Modelling and Control of Manipulators

Homework No. 01

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January 2, 2022

Contents

Abbreviations

$cos(\theta_i) \rightarrow c_i$
$sin(\theta_i) \rightarrow s_i$
$cos(\theta_i + \theta_j) \rightarrow c_{ij}$
$sin(\theta_i + \theta_j) \rightarrow s_{ij}$

1 Introduction

This report will provide the analytical solution of direct and inverse kinematic of ABB IRB-7600 manipulator. Due to the complexity of matrices, MATLAB is used to resolve the complex matrices. The link to the files can be approached by using on the following link:
$https://drive.google.com/drive/folders/1iRiX2OGMy9HkMI5LzrG8tXgPTa5P6t7m?usp = sharing$

2 Denavit-Hartenberg (DH) Parameters for ABB IRB-7600 Manipulator

Placement of axes should follow the rules provided below:

1. Draw the manipulator
2. Identify the axes of rotation or translation (for revolute and prismatic joints)
3. Assign the $_iz$ axis to the i^{th} axis of rotation/translation
4. Find the common perpendiculars between the $_{i-1}z$ and $_iz$ axes (if those axes intersect the common perpendicular is defined along $_{i-1}z \times {_iz}$)
5. Define the $_ix$ axes – the axis $_{i-1}z$ coincides with the common perpendicular between the $_{i-1}z$ and $_iz$ axes and points from $_{i-1}z$ to $_iz$
6. Determine the $_iy$ axes $_iy = {_iz} \times {_ix}$
7. The base coordinate frame 0 should be located in such a way that α_0, a_0 and θ_1 or d_1 are equal to 0, where $_0z = {_1z}$ and $^0_0P = {^0_0P}$

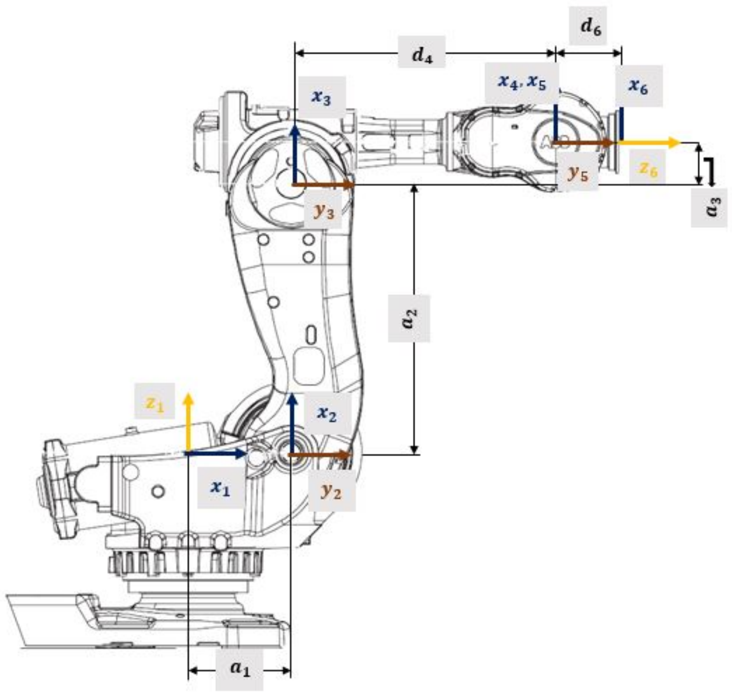

Figure 1: Placement of Axes for ABB IRB-7600 Manipulator

Determination of the link and joint parameters follow the rules stated below:

1. a_{i-1}: the distance between axes $_{i-1}z$ and $_iz$measured along $_{i-1}x$
2. α_{i-1}: the angle between axes $_{i-1}z$ and $_iz$ measured around $_{i-1}x$
3. d_i: the distance between axes $_{i-1}x$ and $_ix$ measured along $_iz$
4. θ_i: the angle between axes $_{i-1}x$ and $_ix$ measured around $_iz$

i	α_{i-1}	a_{i-1}	d_i	θ_i
0	0	0	0	θ_1
1	$-\pi/2$	$a_1 = 410$	0	$\theta_2 := \theta_2 - \pi/2$
2	0	$a_2 = 1075$	0	θ_3
3	$-\pi/2$	$a_3 = 165$	$d_4 = 1056$	θ_4
4	$\pi/2$	0	0	θ_5
5	$-\pi/2$	0	$d_6 = 250$	θ_6

