Homework 3

Instructions

Points: Please see the points for each problem.
Submission: Submit completed homework as a PDF file. Handwritten work or photos of handwritten work must be neat and legible.

Points Summary

Question Number	Points Possible	Points Earned
1	1	______
2	1	______
3	1	______
4	1	______
5	1	______
6	1	______
7	1	______
8	1	______
9	3	______
10	3	______
11	3	______
12	3	______
Total	**20**	______

1 Bayes' Nets Representation

Graph Structure: Conditional Independence

Consider the Bayes' net given below.

Recall:

- $X \perp\!\!\!\perp Y$ reads as "X is independent of Y".
- $X \perp\!\!\!\perp Y \mid \{Z, W\}$ reads as "X is independent of Y given Z and W".

For each statement below, indicate whether it is True or False. (1 point each)

1. **True** **False**
   It is guaranteed that $A \perp\!\!\!\perp B$.

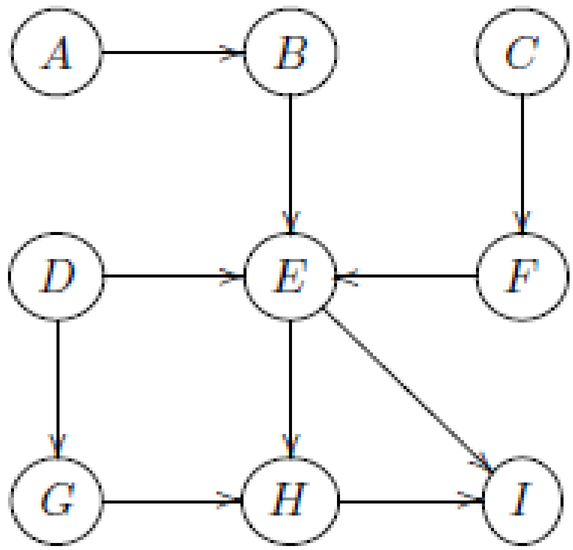


Figure 1: Bayes' net graph structure showing nodes A through I and their directed dependencies

Solution: **Answer:** False

There is a direct edge $A \to B$ in the Bayes' net. While B depends on A (since A is a parent of B), independence is symmetric: $A \perp\!\!\!\perp B$ means both $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Since there is an active path between A and B (the direct edge), they are not independent. Therefore, this statement is **False**.

2. **True** **False**
   It is guaranteed that $A \perp\!\!\!\perp C$.

Solution: **Answer:** True

The only path between A and C is: $A \to B \to E \leftarrow F \leftarrow C$. Node E is a collider on this path. Since E is not observed, the collider blocks the path. With no active paths between A and C, they are independent. Therefore, this statement is **True**.

3. **True** **False**
   It is guaranteed that $A \perp\!\!\!\perp D \mid E$.

Solution: **Answer:** False

There is a path: $A \to B \to E \leftarrow D$. Node E is a collider on this path. When we condition on E (observe E), the collider becomes activated, making the path active. Therefore, A and D are not independent given E. This statement is **False**.

4. **True** **False**
   It is guaranteed that $A \perp\!\!\!\perp I \mid E$.

> **Solution:** **Answer:** True
>
> The path between A and I is: $A \to B \to E \to I$. Node E is on this chain. When we condition on E (observe E), nodes on a chain block the path. Since E blocks the path and there are no other active paths between A and I given E, they are independent. Therefore, this statement is **True**.

5. **True** **False**
   It is guaranteed that $B \perp\!\!\!\perp C \mid I$.

> **Solution:** **Answer:** False
>
> There is a path: $B \to E \to I \leftarrow H \leftarrow E \leftarrow F \leftarrow C$. Node I is a collider on this path. When we condition on I (observe I), the collider becomes activated, making the path active. Therefore, B and C are not independent given I. This statement is **False**.
>
> Alternatively, we can see: $B \to E \to H \to I$ and $C \to F \to E \to H \to I$. When I is observed, it activates the collider, creating an active path through E and H.

6. **True** **False**
   It is guaranteed that $F \perp\!\!\!\perp A \mid H$.

> **Solution:** **Answer:** True
>
> The path between F and A is: $F \to E \to H \leftarrow G \leftarrow D$ (but this doesn't reach A). Let's trace more carefully: $F \to E \leftarrow B \leftarrow A$. On this path, E is a collider. When we condition on H but not on E, the collider at E remains blocked. There is no active path between F and A given H. Therefore, F and A are independent given H. This statement is **True**.

7. **True** **False**
   It is guaranteed that $D \perp\!\!\!\perp I \mid \{E, G\}$.

> **Solution:** **Answer:** True
>
> The paths between D and I are:
>
> - $D \to E \to I$: E is on this chain, so conditioning on E blocks it.
> - $D \to E \to H \to I$: E is on this chain, so conditioning on E blocks it.
> - $D \to G \to H \to I$: G is on this chain, so conditioning on G blocks it.

- $D \to E \to H \leftarrow G$: H is a collider on this path. Since we condition on G but not on H, the collider remains blocked.

All paths between D and I are blocked when conditioning on $\{E, G\}$. Therefore, D and I are independent given $\{E, G\}$. This statement is **True**.

