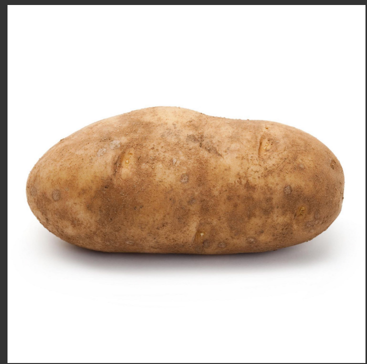
2022 - Nov - 09.

- Double Integral v.s. Iterated Integral.

stick cut



slice cut

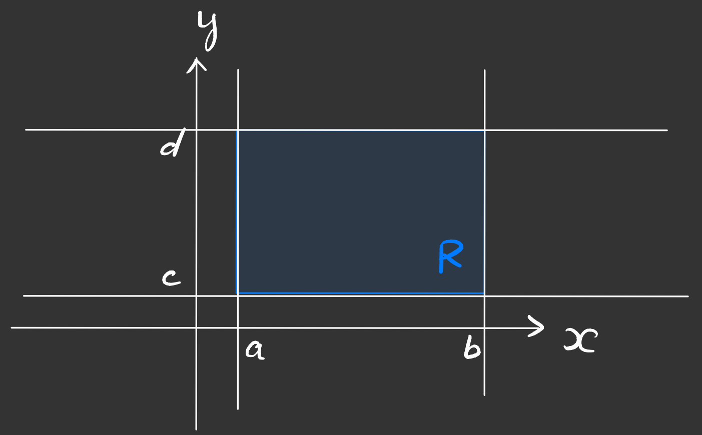
same volumn

" Double Integral "

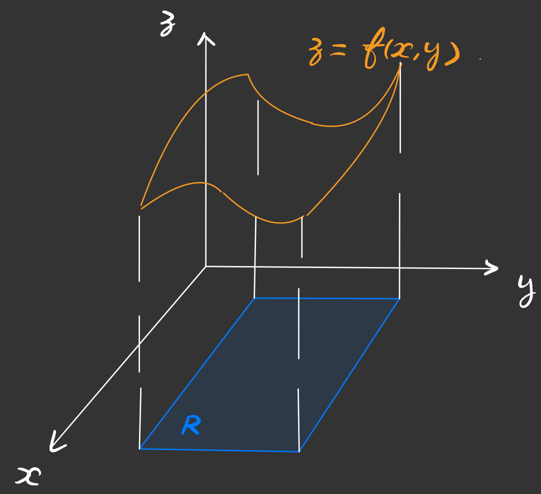
" Iterated Integral "

Rectangular Domain

- $R = \{ (x,y) \in \mathbb{R}^2 , \quad a \le x \le b , \quad c \le y \le d \}$.
- $z = f(x,y) : R \to \mathbb{R}$ is continuous

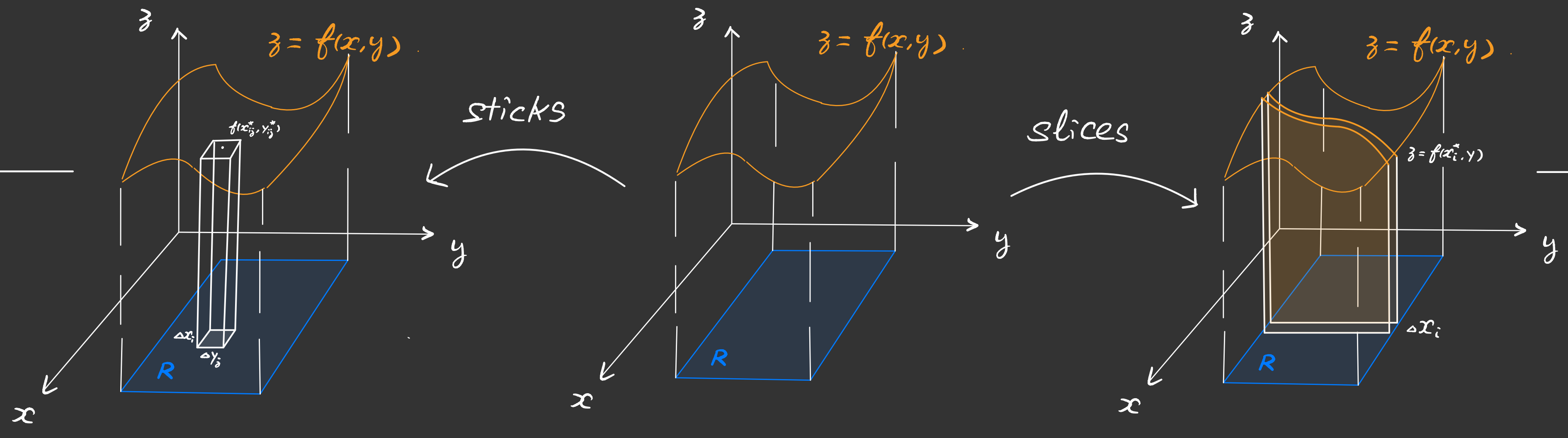


Domain R.



Graph of f.

- How to compute the volumn of the "building" with the blue base and orange roof ?



DOUBLE INTEGRAL

Grid	$a = x_0 < x_1 < x_2 < \cdots < x_m = b,$ $c = y_0 < y_1 < y_2 < \cdots < y_n = d.$
volumn of single stick	$v_{ij} = f(x_{ij}^*, y_{ij}^*) \cdot \Delta x_i \Delta y_j .$
volumn of all sticks	$V_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j \right)$
Limit	$V_{mn} \xrightarrow{m,n \to \infty} \boxed{\iint_R f(x,y)\, dA}$

ITERATED INTEGRAL

Grid	$a = x_0 < x_1 < x_2 < \cdots < x_m = b,$
volumn of single slice	$v_i = \left(\int_c^d f(x_i^*, y)\, dy \right) \Delta x_i$
volumn of all slices	$V_m = \sum_{i=1}^{m} \left(\left(\int_c^d f(x_i^*, y)\, dy \right) \Delta x_i \right)$
Limit	$V_m \xrightarrow{m \to \infty} \boxed{\int_a^b \int_c^d f(x,y)\, dy\, dx}$

- ARE THEY EQUAL ?
- DOES ORDER MATTER IN ITERATED INTEGRAL ?

Fubini - Tonelli THM (Basic Version).

- Suppose $f: R \to \mathbb{R}$ is continuous on $R = \{(x,y) \in \mathbb{R}^2 \mid \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}\}$.

Then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy,$$

Double Integral = Iterated Integral

Order in Iterated integral does NOT matter

RMK

- In practice the thm is important since we don't have powerful tools to compute the double integral by definition
- Somehow for iterated integral, we have a SUPER POWERFUL tool: The FTC

Example

$$\iint_R x^2 y \, dA \qquad R = \{(x,y) \quad 0 \le x \le 3, \quad 1 \le y \le 2\}.$$

- Since $x^2 y$ is continuous on R. the THM of Fubini-Tonelli applies.

