The Charecteristic Equation

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

$$|A_{2\times 2} - \lambda I_{2\times 2}| = 0$$

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$a_{11}a_{22} - a_{11}\lambda - \lambda a_{22} + \lambda^2 - a_{12}a_{21} = 0$$

$a_{11} + a_{22} = trace(A)$

$$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$|A| = a_{11}a_{22} - a_{21}a_{12}$

$$\boxed{\lambda^2 - trace(A)\lambda + |A| = 0}$$

$ax^2 + bx + c = 0$

Sum of roots $= -\frac{b}{a}$

Sum of eigenvalues $= -\frac{(-trace(A))}{1} = trace(A)$

Product of root $= \frac{c}{a}$

Product of eigenvalue $= \frac{|A|}{1} = |A|$

$$\boxed{\lambda_1 + \lambda_2 = trace(A) = \text{sum of diagional} \qquad \lambda_1\lambda_2 = |A|}$$

For $A_{n\times n} \rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n$; $\lambda^n \cdots\cdots \lambda^{n-1} + \lambda + - = 0$

$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$ = sum of eigenvalues = trace $(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$ = Product of eigenvalues $= (-1)^n |A|$

$(-1)^2 = 1$

$A_{3\times3}$ $(-1)^3 = -1$

5.2 EXERCISES

Find the characteristic polynomial and the real eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$
2. $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$
4. $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$ trace$(A) = 11$
5. $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$
6. $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$
7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$
8. $\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for 3 × 3 determinants described

$|A_{2\times2} - \lambda I_2| = 0$

$\lambda^2 - \text{trace}(A)\lambda + |A| = 0$

$\lambda_1 + \lambda_2 = 9 + 2 = 11 = \text{trace}(A)$

$\lambda_1 \lambda_2 = 9 \cdot 2 = 18 = |A|$

4-

$\lambda^2 - (8+3)\lambda + 24 - 6 = 0$

$\lambda^2 - 11\lambda + 18 = 0$

$\lambda = \dfrac{11 \pm \sqrt{121 - 4(18)}}{2}$

$\lambda = \dfrac{11 \pm \sqrt{121 - 72}}{2}$

$\lambda = \dfrac{11 \pm 7}{2}$

$\lambda_1 = \dfrac{11+7}{2}$; $\lambda_2 = \dfrac{11-7}{2}$

$\lambda_1 = 9$; $2 = \lambda_2$

prior to Exercises 15–18 in Section 3.1. [*Note*: Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

9. $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 4 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$ **14.** $\begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}$

$|A_{3\times 3} - \lambda I_3| = 0$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$

trace (A)

$$\lambda^3 - \underbrace{(a_{11}+a_{22}+a_{33})}\lambda^2 + [a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{21}a_{12} - a_{13}a_{31} - a_{23}a_{32}]\lambda - |A| = 0$$

For the matrices in Exercises 15–17, list the real eigenvalues, repeated according to their multiplicities.

15. $\begin{bmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

16. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 3 & 3 & -5 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional:

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$