3 Direct Kinematics

Direct Kinematics follows the below stated formula:

$$ {}^{0}_{n}\mathcal{T} = {}^{0}_{1}\mathcal{T}\,{}^{1}_{2}\mathcal{T}\ldots{}^{n-1}_{n}\mathcal{T} $$

$$ {}^{0}_{n}\mathcal{T} = \prod_{i=1}^{n} {}^{i-1}_{i}\mathcal{T} $$

$$ {}^{0}_{1}\mathcal{T} = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{1}_{2}\mathcal{T} = \begin{pmatrix} c_2 & -s_2 & 0 & a_1 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{2}_{3}\mathcal{T} = \begin{pmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{3}_{4}\mathcal{T} = \begin{pmatrix} c_4 & -s_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s_4 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{4}_{5}\mathcal{T} = \begin{pmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{5}_{6}\mathcal{T} = \begin{pmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ -s_6 & -c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ \mathcal{T}_d = \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{2}\mathcal{T} = \begin{pmatrix} c_1\,c_2 & -c_1\,s_2 & -s_1 & a_1\,c_1 \\ c_2\,s_1 & -s_1\,s_2 & c_1 & a_1\,s_1 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{3}\mathcal{T} = \begin{pmatrix} c_1\,c_2\,c_3 - c_1\,s_2\,s_3 & -c_1\,c_2\,s_3 - c_1\,c_3\,s_2 & -s_1 & a_1\,c_1 + a_2\,c_1\,c_2 \\ c_2\,c_3\,s_1 - s_1\,s_2\,s_3 & -c_2\,s_1\,s_3 - c_3\,s_1\,s_2 & c_1 & a_1\,s_1 + a_2\,c_2\,s_1 \\ -c_2\,s_3 - c_3\,s_2 & s_2\,s_3 - c_2\,c_3 & 0 & -a_2\,s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{4}\mathcal{T} = \begin{pmatrix} s_1\,s_4 + c_4\,\sigma_1 & c_4\,s_1 - s_4\,\sigma_1 & -c_1\,c_2\,s_3 - c_1\,c_3\,s_2 & a_1\,c_1 + a_3\,\sigma_1 - d_4\,(c_1\,c_2\,s_3 + c_1\,c_3\,s_2) + a_2\,c_1\,c_2 \\ c_4\,\sigma_2 - c_1\,s_4 & -c_1\,c_4 - s_4\,\sigma_2 & -c_2\,s_1\,s_3 - c_3\,s_1\,s_2 & a_1\,s_1 + a_3\,\sigma_2 - d_4\,(c_2\,s_1\,s_3 + c_3\,s_1\,s_2) + a_2\,c_2\,s_1 \\ -c_4\,\sigma_3 & s_4\,\sigma_3 & s_2\,s_3 - c_2\,c_3 & -a_2\,s_2 - a_3\,\sigma_3 - d_4\,(c_2\,c_3 - s_2\,s_3) \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

where
$\sigma_1 = c_1\,c_2\,c_3 - c_1\,s_2\,s_3$
$\sigma_2 = c_2\,c_3\,s_1 - s_1\,s_2\,s_3$
$\sigma_3 = c_2\,s_3 + c_3\,s_2$

$$ {}^{0}_{5}\mathcal{T} = \begin{pmatrix} c_5\,\sigma_5 - s_5\,\sigma_1 & -c_5\,\sigma_1 - s_5\,\sigma_5 & s_4\,\sigma_7 - c_4\,s_1 & a_1\,c_1 + a_3\,\sigma_7 - d_4\,\sigma_1 + a_2\,c_1\,c_2 \\ -s_5\,\sigma_2 - c_5\,\sigma_6 & s_5\,\sigma_6 - c_5\,\sigma_2 & c_1\,c_4 + s_4\,\sigma_8 & a_1\,s_1 + a_3\,\sigma_8 - d_4\,\sigma_2 + a_2\,c_2\,s_1 \\ -s_5\,\sigma_4 - c_4\,c_5\,\sigma_3 & c_4\,s_5\,\sigma_3 - c_5\,\sigma_4 & -s_4\,\sigma_3 & -a_2\,s_2 - a_3\,\sigma_3 - d_4\,\sigma_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

