Row Reduction and Echelon Form (Online Lecture 2)

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Examples

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Rank of a Matrix

No of nonzero rows in echelon form of the matrix is called rank of the matrix and it is denoted as $\boxed{\text{Rank}(A)}$.

1- Null matrix have all the rows as zero rows so Rank of null matrix is zero
2- Identity matrix of order 3, I_3, (is in echelon form and) have 3 nonzero rows to Rank(I_3)= 3.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ Echelon form

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\text{Rank}(I_3) = 3$

$\text{Rank}(A) = 0$

Existence and Uniqueness Theorem

LS

Solution   solution does not exist

Existence

$x_1, x_2, x_3, \dots, x_n$

A linear system in n variables having m equations in matrix form can be written as

coefficent matrix   unknown vecto   vector of constant

$$\boxed{A_{m\times n}}\,\boxed{x_{n\times 1}} = \boxed{b_{m\times 1}}$$

The linear system is

$A_d = [A \,\vdots\, b]$

1- $\boxed{\text{Consistent}}$ if and only if $\boxed{\text{Rank}(A)=\text{Rank}(A_d)}$
2- $\boxed{\text{Inconsistent}}$ if and only if $\boxed{\text{Rank}(A)\neq\text{Rank}(A_d)}$

$\{\text{Rank}(A) \leq \text{Rank}(A_d)\}$

where A_d represent the augmented matrix of the linear system.

Uniqueness

$\text{Rank}(A) = \text{Rank}(A_d) \neq$ no of variables
exist infinite many solutions

Further, the solution is $\boxed{\text{unique}}$ if and only if

Rank(A)=Rank(A_d)= No of variables of the system (n in this case)

Linear System

Solution exist
- Consistent system

$\text{Rank}(A) = \text{Rank}(A_d)$

unique solution

$\text{Rank}(A) = \text{Rank}(A_d) =$ no of variable in the system

or no of columns of A

infinite many Solution

$\text{Rank}(A) = \text{Rank}(A_d) \neq$ no of variables

$3 \neq 4$

Solution does not exist
- Inconsistent system

$\text{Rank}(A) \neq \text{Rank}(A_d)$

Example 1 Determine the existence and uniqueness of the solution of the system whose augmented matrix is reduced to the form

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$2x_1 + 3x_2 - x_3 = 1$
$x_2 + x_3 = 0$

$$\underset{A}{\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Solution The augmented matrix of the system is

$$\begin{array}{c} \\ A_d = \\ \\ \end{array} \begin{array}{cccccc} \boxed{x_1} & x_2 & \boxed{x_3} & x_4 & \boxed{x_5} & \boxed{b} \\ \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \\ c_1 & c_2 & c_3 & c_4 & c_5 & \\ P.C & & P.C & & P.C & \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

Pivot columns are those columns that contain Pivot elem

5 - Unknowns
3 - Equation

$A_d = [A \mid b]$

$$\begin{array}{c} \\ A_d = \end{array} \begin{array}{ccc} x_1 & x_2 & x_3 \\ \left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} R_1 \\ R_2 \end{array}$$

$\text{Rank}(A_d) = 3$ = no of non-zero rows in echelon form of A_d

$\text{Rank}(A) = 3$ = no of nonzero rows in echelon form of A

Basic variables are those that are corresponding to pivot columns
x_1, x_3, x_5 are basic

x_2, x_4 are free variables

$\boxed{\text{Rank}(A) = 3 = \text{Rank}(A_d)}$ So System is Consistent

no of variables in LS $= 5 \neq \text{Rank}(A) = \text{Rank}(A_d)$

So Solution is not unique. There exist infinite many solutions of the system

- Linear System has unique solution iff all the variables are basis variables.

Example 2 Determine the existence and uniqueness of the solutions to the system

$$
\begin{aligned}
3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15
\end{aligned}
$$

Step I :-

$$
\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} -5 \\ 9 \\ 15 \end{bmatrix}
$$

$$A \quad \underline{x} = \underline{b}$$

$$
A_d = \left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]
$$

5-unknown

3-Equation

$$
A_d \underset{\approx}{R} \quad
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}
\left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{8}{3} \end{array}\right]
\qquad R_3 + \frac{2}{3}R_2
$$

$$\text{Rank}(A) = 3 = \text{Rank}(A_d)$$

infinite many solutions

$x_1, x_2, x_5 \rightarrow$ Basic variables

$\boxed{x_3, x_4 \rightarrow \text{Free variables}}$