

Homework 4

Instructions

Points: Please see the points for each problem.

Submission: Submit completed homework as a PDF file. Handwritten work or photos of handwritten work must be neat and legible.

Points Summary

Question Number	Points Possible	Points Earned
1	4	_____
2	3	_____
3	3	_____
Total	10	_____

Consider a Markov Model with a binary state $X_t \in \{0, 1\}$. The transition probabilities are given as follows:

X_t	X_{t+1}	$P(X_{t+1} X_t)$
0	0	0.9
0	1	0.1
1	0	0.5
1	1	0.5

1. The prior belief distribution over the initial state X_0 is uniform: $P(X_0 = 1) = P(X_0 = 0) = 0.5$. After one timestep, what is the new belief distribution $P(X_1)$?

Solution: Answer: $P(X_1 = 0) = 0.7, P(X_1 = 1) = 0.3$.

Using the law of total probability:

$$P(X_1 = x) = \sum_{x_0 \in \{0,1\}} P(X_1 = x | X_0 = x_0)P(X_0 = x_0)$$

Given $P(X_0 = 0) = 0.5$ and $P(X_0 = 1) = 0.5$:

$$\begin{aligned} P(X_1 = 0) &= P(X_1 = 0 | X_0 = 0)P(X_0 = 0) + P(X_1 = 0 | X_0 = 1)P(X_0 = 1) \\ &= 0.9(0.5) + 0.5(0.5) \\ &= 0.45 + 0.25 = 0.7 \end{aligned}$$

Since probabilities sum to 1, $P(X_1 = 1) = 1 - 0.7 = 0.3$.

Now, we incorporate sensor readings. The sensor model is parameterized by a number $\beta \in [0, 1]$:

X_t	E_t	$P(E_t X_t)$
0	0	β
0	1	$1 - \beta$
1	0	$1 - \beta$
1	1	β

- At $t = 1$, we get the first sensor reading, $E_1 = 0$. Use your answer from Question 1 to compute $P(X_1 = 0 | E_1 = 0)$. Leave your answer in terms of β .

Solution: Answer: $\frac{0.7\beta}{0.4\beta+0.3}$

Using Bayes' Rule:

$$P(X_1 = 0 | E_1 = 0) = \frac{P(E_1 = 0 | X_1 = 0)P(X_1 = 0)}{P(E_1 = 0)}$$

From Question 1, $P(X_1 = 0) = 0.7$ and $P(X_1 = 1) = 0.3$. From the sensor model:

- $P(E_1 = 0 | X_1 = 0) = \beta$
- $P(E_1 = 0 | X_1 = 1) = 1 - \beta$

Calculate the denominator (evidence):

$$\begin{aligned} P(E_1 = 0) &= P(E_1 = 0 | X_1 = 0)P(X_1 = 0) + P(E_1 = 0 | X_1 = 1)P(X_1 = 1) \\ &= \beta(0.7) + (1 - \beta)(0.3) \\ &= 0.7\beta + 0.3 - 0.3\beta = 0.4\beta + 0.3 \end{aligned}$$

Substitute back into Bayes' Rule:

$$P(X_1 = 0 | E_1 = 0) = \frac{0.7\beta}{0.4\beta + 0.3}$$

- For what range of values of β will a sensor reading $E_1 = 0$ increase our belief that $X_1 = 0$? That is, what is the range of β for which $P(X_1 = 0 | E_1 = 0) > P(X_1 = 0)$?

Solution: Answer: $\beta \in (0.5, 1]$

We want to find β such that:

$$P(X_1 = 0 | E_1 = 0) > P(X_1 = 0)$$

Using the result from Question 2 and $P(X_1 = 0) = 0.7$:

$$\frac{0.7\beta}{0.4\beta + 0.3} > 0.7$$

Since probabilities and β are non-negative, the denominator $0.4\beta + 0.3$ is positive. We can multiply both sides:

$$0.7\beta > 0.7(0.4\beta + 0.3)$$

$$\beta > 0.4\beta + 0.3 \quad (\text{divide by } 0.7)$$

$$0.6\beta > 0.3$$

$$\beta > 0.5$$

Since β is a probability, the range is $\beta \in (0.5, 1]$.