- Integrate x first then y:

$$\iint_R x^2 y \, dA = \int_1^2 \underbrace{\int_0^3 x^2 y \, dx}_{\text{treat } y \text{ as constant}} \, dy.$$

$$= \int_1^2 \left(\frac{1}{3} x^3 y \Big|_0^3 \right) dy$$

$$= \int_1^2 \underbrace{9y \, dy}_{\text{now treat } y \text{ as variable}}$$

$$= \frac{9}{2} y^2 \Big|_1^2$$

$$= \frac{27}{2}.$$

- Integrate y first then x:

$$\iint_R x^2 y \, dA = \int_0^3 \underbrace{\int_1^2 x^2 y \, dy}_{\text{treat } x \text{ as constant}} \, dx$$

$$= \int_0^3 \frac{1}{2} x^2 y^2 \Big|_1^2 \, dx$$

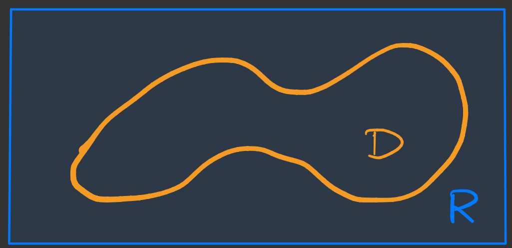
$$= \int_0^3 \underbrace{\frac{3}{2} x^2}_{\text{now treat } x \text{ as variable}} \, dx.$$

$$= \frac{x^3}{2} \Big|_0^3$$

$$= \frac{27}{2}.$$

General Domain

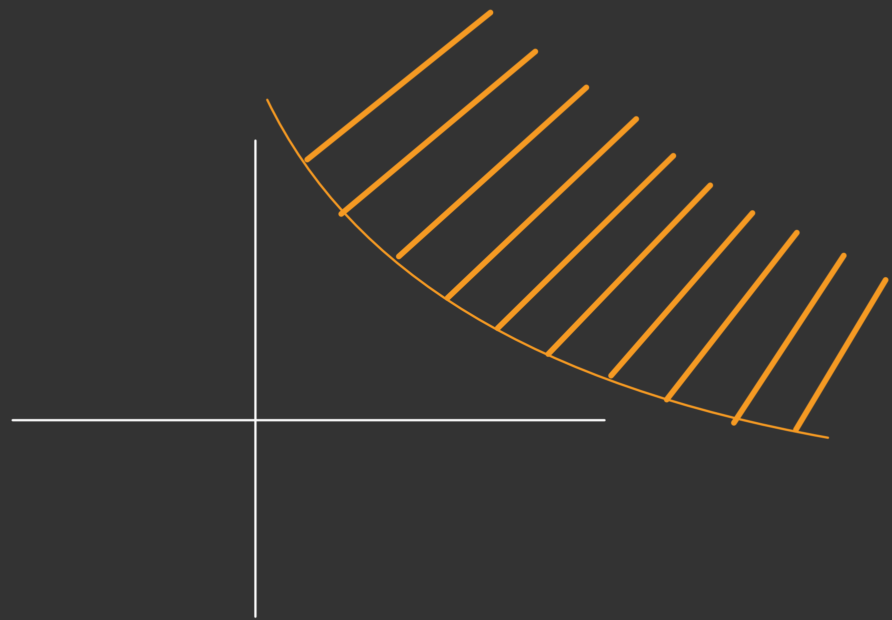
- The previous restriction to rectangular domain is too limited.
- We say a region D is <u>bounded</u> if D can be embedded into a (large) rectangle R.



Example

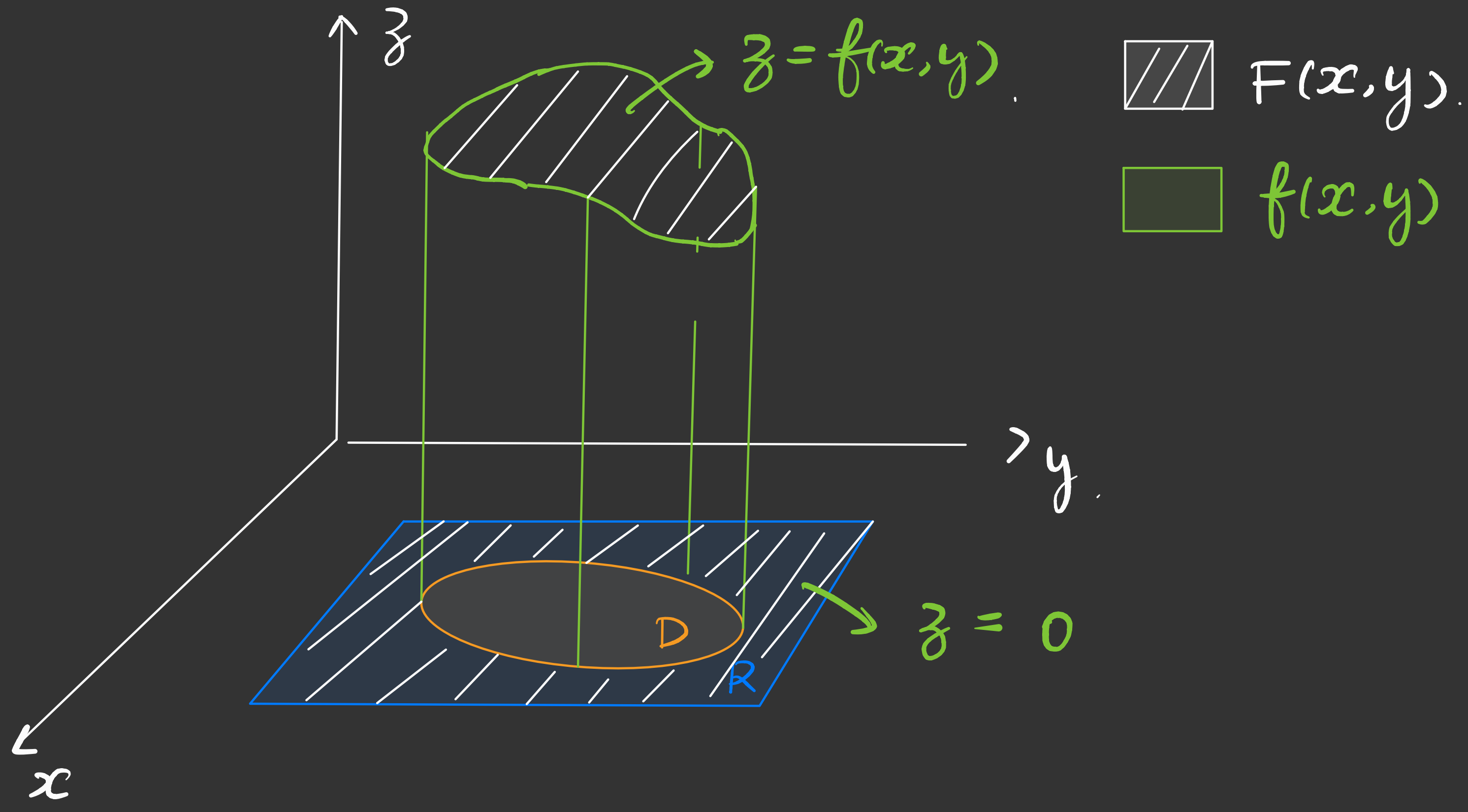
Let $D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 < x < \infty \\ 0 < y < \infty \\ 1 \le xy \end{array} \right\}$.

Then D is <u>NOT</u> bounded.



Recall we only know how to define the double / Iterated integral on rectangles. Let's extend f from D to R.

- Define $F(x,y): R \to \mathbb{R}$, s.t $F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0 & \text{otherwise}. \end{cases}$



Graph of F (shadowed part).

- Now $F(x,y)$ is defined on rectangle. Can we use Fubini-Tonelli?
- Not yet —— $F(x,y)$ is NOT continuous.

As a result, we need a more general version of F-T THM.

Fubini-Tonelli THM (General Version).

