Determinant

$A_{m\times n}$ $m \neq n$ $A_{2\times 3}$ $|A| \to$ define

Determinant of 2 × 2

The determinant of 2x2 Square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ is a real number define as (If a_{ij} are real)

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

Note:-

1- The determinant of a matrix can be zero, +ve or −ve.

2- If $|A| = 0$, then A is called Singular matrix. → A^{-1} exist doesn't exist

3- If $|A| \neq 0$, then A is called non-Singular matrix → A^{-1} exist

$$\boxed{A^{-1} = \frac{\operatorname{adj} A}{|A| \neq 0}}$$

Determinant of $n \times n$ matrix

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}}$$

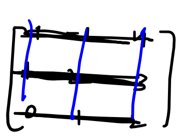
The cofactor expansion down the jth column is

$$\boxed{\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}}$$

$C_{11} = (-1)^{1+1} \left| \; \right|$

$C_{15} =$

R_1



$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \boxed{a_{21}} & \boxed{a_{22}} & \cdots & \boxed{a_{2n}} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \xrightarrow{R_2} |A| = a_{21}C_{21} + a_{22}C_{22} + \cdots + a_{2n}C_{2n}$$

$$|A| = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}$$

EXAMPLE 2 Use a cofactor expansion across the third row to compute det A, where

$$A = \begin{array}{c}\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} R_3 \\ \;\; a_{31} \quad a_{32} \quad a_{33} \end{array}$$

$C_{11} = (-1)^{1+1} \left| \quad \right|$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$

$a_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$

$$\boxed{|A| = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}}$$

$$|A| = 0 \cdot (-1)^{3+1} \begin{vmatrix} 5 & 0 \\ 4 & 1 \end{vmatrix} - 2 (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \, (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$|A| = 2(4-10) = 2(-6) = -12$$

Note:- 1- $|A|$ is unique

2-

$$A = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Determinant of the 3 × 3 matrix (2nd method)

The expansion of a 3 × 3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$- \quad - \quad -$

$+ \quad + \quad +$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21} + a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{31}a_{21}a_{12}$

Example:-

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{matrix} 1 & 5 \\ 2 & 4 \\ 0 & -2 \end{matrix}$$

$|A| = 0 + 0 + 0 - 0 + 2 - 0$

$|A| = -2$

Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to* 4 × 4 *or larger matrices.*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 3 & 4 \\ 2 & -1 & 1 & 1 \\ -1 & 3 & 4 & 1 \end{bmatrix}_{4\times4}$$

$A_{10\times10}$ $\quad A = [1 \;]$

$$|A| = a_{11}\underset{3\times3}{\boxed{C_{11}}} + a_{12}\underset{3\times3}{\boxed{C_{12}}} + a_{13}\underset{3\times3}{\boxed{C_{13}}} + a_{14}\underset{3\times3}{\boxed{C_{14}}}$$

$|A|_{10\times10} \rightarrow$ 10 9×9 determine

10 → cofactor 9x9 →

$A_{10\times10}$ for $|A|$ we have have to calculate 10 9×9 determinant

9×9 we have 9 8×8 deter

8×8 // / 8 7×7 dc

7×7 7 6×6

⋮

Properties of the determinant

1- If A has a row or a column with all elements equal to zero, then $|A|=0$

$|A_{100\times100}|=0$

$A=\begin{bmatrix}1 & 2 & 3\\ 2 & - & -\\ 3 & - & -\end{bmatrix}$ $|A|\neq0$

2- If A has two rows or two columns identical, then $|A|=0$

$A=\begin{bmatrix}1 & 1 & 5\\ 2 & 2 & 10\\ 3 & 3 & 11\end{bmatrix}\Rightarrow|A|=0$

3- $|A|=|A^T|$

↓ 2nd row ↓ 2nd column

$A^T_{odd}=-A_{odd}$ $|A|=0$

$A^T_{5\times5}=-A_{5\times5}\Rightarrow|A|=0$

$A=\begin{bmatrix}0 & -1 & 2\\ 1 & 0 & 3\\ -2 & -3 & 0\end{bmatrix}_{3\times3}\Rightarrow\boxed{|A|=0}$

$A^T=-A$

4- If we interchange two rows or two columns then then the determinant of resulting is $\boxed{-|A|}$

$|A|=\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}=a\Rightarrow -a=\begin{vmatrix}1 & 2 & 3\\ 7 & 8 & 9\\ 4 & 5 & 6\end{vmatrix}$

$$A = \begin{array}{cc} C_{11} & C_{12} \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$|A^T| = 4 - 6 = -2$$

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow |B| = 6 - 4 = 2$$

5- $A = \begin{bmatrix} 20 & 20 & 20 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow |A| = 20 \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

6- $|AB| = |A| \cdot |B|$

$|I_n| = 1$

$$A A^{-1} = I$$
$$|A A^{-1}| = |I| = 1$$
$$|A||A^{-1}| = 1 \qquad \text{If } |A| \neq 0$$
$$\Rightarrow \boxed{|A^{-1}| = \frac{1}{|A|}}$$

If $|A| = 2$; $|A^{-1}| = \frac{1}{2}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $|A| = 4$