where
$\sigma_1 = c_1\,c_2\,s_3 + c_1\,c_3\,s_2$
$\sigma_2 = c_2\,s_1\,s_3 + c_3\,s_1\,s_2$

$\sigma_3 = c_2\, s_3 + c_3\, s_2$
$\sigma_4 = c_2\, c_3 - s_2\, s_3$
$\sigma_5 = s_1\, s_4 + c_4\, \sigma_7$
$\sigma_6 = c_1\, s_4 - c_4\, \sigma_8$
$\sigma_7 = c_1\, c_2\, c_3 - c_1\, s_2\, s_3$
$\sigma_8 = c_2\, c_3\, s_1 - s_1\, s_2\, s_3$

$$ {}^0_6\mathcal{T}_d = \begin{pmatrix} s_6\,\sigma_4 - c_6\,\sigma_2 & s_6\,\sigma_2 + c_6\,\sigma_4 & -c_5\,\sigma_8 - s_5\,\sigma_9 & a_1\,c_1 - d_6\,(c_5\,\sigma_8 + s_5\,\sigma_9) + a_3\,\sigma_{13} - d_4\,\sigma_8 + a_2\,c_1\,c_2 \\ -c_6\,\sigma_1 - s_6\,\sigma_5 & s_6\,\sigma_1 - c_6\,\sigma_5 & s_5\,\sigma_7 - c_5\,\sigma_6 & a_1\,s_1 - d_6\,(c_5\,\sigma_6 - s_5\,\sigma_7) + a_3\,\sigma_{12} - d_4\,\sigma_6 + a_2\,c_2\,s_1 \\ s_4\,s_6\,\sigma_{10} - c_6\,\sigma_3 & s_6\,\sigma_3 + c_6\,s_4\,\sigma_{10} & c_4\,s_5\,\sigma_{10} - c_5\,\sigma_{11} & -a_2\,s_2 - a_3\,\sigma_{10} - d_4\,\sigma_{11} - d_6\,(c_5\,\sigma_{11} - c_4\,s_5\,\sigma_{10}) \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1) $$

where
$\sigma_1 = s_5\,\sigma_6 + c_5\,\sigma_7$
$\sigma_2 = s_5\,\sigma_8 - c_5\,\sigma_9$
$\sigma_3 = s_5\,\sigma_{11} + c_4\,c_5\,\sigma_{10}$
$\sigma_4 = c_4\,s_1 - s_4\,\sigma_{13}$
$\sigma_5 = c_1\,c_4 + s_4\,\sigma_{12}$
$\sigma_6 = c_2\,s_1\,s_3 + c_3\,s_1\,s_2$
$\sigma_7 = c_1\,s_4 - c_4\,\sigma_{12}$
$\sigma_8 = c_1\,c_2\,s_3 + c_1\,c_3\,s_2$
$\sigma_9 = s_1\,s_4 + c_4\,\sigma_{13}$
$\sigma_{10} = c_2\,s_3 + c_3\,s_2$
$\sigma_{11} = c_2\,c_3 - s_2\,s_3$
$\sigma_{12} = c_2\,c_3\,s_1 - s_1\,s_2\,s_3$
$\sigma_{13} = c_1\,c_2\,c_3 - c_1\,s_2\,s_3$

