LQR And LQG Controllers for A Crane

ENPM 667

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Contents

Introduction

The project centers on the control of a crane moving along a one-dimensional track, functioning as a frictionless cart with mass M. The system is actuated by an external force F, serving as the input to the system.

Two loads, denoted as m_1 and m_2, are suspended from cables attached to the crane. Each load is associated with specific cable lengths, l_1 and l_2, contributing to the complexity of the system dynamics.

The initial focus involves utilizing the Lagrangian method to derive the equations of motion for the entire system. This step leads to the formulation of the non-linear state space representation of the crane and load system.

To facilitate control design, the system undergoes linearization around an equilibrium point. Controllability conditions are then established based on the system parameters M, m_1, m_2, l_1 and l_2, providing essential criteria for subsequent control strategy development.

With controllability confirmed, an LQR (Linear Quadratic Regulator) controller is designed for the crane and load system. Simulation responses are recorded for both the original non-linear system and the linearized system. Further stability analysis is performed through Lyapunov analysis, ensuring the efficacy of the designed controller.

Continuing the project, the focus shifts to enhancing system observability by designing Luenberger observers tailored for each observable output vector. These observers aim to estimate internal states based on available output information, ensuring a comprehensive understanding of the system's behavior. Simulations evaluate the Luenberger observers' responses to initial conditions and unit step inputs for both the linearized and original nonlinear systems, providing crucial insights into their performance under various scenarios.

In the subsequent phase, an output feedback controller is crafted using the LQG method, emphasizing efficiency in utilizing the "smallest" output vector for control. The designed controller is then applied to the original nonlinear system, and its performance is rigorously examined through simulation. Exploring reconfiguration possibilities, considerations for achieving asymptotic tracking of a constant reference on the x-axis are undertaken. Furthermore, the controller's robustness is assessed against constant force disturbances on the cart, ensuring its effectiveness in real-world applications. This holistic approach aims to provide a well-rounded understanding of the control strategies and their applicability to the dynamic crane and load system.

A . Equations of Motion

Position of mass m_1 as a function of θ_1

$$x_{m1} = (x - l_1 sin\theta_1)\,\hat{\imath} + (-l_1 \cos\theta_1)\hat{\jmath} \quad (1)$$

Position of mass m_2 as a function of θ_2

$$x_{m2} = (x - l_2 sin\theta_2)\hat{\imath} + (-l_2 cos\theta_2)\hat{\jmath} \quad (2)$$

Differentiating the position 1 w.r.t. time to obtain the velocity equation

$$v_{m1} = (\dot{x} - l_1 \cos(\theta_1)\dot{\theta}_1)\hat{\imath} + \big(l_1 \sin(\theta_1)\,\dot{\theta}_1\big)\hat{\jmath} \quad (3)$$

Differentiating the position 2 w.r.t. time to obtain the velocity equation

$$v_{m2} = (\dot{x} - l_2 \cos(\theta_2)\dot{\theta}_2)\hat{\imath} + \big(l_2 \sin(\theta_2)\,\dot{\theta}_2\big)\hat{\jmath} \quad (4)$$

Deriving the Kinetic Energy of the system using v_{m1} and v_{m2}

$$KE = \frac{1}{2}Mx^2 + \frac{1}{2}m_1(\dot{x} - l_1\dot{\theta}_1 \cos(\theta_1))^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1(\sin(\theta_1)))^2$$
$$+\frac{1}{2}m_2\big(\dot{x} - \dot{\theta}_2 l_2 \cos(\theta_2)\big)^2 + \frac{1}{2}m_2\left(l_2\dot{\theta}_2(\sin(\theta_2))\right)^2 \quad (5)$$

Potential Energy of the system

$$P = -m_1 g l_1 \cos\theta_1 - m_2 g l_2 \cos(\theta_2) = -g[m_1 l_1 cos\theta_1 + m_2 g l_2 \cos\theta_2] \quad (6)$$

Finding the Lagrange Equation

$$L = KE - PE \quad (7)$$

$$L = \frac{1}{2}Mx^2 + \frac{1}{2}m_1(\dot{x} - l_1\dot{\theta}_1 \cos(\theta_1))^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1(\sin(\theta_1)))^2 + \frac{1}{2}m_2\big(\dot{x} -$$
$$\theta_2 l_2 \cos(\theta_2)\big)^2 + \left(l_2\dot{\theta}_2(\sin(\theta_2))\right)^2 - g[m_1 l_1 cos\theta_1 - m_2 l_2 cos\theta_2] \quad (8)$$

Simplifying the equation

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - \dot{x}(m_1 l_1\dot{\theta}_1 \cos(\theta_1) +$$
$$m_2 l_2\dot{\theta}_2 \cos(\theta_2) + g[m_1 l_1 \cos(\theta_1) + m_2 l_2 \cos(\theta_2) \quad (9)$$

Lyapunov Equations related to the state variables considered for the system are as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F \quad (10)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \quad (11)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \quad (12)$$

The computation of the above stated equations gives us the following relations

Relation 1:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F \tag{13}$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + (m_1 + m_2)\dot{x} - m_1 l_1 \dot{\theta}_1 \cos(\theta_1) - m_2 l_2 \dot{\theta}_2 \cos(\theta_2) \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + (m_1 + m_2)\ddot{x} - \left[m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) - m_1 l_1 \dot{\theta}^2 \sin(\theta_1)\right] \tag{15}$$

$$-[m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)] \tag{16}$$

Here, $\frac{\partial L}{\partial x} = 0$

$$[M + m_1 + m_2]\ddot{x} - m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) + m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1) - m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2) = F \tag{17}$$

As we know that,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \tag{18}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 \dot{x} l_1 \cos(\theta_1) \tag{19}$$

Differentiating this equation w.r.t

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 - \left[m_1 l_1 \ddot{x} xos(\theta_1) - m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1)\right] \tag{20}$$

Also here,

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 \dot{\theta}_1 \dot{x} \sin(\theta_1) - m_1 l_1 g sin(\theta_1) \tag{21}$$

Combining these equations

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) - m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) + m_1 l_1 g sin(\theta_1) = 0 \tag{22}$$

Finding the third equation from the Lagrange equation, we perform the following calculations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \tag{23}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 \dot{x}_2 l_2 \cos(\theta_2) \tag{24}$$

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - [m_2 \ddot{x}_2 l_2 \cos(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2)] \tag{25}$$

$$\left(\frac{\partial L}{\partial \theta_2}\right) = m_2 \dot{x}_2 l_2 \dot{\theta}_2 \sin(\theta_2) - m_2 l_2 g sin(\theta_2) \tag{26}$$