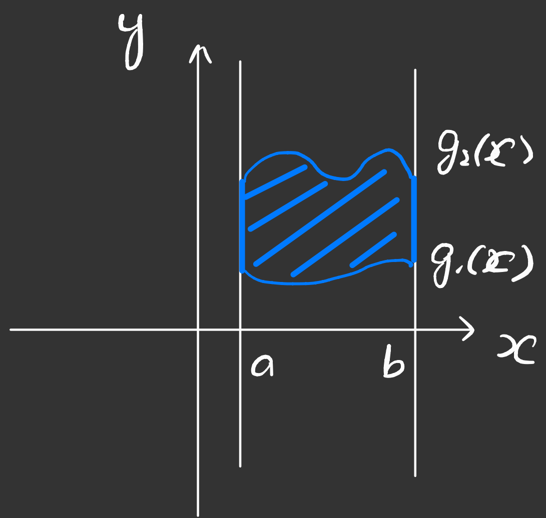
Suppose that

- $F: R \to \mathbb{R}$ is continuous on $R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$ except on finitly many (smooth) curves. (For example, ∂D).
- F: is bounded function: $\exists\, M > 0$ s.t $|F(x,y)| \le M$, $\forall (x,y) \in R$
- The double integral $\iint_R F(x,y)\, dA$ exists.

Then

$$\iint_R F(x,y)\, dA = \int_a^b \int_c^d F(x,y)\, dy\, dx = \int_c^d \int_a^b F(x,y)\, dx\, dy$$

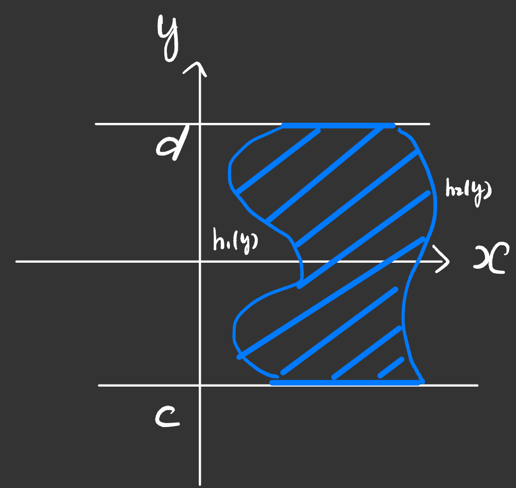
- Although our construction gives definition of double integral with no strong assumption on the region D, in practice we need to restrict our interest in more special domains.



TYPE I

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

where $g_1(x) \le g_2(x)$ are continuous functions on $[a,b]$.

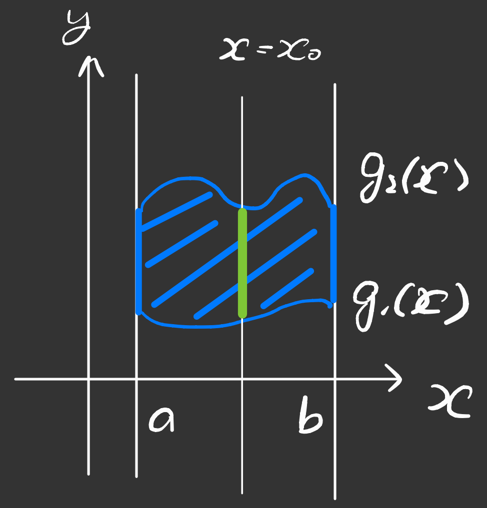


TYPE II

$$D_2 = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

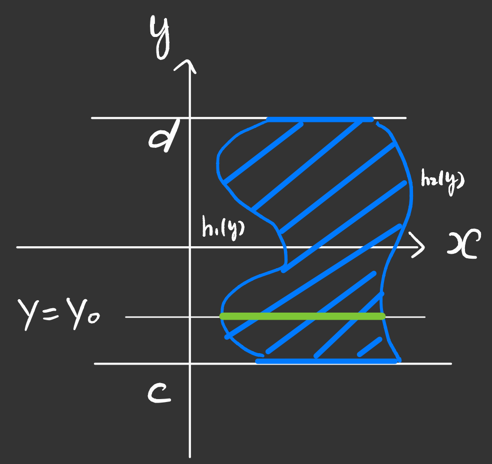
where $h_1(y) \le h_2(y)$ are continuous functions on $[c,d]$

- How to decide if a region is of type I / II ?
- We can use a "scanner"



* Type I criterion.

$\forall x_0 \in [a,b]$, the intersection of $\{x = x_0\} \cap D$ is a SINGLE segment



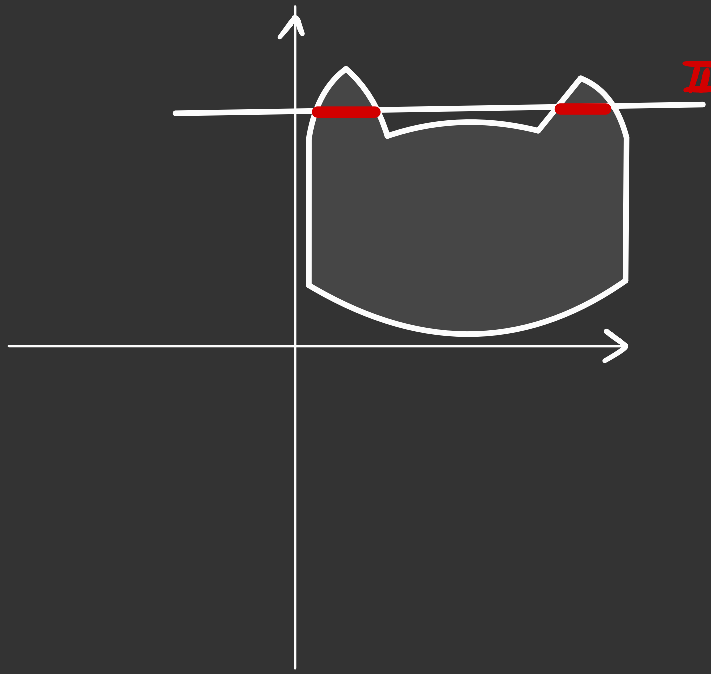
* Type II criterion.

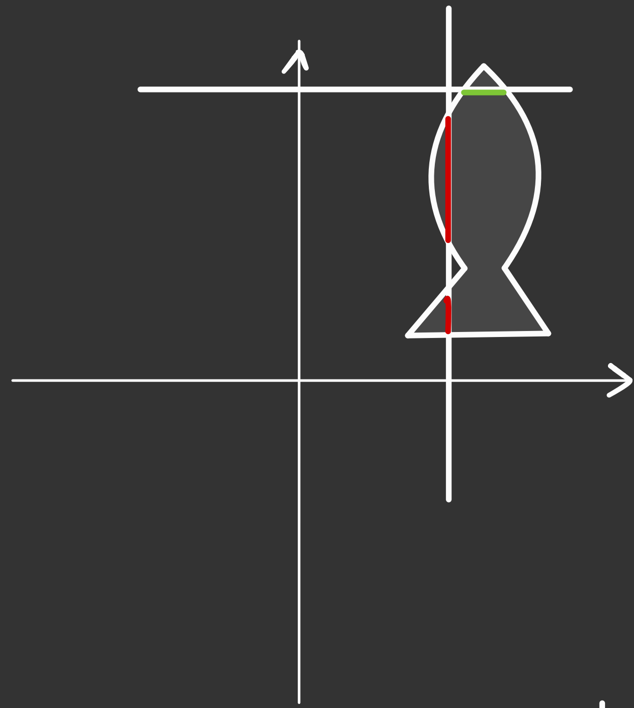
$\forall y_0 \in [c,d]$, the intersection of $\{y = y_0\} \cap D$ is a SINGLE segment

