Types of System of Linear Equations

Consistent and Inconsistent System

$2x - y = 1$
$x + y = 2$

$x + 2 = 5$
$x = 3$

$x + y = 10$
$x = 5 \quad y = 5$
$x = 2 \quad y = 8$

A system of linear equations has

1. no solution, or → Inconsistent
2. exactly one solution, or
3. infinitely many solutions. → Consistent system

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Linear system

Consistent — Unique solution, infinite many

inconsistent — no solution

Examples,

1-
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Unique solution

2-
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

Infinite many solution

3-
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

has no solution

Consistent systems

inconsistent system

Linear System in Matrix Notation/ Augmented Matrix

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

3-eqn

3-variable

$$\left\{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 0x_1 + 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}\right. \Rightarrow \underbrace{\begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \end{array}\!\!\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}}_{\text{coefficent matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{unkwon vector}} = \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$$

$A \quad \underline{x} = \underline{b}$

vector of constant term

In general, a linear system in n variable $x_1, x_2, \dots, x_n$ having m equations can be written as

$$\begin{aligned} \boxed{a_{11}}x_1 + \boxed{a_{12}}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \boxed{a_{22}}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad & \vdots \\ \boxed{a_{m1}}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

n-variables

m- equation

$$\begin{array}{c|cccc} & x_1 & x_2 & & x_n \\ \hline R_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ R_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ R_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline & C_1 & C_2 & & C_n \end{array} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underset{m\times n}{A}\ \underset{n\times 1}{\underline{x}} = \underset{m\times 1}{\underline{b}}$$

$A_{\text{row} \times \text{Column}}$

LS2

$x_1 - x_2 - 5 = 0$

$x_2 - x_3 = 10$

$$\boxed{\begin{aligned} x_1 - x_2 &= 5 \\ x_2 - x_3 &= 10 \end{aligned}} \to \text{3-unkn}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 10 \end{bmatrix}}$$

Augmented Matrix

$$\boxed{A_d} = \begin{array}{ccc:c} C_1 & C_2 & C_3 & b \\ \end{array} \begin{bmatrix} 1 & -1 & 0 & \vdots & 5 \\ 0 & 1 & -1 & \vdots & 10 \end{bmatrix}$$

2×4

General form

$A\underline{x} = \underline{b}$

$$\boxed{A_d = [A \,\vdots\, b]}$$

$$\left.\begin{aligned} x - y &= 2 \\ x + \tfrac{1}{2}y &= 4 \end{aligned}\right]$$

$$A_d = \begin{bmatrix} 1 & -1 & \vdots & 2 \\ 1 & \frac{1}{2} & \vdots & 4 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[2]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

$2x+2y=2$
$x-y=5$

$x-y=5$
$x+y=2$

$2x+2y=4$
$x-y=5$

Example

3-unknowns
3-equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \quad \text{or} \quad \begin{array}{ccccc} c_1 & c_2 & c_3 & b & \\ & & & & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} = A_d$$

3×4

$$\left[\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad R_1 \leftrightarrow R_2 \text{ (interchange)} \quad \left[\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \\ 1 & -2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0/2 & 2/2 & -8/2 & 8/2 \\ -4/3 & 5/3 & 9/3 & -9/3 \\ 2\times 1 & -2\times 2 & 1\times 2 & 0\times 2 \end{array}\right] \quad \begin{array}{l} \frac{1}{2}R_1 \\ 2R_3 \\ \frac{1}{3}R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1+2(0) & -2+2(2) & 1+(-16) & 0+16 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad \boxed{R_2} + \boxed{2R_1}$$

$R_2 - 2R_1$

$R_2 + (-2R_1)$

$$\begin{aligned} x + 2\cancel{y} &= 5 \\ 6x - 2\cancel{y} &= 20 \\ \hline 7x &= 25 \\ x &= 25 \end{aligned}$$

Remarks

- Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system
- The row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange.

Row Reduction and Echelon Form

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.