Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $\boxed{A\mathbf{x} = \mathbf{0}}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

Linear System: Homo $A\underline{x} = \underline{0}$; non-homo $A\underline{x} = \underline{b} \neq \underline{0}$

$A\underline{x} = \underline{b}$, $\underline{b} \neq \underline{0}$ → non-homogeous

Example:-

$$\left.\begin{array}{l} 2x - 3y = 0 \\ -x + 2y = 0 \end{array}\right] \rightarrow \text{are}$$

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\boxed{A\underline{x} = \boxed{0}}$ → Homogenou System

$$\left.\begin{array}{l} 2x_1 - x_2 + 3x_3 = \boxed{1} \\ x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{array}\right] \rightarrow \text{not a homeg.}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$b \neq [^0$

(1) $\begin{bmatrix} 2x - 3y = 0 \\ x + y = 0 \end{bmatrix}$ ← $\boxed{\begin{array}{l} x = 0 \\ y = 0 \end{array}}$ → (1) → $\boxed{\text{Trivial Solution}}$

$2(0) - 3(0) = 0$

$0 = 0$

$0 + 0 = 0$

$0 = 0$

$A\underline{x} = \underline{0}$

$\boxed{\underline{x} = \underline{0}}$ → trivial solution

$$\left.\begin{array}{l} 3x_1 - 2x_2 - x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{array}\right]$$

$\boxed{\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}}$ → trivial

$A\underline{x} = \underline{0}$ → homoge

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$\boxed{A\underline{x} = 0}$

$\boxed{\text{non-trivial exist}}$

non-trivial does not exist

Existence of Nontrivial Solution of Homogenous system

The homogeneous equation $\boxed{A\mathbf{x} = \mathbf{0}}$ has a nontrivial solution if and only if the equation has at least $\boxed{\text{one free variable}}$.

Example:-

$x =$
$y =$

$$\left.\begin{matrix} 2x + 3y = 0 \\ x + y = 0 \end{matrix}\right] \rightarrow \text{Homogen system} ..$$

$\boxed{x = 0, y = 0}$ trivial solution

$$A_d = \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$A_d \overset{R}{\approx} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 0 \end{array}\right] \quad R_1 \Leftrightarrow R_2$$

$$A_d \overset{R}{\equiv} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2-2 & 3-2 & 0-0 \end{array}\right] \quad R_2 - 2R_1$$

$$A_d \overset{R}{\equiv} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cc} \text{P.C} & \text{P.C} \end{array} \end{array}$$

$\boxed{x \text{ and } y \text{ are basic variables}}$

0

EXAMPLE 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

$\boxed{\begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}} \rightarrow$ trivial solution

$\cancel{6}^{2}\left(\frac{4}{\cancel{3}}t\right) + 0 - 8t = 0$

$8t - 8t = 0$

$\boxed{0 = 0}$

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\underline{x} = \underline{0}$ which is homo.... system

$t = 1$

$\boxed{x_1 = \frac{4}{3}t \; ; \; x_2 = 0, \; x_3 = t}$

$\cancel{3}\left(\frac{4}{\cancel{3}}t\right) + 5(0) - 4(t) = 0$

$4t - 4t = 0 \quad \boxed{0 = 0}$

$-\cancel{3}\left(\frac{4}{\cancel{3}}t\right) - 2(0) + 4(t) = 0$

$-4t + 4t = 0 \quad \boxed{0 = 0}$

$$A_d = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

$$A_d \overset{R}{\equiv} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0-0 \\ 6-6 & 1-10 & -8+8 & 0 \end{array}\right] \quad \begin{matrix} R_2 + R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$A_d \overset{R}{\equiv} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0-0 & -9+9 & 0+0 & 0+0 \end{array}\right] \quad R_3 + 3R_2$$

x_1, x_2 are basic variables

$\boxed{x_3 \rightarrow \text{free variables}}$

$$A_d \overset{R}{\equiv} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cc} \text{P.C} & \text{P.C} \end{array} \end{array}$$

So non-trivial exist.

$\boxed{x_3 = t} \quad t \in \mathbb{R}$

$3x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

$3x_1 + 5\cancel{x_2}^{0} - 4\cancel{x_3}^{t} = 0 \qquad 3x_1 = 4t$

$\boxed{x_1 = \frac{4}{3}t \quad t \in \mathbb{R}}$

Applications of Linear System

Network Flow

Systems of linear equations arise naturally when scientists, engineers, or economists study the flow of some quantity through a network. For instance, urban planners and traffic engineers monitor the pattern of traffic flow in a grid of city streets. Electrical engineers calculate current flow through electrical circuits. And economists analyze the distribution of products from manufacturers to consumers through a network of wholesalers and retailers. For many networks, the systems of equations involve hundreds or even thousands of variables and equations.