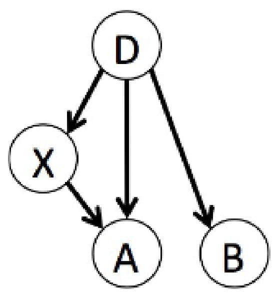
5. **True** **False**
It is guaranteed that $C \perp\!\!\!\perp H \mid G$.

Solution: **Answer:** True

The path between C and H is: $C \to F \to E \to H \leftarrow G$. On this path, H is a collider. When we condition on G but not on H, the collider at H remains blocked. There is no active path between C and H given G. Therefore, C and H are independent given G. This statement is **True**.

2 Bayes' Net Reasoning

$P(A \mid D, X)$			
$+d$	$+x$	$+a$	0.9
$+d$	$+x$	$-a$	0.1
$+d$	$-x$	$+a$	0.8
$+d$	$-x$	$-a$	0.2
$-d$	$+x$	$+a$	0.6
$-d$	$+x$	$-a$	0.4
$-d$	$-x$	$+a$	0.1
$-d$	$-x$	$-a$	0.9



$P(D)$	
$+d$	0.1
$-d$	0.9

$P(X \mid D)$		
$+d$	$+x$	0.7
$+d$	$-x$	0.3
$-d$	$+x$	0.8
$-d$	$-x$	0.2

$P(B \mid D)$		
$+d$	$+b$	0.7
$+d$	$-b$	0.3
$-d$	$+b$	0.5
$-d$	$-b$	0.5

Figure 2: Bayes' net for disease testing with disease D and tests A and B, showing conditional probability tables

1. What is the probability of having disease D and getting a positive result on test A? $P(+d, +a) =$

Solution: To find the joint probability $P(+d, +a)$, we use the chain rule for Bayes' nets. Since test A depends on disease D in the Bayes' net structure, we can factor this as:

$$P(+d, +a) = P(+d) \times P(+a \mid +d)$$

This follows from the chain rule: $P(X, Y) = P(X) \times P(Y \mid X)$ when X is a parent of Y in the Bayes' net.

Calculation:

- $P(+d)$ is the prior probability of having the disease (given in the CPT for node D)
- $P(+a \mid +d)$ is the conditional probability of a positive test A result given the disease is present (given in the CPT for node A)

Multiply these two values from the conditional probability tables to get the final answer.

3. What is the probability of not having disease D and getting a positive result on test A? $P(-d, +a) =$

Solution: This represents a false positive case: the test is positive even though the disease is not present. Using the chain rule:

$$P(-d, +a) = P(-d) \times P(+a \mid -d)$$

Since $P(-d) = 1 - P(+d)$, we can also write:

$$P(-d, +a) = (1 - P(+d)) \times P(+a \mid -d)$$

Explanation:

- $P(-d)$ is the prior probability of not having the disease, which equals $1 - P(+d)$
- $P(+a \mid -d)$ is the conditional probability of a positive test A result given the disease is absent (the false positive rate, given in the CPT for node A)

Multiply these values to get the probability of a false positive result.

3. What is the probability of having disease D given a positive result on test A? $P(+d| + a) =$

Solution: This is a conditional probability that requires Bayes' rule. We want to find $P(+d \mid +a)$.

Using Bayes' rule:

$$P(+d \mid +a) = \frac{P(+a \mid +d) \times P(+d)}{P(+a)}$$

The denominator $P(+a)$ can be found using the law of total probability:

$$P(+a) = P(+a, +d) + P(+a, -d) = P(+a \mid +d) \times P(+d) + P(+a \mid -d) \times P(-d)$$

Step-by-step calculation:

1. Calculate $P(+a, +d) = P(+a \mid +d) \times P(+d)$ (from question 9)
2. Calculate $P(+a, -d) = P(+a \mid -d) \times P(-d)$ (from question 10)
3. Calculate $P(+a) = P(+a, +d) + P(+a, -d)$
4. Apply Bayes' rule: $P(+d \mid +a) = \frac{P(+a,+d)}{P(+a)}$

This gives the posterior probability of having the disease after observing a positive test result.

4. What is the probability of having disease D given a positive result on test B? $P(+d| + b) =$

Solution: Similar to the previous question, we use Bayes' rule to find $P(+d \mid +b)$:

$$P(+d \mid +b) = \frac{P(+b \mid +d) \times P(+d)}{P(+b)}$$

The denominator $P(+b)$ is calculated using the law of total probability:

$$P(+b) = P(+b, +d) + P(+b, -d) = P(+b \mid +d) \times P(+d) + P(+b \mid -d) \times P(-d)$$

Step-by-step calculation:

1. Calculate $P(+b, +d) = P(+b \mid +d) \times P(+d)$ using the CPT values
2. Calculate $P(+b, -d) = P(+b \mid -d) \times P(-d) = P(+b \mid -d) \times (1 - P(+d))$
3. Calculate $P(+b) = P(+b, +d) + P(+b, -d)$
4. Apply Bayes' rule: $P(+d \mid +b) = \frac{P(+b,+d)}{P(+b)}$

This gives the posterior probability of having the disease after observing a positive test B result. Note that this may differ from $P(+d \mid +a)$ if tests A and B have different sensitivity and specificity values.