Vectors and Vector Equations

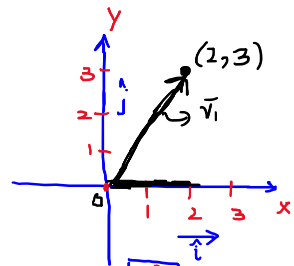
x-component, y-component

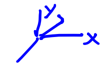
$\underline{V}_1 = 2\hat{i} + 3\hat{j} \equiv \underline{V}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\begin{matrix} \rightarrow \hat{i} \\ \rightarrow \hat{j} \end{matrix}$

$\underline{V}_2 = \hat{i} + 2\hat{j}$

$\underline{V}_3 = 2\hat{i} + \hat{j}$

$\underline{V}_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in \mathbb{R}^{\boxed{2}}$





$\mathbb{R} \rightarrow$ Set of real number

$\boxed{\mathbb{R}^2}$-space $\rightarrow$ 2D-space

$\mathbb{R}^3$-space $\rightarrow$ 3D-space

$\mathbb{R}^{\boxed{n}} \rightarrow$ space — n dimensional

$\underline{V}_4 = \hat{i} + \hat{j} + 4\hat{k}$

$\underline{V}_4 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^{\boxed{3}}$

belong

$\underline{V}_1 + \underline{V}_2 = \hat{i} + 2\hat{j} + 2\hat{i} + \hat{j}$

$= (1+2)\hat{i} + (2+1)\hat{j}$

$= 3\hat{i} + 3\hat{j}$

$\underline{V}_1 + \underline{V}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\underline{V}_1 + \underline{V}_4$

$\underline{V}_1 = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}_{3\times 1} \rightarrow$ not pos

$\underline{V}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$\underline{V}_1 = 2\hat{i} + 3\hat{j} + 0\hat{k} \in \mathbb{R}^3$

$V_1 = 2\hat{i} + 3\hat{j} \in \mathbb{R}^2$

Parallelogram Rule for Addition

If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then **u** + **v** corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See Fig. 3.

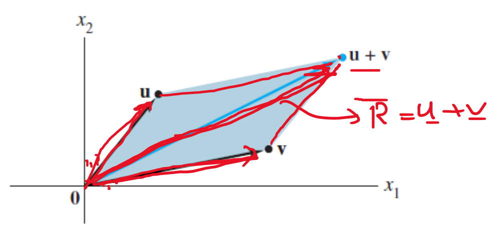


FIGURE 3 The parallelogram rule.

EXAMPLE 2 The vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are displayed in Fig. 4. ■

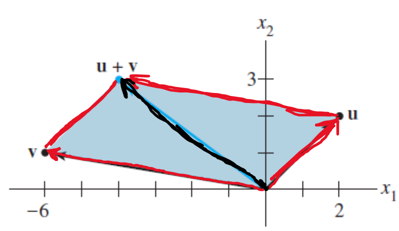
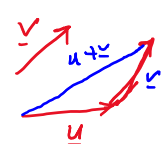
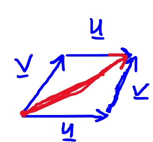


FIGURE 4



The next example illustrates the fact that the set of all scalar multiples of one fixed nonzero vector is a line through the origin, (0, 0).



Linear System in Vector Equation

In general, a linear system in n variable $x_1, x_2, \dots, x_n$ having m equations can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad (1)$$

Equation

n-variables

m-equation

A row x Column

matrix form

$$\begin{array}{cccc} x_1 & x_2 & & x_n \\ \end{array}$$
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 - x_2 &= 5 \\ x_1 + x_2 &= 10 \end{aligned}}$$ 3-unkn

$i + j = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{x_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \boxed{x_2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$ vector form

Vector Equation

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$ const

has the same solution set as the linear system whose augmented matrix is

$$A_d = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \vdots & \mathbf{b} \end{bmatrix} \quad (5)$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 1 \\ x_1 + 4x_2 - x_3 &= 2 \\ x_2 + x_3 &= -1 \end{aligned} \Rightarrow x_1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$x_1\underline{a}_1 + x_2\underline{a}_2 + x_3\underline{a}_3 = \underline{b}$$

$$A_d = \left[\begin{array}{ccc|c} 2 & -3 & 4 & 1 \\ 1 & 4 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \Leftarrow \begin{bmatrix} a_1 & a_2 & a_3 & b \end{bmatrix} = A_d$$

Linearly Independent and dependent Vectors

Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = \boxed{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p}$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$. Property (ii)

$$\underline{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \qquad \boxed{3\underline{u} + 4\underline{v}} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-8 \\ 3+12 \end{bmatrix} = \begin{bmatrix} -2 \\ 15 \end{bmatrix}$$

$\Rightarrow \boxed{\begin{bmatrix} -2 \\ 15 \end{bmatrix}} = 3u + 4v$ is linear combination of vectors $\boxed{\underline{u} \text{ and } \underline{v}}$ with weights 3 & 4

EXAMPLE 5 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{1}$$

If vector equation (1) has a solution, find it.

SOLUTION Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}}_{\mathbf{a}_1} + x_2 \underbrace{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}}_{\mathbf{a}_2} = \underbrace{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}_{\mathbf{b}}$$

$$\underline{V}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \underline{V}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \underline{V}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{2}\underline{V}_1 + \boxed{3}\underline{V}_2 (-4)\underline{V}_3 = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - 4\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix} = \begin{bmatrix} 2+0+4 \\ 4+3-4 \\ 6-3-8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix}$$

$\underline{V}_1 + \underline{V}_2 + \underline{V}_3 =$

$\underline{V}_1 - \underline{V}_2 + 2\underline{V}_3 =$

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact.

The equation $\boxed{A\mathbf{x} = \mathbf{b}}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

$A\underline{x} = \boxed{b}$ $\Rightarrow$ $x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n = b$

Consistent = Solution exist

Solution does not exist = inconsistent

If $\underline{b}$ is linear combination of columns of A

$$A_d = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

Rank(A) = Rank(A)

$\Rightarrow$ $\underline{b}$ is linear combination a_1, a_2, a_3

Linearly Independent and dependent Vectors

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (2)$$