4 Inverse Kinematics

We will compare the matrix ${}^1_4\mathcal{T}$ with ${}^1_4\mathcal{T}_d$ and will find the corresponding $\theta_1, \theta 2$ and θ_3

$$ {}^0_1\mathcal{T}^{-1}\ {}^0_4\mathcal{T} = {}^0_1\mathcal{T}^{-1}\ {}^0_6\mathcal{T}_d\ {}^4_6\mathcal{T}^{-1} $$

$$ {}^1_4\mathcal{T}_d = \begin{pmatrix} c_5\,c_6\,\sigma_6 - c_5\,s_6\,\sigma_4 - s_5\,\sigma_2 & c_6\,\sigma_4 + s_6\,\sigma_6 & c_5\,\sigma_2 + c_6\,s_5\,\sigma_6 - s_5\,s_6\,\sigma_4 & c_1\,p_x + p_y\,s_1 - d_6\,\sigma_2 \\ c_5\,c_6\,\sigma_5 - c_5\,s_6\,\sigma_3 - s_5\,\sigma_1 & c_6\,\sigma_3 + s_6\,\sigma_5 & c_5\,\sigma_1 + c_6\,s_5\,\sigma_5 - s_5\,s_6\,\sigma_3 & c_1\,p_y - p_x\,s_1 - d_6\,\sigma_1 \\ c_5\,c_6\,r_{31} - r_{33}\,s_5 - c_5\,r_{32}\,s_6 & c_6\,r_{32} + r_{31}\,s_6 & c_5\,r_{33} + c_6\,r_{31}\,s_5 - r_{32}\,s_5\,s_6 & p_z - d_6\,r_{33} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

where
$\sigma_1 = c_1\,r_{23} - r_{13}\,s_1$
$\sigma_2 = c_1\,r_{13} + r_{23}\,s_1$
$\sigma_3 = c_1\,r_{22} - r_{12}\,s_1$
$\sigma_4 = c_1\,r_{12} + r_{22}\,s_1$
$\sigma_5 = c_1\,r_{21} - r_{11}\,s_1$
$\sigma_6 = c_1\,r_{11} + r_{21}\,s_1$

$$ {}^1_4\mathcal{T} = \begin{pmatrix} c_4\,c_{23} & -c_{23}\,s_4 & -s_{23} & a_1 + a_2\,c_2 + a_3\,c_{23} - d_4\,s_{23} \\ -s_4 & -c_4 & 0 & 0 \\ -c_4\,s_{23} & s_4\,s_{23} & -c_{23} & -c_{23}\,d_4 - a_2\,s_2 - a_3\,s_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

4.1 Solution for θ_1

By comparing the terms ${}^{1}_{4}\mathcal{T}_{(2,4)}$ with ${}^{1}_{4}\mathcal{T}_{d(2,4)}$, we will get θ_1.

$$a_2\,c_2 + a_3\,c_{23} - d_4\,s_{23} = c_1\,p_x + p_y\,s_1 - d_6\,\sigma_2$$

By rearranging the above equation and taking squares on both sides, we will reduce the equation for θ_1.

$$\theta_1 = \begin{pmatrix} -2\,\mathrm{atan}\left(\frac{p_x - d_6\,r_{13} + \sigma_1}{p_y - d_6\,r_{23}}\right) \\ 2\,\mathrm{atan}\left(\frac{d_6\,r_{13} - p_x + \sigma_1}{p_y - d_6\,r_{23}}\right) \end{pmatrix}$$

where

$$\sigma_1 = \sqrt{{d_6}^2\,{r_{13}}^2 + {d_6}^2\,{r_{23}}^2 - 2\,d_6\,p_x\,r_{13} - 2\,d_6\,p_y\,r_{23} + {p_x}^2 + {p_y}^2}$$

4.2 Solution for θ_3

It can be seen that there are two possible solutions for θ_1

$${}^{0}_{2}\mathcal{T}^{-1}\,{}^{0}_{5}\mathcal{T} = {}^{0}_{2}\mathcal{T}^{-1}\,{}^{0}_{6}\mathcal{T}_d\,{}^{5}_{6}\mathcal{T}^{-1}$$

