dots	B
y_i	0	0	1	\dots	0

If

- ▶ n independent experiments,
- ▶ outcome of each experiment is dichotomous (success/failure),
- ▶ the probability of success π is the same for all experiments,

then, each dichotomous experiment, Y_i , follows a Bernoulli distribution with parameter π :

$$Y_i \sim \text{Bernoulli}(\pi)$$

$$P(Y_i = 1) = \pi$$

$$P(Y_i = 0) = 1 - \pi$$

Logistic regression: GLM for dichotomous variates

Example:

in Jones (*Unpublished BSc dissertation, University of Southampton, 1975*), the main outcome is the **presence/absence of bronchitis** as a function of the **daily number of smoked cigarettes (X_1)** and **level of pollution (X_2)**:

Sample of 212 men in Cardiff: $i = 1$ $i = 2$ $i = 3$ \dots $i = 212$

	y_i	x_{1i}	x_{2i}
	β	β	B
	0	0	1

	0	0	0
	5.15	0	2.5
	67.1	66.9	66.7
			55.4

If

- ▶ n independent experiments,
- ▶ outcome of each experiment is dichotomous (success/failure),
- ▶ the probability of success π is the same for all experiments **given the covariates**,

then, each dichotomous experiment, Y_i , follows a **Bernoulli distribution** with parameter π_i :

$$Y_i \sim \text{Bernoulli}(\pi_i) \text{ where } \pi_i = \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}}}{1 + e^{\mathbf{x}_i^T \boldsymbol{\beta}}}$$

$$P(Y_i = 1) = \pi_i$$

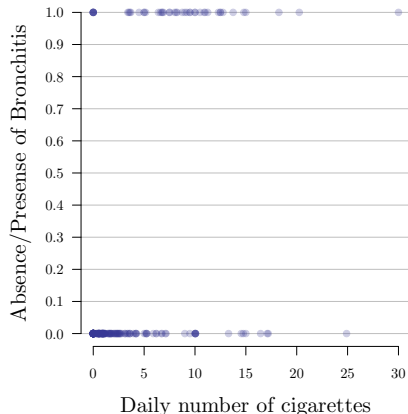
$$P(Y_i = 0) = 1 - \pi_i$$

Logistic regression: predictions and interpretation of β

Example:

Model the probability of presence of bronchitis as a function of the daily number of smoked cigarettes (X_1):

$$P(Y_i = 1) = \pi_i = \frac{e^{\beta_0 + \beta_1 x_{i1}}}{1 + e^{\beta_0 + \beta_1 x_{i1}}}$$

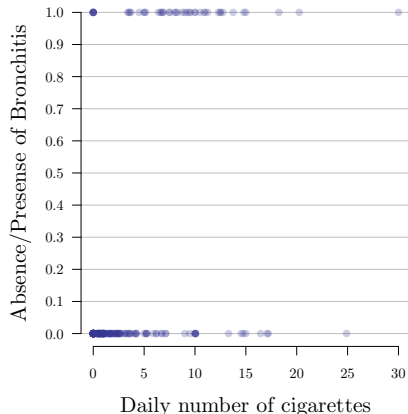


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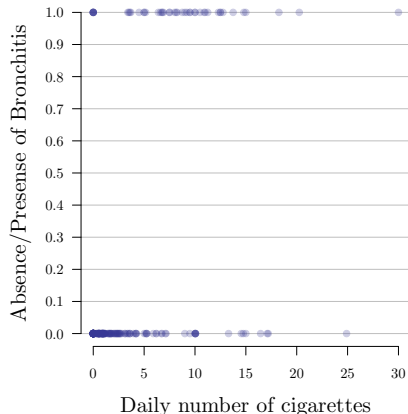
	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	-2.2839832	0.27305505	-8.364552	6.034375e-17
cigs	0.2093618	0.03760466	5.567442	2.585062e-08

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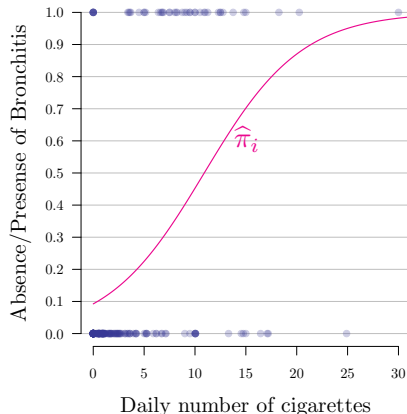
$$\triangleright \hat{\pi}_{0\text{cig}} = \frac{e^{\hat{\beta}_0}}{1 + e^{\hat{\beta}_0}}$$

Logistic regression: predictions and interpretation of β

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Model the probability of presence of bronchitis as a function of the daily number of smoked cigarettes (X_1):