Hence, we can write

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos(\theta_2) + m_2 \theta_2 \dot{x} l_2 \sin(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) + m_2 g l_2 \sin(\theta_2) = 0 \tag{27}$$

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} \cos(\theta_2) + m_2 g l_2 \sin(\theta_2) = 0 \tag{28}$$

Before linearizing about the given equilibrium points, we write the equations for the double differentiation components of some of our state variables

$$\ddot{x} = \frac{1}{M+m_1+m_2}\left[m_1 l_1 \ddot{\theta}_1\ cos\theta_1 + m_2 l_2 \ddot{\theta}_2 cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 sin\theta_2 + F\right] \tag{29}$$

$$\ddot{\theta}_1 = \frac{\ddot{x} cos\theta_1}{l_1} - \frac{g sin\theta_1}{l_1} \tag{30}$$

$$\ddot{\theta}_2 = \frac{\ddot{x} cos\theta_2}{l_2} - \frac{g si\quad_2}{l_2} \tag{31}$$

These equations can be wriiten in the matix form as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{M+m_1+m_2-m_1\cos^2\theta_1 - m_2\cos^2\theta_2} \\ \dot{\theta}_1 \\ \frac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{(M+m_1+m_2-m_1\cos^2\theta_1 - m_2\cos^2\theta_2) l_1} - \frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{(M+m_1+m_2-m_1\cos^2\theta_1 - m_2\cos^2\theta_2) l_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix} \tag{32}$$

B . Non-Linear State Space Representation

Finding the linear approximation to a function at a given point is Linearization.

To approach the system around the equilibrium point specified by $x = 0$ and $\theta_1 = 0$, we assume the following conditions

$$\sin\theta_1 \approx \theta_1$$
$$\sin\theta_2 \approx \theta_2$$
$$\cos\theta_1 \approx 1$$
$$\cos\theta_2 \approx 1$$
$$\dot{\theta}_1^2 = \dot{\theta}_2^2 \approx 0$$

Using the mentioned assumptions ad the taking the equations from (29) (30) and (31). We get the following equations

$$\ddot{x} = \frac{1}{M+m1+m2}[\mathrm{m}_1\mathrm{l}_1\ddot{\theta}_1.1 + \mathrm{m}_2\mathrm{l}_2\,\ddot{\theta}_2 - 0 - 0 + F] \tag{33}$$

$$\ddot{\theta}_1 = \frac{\ddot{x}.1}{l_1} - \frac{g\theta_1}{l_1} \tag{34}$$

$$\ddot{\theta}_2 = \frac{\ddot{x}.1}{l_2} - \frac{g\theta_2}{l_2} \tag{35}$$

The state space form of the linearized system, considering the system states.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-m_1g\theta_1 - m_2g\theta_2 + m_1\ddot{x} + m_2\ddot{x} + F}{M+m_1+m_2} \\ \dot{\theta}_1 \\ \frac{-m_1g\theta_1 - m_2g\theta_2 + F}{Ml_1} - \frac{g\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{-m_1g\theta_1 - m_2g\theta_2 + F}{Ml_2} - \frac{g\theta_2}{l_2} \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g(M+m1)}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{g(M+m2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F \tag{37}$$

C. Controllability Conditions for M, m_1, m_2, l_1, l_2

We need to check if the system is controllable. The time-invariant linear state equation is said to be controllable if and only if the Grammian Matrix of controllability matrix satisfies

$$Rank[A_c] = n, where\ A_c = [B\ AB\ A^2B \dots\ A^{\{n-1\}}B]$$

for the above equation, n=6

```
%Variable symbols
syms M m1 m2 l1 l2 g;
% State space representation of the linearised model
A=[0 1 0 0 0 0;
    0 0 -(m1*g)/M 0 -(m2*g)/M 0;
    0 0 0 1 0 0;
    0 0 -((M+m1)*g)/(M*l1) 0 -(m2*g)/(M*l1) 0;
    0 0 0 0 0 1;
    0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];
disp(A);
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \end{pmatrix}$$

```
B=[0; 1/M; 0; 1/(M*(l1)); 0; 1/(M*l2)];
disp(B);
```

$$\begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M\,l_1} \\ 0 \\ \frac{1}{M\,l_2} \end{pmatrix}$$

```
% The Controllability matrix
C = [B A*B A*A*B A*A*A*B A*A*A*A*B A*A*A*A*A*B]
disp(C);
```

$$\begin{pmatrix} 0 & \frac{1}{M} & 0 & \sigma_2 & 0 & \sigma_1 \\ \frac{1}{M} & 0 & \sigma_2 & 0 & \sigma_1 & 0 \\ 0 & \frac{1}{M\,l_1} & 0 & \sigma_6 & 0 & \sigma_4 \\ \frac{1}{M\,l_1} & 0 & \sigma_6 & 0 & \sigma_4 & 0 \\ 0 & \frac{1}{M\,l_2} & 0 & \sigma_5 & 0 & \sigma_3 \\ \frac{1}{M\,l_2} & 0 & \sigma_5 & 0 & \sigma_3 & 0 \end{pmatrix}$$

where,

$$\sigma 1 = \frac{\left(\frac{g^2 m_1 (M+m_1)}{M^2\, l_1}\right) + \left(\frac{g^2\, m_1 (m_2)}{M^2\, l_2}\right)}{M l_1} + \frac{\left(\frac{g^2 m_2 (M+m_2)}{M^2\, l_2}\right) + \left(\frac{g^2\, m_1 (m_2)}{M^2\, l_1}\right)}{M l_2}$$

$$\sigma 2 = -\frac{g m_1}{M^2\, l_1} - \frac{g m_2}{M^2\, l_2}$$

$$\sigma 3 = \frac{\left(\frac{g^2 m_1 (M+m_2)}{M^2\, l_2^2}\right) + \left(\frac{g^2\, m_1 (M+m_1)}{k7}\right)}{M l_2} + \frac{\left(\frac{g^2 (M+m_2)^2}{M^2\, l_2^2}\right) + \left(\frac{g^2\, m_1 (m_2)}{k7}\right)}{M l_1}$$

$$\sigma 4 = \frac{\left(\frac{g^2 m_1 (M+m_1)}{M^2\, l_1^2}\right) + \left(\frac{g^2\, m_1 (M+m_2)}{k7}\right)}{M l_2} + \frac{\left(\frac{g^2 m_2 (M+m_1)^2}{M^2\, l_1^2}\right) + \left(\frac{g^2\, m_1 (m_2)}{k7}\right)}{M l_1}$$

$$\sigma 5 = -\frac{g (M+m_2)}{M^2\, l_2^2} - \frac{g m_1}{k7}$$

$$\sigma 6 = -\frac{g (M+m_1)}{M^2\, l_1} - \frac{g m_2}{k7}$$

$$\sigma 7 = M^2 l_1 l_2$$