$${}^{2}_{5}\mathcal{T}_d = \begin{pmatrix} c_6\,\sigma_4 - s_6\,\sigma_3 & c_1\,c_2\,r_{13} - r_{33}\,s_2 + c_2\,r_{23}\,s_1 & -c_6\,\sigma_3 - s_6\,\sigma_4 & c_1\,c_2\,p_x - p_z\,s_2 - a_1\,c_2 + c_2\,p_y\,s_1 + d_6\,r_{33}\,s_2 - c_2\,d_6\,r_{23}\,s_1 - c_1\,c_2\,d_6\,r_{13} \\ s_6\,\sigma_1 - c_6\,\sigma_2 & -c_2\,r_{33} - c_1\,r_{13}\,s_2 - r_{23}\,s_1\,s_2 & c_6\,\sigma_1 + s_6\,\sigma_2 & a_1\,s_2 - c_2\,p_z + c_2\,d_6\,r_{33} - c_1\,p_x\,s_2 - p_y\,s_1\,s_2 + c_1\,d_6\,r_{13}\,s_2 + d_6\,r_{23}\,s_1\,s_2 \\ c_6\,\sigma_7 - s_6\,\sigma_6 & \sigma_5 & -c_6\,\sigma_6 - s_6\,\sigma_7 & c_1\,p_y - p_x\,s_1 - d_6\,\sigma_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where

$$\sigma_1 = c_2\,r_{32} + c_1\,r_{12}\,s_2 + r_{22}\,s_1\,s_2$$

$$\sigma_2 = c_2\,r_{31} + c_1\,r_{11}\,s_2 + r_{21}\,s_1\,s_2$$

$$\sigma_3 = c_1\,c_2\,r_{12} - r_{32}\,s_2 + c_2\,r_{22}\,s_1$$

$$\sigma_4 = c_1\,c_2\,r_{11} - r_{31}\,s_2 + c_2\,r_{21}\,s_1$$

$$\sigma_5 = c_1\,r_{23} - r_{13}\,s_1$$

$$\sigma_6 = c_1\,r_{22} - r_{12}\,s_1$$

$$\sigma_7 = c_1\,r_{21} - r_{11}\,s_1$$

$${}^{2}_{5}\mathcal{T} = \begin{pmatrix} c_3\,c_4\,c_5 - s_3\,s_5 & -c_5\,s_3 - c_3\,c_4\,s_5 & c_3\,s_4 & a_2 + a_3\,c_3 - d_4\,s_3 \\ c_3\,s_5 + c_4\,c_5\,s_3 & c_3\,c_5 - c_4\,s_3\,s_5 & s_3\,s_4 & c_3\,d_4 + a_3\,s_3 \\ -c_5\,s_4 & s_4\,s_5 & c_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From ${}^{2}_{5}\mathcal{T}_{d(1,4)} = {}^{2}_{5}\mathcal{T}_{(1,4)}$

$a_2 \;+\; a_3\;c_3 \;-\; d_4\;s_3 = p_x\;c_1\;c_2 - a_1\;c_2 - p_z\;s_2 - d_6\;(r_13\;c_1\;c_2 - r_33\;s_2 + r_23\;c_2\;s_1) + p_y\;c_2\;s_1$

From ${}^{2}_{5}\mathcal{T}_{d(2,4)} = {}^{2}_{5}\mathcal{T}_{(2,4)}$

$d_4\;c_3 \;+\; a_3\;s_3 = d_6\;(r_33\;c_2 + r_13\;c_1\;s_2 + r_23\;s_1\;s_2) - p_z\;c_2 + a_1\;s_2 - p_x\;c_1\;s_2 - p_y\;s_1\;s_2$

Now rearranging the equation

$a_2 \ + \ a_3 \ c_3 \ - \ d_4 \ s_3 = s_2 \ (d_6 \ r_33 - p_z \ ) - c_2 \ (a_1 + c_1 \ (-p_x + d_6 \ r_13 \ ) - p_y \ s_1 + d_6 \ r_23 \ s_1$
$d_4 c_3 \ + \ a_3 s_3 = c_2 \, (d_6 r_{33} - p_z) + s_2 (a_1 + c_1 \, (-p_x + d_6 r_{13}) - p_y s_1 + d_6 r_{23} s_1)$