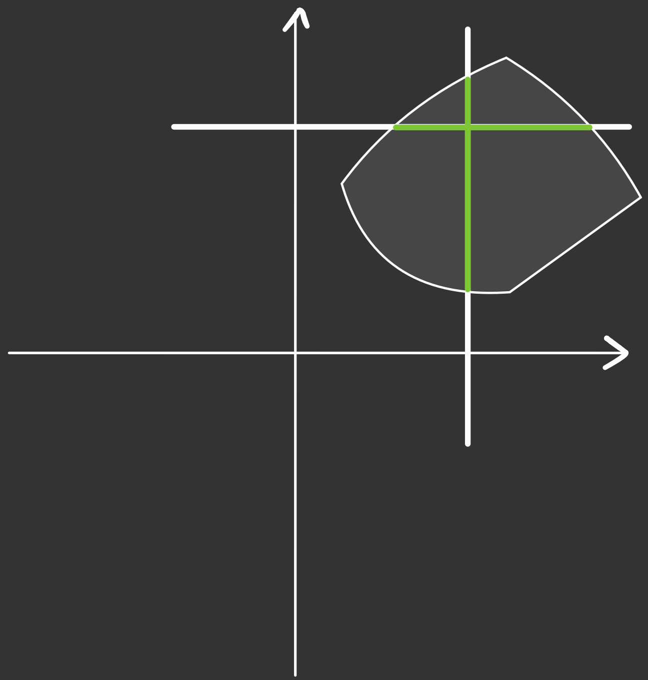
RMK :

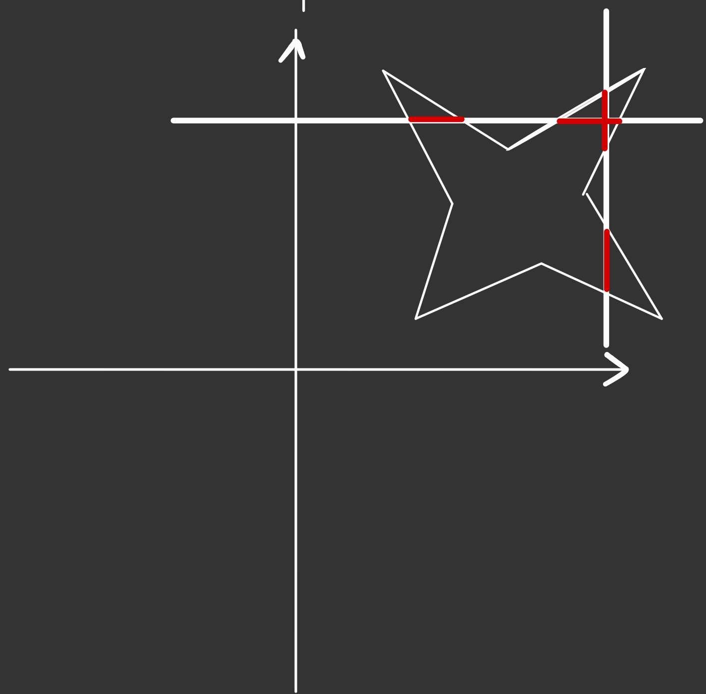
A single point is also considered as a single segment, whose length is 0 though .

Example

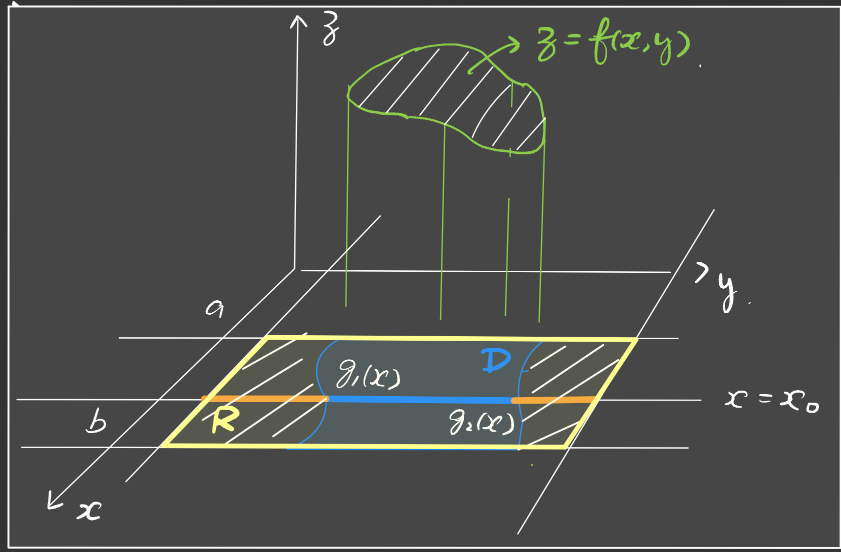








Integral on Type I Region



$$\iint_D f(x,y)\,dA$$

$$\underset{\text{Definition}}{=\!=} \iint_R F(x,y)\,dA$$

$$\underset{\text{Tonelli}}{\overset{\text{Fubini}}{=\!=}} \int_a^b \int_c^d F(x,y)\,dy\,dx$$

$$\underset{\text{of integral}}{\overset{\text{Property}}{=\!=}} \int_a^b \left(\int_c^{g_1(x)} \underbrace{F(x,y)}_{=0} dy + \int_{g_1(x)}^{g_2(x)} \underbrace{F(x,y)}_{=f(x,y)} dy + \int_{g_2(x)}^{d} \underbrace{F(x,y)}_{=0} dy \right) dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Integral on Type II Region

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

The reasoning is similar.
(justify it by yourself!)

Given a double integral $\iint_D f(x,y)\,dA$ on general domain.

- **Step 1** Sketch the region D

- **Step 2**. Decide if D is of type I or type II.

Type I

specify $a, b, g_1(x), g_2(x)$

Type II

specify $c, d, h_1(x), h_2(x)$.

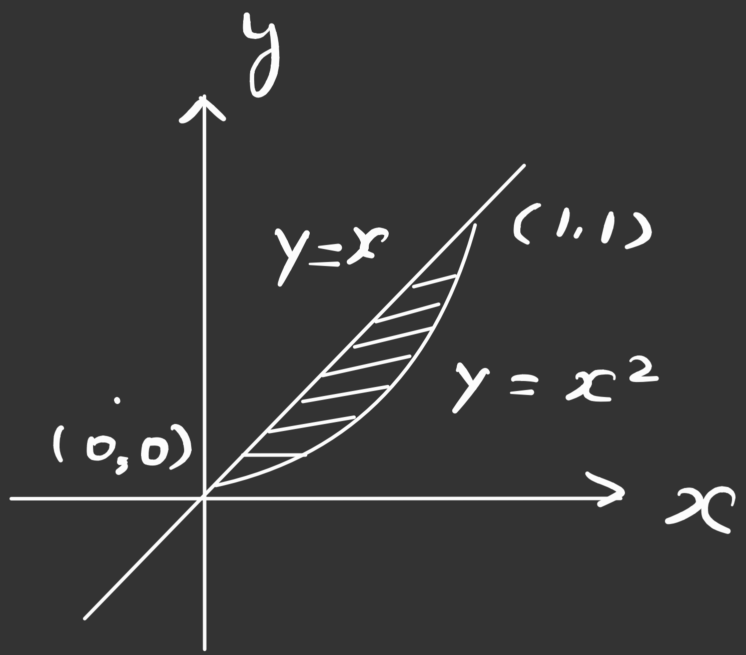
- **Step 3**: Compute (Dirty work ...... with some tricks)

TBA next week.