```
% Rank of Controlability Matrix
Rank = rank(C);
disp(Rank)
```

6

```
% Determinant of Controlability Matrix
disp(simplify(det(C)));
```

$$-\frac{g^6\,(l_1-l_2)^2}{M^6\,l_1^6\,l_2^6}$$

Controllability of the system is determined by examining the full rank of its controllability matrix, assessed through the non-zero nature of its determinant. A non-zero determinant indicates a fully ranked matrix, affirming the system's controllability. In the first case, when $l1 \neq l2$, the matrix achieves full rank, ensuring controllability. However, if $l1$ equals $l2$, the determinant becomes zero, rendering the system uncontrollable.

In the second case, a determinant of zero arises when either $l1 = l2 = 0$ or $l1 = l2$, leading to system uncontrollability. Thus, the conditions for system controllability are established as $l1 \neq l2$, $m1 > 0$, $m2 > 0$, and $M > 0$.

D. Controllability & LQR Controller Design

We have been given the following parameter constants:

$$M = 1000kg, \qquad m_1 = m_2 = 100kg, \qquad l_1 = 20m, \qquad l_2 = 10m$$

Based on the system model above, we have to check if the system is controllable. We can do that using the controllability matrix as shown below:

$$rank([B \;\; AB \;\; A^2B \;\; A^3B \;\; A^4B \;\; A^5B])$$

```
A_val = double(subs(A, {M, m1, m2, l1, l2, g}, {1000, 100, 100, 20, 10, 9.81}));
disp(A_val);
```

```
         0    1.0000         0         0         0         0
         0         0   -0.9810         0   -0.9810         0
         0         0         0    1.0000         0         0
         0         0   -0.5395         0   -0.0491         0
         0         0         0         0         0    1.0000
         0         0   -0.0981         0   -1.0791         0
```

```
B_val = double(subs(B, {M, l1, l2}, {1000, 20, 10}));
disp(B_val);
```

```
   1.0e-03 *

         0
    1.0000
         0
    0.0500
         0
    0.1000
```

```
Control_val = ctrb(A_val, B_val);
disp(Control_val);
```

```
   1.0e-03 *

         0    1.0000         0   -0.1472         0    0.1419
    1.0000         0   -0.1472         0    0.1419         0
         0    0.0500         0   -0.0319         0    0.0227
    0.0500         0   -0.0319         0    0.0227         0
         0    0.1000         0   -0.1128         0    0.1249
    0.1000         0   -0.1128         0    0.1249         0
```

```
if rank(Control_val) == 6
    disp('System is controllable.');
else
    disp('System is not controllable.');
end
```

```
System is controllable.
```

- **We can see that the linearized system is controllable for the given parameter values.** Now we have to design an LQR controller for the above system and simulate the state response for an assumed initial condition. The LQR controller will minimize the below mentioned cost function:

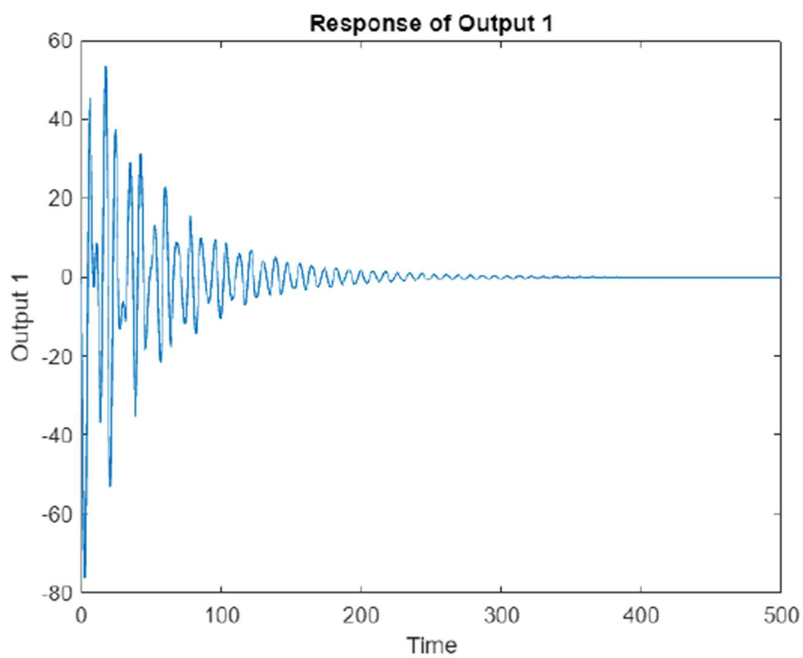
$$J(k,\vec{X}(0) = \int_0^{\infty} \overrightarrow{X^T}(t)Q\vec{X}(t) + \overrightarrow{U^T{}_k}(t)R\overrightarrow{U_k}(t)\,dt$$

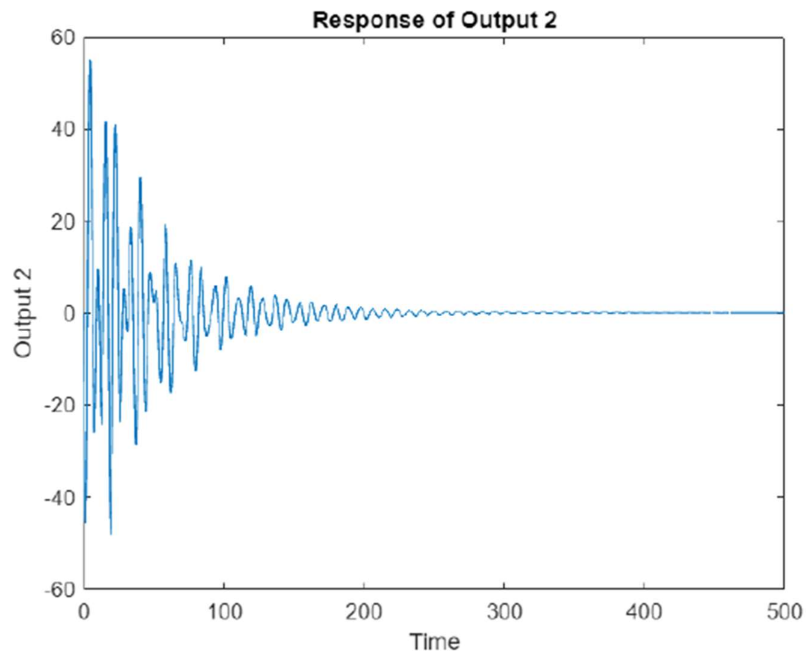
- We will be assuming the values of Q and R so that the LQR controller can provide the best performance. First, we will be simulating the responses for the linearized system.

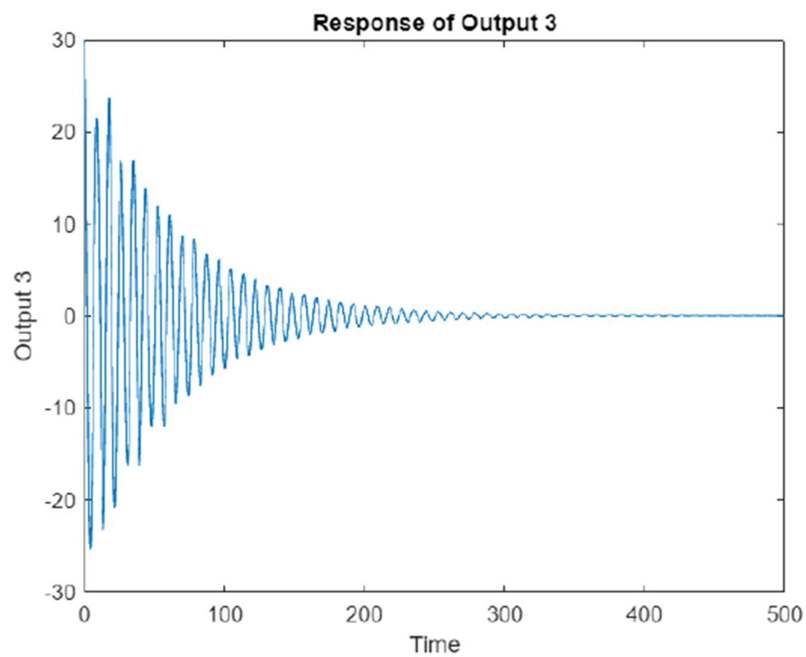
```
Q = 500*eye(6);
R = 0.005;