Introducing the following substitutions

$E = d_6 \ r_33 - p_z$
$F = a_1 + c_1 \ (-p_x + d_6 \ r_13 \ ) - p_y \ s_1 + d_6 \ r_23 \ s_1$

Substituting and squaring both sides the following standard system of equations is obtained:

$(a_2 \ + \ a_3 \ c_3 \ - \ d_4 \ s_3)^2 = (s_2 E - c_2 F)^2$
$(d_4 \ c_3 \ + \ a_3 \ s_3)^2 \ = \ (c_2 \ E + s_2 \ F)^2$

Combining the two equations after squaring

$a_3 \ c_3 + \frac{-E^2 - F^2 + a_2^2 + a_3^2 + d_4^2}{2a_2} = d_4 s_3$
$G = \frac{-E^2 - F^2 + a_2^2 + a_3^2 + d_4^2}{2a_2}$

Thus, we get

$a_3 c_3 + G = d_4 s_3$

From this solving for θ_3

$$\theta_3 = \begin{pmatrix} \arctan\left(\frac{G d_4 - \sqrt{{a_3}^2 (-G^2 + {a_3}^2 + {d_4}^2)}}{({a_3}^2 + {d_4}^2)\sqrt{1 - \frac{\left(-G d_4 + \sqrt{{a_3}^2(-G^2 + {a_3}^2 + {d_4}^2)}\right)^2}{{a_3}^2 + {d_4}^2)^2}}}\right) \\ \arctan\left(\frac{G d_4 + \sqrt{{a_3}^2 (-G^2 + {a_3}^2 + {d_4}^2)}}{({a_3}^2 + {d_4}^2)\sqrt{1 - \frac{\left(G d_4 + \sqrt{{a_3}^2(-G^2 + {a_3}^2 + {d_4}^2)}\right)^2}{({a_3}^2 + {d_4}^2)^2}}}\right) \end{pmatrix}$$

4.3 Solution for θ_2

Now, by comparing the terms ${}^1_4\mathcal{T}_{(3,4)}$ with ${}^1_4\mathcal{T}_{d(3,4)}$, we will get θ_2.

$$-c_{23}\, d_4 - a_2\, s_2 - a_3\, s_{23} = p_z - d_6\, r_{33}$$
$$-c_{23}\, d_4 - a_3\, s_{23} = p_z - d_6\, r_{33} + a_2\, s_2$$
$$-(c_2\, c_3 - s_2\, s_3)\, d_4 - a_3\, (s_2\, c_3 + c_3\, s_2) = p_z - d_6\, r_{33} + a_2\, s_2$$

Rearranging both sides and taking squares, we will get θ_2 .

$$\theta_2 = \begin{pmatrix} 2\,\mathrm{atan}\left(\frac{a_2 + \sigma_1 + a_3\, c_3 - d_4\, s_3}{c_3\, d_4 - p_z + a_3\, s_3 + d_6\, r_{33}}\right) \\ 2\,\mathrm{atan}\left(\frac{a_2 - \sigma_1 + a_3\, c_3 - d_4\, s_3}{c_3\, d_4 - p_z + a_3\, s_3 + d_6\, r_{33}}\right) \end{pmatrix}$$

where

$$\sigma_1 = \sqrt{{a_2}^2 + 2\, a_2\, a_3\, c_3 - 2\, a_2\, d_4\, s_3 + {a_3}^2\, {c_3}^2 + {a_3}^2\, {s_3}^2 + {c_3}^2\, {d_4}^2 + {d_4}^2\, {s_3}^2 - {d_6}^2\, {r_{33}}^2 + 2\, d_6\, p_z\, r_{33} - {p_z}^2}$$

There will be multiple possible solutions for θ_2 depending on θ_3

It should be noted that the θ_2 we get here should be transformed into θ_2'.

$$\theta_2 := \theta_2 - \frac{\pi}{2}$$

From this point θ_2 should be considered as updated.