Example

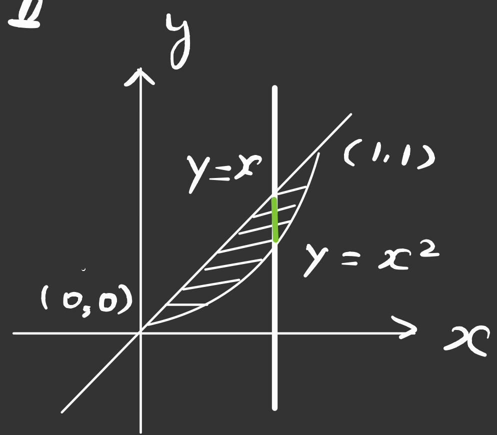
Compute $\iint_D f(x,y)\,dA$, where D is the region enclosed by $y = x^2$ and $y = x$, $f(x,y) = x^2y + 2y^2$.

- Sketch the region.



- Both Type I & II

$$D = \left\{ (x,y) \;\middle|\; \begin{matrix} 0 \le x \le 1 \\ x^2 \le y \le x \end{matrix} \right\}$$



$$D = \left\{ (x,y) \;\middle|\; \begin{matrix} 0 \le y \le 1 \\ y \le x \le \sqrt{y} \end{matrix} \right\}.$$

As Type I region:

$$\iint_D x^2y + 2y^2 \, dA$$

$$= \int_0^1 \int_{x^2}^{x} (x^2y + 2y^2) \, dy \, dx$$

$$= \int_0^1 \left(\frac{x^2}{2} y^2 + \frac{2}{3} y^3 \right) \Bigg|_{x^2}^{x} dx$$

$$= \int_0^1 \frac{x^4}{2} + \frac{2}{3} x^3 - \frac{7}{6} x^6 \, dx$$

$$= \frac{1}{10} x^5 + \frac{1}{6} x^4 - \frac{1}{6} x^7 \Bigg|_0^1$$

$$= \frac{1}{10}$$

As Type II region:

$$\iint_D x^2y + 2y^2 \, dA$$

$$= \int_0^1 \int_y^{y^{\frac{1}{2}}} x^2y + 2y^2 \, dx \, dy.$$

$$= \int_0^1 \left(\frac{1}{3} y x^3 + 2y^2 x \right) \Bigg|_y^{y^{\frac{1}{2}}} dy$$

$$= \int_0^1 \frac{7}{3} y^{\frac{5}{2}} - \frac{1}{3} y^4 - 2y^3 \, dy$$

$$= \frac{2}{3} y^{\frac{7}{2}} - \frac{1}{15} y^5 - \frac{1}{2} y^4 \Bigg|_0^1$$

$$= \frac{1}{10}$$

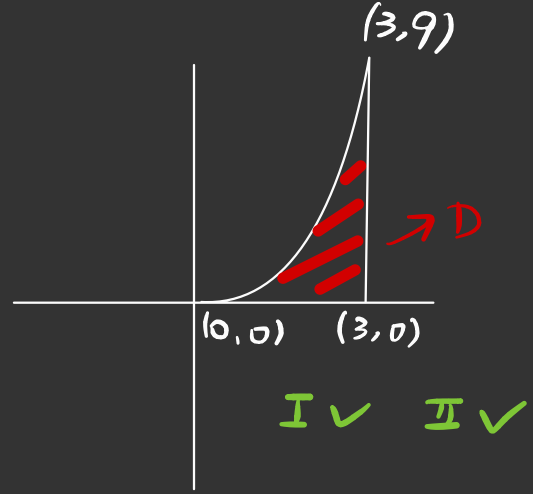
Techniques For Double Integral.

2022 Nov 16

- Change of Order for Iterated Integral.

Example.

Compute $\displaystyle\int_0^9 \int_{\sqrt{y}}^3 \sin(x^3)\, dx\, dy$.



- Tough to compute: anti-derivative of $\sin(x^3)$ mixed with $\sqrt{y}$ — catastrofique!

Let's use double integral as an intermediate media.

By Fubini-Tonelli THM.

$$\int_0^9 \int_{\sqrt{y}}^3 \sin(x^3)\, dx\, dy = \iint_D \sin(x^3)\, dA = \int_0^3 \int_0^{x^2} \sin(x^3)\, dy\, dx$$

$$= \int_0^3 x^2 \sin(x^3)\, dx$$

$$\overset{u=x^3}{=\!=} \int_0^{27} \frac{1}{3} \sin u\, du$$

$$= \frac{1}{3}(-\cos u)\Big|_0^{27} = \frac{1}{3}(1-\cos 27)$$

Change of Variables

- Recall the 1D case.

$$\int_a^b f(x)\,dx \xlongequal{x = g(u)} \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\cdot g'(u)\,du.$$

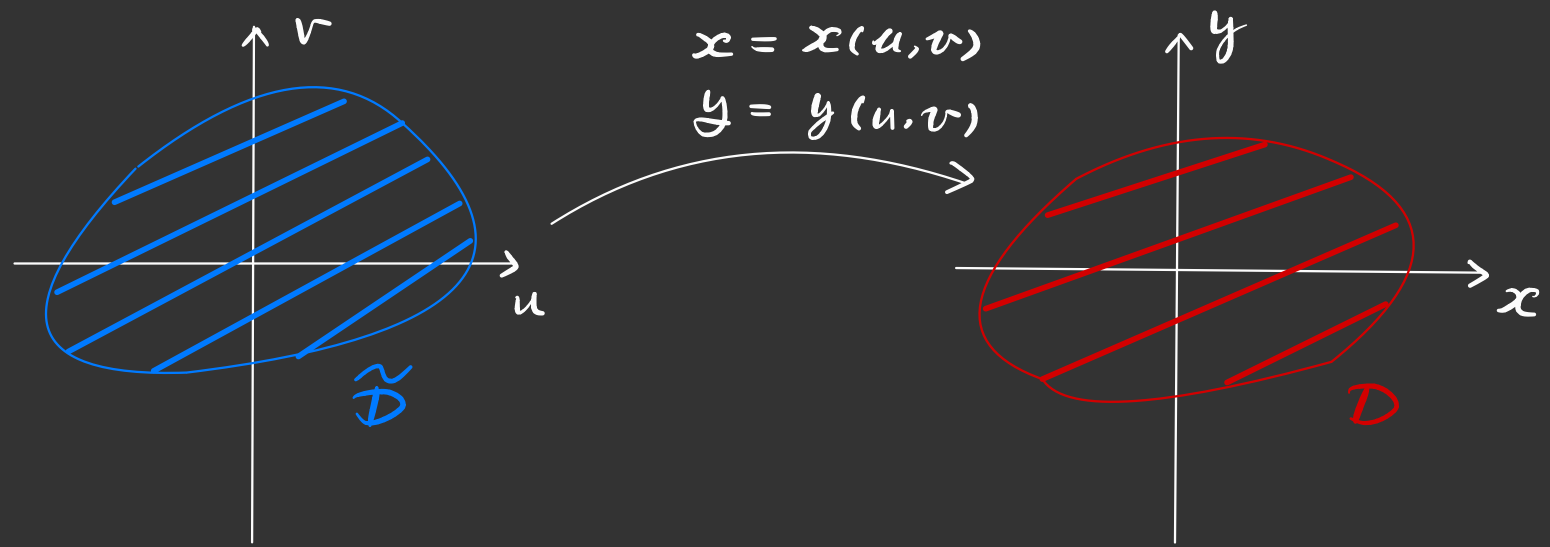
3 Things have changed:

1. Interval: $[a,b] \rightarrow [g^{-1}(a), g^{-1}(b)]$.
2. Integrand: $f(x) \rightarrow f(g(u))$.
3. length element: $dx \rightarrow g'(u)du$.