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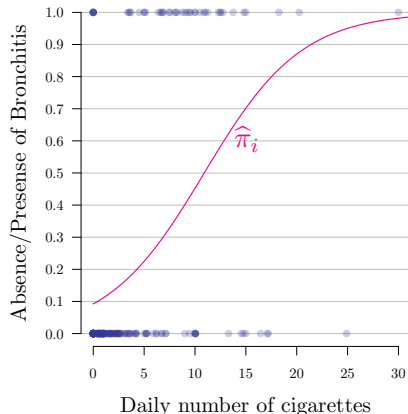
$$\begin{aligned} \bullet \quad \hat{\pi}_{0\text{cig}} &= \frac{e^{\hat{\beta}_0}}{1 + e^{\hat{\beta}_0}} \Leftrightarrow \frac{\hat{\pi}_{0\text{cig}}}{1 - \hat{\pi}_{0\text{cig}}} = e^{\hat{\beta}_0} \\ \bullet \quad \hat{\pi}_{1\text{cig}} &= \frac{e^{\hat{\beta}_0 + \hat{\beta}_1}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1}} \Leftrightarrow \frac{\hat{\pi}_{1\text{cig}}}{1 - \hat{\pi}_{1\text{cig}}} = e^{\hat{\beta}_0 + \hat{\beta}_1} \\ \bullet \quad \frac{\frac{\hat{\pi}_{1\text{cig}}}{1 - \hat{\pi}_{1\text{cig}}}}{\frac{\hat{\pi}_{0\text{cig}}}{1 - \hat{\pi}_{0\text{cig}}}} &= \frac{e^{\hat{\beta}_0 + \hat{\beta}_1}}{e^{\hat{\beta}_0}} = e^{\hat{\beta}_1} \quad (\text{Odd ratio}) \end{aligned}$$

Logistic regression: predictions and interpretation of β

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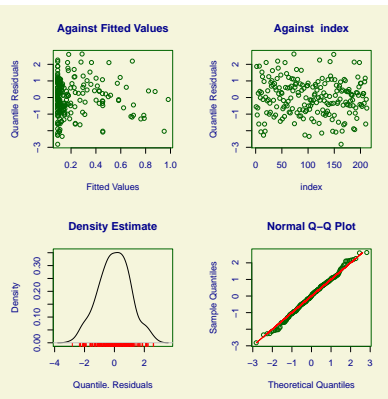
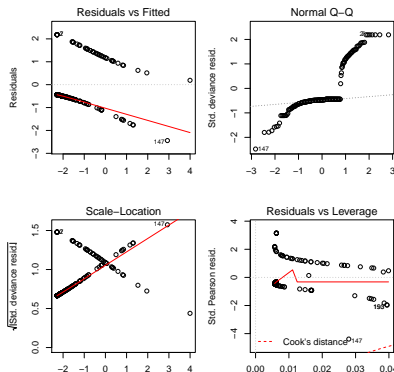
$$\blacktriangleright \hat{\pi}_{1\text{cig}} = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1}} \Leftrightarrow \frac{\hat{\pi}_{1\text{cig}}}{1 - \hat{\pi}_{1\text{cig}}} = e^{\hat{\beta}_0 + \hat{\beta}_1}$$

$$\blacktriangleright \frac{\frac{\hat{\pi}_{1\text{cig}}}{1 - \hat{\pi}_{1\text{cig}}}}{\frac{\hat{\pi}_{0\text{cig}}}{1 - \hat{\pi}_{0\text{cig}}}} = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1}}{e^{\hat{\beta}_0}} = e^{\hat{\beta}_1} \text{ (Odd ratio)}$$

- $\blacktriangleright \text{H0}_1: \beta_0 = 0 \text{ versus H1}_1: \beta_0 \neq 0$
- $\text{H0}_2: \beta_1 = 0 \text{ versus H1}_2: \beta_1 \neq 0$

Logistic regression: model check

- ▶ pearson residuals $(y_i - \hat{\pi}) / \sqrt{\text{Var}(\hat{\pi})}$,
- ▶ deviance residuals [Default in R],
- ▶ randomised normalised quantile residuals [Default in package `gamlss()`]



Distribution for count data: Poisson

Example:

Interest for the number of high school students diagnosed with an infectious disease

Sample of 115 days: $t = 1$ $t = 2$ $t = 3$ \dots $t = 115$

y_i 6 8 12 \dots 0

If, during a time interval or in a given area,

- ▶ events occur independently,
- ▶ at the same rate,
- ▶ and the probability of an event to occur in a small interval (area) is proportional to the length of the interval (size of the area),

then,

- ▶ a count occurring in a fixed time interval or in a given area, Y , may be modelled by means of a Poisson distribution with parameter μ :

$$Y \sim \text{Poisson}(\mu) \text{ where } \mu = E[Y] = \text{Var}[Y],$$

- ▶ the probability of observing x events during a fixed time interval or in a given area is given by

$$P(Y = y|\mu) = \frac{\mu^y e^{-\mu}}{y!}.$$

Poisson regression: GLM for count data

Example:

Interest for the number of high school students diagnosed with an infectious disease as a function of the number of days from the initial outbreak

