Determinant

$A_{m\times n}$ $m \neq n$ $A_{2\times 3}$ $|A| \to$ define

Determinant of 2 × 2

The determinant of 2×2 square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ is a real number define as

If a_{ij} are real

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

Note:–

1– The determinant of a matrix can be zero, +ve or –ve.

2– If $|A| = 0$, then A is called Singular matrix. → A^{-1} exist doesnot exist

3– If $|A| \neq 0$, then A is called non-Singular matrix → A^{-1} exist

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

Determinant of $n \times n$ matrix

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}}$$

The cofactor expansion down the jth column is

$$\boxed{\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}}$$

R_1

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \xleftarrow{R_2}$$

$\Rightarrow |A| = a_{21}C_{21} + a_{22}C_{22} + \cdots + a_{2n}C_{2n}$

$|A| = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}$

EXAMPLE 2 Use a cofactor expansion across the third row to compute det A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} R_3$$

$a_{31} \quad a_{32} \quad a_{33}$

$C_{11} = (-1)^{1+1} \left| \quad \right|$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}$

$\boxed{|A| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}}$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$

$|A| = 0 \cdot (-1)^{3+1} \begin{vmatrix} 5 & 0 \\ 4 & 1 \end{vmatrix} - 2^{(-1)} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \, (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$

$a_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$

$|A| = 2(4-10) = 2(-6) = -12$

Note:- 1- $|A|$ is unique

2-

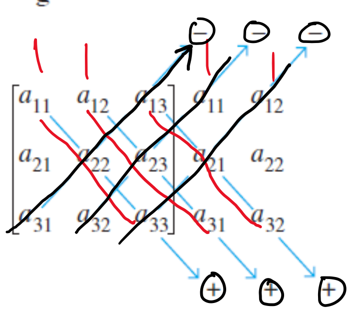
$$A = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Determinant of the 3 × 3 matrix (2nd method)

Example:-

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{matrix} 1 & 5 \\ 2 & 4 \\ 0 & -2 \end{matrix}$$

The expansion of a 3 × 3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:



$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{31}a_{21}a_{12}$

$|A| = 0 + 0 + 0 - 0 + 2 - 0$

$|A| = -2$

Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to* 4 × 4 *or larger matrices*.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 3 & 4 \\ 2 & -1 & 1 & 1 \\ -1 & 3 & 4 & 1 \end{bmatrix}_{4\times4}$$

$A_{10\times10}$

$A = [\,1 \,]$

$|A| = a_{11}\boxed{C_{11}} + a_{12}\boxed{C_{12}} + a_{13}\boxed{C_{13}} + a_{14}\boxed{C_{14}}$

$3\times3 \quad 3\times3 \quad 3\times3 \quad 3\times3$

$|A|_{10\times10} \rightarrow 10 \quad 9\times9$ determine

$A_{10\times10}$ for $|A|$ we have have to calculate 10 9×9 determint

9×9 we have 9 8×8 deter

8×8 " " " 8 7×7 dc

7×7 7 6×6

⋮

Properties of the determinant

1 – If A has a row or a column with all elements equal to zero, then

$$\boxed{|A| = 0}$$

$|A_{60\times100}| = 0$

2 - If A has two rows or two columns identical, then $|A| = 0$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & 10 \\ 3 & 3 & 11 \end{bmatrix} \Rightarrow |A| = 0$$

3- $|A| = |A^T|$

2nd row 2nd column

$$\boxed{A^T_{5\times5} = -A_{5\times5} \Rightarrow |A| = 0}$$

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}_{3\times3} \Rightarrow |A| = 0$$

4 - If we interchange two rows or two columns then then the determinant of resulting is $\boxed{-|A|}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = a \Rightarrow -a = \left|\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}\right|$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$|A^T| = 4 - 6 = -2$$

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow |B| = 6 - 4 = 2$$

5- $A = \begin{bmatrix} 20 & 20 & 20 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow |A| = 20 \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

6- $\underline{|AB| = |A| \cdot |B|}$

$A A^{-1} = I$ $\qquad |I_n| = 1$

$|A A^{-1}| = |I| = 1$

$|A||A^{-1}| = 1$ $\qquad$ If $|A| \neq 0$

$\Rightarrow |A^{-1}| = \frac{1}{|A|}$

If $|A| = 2$; $|A^{-1}| = \frac{1}{2}$

7- If A is diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11} a_{22} a_{33}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow |A| = 1 \cdot 2 \cdot 3 = 6$$

upper triangle & lowe matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

upper triangle matrix

lower triangular matx

$|A|$ = product of diagonal element

$$|A| = 1 \cdot 2 \cdot 3 = 6$$

$$|B| = 1 \cdot 1 \cdot 3 = 3$$