- For 2D case it is extremely similar.

$$\iint_D f(x,y)\,dA \xrightarrow[]{\substack{x = x(u,v)\\ y = y(u,v)}} \ ?$$

① Change of Domain

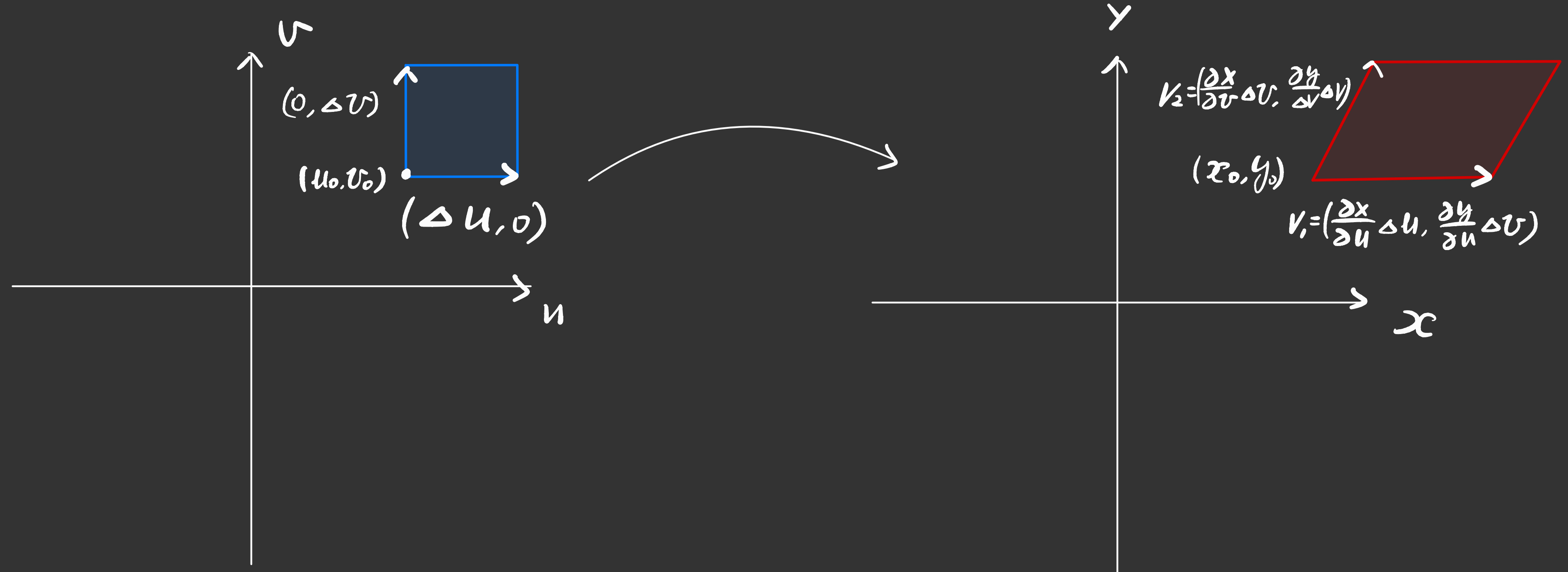


② Change of Integrand

$$f(x(u,v), y(u,v)) \longrightarrow f(x,y)$$

③ Change of Area form

$$? \longrightarrow dA$$



Using Cross Product (Recall the geometric meaning)

- $(\Delta u, 0, 0) \times (0, \Delta v, 0) = (0, 0, \underbrace{\boxed{\Delta u \cdot \Delta v}}_{d\tilde{A}})$
- $\left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, 0\right) \times \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, 0\right) = (0, 0, \underbrace{\boxed{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\Delta u \Delta v}}_{dA})$

Thus

$$\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right| d\tilde{A} \longrightarrow dA$$

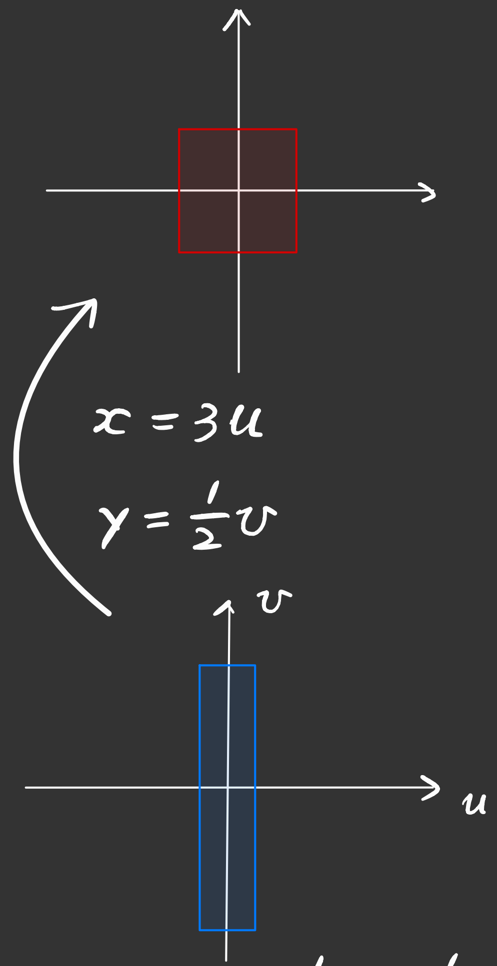
The absolute value makes sure the area is positive.

Example

Let $D = \{(x,y) \mid -1 \le x \le 1,\ -1 \le y \le 1\}$.

Compute $\displaystyle\iint_D 1 \, dA$

$$D = \left\{(x,y) \;\middle|\; \begin{array}{l} -1 \le x \le 1, \\ -1 \le y \le 1 \end{array} \right\}$$



$$\tilde{D} = \left\{(u,v) \;\middle|\; \begin{array}{l} -\frac{1}{3} \le u \le \frac{1}{3} \\ -2 \le v \le 2 \end{array} \right\}$$

- <u>No change of variable.</u>

$$\iint_D 1 \, dA = 1 \cdot \text{Area}(D) = 2 \times 2 = 4$$

SAME RESULT

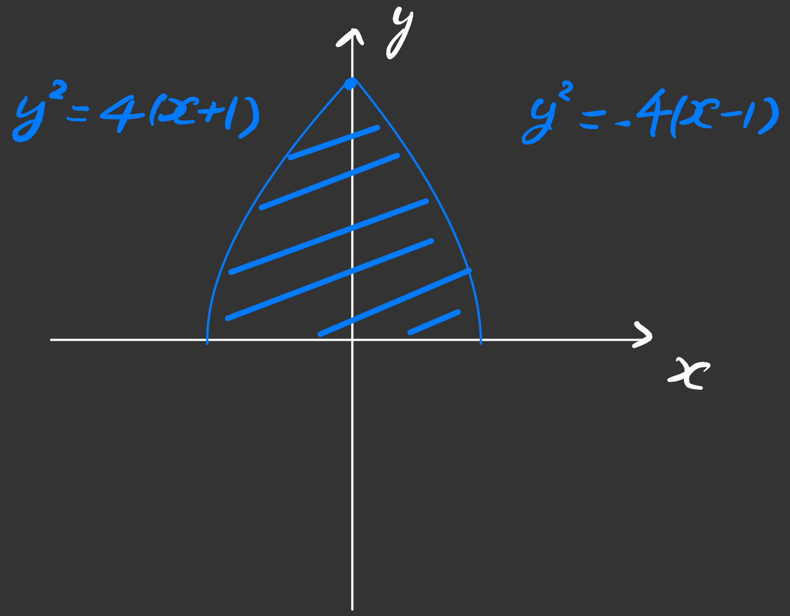
- <u>Try change of variable</u>

$$\begin{cases} x = 3u \\ y = \frac{1}{2}v \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{3}{2} > 0$$

$$\iint_{\tilde{D}} 1 \cdot \frac{3}{2} \, d\tilde{A} = \frac{3}{2} \cdot \text{Area}(\tilde{D}) = \frac{3}{2} \cdot \left(\frac{2}{3} \times 4\right) = 4$$

Example

Compute $\iint_D y \, dA$.



