Types of System of Linear Equations

Consistent and Inconsistent System

$2x - y = 1$
$x + y = 2$

$x + 2 = 5$
$x = 3$

$x + y = 10$
$x = 5 \quad y = 5$
$x = 2 \quad y = 8$

A system of linear equations has

1. no solution, or → Inconsistent
2. exactly one solution, or
3. infinitely many solutions. → consistent system

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Linear system

Consistent — Unique solution, infinite many

inconsistent — no solution

Examples,

1-
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Unique solution

2-
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

Infinite many solution

Consistent systems

3-
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

has no solution

inconsistent system

Linear System in Matrix Notation/ Augmented Matrix

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

3-eqn
3-variable

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 0x_1 + 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \Rightarrow \begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\underset{\text{Coefficent maxtix}}{\underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}}_{A}} \underset{\text{unkwon vector}}{\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}}} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}}_{\underline{b}} \rightarrow \text{vector of constant term}$$

In general, a linear system in n variable $x_1, x_2, \dots, x_n$ having m equations can be written as

$$\begin{aligned} \boxed{a_{11}} x_1 + \boxed{a_{12}} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + \boxed{a_{22}} x_2 + \cdots + a_{2n} x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ \boxed{a_{m1}} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m \end{aligned}$$

n-variables

m-equation

$$\begin{array}{c|cccc} & x_1 & x_2 & & x_n \\ R_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ R_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ R_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ & C_1 & C_2 & & C_n \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A (matrix), $\underline{x}$, $\underline{b}$

$$\underset{m\times n}{A}\ \underset{n\times 1}{\underline{x}} = \underset{m\times 1}{\underline{b}}$$

A row x Column

LS2

$x_1 - x_2 - 5 = 0$

$x_2 - x_3 = 10$

$$\boxed{\begin{aligned} x_1 - x_2 &= 5 \\ x_2 - x_3 &= 10 \end{aligned}} \rightarrow \text{3-unkn}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 10 \end{bmatrix}}$$

Augmented Matrix

$$\boxed{A_d} = \begin{array}{cccc} C_1 & C_2 & C_3 & b \end{array}$$

$$\underset{2\times 4}{\left[\begin{array}{ccc:c} 1 & -1 & 0 & 5 \\ 0 & 1 & -1 & 10 \end{array}\right]}$$

General form

$A \underline{x} = \underline{b}$

$$\boxed{A_d = [A \,\vdots\, \underline{b}]}$$

$x - y = 2$

$x + \frac{1}{2} y = 4$

$$A_d = \left[\begin{array}{cc:c} 1 & -1 & 2 \\ 1 & \frac{1}{2} & 4 \end{array}\right]$$

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[2]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

$2x+y=2$
$x-y=5$

$x-y=5$
$x+y=2$

$2x+2y=4$
$x-y=5$

Example

3-unknowns
3-equations

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{aligned}
\quad \text{or} \quad
\begin{array}{c}
\begin{array}{cccc} c_1 & c_2 & c_3 & b \end{array} \\
\left[\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{array}\right]
\end{array}
\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} = A_d
$$

3×4

$$
\left[\begin{array}{ccc|c}
0 & 2 & -8 & 8 \\
1 & -2 & 1 & 0 \\
-4 & 5 & 9 & -9
\end{array}\right]
\qquad R_1 \xleftrightarrow{\text{Interchange}} R_2 \qquad
\left[\begin{array}{ccc|c}
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9 \\
1 & -2 & 1 & 0
\end{array}\right]
$$

$$
\left[\begin{array}{ccc|c}
0/2 & 2/2 & -8/2 & 8/2 \\
-4/3 & 5/3 & 9/3 & -9/3 \\
2\times 1 & -2\times 2 & 1\times 2 & 0\times 2
\end{array}\right]
\qquad
\begin{array}{l} \frac{1}{2}R_1 \\ 2R_3 \\ \frac{1}{3}R_2 \end{array}
$$

$$
\left[\begin{array}{ccc|c}
0 & 2 & -8 & 8 \\
1+2(0) & -2+2(2) & 1+(-16) & 0+16 \\
-4 & 5 & 9 & -9
\end{array}\right]
\qquad \boxed{R_2} + \boxed{2R_1}
$$

$R_2 - 2R_1$

$R_2 + (-2R_1)$

$$
\begin{aligned}
x + \cancel{2y} &= 5 \\
6x - \cancel{2y} &= 20 \\
\hline
7x &= 25 \\
x &= 25
\end{aligned}
$$

Remarks

- Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system
- The row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange.