4.4 Solution for θ_5

Now, We will compare the matrix ${}^3_6\mathcal{T}$ with ${}^3_6\mathcal{T}_d$ and will find the corresponding $\theta_4, \theta 5$ and θ_6

$${}^0_3\mathcal{T}^{-1}\ {}^0_6\mathcal{T} = {}^0_3\mathcal{T}^{-1}\ {}^0_6\mathcal{T}_d$$

$$
{}^{3}_{6}\mathcal{T}_d = \begin{pmatrix} r_{11}\,\sigma_1 + r_{21}\,\sigma_3 - r_{31}\,\sigma_6 & r_{12}\,\sigma_1 + r_{22}\,\sigma_3 - r_{32}\,\sigma_6 & r_{13}\,\sigma_1 + r_{23}\,\sigma_3 - r_{33}\,\sigma_6 & p_x\,\sigma_1 - a_1\,c_{23} - a_2\,c_3 + p_y\,\sigma_3 - p_z\,\sigma_6 \\ -r_{11}\,\sigma_2 - r_{21}\,\sigma_4 - r_{31}\,\sigma_5 & -r_{12}\,\sigma_2 - r_{22}\,\sigma_4 - r_{32}\,\sigma_5 & -r_{13}\,\sigma_2 - r_{23}\,\sigma_4 - r_{33}\,\sigma_5 & a_2\,s_3 + a_1\,s_{23} - p_x\,\sigma_2 - p_y\,\sigma_4 - p_z\,\sigma_5 \\ c_1\,r_{21} - r_{11}\,s_1 & c_1\,r_{22} - r_{12}\,s_1 & c_1\,r_{23} - r_{13}\,s_1 & c_1\,p_y - p_x\,s_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}
$$

where

$\sigma_1 = c_1\,c_2\,c_3 - c_1\,s_2\,s_3$
$\sigma_2 = c_1\,c_2\,s_3 + c_1\,c_3\,s_2$
$\sigma_3 = c_2\,c_3\,s_1 - s_1\,s_2\,s_3$
$\sigma_4 = c_2\,s_1\,s_3 + c_3\,s_1\,s_2$
$\sigma_5 = c_2\,c_3 - s_2\,s_3$
$\sigma_6 = c_2\,s_3 + c_3\,s_2$

$$
{}^{3}_{6}\mathcal{T} = \begin{pmatrix} c_4\,c_5\,c_6 - s_4\,s_6 & -c_6\,s_4 - c_4\,c_5\,s_6 & -c_4\,s_5 & a_3 - c_4\,d_6\,s_5 \\ c_6\,s_5 & -s_5\,s_6 & c_5 & d_4 + c_5\,d_6 \\ -c_4\,s_6 - c_5\,c_6\,s_4 & c_5\,s_4\,s_6 - c_4\,c_6 & s_4\,s_5 & d_6\,s_4\,s_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

By comparing the element ${}^{3}_{6}\mathcal{T}_{(2,3)}$ with ${}^{3}_{6}\mathcal{T}_{d(2,3)}$, we will get θ_5.

$$
c_5 = -r_{13}\,\sigma_2 - r_{23}\,\sigma_4 - r_{33}\,\sigma_5
$$

where

$\sigma_2 = c_1\,c_2\,s_3 + c_1\,c_3\,s_2$
$\sigma_4 = c_2\,s_1\,s_3 + c_3\,s_1\,s_2$
$\sigma_5 = c_2\,c_3 - s_2\,s_3$

$$
\theta_5 = \begin{pmatrix} \operatorname{acos}\left(-r_{13}\,(c_1\,c_2\,s_3 + c_1\,c_3\,s_2) - r_{23}\,(c_2\,s_1\,s_3 + c_3\,s_1\,s_2) - r_{33}\,(c_2\,c_3 - s_2\,s_3)\right) \\ -\operatorname{acos}\left(-r_{13}\,(c_1\,c_2\,s_3 + c_1\,c_3\,s_2) - r_{23}\,(c_2\,s_1\,s_3 + c_3\,s_1\,s_2) - r_{33}\,(c_2\,c_3 - s_2\,s_3)\right) \end{pmatrix}
$$