Method 1

$$\iint_D y \, dA \underset{\text{Tonelli}}{\overset{\text{Fubini}}{=\!=}} \int_0^2 \int_{\frac{y^2-4}{4}}^{\frac{y^2-4}{-4}} y \, dx \, dy$$

$$\overset{\text{symmetry}}{=\!=} 2 \int_0^2 \int_0^{1-\frac{y^2}{4}} y \, dx \, dy$$

$$\underset{\text{Leibniz}}{\overset{\text{Newton}}{=\!=}} 2 \int_0^2 xy \Big|_0^{1-\frac{y^2}{4}} dy$$

$$=\!= 2 \int_0^2 y - \frac{y^3}{4} \, dy$$

$$\underset{\text{Leibniz}}{\overset{\text{Newton}}{=\!=}} 2 \left(\frac{y^2}{2} - \frac{y^4}{16} \right) \Big|_0^2$$

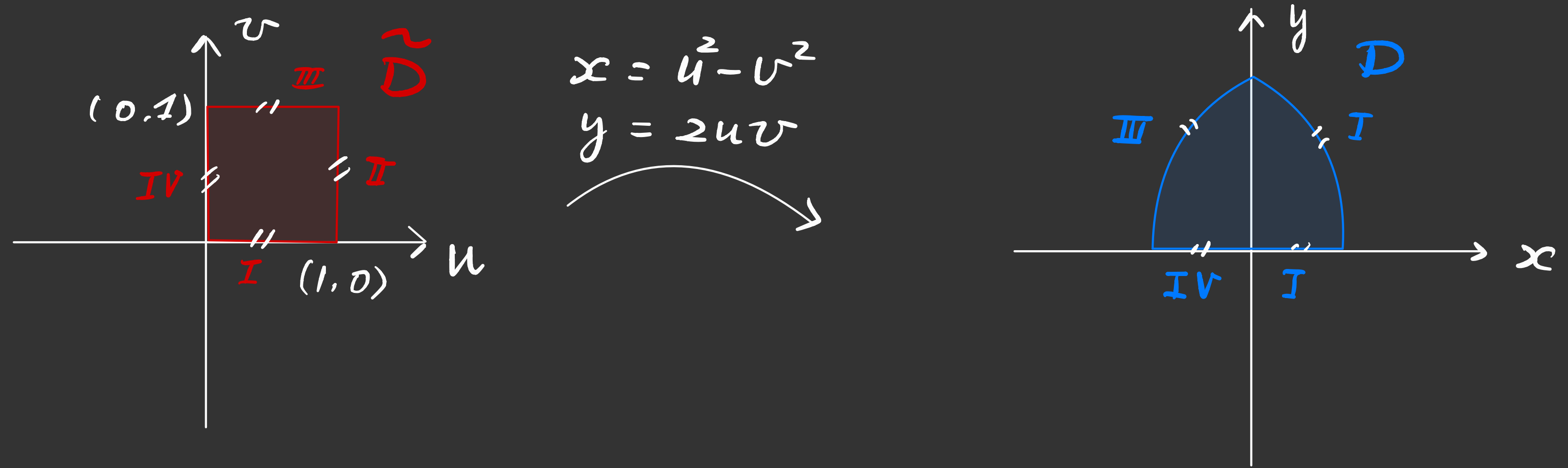
$$=\!= 2$$

Method 2. (Change of variable).

Let's consider the change of variables

$$F: \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

If $\tilde{D} = \left\{ (u,v) \,\middle|\, \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1 \end{array} \right\}$, what's $\tilde{D} \xrightarrow{F} ?$



I	$0 \le u \le 1,\ v = 0$	$\longrightarrow$	$x = u^2,\ y = 0$	I
II	$u = 1 \quad 0 \le v \le 1$	$\longrightarrow$	$x = 1 - v^2,\ y = 2v \rightarrow x = 1 - \left(\frac{y}{2}\right)^2$	II
III	$0 \le u \le 1 \quad v = 1$	$\longrightarrow$	$x = u^2 - 1,\ y = 2u \rightarrow x = \left(\frac{y}{2}\right)^2 - 1$	III
IV	$u = 0 \quad 0 \le v \le 1$	$\longrightarrow$	$x = -v^2,\ y = 0$	IV

As a result, by the change of variables formula.

$$\iint_D y \, dA = \iint_{\tilde{D}} 2uv \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot d\tilde{A}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2) \geq 0$$

$$= \iint_{\tilde{D}} 2uv \, 4(u^2+v^2) \, d\tilde{A}$$

$$\overset{\text{Fubini}}{\underset{\text{Tonelli}}{=}} \int_0^1 \int_0^1 8u^3v + 8uv^3 \, du \, dv$$

$$\overset{\text{Newton}}{\underset{\text{Leibniz}}{=}} \int_0^1 \left(2u^4v + 4u^2v^3 \Big|_0^1 \right) dv$$

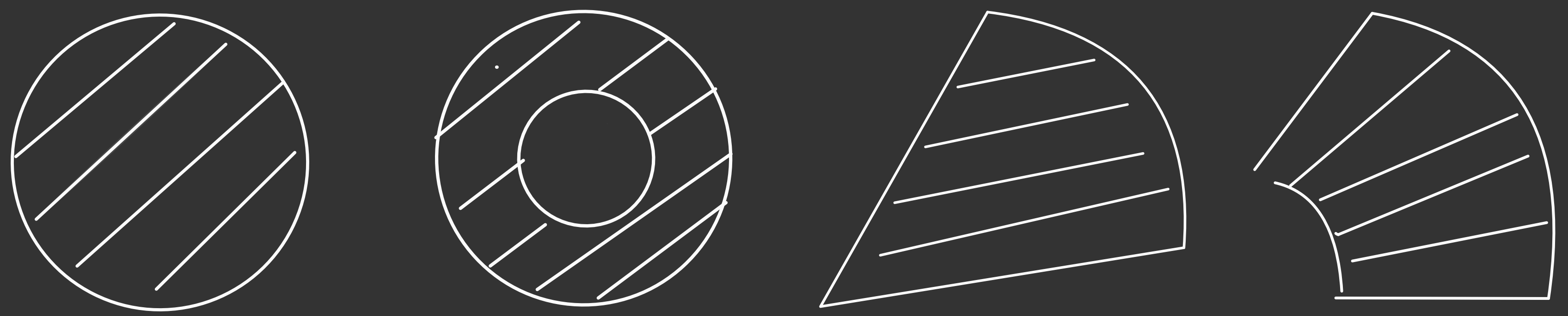
$$= \int_0^1 2v + 4v^3 \, dv.$$

$$\overset{\text{Newton}}{\underset{\text{Leibniz}}{=}} (v^2+v^4) \Big|_0^1$$

$$= 2.$$

Polar Coordinates.

A particular change of variable, i.e. the polar coordinates might be useful when dealing with domains like.



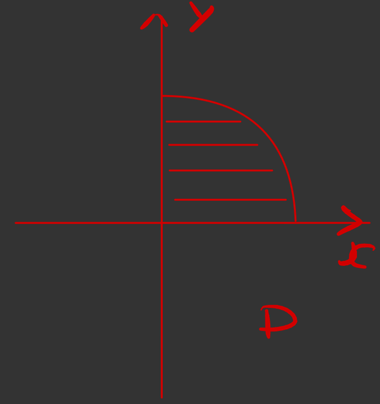
$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence $\boxed{dx\,dy} \longrightarrow \boxed{r\,dr\,d\theta}$.