Row Reduction and Echelon Form (Online Lecture 2)

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Examples

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Rank of a Matrix

No of nonzero rows in echelon form of the matrix is called rank of the matrix and it is denoted as $\boxed{\text{Rank(A)}}$.

1- Null matrix have all the rows as zero rows so Rank of null matrix is zero
2- Identity matrix of order 3, I_3, (is in echelon form and) have 3 nonzero rows to Rank(I_3)= 3.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ Echelon form

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rank $(I_3) = 3$

Rank$(A) = 0$

Existence and Uniqueness Theorem

LS
Solution
Solution does not exist

Existence

$x_1, x_2, x_3, \cdots, x_n$

A linear system in n variables having m equations in matrix form can be written as

coefficient matrix
unknown vector
vector of constant

$$A_{\boxed{m\times n}}\, x_{\boxed{n\times 1}} = b_{\boxed{m\times 1}}$$

$A_d = [A \,\vdots\, b]$

The linear system is

1- Consistent if and only if Rank(A)=Rank(A_d)
2- Inconsistent if and only if Rank(A)≠Rank(A_d)

$\text{Rank}(A) \leq \text{Rank}(A_d)$

where A_d represent the augmented matrix of the linear system.

Uniqueness

Rank(A) = Rank(A_d) ≠ no of variables
exist infinite many solutions

Further, the solution is unique if and only if

Rank(A)=Rank(A_d)= No of variables of the system (n in this case)

Linear System

Solution exist
- Consistent system

Rank(A) = Rank(A_d)

unique solution

Rank(A) = Rank(A_d) =
no of variable in the system
or no of columns of A

infinite many Solution

Rank(A) = Rank(A_d) ≠ no of variables

3 ≠ 4

Solution does not exist
- Inconsistent system

Rank(A) ≠ Rank(A_d)

Example 1 Determine the existence and uniqueness of the solution of the system whose augmented matrix is reduced to the form

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$2x_1 + 3x_2 - x_3 = 1$
$x_2 + x_3 = 0$

$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A \quad x = b$

Solution The augmented matrix of the system is

$$\boxed{A_d} = \begin{array}{c} \begin{array}{cccccc} \boxed{x_1} & x_2 & \boxed{x_3} & x_4 & \boxed{x_5} & \boxed{b} \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \\ \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & \uparrow \\ P.C & & P.C & & P.C & \end{array} \end{array}$$

Pivot columns are those columns that contain pivot elem

$\rightarrow$ 5 - Unknowns, 3 - Equation

$A_d = [A \mid b]$

$$A_d = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \end{array} \end{array}$$

$\text{Rank}(A_d) = 3$ = no of non-zero rows in echelon form of A_d

$\text{Rank}(A) = 3$ = no of nonzero rows in echelon form of A

$\boxed{\text{Rank}(A) = 3 = \text{Rank}(A_d)}$ So system is consistent

Basic variables are those that are corresponding to pivot columns
x_1, x_3, x_5 are basic

x_2, x_4 are free variables

no of variables in LS $= 5 \neq \text{Rank}(A) = \text{Rank}(A_d)$

So Solution is not unique. There exist infinite many solutions of the system

- Linear System has unique solution iff all the variables are basis variables.