$A^{-1} =$ ___ $|A^{-1}| = \frac{1}{4}$

7- If A is diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11} a_{22} a_{33}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow |A| = 1 \cdot 2 \cdot 3 = 6$$

upper triangle & lowe matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

upper triangle matrix

lower triangular matx

$|A|$ = product of diagonal element

$|A| = 1 \cdot 2 \cdot 3 = 6$

$|B| = 1 \cdot 1 \cdot 3 = 3$

Determinant by Row Operation

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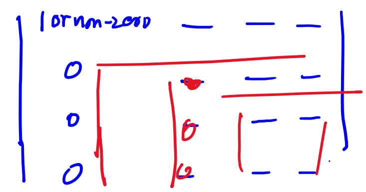
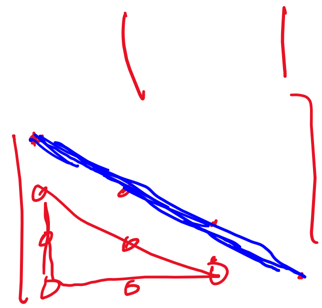
EXAMPLE 2 Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution

$$\det(A) = |A| = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$



$$|A| = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 15 & -13 & -14 \\ 0 & -24 & 19 & 22 \\ 0 & 4 & -6 & -2 \end{vmatrix} \begin{array}{l} \frac{1}{2}R_1 \\ R_2 - 3R_1 \\ R_3 + 3R_1 \\ R_4 - R_1 \end{array}$$



Expending by column 1

$$a_{11}C_{11} + \cancel{a_{21}C_{21}}^{0} + \cancel{a_{31}C_{31}}^{0} + \cancel{a_{41}C_{41}}^{0}$$

$$|A| = 2 \cdot 1 \begin{vmatrix} 15 & -13 & -14 \\ -24 & 19 & 22 \\ 4 & -6 & -2 \end{vmatrix}$$

$$|A| = 2 \begin{vmatrix} 15 & -13 & -14 \\ -24 & 19 & 22 \\ 4 & -6 & -2 \end{vmatrix} = 2 \cdot 2 \begin{vmatrix} 15 & -13 & -14 \\ -24 & 19 & 22 \\ 2 & -3 & -1 \end{vmatrix}$$

$$|A| = -4 \begin{vmatrix} 2 & -3 & -1 \\ -24 & 19 & 22 \\ 15 & -13 & -14 \end{vmatrix} = -4 \begin{vmatrix} 2 & -3 & 1 \\ 0 & \_ & \_ \\ 0 & \_ & \_ \end{vmatrix} \begin{array}{l} R_2 + 12R_1 \\ R_3 - \frac{15}{2}R_1 \end{array}$$

EXAMPLE 3 Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}_{4\times 4}$. $2R_1 + R_3$

SOLUTION Add 2 times row 1 to row 3 to obtain

$$\det A = \det \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

because the second and third rows of the second matrix are equal. ■

Cramer's Rule

n-equations
n - variables
$|A| \neq 0$
non-singular

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

Linear system with n equations and n variables

In matrix form

$$\underset{n\times n}{A}\ \underset{n\times 1}{x} = \underset{n\times 1}{b}$$

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

x_1, x_2

$$\boxed{x_i} = \frac{\boxed{\det A_i(\mathbf{b})}}{\boxed{\det A}}, \qquad i = 1, 2, \ldots, n \tag{1}$$

where

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \ \cdots \ \underset{\text{col } i}{\mathbf{b}} \ \cdots \ \mathbf{a}_n]$$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)}$$

replace 2nd column of A by b

$$A_2(b) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$A_3(b) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$$

replace 1st column of A by b

$$A_1(b) = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$x_1 = \frac{|A_1(b)|}{|A|}$$

$$x_2 = \frac{|A_2(b)|}{|A|}$$

$$x_3 = |A_3(b)| / |A|$$

EXAMPLE 1 Use Cramer's rule to solve the system

2-eqn
$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$$

M P = —
M P = —

$$\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\underset{2\times 2}{A}\ \underset{2\times 1}{\underline{x}} = \underline{b}$$

$|A| = 12 - (10) = 2 \neq 0$ So 'A is non-singular

$$x_1 = \frac{|A_1(b)|}{|A|} = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{2} = \frac{24+16}{2} = \frac{40}{2} = 20$$

$$x_2 = \frac{|A_2(b)|}{|A|} = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{2} = \frac{24+30}{2} = \frac{54}{2} = 27$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$x_1 = \frac{A_1(b)}{|A|} = \frac{\begin{vmatrix} 10 & 2 & 3 \\ 20 & 1 & 1 \\ 30 & 2 & 3 \end{vmatrix}}{|A|}$$

$$x_2 = A_2(b) = \frac{\begin{vmatrix} 1 & 10 & 3 \\ 1 & 20 & 1 \\ -1 & 30 & 3 \end{vmatrix}}{|A|}$$

$$\begin{array}{ccc|cc} & & & - & - \\ 1 & -1 & 1 & 1 & -1 \\ 2 & 2 & 1 & 2 & 2 \\ 3 & 1 & -1 & 3 & 1 \\ & & + & + & + \end{array}$$

$$|A| = -2 - 3 + \cancel{2} - 6 - 1 - \cancel{2} =$$

$$= -12$$