Example.

Compute $\displaystyle\iint_D \ln(1+x^2+y^2)\, dA$

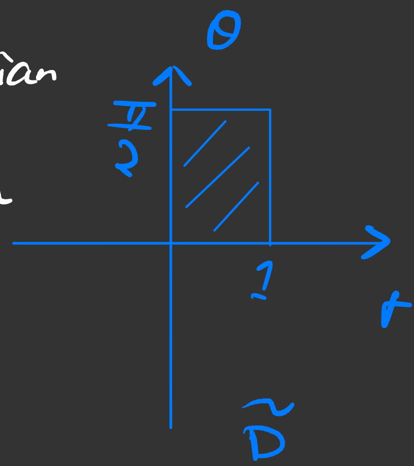
$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1,\ x^2+y^2 \le 1\}.$



Using the polar coordinates.

Jacobian

$$\iint_D \ln(1+x^2+y^2)\, dA = \iint_{\tilde{D}} \ln(1+r^2)\cdot \boxed{r}\cdot d\tilde{A}$$



$$= \int_0^{\frac{\pi}{2}} \int_0^1 \ln(1+r^2)\, r\, dr\, d\theta$$

$u = 1+r^2$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\int_1^2 \ln u\, du\right) d\theta = \frac{\pi}{4}\cdot\left(u\cdot \ln u \Big|_1^2 - \int_1^2 1\, du\right)$$

Integration by parts.

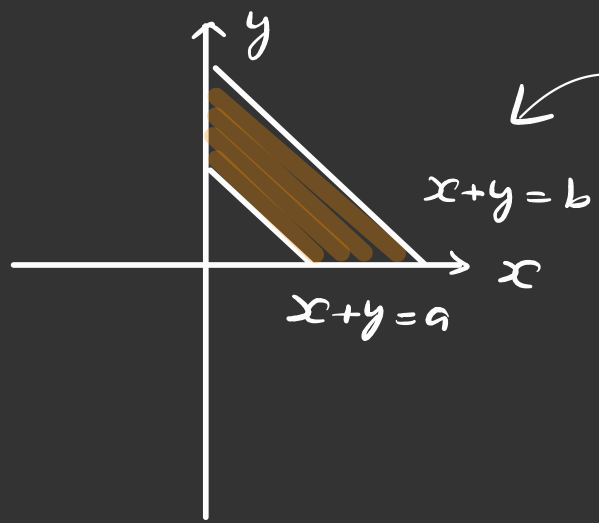
$$= \frac{\pi}{4}\,(2\ln 2 - 1).$$

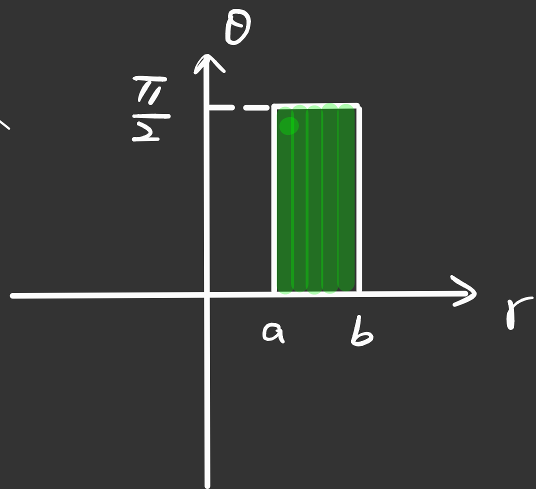
Example

Compute the volumn of the region enclosed by

$z^2 = xy \qquad x+y=a \qquad x+y=b \qquad (0<a<b)$.

$$V = 2\iint_D \sqrt{xy}\, dA$$





- Let $x = r\cos^2\varphi$,

  $y = r\sin^2\varphi$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = 2r\sin\varphi\cos\varphi$$

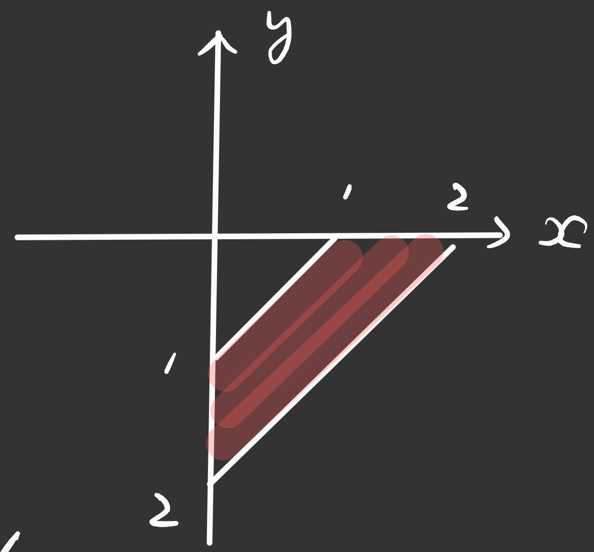
$$V = 2\int_a^b\int_0^{\frac{\pi}{2}} 2r\sin\varphi\cos\varphi \cdot r\sin\varphi\cos\varphi\, dr\, d\varphi = 2\int_a^b 2r^2\,dr \int_0^{\frac{\pi}{2}} \sin^2\varphi\cos^2\varphi\, d\varphi$$

$$= 2\cdot \frac{2}{3}r^3\Big|_a^b \cdot \int_0^{\frac{\pi}{2}} \frac{1}{4}\sin^2 2\varphi\, d\varphi = \frac{4}{3}(b^3-a^3)\cdot\frac{1}{8}\int_0^{\frac{\pi}{2}} 1-\cos4\varphi\, d\varphi$$

$$= \frac{4}{3}(b^3-a^3)\cdot\frac{\pi}{16} = \boxed{\frac{\pi}{12}(b^3-a^3)}$$

Example

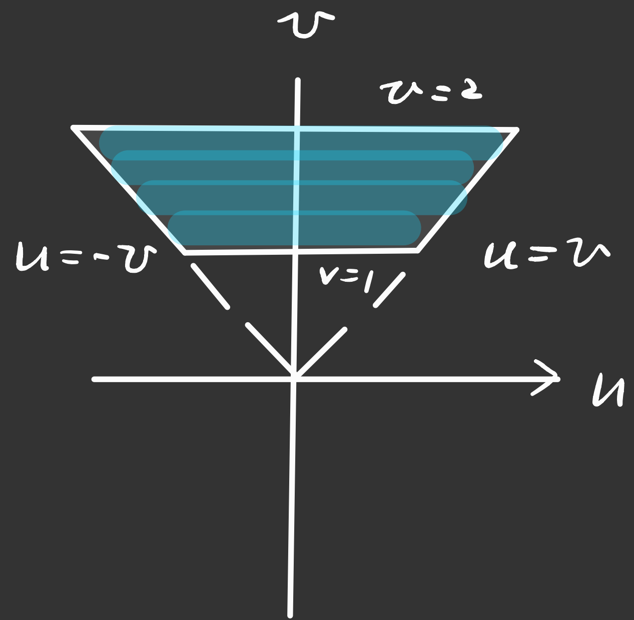
Compute $\iint e^{\frac{x+y}{x-y}}\, dA$



Method 1.

$$\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} < 0$$



$$\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \cdot \left|-\frac{1}{2}\right| \, du\, dv$$

$$= \frac{1}{2} \int_1^2 v e^{\frac{u}{v}} \Big|_{-v}^{v} \, dv$$

$$= \frac{1}{2}(e - e^{-1}) \int_1^2 v \, dv$$

$$= \frac{1}{2}(e - e^{-1}) \frac{1}{2} v^2 \Big|_1^2 = \frac{3}{4}(e - e^{-1}).$$