Example 2 Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Step1:-

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ 15 \end{bmatrix}$$

$$A \; x = b$$

$$A_d = \left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

5-unknown
3- Equation

$$A_d \underset{\approx}{R} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{8}{3} \end{array}\right] \quad R_3 + \frac{2}{3}R_2$$

$$\text{Rank}(A) = 3 = \text{Rank}(A_d)$$

infinite many solutions

$x_1, x_2, x_5 \to$ Basic variables

$\boxed{x_3, x_4 \to \text{Free variables}}$

Solution of Linear System

Solution from Echelon from of the Matrix

EXAMPLE 4 Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Gauss Elimination and Gauss Jorden Methods for Linear System

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

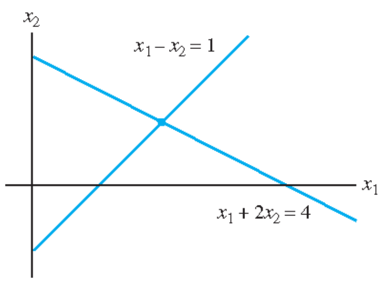
1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. $$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$$

2. $$\begin{aligned} 3x_1 + 6x_2 &= -3 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 2x_2 = 4$ and on the line $x_1 - x_2 = 1$. See the figure.



4. Find the point of intersection of the lines $x_1 + 2x_2 = -13$ and $3x_1 - 2x_2 = 1$

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. $$\begin{bmatrix} 1 & -4 & -3 & 0 & 7 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

6. $$\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 1 & 2 \end{bmatrix}$$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. $$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

8. $$\begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

9. $$\begin{bmatrix} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

10. $$\begin{bmatrix} 1 & 3 & 0 & -2 & -7 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Solve the systems in Exercises 11–14.

11. $$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

12. $$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$$

13. $$\begin{aligned} x_1 \qquad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

14. $$\begin{aligned} 2x_1 \qquad - 6x_3 &= -8 \\ x_2 + 2x_3 &= 3 \\ 3x_1 + 6x_2 - 2x_3 &= -4 \end{aligned}$$

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

15. $$\begin{aligned} x_1 - 6x_2 \qquad\qquad &= 5 \\ x_2 - 4x_3 + x_4 &= 0 \\ -x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\ -x_2 + 5x_3 + 4x_4 &= 0 \end{aligned}$$

16. $$\begin{aligned} 2x_1 \qquad\qquad - 4x_4 &= -10 \\ 3x_2 + 3x_3 \qquad &= 0 \\ x_3 + 4x_4 &= -1 \\ -3x_1 + 2x_2 + 3x_3 + x_4 &= 5 \end{aligned}$$

17. Do the three lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection? Explain.

18. Do the three planes $2x_1 + 4x_2 + 4x_3 = 4$, $x_2 - 2x_3 = -2$, and $2x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$ **20.** $\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}$ **22.** $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$

In Exercises 23 and 24, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

23. a. Every elementary row operation is reversible.

b. A 5×6 matrix has six rows.

c. The solution set of a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

d. Two fundamental questions about a linear system involve existence and uniqueness.

24. a. Two matrices are row equivalent if they have the same number of rows.

b. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

c. Two equivalent linear systems can have different solution sets.

d. A consistent system of linear equations has one or more solutions.

25. Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:
$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

26. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} 2x_1 + 4x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

27. Suppose a, b, c, and d are constants such that a is not zero and the system below is consistent for all possible values of f and g. What can you say about the numbers a, b, c, and d? Justify your answer.

$$\begin{aligned} ax_1 + bx_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

28. Construct three different augmented matrices for linear systems whose solution set is $x_1 = 3$, $x_2 = -2$, $x_3 = -1$.

In Exercises 29–32, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

29. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & -5 \\ 0 & -2 & 5 \end{bmatrix}$

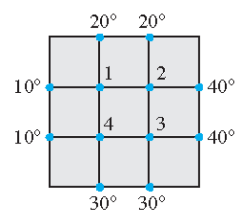
30. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{bmatrix}$

31. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 4 & -12 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 15 \end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.[3] For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



33. Write a system of four equations whose solution gives estimates for the temperatures $T_1, \ldots, T_4$.

34. Solve the system of equations from Exercise 33. [*Hint*: To speed up the calculations, interchange rows 1 and 4 before starting "replace" operations.]

[3] See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.