[K, P, lamda] = lqr(sys1, Q, R);

disp(K)
```

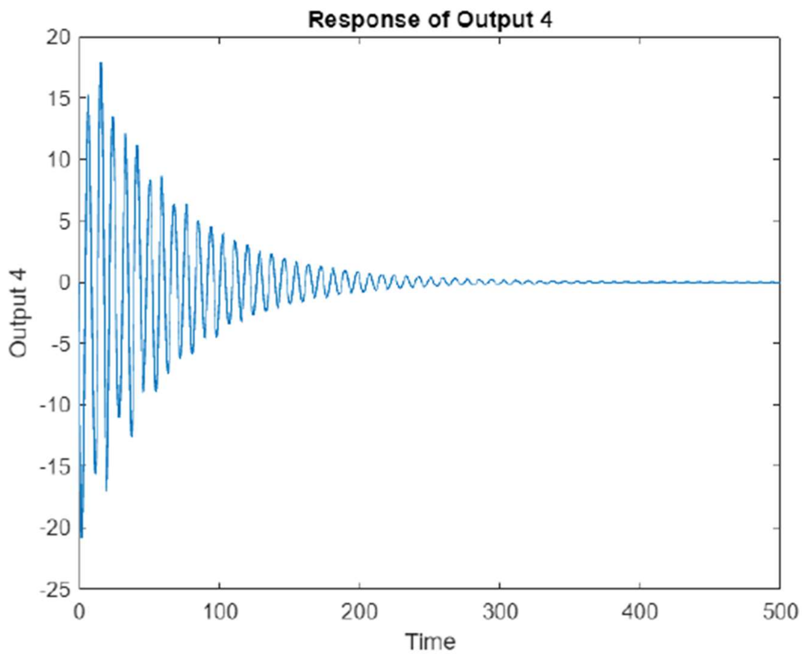
```
  316.2278  926.8775  -41.7027 -683.4756   44.1102 -334.2650
```

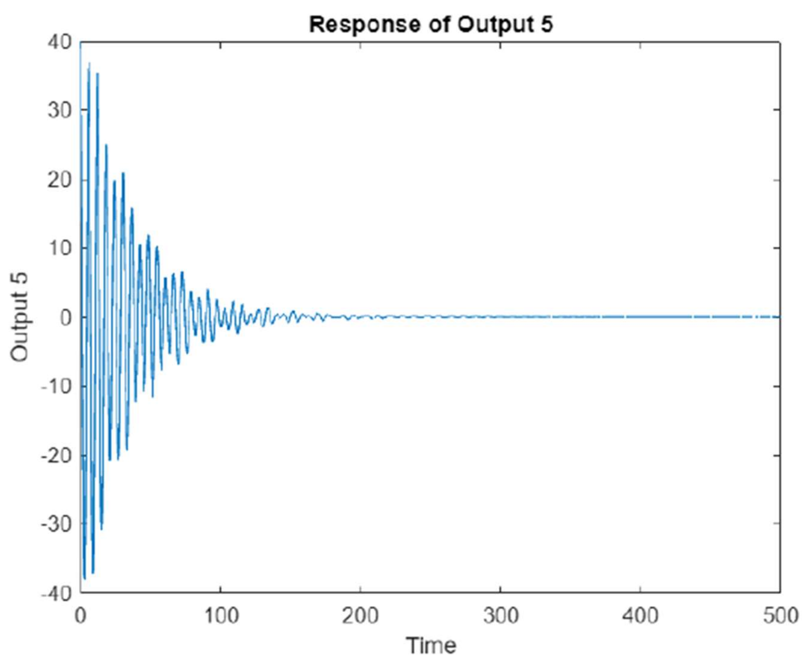
```
sys2 = ss((A_val-B_val*K), B_val, C, D);

