

## Row Reduction and Echelon Form (Online Lecture 2)

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

### Examples

$$\text{c. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{d. } \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

### Rank of a Matrix

No of nonzero rows in echelon form of the matrix is called rank of the matrix and it is denoted as  $\text{Rank}(A)$ .

- 1- Null matrix have all the rows as zero rows so Rank of null matrix is zero
- 2- Identity matrix of order 3,  $I_3$ , (is in echelon form and) have 3 nonzero rows to  $\text{Rank}(I_3) = 3$ .

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Rank}(A) = 0$$

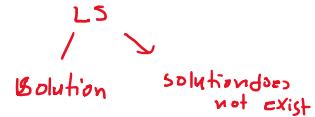
$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Echelon form}$$

$$\text{Rank}(I_3) = 3$$

## Existence and Uniqueness Theorem

### Existence

$$x_1, x_2, x_3, \dots, x_n$$



A linear system in  $n$  variables having  $m$  equations in matrix form can be written as

$$A_{m \times n} x_{n \times 1} = b_{m \times 1}$$

coefficient matrix      unknown vector      vector of constant

The linear system is

$$A_d = [A : b]$$

- 1- **Consistent** if and only if  $\text{Rank}(A) = \text{Rank}(A_d)$
- 2- **Inconsistent** if and only if  $\text{Rank}(A) \neq \text{Rank}(A_d)$

$$\{ \text{Rank}(A) \leq \text{Rank}(A_d) \}$$

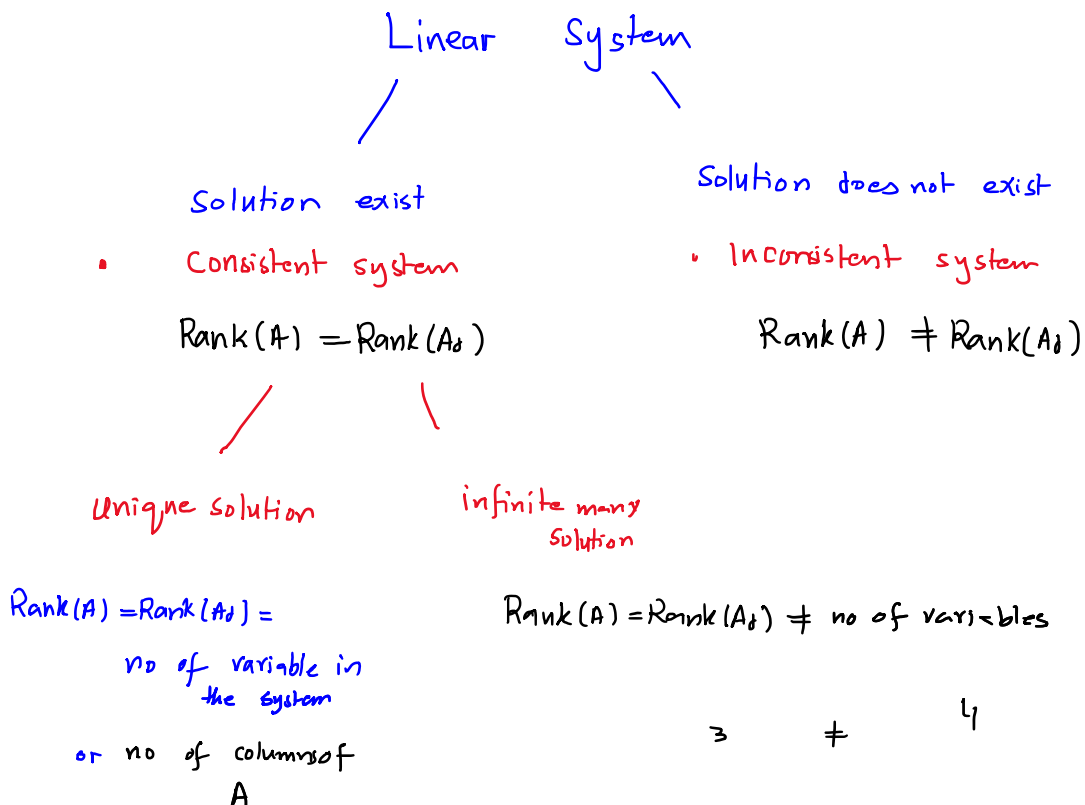
where  $A_d$  represent the augmented matrix of the linear system.

### Uniqueness

$\text{Rank}(A) = \text{Rank}(A_d) \neq \text{no of variables}$   
 exist infinite many solutions

Further, the solution is **unique** if and only if

$$\text{Rank}(A) = \text{Rank}(A_d) = \text{No of variables of the system (n in this case)}$$



**Example 1** Determine the existence and uniqueness of the solution of the system whose augmented matrix is reduced to the form

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

**Solution** The augmented matrix of the system is

Pivot columns are those columns that contain pivot element

$$A_d = \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline \textcircled{1} & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & \textcircled{2} & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 7 \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$c_1$   $c_2$   $c_3$   $c_4$   $c_5$   $\uparrow$   
P.C. P.C. P.C.

$2x_1 + 3x_2 - x_3 = 1$   
 $x_2 + x_3 = 0$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$Ax = b$

5-Unknowns  
3-Equation

$A_d = [A \mid b]$

$$A_d = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \end{array}$$

$\text{Rank}(A_d) = 3 = \text{no of non-zero rows in echelon form of } A_d$

$\text{Rank}(A) = 3 = \text{no of non-zero rows in echelon form of } A$

$\text{Rank}(A) = 3 = \text{Rank}(A_d)$  So system is consistent

Basic variables are those that are corresponding to pivot columns  
 $x_1, x_3, x_5$  are basic

$x_2, x_4$  are free variables

no of variables in L.S.  $= 5 \neq \text{Rank}(A) = \text{Rank}(A_d)$

So solution is not unique. There exist infinite many solutions of the system

- Linear system has unique solution iff all the variables are basic variables.

**Example 2** Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Step 1:-

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ 15 \end{bmatrix}$$

$$A \underline{x} = \underline{b}$$

$$A_d = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

5-unknown  
3-Equation

$$A_d \xrightarrow{R} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & & \\ \textcircled{3} & -7 & 8 & -5 & 8 & 1 & 9 \\ 0 & \textcircled{3} & -6 & 6 & 4 & - & -5 \\ 0 & 0 & 0 & 0 & \textcircled{\frac{2}{3}} & - & \frac{8}{3} \end{bmatrix} \quad \left( R_3 + \frac{2}{3} R_2 \right)$$

$$\text{Rank}(A) = 3 = \text{Rank}(A_d)$$

$x_1, x_2, x_5 \rightarrow$  Basic variables

$x_3, x_4 \rightarrow$  Free variables

Infinite many  
solutions