A *network* consists of a set of points called *junctions*, or *nodes*, with lines or arcs called *branches* connecting some or all of the junctions. The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

EXAMPLE 2 The network in Fig. 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

$0 < t < 500$

$x_5 = 100$

At junction A

inflow outflow

$\boxed{800 = x_1 + x_2}$

At junction B

$\boxed{x_2 + x_4 = x_3 + 300}$

At junction C

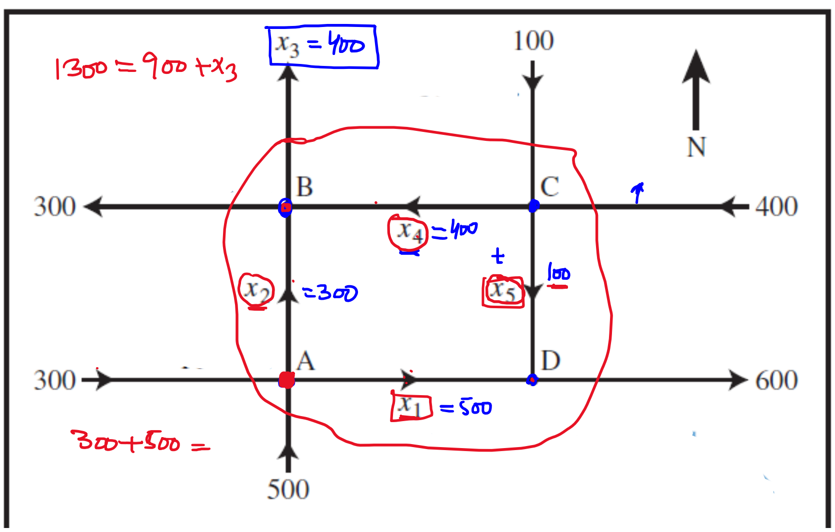
$\boxed{500 = x_4 + x_5}$

At junction D

$\boxed{x_1 + x_5 = 600}$

$1300 = 900 + x_3$

$\boxed{x_3 = 400}$



At any junction in the network

$\boxed{\text{inflow} = \text{outflow}}$

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$\boxed{x_2} + x_4$	$=$	$300 + \boxed{x_3}$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

$x_1 + x_2 = 800$

$x_2 \boxed{- x_3} + x_4 = 300$

$x_4 + x_5 = 500$

$x_1 + x_5 = 600$

$x_3 = 500$

Total

$\boxed{x_3 = 400}$ — 5

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \boxed{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 800 \\ 300 \\ 500 \\ 600 \\ 400 \end{bmatrix}}_{\underline{b}}$$

$$A\underline{x} = \underline{b}$$

$\boxed{A\underline{x} = \underline{0}} \to$ homo

$\boxed{A\underline{x} = \underline{b}} \to$ non-homo

$$A_d = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1-1 & 0-1 & 0-0 & 0-0 & 1-0 & 600 - 800 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array}\right] \quad R_4 - R_1$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$A_d \underset{=}{\overset{R}{=}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & \boxed{800} \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_4 + R_2 \\ R_5 \leftrightarrow R_3 \\ R_4 + R_3 \\ R_5 - R_4 \end{array}$$

$$\begin{array}{ccccc} P.C & P.C & P.C & P.C & \cancel{P.C} \end{array}$$

$0 = 0$

$x_5 = t = 100$

$\boxed{x_3 = 400}$

$\boxed{x_4 = 500 - t} = \boxed{400}$

$x_2 - x_3 \overset{+x_4}{} = 300$

$x_2 = 300 + x_3 - x_4$

$x_2 = 700 - (500 - t)$

$\boxed{x_2 = 200 + t} = \boxed{300}$

Rank $(A) = \underline{4} =$ Rank (A_d) $\to$ System is consistence

$t \leq 500$ ½

$\boxed{x_1, x_2, x_3, x_4} \to$ Basic

$\boxed{x_5 = \text{free variable}}$

$\boxed{x_5 = t}$: $t = \{0, 1, 2, \dots\}$

no of variables $\neq$ Rank $(A) =$ Rank (A_d)

5 4

$\Downarrow$

$x_4 + x_5 = 500$

$x_4 = 500 - x_5$

$\boxed{x_4 = 500 - \boxed{t}} \to$

$x_1 + x_2 = 800$

$x_1 = 800 - x_2$

$x_1 = 800 - (200 + t)$

$\boxed{x_1 = 600 - t}$

$\boxed{t = 100}$

$x_4 \geq 0$

$500 - t \geq 0$

$\boxed{500 \geq t}$

+ve

$$\begin{cases} x_5 = t \\ \boxed{x_4 = 200 - t} \\ x_3 = s \\ x_2 = 100 - t \\ x_1 = \underline{200 - s + t} \end{cases}$$

$200 - 50 + 100$

$t \leq 200$

$\boxed{t < 100}$

$t < 100$

$\boxed{t \leq 200}$ $t = 100$

$\boxed{s < 100}$ 500