We can convert cos inverse in terms of tan inverse

$$
\theta_5 = \begin{pmatrix} \arctan\frac{\sqrt{1-(-r_{13}\,(c_1\,c_2\,s_3+c_1\,c_3\,s_2)-r_{23}\,(c_2\,s_1\,s_3+c_3\,s_1\,s_2)-r_{33}\,(c_2\,c_3-s_2\,s_3))^2}}{(-r_{13}\,(c_1\,c_2\,s_3+c_1\,c_3\,s_2)-r_{23}\,(c_2\,s_1\,s_3+c_3\,s_1\,s_2)-r_{33}\,(c_2\,c_3-s_2\,s_3))} \\ -\arctan\frac{\sqrt{1-(-r_{13}\,(c_1\,c_2\,s_3+c_1\,c_3\,s_2)-r_{23}\,(c_2\,s_1\,s_3+c_3\,s_1\,s_2)-r_{33}\,(c_2\,c_3-s_2\,s_3))^2}}{(-r_{13}\,(c_1\,c_2\,s_3+c_1\,c_3\,s_2)-r_{23}\,(c_2\,s_1\,s_3+c_3\,s_1\,s_2)-r_{33}\,(c_2\,c_3-s_2\,s_3))} \end{pmatrix}
$$

4.5 Solution for θ_4

Now, by comparing the terms ${}^{3}_{6}\mathcal{T}_{(3,3)}$ with ${}^{3}_{6}\mathcal{T}_{d(3,3)}$, we will get θ_4.

$$
s_4\,s_5 = c_1\,r_{23} - r_{13}\,s_1
$$
$$
s_4 = \frac{c_1\,r_{23} - r_{13}\,s_1}{s_5}
$$

$$
\theta_4 = \begin{pmatrix} \operatorname{asin}\left(\frac{c_1\,r_{23}-r_{13}\,s_1}{s_5}\right) \\ \pi - \operatorname{asin}\left(\frac{c_1\,r_{23}-r_{13}\,s_1}{s_5}\right) \end{pmatrix}
$$

We can write sin inverse in terms of tan inverse as under

$$
\theta_4 = \begin{pmatrix} \arctan\frac{\left(\frac{c_1\,r_{23}-r_{13}\,s_1}{s_5}\right)}{\sqrt{1-\left(\frac{c_1\,r_{23}-r_{13}\,s_1}{s_5}\right)^2}} \\ \pi - \arctan\frac{\left(\frac{c_1\,r_{23}-r_{13}\,s_1}{s_5}\right)}{\sqrt{1-\left(\frac{c_1\,r_{23}-r_{13}\,s_1}{s_5}\right)^2}} \end{pmatrix}
$$

4.6 Solution for θ_6

Now, by comparing the terms ${}^{1}_{4}\mathcal{T}_{(3,1)}$ with ${}^{1}_{4}\mathcal{T}_{d(3,1)}$, we will get θ_6.

$$-c_4\,s_6 - c_5\,c_6\,s_4 = c_1\,r_{21} - r_{11}\,s_1$$

Squaring both sides we get

$$(-c_4\,s_6 - c_5\,c_6\,s_4)^2 = (c_1\,r_{21} - r_{11}\,s_1)^2$$

By solving the equations analytically, we get

$$\theta_6 = \begin{pmatrix} 2\,\mathrm{atan}\left(\frac{c_4-\sigma_1}{c_5\,s_4-c_1\,r_{21}+r_{11}\,s_1}\right) \\ 2\,\mathrm{atan}\left(\frac{c_4+\sigma_1}{c_5\,s_4-c_1\,r_{21}+r_{11}\,s_1}\right) \end{pmatrix}$$

where

$\sigma_1 = \sqrt{-{c_1}^2\,{r_{21}}^2 + 2\,c_1\,r_{11}\,r_{21}\,s_1 + {c_4}^2 + {c_5}^2\,{s_4}^2 - {r_{11}}^2\,{s_1}^2}$

5 Conclusion

It is to be concluded that if we follow Denavit-Hartenberg method, we can end up finding the kinematic model, that is Direct and Inverse Kinematics, of ABB IRB-7600 Robot. There are two ways regarding the placement of axes and determining the parameters but the report solely follows the method described in the lectures which is also stated in the report.