[y, t] = initial(sys2, X_0, t);

for i = 1:size(y, 2)
    figure;
    plot(t, y(:, i));
    title(['Response of Output ', num2str(i)]);
    xlabel('Time');
    ylabel(['Output ', num2str(i)]);
end
```

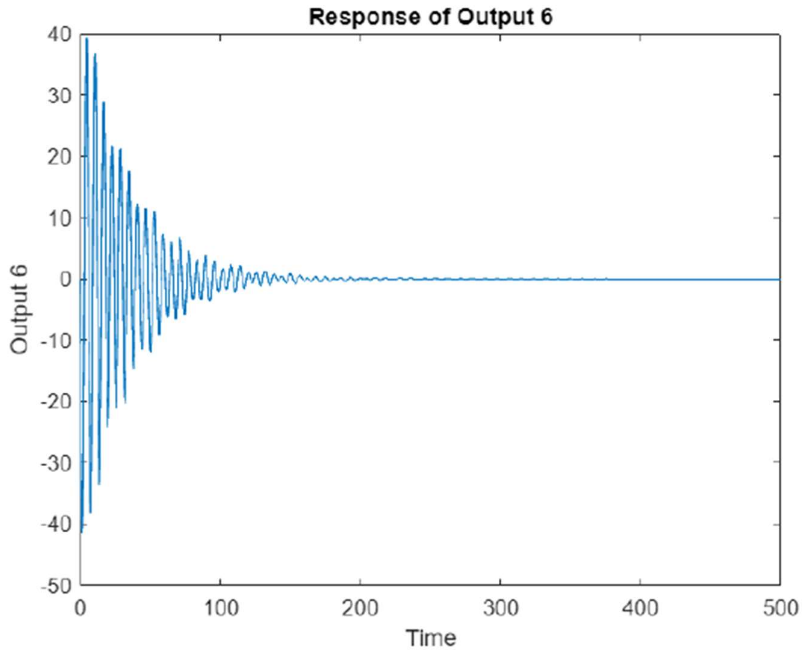



Response of Output 2
Output 2
Time
60
40
20
0
-20
-40
-60
0
100
200
300
400
500


Response of Output 3
Output 3
Time
30
20
10
0
-10
-20
-30
0
100
200
300
400
500

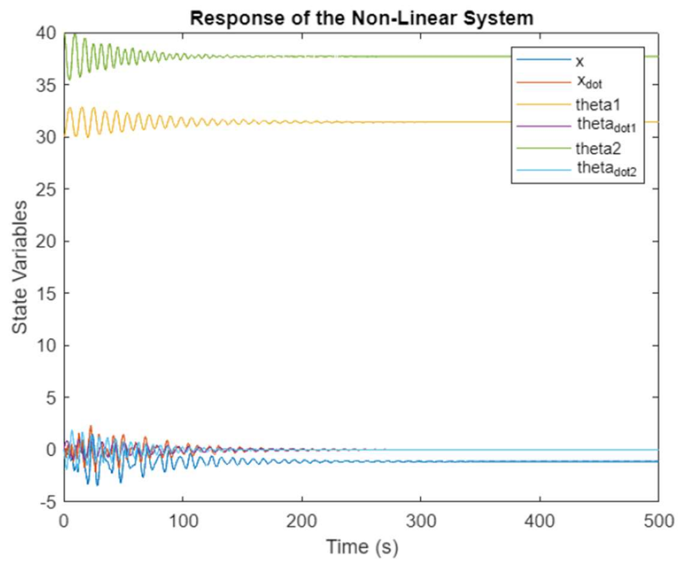

Response of Output 4
Output 4
Time
20
15
10
5
0
-5
-10
-15
-20
-25
0
100
200
300
400
500


Response of Output 5
Output 5
Time
40
30
20
10
0
-10
-20
-30
-40
0
100
200
300
400
500



- Next, we will simulate the state responses for the non-linear system as shown below:

```
% Initial conditions
x0 = [0; 0; 30; 0; 40; 0]; % [x; x_dot; theta1; theta_dot_1; theta2; theta_dot_2]

% Time span
tspan = [0 500]; % For example, from 0 to 10 seconds

[t, states] = ode45(@systemDynamics, tspan, x0);

% Plotting states
plot(t, states);
legend('x', 'x_dot', 'theta1', 'theta_dot_1', 'theta2', 'theta_dot_2');
xlabel('Time (s)');
ylabel('State Variables');
title('Response of the Non-Linear System');
```




```
function dxdt = systemDynamics(~, x)
    % Declaring the constants
    M = 1000; % Mass of cart
    m1 = 100; % Mass of pendulum 1
    m2 = 100; % Mass of pendulum 2
    l1 = 20; % Length of pendulum 1
    l2 = 10; % Length of pendulum 2
    g = 9.81; % Acceleration due to gravity
    K = [316.2278, 926.8775, -41.7027, -683.4756, 44.1102, -334.2650];
    F = -K*x; % External Force (control input from LQR)

    % State variables
    x_dot = x(2);
    theta1 = x(3);
    theta_dot_1 = x(4);
    theta2 = x(5);
    theta_dot_2 = x(6);

    % State equations
    f1 = x_dot;
    f2 = (-m1*g*sin(theta1)*cos(theta1)-m2*g*sin(theta2)*cos(theta2)-
m1*l1*(theta_dot_1^2)*sin(theta1)-m2*l2*(theta_dot_2^2)*sin(theta2)+F)/(M+m1+m2-
m1*((cos(theta1))^2)-m2*((cos(theta2))^2));
    f3 = theta_dot_1;
    f4 = (-m1*g*sin(theta1)*cos(theta1)-m2*g*sin(theta2)*cos(theta2)-
m1*l1*(theta_dot_1^2)*sin(theta1)-m2*l2*(theta_dot_2^2)*sin(theta2)+F)/((M+m1+m2-
m1*((cos(theta1))^2)-m2*((cos(theta2))^2))*l1)-(g*sin(theta1))/l1;
    f5 = theta_dot_2;
    f6 = (-m1*g*sin(theta1)*cos(theta1)-m2*g*sin(theta2)*cos(theta2)-
m1*l1*(theta_dot_1^2)*sin(theta1)-m2*l2*(theta_dot_2^2)*sin(theta2)+F)/((M+m1+m2-
m1*((cos(theta1))^2)-m2*((cos(theta2))^2))*l2)-(g*sin(theta2))/l2;

    % Return the derivatives as a vector
    dxdt = [f1; f2; f3; f4; f5; f6];
end
```

- Finally, we use the Lyapunov's Indirect Method to check the stability of the non-linear system about the given equilibrium point ($x = 0,\ \theta_1 = 0,\ \theta_2 = 0$).