Sample of 115 days: $t = 1$ $t = 2$ $t = 3$ \dots $t = 115$

y_t	6	8	12	\dots	0
t	1	2	3	\dots	115

If, during a time interval or in a given area,

- ▶ events occur independently given the covariates,
- ▶ at the same rate given the covariates,
- ▶ and the probability of an event to occur in a small interval (area) is proportional to the length of the interval (size of the area) given the covariates,

then,

- ▶ each count occurring in a fixed time interval or in a given area, Y_t , may be modelled by means of a Poisson distribution with parameter μ_t :

$$Y_t \sim \text{Poisson}(\mu_t) \text{ where } \mu_t = \mathbb{E}[Y] = \text{Var}[Y] = e^{\mathbf{x}_t^T \boldsymbol{\beta}},$$

- ▶ the probability of observing y during the fixed time interval or in the given area is given by

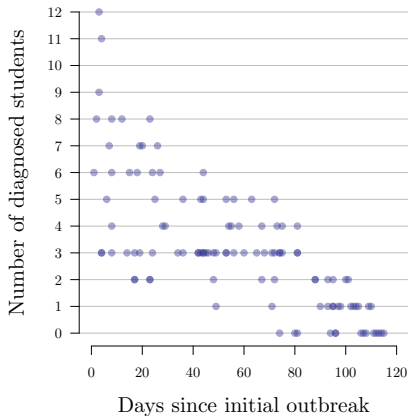
$$P(Y_t = y_t | \mu_t) = \frac{\mu_t^{y_t} e^{-\mu_t}}{y_t!}.$$

Poisson regression: predictions and interpretation of β

Example:

Model the mean count of diagnosed students, μ_t , as a function of the number of days from the outbreak (T) :

$$\mu_t = \frac{e^{\beta_0 + \beta_1 t}}{1 + e^{\beta_0 + \beta_1 t}}$$

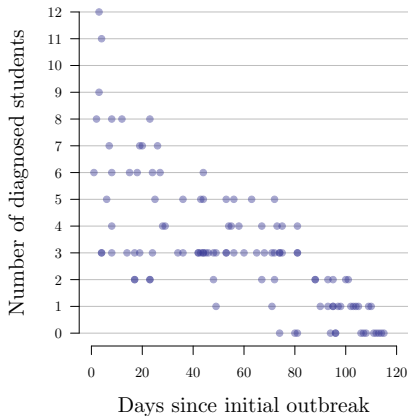


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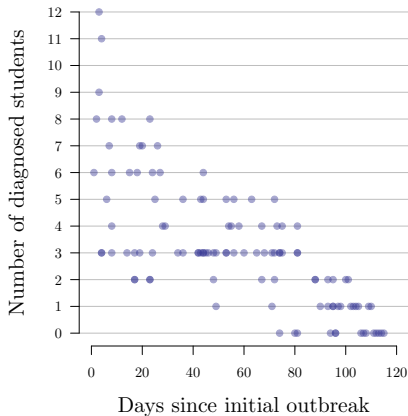
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(Intercept)	1.99023497	0.083935207	23.71156	2.739875e-124
day	-0.01746317	0.001726709	-10.11356	4.810392e-24

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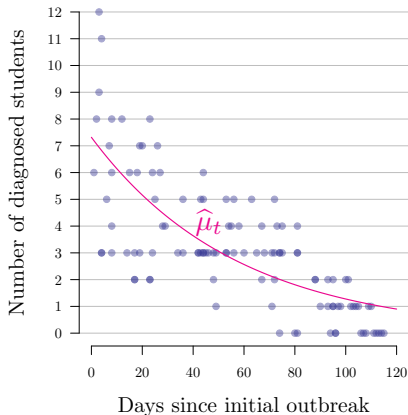
► $\hat{\mu}_{\text{day}0} = e^{\hat{\beta}_0}$

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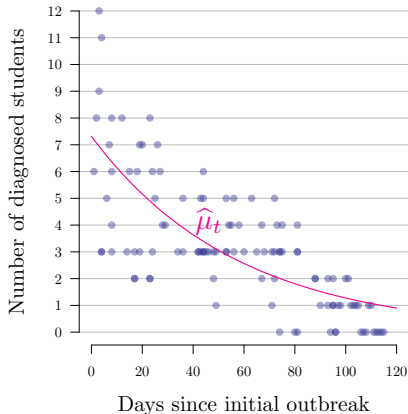
- ▶ $\hat{\mu}_{\text{day0}} = e^{\hat{\beta}_0}$
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- ▶ $\frac{\hat{\mu}_{\text{day1}}}{\hat{\mu}_{\text{day0}}} = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1}}{e^{\hat{\beta}_0}} = e^{\hat{\beta}_1}$

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- ▶ $\frac{\hat{\mu}_{\text{day}1}}{\hat{\mu}_{\text{day}0}} = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1}}{e^{\hat{\beta}_0}} = e^{\hat{\beta}_1}$
- ▶ $H0_1: \beta_0 = 0$ versus $H1_1: \beta_0 \neq 0$
 $H0_2: \beta_1 = 0$ versus $H1_2: \beta_1 \neq 0$

Poisson regression: model check

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