Chapter 05: Eigenvalues and Eigenvectors

Given $A_{n\times n}$

$$\boxed{A\underline{x} = \lambda \underline{x}}$$

$\underline{x}_{n\times 1} \neq 0$ — vector → eigenvector of A

λ — Scalar, Real number → eigenvalue of A

$A_{2\times 2}$, $x_{2\times 1}$, λ

$A\underline{x} = \lambda \underline{x}$ (λ: eigenvalue, $\underline{x}$: eigenvector)

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \lambda\mathbf{x}}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}_{2\times 2}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}_{2\times 1}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}_{2\times 1}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

SOLUTION

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\boxed{A\underline{x} = \lambda\underline{x}}$$

$$A\underline{u} = -4\underline{u}$$

$$\boxed{\lambda = -4}$$

$$A\underline{v} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad A\underline{v} \neq \lambda \underline{v}$$

$A\underline{u} = \lambda \underline{u}$

$\underline{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is eigenvector of A and $\lambda = -4$

$\underline{v}$ is not an eigenvector of A

$$A\underline{v} = 5\underline{v} \qquad A\underline{x} = \lambda\underline{x}$$

The Characteristic Equation (to find the eigenvalues)

$A\underset{\sim}{y} = \lambda \underset{\sim}{y}$

$A_{n\times n}$

For a given $n\times n$ matrix A, the eigenvalue equation

A → known
λ → unknown
x → unknown

$$\boxed{A\underset{\sim}{x} = \lambda \underset{\sim}{x}}$$

$$A\underset{\sim}{x}_{n\times 1} - \lambda \underset{\sim}{x}_{n\times 1} = \underset{\sim}{0}_{n\times 1}$$

$\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to n \text{ times}$

$I_n \underset{\sim}{x} = \underset{\sim}{x}$

$$A_{n\times n}\underset{\sim}{x} - \lambda I_n \underset{\sim}{x} = \underset{\sim}{0}$$

$\underset{\sim}{0}_{3\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Step 1:- eigenvalues calcu

Step 2:- eigenvector

$$(\underbrace{A_{n\times n} - \lambda I_{n\times n}}_{\text{Coefficient}})\underbrace{\underset{\sim}{x}}_{\text{unknown}} = \underset{\sim}{0} \to \text{Homogeneous system}$$

$\underset{\sim}{x} \neq \underset{\sim}{0}$

has non-trivial $\Leftrightarrow$ $|A_{n\times n} - \lambda I_{n\times 1}| = 0$ → characteristic equation of A

> A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation
> $$\det(A - \lambda I) = 0$$

EXAMPLE 1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}_{2\times 2}$.

Solution:-

For $A_{2\times 2}$ matrix the characteristic equation

$$|A_{2\times 2} - \lambda I_{2\times 2}| = 0$$

$$|A_{n\times n} - \lambda I_n| = 0$$

$$\left| \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-6-\lambda)-9=0$$

$$-12-2\lambda+6\lambda+\lambda^2-9=0$$

$$\lambda^2+4\lambda-21=0$$

real | Complex

repeated

Factorizatn

Disc $= b^2-4ac$

$= 0$

$= 1$

$= 4$

$= 9$

$= 16$

$\vdots$

$$\lambda = \frac{-4 \pm \sqrt{16-4(-21)}}{2}$$

$$\lambda = \frac{-4 \pm \sqrt{16+84}}{2}$$

$$\lambda = \frac{-4 \pm 10}{2}$$

$\lambda_1 = \frac{-4+10}{2}$; $\lambda_2 = \frac{-4-10}{2}$

$\boxed{\lambda_1 = 3 \;;\; \lambda_2 = -7}$

eigenvalues of A

$A_{2\times 2} \rightarrow$ 2 - eigenvalues

Complex real $2/3$ $\sqrt{2}$

$\lambda = 1, 2, 1$

$A_{n\times n} \rightarrow \lambda^n + \lambda^{n-1} + \dots + 1 = 0$

n - eigenvalue

EXAMPLE 3 Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$|A - \lambda I_3| = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & -1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

Solution:- The characteristic equation

$$|A_{4\times4} - \lambda I_{4\times4}| = 0$$

$$\left| \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

upper triangular ←

$$\begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$$

$$\lambda = \underline{5, 5}, \underline{3, 1}$$

$$\boxed{\lambda^3 - 3\lambda^2 + \lambda + 1 = 0} \; ?$$

$x^3 - 3x^2 + x + 1 = 0$

$\boxed{\lambda = 1}$

$1^3 - 3 + 1 + 1 = 0$

$0 = 0$

$\lambda = \pm 1$

$$\begin{array}{c|cccc} 1 & 1 & -3 & 1 & 1 \\ & & 1 & -2 & -1 \\ \hline & 1 & -2 & -1 & 0 \end{array}$$

$\lambda^2 - 2\lambda - 1 = 0$

$\lambda^3 - \lambda^2 + \lambda + 6 = 0$

$6 = 1 \times 6$

$6 = -1 \times (-6)$

$6 = -2 \times -3$

$6 = 2 \times 3$

$\lambda = 1$ $\quad 1^3 - 1^2 + 1 + 6 \neq 0$

$\lambda = -1$ $\quad -1 - 1 - 1 + 6 \neq 0$

$6 = \pm 1, \pm 2, \pm 3, \pm 6$

$\lambda = 2$ $\quad 2^3 - 2^2 + 2 + 6 = 0$

$8 - 4 + \cancel{2} + \cancel{6} \neq 0$

$9 = \pm 1, \pm 3, \pm 9$

$\lambda = -2$ $\quad -8 - 4 - 2 + 6 \neq 0$

$\lambda = 3$

$\lambda = -3$

$\lambda = 6$

$\lambda = -5$

$1 = \text{ans}$

$(\text{ans})^3 - (\text{ans})^\lambda + (\text{ans}) + 6 = ?$

AC

$-1 = \text{ans}$

$= -$

-1