```
% Linearized A & B matrices
A = [0, 1, 0, 0, 0, 0; 0, 0, -(m1*g)/M, 0, -(m2*g)/M, 0; 0, 0, 0, 1, 0, 0; 0, 0,
-((M+m1)*g)/(M*l1), 0, -(m2*g)/(M*l1), 0; 0, 0, 0, 0, 0, 1; 0, 0, -(m1*g)/(M*l2),
0, -(g*(M+m2))/(M*l2), 0];
B = [0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];

% Eigenvalues of closed-loop system
poles = eig(A - B*K);
disp(poles)
```

```
  -0.3838 + 0.3543i
  -0.3838 - 0.3543i
  -0.0298 + 1.0347i
  -0.0298 - 1.0347i
  -0.0161 + 0.7213i
  -0.0161 - 0.7213i
```

- **We can see from the above result that all the poles of the system linearized at the equilibrium point have negative real parts. Hence the system is locally stable at the equilibrium point.**

E. Observability

- We have been given the following outputs:

  Output-1: $x(t)$

  Output-2: $\theta_1(t), \theta_2(t)$

  Output-3: $x(t), \theta_2(t)$

  Output-4: $x(t), \theta_1(t), \theta_2(t)$

- We now have to check for which of these outputs, the system is controllable. To check that, we use the controllability matrix as follows:

$$rank([C;\ CA;\ CA^2;\ CA^3;\ CA^4;\ CA^5])$$

```
% Declaring the constants
M = 1000; % Mass of cart
m1 = 100; % Mass of pendulum 1
m2 = 100; % Mass of pendulum 2
l1 = 20; % Length of pendulum 1
l2 = 10; % Length of pendulum 2
g = 9.81; % Acceleration due to gravity

% Linearized A matrix
A = [0, 1, 0, 0, 0, 0; 0, 0, -(m1*g)/M, 0, -(m2*g)/M, 0; 0, 0, 0, 1, 0, 0; 0, 0,
-((M+m1)*g)/(M*l1), 0, -(m2*g)/(M*l1), 0; 0, 0, 0, 0, 0, 1; 0, 0, -(m1*g)/(M*l2),
0, -(g*(M+m2))/(M*l2), 0];

% C matrices for each case
C1 = [1 0 0 0 0 0];
C2 = [0 0 1 0 0 0; 0 0 0 0 1 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

% Observability matrices for each case
O1 = [C1; C1*A; C1*(A^2); C1*(A^3); C1*(A^4); C1*(A^5)];
O2 = [C2; C2*A; C2*(A^2); C2*(A^3); C2*(A^4); C2*(A^5)];
O3 = [C3; C3*A; C3*(A^2); C3*(A^3); C3*(A^4); C3*(A^5)];
O4 = [C4; C4*A; C4*(A^2); C4*(A^3); C4*(A^4); C4*(A^5)];

% Rank check for observability
% Output 1
if rank(O1) == 6
    disp('System is observable for Output 1.');
else
    disp('System is not observable for Output 1.');
end
```

```
System is observable for Output 1.
```

```
% Output 2
if rank(O2) == 6
    disp('System is observable for Output 2.');
else
    disp('System is not observable for Output 2.');
end
```

```
System is not observable for Output 2.
```

```
% Output 3
if rank(O3) == 6
    disp('System is observable for Output 3.');
else
    disp('System is not observable for Output 3.');
end
```

```
System is observable for Output 3.
```

```
% Output 4
if rank(O4) == 6
    disp('System is observable for Output 4.');
else
    disp('System is not observable for Output 4.');
end
```

```
System is observable for Output 4.
```

- We can see that Outputs: 1, 3 & 4 are observable.

F. Luenberger Observer

- We now have to design the Luenberger observer for each of the observable outputs that we derived previously and simulate their state responses.

```
% poles of (A - BK)
closed_loop_poles = [-0.3838 + 0.3543i; -0.3838 - 0.3543i; -0.0298 + 1.0347i;
-0.0298 - 1.0347i; -0.0161 + 0.7213i; -0.0161 - 0.7213i];

% assumed factor between the closed-loop and observer poles
scale_factor = 5;

% desired poles for the Luenberger Observer
observer_poles = scale_factor*closed_loop_poles;

% Declaring the constants
M = 1000; % Mass of cart
m1 = 100; % Mass of pendulum 1
m2 = 100; % Mass of pendulum 2
l1 = 20; % Length of pendulum 1
l2 = 10; % Length of pendulum 2
g = 9.81; % Acceleration due to gravity

K = [316.2278, 926.8775, -41.7027, -683.4756, 44.1102, -334.2650];

% Linearized A, B & D matrices
A = [0, 1, 0, 0, 0, 0; 0, 0, -(m1*g)/M, 0, -(m2*g)/M, 0; 0, 0, 0, 1, 0, 0; 0, 0,
-((M+m1)*g)/(M*l1), 0, -(m2*g)/(M*l1), 0; 0, 0, 0, 0, 0, 1; 0, 0, -(m1*g)/(M*l2),
0, -(g*(M+m2))/(M*l2), 0];
B = [0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];
D = 0;

% C matrices only for observable cases
C1 = [1 0 0 0 0 0];
C3 = [1 0 0 0 0 0; 0 0 0 0 1 0];
C4 = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];

% Observer pole placement and calculation of Observer Gain matrices
L1 = place(A', C1', observer_poles)';
disp(L1)
```

```
   1.0e+03 *

    0.0043
    0.0468
   -2.7310
   -4.2985
    2.5707
    3.7119
```

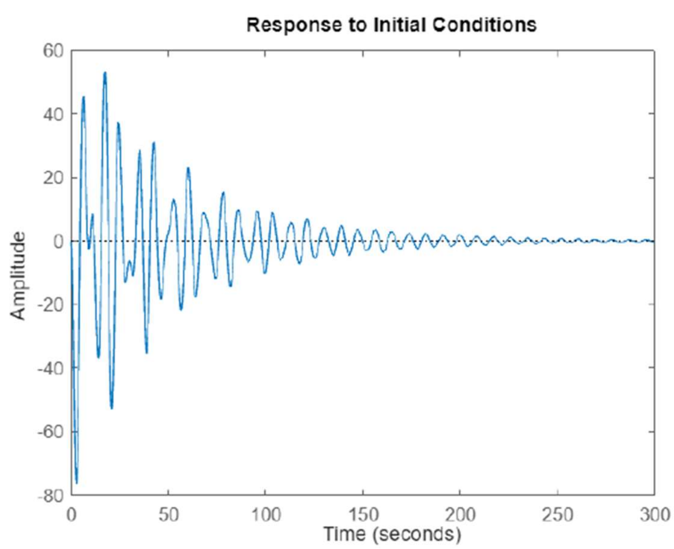
```
L3 = place(A', C3', observer_poles)';
disp(L3)
```

```
   3.9304   -1.2472
  20.8780   -3.8187
 -53.6598    2.3590
 -87.4060    4.1625

   1.1203    0.3666
   2.9518   23.0962
```

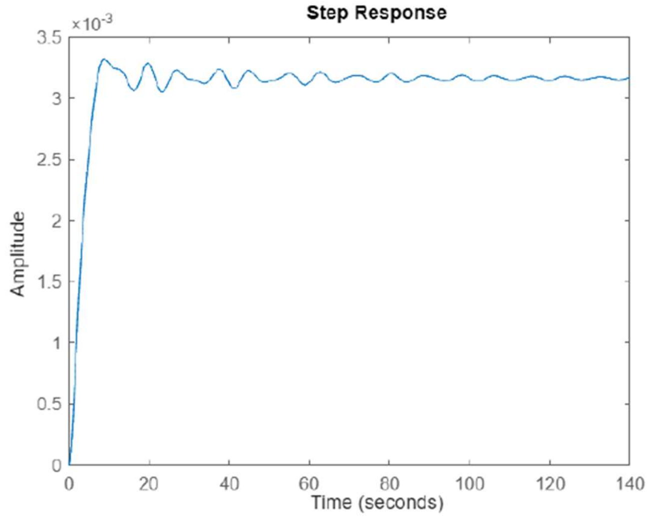
```
L4 = place(A', C4', observer_poles)';
disp(L4)
```

```
    0.2941    0.4867   -0.8944
   12.7650   -1.8668   -3.5660
    0.4826    1.9266   -3.2976
   -0.9247    9.1733   -9.3869
    0.8951    3.2668    2.0763
    2.5137    9.2072    8.3581
```

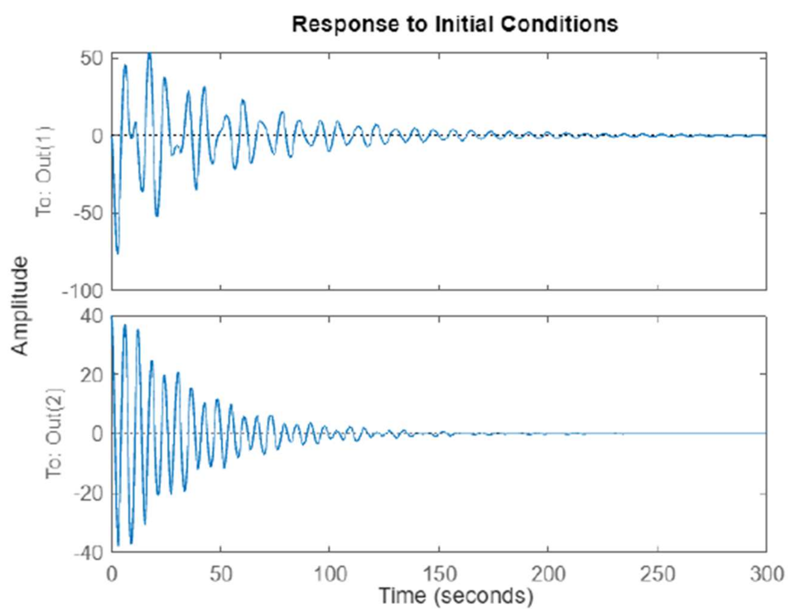
```
% designing the system matrix with controller and observer
% System 1
Ac1 = [(A-B*K), B*K; zeros(size(A)), (A-L1*C1)];
Bc = [B; zeros(size(B))];
Cc1 = [C1 zeros(size(C1))];
sys1 = ss(Ac1, Bc, Cc1, D);

% defining intial conditions
x0 = [0; 0; 30; 0; 40; 0];
X0 = [x0; 0; 0; 0; 0; 0; 0];

% Plotting response
figure
initial(sys1, X0)
```



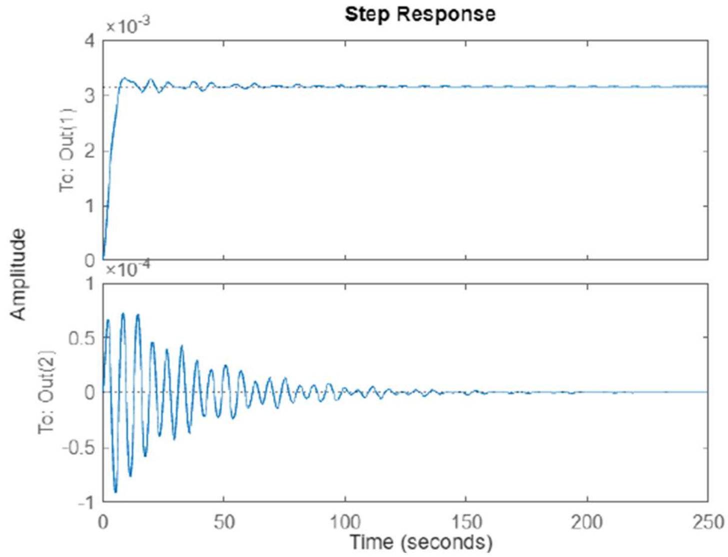
```
% Unit step response
figure
step(sys1)
```



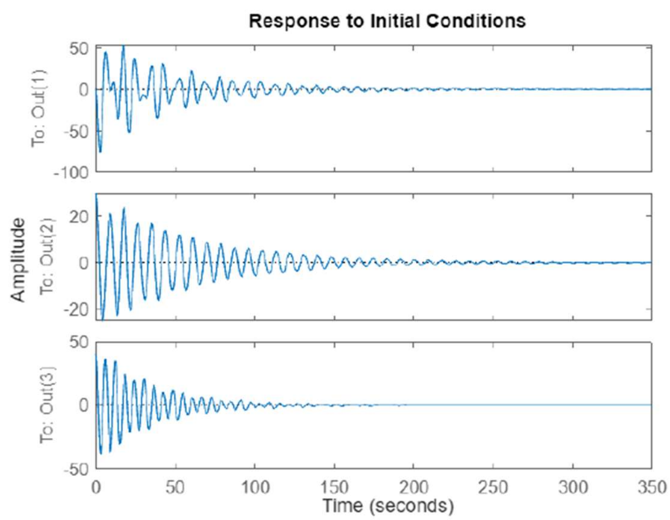

```
% System 2
Ac3 = [(A-B*K), B*K; zeros(size(A)), (A-L3*C3)];
Cc3 = [C3 zeros(size(C3))];
sys2 = ss(Ac3, Bc, Cc3, D);

% Plotting response
figure
initial(sys2, X0)
```



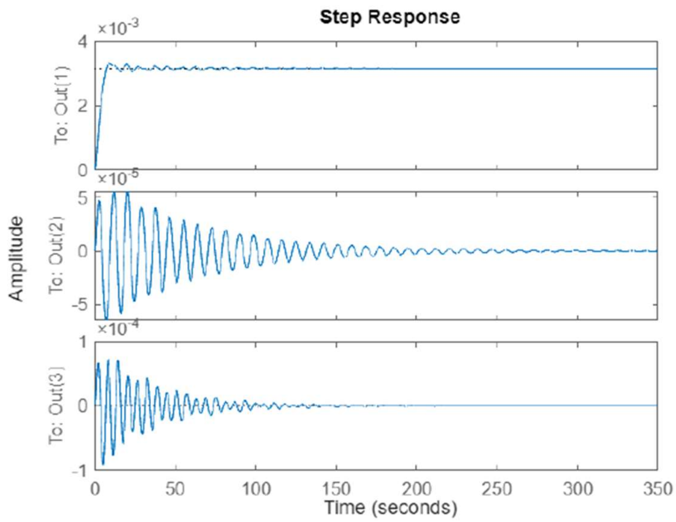

```
% Unit step response
figure
step(sys2)
```




```
% System 3
Ac4 = [(A-B*K), B*K; zeros(size(A)), (A-L4*C4)];
Cc4 = [C4 zeros(size(C4))];
sys3 = ss(Ac4, Bc, Cc4, D);

% Plotting response
figure
initial(sys3, X0)
```




```
% Unit step response
figure
step(sys3)
```



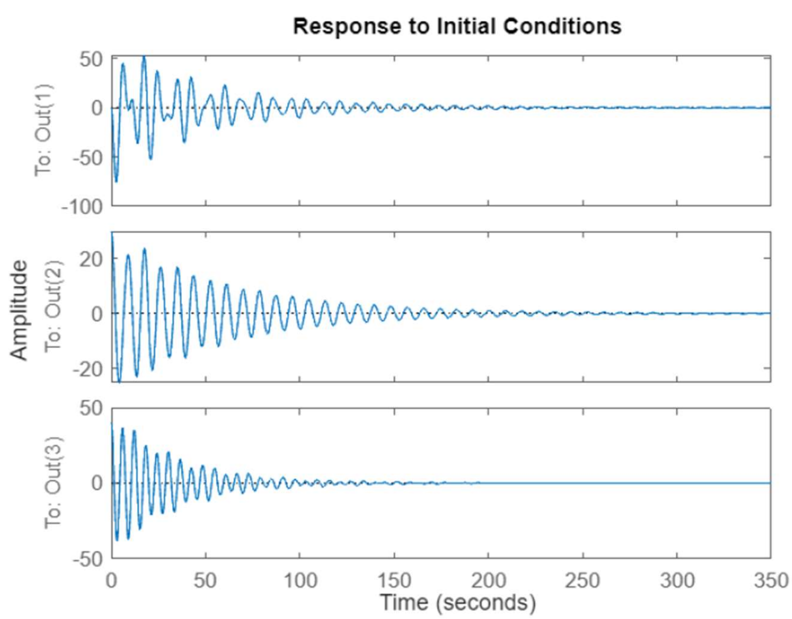
Step Response
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
Time (seconds)

G. LQG Controller Design

- For designing the LQG controller, **we have chosen the Output-3 from the previous observer design as this output is able to account for all the primary parameters:** $x(t)$, $\theta_1(t)$, $\theta_2(t)$
- The LQG controller consists of the K-Gain component derived from the LQR controller and the L-Gain component derived from the Kalman Filter. We have designed and simulated the LQG controller as below:

```
% Declaring the constants
M = 1000; % Mass of cart
m1 = 100; % Mass of pendulum 1
m2 = 100; % Mass of pendulum 2
l1 = 20; % Length of pendulum 1
l2 = 10; % Length of pendulum 2
g = 9.81; % Acceleration due to gravity

% Linearized A, B & D matrices
A = [0, 1, 0, 0, 0, 0; 0, 0, -(m1*g)/M, 0, -(m2*g)/M, 0; 0, 0, 0, 1, 0, 0; 0, 0,
-((M+m1)*g)/(M*l1), 0, -(m2*g)/(M*l1), 0; 0, 0, 0, 0, 0, 1; 0, 0, -(m1*g)/(M*l2),
0, -(g*(M+m2))/(M*l2), 0];
B = [0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)];
C = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 1 0];        % Our choice of output as it
covers x, theta1, theta2
D = 0;
```

```
% LQR cost parameters
Q = 500*eye(6);
R = 0.005;

% State space
sys = ss(A, B, C, D);

% K Gain
[K, ~, ~] = lqr(sys, Q, R);

% Noise Covariance in Kalman Filter
Ed = 1;
Ev = 1;

% Kalman Gain matrix
[~, L, ~] = kalman(sys, Ed, Ev);

% designing the system with LQR controller & Kalman Filter
% System 1
Ac = [(A-B*K), B*K; zeros(size(A)), (A-L*C)];
Bc = [B; zeros(size(B))];
Cc = [C zeros(size(C))];
sys1 = ss(Ac, Bc, Cc, D);

% initial conditions
X0 = [0; 0; 30; 0; 40; 0; 0; 0; 0; 0; 0; 0];

% plotting response
figure
initial(sys1, X0)
```



- For Optimal Reference Tracking, the cost of the controller changes to the following:

$$J(k, \vec{X}(0) = \int_0^{\infty} \left(\vec{X}(t) - \overrightarrow{X_d}(t)\right)^T Q \left(\vec{X}(t) - \overrightarrow{X_d}(t)\right) + \left(\overrightarrow{U_k}(t) - U_{\infty}(t)\right)^T R(\overrightarrow{U_k}(t) - U_{\infty}(t))\, dt$$

Therefore, we will have to modify the state variables and inputs as per the above cost function for our controller to asymptotically track a constant reference.

- The LQG controller is robust against Gaussian noises. **Therefore, our controller design can successfully handle constant force disturbances.**

- The link for the simulation videos has been provided below: https://drive.google.com/drive/folders/1kfNRXD6JaNHnlCaHx39NJaKwQwDxAbXB?usp=drive_link